ON THE DIOPHANTINE EQUATION \( x^2 = 2y^4 - 1 \)

Florentin Smarandache
University of New Mexico
200 College Road
Gallup, NM 87301, USA

Abstract.
Another method of solving this equation, different from Ljunggren’s, is given in this paper.

Keywords: diophantine equation, quadratic equation, recurrent solution, characteristic equation of a matrix, eigenvalues

1991 MSC: 11D09

Introduction.

In his book of unsolved problems R. K. Guy [2] informs us that the equation \( x^2 = 2y^4 - 1 \) has in positive integers the only solutions \((1, 1)\) and \((239, 13)\); (Ljunggren has shown it by a difficult proof). But Mordell asked for a simple proof.
In this note we find another method of solving. Note \( t = y^2 \). The general integer solution for \( x^2 - 2t^2 + 1 = 0 \) is:

\[
\begin{align*}
   x_{n+1} &= 3x_n + 4t_n, \\
   t_{n+1} &= 2x_n + 3t_n,
\end{align*}
\]

for all \( n \in \mathbb{N} \), where \( (x_0, y_0) = (1, \varepsilon) \) with \( \varepsilon = \pm 1 \) (see [6]),

\[
\begin{pmatrix}
   x_n \\
   y_n
\end{pmatrix}
= \begin{pmatrix}
   3 & 4 \\
   2 & 3
\end{pmatrix}^n
\begin{pmatrix}
   1 \\
   \varepsilon
\end{pmatrix},
\]

for all \( n \in \mathbb{N} \),

where a matrix at the power zero is equal to the unit matrix I.

Let \( A = \begin{pmatrix}
   3 & 4 \\
   2 & 3
\end{pmatrix} \), and \( \lambda \in \mathbb{R} \). Then \( \det (A - \lambda I) = 0 \) involves \( \lambda_{1,2} = 3 \pm \sqrt{2} \), whence if \( v \) is a vector of dimension two then: \( Av = \lambda_{1,2} \cdot v \) involves \( v_{1,2} = (2, \pm \sqrt{2}) \).

Let \( P = \begin{pmatrix}
   \sqrt{2} & 2 \\
   \sqrt{2} & 3
\end{pmatrix} \) and \( D = \begin{pmatrix}
   3 + 2\sqrt{2} & 0 \\
   0 & 3 - 2\sqrt{2}
\end{pmatrix} \). We have \( P^{-1}A \cdot P = D \), or \( A^n = P \cdot D^n \cdot P^{-1} = \begin{pmatrix}
   (1/2)(a+b) & (\sqrt{2}/2)(a-b) \\
   (\sqrt{2}/4)(a-b) & (1/2)(a+b)
\end{pmatrix} \)

where \( a = (3 + 2\sqrt{2})^n \) and \( b = (3 - 2\sqrt{2})^n \). Hence, we find:
\[
\begin{align*}
x_n &= \frac{1 + \varepsilon \sqrt{2}}{2} (3 + 2\sqrt{2})^n + \frac{1 - \varepsilon \sqrt{2}}{2} (3 - 2\sqrt{2})^n \\
t_n &= \frac{2\varepsilon + \sqrt{2}}{4} (3 + 2\sqrt{2})^n + \frac{2\varepsilon - \sqrt{2}}{4} (3 - 2\sqrt{2})^n
\end{align*}
\]

Or \( y_n = \frac{2\varepsilon + \sqrt{2}}{4} (3 + 2\sqrt{2})^n + \frac{2\varepsilon - \sqrt{2}}{4} (3 - 2\sqrt{2})^n \), \( \forall n \in \mathbb{N} \). For

\( n = 0, 0 = 1 \) it obtains \( y_0^2 = 1 \) (whence \( x_0 = 1 \)), and for \( n = 3, 0 = 1 \) it obtains \( y_3^2 = 169 \) (whence \( x_3 = 239 \)).

\[
(1) \quad y_n = \varepsilon \sum_{k=0}^{n/2} \binom{n}{2k} 3^{n-2k} 2^{3k} + \sum_{k=0}^{(n-1)/2} \binom{n}{2k+1} 3^{n-2k-1} 2^{3k+1}
\]

We still must prove that \( y_n^2 \) is a perfect square if and only if \( n = 0, 3 \).

We can use a similar method for the diophantine equation \( x^2 = Dy^4 + 1 \), or more generally: \( CX^{2a} = Dy^{2b} + E \),

with \( a, b \in \mathbb{N}^* \) and \( C, D, E \in \mathbb{Z}^* \), noting \( x^a = U \), \( y^b = V \), and applying the results of [6], but the relation (1) becomes very intricate.

References:


[4] W. Ljunggren, Some remarks on the diophantine equation $x^2 - Dy^4 = 1$ and $x^4 - Dy^2 = 1$ @ J. London Math. Soc. 41 (1966), 542-544; MR 33#5555.


["Gamma", Brasov, Anul IX, November 1986, Nr. 1.]