Soft Computing
https://doi.org/10.1007/s00500-023-08338-y

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VIKOR and TOPSIS framework with a truthful-distance measure for the (*t*, *s*)-regulated interval-valued neutrosophic soft set

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Accepted: 26 April 2023

Abstract
This article introduces the structure of the (t, s)-regulated interval-valued neutrosophic soft set (abbr. (t, s)-INSS). The structure of (t, s)-INSS is shown to be capable of handling the sheer heterogeneity and complexity of real-life situations, i.e. multiple inputs with various natures (hence neutrosophic), uncertainties over the input strength (hence interval-valued), the existence of different opinions (hence soft), and the perception at different strictness levels (hence (t, s)-regulated). Besides, a novel distance measure for the (t, s)-INSS model is proposed, which is truthful to the nature of each of the three membership (truth, indeterminacy, falsity) values present in a neutrosophic system. Finally, a Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) and a *Viekriterijumsko Kompromisno Rangiranje* (VIKOR) algorithm that works on the (t, s)-INSS are introduced. The design of the proposed algorithms consists of TOPSIS and VIKOR frameworks that deploy a novel distance measure truthful to its intuitive meaning. The conventional method of TOPSIS and VIKOR will be generalized for the structure of (t, s)-INSS. The parameters t and s in the (t, s)-INSS model take the role of strictness in accepting a collection of data subject to the amount of mutually contradicting information present in that collection of data. The proposed algorithm will then be subjected to rigorous testing to justify its consistency with human intuition, using numerous examples which are specifically made to tally with the various human intuitions. Both the proposed algorithms are shown to be consistent with human intuitions through all the tests that were conducted. In comparison, all other works in the previous literature failed to comply with all the tests for consistency with human intuition. The (t, s)-INSS model is designed to be a conclusive generalization of Pythagorean fuzzy sets, interval neutrosophic sets, and fuzzy soft sets. This combines the advantages of all the three previously established structures, as well as having user-customizable parameters t and s, thereby enabling the (t, s)-INSS model to handle data of an unprecedentedly heterogeneous nature. The distance measure is a significant improvement over the current disputable distance measures, which handles the three types of membership values in a neutrosophic system as independent components, as if from a Euclidean vector. Lastly, the proposed algorithms were applied to data relevant to the ongoing COVID-19 pandemic which proves indispensable for the practical implementation of artificial intelligence.

Keywords Pythagorean fuzzy set · Neutrosophic set · MADM · TOPSIS · VIKOR · Aggregation operator

1 Introduction

A multi-attribute decision-making (MADM) problem involves the determination of the best alternative among several given alternatives, where each alternative has multiple attributes to be considered at once. Among the vast literature on MADM in general, beginning with the initial work by Hwang and Yoon (Hwang and Yoon 1981), all data presented in an MADM problem are usually presented by a matrix of the form:

$$\mathbf{D} = \begin{pmatrix} x_{1,1} & x_{1,2} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & \cdots & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m,1} & x_{m,2} & \cdots & x_{m,n} \end{pmatrix},$$

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Published online: 22 May 2023

where each row $\mathbf{a}_i = \begin{pmatrix} x_{i,1} & x_{i,2} \cdots x_{i,n} \end{pmatrix}$ represents a set of possible alternative and each column $\mathbf{b}_j = \begin{pmatrix} x_{1,j} \\ x_{2,j} \\ \vdots \\ x_{m,j} \end{pmatrix}$ represents an attribute to be considered. Such elements $x_{i,j}$ in $\mathbf{D}$, though conventionally consists of real numbers, are generalized to much more complicated structures containing multiple subentries, such as intervals in$\mathbb{R}$. Xiao et al. (2013) introduced MADM problems on interval-valued fuzzy soft set, for which $x_{i,j}$ takes the form of a closed interval $[a, b]$ where$0 \leq a \leq b \leq 1$, i.e. each $x_{i,j}$ is now an element of an interval-valued fuzzy soft set. Pythagorean fuzzy sets are an important class of fuzzy sets and have been increasingly studied since its inception by Yager (2013). A comprehensive overview of some of the most important studies related to Pythagorean fuzzy sets can be found in Peng and Selvachandran (2019). Yager (2014) introduced MADM problems on Pythagorean fuzzy sets, for which $x_{i,j} = \langle \mu_{i,j}, \nu_{i,j} \rangle$ for some $0 \leq \mu_{i,j}^2 + \nu_{i,j}^2 \leq 1$. Peng et al. (2015) then extended the theory of MADM to Pythagorean fuzzy soft set, which served as a further generalization of Pythagorean fuzzy sets. Ullah et al. (2020) proposed the generalized structure of complex Pythagorean fuzzy sets and proposed its application in pattern recognition. Mahmood et al. (2018) introduced an approach to MADM using spherical fuzzy sets in which $x_{i,j} = \langle \mu_{i,j}, \nu_{i,j}, w_{i,j} \rangle$ for some$0 \leq \mu_{i,j}^2 + \nu_{i,j}^2 + w_{i,j}^2 \leq 1$. Such a condition of $0 \leq \mu_{i,j}^2 + \nu_{i,j}^2 + w_{i,j}^2 \leq 1$ was loosened to $0 \leq \mu_{i,j}^2 + \nu_{i,j}^2 + w_{i,j}^2 \leq 2$ in the work of Jansi et al. (2019) as they introduced Pythagorean neutrosophic sets, though not as loose as the condition $0 \leq \mu_{i,j} + \nu_{i,j} + w_{i,j} \leq 3$ for the classical single-valued neutrosophic sets mentioned in Gündoğdu and Kahraman (2019). Al-Shahmi (2022) introduced the concept of $(2, 1)$-fuzzy sets for which $x_{i,j} = \langle \mu_{i,j}, \nu_{i,j} \rangle$ for some$0 \leq \mu_{i,j}^2 + \nu_{i,j} \leq 1$. The concept was then generalized to (m, n)-fuzzy sets by Al-Shahmi and Mhemdi (2023) in which $x_{i,j} = \langle \mu_{i,j}, \nu_{i,j} \rangle$ for some $0 \leq \mu_{i,j}^m + \nu_{i,j}^n \leq 1$, who then applied this model to a MADM problem related to the selection of an optimal company for investment. Al-Shahmi et al. (2023) then further generalized the concept of (m, n)-fuzzy sets to (m, n)-fuzzy soft sets.

It was observed that a major portion of the recent works on MADM problems on complicated fuzzy logic structure, notably on the formation of aggregation operators, was produced by Jana and Pal or Jana et al. These works are expounded below. Firstly, Jana and Pal (2019) innovated an aggregation operator for MADM on single-valued neutrosophic soft sets which claimed to be more robust than the previous aggregation operators used in the literature. In addition, Jana et al. (2019a) also innovated an aggregation operator for MADM on bipolar fuzzy soft sets. Jana et al. (2019b) have also introduced Pythagorean fuzzy Dombi aggregation operators for MADM on Pythagorean fuzzy sets. Jana et al. (2020) established trapezoidal neutrosophic aggregation operators and studied their application to the MADM process. Jana and Pal (2021) also introduced Pythagorean fuzzy Dombi power aggregation operators for MADM on Pythagorean fuzzy sets, which serves as a further generalization to their previous works in Jana et al. (2019b). In recent times, Jana et al. (2021) introduced a MADM approach based on SVTrN Dombi aggregation functions.

Besides the works by Jana and Pal and Jana et al. mentioned above, there exist other important works on the MADM based on Pythagorean fuzzy sets and neutrosophic-based models in the existing literature. These works are expounded below. Tang et al. (2018) established the notion of dual hesitant fuzzy Frank aggregation operators, while Hadi et al. (2021) constructed a new approach to MADM using Fermatean fuzzy Hamacher aggregation operators, while Jan et al. (Jan et al. 2021) introduced some new Fermatean fuzzy Hamacher aggregation operators. Akram et al. (2021) established a complex intuitionistic fuzzy Hamacher aggregation operator and applied this to solve some MADM problems, Wang and Li (2019) introduced the notion of Bonferroni mean aggregation operators, whereas Ayub et al. (2021) proposed the concept of cubic fuzzy Heronian mean Dombi aggregation operators. Munir et al. (2021) constructed several new aggregation operators for MADM on t-spherical fuzzy sets, whereas Xing et al. (2019) constructed new aggregation operators for MADM on q-rung orthopair fuzzy sets.

There are two well-established frameworks for MADM, namely the Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) and *Viekriterijumsko Kompromisno Rangiranje* (VIKOR), both of which have been subjected to intensive studies by many researchers. The TOPSIS method is based on assessing the distances to the positive ideal solution (PIS) and the negative ideal solution (NIS) for each alternative, then a preference order of all alternatives is calculated in a certain way subject to those PIS and NIS as inputs. The study of the TOPSIS algorithm onto a fuzzy algebraic structure has been a common topic since the early 2010s, as society moves towards the era of Industrial Revolution 4.0 which is characterized by the widespread use of artificial intelligence. As a result, such work produced in recent years will enable even better design of AI systems as it is not capable of handling real-life data which are vague, heterogeneous, and subject to different parameters.

In intuitionistic fuzzy sets, the restriction $0 \leq \mu + \nu \leq 1$ was imposed on its two membership functions. Such restrictions were made more lenient in the innovation of Pythagorean fuzzy sets with the restriction $0 \leq \mu^2 + \nu^2 \leq 1$. On the other hand, neutrosophic sets post no restrictions on their three membership functions, which present a serious risk of having too much of contradicting information (e.g. truth and false membership functions both equal to 1) that jeopardize the usefulness of the input data.

A chronological summary of the recent works done on TOPSIS is presented below. The framework of TOPSIS first appeared in the work of Boran et al. (2009) on MADM for intuitionistic fuzzy sets. Zhang and Xu (2014) proposed a TOPSIS framework for MADM on Pythagorean fuzzy sets, which serves as a further generalization to the works by Boran et al. (2009). Eraslan and Karaaslan (2015) proposed a TOPSIS framework for MADM for fuzzy soft sets. This framework incorporates many features in dealing with each parameter in a soft set. Abootalebi et al. (2019) presented an improvement in the TOPSIS method for MADM, while Akram and Arshad (2018) proposed a TOPSIS framework for bipolar fuzzy sets in 2018. This was further generalized by Adeel et al. (2019) who established another TOPSIS framework for m-polar fuzzy linguistic sets. Kumar and Garg (2018) proposed another framework for intuitionistic fuzzy sets which utilized complex numbers in their algorithm. Meanwhile, Peng and Dai (2018) proposed multiple MADM algorithms based on TOPSIS and MABAC in a single-valued neutrosophic environment and introduced a new similarity measure.

Akram et al. (2019) proposed another TOPSIS framework for Pythagorean fuzzy sets. In addition, Kacprzak (2019) proposed one for ordered fuzzy numbers, whereas Fu and Liao (2019) proposed an algorithm for converting linguistic feedback into numerical entities to be fed into the TOPSIS algorithm. Zeng et al. (2020) proposed a TOPSIS framework for interval-valued intuitionistic fuzzy sets, while Zulqarnain et al. (2021) proposed a TOPSIS framework for interval-valued intuitionistic fuzzy soft sets (IVIFSS) which is a further generalization of the work in Zeng et al. (2020) and also discussed a new type of correlation coefficient under IVIFSSs.

On the other hand, VIKOR is capable of computing compromise solutions for a problem with inconsistent criteria, which can help the decision-makers to get a final decision (Opricovic and Tzeng 2007; Opricovic 2011). Opricovic and Tzeng (2003) first introduced VIKOR for fuzzy sets in the literature. However, the amount of literature dedicated to VIKOR is much less than that for TOPSIS. This is believed to be caused by the existence of an extra parameter in the algorithm itself which has a major role in determining the behaviour of the algorithm. The extent of the influence of this extra parameter in the VIKOR algorithm is shown in Fig. 1 in which the ranking of the alternatives is plotted against the values of this extra parameter.

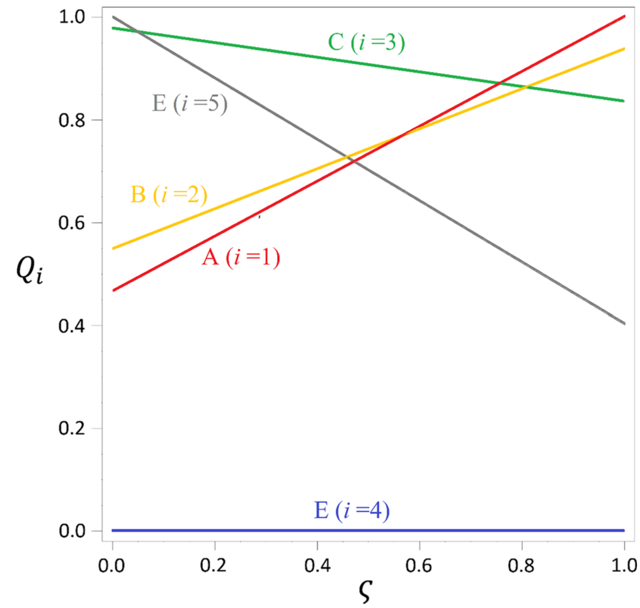

Fig. 1 Values of ***Qi*** for the 5 companies A, B, C, D, and E

A chronological summary of the recent works done on VIKOR is likewise presented below. Kim and Ahn (2019) extended VIKOR to accommodate incomplete information, while Eroglu and Sahin (2020) improvised a new distance measure for VIKOR in 2020. Gou et al. (2021) also improved the VIKOR method by enabling it to accept linguistic terms. Besides these works, Riaz and Tehrim (2021) and Zhou and Chen (2021) proposed VIKOR frameworks for bipolar fuzzy sets and Pythagorean fuzzy sets, respectively, with Zhou and Chen (2021) also introducing a new distance measure for VIKOR. Thus, it can be observed that Kim and Ahn (2019) and Gou et al. (2021) had both worked on improvised MADM algorithms that can deal with incomplete or even linguistic data.

This study focuses on the TOPSIS and VIKOR algorithms, both of which were chosen due to their inherently stable natures of computations which possess the lowest risks of division-by-zero among all MADM algorithms in the existing literature, regardless of the extremities of the input values. Moreover, the calculation procedure of both the TOPSIS and VIKOR algorithms does not involve any high-end operators, such as numerical integrations or numerical solutions to differential equations, which are computationally burdensome for computer programmes. Therefore, the TOPSIS and VIKOR algorithms are deemed to be among the most versatile algorithms that can be implemented using any kind of computer hardware system regardless of their hardware specifications.

In view of the above, this study seeks to answer the following research questions.

(i) How to implement a distance measure, a score function, and an accuracy function that is truthful to the mutually contradicting nature of the membership and non-membership degrees, and captures the dampening effect of the degree of indeterminacy on the membership and non-membership degrees?
(ii) How to adapt the TOPSIS and VIKOR algorithms onto the structure of the (t, s)-INSS model to handle the four different facets of complexities most commonly encountered in real-life scenarios?
(a) The existence of multiple types of inputs with varying natures.
(b) The uncertainties over the strength of an input.
(c) The existence of different opinions among the experts.
(d) The existence of customizable perceptions due to the varying personalities of the end users.

This study serves to contribute to the existing literature and body of knowledge related to TOPSIS and VIKOR mainly in the following two areas:

(i) The improvement in the current TOPSIS and VIKOR algorithms through the innovation and adaptation of a novel distance measure which enables the membership and non-membership functions to mutually interact and neutralize each other's impact, as well as enabling the indeterminacy function to dampen the effect of the membership and non-membership functions, to be truthful to their intuitive meaning.
(ii) The adaptation of the improved TOPSIS and VIKOR algorithm onto the structure of the (t, s)-regulated interval-valued neutrosophic soft set model (abbr. (t, s)-INSS), which is a generalization of all models in the existing literature from which the TOPSIS and VIKOR algorithms were adopted.

Firstly, the innovation of this novel distance measure proves indispensable and vital to the development of the TOPSIS and VIKOR algorithms. This is because among all the previous works on MADM that involve fuzzy system structures, it was observed that the distance measure used always suffered from the same severe flaw, i.e. all the different entities in that fuzzy system, such as the membership and non-membership functions, were merely treated independently like the components in a vector. Therefore, in such cases, there is no interaction between the membership and non-membership functions, which are known to be mutually opposite in nature. Such ways of measuring distance defeat the intuitive meaning of membership and non-membership functions. Furthermore, for the fuzzy systems with a degree of indeterminacy as one of the entities, the degree of indeterminacy was not given a distinctive role from that of the membership and non-membership functions, which again defeats the purpose of having the degree of indeterminacy as another entity in addition to the membership and non-membership functions. Even in the work of Ganie and Singh (2021) in which the authors defined a set of general axioms that must be satisfied for all distance measures, such severe discrepancies persist.

Secondly, the structure of the (t, s)-INSS model that was innovated in this study is to cater to the following four main facets of complexities, as commonly encountered in many real-life situations, which are listed as follows.

(i) Multiple types of inputs with various natures, and those inputs often interfere with one another, especially by mutual cancellation or with one dampening the effect of the other. For example, in the scenario of voting "like" or "dislike": When 10% of the population voted "like", another equal 10% voted "dislike" and the rest did not vote, such case may be interpreted as having no difference from the case where no one voted at all.
(ii) Uncertainties over the strength of an input. For example, in the scenario of weather forecasting, the forecasted humidity of a particular region at a certain period is often stated as an interval (range), say 60% to 80%, instead of a single value.
(iii) Existence of different opinions by experts which gives rise to different collections of data, all of which must be considered. For example, in medical diagnosis, several medical experts may concurrently access the medical report of a patient, and they may have different opinions over an observation (say, one computerized tomography (CT) scan image), which gives rise to a set of different output values for that single observation.
(iv) Customizable perception based on the personality of the end users. For example, in evaluating the air pollution over a particular region in a month, some may give equal attention to all the days in that month, whereas another person may only look at the single day which is the most polluted in that month. In enabling such customization, extra parameter(s) must be considered, and the range of values must be corresponding to the real-life personalities of the end users.

The neutrosophic structure of the (t, s)-INSS model enables it to cater to multiple types of inputs with various natures, its interval-valued structure can handle uncertainties over the input strength, its soft structure can cater to the differing opinions of the experts/users, and its user-customizable parameters t and s can be used to model the different perceptions of the users. On the other hand, the fuzzy-based algorithm found in all the observed works in the literature is only capable of handling some of the four facets of complexities mentioned above (predominantly only (i)). Moreover, for most studies in the existing literature, even the structure of the chosen fuzzy systems was not related to any data complexities in real-life scenarios.

