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An equation involving the Smarandache function and its positive integer solutions[1]

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Abstract For any positive integer $n \geq 2$, the Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. The main purpose of this paper is using the elementary method to study the solvability of the equation $S(x) = n$, and give an exact calculating formula for the number of all solutions of the equation.

Keywords Smarandache function, equation, positive integer solution.

§1. Introduction

For any positive integer n, the famous Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : \ n \mid m!, \ m \in N\}$. From the definition of $S(n)$ one can easily deduce that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the factorization of n into prime powers, then $S(n) = \max\limits_{1 \leq i \leq k} \{S(p_i^{\alpha_i})\}$. From this formula we can get $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $S(11) = 11$, $S(12) = 4$, $S(13) = 13$, $S(14) = 7$, $S(15) = 5$, $S(16) = 6$, $\cdots$. About the other elementary properties of $S(n)$, many people had studied it, and obtained a series important results, see references [2], [3], [4] and [5]. For example, Dr. Xu Zhefeng [4] proved the following conclusion: Let $P(n)$ denotes the largest prime divisor of n, then for any real number $x > 1$, we have the asymptotic formula:

$$\sum_{n \leq x} (S(n) - P(n))^2 = \frac{2\zeta\left(\frac{3}{2}\right) x^{\frac{3}{2}}}{3 \ln x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ denotes the Riemann zeta-function.

Charles Ashbacher [5] studied the solvability of the equation $s(m) = n!$, and proved that for any positive integer n and prime p with $p \leq n$, there are some integers k such that

$$S\left(p^k\right) = n!.$$

In this paper, we using the elementary method to study the solvability of the equation $S(p^k) = n$ (where p be a prime), and give an exact calculating formula for the number of all solutions of the equation. That is, we shall prove the following two conclusions:

[1]This work is supported by the Shaanxi Provincial Education Department Foundation 07JK267.

Theorem 1. Let $n \geq 2$ be a positive integer, p be any prime with $p^\alpha \parallel n$. Then there are exact α positive integers k such that the equation

$$S\left(p^k\right) = n,$$

where $p^\alpha \parallel n$ denotes that $p^\alpha \mid n$ and $p^{\alpha+1} \nmid n$.

Theorem 2. Let n be a fixed integer with $n \geq 2$. $A(k)$ denotes the number of all solutions of the equation $S(x) = k$. Then we have the calculating formula

$$\sum_{k=1}^{n} A(k) = \prod_{p \leq n} \left(1 + \frac{n - \beta(n,p)}{p-1}\right),$$

where $\prod\limits_{p \leq n}$ denotes the product over all primes p less than or equal to n, $\beta(n,p)$ denotes the sum of the base p digits of n.

It is clear that from our Theorem 1 we may immediately deduce Charles Ashbacher's result in reference [5]. In fact, we can get following more accurate result:

Corollary. Let n be a positive integer, then for any prime p with $p \leq n$, there are exact

$$\frac{n - \beta(n,\ p)}{p-1}$$

integers k such that the equation $S\left(p^k\right) = n!$, where $\beta(n,p)$ is defined as in Theorem 2.

Of course, our Corollary is also holds if $p > n$. In this case, $\beta(n,p) = n$, so that $\dfrac{n - \beta(n,\ p)}{p-1} = 0$. Therefore, the equation $S\left(p^k\right) = n!$ has no positive integer solution.

