SOLVING PROBLEMS BY USING A FUNCTION IN THE NUMBER THEORY

Let $n \geq 1$, $h \geq 1$, and $a \geq 2$ be integers. For which values of a and n is $(n + h)!$ a multiple of a^n ?
(A generalization of the problem $n^0 = 1270$, Mathematics Magazine, Vol. 60, No. 3, June 1987, p. 179, proposed by Roger B. Eggleton, The University of Newcastle, Australia.)

Solution

(For $h = 1$ the problem $n^0 = 1270$ is obtained.)

§1. Introduction

We have constructed a function η (see [1]) having the following properties:

(a) For each non-null integer n, $\eta(n)!$ is a multiple of n;

(b) $\eta(n)$ is the smallest natural number with the property (a).

It is easy to prove:

Lemma 1. $(\forall)$ $k, p \in N^*$, $p \neq 1$, k is uniquely written in the form:

$$k = t_1 a_{n_1}^{(p)} + \dots + t_\ell a_{n_\ell}^{(p)} ,$$

where $a_{n_i}^{(p)} = (p^{n_i} - 1) / (p - 1)$, $i = 1, 2, \dots, \ell$,

$n_1 > n_2 > \dots > n_\ell > 0$ and $1 \le t_j \le p - 1$, $j = 1, 2, \dots, \ell - 1$, $1 \le t_\ell \le p$, $n_i, t_i \in N$, $i = 1, 2, \dots, \ell$, $\ell \in N^*$.

We have constructed the function η_p, p prime > 0, $\eta_p : N^* \to N^*$, thus:

$$(\forall)\ n \in N^*,\ \eta_p\left(a_n^{(p)}\right) = p^n, \text{ and}$$

$$\eta_p\left(t_1 a_{n_1}^{(p)} + \dots + t_\ell a_{n_\ell}^{(p)}\right) =$$

$$= t_1 \eta_p\left(a_{n_1}^{(p)}\right) + \dots + t_\ell \eta_p\left(a_{n_\ell}^{(p)}\right) .$$

Of course:

Lemma 2.

(a) $(\forall)\ k \in N^*,\ \eta_p(k)! = \dot{M}p^k$.

(b) $\eta_p(k)$ is the smallest number with the property (a). Now, we construct another function:

$\eta : Z \setminus \{0\} \to N$ defined as follows:

$$\begin{cases} \eta(\pm 1) = 0, \\ (\forall)\ n = \epsilon\, p_1^{\alpha_1} \dots p_s^{\alpha_s} \text{ with } \epsilon = \pm 1,\ p_i \text{ prime and} \\ p_i \neq p_j \text{ for } i \neq j, \text{ all } \alpha_i \in N^*,\ \eta(n) = \\ = \max\limits_{1 \leq i \leq s} \{\eta_p\ (\alpha_i)\}. \end{cases}$$

It is not difficult to prove η has the demanded properties of §1.

§2. Now, let $a = p_1^{\alpha_1} \dots p_s^{\alpha_s}$, with all $\alpha_i \in N^*$ and all p_i distinct primes. By the previous theory we have:

$$\eta(a) = \max_{1 \leq i \leq s} \{n_{p_i}\ (\alpha_i)\} = \eta_p\ (\alpha) \text{ (by notation)}.$$

Hence $\quad \eta(a) = \eta(p^\alpha),\ \eta(p^\alpha)\,! = Mp^\alpha$.

We know:

$$(t_1 p^{n_1} + \dots + t_\ell\, p^{n_\ell})\,! = Mp^{t_1 \frac{p^{n_1}-1}{p-1} + \dots + t_\ell \frac{p^{n_\ell}-1}{p-1}}.$$

We put:

$$t_1\, p^{n_1} + \dots + t_\ell\, p^{n_\ell} = n + h$$

and $t_1 \dfrac{p^{n_1}-1}{p-1} + \ldots + t_\ell \dfrac{p^{n_\ell}-1}{p-1} = \alpha\, n.$

Whence

$$\frac{1}{\alpha}\left[\frac{p^{n_1}-1}{p-1} + \ldots + t_\ell \frac{p^{n_\ell}-1}{p-1} \right] \geq t_1\, p^{n_1} + \ldots + t_\ell\, p^{n_\ell} - h$$

or

$$(1) \quad \alpha\ (p - 1)\ h \geq (\alpha\ p - \alpha - 1)\ [t_1\, p^{n_1} + \ldots + t_\ell\, p^{n_\ell}] +$$

$$+ (t_1 + \ldots + t_\ell).$$

On this condition we take $n_0 = t_1\, p^{n_1} + \ldots + t_\ell\, p^{n_\ell} - h$

(see Lemma 1), hence $n = \begin{cases} n_0, & n_0 > 0; \\ 1, & n_0 \leq 0 \end{cases}$.

Consider giving $a \neq 2$, we have a finite number of n. There are an infinite number of n if and only if $\alpha\ p - \alpha - 1 = 0$, i.e., $\alpha = 1$ and $p = 2$, i.e., $a = 2$.