Therefore, in Sect. 2, the basic definition of the model will be established, and the characteristics of the proposed model will be highlighted. In Sect. 3, the distance measure for the (t, s)-INSS model will be first established before deriving the TOPSIS and VIKOR decision-making algorithms. This is then followed by a rigorous study of the distance measure, score function, and accuracy function of the (t, s)-INSS model, which also forms part of the preparation for the comparative studies in Sect. 6. Then in Sect. 4, the TOPSIS and VIKOR decision-making algorithms for the (t, s)-INSS model that incorporated the distance measure and score function established in Sect. 3 will be introduced. In Sect. 5, the applicability of the newly introduced (t, s)-INSS model and the corresponding TOPSIS and VIKOR decision-making algorithms will then be demonstrated using a MADM problem related to the COVID-19 pandemic. In Sect. 6, a comparative study that uses the seven criteria established in Sects. 2 and 3 will be presented. More than 20 recent studies in the existing literature will be examined through rigorous computations followed by a thorough analysis. Finally, the potential future direction of this study, which is to apply the newly established model and decision-making algorithms to real-life data sets, will be outlined in Sect. 7

2 Preliminaries

This section presents a brief overview of Pythagorean fuzzy sets and its extensions or related models, which leads to the formation of the proposed (t, s)-regulated interval-valued neutrosophic soft set (abbr. (t, s)-INSS). Firstly, five preliminary fuzzy-based structures are expounded in Definitions 2.1 to 2.5, leading to the establishment of the (t, s)-INSS model in Definition 2.6.

The first source of the proposed (t, s)-INSS model comes from the Pythagorean interval-valued fuzzy set (abbr. PyIFS), which is presented in Definition 2.1.

Definition 2.1 Mohagheghi et al. (2020) Let U be a universal set. Define

$$A = \left\{ \left(x, \left\langle \left[\mu_A^L(x), \mu_A^U(x)\right], \left[\nu_A^L(x), \nu_A^U(x)\right] \right\rangle \right) : x \in U \right\}$$

where $\mu_A^L(x), \mu_A^U(x), \nu_A^L(x), \nu_A^U(x) : U \to [0, 1]$, with $\mu_A^L(x) \le \mu_A^U(x)$, $\nu_A^L(x) \le \nu_A^U(x)$ and $0 \le \left(\mu_A^U(x)\right)^2 + \left(\nu_A^U(x)\right)^2 \le 1$ for all x. Then:

(i) A is said to be a *Pythagorean interval-valued fuzzy set* (abbr. PyIFS) on U.

(ii) $\mu_A = \left[\mu_A^L, \mu_A^U\right]$ is said to be the *degree of membership* of A.

(iii) $\nu_A = \left[\nu_A^L, \nu_A^U\right]$ is said to be the *degree of non-membership* of A.

Remark 2.1.1 The complement of A, denoted as A^*, is defined as

$$A^* = \left\{ \left(x, \left\langle \left[\nu_A^L(x), \nu_A^U(x)\right], \left[\mu_A^L(x), \mu_A^U(x)\right] \right\rangle \right) : x \in U \right\}.$$

Therefore, the non-membership degree of A is the membership degree of A^*. This reflects the mutually opposite nature of the membership and non-membership functions.

Despite being interval-valued, and with the restriction $0 \le \left(\mu_A^U(x)\right)^2 + \left(\nu_A^U(x)\right)^2 \le 1$, the PyIFS model defined in Definition 2.1 only possesses two types of entities, i.e. the membership and non-membership functions. However, in real-life scenarios, there are always grey areas and multiple factors that affect the decision-making process. For example, in an investment scenario, besides the decision of investors on whether to invest or not to invest in a company, many other factors need to be considered in the decision-making process. This includes the perception of a company by investors that would decide whether the investors should act bullishly or bearishly towards that company, as well as other scenarios such as investors being unable to decide as they have insufficient information to facilitate their decision-making, which leads to a certain degree of indeterminacy that must be dealt with. Hence, the introduction of the degree of indeterminacy, in addition to the membership and non-membership degrees, gives rise to the concept of spherical interval-valued fuzzy sets (abbr. SIFS). The formal definition of the SIFS model is presented in Definition 2.2.

Definition 2.2 Gündoğdu and Kahraman (2019) Let U be a universal set. Define:

$$A = \left\{ \left(x, \left\langle \left[\mu_A^L(x), \mu_A^U(x)\right], \left[\eta_A^L(x), \eta_A^U(x)\right], \left[\nu_A^L(x), \nu_A^U(x)\right] \right\rangle \right) : x \in U \right\},$$

where $\mu_A^L(x), \mu_A^U(x), \eta_A^L(x), \eta_A^U(x), \nu_A^L(x), \nu_A^U(x) : U \to [0, 1]$; and with $\mu_A^L(x) \le \mu_A^U(x)$, $\eta_A^L(x) \le \eta_A^U(x)$, $\nu_A^L(x) \le \nu_A^U(x)$ and $0 \le \left(\mu_A^U(x)\right)^2 + \left(\eta_A^U(x)\right)^2 + \left(\nu_A^U(x)\right)^2 \le 1$ for all x. Then:

(i) A is said to be a *spherical interval-valued fuzzy set* (abbr. SIFS) on U.
(ii) $\mu_A = [\mu_A^L, \mu_A^U]$ is said to be the *degree of membership* of A.
(iii) $\eta_A = [\eta_A^L, \eta_A^U]$ is said to be the *degree of indeterminacy* of A.
(iv) $\nu_A = [\nu_A^L, \nu_A^U]$ is said to be the *degree of non-membership* of A.

Therefore, in such a case, the structure $\{(x, \langle \mu_A(x), \eta_A(x), \nu_A(x)\rangle) : x \in U\}$ *with* $0 \le (\mu_A(x))^2 + (\eta_A(x))^2 + (\nu_A(x))^2$ *from spherical fuzzy sets in Mahmood et al.* (2018) *is generalized to* $\{(x, \langle [\mu_A^L(x), \mu_A^U(x)], [\eta_A^L(x), \eta_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]\rangle) : x \in U\}$ *for SIFS while retaining a condition of* $0 \le (\mu_A^U(x))^2 + (\eta_A^U(x))^2 + (\nu_A^U(x))^2 \le 1$.

The restriction $0 \le (\mu_A^U(x))^2 + (\nu_A^U(x))^2 \le 1$ for the PyIFS model given in Definition 2.1 is generalized to $0 \le (\mu_A^U(x))^2 + (\eta_A^U(x))^2 + (\nu_A^U(x))^2 \le 1$ for the SIFS model in Definition 2.2. There exist some other studies on single-valued fuzzy systems with three entities that had taken on another approach, i.e. the generalization of the restriction $0 \le (\mu_A(x))^2 + (\eta_A(x))^2 + (\nu_A(x))^2 \le 1$ on conventional spherical fuzzy sets (abbr. SFS) which is a predecessor of the SIFS model. One such generalization is as given in Definition 2.3.

Definition 2.3 Jansi et al. (2019) Let U be a universal set. Define.

$A = \{(x, \langle \mu_A(x), \eta_A(x), \nu_A(x)\rangle) : x \in U\}$,

where $\mu_A(x), \eta_A(x), \nu_A(x) : U \to [0,1]$, and $0 \le (\mu_A(x))^2 + (\eta_A(x))^2 + (\nu_A(x))^2 \le 2$ for all x. Then:

(i) A is said to be a *Pythagorean neutrosophic set* (abbr. PNS) on U.
(ii) μ_A is said to be the *degree of membership* of A.
(iii) η_A is said to be the *degree of indeterminacy* of A.
(iv) ν_A is said to be the *degree of non-membership* of A.

Therefore, in such a case, the condition $0 \le (\mu_A(x))^2 + (\eta_A(x))^2 + (\nu_A(x))^2 \le 1$ on the conventional SFS model was generalized to $0 \le (\mu_A(x))^2 + (\eta_A(x))^2 + (\nu_A(x))^2 \le 2$ by Jansi et al. (2019). This enables a greater degree of freedom among the μ_A, η_A and ν_A, while still preventing too much contradicting data as in the case of $0 \le \mu_A(x) + \eta_A(x) + \nu_A(x) \le 3$ on the classical single-valued neutrosophic sets (abbr. SVNS) model, which no longer posts any restrictions on the values of μ_A, η_A and ν_A. It is also worth noting that such restriction serves as the first line of filtering against any data sets found to be containing too much of contradictory information.

The concept of Pythagorean neutrosophic soft set (abbr. PyNSS), which consists of adding the soft structure to the PyNS model, is presented in Definition 2.4.

Definition 2.4 Ajay and Chellamani (2022) Let U be a universal set. Let E be a set of parameters. Let $F : E \to \{A(\varepsilon) : \varepsilon \in E\}$, where $\{A(\varepsilon) : \varepsilon \in E\}$ is a collection of PNS on U, then (E, F) is said to be a *Pythagorean neutrosophic soft set* (abbr. PNSS) on U.

Following from Definitions 2.1, 2.2, and 2.3, Definition 2.5 that further generalizes the restrictions for various degrees of strictness in preventing excessive contradictory information, as seen in the PyNS and conventional SFS models, is now established in this study. Subsequently, Definition 2.6 which further implements the interval-valued structure seen in the PyIFS and SIFS models is then established.

Definition 2.5 Let U be a universal set, $t \in [0, \infty)$ and $s \in [1, 3]$. Define:

$A = \{(x, \langle \mu_A(x), \eta_A(x), \nu_A(x)\rangle) : x \in U\}$,

where $\mu_A(x), \eta_A(x), \nu_A(x) : U \to [0,1]$; and with $0 \le (\mu_A(x))^t + (\eta_A(x))^t + (\nu_A(x))^t \le s$ for all $x \in U$. Then A is said to be a (t,s)*-regulated neutrosophic set* (abbr. (t,s)-NS) on U.

Remark 2.5.1 Definition 2.5 holds true for any real values of t, where $t \in [0, \infty)$. However, the definition only holds true for values of s between 1 and 3, i.e. $s \in [1,3]$. The value of s must be at least 1 to enable the (t,s)-NS to remain as a generalization of the classical crisp set by allowing $(\mu_A(x), \eta_A(x), \nu_A(x)) = (1,0,0)$ to produce $(\mu_A(x))^t + (\eta_A(x))^t + (\nu_A(x))^t = 1 + 0 + 0 = 1$. On the other hand, the value of s cannot exceed 3 because $\mu_A(x), \eta_A(x), \nu_A(x) \in [0,1]$, so $0 \le (\mu_A(x))^t + (\eta_A(x))^t + (\nu_A(x))^t \le 3$ is already the maximum value that can be reached when $\mu_A(x)$, $\eta_A(x)$ and $\nu_A(x)$ having the maximum value of 1. For example, if $0 \le (\mu_A(x))^2 + (\eta_A(x))^2 + (\nu_A(x))^2 \le 3$ holds for all $x \in U$, then A is said to be a $(2,3)$-regulated neutrosophic set (abbr. $(2,3)$-NS) on U.

Remark 2.5.2 The conventional SFS model is thus a $(2,1)$-NS, whereas the PyNS model defined in Definition 2.3 is thus a $(2,2)$-NS. The conventional single-valued neutrosophic set (abbr. SVNS) is thus a $(1,3)$-NS.

Definition 2.6 Let U be a universal set, $t \in [0, \infty)$ and $s \in [1,3]$. Define:

$A = \{(x, \langle [\mu_A^L(x), \mu_A^U(x)], [\eta_A^L(x), \eta_A^U(x)], [\nu_A^L(x), \nu_A^U(x)]\rangle) : x \in U\}$, where $\mu_A^L(x), \mu_A^U(x), \eta_A^L(x), \eta_A^U(x), \nu_A^L(x), \nu_A^U(x) : U \to [0,1]$; and with $\mu_A^L(x) \le \mu_A^U(x)$, $\eta_A^L(x) \le \eta_A^U(x)$, $\nu_A^L(x) \le \nu_A^U(x)$ and $0 \le \left(\mu_A^U(x)\right)^t + \left(\eta_A^U(x)\right)^t + \left(\nu_A^U(x)\right)^t \le s$ for all $x \in U$. Then A is said to be a *(t,s)-regulated interval-valued neutrosophic set* (abbr. (t,s)-INS) on U.

Remark 2.6.1 The SIFS model is thus a $(2,1)$-INS.

The proposed concept of (t,s)-INSS which further implements the interval-valued structure from Definition 2.4 onto Definition 2.6 is presented in Definition 2.7.

Definition 2.7 Let U be a universal set, $t \in [0,\infty)$, $s \in [1,3]$ and E be a set of parameters. Let $F : E \to \{A(\varepsilon) : \varepsilon \in E\}$, where $\{A(\varepsilon) : \varepsilon \in E\}$ is a collection of (t,s)-INS on U. Then (E,F) is said to be a *(t,s)-regulated interval-valued neutrosophic soft set* (abbr. (t,s)-INSS) on U.

Remark 2.7.1 In other words,

$$(E,F) = \left\{ \left(x, \left\{ \langle \varepsilon, \left[\mu^L_{A(\varepsilon)}(x), \mu^U_{A(\varepsilon)}(x)\right], \left[\eta^L_{A(\varepsilon)}(x), \eta^U_{A(\varepsilon)}(x)\right], \left[\nu^L_{A(\varepsilon)}(x), \nu^U_{A(\varepsilon)}(x)\right] \rangle : \varepsilon \in E \right\} \right) : x \in U \right\}.$$

Lastly, the concepts of (t,s)-regulated interval-valued neutrosophic soft number (abbr. (t,s)-INSn) and (t,s)-regulated interval-valued neutrosophic soft matrix (abbr. (t,s)-INSM) are established in Definitions 2.8 and 2.9, respectively.

Definition 2.8 Let $t \in [0,\infty)$, $s \in [1,3]$ and $\mathfrak{n} = \left\{ \langle \varepsilon, [\dot\mu^L_\varepsilon, \dot\mu^U_\varepsilon], [\dot\eta^L_\varepsilon, \dot\eta^U_\varepsilon], [\dot\nu^L_\varepsilon, \dot\nu^U_\varepsilon] \rangle : \varepsilon \in E \right\}$, where $\dot\mu^L_\varepsilon, \dot\mu^U_\varepsilon, \dot\eta^L_\varepsilon, \dot\eta^U_\varepsilon, \dot\nu^L_\varepsilon, \dot\nu^U_\varepsilon \in [0,1]$, and with $\dot\mu^L_\varepsilon \le \dot\mu^U_\varepsilon$, $\dot\eta^L_\varepsilon \le \dot\eta^U_\varepsilon$, $\dot\nu^L_\varepsilon \le \dot\nu^U_\varepsilon$ and $0 \le \left(\dot\mu^U_\varepsilon\right)^t + \left(\dot\eta^U_\varepsilon\right)^t + \left(\dot\nu^U_\varepsilon\right)^t \le s$. Then $\mathfrak{n}$ is said to be a *(t,s)-regulated interval-valued neutrosophic soft number* (abbr. (t,s)-INSn).

Definition 2.9 Let $t \in [0,\infty)$ and $s \in [1,3]$. Let $\mathbf{M}$ be defined as follows:

$$\mathbf{M} = \begin{pmatrix} \mathfrak{n}_{1,1} & \mathfrak{n}_{1,2} & \cdots & \mathfrak{n}_{1,n} \\ \mathfrak{n}_{2,1} & \mathfrak{n}_{2,2} & \cdots & \mathfrak{n}_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{n}_{m,1} & \mathfrak{n}_{m,2} & \cdots & \mathfrak{n}_{m,n} \end{pmatrix},$$

where each $\mathfrak{n}_{i,j}$ is a (t,s)-INSn. Then $\mathbf{M}$ is said to be a *(t,s)-regulated interval-valued neutrosophic soft matrix* (abbr. (t,s)-INSM) and $\langle \varepsilon, [\dot\mu^L_\varepsilon, \dot\mu^U_\varepsilon], [\dot\eta^L_\varepsilon, \dot\eta^U_\varepsilon], [\dot\nu^L_\varepsilon, \dot\nu^U_\varepsilon] \rangle$, where $\mathfrak{n} = \left\{ \langle \varepsilon, [\dot\mu^L_\varepsilon, \dot\mu^U_\varepsilon], [\dot\eta^L_\varepsilon, \dot\eta^U_\varepsilon], [\dot\nu^L_\varepsilon, \dot\nu^U_\varepsilon] \rangle : \varepsilon \in E \right\}$.

Therefore, any TOPSIS or VIKOR algorithms that have been established in the past studies in the literature must first be capable of accepting the following complexities in their inputs, in order to work with the (t,s)-INSS structure.

(i) There must exist *at least three entities*, to cater for the membership, indeterminacy, and non-membership degrees.
(ii) All the entities must be *interval-valued*.
(iii) The set structure must consist of multiple versions, governed by a parameter $\varepsilon \in E$ (hence being *soft*).
(iv) The restriction to regulate the amount of contradicting information present must be *user customizable*.

Hence, these complexities shall serve as the first four indicators for the comparative studies (that will be presented and discussed in Sect. 6) that will discuss and analyse the various TOPSIS and VIKOR algorithms that have been established in the existing literature.

3 Comparison and verification of the proposed distance measure

Before forming the TOPSIS and VIKOR algorithms for the (t,s)-INSS model, the establishment of the distance measure, score function, and accuracy function for the (t,s)-INSn is necessary. However, the chosen distance measure, score function, and accuracy function must be subjected to rigorous verification to justify its meaning in interpreting real-life events. In particular, the three functions that were established, namely the distance measure, score function, and accuracy function, must enable the degrees of membership, indeterminacy, and non-membership to take their characteristic effects consistent with their natures as understood by most experts. This consistency with human judgement/intuition can only be satisfied by enabling the mutual cancellation between the membership and the non-membership degrees, and by allowing the degree of indeterminacy to dampen the effect of the membership and the non-membership degrees in a (t,s)-INSn.