§2. Proof of the theorems

In this section, we shall use the elementary method to complete the proof of our Theorems. First we prove Theorem 1. Let $n \geq 2$ be an integer, for any prime p with $p^\alpha \parallel n$, if $\alpha = 0$, then $(p,\ n) = 1$, and the equation $S\left(p^k\right) = n$ has no positive integer solution k. Otherwise, $p \mid n$, this contradiction with $(p,\ n) = 1$. So the number of all positive integer solutions of the equation $S\left(p^k\right) = n$ is $\alpha = 0$. That is means, Theorem 1 is true if $\alpha = 0$. Now we assume that $\alpha \geq 1$, and let $p^{\alpha(n,p)} \parallel n!$. Then from the elementary number theory textbook (see [6], [7]) and reference [8] we know that

$$\alpha(n,p) = \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right] = \frac{n - \beta(n,p)}{p-1}, \tag{1}$$

where $\beta(n,p)$ denotes the sum of the base p digits of n. That is, if $n = a_1 p^{\alpha_1} + a_2 p^{\alpha_2} + \cdots + a_s p^{\alpha_s}$ with $\alpha_s > \alpha_{s-1} > \cdots > \alpha_1 \geq 0$ and $1 \leq a_i \leq p-1$, $i = 1,\ 2,\ \cdots,\ s$, then $\beta(n,p) = \sum\limits_{i=1}^{s} \alpha_i$.

Now for all positive integers $k = \alpha(n,p) - \alpha + 1,\ \alpha(n,p) - \alpha + 2,\ \cdots,\ \alpha(n,p) - 1$ and $\alpha(n,p)$, we have $S\left(p^k\right) = n$. Since this time, from (1) we know that $p^k \mid n!$, but $p^k \nmid (n-1)!$, because $p^{\alpha(n,p)-\alpha} \parallel (n-1)!$. From the definition of the Smarandache function $S(n)$ we know that if $k \leq \alpha(n,p) - \alpha$, then $p^k \mid (n-1)!$, if $k > \alpha(n,p)$, then $p^k \nmid n!$. Therefore, the equation $S\left(p^k\right) = n$ has exact α positive integer solutions. This proves Theorem 1.

Now we prove Theorem 2. For any positive $k \geq 2$, if x satisfy the equation $S(x) = k$, then $x \mid k!$ and $x \dagger (k-1)!$. So the number of all solutions of the equation $S(x) = k$ is equal to

$$\sum_{d|k!} 1 - \sum_{d|(k-1)!} 1 = d(k!) - d((k-1)!),$$

where $d(m)$ is the Dirichlet divisor function. Therefore,

$$\sum_{k \leq n} A(k) = A(1) + \sum_{2 \leq k \leq n} A(k) = 1 + \sum_{2 \leq k \leq n} [d(k!) - d((k-1)!] = d(n!).$$

Note that (1), from the definition and properties of $d(n)$ we may immediately get

$$\sum_{k \leq n} A(k) = d(n!) = \prod_{p \leq n} \left(1 + \frac{n - \beta(n,p)}{p-1}\right),$$

where $\prod\limits_{p \leq n}$ denotes the product over all primes p less than or equal to n, $\beta(n,p)$ denotes the sum of the base p digits of n.

This completes the proof of Theorem 2.

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On the Smarandache kn-digital subsequence

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Abstract For any positive integer n and any fixed positive integer $k \geq 2$, the Smarandache kn-digital subsequence $\{S_k(n)\}$ is defined as the numbers $S_k(n)$, which can be partitioned into two groups such that the second is k times bigger than the first. The main purpose of this paper is using the elementary method to study the convergent properties of the infinite series involving the Smarandache kn-digital subsequence $\{S_k(n)\}$, and obtain some interesting conclusions.

Keywords The Smarandache kn-digital sequence, infinite series, convergence.

§1. Introduction

For any positive integer n and any fixed positive integer $k \geq 2$, the Smarandache kn-digital subsequence $\{S_k(n)\}$ is defined as the numbers $S_k(n)$, which can be partitioned into two groups such that the second is k times bigger than the first. For example, the Smarandache $3n$-digital subsequence are: $S_3(1) = 13$, $S_3(2) = 26$, $S_3(3) = 39$, $S_3(4) = 412$, $S_3(5) = 515$, $S_3(6) = 618$, $S_3(7) = 721$, $S_3(8) = 824$, $S_3(9) = 927$, $S_3(10) = 1030$, $S_3(11) = 1133$, $S_3(12) = 1236$, $S_3(13) = 1339$, $S_3(14) = 1442$, $S_3(15) = 1545$, $S_3(16) = 1648$, $S_3(17) = 1751$, $S_3(18) = 1854$, $S_3(19) = 1957$, $S_3(20) = 2060$, $S_3(21) = 2163$, $S_3(22) = 2266$, $\cdots$.

