ON THE DIOPHANTINE EQUATION $x^2 = 2y^4 - 1$

Florentin Smarandache
University of New Mexico
200 College Road
Gallup, NM 87301, USA

Abstract.
Another method of solving this equation, different from Ljunggren's, is given in this paper.

Keywords: diophantine equation, quadratic equation, recurrent solution, characteristic equation of a matrix, eigenvalues

1991 MSC: 11D09

Introduction.

In his book of unsolved problems R. K. Guy [2] informs us that the equation $x^2 = 2y^4 - 1$ has in positive integers the only solutions (1, 1) and (239, 13); (Ljunggren has shown it by a difficult proof). But Mordell asked for a simple proof.

In this note we find another method of solving. Note $t = y^2$. The general integer solution for $x^2 - 2t^2 + 1 = 0$ is:

$$\begin{cases} x_{n+1} = 3x_n + 4t_n \\ t_{n+1} = 2x_n + 3t_n, \end{cases}$$

for all $n \in N$, where $(x_0, y_0) = (1, \varepsilon)$ with $\varepsilon = \pm 1$ (see [6]),

$$\text{or} \quad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}^n \cdot \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix}, \text{ for all } n \in N,$$

where a matrix at the power zero is equal to the unit matrix I.

Let $A = \begin{pmatrix} 3 & 4 \\ 2 & 3 \end{pmatrix}$, and $\lambda \in R$. Then $\det(A - \lambda \cdot I) = 0$ involves $\lambda_{1,2} = 3 \pm \sqrt{2}$, whence if v is a vector of dimension two then: $Av = \lambda_{1,2} \cdot v$ involves $v_{1,2} =$
$= (2, \pm \sqrt{2})$.

Let $P = \begin{pmatrix} 2 & 2 \\ \sqrt{2} & -\sqrt{2} \end{pmatrix}$ and $D = \begin{pmatrix} 3+2\sqrt{2} & 0 \\ 0 & 3-2\sqrt{2} \end{pmatrix}$. We have $P^{-1} \cdot A \cdot P = D$, or $A^n = P \cdot D^n \cdot P^{-1} = \begin{pmatrix} (1/2)(a+b) & (\sqrt{2}/2)(a-b) \\ (\sqrt{2}/4)(a-b) & (1/2)(a+b) \end{pmatrix}$

where $a = (3 + 2\sqrt{2})^n$ and $b = (3 - 2\sqrt{2})^n$. Hence, we find:

$$\begin{pmatrix} x_n \\ \\ t_n \end{pmatrix} = \begin{pmatrix} \frac{1+\varepsilon\sqrt{2}}{2}(3+2\sqrt{2})^n + \frac{1-\varepsilon\sqrt{2}}{2}(3-2\sqrt{2})^n \\ \\ \frac{2\varepsilon+\sqrt{2}}{4}(3+2\sqrt{2})^n + \frac{2\varepsilon-\sqrt{2}}{4}(3-2\sqrt{2})^n \end{pmatrix}, \ n \in N.$$

Or $y_n^2 = \frac{2\varepsilon+\sqrt{2}}{4}(3+2\sqrt{2})^n + \frac{2\varepsilon-\sqrt{2}}{4}(3-2\sqrt{2})^n$, $n \in N$. For

$n = 0$, $0 = 1$ it obtains $y_0^2 = 1$ (whence $x_0 = 1$), and for

$n = 3$, $0 = 1$ it obtains $y_3^2 = 169$ (whence $x_3 = 239$).

$$(1)\quad Y_n^2 = \varepsilon\cdot \sum_{k=0}^{n/2} \binom{n}{2k} \cdot 3^{n-2k} \cdot 2^{3k} + \sum_{k=0}^{(n-1)/2} \binom{n}{2k+1} \cdot 3^{n-2k-1} \cdot 2^{3k+1}.$$

We still must prove that y_n^2 is a perfect square if and only if $n = 0, 3$.

We can use a similar method for the diophantine equation $x^2 = Dy^4 \pm 1$, or more generally: $CX^{2a} = Dy^{2b} + E$, with $a, b \in N^*$ and $C, D, E \in Z^*$, noting $x^a = U$, $y^b = V$, and applying the results of [6], but the relation (1) becomes very intricate.

References:

[1] J. H. E. Cohn, AThe diophantine equation $y^2 = Dx^4 + 1$@, Math. Scand. 42 (1978), 180-188, MR 80a: 10031.

[2] R. K. Guy, AUnsolved problems in number theory@, Springer-Verlag, 1981, Problem D6, 84-85.

[3] W. Ljunggren, AZur theorie der Gleichung $x^2 + 1 = Dy^4$@, Avh. Norske Vid. Akad. Oslo, I, 5 (1942) #5, p. 27; MR 8, 6.

[4] W. Ljunggren, ASome remarks on the diophantine equation $x^2 - Dy^4 = 1$ and $x^4 - Dy^2 = 1$@, J. London Math. Soc. 41 (1966), 542-544; MR 33#5555.

[5] L. J. Mordell, AThe diophantine equation $y^2 = Dx^4 + 1$@, J. London Math. Soc. 39 (1964) 161-164; MR 29#65.

[6] F. Smarandache, AA method to solve diophantine equations of two unknowns of second degree@, "Gaceta Matematica", 2ª Serie, Volumen 1, Numero 2, 1988, pp. 151-7; translated in Spanish by Francisco Bellot Rosado.

["Gamma", Brasov, Anul IX, November 1986, Nr. 1.]