Among the previous studies in the existing literature on the interval-valued fuzzy systems, all the interval-valued fuzzy systems that use distance measures were observed to solely rely on conventional distance measures in classical data science, such as those based on the Euclidean or Hamming distance measures, which merely treats each component as a separate individual entry. This shortcoming persists even among all instances of distance measures mentioned in the work by Ganie and Singh (2021). Therefore, given two arbitrary (t,s)-INSn: $\mathfrak{n}_1 = \left\{ \langle \varepsilon, \left[\dot\mu^L_{\varepsilon,1}, \dot\mu^U_{\varepsilon,1}\right], \left[\dot\eta^L_{\varepsilon,1}, \dot\eta^U_{\varepsilon,1}\right], \left[\dot\nu^L_{\varepsilon,1}, \dot\nu^U_{\varepsilon,1}\right] \rangle : \varepsilon \in E \right\}$ and $\mathfrak{n}_2 = \left\{ \langle \varepsilon, \left[\dot\mu^L_{\varepsilon,2}, \dot\mu^U_{\varepsilon,2}\right], \left[\dot\eta^L_{\varepsilon,2}, \dot\eta^U_{\varepsilon,2}\right], \left[\dot\nu^L_{\varepsilon,2}, \dot\nu^U_{\varepsilon,2}\right] \rangle : \varepsilon \in E \right\}$, all observed attempts to define distance measures on interval-valued fuzzy systems using the Euclidean concept, as seen among all past studies in the existing literature, had always yielded the following result (or results similar to it).

$$d_e(\mathfrak{n}_1,\mathfrak{n}_2)=\frac{1}{|E|}\sum_{\varepsilon\in E}\sqrt{\left(\frac{\dot{\mu}^L_{\varepsilon,1}+\dot{\mu}^U_{\varepsilon,1}}{2}-\frac{\dot{\mu}^L_{\varepsilon,2}+\dot{\mu}^U_{\varepsilon,2}}{2}\right)^2+\left(\frac{\dot{\eta}^L_{\varepsilon,1}+\dot{\eta}^U_{\varepsilon,1}}{2}-\frac{\dot{\eta}^L_{\varepsilon,2}+\dot{\eta}^U_{\varepsilon,2}}{2}\right)^2+\left(\frac{\dot{v}^L_{\varepsilon,1}+\dot{v}^U_{\varepsilon,1}}{2}-\frac{\dot{v}^L_{\varepsilon,2}+\dot{v}^U_{\varepsilon,2}}{2}\right)^2}.$$

This formula gives identical roles to μ, η, and v, as if they are entries in a vector. Likewise, all observed attempts to define distance measures on interval-valued fuzzy systems using the Hamming distance measure, as seen among all past studies in the existing literature, had always yielded the following result (or results similar to it).

$$d_h(\mathfrak{n}_1,\mathfrak{n}_2)=\frac{1}{|E|}\sum_{\varepsilon\in E}\left(\left|\frac{\dot{\mu}^L_{\varepsilon,1}+\dot{\mu}^U_{\varepsilon,1}}{2}-\frac{\dot{\mu}^L_{\varepsilon,2}+\dot{\mu}^U_{\varepsilon,2}}{2}\right|+\left|\frac{\dot{\eta}^L_{\varepsilon,1}+\dot{\eta}^U_{\varepsilon,1}}{2}-\frac{\dot{\eta}^L_{\varepsilon,2}+\dot{\eta}^U_{\varepsilon,2}}{2}\right|+\left|\frac{\dot{v}^L_{\varepsilon,1}+\dot{v}^U_{\varepsilon,1}}{2}-\frac{\dot{v}^L_{\varepsilon,2}+\dot{v}^U_{\varepsilon,2}}{2}\right|\right).$$

This formula also gives identical roles to μ, η and v.

Both the two mainstream choices of defining distance measures which are observed in most of the studies in the existing literature, failed to acknowledge the role of η as a measure of indeterminacy, and the roles of μ and v as measures of determinacies of opposite natures. Even in the case of Liu and Jiang (2020), their new distance measure on interval-valued fuzzy systems is defined as:

$$d_j(\mathfrak{n}_1,\mathfrak{n}_2)=\frac{1}{|E|}\sum_{\varepsilon\in E}\sqrt{\begin{array}{c}\left(\frac{\dot{\mu}^L_{\varepsilon,1}+\dot{\mu}^U_{\varepsilon,1}}{2}-\frac{\dot{\mu}^L_{\varepsilon,2}+\dot{\mu}^U_{\varepsilon,2}}{2}\right)^2+\left(\frac{\dot{\eta}^L_{\varepsilon,1}+\dot{\eta}^U_{\varepsilon,1}}{2}-\frac{\dot{\eta}^L_{\varepsilon,2}+\dot{\eta}^U_{\varepsilon,2}}{2}\right)^2+\left(\frac{\dot{v}^L_{\varepsilon,1}+\dot{v}^U_{\varepsilon,1}}{2}-\frac{\dot{v}^L_{\varepsilon,2}+\dot{v}^U_{\varepsilon,2}}{2}\right)^2+\\ \left(\frac{\dot{\mu}^L_{\varepsilon,1}+\dot{\mu}^U_{\varepsilon,1}}{2}-\frac{\dot{\mu}^L_{\varepsilon,2}+\dot{\mu}^U_{\varepsilon,2}}{2}\right)\left(\frac{\dot{\eta}^L_{\varepsilon,1}+\dot{\eta}^U_{\varepsilon,1}}{2}-\frac{\dot{\eta}^L_{\varepsilon,2}+\dot{\eta}^U_{\varepsilon,2}}{2}\right)+\left(\frac{\dot{v}^L_{\varepsilon,1}+\dot{v}^U_{\varepsilon,1}}{2}-\frac{\dot{v}^L_{\varepsilon,2}+\dot{v}^U_{\varepsilon,2}}{2}\right)\left(\frac{\dot{\eta}^L_{\varepsilon,1}+\dot{\eta}^U_{\varepsilon,1}}{2}-\frac{\dot{\eta}^L_{\varepsilon,2}+\dot{\eta}^U_{\varepsilon,2}}{2}\right)\end{array}}$$

which still failed to do much justice to the distinctive roles of μ,η and v.

Therefore, in addressing such shortcomings that have been observed in the past studies in the existing literature, a set of axioms is introduced in Definition 3.1, followed by the introduction of the Q distance measure in Definition 3.2. Subsequently, in Lemma 3.3, the Q distance measure is verified to comply with the axioms that were established in Definition 3.1.

Definition 3.1 A measure d is said to be a truthful-distance measure on (t,s)-INSn if it satisfies all the following axioms for all (t,s)-INSn from any given set of parameters E, and for any given (t,s).

(i) $1=d(\{\langle\varepsilon,[1,1],[0,0],[0,0]\rangle:\varepsilon\in E\}, \{\langle\varepsilon,[0,0],[0,0],[1,1]\rangle:\varepsilon\in E\})\geq d(\mathfrak{a},\mathfrak{b})\geq d(\mathfrak{a},\mathfrak{a})=0$ for all (t,s)-INSn $\mathfrak{a}$ and $\mathfrak{b}$.

(ii) $d(\mathfrak{a},\mathfrak{b})=d(\mathfrak{b},\mathfrak{a})$ for all (t,s)-INSn $\mathfrak{a}$ and $\mathfrak{b}$.

(iii) $d\big(\{\langle\varepsilon,[\mu_{\varepsilon,1},\mu_{\varepsilon,1}],[\eta_\varepsilon,\eta_\varepsilon],[v_{\varepsilon,1},v_{\varepsilon,1}]\rangle:\varepsilon\in E\}, \{\langle\varepsilon,[\mu_{\varepsilon,2},\mu_{\varepsilon,2}],[\eta_\varepsilon,\eta_\varepsilon],[v_{\varepsilon,2},v_{\varepsilon,2}]\rangle:\varepsilon\in E\}\big)\geq d\big(\{\langle\varepsilon,[\mu_{\varepsilon,3},\mu_{\varepsilon,3}],[\eta_\varepsilon,\eta_\varepsilon],[v_{\varepsilon,3},v_{\varepsilon,3}]\rangle:\varepsilon\in E\},\{\langle\varepsilon,[\mu_{\varepsilon,4},\mu_{\varepsilon,4}],[\eta_\varepsilon,\eta_\varepsilon],[v_{\varepsilon,4},v_{\varepsilon,4}]\rangle:\varepsilon\in E\}\big)$ if $\left|(\mu_{\varepsilon,1}-v_{\varepsilon,1})-(\mu_{\varepsilon,2}-v_{\varepsilon,2})\right|>\left|(\mu_{\varepsilon,3}-v_{\varepsilon,3})-(\mu_{\varepsilon,4}-v_{\varepsilon,4})\right|$ for all ε.

(iv) $d\big(\{\langle\varepsilon,[\mu'_\varepsilon,\mu'_\varepsilon],[\eta_{\varepsilon,2},\eta_{\varepsilon,2}],[v'_\varepsilon,v'_\varepsilon]\rangle:\varepsilon\in E\}, \{\langle\varepsilon,[\mu_\varepsilon,\mu_\varepsilon],[\eta_{\varepsilon,2},\eta_{\varepsilon,2}],[v_\varepsilon,v_\varepsilon]\rangle:\varepsilon\in E\}\big)\geq d\big(\{\langle\varepsilon,[\mu'_\varepsilon,\mu'_\varepsilon],[\eta_{\varepsilon,1},\eta_{\varepsilon,1}],[v'_\varepsilon,v'_\varepsilon]\rangle:\varepsilon\in E\},\{\langle\varepsilon,[\mu_\varepsilon,\mu_\varepsilon],[\eta_{\varepsilon,1},\eta_{\varepsilon,1}],[v_\varepsilon,v_\varepsilon]\rangle:\varepsilon\in E\}\big)$ if $\eta_{\varepsilon,2}<\eta_{\varepsilon,1}$ for all ε.

The motive behind the establishment for each axiom is as follows.

(i) As $\{\langle \varepsilon, [1,1], [0,0], [0,0] \rangle : \varepsilon \in E\}$ represent the "purest form of positivity", while $\{\langle \varepsilon, [0,0], [0,0], [1,1] \rangle : \varepsilon \in E\}$ represent the "purest form of negativity", $\{\langle \varepsilon, [1,1], [0,0], [0,0] \rangle : \varepsilon \in E\}$ and $\{\langle \varepsilon, [0,0], [0,0], [1,1] \rangle : \varepsilon \in E\}$ should be the furthest apart among any two arbitrary (t,s)-INSn.

(ii) Distance has no directions, so the distance "from $\mathfrak{a}$ to $\mathfrak{b}$" should be regarded as the same with the distance "from $\mathfrak{b}$ to $\mathfrak{a}$".

(iii) By allowing the cancellation between the positive and the negative membership, the mutually opposite nature of "positivity" and "negativity" is thus established.

(iv) By allowing the neutral membership to dampen the difference between (t,s)-INSn, the intuitive nature of "indeterminacy" is thus established.

Definition 3.2 Let E be a set of parameters. Let $\mathfrak{n}_1 = \left\{\langle \varepsilon, \left[\dot{\mu}^L_{\varepsilon,1}, \dot{\mu}^U_{\varepsilon,1}\right], \left[\dot{\eta}^L_{\varepsilon,1}, \dot{\eta}^U_{\varepsilon,1}\right], \left[\dot{\nu}^L_{\varepsilon,1}, \dot{\nu}^U_{\varepsilon,1}\right] \rangle : \varepsilon \in E\right\}$ and $\mathfrak{n}_2 = \left\{\langle \varepsilon, \left[\dot{\mu}^L_{\varepsilon,2}, \dot{\mu}^U_{\varepsilon,2}\right], \left[\dot{\eta}^L_{\varepsilon,2}, \dot{\eta}^U_{\varepsilon,2}\right], \left[\dot{\nu}^L_{\varepsilon,2}, \dot{\nu}^U_{\varepsilon,2}\right] \rangle : \varepsilon \in E\right\}$ be two (t,s)-INSn for some $t \in [0,\infty)$ and $s \in [1,3]$. The Q *distance measure* between $\mathfrak{n}_1$ and $\mathfrak{n}_2$ under E is defined as follows:

$$d_Q(\mathfrak{n}_1, \mathfrak{n}_2) = \sum_{\varepsilon \in E} \frac{1}{8|E|} \begin{pmatrix} \left|\left(\dot{\mu}^U_{\varepsilon,1} - \dot{\nu}^U_{\varepsilon,1}\right) - \left(\dot{\mu}^U_{\varepsilon,2} - \dot{\nu}^U_{\varepsilon,2}\right)\right|\left(2 - \dot{\eta}^U_{\varepsilon,1} - \dot{\eta}^U_{\varepsilon,2}\right) \\ + \left|\left(\dot{\mu}^L_{\varepsilon,1} - \dot{\nu}^L_{\varepsilon,1}\right) - \left(\dot{\mu}^L_{\varepsilon,2} - \dot{\nu}^L_{\varepsilon,2}\right)\right|\left(2 - \dot{\eta}^L_{\varepsilon,1} - \dot{\eta}^L_{\varepsilon,2}\right) \end{pmatrix}.$$

Without loss of generality, $\dot{\mu}_{\varepsilon,1} = \left[\dot{\mu}^L_{\varepsilon,1}, \dot{\mu}^U_{\varepsilon,1}\right]$ is the ε-relative degree of membership for $\mathfrak{n}_1$, $\dot{\eta}_{\varepsilon,1} = \left[\dot{\eta}^L_{\varepsilon,1}, \dot{\eta}^U_{\varepsilon,1}\right]$ is the ε-relative degree of indeterminacy for $\mathfrak{n}_1$, whereas $\dot{\nu}_{\varepsilon,1} = \left[\dot{\nu}^L_{\varepsilon,1}, \dot{\nu}^U_{\varepsilon,1}\right]$ is the ε-relative degree of non-membership for $\mathfrak{n}_1$.

Lemma 3.3 *The Q distance measure defined in Definition 3.2 is a truthful-distance measure for (t,s)-INSn.*

Proof Let $\mathfrak{n}_1 = \left\{\langle \varepsilon, \left[\dot{\mu}^L_{\varepsilon,1}, \dot{\mu}^U_{\varepsilon,1}\right], \left[\dot{\eta}^L_{\varepsilon,1}, \dot{\eta}^U_{\varepsilon,1}\right], \left[\dot{\nu}^L_{\varepsilon,1}, \dot{\nu}^U_{\varepsilon,1}\right] \rangle : \varepsilon \in E\right\}$ and $\mathfrak{n}_2 = \left\{\langle \varepsilon, \left[\dot{\mu}^L_{\varepsilon,2}, \dot{\mu}^U_{\varepsilon,2}\right], \left[\dot{\eta}^L_{\varepsilon,2}, \dot{\eta}^U_{\varepsilon,2}\right], \left[\dot{\nu}^L_{\varepsilon,2}, \dot{\nu}^U_{\varepsilon,2}\right] \rangle : \varepsilon \in E\right\}$ be two arbitrary (t,s)-INSn for some $t \in [0,\infty)$ and $s \in [1,3]$. So, by Definition 3.2, we have:

$$d_Q(\mathfrak{n}_1, \mathfrak{n}_2) = \sum_{\varepsilon \in E} \frac{1}{8|E|} \begin{pmatrix} \left|\left(\dot{\mu}^U_{\varepsilon,1} - \dot{\nu}^U_{\varepsilon,1}\right) - \left(\dot{\mu}^U_{\varepsilon,2} - \dot{\nu}^U_{\varepsilon,2}\right)\right|\left(2 - \dot{\eta}^U_{\varepsilon,1} - \dot{\eta}^U_{\varepsilon,2}\right) \\ + \left|\left(\dot{\mu}^L_{\varepsilon,1} - \dot{\nu}^L_{\varepsilon,1}\right) - \left(\dot{\mu}^L_{\varepsilon,2} - \dot{\nu}^L_{\varepsilon,2}\right)\right|\left(2 - \dot{\eta}^L_{\varepsilon,1} - \dot{\eta}^L_{\varepsilon,2}\right) \end{pmatrix}.$$

(i) As $0 \le \dot{\eta}^L_{\varepsilon,1}, \dot{\eta}^U_{\varepsilon,1}, \dot{\eta}^L_{\varepsilon,2}, \dot{\eta}^U_{\varepsilon,2} \le 1$,

$$\begin{aligned} &\sum_{\varepsilon \in E} \frac{1}{8|E|} \begin{pmatrix} \left|\left(\dot{\mu}^U_{\varepsilon,1} - \dot{\nu}^U_{\varepsilon,1}\right) - \left(\dot{\mu}^U_{\varepsilon,2} - \dot{\nu}^U_{\varepsilon,2}\right)\right|\left(2 - \dot{\eta}^U_{\varepsilon,1} - \dot{\eta}^U_{\varepsilon,2}\right) \\ + \left|\left(\dot{\mu}^L_{\varepsilon,1} - \dot{\nu}^L_{\varepsilon,1}\right) - \left(\dot{\mu}^L_{\varepsilon,2} - \dot{\nu}^L_{\varepsilon,2}\right)\right|\left(2 - \dot{\eta}^L_{\varepsilon,1} - \dot{\eta}^L_{\varepsilon,2}\right) \end{pmatrix} \\ &\le \sum_{\varepsilon \in E} \frac{1}{8|E|} \begin{pmatrix} 2\left|\left(\dot{\mu}^U_{\varepsilon,1} - \dot{\nu}^U_{\varepsilon,1}\right) - \left(\dot{\mu}^U_{\varepsilon,2} - \dot{\nu}^U_{\varepsilon,2}\right)\right| \\ +2\left|\left(\dot{\mu}^L_{\varepsilon,1} - \dot{\nu}^L_{\varepsilon,1}\right) - \left(\dot{\mu}^L_{\varepsilon,2} - \dot{\nu}^L_{\varepsilon,2}\right)\right| \end{pmatrix} \\ &= \sum_{\varepsilon \in E} \frac{1}{|E|} \begin{pmatrix} \frac{1}{4}\left|\left(\dot{\mu}^U_{\varepsilon,1} - \dot{\nu}^U_{\varepsilon,1}\right) - \left(\dot{\mu}^U_{\varepsilon,2} - \dot{\nu}^U_{\varepsilon,2}\right)\right| \\ +\frac{1}{4}\left|\left(\dot{\mu}^L_{\varepsilon,1} - \dot{\nu}^L_{\varepsilon,1}\right) - \left(\dot{\mu}^L_{\varepsilon,2} - \dot{\nu}^L_{\varepsilon,2}\right)\right| \end{pmatrix} \\ &= \frac{1}{|E|} \sum_{\varepsilon \in E} \begin{pmatrix} \frac{1}{4}\left|\left(\dot{\mu}^U_{\varepsilon,1} - \dot{\nu}^U_{\varepsilon,1}\right) - \left(\dot{\mu}^U_{\varepsilon,2} - \dot{\nu}^U_{\varepsilon,2}\right)\right| \\ +\frac{1}{4}\left|\left(\dot{\mu}^L_{\varepsilon,1} - \dot{\nu}^L_{\varepsilon,1}\right) - \left(\dot{\mu}^L_{\varepsilon,2} - \dot{\nu}^L_{\varepsilon,2}\right)\right| \end{pmatrix}. \end{aligned}$$

As $0 \le \dot{\mu}^L_{\varepsilon,1}, \dot{\mu}^U_{\varepsilon,1}, \dot{\nu}^L_{\varepsilon,1}, \dot{\nu}^U_{\varepsilon,1}, \dot{\mu}^L_{\varepsilon,2}, \dot{\mu}^U_{\varepsilon,2}, \dot{\nu}^L_{\varepsilon,2}, \dot{\nu}^U_{\varepsilon,2} \le 1$ for all ε. So $-1 \le \left(\dot{\mu}^L_{\varepsilon,1} - \dot{\nu}^L_{\varepsilon,1}\right) \le 1$, $-1 \le \left(\dot{\mu}^L_{\varepsilon,2} - \dot{\nu}^L_{\varepsilon,2}\right) \le 1$, $-1 \le \left(\dot{\mu}^U_{\varepsilon,1} - \dot{\nu}^U_{\varepsilon,1}\right) \le 1$ and $-1 \le \left(\dot{\mu}^U_{\varepsilon,2} - \dot{\nu}^U_{\varepsilon,2}\right) \le 1$ will follow. These further imply that $0 \le \left|\left(\dot{\mu}^U_{\varepsilon,1} - \dot{\nu}^U_{\varepsilon,1}\right) - \left(\dot{\mu}^U_{\varepsilon,2} - \dot{\nu}^U_{\varepsilon,2}\right)\right| \le 2$ and $0 \le \left|\left(\dot{\mu}^L_{\varepsilon,1} - \dot{\nu}^L_{\varepsilon,1}\right) - \left(\dot{\mu}^L_{\varepsilon,2} - \dot{\nu}^L_{\varepsilon,2}\right)\right| \le 2$.