The Smarandache $4n$-digital subsequence are: $S_4(1) = 14$, $S_4(2) = 28$, $S_4(3) = 312$, $S_4(4) = 416$, $S_4(5) = 520$, $S_4(6) = 624$, $S_4(7) = 728$, $S_4(8) = 832$, $S_4(9) = 936$, $S_4(10) = 1040$, $S_4(11) = 1144$, $S_4(12) = 1248$, $S_4(13) = 1352$, $S_4(14) = 1456$, $S_4(15) = 1560$, $\cdots$.

The Smarandache $5n$-digital subsequence are: $S_5(1) = 15$, $S_5(2) = 210$, $S_5(3) = 315$, $S_5(4) = 420$, $S_5(5) = 525$, $S_5(6) = 630$, $S_5(7) = 735$, $S_5(8) = 840$, $S_5(9) = 945$, $S_5(10) = 1050$, $S_5(11) = 1155$, $S_5(12) = 1260$, $S_5(13) = 1365$, $S_5(14) = 1470$, $S_5(15) = 1575$, $\cdots$.

These subsequences are proposed by Professor F.Smarandache, he also asked us to study the properties of these subsequences. About these problems, it seems that none had studied them, at least we have not seen any related papers before. The main purpose of this paper is using the elementary method to study the convergent properties of one kind infinite series involving the Smarandache kn-digital subsequence, and prove the following conclusion:

Theorem. Let z be a real number. If $z > \frac{1}{2}$, then the infinite series

$$f(z,k) = \sum_{n=1}^{+\infty} \frac{1}{S_k^z(n)} \tag{1}$$

is convergent; If $z \leq \frac{1}{2}$, then the infinite series (1) is divergent.

In these Smarandache kn-digital subsequences, it is very hard to find a complete square number. So we believe that the following conclusion is correct:

Conjecture. There does not exist any complete square number in the Smarandache kn-digital subsequence, where $k = 3,\ 4,\ 5$. That is, for any positive integer m, $m^2 \notin \{S_k(n)\}$, where $k = 3,\ 4,\ 5$.

§2. Proof of the theorem

In this section, we shall use the elementary method to complete the proof of our Theorem. We just prove that the theorem is holds for Smarandache $3n$-digital subsequence. Similarly, we can deduce that the theorem is also holds for any other positive integer $k \geq 4$. For any element $S_3(a)$ in $\{S_3(n)\}$, let $3a = b_k b_{k-1} \cdots b_2 b_1$, where $1 \leq b_k \leq 9$, $0 \leq b_i \leq 9$, $i = 1,\ 2,\ \cdots,\ k-1$. Then from the definition of the Smarandache $3n$-digital subsequence we have

$$S_3(a) = a \cdot 10^k + 3 \cdot a = a \cdot \left(10^k + 3\right). \tag{2}$$

On the other hand, let $a = a_s a_{s-1} \cdots a_2 a_1$, where $1 \leq a_s \leq 9$, $0 \leq a_i \leq 9$, $i = 1,\ 2,\ \cdots,\ s-1$. It is clear that if $a \leq \underbrace{33\cdots33}_{s}$, then $s = k$; If $a \geq \underbrace{33\cdots34}_{s}$, then $s = k-1$. So from the definition of $S_3(a)$ and the relationship of s and k we have