§3. Particular Case

If $h = 1$ and $a \neq 2$, because

$$t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} \geq p^{n_\ell} > 1$$

and $t_1 + \ldots + t_\ell \geq 1$, it follows from (1) that:

(1') $(\alpha p - \alpha) > (\alpha p - \alpha - 1) \cdot 1 + 1 = \alpha p - \alpha$,

which is impossible. If h = 1 and a = 2 then $\alpha = 1$, $p = 2$, or

(1") $1 \geq t_1 + \ldots + t_\ell$,

hence $\ell = 1$, $t_1 = 1$ whence $n = t_1 p^{n_1} + \ldots + t_\ell p^{n_\ell} - h =$

$= 2^{n_1} - 1$, $n_1 \in N^*$ (the solution to problem 1270).

Example 1. Let h = 16 and a = $3^4 \cdot 5^2$. Find all n such that

$$(n + 16)\,! = M\ 2025^n.$$

Solution

$\eta\ (2025) = \max\ \{\eta_3\ (4),\ \eta_5\ (2)\} = \max\ \{9,\ 10\} = 10 =$
$= \eta_5\ (2) = \eta\ (5^2)$. Whence $\alpha = 2$, $p = 5$. From (1) we have:

$$128 \geq 7[t_1 5^{n_1} + \ldots + t_\ell\ 5^{n_\ell}] + t_1 + \ldots + t_\ell.$$

Because $5^4 > 128$ and $7\ [t_1\ 5^{n_1} + \ldots + t_\ell\ 5^{n_\ell}] < 128$ we find $\ell = 1$,

$$128 \geq 7\, t_1\, 5^{n_1} + t_1\,,$$

whence $n_1 \leq 1$, i.e., $n_1 = 1$, and $t_1 = 1, 2, 3$. Then $n_0 = t_1\, 5 - 16 < 0$, hence we take $n = 1$.

Example 2

$(n + 7)! = \mathcal{M}\, 3^n$ when $n = 1, 2, 3, 4, 5$.

$(n + 7)! = \mathcal{M}\, 5^n$ when $n = 1$.

$(n + 7)! = \mathcal{M}\, 7^n$ when $n = 1$.

But $(n + 7)! \neq \mathcal{M}\, p^n$, for p prime > 7, $(\forall)\, n \in N^*$.

$(n + 7)! = \mathcal{M}\, 2^n$ when

$$n_0 = t_1\, 2^{n_1} + \ldots + t_\ell\, 2^{n_\ell} - 7,$$

$$t_1, \ldots, t_{\ell-1} = 1,$$

$$1 \leq t_\ell \leq 2,\ t_1 + \ldots + t_\ell \leq 7$$

and
$$n = \begin{cases} n_0, & n_0 > 0; \\ 1, & n_0 \leq 0. \end{cases}$$

etc.

Exercise for Readers

If $n \in N^*$, $a \in N^* \setminus \{1\}$, find all values of a and n such that:

$(n + 7)!$ be a multiple of a^n.

Some Unsolved Problems (see [2])

Solve the diophantine equations:

(1) $\eta(x) \cdot \eta(y) = \eta(x + y)$.

(2) $\eta(x) = y!$ (A solution: $x = 9$, $y = 3$).

(3) Conjecture: the equation $\eta(x) = \eta(x + 1)$ has no solution.

References

[1] Florentin Smarandache, "A Function in the Number Theory," Analele Univ. Timisoara, Fasc. 1, Vol. XVIII, pp. 79-88, 1980, MR: 83c: 10008.

[2] Idem, Un Infinity of Unsolved Problems Concerning a Function in Number Theory, International Congress of Mathematicians, Univ. of Berkeley, CA, August 3-11, 1986.

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[A comment about this generalization was published in "Mathematics Magazine", Vol. 61, No. 3, June 1988, p. 202: "Smarandache considered the general problem of finding positive integers n, a, and k, so that $(n + k)!$ should be a multiple of a^n. Also, for positive integers p and k, with p prime, he found a formula for determining the smallest integer f(k) with the property that $(f(k))!$ is a multiple of p^k."]

SOME LINEAR EQUATIONS INVOLVING A FUNCTION IN THE NUMBER THEORY

We have constructed a function η which associates to eac non-null integer m the smallest positive n such that n! is a multiple of m.

(a) Solve the equation $\eta(x) = n$, where $n \in N$.

*(b) Solve the equation $\eta(mx) = x$, where $m \in Z$. Discussion.

(c) Let $\eta^{(i)}$ note $\eta \circ \eta \circ \ldots \circ \eta$ of i times. Prove tha there is a k for which

$$\eta^{(k)}(m) = \eta^{(k+1)}(m) = n_m, \text{ for all } m \in Z^* \setminus \{1\}.$$

**Find n_m and the smallest k with this property.

Solution

(a) The cases n = 0, 1 are trivial.

We note the increasing sequence of primes less or equal than n by $p_1, p_2, \ldots, p_k$, and

$$\beta_t = \sum_{h \geq 1} [n/p_t^h], \quad t = 1, 2, \ldots, k;$$

where [y] is the greatest integer less or equal than y.

Let $n = p_{i_1}^{\alpha_{i_1}} \dots p_{i_s}^{\alpha_{i_s}}$, where all p_{i_j} are distinct primes and all α_{i_j} are from N.