Therefore, it follows that:

Table 1 Findings of the magazine editors α and β after evaluating the financial performance of companies u, v and w, for the scenario discussed in Example 3.4

	u	v	w
α	5–10% of the investors showed bullish behaviour towards the company. 5–10% investors showed bearish behaviour towards the company	20–25% of the investors showed bullish behaviour towards the company. 20–25% investors showed bearish behaviour towards the company	20–25% of the investors showed bullish behaviour towards the company. No investors showed bearish behaviour towards the company
β	5–15% of the investors showed bullish behaviour towards the company. 5–15% of the investors showed bearish behaviour towards the company	15–25% of the investors showed bullish behaviour towards the company. 15–25% of the investors showed bearish behaviour towards the company	15–25% of the investors showed bullish behaviour towards the company. No investors showed bearish behaviour towards the company

$$\begin{aligned}
&\sum_{\varepsilon\in E}\frac{1}{8|E|}\begin{pmatrix} \left|\left(\dot{\mu}^U_{\varepsilon,1}-\dot{\nu}^U_{\varepsilon,1}\right)-\left(\dot{\mu}^U_{\varepsilon,2}-\dot{\nu}^U_{\varepsilon,2}\right)\right|\left(2-\dot{\eta}^U_{\varepsilon,1}-\dot{\eta}^U_{\varepsilon,2}\right)\\ +\left|\left(\dot{\mu}^L_{\varepsilon,1}-\dot{\nu}^L_{\varepsilon,1}\right)-\left(\dot{\mu}^L_{\varepsilon,2}-\dot{\nu}^L_{\varepsilon,2}\right)\right|\left(2-\dot{\eta}^L_{\varepsilon,1}-\dot{\eta}^L_{\varepsilon,2}\right)\end{pmatrix}\\
&\quad\le\frac{1}{|E|}\sum_{\varepsilon\in E}\begin{pmatrix}\frac{1}{4}\left|\left(\dot{\mu}^U_{\varepsilon,1}-\dot{\nu}^U_{\varepsilon,1}\right)-\left(\dot{\mu}^U_{\varepsilon,2}-\dot{\nu}^U_{\varepsilon,2}\right)\right|\\ +\frac{1}{4}\left|\left(\dot{\mu}^L_{\varepsilon,1}-\dot{\nu}^L_{\varepsilon,1}\right)-\left(\dot{\mu}^L_{\varepsilon,2}-\dot{\nu}^L_{\varepsilon,2}\right)\right|\end{pmatrix}\\
&\quad\le\frac{1}{|E|}\sum_{\varepsilon\in E}\left(\frac{1}{4}(2)+\frac{1}{4}(2)\right)\\
&\quad=\frac{1}{|E|}|E|\\
&\quad=1.
\end{aligned}$$

On the other hand, as $0\le\dot{\eta}^L_{\varepsilon,1},\dot{\eta}^U_{\varepsilon,1},\dot{\eta}^L_{\varepsilon,2},\dot{\eta}^U_{\varepsilon,2}\le 1$, we have $2-\dot{\eta}^U_{\varepsilon,1}-\dot{\eta}^U_{\varepsilon,2}\ge 0$ and $2-\dot{\eta}^L_{\varepsilon,1}-\dot{\eta}^L_{\varepsilon,2}\ge 0$. As $\left|\left(\dot{\mu}^U_{\varepsilon,1}-\dot{\nu}^U_{\varepsilon,1}\right)-\left(\dot{\mu}^U_{\varepsilon,2}-\dot{\nu}^U_{\varepsilon,2}\right)\right|$, $\left|\left(\dot{\mu}^L_{\varepsilon,1}-\dot{\nu}^L_{\varepsilon,1}\right)-\left(\dot{\mu}^L_{\varepsilon,2}-\dot{\nu}^L_{\varepsilon,2}\right)\right|\ge 0$ too, and therefore it follows that $0\le \frac{1}{8|E|}\begin{pmatrix} \left|\left(\dot{\mu}^U_{\varepsilon,1}-\dot{\nu}^U_{\varepsilon,1}\right)-\left(\dot{\mu}^U_{\varepsilon,2}-\dot{\nu}^U_{\varepsilon,2}\right)\right|\left(2-\dot{\eta}^U_{\varepsilon,1}-\dot{\eta}^U_{\varepsilon,2}\right)\\ +\left|\left(\dot{\mu}^L_{\varepsilon,1}-\dot{\nu}^L_{\varepsilon,1}\right)-\left(\dot{\mu}^L_{\varepsilon,2}-\dot{\nu}^L_{\varepsilon,2}\right)\right|\left(2-\dot{\eta}^L_{\varepsilon,1}-\dot{\eta}^L_{\varepsilon,2}\right)\end{pmatrix}$ for all ε. Hence, $0\le d_Q(\mathfrak{n}_1,\mathfrak{n}_2)\le 1$ holds.

$$\begin{aligned}
&d_Q(\{\langle\varepsilon,[1,1],[0,0],[0,0]\rangle:\varepsilon\in E\},\{\langle\varepsilon,[0,0],[0,0],[1,1]\rangle:\varepsilon\in E\})\\
&=\sum_{\varepsilon\in E}\frac{1}{8|E|}\begin{pmatrix}|(1-0)-(0-1)|(2-0-0)\\ +|(1-0)-(0-1)|(2-0-0)\end{pmatrix}\\
&=1
\end{aligned}$$

$$\begin{aligned}
&d_Q(\mathfrak{n}_1,\mathfrak{n}_1)\\
&=\sum_{\varepsilon\in E}\frac{1}{8|E|}\begin{pmatrix} \left|\left(\dot{\mu}^U_{\varepsilon,1}-\dot{\nu}^U_{\varepsilon,1}\right)-\left(\dot{\mu}^U_{\varepsilon,1}-\dot{\nu}^U_{\varepsilon,1}\right)\right|\left(2-\dot{\eta}^U_{\varepsilon,1}-\dot{\eta}^U_{\varepsilon,1}\right)\\ +\left|\left(\dot{\mu}^L_{\varepsilon,1}-\dot{\nu}^L_{\varepsilon,1}\right)-\left(\dot{\mu}^L_{\varepsilon,1}-\dot{\nu}^L_{\varepsilon,1}\right)\right|\left(2-\dot{\eta}^L_{\varepsilon,1}-\dot{\eta}^L_{\varepsilon,1}\right)\end{pmatrix}=0.
\end{aligned}$$

(ii), (iii), (iv) The proofs follow directly from Definition 3.2.

The proposed Q distance measure from Definition 3.2 is now compared with the two mainstream choices of distance measures, i.e. the Euclidean and Hamming distance measures, through Examples 3.4 and 3.5 on the case study of the perception of investors on a company. Example 3.4 highlights the effect of cancellation by entities of opposite natures, whereas Example 3.5 highlights the effect of the degree of indeterminacy in dampening the membership and non-membership degrees.

Example 3.4 Suppose there are two magazine editors, denoted by α and β who are evaluating the financial performance of three companies, denoted by u, v, and w, based on the perception of all the investors (whether bullish or bearish). Table 1 summarizes the findings of both the magazine editors.

This case can be modelled by the following three (t,s)-INSn, for some $t\in[0,\infty)$ and $s\in[1,3]$, as follows.

$$\begin{aligned}
\mathfrak{n}_u&=\{\langle\alpha,[0.05,0.10],[0.00,0.00],[0.05,0.10]\rangle,\\
&\qquad\langle\beta,[0.05,0.15],[0.00,0.00],[0.05,0.15]\rangle\}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{n}_v&=\{\langle\alpha,[0.20,0.25],[0.00,0.00],[0.20,0.25]\rangle,\\
&\qquad\langle\beta,[0.15,0.25],[0.00,0.00],[0.15,0.25]\rangle\}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{n}_w&=\{\langle\alpha,[0.20,0.25],[0.00,0.00],[0.00,0.00]\rangle,\\
&\qquad\langle\beta,[0.15,0.25],[0.00,0.00],[0.00,0.00]\rangle\}
\end{aligned}$$

for which $E=\{\alpha,\beta\}$.

Hence, by human intuition, it is evident that w performs significantly better than u and v because there only exist investors acting bullishly towards w, whereas u and v are given equal proportions of bullish and bearish behaviours by the investors. Moreover, as u and v are given equal proportions of bullish and bearish behaviours, the performance difference between u and v is not significant. Therefore, the distance between u and w should be regarded as significantly farther than the distance between u and v.

However, all conventional distance measures observed in the past studies, including those proposed in Ganie and Singh (2021), will end up treating all entities in the three (t,s)-INSn in the same manner as if they are components of a vector. For instance, the Hamming distance measure produces the following values:

$$d_h(\text{п}_u, \text{п}_w) = \frac{1}{2}\left(\begin{array}{l} \left|\frac{0.05+0.10}{2} - \frac{0.20+0.25}{2}\right| + \left|\frac{0.00+0.00}{2} - \frac{0.00+0.00}{2}\right| + \left|\frac{0.05+0.10}{2} - \frac{0.00+0.00}{2}\right| \\ + \left|\frac{0.05+0.15}{2} - \frac{0.15+0.25}{2}\right| + \left|\frac{0.00+0.00}{2} - \frac{0.00+0.00}{2}\right| + \left|\frac{0.05+0.15}{2} - \frac{0.00+0.00}{2}\right| \end{array}\right)$$
$$= 0.2125.$$

$$d_h(\text{п}_u, \text{п}_v) = \frac{1}{2}\left(\begin{array}{l} \left|\frac{0.05+0.10}{2} - \frac{0.20+0.25}{2}\right| + \left|\frac{0.00+0.00}{2} - \frac{0.00+0.00}{2}\right| + \left|\frac{0.05+0.10}{2} - \frac{0.20+0.25}{2}\right| \\ + \left|\frac{0.05+0.15}{2} - \frac{0.15+0.25}{2}\right| + \left|\frac{0.00+0.00}{2} - \frac{0.00+0.00}{2}\right| + \left|\frac{0.05+0.15}{2} - \frac{0.15+0.25}{2}\right| \end{array}\right)$$
$$= 0.2500$$

whereas the Euclidean distance measure produces the following values:

$$d_e(\text{п}_u, \text{п}_w) = \frac{1}{2}\sqrt{\begin{array}{l} \left(\frac{0.05+0.10}{2} - \frac{0.20+0.25}{2}\right)^2 + \left(\frac{0.00+0.00}{2} - \frac{0.00+0.00}{2}\right)^2 + \left(\frac{0.05+0.10}{2} - \frac{0.00+0.00}{2}\right)^2 \\ + \left(\frac{0.05+0.15}{2} - \frac{0.15+0.25}{2}\right)^2 + \left(\frac{0.00+0.00}{2} - \frac{0.00+0.00}{2}\right)^2 + \left(\frac{0.05+0.15}{2} - \frac{0.00+0.00}{2}\right)^2 \end{array}}$$
$$= 0.1097$$

$$d_e(\text{п}_u, \text{п}_v) = \frac{1}{2}\sqrt{\begin{array}{l} \left(\frac{0.05+0.10}{2} - \frac{0.20+0.25}{2}\right)^2 + \left(\frac{0.00+0.00}{2} - \frac{0.00+0.00}{2}\right)^2 + \left(\frac{0.05+0.10}{2} - \frac{0.20+0.25}{2}\right)^2 \\ + \left(\frac{0.05+0.15}{2} - \frac{0.15+0.25}{2}\right)^2 + \left(\frac{0.00+0.00}{2} - \frac{0.00+0.00}{2}\right)^2 + \left(\frac{0.05+0.15}{2} - \frac{0.15+0.25}{2}\right)^2 \end{array}}$$
$$= 0.1275$$

In both the cases above, v is deduced to be farther from u than w is from u, which contradicts the human judgement/intuition mentioned previously.

On the other hand, the proposed Q distance measure in Definition 3.2 proves to be capable of fulfilling the human judgement/intuition, and this can be seen from the values of d_Q given below.

$$d_Q(\text{п}_u, \text{п}_w) = \frac{1}{8(2)}\left(\begin{array}{l} |(0.05-0.05)-(0.20-0.00)|(2-0.00-0.00) \\ +|(0.10-0.10)-(0.25-0.00)|(2-0.00-0.00) \\ +|(0.05-0.05)-(0.15-0.00)|(2-0.00-0.00) \\ +|(0.15-0.15)-(0.25-0.00)|(2-0.00-0.00) \end{array}\right)$$
$$= 0.1063$$

$$d_Q(\text{п}_u, \text{п}_v) = \frac{1}{8(2)}\left(\begin{array}{l} |(0.05-0.05)-(0.20-0.20)|(2-0.00-0.00) \\ +|(0.10-0.10)-(0.25-0.25)|(2-0.00-0.00) \\ +|(0.05-0.05)-(0.15-0.15)|(2-0.00-0.00) \\ +|(0.15-0.15)-(0.25-0.25)|(2-0.00-0.00) \end{array}\right)$$
$$= 0.0000$$

Therefore, $d_Q(\text{п}_u, \text{п}_w) > d_Q(\text{п}_u, \text{п}_v)$. This is because the terms $\left(\dot{\mu}^U_{\varepsilon,1} - \dot{\nu}^U_{\varepsilon,1}\right)$, $\left(\dot{\mu}^L_{\varepsilon,1} - \dot{\nu}^L_{\varepsilon,1}\right)$, $\left(\dot{\mu}^U_{\varepsilon,2} - \dot{\nu}^U_{\varepsilon,2}\right)$ and $\left(\dot{\mu}^L_{\varepsilon,2} - \dot{\nu}^L_{\varepsilon,2}\right)$ from Definition 3.2 enable the mutual cancellation of the degree of membership with the degree of non-membership, which are known to be mutually opposite in nature.

Example 3.5 Suppose the two magazine editors α, β (from Example 3.4) are now evaluating the financial performance of four other companies, denoted by p, q, r, and s, again based on the perception by all the investors (whether bullish or bearish). Table 2 summarizes the findings of the editors.

This case can be modelled by the following four (t, s)-INSn, for some $t \in [0, \infty)$ and $s \in [1, 3]$, as follows.

$$\text{n}_p = \{\langle\alpha, [0.20, 0.25], [0.00, 0.00], [0.00, 0.00]\rangle, \langle\beta, [0.10, 0.15], [0.00, 0.00], [0.00, 0.00]\rangle\}$$

$$\text{n}_q = \{\langle\alpha, [0.00, 0.00], [0.00, 0.00], [0.10, 0.15]\rangle, \langle\beta, [0.00, 0.00], [0.00, 0.00], [0.05, 0.10]\rangle\}$$

$$\text{n}_r = \{\langle\alpha, [0.20, 0.25], [0.30, 0.35], [0.00, 0.00]\rangle, \langle\beta, [0.10, 0.15], [0.25, 0.35], [0.00, 0.00]\rangle\}$$

$$\text{n}_s = \{\langle\alpha, [0.00, 0.00], [0.35, 0.40], [0.10, 0.15]\rangle, \langle\beta, [0.00, 0.00], [0.20, 0.25], [0.05, 0.10]\rangle\}$$

Likewise, $E = \{\alpha, \beta\}$.

Hence, by human intuition, it is evident that r and s should be regarded as having less mutual difference compared to p and q, because there only exist investors who are unsure of their own actions towards r and s. Therefore, the distance between r and s should be regarded as significantly closer than the distance between p and q due to the higher degree of uncertainty.