$$\begin{aligned} f(z,3) &= \sum_{n=1}^{+\infty} \frac{1}{S_3^z(n)} = \sum_{i=1}^{3} \frac{1}{S_3^z(i)} + \sum_{i=4}^{33} \frac{1}{S_3^z(i)} + \sum_{i=34}^{333} \frac{1}{S_3^z(i)} + \sum_{i=34}^{333} \frac{1}{S_3^z(i)} + \sum_{i=334}^{3333} \frac{1}{S_3^z(i)} + \cdots \\ &= \sum_{i=1}^{3} \frac{1}{i^z \cdot 13^z} + \sum_{i=4}^{33} \frac{1}{i^z \cdot 103^z} + \sum_{i=34}^{333} \frac{1}{i^z \cdot 1003^z} + \sum_{i=334}^{3333} \frac{1}{i^z \cdot 10003^z} + \cdots \\ &\leq \sum_{k=1}^{+\infty} \frac{3 \cdot 10^{k-1}}{10^{z(k-2)} \cdot 10^{zk}} \leq 3 \cdot \sum_{k=0}^{+\infty} \frac{10^k}{10^{z(k-1)} \cdot 10^{z(k+1)}} = 3 \cdot \sum_{k=0}^{+\infty} \frac{1}{10^{k\cdot(2z-1)}}. \end{aligned} \tag{3}$$

Now if $z > \frac{1}{2}$, then from (3) and the properties of the geometric progression we know that $f(z,3)$ is convergent.

If $z \leq \frac{1}{2}$, then from (3) we also have

$$\begin{aligned} f(z,3) &= \sum_{n=1}^{+\infty} \frac{1}{S_3^z(n)} = \sum_{i=1}^{3} \frac{1}{S_3^z(i)} + \sum_{i=4}^{33} \frac{1}{S_3^z(i)} + \sum_{i=34}^{333} \frac{1}{S_3^z(i)} + \sum_{i=34}^{333} \frac{1}{S_3^z(i)} + \sum_{i=334}^{3333} \frac{1}{S_3^z(i)} + \cdots \\ &= \sum_{i=1}^{3} \frac{1}{i^z \cdot 13^z} + \sum_{i=4}^{33} \frac{1}{i^z \cdot 103^z} + \sum_{i=34}^{333} \frac{1}{i^z \cdot 1003^z} + \sum_{i=334}^{3333} \frac{1}{i^z \cdot 10003^z} + \cdots \\ &\geq \sum_{k=1}^{+\infty} \frac{3 \cdot 10^{k-1}}{10^{z(k-1)} \cdot 10^{z(k+1)}} \geq 3 \cdot \sum_{k=0}^{+\infty} \frac{10^k}{10^{zk} \cdot 10^{z(k+2)}} = 3 \cdot \sum_{k=0}^{+\infty} \frac{1}{10^{(2zk+2z-k)}}. \end{aligned} \tag{4}$$

Then from the properties of the geometric progression and (4) we know that the series $f(z,3)$ is divergent if $z \leq \frac{1}{2}$. This proves our theorem for $k = 3$.

Similarly, we can deduce the other cases. For example, if $k = 4$, then we have

$$\begin{aligned} f(z,4) &= \sum_{n=1}^{+\infty} \frac{1}{S_4^z(n)} = \sum_{i=1}^{2} \frac{1}{S_4^z(i)} + \sum_{i=3}^{24} \frac{1}{S_4^z(i)} + \sum_{i=25}^{249} \frac{1}{S_4^z(i)} + \sum_{i=250}^{2499} \frac{1}{S_4^z(i)} + \sum_{i=2500}^{24999} \frac{1}{S_3^z(i)} + \cdots \\ &= \sum_{i=1}^{2} \frac{1}{i^z \cdot 14^z} + \sum_{i=3}^{24} \frac{1}{i^z \cdot 104^z} + \sum_{i=25}^{249} \frac{1}{i^z \cdot 1004^z} + \sum_{i=250}^{2499} \frac{1}{i^z \cdot 10004^z} + \cdots \\ &\leq \sum_{k=1}^{+\infty} \frac{225 \cdot 10^{k-2}}{10^{z(k-2)} \cdot 10^{zk}} \leq 225 \cdot \sum_{k=0}^{+\infty} \frac{10^k}{10^{zk} \cdot 10^{z(k+2)}} = 225 \cdot \sum_{k=0}^{+\infty} \frac{1}{10^{k\cdot(2z-1)+2z}} \end{aligned} \tag{5}$$