Of course we have $n \leq x \leq n!$

Thus $x = p_1^{\sigma_1} \dots p_k^{\sigma_k}$ where $0 \leq \sigma_t \leq \beta_t$ for all $t = 1, 2, \dots, k$ and there exists at least a $j \in \{1, 2, \dots, s\}$ for which

$$\sigma_{i_j} \in \{\beta_{i_j}, \beta_{i_j} - 1, \dots, \beta_{i_j} - \alpha_{i_j} + 1\}.$$

Clearly n! is a multiple of x, and is the smallest one.

(b) See [1] too. We consider $m \in N^*$.

<u>Lemma 1</u>. $\eta(m) \leq m$, and $\eta(m) = m$ if and only if $m = 4$ or m is a prime.

Of course m! is a multiple of m.

If $m \neq 4$ and m is not a prime, the Lemma is equivalent to there are m_1, m_2 such that $m = m_1 \cdot m_2$ with $1 < m_1 \leq m_2$ and ($2 m_2 < m$ or $2 m_1 < m$). Whence $\eta(m) \leq 2 m_2 < m$, respectively $\eta(m) \leq \max(m_2, 2m) < m$.

<u>Lemma 2</u>. Let p be a prime ≥ 5. Then $\eta(p x) = x$ if and only if x is a prime $> p$, or $x = 2p$.

Proof: $\eta(p) = p$. Hence $x > p$.

Analogously: x is not a prime and $x \neq 2p \Rightarrow x = x_1 x_2$, $1 < x_1 \leq x_2$ and ($2 x_2 < x_1$, $x_2 \neq p_1$, and $2 x_1 < x$) $\Rightarrow \eta(p x) \leq$

$\leq \max \{p, 2x_2\} < x$ respectively $\eta(px) \leq \max \{p, 2x_1, x_2\} < x$.

Observations

$\eta(2x) = x \Leftrightarrow x = 4$ or x is an odd prime.

$\eta(3x) = x \Leftrightarrow x = 4, 6, 9$ or x is a prime > 3.

Lemma 3. If $(m, x) = 1$ then x is a prime $> \eta(m)$.

Of course, $\eta(mx) = \max \{\eta(m), \eta(x)\} = \eta(x) = x$. And $x \neq \eta(m)$, because if $x = \eta(m)$ then $m \cdot \eta(m)$ divides $\eta(m)!$ that is m divides $(\eta(m) - 1)!$ whence $\eta(m) \leq \eta(m) - 1$.

Lemma 4. If x is not a prime then $\eta(m) < x \leq 2\eta(m)$ and $x = 2\eta(m)$ if and only if $\eta(m)$ is a prime.

Proof: If $x > 2\eta(m)$ there are x_1, x_2 with $1 < x_1 \leq x_2$, $x = x_1 x_2$. For $x_1 < \eta(m)$ we have $(x - 1)!$ is a multiple of $m x$. Same proof for other cases.

Let $x = 2\eta(m)$; if $\eta(m)$ is not a prime, then $x = 2ab$, $1 < a \leq b$, but the product $(\eta(m) + 1)(\eta(m) + 2) \ldots (2\eta(m) - 1)$ is divided by x.

If $\eta(m)$ is a prime, $\eta(m)$ divides m, whence $m \cdot 2\eta(m)$ is divided by $\eta(m)^2$, it results in $\eta(m \cdot 2\eta(m)) \geq 2 \cdot \eta(m)$, but $(\eta(m) + 1)(\eta(m) + 2) \ldots (2\eta(m))$ is a multiple of $2\eta(m)$, that is $\eta(m \cdot 2\eta(m)) = 2\eta(m)$.

Conclusion

All x, prime number > η (m), are solutions.

If η (m) is prime, then x = 2 η (m) is a solution.

*If x is not a prime, η (m) < x < 2 η (m), and x does not divide (x - 1)!/m then x is a solution (semi-open question). If m = 3 it adds x = 9 too. (No other solution exists yet.)

(c)

Lemma 5. η (a b) $\leq \eta$ (a) + η (b).

Of course, η (a) = a' and η (b) = b' involves (a' + + b')! = b'! (b' + 1) ... (b' + a'). Let a' $\leq$ b'. Then η(ab) $\leq$ a' + b', because the product of a' consecutive positive integers is a multiple of a'!

Clearly, if m is a prime then k = 1 and n_m = m.

If m is not a prime then η (m) < m, whence there is a k for which $\eta^{(k)}$ (m) = $\eta^{(k+1)}$ (m).

If m $\neq$ 1 then 2 $\leq n_m \leq$ m.

Lemma 6. n_m = 4 or n_m is a prime.

If $n_m = n_1 n_2$, $1 < n_1 \leq n_2$, then η (n_m) < n_m. Absurd.

$n_m \neq 4$.

(**) This question remains open.

Reference

[1] F. Smarandache, A Function in the Number Theory, An. Univ. Timisoara, seria st. mat., Vol. XVIII, fasc. 1, pp. 79-88, 1980; Mathematical Reviews: 83c: 10008.

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