Likewise for this case, the Hamming distance measure produces the following values:

$$d_h(\text{n}_r, \text{n}_s) = \frac{1}{2}\left(\begin{array}{l} \left|\frac{0.20+0.25}{2} - \frac{0.00+0.00}{2}\right| + \left|\frac{0.30+0.35}{2} - \frac{0.35+0.40}{2}\right| + \left|\frac{0.00+0.00}{2} - \frac{0.10+0.15}{2}\right| \\ + \left|\frac{0.10+0.15}{2} - \frac{0.00+0.00}{2}\right| + \left|\frac{0.25+0.35}{2} - \frac{0.20+0.25}{2}\right| + \left|\frac{0.00+0.00}{2} - \frac{0.05+0.10}{2}\right| \end{array}\right) = 0.3375$$

$$d_h(\text{n}_p, \text{n}_q) = \frac{1}{2}\left(\begin{array}{l} \left|\frac{0.20+0.25}{2} - \frac{0.00+0.00}{2}\right| + \left|\frac{0.00+0.00}{2} - \frac{0.00+0.00}{2}\right| + \left|\frac{0.00+0.00}{2} - \frac{0.10+0.15}{2}\right| \\ + \left|\frac{0.10+0.15}{2} - \frac{0.00+0.00}{2}\right| + \left|\frac{0.00+0.00}{2} - \frac{0.00+0.00}{2}\right| + \left|\frac{0.00+0.00}{2} - \frac{0.05+0.10}{2}\right| \end{array}\right) = 0.2750$$

whereas the Euclidean distance measure produces the following values:

$$d_e(\text{n}_r, \text{n}_s) = \frac{1}{2}\sqrt{\begin{array}{l} \left(\frac{0.20+0.25}{2} - \frac{0.00+0.00}{2}\right)^2 + \left(\frac{0.30+0.35}{2} - \frac{0.35+0.40}{2}\right)^2 + \left(\frac{0.00+0.00}{2} - \frac{0.10+0.15}{2}\right)^2 \\ + \left(\frac{0.10+0.15}{2} - \frac{0.00+0.00}{2}\right)^2 + \left(\frac{0.25+0.35}{2} - \frac{0.20+0.25}{2}\right)^2 + \left(\frac{0.00+0.00}{2} - \frac{0.05+0.10}{2}\right)^2 \end{array}} = 0.1546$$

$$d_e(\text{n}_p, \text{n}_q) = \frac{1}{2}\sqrt{\begin{array}{l} \left(\frac{0.20+0.25}{2} - \frac{0.00+0.00}{2}\right)^2 + \left(\frac{0.00+0.00}{2} - \frac{0.00+0.00}{2}\right)^2 + \left(\frac{0.00+0.00}{2} - \frac{0.10+0.15}{2}\right)^2 \\ + \left(\frac{0.10+0.15}{2} - \frac{0.00+0.00}{2}\right)^2 + \left(\frac{0.00+0.00}{2} - \frac{0.00+0.00}{2}\right)^2 + \left(\frac{0.00+0.00}{2} - \frac{0.05+0.10}{2}\right)^2 \end{array}} = 0.1479$$

Table 2 Findings of the magazine editors α and β after evaluating the financial performance of companies p, q, r and s, for the scenario discussed in Example 3.5

	p	q	r	s
α	20–25% of the investors shown bullish behaviour towards the company. No investors express their unsureness of actions towards the company	10–15% of the investors shown bearish behaviour towards the company. No investors express their unsureness of actions towards the company	20–25% of the investors shown bullish behaviour towards the company. 30–35% of the investors expressed that they are not sure of their actions towards the company	10–15% of the investors shown bearish behaviour towards the company. 35–40% of the investors expressed that they are not sure of their actions towards the company
β	10–15% of the investors shown bullish behaviour towards the company. No investors express their unsureness of actions towards the company	5–10% of the investors shown bearish behaviour towards the company. No investors express their unsureness of actions towards the company	10–15% of the investors shown bullish behaviour towards the company. 25–35% of the investors expressed that they are not sure of their actions towards the company	5–10% of the investors shown bearish behaviour towards the company. 20–25% of the investors expressed that they are not sure of their actions towards the company

In both the cases above, r and s are deemed farther apart from each other, compared to p and q, which contradicts the human judgement/intuition mentioned previously.

On the other hand, the proposed Q distance measure in Definition 3.2 proves to be capable of fulfilling the human judgement/intuition, as evident from the values of d_Q given below.

$$d_Q(\mathfrak{n}_r,\mathfrak{n}_s)=\frac{1}{8(2)}\left(\begin{array}{l}|(0.25-0.00)-(0.00-0.15)|(2-0.35-0.40)\\+|(0.20-0.00)-(0.00-0.10)|(2-0.30-0.35)\\+|(0.15-0.00)-(0.00-0.10)|(2-0.35-0.25)\\+|(0.10-0.00)-(0.00-0.05)|(2-0.25-0.20)\end{array}\right)$$
$$=0.0930$$

$$d_Q(\mathfrak{n}_p,\mathfrak{n}_q)=\frac{1}{8(2)}\left(\begin{array}{l}|(0.25-0.00)-(0.00-0.15)|(2-0.00-0.00)\\+|(0.20-0.00)-(0.00-0.10)|(2-0.00-0.00)\\+|(0.15-0.00)-(0.00-0.10)|(2-0.00-0.00)\\+|(0.10-0.00)-(0.00-0.05)|(2-0.00-0.00)\end{array}\right)$$
$$=0.1375$$

Therefore, $d_Q(\mathfrak{n}_r,\mathfrak{n}_s)<d_Q(\mathfrak{n}_p,\mathfrak{n}_q)$. This is because the terms $\left(2-\dot{\eta}^U_{\varepsilon,1}-\dot{\eta}^U_{\varepsilon,2}\right)$ and $\left(2-\dot{\eta}^L_{\varepsilon,1}-\dot{\eta}^L_{\varepsilon,2}\right)$ from Definition 3.2 enable the dampening of the membership and non-membership degrees by the degree of indeterminacy.

Definition 3.6 Let E be a set of parameters, and $\mathfrak{n}=\left\{\langle\varepsilon,\left[\dot{\mu}^L_\varepsilon,\dot{\mu}^U_\varepsilon\right],\left[\dot{\eta}^L_\varepsilon,\dot{\eta}^U_\varepsilon\right],\left[\dot{\nu}^L_\varepsilon,\dot{\nu}^U_\varepsilon\right]\rangle:\varepsilon\in E\right\}$ be a (t,s)-INSn for some $t\in[0,\infty)$ and $s\in[1,3]$. The Q *score function* of $\mathfrak{n}$ under E is defined as follows:

$$s_Q(\mathfrak{n})=\sum_{\varepsilon\in E}\frac{1}{|E|}\left(\frac{2+\left(\dot{\mu}^U_\varepsilon-\dot{\nu}^U_\varepsilon\right)\left(1-\dot{\eta}^U_\varepsilon\right)+\left(\dot{\mu}^L_\varepsilon-\dot{\nu}^L_\varepsilon\right)\left(1-\dot{\eta}^L_\varepsilon\right)}{4}\right).$$

Like the Q distance measure, the Q score function enables the degree of indeterminacy to take its characteristic effect consistent with its nature as understood by most experts, as shown in Example 3.7.

Example 3.7 Consider again the four companies p, q, r, and s from Example 3.5. Let the four (t,s)-INSn denoted by $\mathfrak{n}_p$, $\mathfrak{n}_q$, $\mathfrak{n}_r$ and $\mathfrak{n}_s$ be defined as follows:

$$\mathfrak{n}_p=\{\langle\alpha,[0.20,0.25],[0.00,0.00],[0.00,0.00]\rangle,\ \langle\beta,[0.10,0.15],[0.00,0.00],[0.00,0.00]\rangle\}$$

$$\mathfrak{n}_q=\{\langle\alpha,[0.00,0.00],[0.00,0.00],[0.10,0.15]\rangle,\ \langle\beta,[0.00,0.00],[0.00,0.00],[0.05,0.10]\rangle\}$$

$$\mathfrak{n}_r=\{\langle\alpha,[0.20,0.25],[0.30,0.35],[0.00,0.00]\rangle,\ \langle\beta,[0.10,0.15],[0.25,0.35],[0.00,0.00]\rangle\}$$

$$\mathfrak{n}_s=\{\langle\alpha,[0.00,0.00],[0.35,0.40],[0.10,0.15]\rangle,\ \langle\beta,[0.00,0.00],[0.20,0.25],[0.05,0.10]\rangle\}$$

with $E=\{\alpha,\beta\}$.

In this example, the membership and non-membership degrees in p are chosen to be the same as the membership and non-membership degrees in r, whereas the membership and non-membership degrees in q are chosen to be the same as the membership and non-membership degrees in s. Therefore, as p is compared against q, and r is compared against s, it is only the degree of indeterminacy that differentiates the distance between r and s with the distance between p and q.

Therefore, by human intuition, the order of preference, from the most preferred to the least preferred, should be p, r, s and q. This is because the degree of indeterminacy in r and s (there only exist investors who are not sure of their actions/perceptions towards r and s) should bring them closer to neutrality.

Therefore, r should be regarded as being "not as good as p". On the other hand, s should be regarded as being "not as bad as q". This is because the presence of uncertainties makes s more worth investing in than q, as it is still possible to generate profits, in case if the investor must choose one company between s and q.

The existing formulas for the score functions proposed in the past studies in the existing literature will simply minus both the indeterminacy and non-membership degrees from the membership degree yielding the following results (or results similar to it).

$$\mathrm{s}(\pi_p) = \frac{1}{2}\left(\frac{0.20+0.25}{2} - \frac{0.00+0.00}{2} - \frac{0.00+0.00}{2} + \frac{0.10+0.15}{2} - \frac{0.00+0.00}{2} - \frac{0.00+0.00}{2}\right) = 0.1750$$

$$\mathrm{s}(\pi_q) = \frac{1}{2}\left(\frac{0.00+0.00}{2} - \frac{0.00+0.00}{2} - \frac{0.10+0.15}{2} + \frac{0.00+0.00}{2} - \frac{0.00+0.00}{2} - \frac{0.05+0.10}{2}\right) = -0.1000$$

$$\mathrm{s}(\pi_r) = \frac{1}{2}\left(\frac{0.20+0.25}{2} - \frac{0.30+0.35}{2} - \frac{0.00+0.00}{2} + \frac{0.10+0.15}{2} - \frac{0.25+0.35}{2} - \frac{0.00+0.00}{2}\right) = -0.1375$$

$$\mathrm{s}(\pi_s) = \frac{1}{2}\left(\frac{0.00+0.00}{2} - \frac{0.35+0.40}{2} - \frac{0.10+0.15}{2} + \frac{0.00+0.00}{2} - \frac{0.20+0.25}{2} - \frac{0.05+0.10}{2}\right) = -0.4000$$

From the results above, it can be deduced that r and s are performing poorer than q, which contradicts human intuitions/judgement. This is because the degree of indeterminacy is not given an independent role true to its intuitive meaning in the conventional formulas for score function in the existing literature. Instead, the degree of indeterminacy is given the exact same role as the degree of non-membership, which defeats the fundamental purpose of having the degree of indeterminacy as the third entity.

On the other hand, the proposed Q score function in Definition 3.6 proves to be capable of fulfilling the human judgement/intuition which is evident from the values given below.

$$\begin{aligned}\mathrm{s}_Q(\pi_p) &= \frac{1}{2}\left(\frac{2+(0.25-0.00)(1-0.00)+(0.20-0.00)(1-0.00)}{4} + \frac{2+(0.15-0.00)(1-0.00)+(0.10-0.00)(1-0.00)}{4}\right)\\ &= 0.5875\end{aligned}$$

$$\begin{aligned}\mathrm{s}_Q(\pi_q) &= \frac{1}{2}\left(\frac{2+(0.00-0.10)(1-0.00)+(0.00-0.15)(1-0.00)}{4} + \frac{2+(0.00-0.05)(1-0.00)+(0.00-0.10)(1-0.00)}{4}\right)\\ &= 0.4500\end{aligned}$$

$$\begin{aligned}\mathrm{s}_Q(\pi_r) &= \frac{1}{2}\left(\frac{2+(0.25-0.00)(1-0.35)+(0.20-0.00)(1-0.30)}{4} + \frac{2+(0.15-0.00)(1-0.35)+(0.10-0.00)(1-0.25)}{4}\right)\\ &= 0.5593\end{aligned}$$

$$\begin{aligned}\mathrm{s}_Q(\pi_s) &= \frac{1}{2}\left(\frac{2+(0.00-0.10)(1-0.40)+(0.00-0.15)(1-0.35)}{4} + \frac{2+(0.00-0.05)(1-0.25)+(0.00-0.10)(1-0.20)}{4}\right)\\ &= 0.4656\end{aligned}$$

From the results above, we have $\mathrm{s}_Q(\pi_p) > \mathrm{s}_Q(\pi_r) > \mathrm{s}_Q(\pi_s) > \mathrm{s}_Q(\pi_q)$. This is because the terms $\left(1-\dot{\eta}_\varepsilon^U\right)$ and $\left(1-\dot{\eta}_\varepsilon^L\right)$ in Definition 3.6 enable the dampening of the membership and non-membership degrees by the degree of indeterminacy.

Definition 3.8 Let E be a set of parameters. Let $\pi = \left\{\langle \varepsilon, \left[\dot{\mu}_\varepsilon^L, \dot{\mu}_\varepsilon^U\right], \left[\dot{\eta}_\varepsilon^L, \dot{\eta}_\varepsilon^U\right], \left[\dot{\nu}_\varepsilon^L, \dot{\nu}_\varepsilon^U\right]\rangle : \varepsilon \in E\right\}$ be a (t, s)-INSn for some $t \in [0, \infty)$ and $s \in [1, 3]$. The Q *accuracy function* of π under E is defined as follows:

$$\mathrm{a}_Q(\pi) = \sum_{\varepsilon \in E} \frac{1}{|E|}\left(\frac{2-\dot{\eta}_\varepsilon^L-\dot{\eta}_\varepsilon^U}{\max\left\{\dot{\mu}_\varepsilon^U+\dot{\nu}_\varepsilon^U, 1\right\}+\max\left\{\dot{\mu}_\varepsilon^L+\dot{\nu}_\varepsilon^L, 1\right\}}\right).$$

Like the Q score function, the Q accuracy function provides a faithful interpretation of indeterminacy. Moreover, the membership and non-membership degrees will only influence the accuracy when $\dot{\mu}_\varepsilon^U + \dot{\nu}_\varepsilon^U > 1$, which is understood to be an obvious sign of the existence of mutually contradicting information.

Therefore, the reliability of any TOPSIS or VIKOR algorithms depend greatly on the choice of its chosen distance measures and score functions which must be shown to be consistent with human intuition/judgement for a given scenario, as demonstrated in Examples 3.4, 3.5, and 3.7. These examples shall likewise serve as another three indicators, alongside the first four indicators from Sect. 2, for the comparative study which will be presented and discussed in Sect. 6. The comparative study will discuss and analyse various TOPSIS and VIKOR algorithms that have been established in past studies to examine if these existing algorithms with their respective chosen distance measure d and score function s are capable of yielding results that fulfil the following conditions:

(i) $d(\mathfrak{n}_u, \mathfrak{n}_w) > d(\mathfrak{n}_u, \mathfrak{n}_v)$ (from Example 3.4)
(ii) $d(\mathfrak{n}_r, \mathfrak{n}_s) < d(\mathfrak{n}_p, \mathfrak{n}_q)$ (from Example 3.5)
(iii) $s(\mathfrak{n}_p) > s(\mathfrak{n}_r) > s(\mathfrak{n}_s) > s(\mathfrak{n}_q)$ (from Example 3.7)

all of which are the relationships that are deemed consistent with human knowledge, intuition, and judgement.

4 Methodology for TOPSIS and VIKOR for the proposed (t, s) -INSS

In this section, the procedure of computations for the TOPSIS and VIKOR algorithms will be presented in which the newly introduced distance measure and score function will be used.

4.1 All the Inputs

Given E as a set of parameters, and the decision matrix, $\mathbf{M}$ defined as follows:

$$\mathbf{M} = \begin{pmatrix} \mathfrak{n}_{1,1} & \mathfrak{n}_{1,2} & \cdots & \mathfrak{n}_{1,n} \\ \mathfrak{n}_{2,1} & \mathfrak{n}_{2,2} & \cdots & \mathfrak{n}_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ \mathfrak{n}_{m,1} & \mathfrak{n}_{m,2} & \cdots & \mathfrak{n}_{m,n} \end{pmatrix},$$

where $t_0 \in [0, \infty)$ and $s_0 \in [1, 3]$ are such that all $\mathfrak{n}_{i,j} = \left\{\langle \varepsilon, \left[\dot{\mu}^L_{\varepsilon,(i,j)}, \dot{\mu}^U_{\varepsilon,(i,j)}\right], \left[\dot{\eta}^L_{\varepsilon,(i,j)}, \dot{\eta}^U_{\varepsilon,(i,j)}\right], \left[\dot{\nu}^L_{\varepsilon,(i,j)}, \dot{\nu}^U_{\varepsilon,(i,j)}\right]\rangle : \varepsilon \in E\right\}$ are (t_0, s_0)-INSn representing the assessment result of the ith alternative on the jth attribute among all the parameters involved (i.e. $\mathbf{M}$ is a (t_0, s_0)-INSM). The vector $\mathbf{h} = (h_1, h_2, \ldots, h_n)$ is the subjective weight vector for each of the attributes.

4.2 TOPSIS

Step T1: Compute the objective weight vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$.

$$\theta_k = \frac{\sum_{i=1}^m \sum_{l=1}^m d\left(\mathfrak{n}_{i,k}, \mathfrak{n}_{l,k}\right)}{\sum_{j=1}^n \sum_{i=1}^m \sum_{l=1}^m d\left(\mathfrak{n}_{i,j}, \mathfrak{n}_{l,j}\right)}$$

for all $1 \leq k \leq n$; In the demonstration presented in this article, d is taken to be the Q distance, d_Q.

Remark The objective weight vector is therefore an entity that is normalized so that $\sum_{h=1}^n \theta_h = 1$. In the formula, the numerator $\sum_{i=1}^m \sum_{l=1}^m d\left(\mathfrak{n}_{i,k}, \mathfrak{n}_{l,k}\right)$ determines the relative size of each θ_k for all k, whereas the denominator $\sum_{j=1}^n \sum_{i=1}^m \sum_{l=1}^m d\left(\mathfrak{n}_{i,j}, \mathfrak{n}_{l,j}\right)$ serves to normalize all θ_k in $\boldsymbol{\theta}$ so that $\sum_{h=1}^n \theta_h = 1$.

Step T2: Compute the integrated weight vector $\boldsymbol{\omega} = (\omega_1, \omega_2, \ldots, \omega_n)$ from the subjective weight vector $\mathbf{h} = (h_1, h_2, \ldots, h_n)$ and the objective weight vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_n)$.

$$\omega_k = \frac{h_k \theta_k}{\sum_{j=1}^n h_j \theta_j}$$

for all $1 \leq k \leq n$.