and

$$\begin{aligned} f(z,4) &= \sum_{n=1}^{+\infty} \frac{1}{S_4^z(n)} = \sum_{i=1}^{2} \frac{1}{S_4^z(i)} + \sum_{i=3}^{24} \frac{1}{S_4^z(i)} + \sum_{i=25}^{249} \frac{1}{S_4^z(i)} + \sum_{i=250}^{2499} \frac{1}{S_4^z(i)} + \sum_{i=2500}^{24999} \frac{1}{S_3^z(i)} + \cdots \\ &= \sum_{i=1}^{2} \frac{1}{i^z \cdot 14^z} + \sum_{i=3}^{24} \frac{1}{i^z \cdot 104^z} + \sum_{i=25}^{249} \frac{1}{i^z \cdot 1004^z} + \sum_{i=250}^{2499} \frac{1}{i^z \cdot 10004^z} + \cdots \\ &\geq \sum_{k=1}^{+\infty} \frac{2 \cdot 10^k}{10^{z(k-1)} \cdot 10^{zk}} \geq 20 \cdot \sum_{k=0}^{+\infty} \frac{10^k}{10^{zk} \cdot 10^{z(k+1)}} = 20 \cdot \sum_{k=0}^{+\infty} \frac{1}{10^{k\cdot(2z-1)+z}}. \end{aligned} \tag{6}$$

From (5), (6) and the properties of the geometric progression we know that the theorem is holds for the Smarandache $4n$-digital subsequence.

This completes the proof of Theorem.

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On the Pseudo-Smarandache-Squarefree function and Smarandache function

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Abstract For any positive integer n, Pseudo-Smarandache- Squarefree function $Z_w(n)$ is defined as $Z_w(n) = \min\{m : n|m^n, m \in N\}$. Smarandache function $S(n)$ is defined as $S(n) = \min\{m : n|m!, m \in N\}$. The main purpose of this paper is using the elementary methods to study the mean value properties of the Pseudo-Smarandache-Squarefree function and Smarandache function, and give two sharper asymptotic formulas for it.

Keywords Pseudo-Smarandache-Squarefree function$Z_w(n)$, Smarandache function $S(n)$, me an value, asymptotic formula.

§1. Introduction and result

For any positive integer n, the famous Smarandache function $S(n)$ is defined as $S(n) = \min\{m : n|m!, m \in N\}$, Pseudo-Smarandache-Squarefree function $Z_w(n)$ is defined as the smallest positive integer m such that $n \mid m^n$. That is,

$$Z_w(n) = \min\{m : n|m^n, m \in N\}.$$

For example $Z_w(1) = 1$, $Z_w(2) = 2$, $Z_w(3) = 3$, $Z_w(4) = 2$, $Z_w(5) = 5$, $Z_w(6) = 6$, $Z_w(7) = 7$, $Z_w(8) = 2$, $Z_w(9) = 3$, $Z_w(10) = 10$, $\cdots$. About the elementary properties of $Z_w(n)$, some authors had studied it, and obtained some interesting results. For example, Felice Russo [1] obtained some elementary properties of $Z_w(n)$ as follows:

Property 1. For any positive integer $k > 1$ and prime p, we have $Z_w(p^k) = p$.

Property 2. For any positive integer n, we have $Z_w(n) \leq n$.

Property 3. The function $Z_w(n)$ is multiplicative. That is, if $GCD(m, n) = 1$, then $Z_w(m \cdot n) = Z_w(m) \cdot Z_w(n)$.