Step T3: Compute the positive ideal solutions $\boldsymbol{\Gamma}^+ = \left(\mathfrak{n}_1^+, \mathfrak{n}_2^+, \ldots, \mathfrak{n}_n^+\right)$ and the negative ideal solutions $\boldsymbol{\Gamma}^- = \left(\mathfrak{n}_1^-, \mathfrak{n}_2^-, \ldots, \mathfrak{n}_n^-\right)$ using the following equations.

$$\mathfrak{n}_\varrho^+ = \left\{\left\langle \varepsilon, \left[\mu^{L+}_{\varepsilon,\varrho}, \mu^{U+}_{\varepsilon,\varrho}\right], \left[\dot{\eta}^L_{\varepsilon,\varrho}, \dot{\eta}^U_{\varepsilon,\varrho}\right], \left[\nu^{L-}_{\varepsilon,\varrho}, \nu^{U-}_{\varepsilon,\varrho}\right]\right\rangle : \varepsilon \in E\right\}$$

$$= \left\{\left\langle \varepsilon, \left[\max_i \dot{\mu}^L_{\varepsilon,(i,\varrho)}, \max_i \dot{\mu}^U_{\varepsilon,(i,\varrho)}\right], \left[\dot{\eta}^L_{\varepsilon,\varrho}, \dot{\eta}^U_{\varepsilon,\varrho}\right], \left[\min_i \dot{\nu}^L_{\varepsilon,(i,\varrho)}, \min_i \dot{\nu}^U_{\varepsilon,(i,\varrho)}\right]\right\rangle : \varepsilon \in E\right\}$$

$$\mathfrak{n}_\varrho^- = \left\{\left\langle \varepsilon, \left[\mu^{L-}_{\varepsilon,\varrho}, \mu^{U-}_{\varepsilon,\varrho}\right], \left[\dot{\eta}^L_{\varepsilon,\varrho}, \dot{\eta}^U_{\varepsilon,\varrho}\right], \left[\nu^{L+}_{\varepsilon,\varrho}, \nu^{U+}_{\varepsilon,\varrho}\right]\right\rangle : \varepsilon \in E\right\}$$

$$= \left\{\left\langle \varepsilon, \left[\min_i \dot{\mu}^L_{\varepsilon,(i,\varrho)}, \min_i \dot{\mu}^U_{\varepsilon,(i,\varrho)}\right], \left[\dot{\eta}^L_{\varepsilon,\varrho}, \dot{\eta}^U_{\varepsilon,\varrho}\right], \left[\max_i \dot{\nu}^L_{\varepsilon,(i,\varrho)}, \max_i \dot{\nu}^U_{\varepsilon,(i,\varrho)}\right]\right\rangle : \varepsilon \in E\right\}$$

and $\psi_{\varepsilon,\varrho} = \max_i \dot{\mu}^U_{\varepsilon,(i,\varrho)} - \min_i \dot{\nu}^L_{\varepsilon,(i,\varrho)}$, $\phi_{\varepsilon,\varrho} = \min_i \dot{\mu}^L_{\varepsilon,(i,\varrho)} - \max_i \dot{\nu}^U_{\varepsilon,(i,\varrho)}$ for all ε, ϱ, where,

$$\dot{\eta}^U_{\varepsilon,\varrho} = \begin{cases} \min_i \dot{\eta}^U_{\varepsilon,(i,\varrho)}, & \phi_{\varepsilon,\varrho} > 0 \\ \frac{|\psi_{\varepsilon,\varrho}|}{|\phi_{\varepsilon,\varrho}| + |\psi_{\varepsilon,\varrho}|} \min_i \dot{\eta}^U_{\varepsilon,(i,\varrho)} \\ \quad + \frac{|\phi_{\varepsilon,\varrho}|}{|\phi_{\varepsilon,\varrho}| + |\psi_{\varepsilon,\varrho}|} \max_i \dot{\eta}^U_{\varepsilon,(i,\varrho)}, & \psi_{\varepsilon,\varrho} > 0 \geq \phi_{\varepsilon,\varrho} \\ \max_i \dot{\eta}^U_{\varepsilon,(i,\varrho)}, & 0 \geq \psi_{\varepsilon,\varrho} \end{cases},$$

$$\grave{\eta}^{L}_{\varepsilon,\varrho} = \begin{cases} \min_i \dot{\eta}^{L}_{\varepsilon,(i,\varrho)}, & \phi_{\varepsilon,\varrho} > 0 \\ \dfrac{|\psi_{\varepsilon,\varrho}|}{|\phi_{\varepsilon,\varrho}| + |\psi_{\varepsilon,\varrho}|} \min_i \dot{\eta}^{L}_{\varepsilon,(i,\varrho)} & \\ \quad + \dfrac{|\phi_{\varepsilon,\varrho}|}{|\phi_{\varepsilon,\varrho}| + |\psi_{\varepsilon,\varrho}|} \max_i \dot{\eta}^{L}_{\varepsilon,(i,\varrho)}, & \psi_{\varepsilon,\varrho} > 0 \geq \phi_{\varepsilon,\varrho} \\ \max_i \dot{\eta}^{L}_{\varepsilon,(i,\varrho)} & 0 \geq \psi_{\varepsilon,\varrho} \end{cases},$$

$$\grave{\eta}^{U}_{\varepsilon,\varrho} = \begin{cases} \max_i \dot{\eta}^{U}_{\varepsilon,(i,\varrho)}, & \phi_{\varepsilon,\varrho} \geq 0 \\ \dfrac{|\psi_{\varepsilon,\varrho}|}{|\phi_{\varepsilon,\varrho}| + |\psi_{\varepsilon,\varrho}|} \max_i \dot{\eta}^{U}_{\varepsilon,(i,\varrho)} & \\ \quad + \dfrac{|\phi_{\varepsilon,\varrho}|}{|\phi_{\varepsilon,\varrho}| + |\psi_{\varepsilon,\varrho}|} \min_i \dot{\eta}^{U}_{\varepsilon,(i,\varrho)}, & \psi_{\varepsilon,\varrho} \geq 0 > \phi_{\varepsilon,\varrho} \\ \min_i \dot{\eta}^{U}_{\varepsilon,(i,\varrho)}, & 0 > \psi_{\varepsilon,\varrho} \end{cases},$$

$$\grave{\eta}^{L}_{\varepsilon,\varrho} = \begin{cases} \max_i \dot{\eta}^{L}_{\varepsilon,(i,\varrho)}, & \phi_{\varepsilon,\varrho} \geq 0 \\ \dfrac{|\psi_{\varepsilon,\varrho}|}{|\phi_{\varepsilon,\varrho}| + |\psi_{\varepsilon,\varrho}|} \max_i \dot{\eta}^{L}_{\varepsilon,(i,\varrho)} & \\ \quad + \dfrac{|\phi_{\varepsilon,\varrho}|}{|\phi_{\varepsilon,\varrho}| + |\psi_{\varepsilon,\varrho}|} \min_i \dot{\eta}^{L}_{\varepsilon,(i,\varrho)}, & \psi_{\varepsilon,\varrho} \geq 0 > \phi_{\varepsilon,\varrho} \\ \min_i \dot{\eta}^{L}_{\varepsilon,(i,\varrho)}, & 0 > \psi_{\varepsilon,\varrho} \end{cases}$$

for all ε, ϱ.

Remark It is clear that $\psi_{\varepsilon,\varrho} \geq \phi_{\varepsilon,\varrho}$ holds for all ε, ϱ. Moreover, $\psi_{\varepsilon,\varrho} = \phi_{\varepsilon,\varrho} = 0$ if and only if.

$$\max_i \dot{\mu}^{U}_{\varepsilon,(i,\varrho)} = \min_i \dot{v}^{L}_{\varepsilon,(i,\varrho)} = \min_i \dot{\mu}^{L}_{\varepsilon,(i,\varrho)} = \max_i \dot{v}^{U}_{\varepsilon,(i,\varrho)}$$

Step T4: For all $1 \leq p \leq m$, using the Q distance measure, calculate the distance between each alternative with $\mathbf{\Gamma}^{+}$ and $\mathbf{\Gamma}^{-}$ as follows:

$$D_p^{+} = \sum_{j=1}^{n} \omega_j d_Q\left(\mathfrak{n}_{p,j}, \mathfrak{n}_j^{+}\right);\ D_p^{-} = \sum_{j=1}^{n} \omega_j d_Q\left(\mathfrak{n}_{p,j}, \mathfrak{n}_j^{-}\right)$$

Step T5: For all $1 \leq p \leq m$, calculate the relative closeness coefficient (C_p) of the alternative as follows:

$$C_p = \frac{D_p^{-}}{\max_i D_i^{-}} - \frac{D_p^{+}}{\min_i D_i^{+}}$$

This forms an ordered set $\mathfrak{C} = (C_1, C_2, \ldots, C_m)$.

Step T6: Rank the alternatives.

The alternatives will be ranked according to the descending order of the closeness coefficient C_p in $\mathfrak{C}$, whereby the alternative with the highest relative closeness coefficient value would be selected as the best/optimal alternative.

Remark Steps T3 and T4 involve the newly introduced distance measure and score function.

4.3 VIKOR

Step V1–V3: Same as Step T1–T3.

Step V4: For all $1 \leq p \leq m$, calculate the utility measure (S_p) as well as the regret measure (R_p) of the alternative, using the following formulas:

$$S_p = \sum_{j=1}^{n} \omega_j \frac{d_Q\left(\mathfrak{n}_{p,j}, \mathfrak{n}_j^{+}\right)}{d_Q\left(\mathfrak{n}_j^{-}, \mathfrak{n}_j^{+}\right)};\ R_p = \max_j\ \omega_j \frac{d_Q\left(\mathfrak{n}_{p,j}, \mathfrak{n}_j^{+}\right)}{d_Q\left(\mathfrak{n}_j^{-}, \mathfrak{n}_j^{+}\right)}$$

for all $1 \leq p \leq m$.

Step V5: Calculate Q_p as follows:

$$Q_p = \varsigma\left(\frac{S_p - \min_i S_i}{\max_i S_i - \min_i S_i}\right) + (1-\varsigma)\left(\frac{R_p - \min_i R_i}{\max_i R_i - \min_i R_i}\right),$$

where $0 \leq \varsigma \leq 1$ is to be chosen. This forms an ordered set $\mathfrak{Q} = (Q_1, Q_2, \ldots, Q_m)$.

Remark Therefore, R_p and S_p in Step V4 represent two entirely different ways of decision-making, whereas ς in Step V5 serves as an extra degree of freedom in deciding the amount of attention given to R_p and S_p.

Step V6: Rank the alternatives.

The alternatives will be ranked according to the ascending order of Q_p in $\mathfrak{Q}$, whereby the alternative with the lowest value of Q_p would be selected as the best/optimal alternative.

Remark Steps V4 and V5 involve the newly introduced distance measure and score function.

4.4 Notable differences between the VIKOR and TOPSIS algorithms

For an attribute, VIKOR compares the distance between each alternative with the best alternative for that attribute, against the distance between the best and the worst alternative for that attribute. On the other hand, TOPSIS compares the distance between each alternative with the best alternative for that attribute, against the distance between each alternative with the worst alternative for that attribute.

It is also worth noting that:

$$\begin{aligned} \mathfrak{n}_{\varrho}^{+} &= \left\{\left\langle \varepsilon, \left[\mu^{L+}_{\varepsilon,\varrho}, \mu^{U+}_{\varepsilon,\varrho}\right], \left[\grave{\eta}^{L}_{\varepsilon,\varrho}, \grave{\eta}^{U}_{\varepsilon,\varrho}\right], \left[v^{L-}_{\varepsilon,\varrho}, v^{U-}_{\varepsilon,\varrho}\right] \right\rangle : \varepsilon \in E\right\} \\ &= \left\{\left\langle \varepsilon, \left[\max_i \dot{\mu}^{L}_{\varepsilon,(i,\varrho)}, \max_i \dot{\mu}^{U}_{\varepsilon,(i,\varrho)}\right], \left[\grave{\eta}^{L}_{\varepsilon,\varrho}, \grave{\eta}^{U}_{\varepsilon,\varrho}\right], \left[\min_i \dot{v}^{L}_{\varepsilon,(i,\varrho)}, \min_i \dot{v}^{U}_{\varepsilon,(i,\varrho)}\right] \right\rangle : \right. \\ &\qquad \left. \varepsilon \in E\right\} \end{aligned}$$

need not be $\mathfrak{n}_{p,j}$ for any p. This is because there may exist unique i_1 and i_2, with $i_1 \neq i_2$, where $\dot{\mu}^L_{\varepsilon,(i_1,\varrho)} = \max_i \dot{\mu}^L_{\varepsilon,(i,\varrho)}$ and $\dot{v}^L_{\varepsilon,(i_2,\varrho)} = \min_i \dot{v}^L_{\varepsilon,(i,\varrho)}$. Likewise,

$$\text{ɲ}^-_{\varrho} = \left\{ \left\langle \varepsilon, \left[\mu^{L-}_{\varepsilon,\varrho}, \mu^{U-}_{\varepsilon,\varrho}\right], \left[\grave{\eta}^L_{\varepsilon,\varrho}, \grave{\eta}^U_{\varepsilon,\varrho}\right], \left[v^{L+}_{\varepsilon,\varrho}, v^{U+}_{\varepsilon,\varrho}\right] \right\rangle : \varepsilon \in E \right\}$$
$$= \left\{ \left\langle \varepsilon, \left[\min_i \dot{\mu}^L_{\varepsilon,(i,\varrho)}, \min_i \dot{\mu}^U_{\varepsilon,(i,\varrho)}\right], \left[\grave{\eta}^L_{\varepsilon,\varrho}, \grave{\eta}^U_{\varepsilon,\varrho}\right], \left[\max_i \dot{v}^L_{\varepsilon,(i,\varrho)}, \max_i \dot{v}^U_{\varepsilon,(i,\varrho)}\right] \right\rangle : \varepsilon \in E \right\}$$

need not be $\text{ɲ}_{p,j}$ for any p. As a result, both $\max\left\{ d_Q\left(\text{ɲ}_{p,j}, \text{ɲ}^+_j\right) : p = 1, 2, \ldots, m \right\}$ and $\max\left\{ d_Q\left(\text{ɲ}_{p,j}, \text{ɲ}^-_j\right) : p = 1, 2, \ldots, m \right\}$ need not reach $d_Q\left(\text{ɲ}^-_j, \text{ɲ}^+_j\right)$.

In conclusion, the VIKOR and TOPSIS algorithms are two different methods in accessing the closeness of each $\text{ɲ}_{p,j}$ with ɲ^+_j and ɲ^-_j.

5 Company performance assessment during pandemic: a MADM approach

The recent COVID-19 pandemic has brought a very severe impact on the global economy, and one of the most affected sectors was the stock exchange. Investors worldwide are facing unprecedented uncertainties in evaluating the performance of companies. Consequently, for companies that plan to invest in other companies, it has also become much more difficult for its staff to evaluate the performance of the companies, due to such uncertainties.

One of the main difficulties in decision-making by the stakeholders during the pandemic period arises due to the indeterminacy of the number of new COVID-19 cases in the future as well as the implementation of lockdowns and border restrictions in different areas/regions. Most people have different opinions and perceptions about the future of the COVID-19 pandemic. Some believe that the spread of the virus will come to a halt soon, while others worry that the spreading rate of the virus will increase along with the severity of the impact of the virus on human health, thus making lockdowns and border restrictions inevitable in the future. The existence of such multiple opinions gives rise to multiple possible scenarios of how the future may unfold, all of which need to be considered carefully.

Another difficulty that arises in decision-making is the existence of many different perceptions of the target company, such as the potential profitability, liquidity, and potential risks, as well as the extent of uncertainties involved. To make it worse, the assessment of potential profitability, liquidity, and risks will mutually influence one another through mutual cancellation. The existence of risk for a target company will negatively affect the stakeholders' perception of the company, which would nullify or even overturn the positive perception that would arise from its potential profitability.

Yet another difficulty that arises in this scenario is the absence of crisp and exact values. During a quality assessment, in which the potential profitability, liquidity, and risks are evaluated, there are bound to be differing perceptions and opinions among the stakeholders. Therefore, the potential profit that is generated would prove too complex to be represented by a single crisp value, such as "\$100,000". Instead, an interval would usually be given to denote the potential profits that are generated, such as "\$70,000–\$160,000".

Furthermore, a stakeholder company always has its tolerance limit in dealing with contradictory information. In the case where too much mutually contradicting information is present in the data or certain sections of the data, the whole data set may be discarded, or the affected section of the data set may be discarded. For example, for a target company that is deemed to be "generating the most profit" while at the same time is "the riskiest" and "the most uncertain", that target company may be discarded from consideration due to the excessive amount of contradictory information present in its data set, which brings the legitimacy of the data itself into question.

Therefore, in this scenario, the proposed (t, s)-INSn structure is chosen to model the perception of a target company by a stakeholder. The different opinions on the future of the pandemic would be modelled by the soft structure, and the multiple and mutually interacting perceptions of the target company would be modelled by the neutrosophic structure (including the potential profitability, number of uncertainties about the company, potential risks associated with the company), the interval-valued structure would be used to model the data as it is not practical to denote the data related to the potential profits that are generated, the uncertainties and the risks using crisp and exact values, whereas the tolerance towards contradicting information is modelled by the (t, s)-regulation.

5.1 An introduction of the scenario

Suppose there is a stakeholder company that plans to choose the best target company out of five companies denoted by A, B, C, D, and E to invest a sum of money during the ongoing COVID-19 pandemic. Members of the top management have observed the recent patterns in the stock price movements of the five target companies. Based on the patterns in the stock price movements, the five target companies were based on five attributes, namely momentum, value, growth, volatility, and quality. Moreover, members of the top management have considered four different types of possible scenarios, namely (i) no new cases; (ii) new cases but without lockdowns; (iii) new cases and with lockdowns; and (iv) another wave of the pandemic. Thus, under each possible scenario of the pandemic, the members of the top management provided feedback for each of the five target companies for each of the five attributes that were evaluated, where each person can provide feedback on one or more of the following:

(i) The potential profit generated: From "no hope to generate any profits" to "most potential to generate the highest profits".

(ii) The number/extent of uncertainties involved: From "most certain" to "most uncertain".

(iii) The potential risks involved: From "no risk at all" to "most risky".

Thus, it is possible for some members of the top management to simultaneously conclude that one target company is the most potential at generating profits and is at the same time the riskiest one.

5.2 The Input (t_0, s_0)-INSM

Considering the constrained space,

$$\langle \varepsilon, \left[\dot{\mu}^L_{\varepsilon,(p,q)}, \dot{\mu}^U_{\varepsilon,(p,q)}\right], \left[\dot{\eta}^L_{\varepsilon,(p,q)}, \dot{\eta}^U_{\varepsilon,(p,q)}\right], \left[\dot{\nu}^L_{\varepsilon,1}, \dot{\nu}^U_{\varepsilon,(p,q)}\right] \rangle$$

is presented as follows:

$$\left\langle \varepsilon, \begin{array}{c} \left[\dot{\mu}^L_{\varepsilon,(p,q)}, \dot{\mu}^U_{\varepsilon,(p,q)}\right], \\ \left[\dot{\eta}^L_{\varepsilon,(p,q)}, \dot{\eta}^U_{\varepsilon,(p,q)}\right], \\ \left[\dot{\nu}^L_{\varepsilon,(p,q)}, \dot{\nu}^U_{\varepsilon,(p,q)}\right] \end{array} \right\rangle$$

in the following tables, for all $\varepsilon \in \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$, $p \in \{\mathrm{A, B, C, D, E}\}$ and

$$q \in \{\text{Momentum, Value, Growth, Volatility, Quality}\}$$

. The values given in the four tables below for all $\varepsilon \in \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ are the raw input data that will be used for the computations in the subsequent steps in the decision-making process.

ε_1: No new cases

Attributes Company	Momentum	Value	Growth	Volatility	Quality
A	$\langle \varepsilon_1, [0.32, 0.57], [0.01, 0.05], [0.40, 0.54] \rangle$	$\langle \varepsilon_1, [0.50, 0.52], [0.08, 0.11], [0.10, 0.43] \rangle$	$\langle \varepsilon_1, [0.51, 0.57], [0.45, 0.59], [0.24, 0.28] \rangle$	$\langle \varepsilon_1, [0.07, 0.07], [0.22, 0.50], [0.11, 0.61] \rangle$	$\langle \varepsilon_1, [0.42, 0.51], [0.13, 0.56], [0.15, 0.58] \rangle$
B	$\langle \varepsilon_1, [0.07, 0.15], [0.31, 0.46], [0.32, 0.37] \rangle$	$\langle \varepsilon_1, [0.02, 0.58], [0.57, 0.61], [0.01, 0.47] \rangle$	$\langle \varepsilon_1, [0.05, 0.35], [0.48, 0.70], [0.22, 0.22] \rangle$	$\langle \varepsilon_1, [0.14, 0.20], [0.09, 0.52], [0.02, 0.58] \rangle$	$\langle \varepsilon_1, [0.52, 0.54], [0.43, 0.44], [0.12, 0.49] \rangle$
C	$\langle \varepsilon_1, [0.15, 0.41], [0.45, 0.54], [0.00, 0.37] \rangle$	$\langle \varepsilon_1, [0.35, 0.54], [0.14, 0.43], [0.15, 0.45] \rangle$	$\langle \varepsilon_1, [0.02, 0.17], [0.07, 0.40], [0.03, 0.79] \rangle$	$\langle \varepsilon_1, [0.24, 0.43], [0.04, 0.13], [0.28, 0.58] \rangle$	$\langle \varepsilon_1, [0.22, 0.67], [0.06, 0.47], [0.06, 0.38] \rangle$
D	$\langle \varepsilon_1, [0.61, 0.63], [0.19, 0.24], [0.01, 0.54] \rangle$	$\langle \varepsilon_1, [0.45, 0.76], [0.41, 0.48], [0.31, 0.70] \rangle$	$\langle \varepsilon_1, [0.48, 0.52], [0.35, 0.80], [0.34, 0.57] \rangle$	$\langle \varepsilon_1, [0.54, 0.58], [0.13, 0.45], [0.35, 0.37] \rangle$	$\langle \varepsilon_1, [0.06, 0.32], [0.15, 0.72], [0.10, 0.19] \rangle$
E	$\langle \varepsilon_1, [0.12, 0.35], [0.33, 0.39], [0.38, 0.54] \rangle$	$\langle \varepsilon_1, [0.21, 0.29], [0.52, 0.68], [0.19, 0.64] \rangle$	$\langle \varepsilon_1, [0.19, 0.25], [0.02, 0.62], [0.30, 0.36] \rangle$	$\langle \varepsilon_1, [0.53, 0.66], [0.09, 0.36], [0.01, 0.20] \rangle$	$\langle \varepsilon_1, [0.37, 0.61], [0.30, 0.50], [0.40, 0.60] \rangle$

ε_2: New cases but without lockdowns

Attributes Company	Momentum	Value	Growth	Volatility	Quality
A	$\langle \varepsilon_2, [0.51, 0.80], [0.22, 0.35], [0.19, 0.35] \rangle$	$\langle \varepsilon_2, [0.38, 0.60], [0.04, 0.18], [0.55, 0.61] \rangle$	$\langle \varepsilon_2, [0.02, 0.48], [0.29, 0.50], [0.05, 0.47] \rangle$	$\langle \varepsilon_2, [0.45, 0.55], [0.07, 0.43], [0.05, 0.70] \rangle$	$\langle \varepsilon_2, [0.11, 0.53], [0.02, 0.60], [0.23, 0.25] \rangle$
B	$\langle \varepsilon_2, [0.28, 0.70], [0.25, 0.43], [0.11, 0.42] \rangle$	$\langle \varepsilon_2, [0.15, 0.30], [0.09, 0.43], [0.20, 0.45] \rangle$	$\langle \varepsilon_2, [0.01, 0.33], [0.04, 0.63], [0.43, 0.54] \rangle$	$\langle \varepsilon_2, [0.03, 0.35], [0.43, 0.50], [0.35, 0.61] \rangle$	$\langle \varepsilon_2, [0.43, 0.60], [0.23, 0.38], [0.09, 0.29] \rangle$
C	$\langle \varepsilon_2, [0.04, 0.64], [0.16, 0.20], [0.13, 0.58] \rangle$	$\langle \varepsilon_2, [0.66, 0.80], [0.25, 0.31], [0.36, 0.49] \rangle$	$\langle \varepsilon_2, [0.21, 0.25], [0.44, 0.69], [0.07, 0.54] \rangle$	$\langle \varepsilon_2, [0.16, 0.16], [0.37, 0.54], [0.47, 0.58] \rangle$	$\langle \varepsilon_2, [0.41, 0.69], [0.07, 0.27], [0.15, 0.30] \rangle$
D	$\langle \varepsilon_2, [0.26, 0.36], [0.12, 0.20], [0.04, 0.62] \rangle$	$\langle \varepsilon_2, [0.54, 0.60], [0.11, 0.39], [0.23, 0.59] \rangle$	$\langle \varepsilon_2, [0.55, 0.60], [0.01, 0.15], [0.26, 0.39] \rangle$	$\langle \varepsilon_2, [0.42, 0.61], [0.05, 0.48], [0.13, 0.16] \rangle$	$\langle \varepsilon_2, [0.38, 0.47], [0.10, 0.57], [0.35, 0.61] \rangle$
E	$\langle \varepsilon_2, [0.11, 0.13], [0.15, 0.47], [0.03, 0.66] \rangle$	$\langle \varepsilon_2, [0.35, 0.41], [0.01, 0.36], [0.13, 0.72] \rangle$	$\langle \varepsilon_2, [0.12, 0.17], [0.04, 0.60], [0.08, 0.55] \rangle$	$\langle \varepsilon_2, [0.34, 0.65], [0.05, 0.15], [0.28, 0.38] \rangle$	$\langle \varepsilon_2, [0.10, 0.26], [0.20, 0.59], [0.32, 0.65] \rangle$

ε_3: New cases and with lockdowns

Attributes Company	Momentum	Value	Growth	Volatility	Quality
A	$\langle \varepsilon_3, [0.22, 0.52], [0.24, 0.52], [0.38, 0.43] \rangle$	$\langle \varepsilon_3, [0.09, 0.35], [0.28, 0.48], [0.52, 0.57] \rangle$	$\langle \varepsilon_3, [0.13, 0.56], [0.20, 0.25], [0.57, 0.59] \rangle$	$\langle \varepsilon_3, [0.09, 0.58], [0.06, 0.18], [0.46, 0.49] \rangle$	$\langle \varepsilon_3, [0.54, 0.56], [0.42, 0.45], [0.07, 0.18] \rangle$
B	$\langle \varepsilon_3, [0.04, 0.55], [0.17, 0.61], [0.24, 0.49] \rangle$	$\langle \varepsilon_3, [0.14, 0.55], [0.09, 0.36], [0.05, 0.56] \rangle$	$\langle \varepsilon_3, [0.32, 0.71], [0.13, 0.28], [0.20, 0.49] \rangle$	$\langle \varepsilon_3, [0.24, 0.48], [0.25, 0.62], [0.28, 0.55] \rangle$	$\langle \varepsilon_3, [0.13, 0.63], [0.30, 0.54], [0.39, 0.40] \rangle$
C	$\langle \varepsilon_3, [0.36, 0.83], [0.17, 0.19], [0.19, 0.20] \rangle$	$\langle \varepsilon_3, [0.01, 0.12], [0.20, 0.58], [0.16, 0.64] \rangle$	$\langle \varepsilon_3, [0.15, 0.54], [0.17, 0.34], [0.51, 0.62] \rangle$	$\langle \varepsilon_3, [0.30, 0.55], [0.10, 0.55], [0.23, 0.35] \rangle$	$\langle \varepsilon_3, [0.33, 0.48], [0.08, 0.63], [0.22, 0.55] \rangle$
D	$\langle \varepsilon_3, [0.68, 0.76], [0.07, 0.31], [0.10, 0.29] \rangle$	$\langle \varepsilon_3, [0.32, 0.47], [0.28, 0.33], [0.45, 0.47] \rangle$	$\langle \varepsilon_3, [0.04, 0.06], [0.34, 0.58], [0.26, 0.60] \rangle$	$\langle \varepsilon_3, [0.13, 0.20], [0.49, 0.50], [0.42, 0.56] \rangle$	$\langle \varepsilon_3, [0.18, 0.54], [0.29, 0.46], [0.18, 0.62] \rangle$

Attributes Company	Momentum	Value	Growth	Volatility	Quality
E	$\left\langle \varepsilon_3, \begin{matrix}[0.12,0.68],\\ [0.21,0.56],\\ [0.15,0.24]\end{matrix} \right\rangle$	$\left\langle \varepsilon_3, \begin{matrix}[0.57,0.57],\\ [0.58,0.69],\\ [0.10,0.25]\end{matrix} \right\rangle$	$\left\langle \varepsilon_3, \begin{matrix}[0.41,0.70],\\ [0.03,0.35],\\ [0.26,0.47]\end{matrix} \right\rangle$	$\left\langle \varepsilon_3, \begin{matrix}[0.49,0.57],\\ [0.23,0.56],\\ [0.01,0.19]\end{matrix} \right\rangle$	$\left\langle \varepsilon_3, \begin{matrix}[0.41,0.61],\\ [0.53,0.56],\\ [0.37,0.56]\end{matrix} \right\rangle$

ε_4: Another wave of pandemic

Attributes Company	Momentum	Value	Growth	Volatility	Quality
A	$\left\langle \varepsilon_4, \begin{matrix}[0.07,0.38],\\ [0.01,0.60],\\ [0.34,0.34]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.13,0.14],\\ [0.20,0.27],\\ [0.03,0.72]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.63,0.70],\\ [0.30,0.34],\\ [0.26,0.62]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.38,0.41],\\ [0.16,0.38],\\ [0.45,0.65]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.23,0.53],\\ [0.24,0.28],\\ [0.12,0.64]\end{matrix} \right\rangle$
B	$\left\langle \varepsilon_4, \begin{matrix}[0.63,0.66],\\ [0.38,0.38],\\ [0.13,0.57]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.02,0.56],\\ [0.27,0.32],\\ [0.44,0.53]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.38,0.61],\\ [0.15,0.16],\\ [0.29,0.36]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.11,0.39],\\ [0.02,0.56],\\ [0.01,0.54]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.16,0.29],\\ [0.20,0.35],\\ [0.14,0.71]\end{matrix} \right\rangle$
C	$\left\langle \varepsilon_4, \begin{matrix}[0.22,0.24],\\ [0.18,0.56],\\ [0.19,0.66]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.03,0.56],\\ [0.33,0.63],\\ [0.06,0.11]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.31,0.61],\\ [0.16,0.37],\\ [0.20,0.31]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.07,0.28],\\ [0.31,0.46],\\ [0.12,0.51]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.06,0.58],\\ [0.16,0.43],\\ [0.20,0.29]\end{matrix} \right\rangle$
D	$\left\langle \varepsilon_4, \begin{matrix}[0.60,0.61],\\ [0.14,0.18],\\ [0.30,0.56]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.09,0.54],\\ [0.15,0.51],\\ [0.29,0.48]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.13,0.60],\\ [0.47,0.60],\\ [0.03,0.30]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.20,0.41],\\ [0.24,0.31],\\ [0.20,0.69]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.19,0.74],\\ [0.02,0.35],\\ [0.35,0.52]\end{matrix} \right\rangle$
E	$\left\langle \varepsilon_4, \begin{matrix}[0.14,0.35],\\ [0.13,0.32],\\ [0.16,0.66]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.26,0.59],\\ [0.05,0.32],\\ [0.20,0.57]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.10,0.52],\\ [0.18,0.41],\\ [0.08,0.50]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.27,0.29],\\ [0.27,0.38],\\ [0.13,0.43]\end{matrix} \right\rangle$	$\left\langle \varepsilon_4, \begin{matrix}[0.51,0.72],\\ [0.33,0.49],\\ [0.14,0.17]\end{matrix} \right\rangle$

Moreover, let the list of subjective weights for the five attributes be denoted by $\mathbf{h} = (0.2, 0.2, 0.3, 0.2, 0.1)$.

Now the results are presented alongside the calculation outputs for each step of the TOPSIS and VIKOR decision-making algorithms.

Step T1 and V1: $\boldsymbol{\theta} = (0.228, 0.189, 0.179, 0.216, 0.188)$.

Step T2 and V2: $\boldsymbol{\omega} = (0.229, 0.190, 0.269, 0.217, 0.095)$.

Step T3 and V3:

$$\mathfrak{n}_1^+ = \left\{ \left\langle \varepsilon_1, \begin{matrix}[0.610,0.630],\\ [0.198,0.259],\\ [0.000,0.370]\end{matrix} \right\rangle, \left\langle \varepsilon_2, \begin{matrix}[0.510,0.800],\\ [0.178,0.320],\\ [0.030,0.350]\end{matrix} \right\rangle, \left\langle \varepsilon_3, \begin{matrix}[0.680,0.830],\\ [0.135,0.350],\\ [0.100,0.200]\end{matrix} \right\rangle, \left\langle \varepsilon_4, \begin{matrix}[0.630,0.660],\\ [0.205,0.401],\\ [0.130,0.340]\end{matrix} \right\rangle \right\}$$

$$\mathfrak{n}_2^+ = \left\{ \left\langle \varepsilon_1, \begin{matrix}[0.500,0.760],\\ [0.313,0.381],\\ [0.010,0.430]\end{matrix} \right\rangle, \left\langle \varepsilon_2, \begin{matrix}[0.660,0.800],\\ [0.120,0.295],\\ [0.130,0.450]\end{matrix} \right\rangle, \left\langle \varepsilon_3, \begin{matrix}[0.570,0.570],\\ [0.358,0.527],\\ [0.050,0.250]\end{matrix} \right\rangle, \left\langle \varepsilon_4, \begin{matrix}[0.260,0.590],\\ [0.206,0.470],\\ [0.030,0.110]\end{matrix} \right\rangle \right\}$$

$$\mathfrak{n}_3^+ = \left\{ \left\langle \varepsilon_1, \begin{matrix}[0.510,0.570],\\ [0.290,0.635],\\ [0.030,0.220]\end{matrix} \right\rangle, \left\langle \varepsilon_2, \begin{matrix}[0.550,0.600],\\ [0.223,0.418],\\ [0.050,0.390]\end{matrix} \right\rangle, \left\langle \varepsilon_3, \begin{matrix}[0.410,0.710],\\ [0.195,0.426],\\ [0.200,0.470]\end{matrix} \right\rangle, \left\langle \varepsilon_4, \begin{matrix}[0.630,0.700],\\ [0.290,0.352],\\ [0.030,0.300]\end{matrix} \right\rangle \right\}$$

$$\mathfrak{n}_4^+ = \left\{ \left\langle \varepsilon_1, \begin{matrix}[0.540, 0.660],\\ [0.122, 0.307],\\ [0.010, 0.200]\end{matrix} \right\rangle, \left\langle \varepsilon_2, \begin{matrix}[0.450, 0.650],\\ [0.250, 0.356],\\ [0.050, 0.160]\end{matrix} \right\rangle, \left\langle \varepsilon_3, \begin{matrix}[0.490, 0.580],\\ [0.254, 0.379],\\ [0.010, 0.190]\end{matrix} \right\rangle, \left\langle \varepsilon_4, \begin{matrix}[0.380, 0.410],\\ [0.196, 0.462],\\ [0.010, 0.430]\end{matrix} \right\rangle \right\}$$

$$\mathfrak{n}_5^+ = \left\{ \left\langle \varepsilon_1, \begin{matrix}[0.520, 0.670],\\ [0.234, 0.571],\\ [0.060, 0.190]\end{matrix} \right\rangle, \left\langle \varepsilon_2, \begin{matrix}[0.430, 0.690],\\ [0.120, 0.428],\\ [0.090, 0.250]\end{matrix} \right\rangle, \left\langle \varepsilon_3, \begin{matrix}[0.540, 0.630],\\ [0.290, 0.534],\\ [0.070, 0.180]\end{matrix} \right\rangle, \left\langle \varepsilon_4, \begin{matrix}[0.510, 0.740],\\ [0.179, 0.387],\\ [0.120, 0.170]\end{matrix} \right\rangle \right\}$$

$$\mathfrak{n}_1^- = \left\{ \left\langle \varepsilon_1, \begin{matrix}[0.070, 0.150],\\ [0.262, 0.331],\\ [0.400, 0.540]\end{matrix} \right\rangle, \left\langle \varepsilon_2, \begin{matrix}[0.040, 0.130],\\ [0.192, 0.350],\\ [0.190, 0.660]\end{matrix} \right\rangle, \left\langle \varepsilon_3, \begin{matrix}[0.040, 0.520],\\ [0.175, 0.450],\\ [0.380, 0.490]\end{matrix} \right\rangle, \left\langle \varepsilon_4, \begin{matrix}[0.070, 0.240],\\ [0.185, 0.379],\\ [0.340, 0.660]\end{matrix} \right\rangle \right\}$$