The main purpose of this paper is using the elementary methods to study the mean value properties of $Z_w(S(n))$ and $S(n) \cdot Z_w(n)$, and give two sharper asymptotic formulas for it. That is, we shall prove the following conclusions:

Theorem 1. Let $k \geq 2$ be any fixed positive integer. Then for any real number $x \geq 2$,

we have the asymptotic formula

$$\sum_{n\le x} Z_w(S(n)) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x} + O\Big(\frac{x^2}{\ln^{k+1} x}\Big),$$

where c_i $(i = 2, 3\cdots k)$ are computable constants.

Theorem 2. Let $k \ge 2$ be any fixed positive integer. Then for any real number $x \ge 2$, we have the asymptotic formula

$$\sum_{n\le x} Z_w(n)\cdot S(n) = \frac{\zeta(2)\cdot\zeta(3)}{3\zeta(4)}\prod_p\Big(1-\frac{1}{p+p^3}\Big)\cdot\frac{x^3}{\ln x} + \sum_{i=2}^{k}\frac{e_i\cdot x^3}{\ln^i x} + O\Big(\frac{x^3}{\ln^{k+1} x}\Big),$$

where $\zeta(n)$ is the Riemann zeta-function, $\prod\limits_p$ denotes the product over all primes, e_i $(i = 2, 3\cdots k)$ are computable constants.

§2. A simple lemma

To complete the proof of the theorem, we need the following :

Lemma. For any real number $x \ge 2$ and $s \ge 2$, we have the asymptotic formula

$$\sum_{n\le\sqrt{x}}\frac{Z_w(n)}{n^s} = \frac{\zeta(s)\cdot\zeta(s-1)}{\zeta(2(s-1))}\cdot\prod_p\Big(1-\frac{1}{p+p^s}\Big) + O\Big(x^{1-\frac{s}{2}}\Big).$$

Proof. Note that Property 1 and 3, by the Euler product formula (See Theorem 11.7 of [2]), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{Z_w(n)}{n^s} &= \prod_p\Big(1+\frac{Z_w(p)}{p^s}+\frac{Z_w(p^2)}{p^{2s}}+\cdots\Big)\\ &= \prod_p\Big(1+\frac{p}{p^s}+\frac{p}{p^{2s}}+\cdots\Big)\\ &= \prod_p\Big(1+\frac{1}{p^{s-1}}\cdot\frac{1}{1-p^{-s}}\Big)\\ &= \frac{\zeta(s)\zeta(s-1)}{\zeta(2(s-1))}\prod_p\Big(1-\frac{1}{p+p^s}\Big). \qquad (1)\end{aligned}$$

From (1) we have

$$\sum_{n\le\sqrt{x}}\frac{Z_w(n)}{n^s} = \sum_{n=1}^{\infty}\frac{Z_w(n)}{n^s} - \sum_{n>\sqrt{x}}\frac{Z_w(n)}{n^s} = \frac{\zeta(s)\zeta(s-1)}{\zeta(2(s-1))}\prod_p\Big(1-\frac{1}{p+p^s}\Big) + O\Big(x^{1-\frac{s}{2}}\Big).$$

Specially, if $s = 3$, then we have the asymptotic formula

$$\sum_{n\le\sqrt{x}}\frac{Z_w(n)}{n^s} = \frac{\zeta(3)\zeta(2)}{\zeta(4)}\prod_p\Big(1-\frac{1}{p+p^3}\Big) + O\Big(\frac{1}{\sqrt{x}}\Big).$$

This completes the proof of Lemma.

§3. Proof of the theorems

In this section, we shall use the elementary methods to complete the proof of the theorems.

First we prove Theorem 1. We separate all integer n in the interval $[1, x]$ into two subsets A and B as follows:

A: $p \mid n$ and $p > \sqrt{n}$, where p is a prime. B: other positive integer n such that $n \in [1, x] \setminus A$.