$$\mathfrak{n}_2^- = \left\{ \left\langle \varepsilon_1, \begin{matrix}[0.020, 0.290],\\ [0.337, 0.409],\\ [0.310, 0.700]\end{matrix} \right\rangle, \left\langle \varepsilon_2, \begin{matrix}[0.150, 0.300],\\ [0.140, 0.315],\\ [0.550, 0.720]\end{matrix} \right\rangle, \left\langle \varepsilon_3, \begin{matrix}[0.010, 0.120],\\ [0.312, 0.493],\\ [0.520, 0.640]\end{matrix} \right\rangle, \left\langle \varepsilon_4, \begin{matrix}[0.020, 0.140],\\ [0.174, 0.430],\\ [0.440, 0.720]\end{matrix} \right\rangle \right\}$$

$$\mathfrak{n}_3^- = \left\{ \left\langle \varepsilon_1, \begin{matrix}[0.020, 0.170],\\ [0.210, 0.565],\\ [0.340, 0.790]\end{matrix} \right\rangle, \left\langle \varepsilon_2, \begin{matrix}[0.010, 0.170],\\ [0.227, 0.422],\\ [0.430, 0.550]\end{matrix} \right\rangle, \left\langle \varepsilon_3, \begin{matrix}[0.040, 0.060],\\ [0.175, 0.404],\\ [0.570, 0.620]\end{matrix} \right\rangle, \left\langle \varepsilon_4, \begin{matrix}[0.100, 0.520],\\ [0.330, 0.408],\\ [0.290, 0.620]\end{matrix} \right\rangle \right\}$$

$$\mathfrak{n}_4^- = \left\{ \left\langle \varepsilon_1, \begin{matrix}[0.070, 0.070],\\ [0.138, 0.343],\\ [0.350, 0.610]\end{matrix} \right\rangle, \left\langle \varepsilon_2, \begin{matrix}[0.030, 0.160],\\ [0.230, 0.334],\\ [0.470, 0.700]\end{matrix} \right\rangle, \left\langle \varepsilon_3, \begin{matrix}[0.090, 0.200],\\ [0.296, 0.421],\\ [0.460, 0.560]\end{matrix} \right\rangle, \left\langle \varepsilon_4, \begin{matrix}[0.070, 0.280],\\ [0.134, 0.408],\\ [0.450, 0.690]\end{matrix} \right\rangle \right\}$$

$$\mathfrak{n}_5^- = \left\{ \left\langle \varepsilon_1, \begin{matrix}[0.060, 0.320],\\ [0.256, 0.589],\\ [0.400, 0.600]\end{matrix} \right\rangle, \left\langle \varepsilon_2, \begin{matrix}[0.100, 0.260],\\ [0.130, 0.442],\\ [0.350, 0.650]\end{matrix} \right\rangle, \left\langle \varepsilon_3, \begin{matrix}[0.130, 0.480],\\ [0.320, 0.546],\\ [0.390, 0.620]\end{matrix} \right\rangle, \left\langle \varepsilon_4, \begin{matrix}[0.060, 0.290],\\ [0.171, 0.383],\\ [0.350, 0.710]\end{matrix} \right\rangle \right\}$$

Then, for the TOPSIS decision-making algorithm, we have the following.

Step T4:

$$D_1^+ = 0.5996, D_2^+ = 0.5854, D_3^+ = 0.5672, D_4^+ = 0.4402, D_5^+ = 0.4998$$

$$D_1^- = 0.5129, D_2^- = 0.4755, D_3^- = 0.5100, D_4^- = 0.6365, D_5^- = 0.5823$$

Step T5: $\mathfrak{C} = (-0.5562, -0.5827, -0.4873, 0.0000, -0.2205)$

Step T6: Therefore, the ranking of the five companies arranged in descending order (the best to the worst) before the data validation on each company through the (t, s)-regulation is as follows

D > E > C > A > B

On the other hand, for the VIKOR decision-making algorithm, we have the following.

Step V4:

$$S_1 = 0.5470, S_2 = 0.5385, S_3 = 0.5243, S_4 = 0.4091, S_5 = 0.4645$$

$$R_1 = 0.1355, R_2 = 0.1404, R_3 = 0.1656, R_4 = 0.1082, R_5 = 0.1667$$

Step V5: The results yielded for each Q_i across all $0 \leq \varsigma \leq 1$ are as shown in Fig. 1. Contrary to the TOPSIS decision-making algorithm, the lower the value of Q_i the better is the choice. The ranking obtained are as given below:

$$Q_1 = 0.466 + 0.534\varsigma, Q_2 = 0.549 + 0.389\varsigma, Q_3 = 0.978 - 0.143\varsigma, Q_4 = 0.000 + 0.000\varsigma, Q_5 = 1.000 - 0.598\varsigma$$

Therefore, the ranking of the five companies arranged in descending order (the best to the worst) before the data validation on each company through the (t, s)-regulation, for instance when $\varsigma = 0.4$, is presented as follows:

D > A > B > E > C

Thus, D is unanimously concluded to be the best choice under all values of ς, before the data validation on each company is done through the implementation of the (t,s)-regulation.

Finally, for each of the five companies, all their respective entries are examined against the choice of (t,s) by the user. Any company who possesses entries violating the condition $0 \leq \left(\mu_A^U(x)\right)^t + \left(\eta_A^U(x)\right)^t + \left(v_A^U(x)\right)^t \leq s$ is ignored due to the presence of overly contradicting data.

Therefore, for a lenient user who chooses $(t,s) = (1,3)$, which is the most lenient as $s = 3$ implies $0 \leq \left(\mu_A^U(x)\right)^t + \left(\eta_A^U(x)\right)^t + \left(v_A^U(x)\right)^t \leq 3$ regardless of the value of t, all five companies will be considered, resulting in company D being chosen for both the TOPSIS and VIKOR algorithms.

On the other hand, suppose there exist another more nitpicking user who chooses $(t,s) = (1,1.70)$. For the five companies $\{\mathrm{A},\mathrm{B},\mathrm{C},\mathrm{D},\mathrm{E}\}$, we have the following.

$$\max\left\{\left(\dot{\mu}^U_{\varepsilon,(\mathrm{A},q)}(x)\right)^1 + \left(\dot{\eta}^U_{\varepsilon,(\mathrm{A},q)}(x)\right)^1 + \left(\dot{v}^U_{\varepsilon,(\mathrm{A},q)}(x)\right)^1 : q \in \left\{\begin{array}{c}\text{Momentum, Value,}\\ \text{Growth, Volatility,}\\ \text{Quality}\end{array}\right\}, \varepsilon \in \{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\}\right\} = 1.68 \leq 1.70$$

$$\max\left\{\left(\dot{\mu}^U_{\varepsilon,(\mathrm{B},q)}(x)\right)^1 + \left(\dot{\eta}^U_{\varepsilon,(\mathrm{B},q)}(x)\right)^1 + \left(\dot{v}^U_{\varepsilon,(\mathrm{B},q)}(x)\right)^1 : q \in \left\{\begin{array}{c}\text{Momentum, Value,}\\ \text{Growth, Volatility,}\\ \text{Quality}\end{array}\right\}, \varepsilon \in \{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\}\right\}$$
$$= 1.66 \leq 1.70$$

$$\max\left\{\left(\dot{\mu}^U_{\varepsilon,(\mathrm{C},q)}(x)\right)^1 + \left(\dot{\eta}^U_{\varepsilon,(\mathrm{C},q)}(x)\right)^1 + \left(\dot{v}^U_{\varepsilon,(\mathrm{C},q)}(x)\right)^1 : q \in \left\{\begin{array}{c}\text{Momentum, Value,}\\ \text{Growth, Volatility,}\\ \text{Quality}\end{array}\right\}, \varepsilon \in \{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\}\right\}$$
$$= 1.66 \leq 1.70$$

$$\max\left\{\left(\dot{\mu}^U_{\varepsilon,(\mathrm{D},q)}(x)\right)^1 + \left(\dot{\eta}^U_{\varepsilon,(\mathrm{D},q)}(x)\right)^1 + \left(\dot{v}^U_{\varepsilon,(\mathrm{D},q)}(x)\right)^1 : q \in \left\{\begin{array}{c}\text{Momentum, Value,}\\ \text{Growth, Volatility,}\\ \text{Quality}\end{array}\right\}, \varepsilon \in \{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\}\right\}$$
$$= 1.94 > 1.70$$

$$\max\left\{\left(\dot{\mu}^U_{\varepsilon,(\mathrm{E},q)}(x)\right)^1 + \left(\dot{\eta}^U_{\varepsilon,(\mathrm{E},q)}(x)\right)^1 + \left(\dot{v}^U_{\varepsilon,(\mathrm{E},q)}(x)\right)^1 : q \in \left\{\begin{array}{c}\text{Momentum, Value,}\\ \text{Growth, Volatility,}\\ \text{Quality}\end{array}\right\}, \varepsilon \in \{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4\}\right\}$$
$$= 1.73 > 1.70$$

This results in Companies D and E are discarded due to the presence of excessive contradicting information present in their data. As a result, the user will choose Company C after Step T6 if he/she uses the TOPSIS decision-making algorithm, whereas his/her choice will depend on his/her preference of ς if he/she chooses the VIKOR decision-making algorithm.

6 Comparative study

The seven indicators derived in Sects. 2 and 3 are as follows.

(i) There must exist at least three entities to cater to the membership, indeterminacy and non-membership degrees.
(ii) All the entities must be interval-valued.

Table 3 Results of the comparative study

Indicators Source of the algorithm (references)	(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)
Ullah et al. (2020), Jana et al. (2019b), Tang et al. (2018), Hadi et al. (2021), Jan et al. (2021), Akram et al. (2021), Wang and Li (2019), Ayub et al. (2021), Boran et al. (2009), Kumar and Garg (2018), Akram et al. (2019) and Al-Shahmi (2022)	N	N	N	N	–	–	–
Peng et al. (2015), Jana and Pal (2019), Jana et al. (2019a), Eraslan and Karaaslan (2015), Riaz and Tehrim (2021) and Zhou and Chen (2021)	N	N	Y	N	–	–	–
Mohagheghi et al. (2020), Liu and Jiang (2020) and Hu et al. (2019)	N	Y	N	N	–	–	–
Xing et al. (2019), Zhang and Xu (2014) and Al-Shahmi and Mhemdi (2023)	N	N	N	Y	–	–	–
Al-Shahmi et al. (2023)	N	N	Y	Y	–	–	–
Fu and Liao (2019)	N	Y	N	Y	–	–	–
Zulqarnain et al. (2021) and Kim and Ahn (2019)	N	Y	Y	Y	–	–	–
Xiao et al. (2013)	N	Y	Y	N	–	–	–
Yager (2014), Mahmood et al. (2018), Jansi et al. (2019), Jana and Pal (2021), Jana et al. (2021), Munir et al. (2021), Adeel et al. (2019), Peng and Dai (2018), Kacprzak (2019) and Eroglu and Sahin (2020)	Y	N	N	N	N	N	N
Ganie and Singh (2021)	Y	N	N	Y	N	N	N
Jana et al. (2020) and Akram and Arshad (2018) and Gündoğdu and Kahraman (2019)	Y	Y	N	N	N	N	N
The newly established TOPSIS and VIKOR algorithms	Y	Y	Y	Y	Y	Y	Y

(iii) The set structure must consist of multiple versions, governed by a parameter $\varepsilon \in E$ (hence being soft).

(iv) The restriction to regulate the amount of contradicting information present must be user customizable.

(v) $d(\text{п}_u, \text{п}_w) > d(\text{п}_u, \text{п}_v)$ must be obtained through the scenario presented in Example 3.4.

(vi) $d(\text{п}_r, \text{п}_s) < d(\text{п}_p, \text{п}_q)$ must be obtained through the scenario presented in Example 3.5.

(vii) $\mathrm{s}(\text{п}_p) > \mathrm{s}(\text{п}_r) > \mathrm{s}(\text{п}_s) > \mathrm{s}(\text{п}_q)$ must be obtained through the scenario presented in Example 3.7.

These seven indicators are thus used for the comparative study in which various TOPSIS- and VIKOR-based MADM algorithms in the existing literature will be compared, discussed, and analysed. Conditions (v), (vi), and (vii) involve rigorous computations as seen in Examples 3.4, 3.5 and 3.7.

In most of the cases where an algorithm falls short of even indicators (ii), (iii), or (iv), before considering indicators (v), (vi), and (vii), suitable generalizations will be applied (and thus advantages given deliberately), whenever possible, so that indicators (v), (vi), and (vii) may be examined through calculations involving the inputs from Examples 3.4, 3.5, and 3.7. For instance, in case the existing algorithm does not accept interval values, then the average of μ_ε^U and μ_ε^L will be taken for each ε, to arrive at μ_ε (without loss of generality). Moreover, in case the existing algorithm does not accept soft sets, then the average of μ_α^U and μ_β^U will be taken to arrive at μ^U (without loss of generality). Nevertheless, if even indicator (i) is not fulfilled (i.e. less than three entities), then indicators (v), (vi), and (vii) will not be considered at all due to the complete absence of at least one entity among the membership, indeterminacy, and non-membership degrees.

Among all the references that were studied, the following results are thus obtained, for all the seven indicators and more than 20 algorithms from the corresponding references. The results of the comparative study presented in Table 3 are the results obtained even after giving advantages to other works during the computations for criteria (v), (vi), and (vii).

Remarks

(i) The references listed in Table 3 have been categorized based on the indicators that they fulfil.

(ii) Some of the references listed in Table 3 have studied other decision-making algorithms in addition to the TOPSIS and VIKOR algorithms. However, only the TOPSIS and VIKOR decision-making algorithms that were proposed in the references listed in Table 3 have been considered in this comparative study.

Therefore, the proposed TOPSIS and VIKOR decision-making algorithms have proven to be the only decision-making algorithms that can handle such complexities of real-life data set yet remain faithful to human intuition/judgement.

It is also worth mentioning that the decision-making algorithms established in this study only consider $0 \le \left(\mu_A^U(x)\right)^t + \left(\eta_A^U(x)\right)^t + \left(\nu_A^U(x)\right)^t \le s$ during the final steps as shown at the end of Sect. 6, which proves much more useful than all the observed works in the existing literature that have deployed the all-or-nothing ruling, i.e. the entire data set must satisfy the condition $0 \le \left(\mu_A^U(x)\right)^t + \left(\eta_A^U(x)\right)^t + \left(\nu_A^U(x)\right)^t \le s$, else, the entire data set will be discarded and nothing will be calculated. Such a ruling is evident from the hypothetical data set that were crafted/derived in these studies, in which the data sets were constructed to explicitly fulfil the condition

$$0 \le \left(\mu_A^U(x)\right)^t + \left(\eta_A^U(x)\right)^t + \left(\nu_A^U(x)\right)^t \le s.$$

However, in real-life situations, it can never be guaranteed that the fuzzified input will always fulfil the condition $0 \le \left(\mu_A^U(x)\right)^t + \left(\eta_A^U(x)\right)^t + \left(\nu_A^U(x)\right)^t \le s$ for a particular choice of (t, s).

7 Conclusion

In this study, the (t, s)-regulated interval-valued neutrosophic soft set (abbr. (t, s)-INSS) was introduced as a new concept towards decision-making under many facets of complexities that exist in most real-life data. The distance measure, score function, and accuracy function were proposed for the (t, s)-INSS model, all of which were proven to be consistent with human intuition and understanding of the degrees of membership, indeterminacy, and non-membership. The newly introduced distance measure and score function were then used to establish the TOPSIS and VIKOR decision-making algorithms for the (t, s)-INSS model. Through the comparative study, the proposed TOPSIS and VIKOR decision-making algorithms for the (t, s)-INSS model, which uses the newly introduced distance measure and score function, were proved to be the only methods that could handle the many aspects of data complexities that are present in most in real-life data, while at the same time remaining faithful to human intuition and understanding on the degrees of membership, indeterminacy, and non-membership. The newly introduced TOPSIS and VIKOR decision-making algorithms for the (t, s)-INSS model were thereby proven to be superior to the other TOPSIS and VIKOR decision-making algorithms in the existing literature.

Currently, the main limitation of the proposed framework lies in its reliance on an undersized, hypothetical data set, which is significantly deviated from the data obtained in real life, both in structure and in calibre. It is desirable to obtain real-life data sets for different types of real-life scenarios that necessitate the use of strict and/or lenient choices for the values of t and s in the (t, s)-INSS model and portray the roles of t and s in filtering out overly contradicting information.

Future studies in this area would involve applying the (t, s)-INSS model, and the accompanying TOPSIS and VIKOR decision-making algorithms to real-life data sets to further affirm its usefulness. In addition, more rigorous validity tests will be conducted across all possible ranges of values of the (t, s)-INSS model to conclude and affirm the consistency of the results obtained via the newly established TOPSIS and VIKOR decision-making algorithms with human intuition and understanding. In addition, appropriate fuzzification techniques will also be introduced to convert the vast amount of heterogeneous raw data into the appropriate (t, s)-INSM, which will then be fed into the programme for the computation of the results using the TOPSIS and VIKOR decision-making algorithms as well as other well-known MADM methods. These methods of fuzzification will likewise be subjected to changes based on the different personalities of investors looking to invest in the stock exchange and other forms of financial trading.

Acknowledgements The authors would like to thank the editors and the anonymous reviewers for their valuable comments and suggestions which have helped to improve the quality and clarity of the paper.

Author contributions All authors contributed to the conception and design of the study. SGQ contributed to conceptualization, methodology, writing—original draft, data curation, visualization, formal analysis, and investigation. HG contributed to writing—review and editing, investigation, and validation. GS contributed to investigation, validation, writing—review and editing, resources, supervision, and project administration. PM contributed to writing—original draft, visualization, and investigation. AK contributed to writing—original draft, visualization, and investigation. FS contributed to writing—review and editing, and validation. The final draft was edited and proofread by GS and HG. All authors commented on the previous versions of the manuscript. All authors have read and approved the final manuscript.

Funding The study was funded by the Yayasan Tun Ismail Professorial Chair Programme from Permodalan Nasional Berhad (PNB), Malaysia, research grant under grant no. YTIPCP-FBM-2021/001.

Availability of data and materials This manuscript has no associated data.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

Ethical approval Authors' declaration: This manuscript is the authors' original work and has not been published elsewhere. All authors have checked the manuscript and have agreed to this submission. This article does not contain any studies with human participants or animals performed by any of the authors.

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