From the definition of the subsets A and B we have

$$\sum_{n \le x} Z_w(S(n)) = \sum_{n \in A} Z_w(S(n)) + \sum_{n \in B} Z_w(S(n)). \tag{2}$$

From Property 1 and the definition of the function $S(n)$ and the subset A we know that if $n \in A$, then we have

$$\sum_{n \in A} Z_w(S(n)) = \sum_{\substack{pn \le x \\ p > n}} Z_w(S(pn)) = \sum_{\substack{pn \le x \\ p > n}} Z_w(p) = \sum_{\substack{pn \le x \\ p > n}} p = \sum_{n \le \sqrt{x}} \sum_{n < p \le \frac{x}{n}} p. \tag{3}$$

By the Abel's summation formula (see Theorem 4.2 of [2]) and the Prime Theorem (see Theorem 3.2 of [3]):

$$\pi(x) = \sum_{p \le x} 1 = \sum_{i=1}^{k} \frac{a_i \cdot x}{\ln^i x} + O\Big(\frac{x}{\ln^{k+1} x}\Big),$$

where a_i $(i = 1, 2, 3 \cdots k)$ are computable constants and $a_1 = 1$, we have

$$\begin{aligned} \sum_{n < p \le \frac{x}{n}} p &= \frac{x}{n} \pi\Big(\frac{x}{n}\Big) - n\pi(n) - \int_n^{\frac{x}{n}} \pi(t) \mathrm{d}t \\ &= \frac{x^2}{2n^2 \cdot \ln x} + \sum_{i=2}^{k} \frac{b_i \cdot x^2 \cdot \ln^i n}{n^2 \cdot \ln^i x} + O\Big(\frac{x^2}{n^2 \cdot \ln^{k+1} x}\Big), \end{aligned} \tag{4}$$

where b_i $(i = 2, 3 \cdots k)$ are computable constants.

Note that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi 2}{6}$ and $\sum_{n=1}^{\infty} \frac{\ln^i n}{n^2}$ is convergent for all $i = 1, 2, \cdots, k$. Combining (3) and (4) we have

$$\begin{aligned} \sum_{n \in A} Z_w(S(n)) &= \sum_{n \le \sqrt{x}} \left(\frac{x^2}{2n^2 \cdot \ln x} + \sum_{i=2}^{k} \frac{b_i \cdot x^2 \cdot \ln^i n}{n^2 \cdot \ln^i x} + O\left(\frac{x^2}{n^2 \ln^{k+1} x} \right) \right) \\ &= \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O\Big(\frac{x^2}{\ln^{k+1} x}\Big), \end{aligned} \tag{5}$$

where c_i $(i = 2, 3 \cdots k)$ are computable constants.

Now we estimate the error terms in set B. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the factorization of n into prime powers. If $n \in B$, then we have

$$S(n) = \max_{1 \le i \le s} \Big(S(p_i^{\alpha_i}) \Big) \le \max_{1 \le i \le s} (\alpha_i p_i) \le \sqrt{n} \ln n \ll n^{\frac{5}{6}}. \tag{6}$$

From (6) and Property 2 we have

$$\sum_{n\in B} Z_w(S(n)) \ll \sum_{n\le x} n^{\frac{5}{6}} \ll x^{\frac{11}{6}}. \tag{7}$$

Combining (2), (5) and (7) we have

$$\sum_{n\le x} Z_w(S(n)) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x} + O\Big(\frac{x^2}{\ln^{k+1} x}\Big).$$

This proves Theorem 1.

Now we prove Theorem 2. From Property 1 and Property 3 we have

$$\sum_{n\in A} S(n)\cdot Z_w(n) = \sum_{\substack{pn\le x\\ n<p}} S(pn)\cdot Z_w(pn) = \sum_{n\le\sqrt{x}}\sum_{n<p\le\frac{x}{n}} p^2\cdot Z_w(n). \tag{8}$$

$$\begin{aligned}\sum_{n<p\le\frac{x}{n}} p^2 &= \frac{x^2}{n^2}\pi\Big(\frac{x}{n}\Big) - n^2\pi(n) - 2\int_n^{\frac{x}{n}} t\cdot\pi(t)\mathrm{d}t\\ &= \frac{x^3}{3n^3\cdot\ln x} + \sum_{i=2}^{k}\frac{d_i\cdot x^3\cdot\ln^i n}{n^3\cdot\ln^i x} + O\Big(\frac{x^3}{n^3\cdot\ln^{k+1} x}\Big),\end{aligned} \tag{9}$$

where d_i $(i=2,3\cdots k)$ are computable constants.

Note that the lemma and $\sum_{n=1}^{\infty}\frac{\ln^i n\cdot Z_w(n)}{n^3}$ is convergent for all $i=1,2,\cdots,k$. Combining (8) and (9) we have

$$\begin{aligned}\sum_{n\in A} S(n)\cdot Z_w(n) &= \sum_{n\le\sqrt{x}}\left(\frac{x^3}{3n^3\cdot\ln x} + \sum_{i=2}^{k}\frac{d_i\cdot x^3\cdot\ln^i n}{n^3\cdot\ln^i x} + O\left(\frac{x^3}{n^3\cdot\ln^{k+1} x}\right)\right)\cdot Z_w(n)\\ &= \frac{\zeta(2)\zeta(3)}{3\zeta(4)}\prod_p\Big(1-\frac{1}{p+p^3}\Big)\cdot\frac{x^3}{\ln x} + \sum_{i=2}^{k}\frac{e_i\cdot x^3}{\ln^i x} + O\Big(\frac{x^3}{\ln^{k+1} x}\Big),\end{aligned} \tag{10}$$

where e_i $(i=2,3\cdots k)$ are computable constants.

If $n\in B$, then we have

$$\sum_{n\in B} S(n)\cdot Z_w(n) \ll \sum_{n\le x}\sqrt{n}\ln n\cdot n \ll x^{\frac{5}{2}}\ln x. \tag{11}$$

Combining (10) and (11) we have

$$\sum_{n\le x} Z_w(n)\cdot S(n) = \frac{\zeta(2)\cdot\zeta(3)}{3\zeta(4)}\prod_p\Big(1-\frac{1}{p+p^3}\Big)\cdot\frac{x^3}{\ln x} + \sum_{i=2}^{k}\frac{e_i\cdot x^3}{\ln^i x} + O\Big(\frac{x^3}{\ln^{k+1} x}\Big).$$

This proves Theorem 2.

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On the Smarandache prime-digital subsequence sequences

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Abstract The main purpose of the paper is using the elementary method to study the properties of the Smarandache Prime-Digital Subsequence, and give an interesting limit Theorem. This solved a problem proposed by Charles.

Keywords Smarandache $SPDS$ subsequence, $\pi(n)$ function, limit.

§1. Introduction and results

For any positive integer n, the Smarandache Prime-Digital Subsequence ($SPDS$) is defined as follows:

A positive integer n is an element of $SPDS$, if it satisfies the following properties:

a) m is a prime.

b) All of the digits of m are prime, i.e, they are all elements of the set $\{2,\ 3,\ 5,\ 7\}$.

For example, the first few values of $SPDS$ are:

$2,\ 3,\ 5,\ 7,\ 23,\ 53,\ 73,\ 223,\ 227,\ 233,\ 257,\ 277,\ 377,\ 353,\ 373,\ 523, \cdots$.

This sequence was introduced by professor F.Smarandache in reference [1], where he asked us to studied its elementary properties. In reference [2], Charles Ashbacher had studied this problem, and obtained some interesting results. At the same time, he also proposed the following Conjecture and Unsolved problems.

Conjecture. Sequence $SPDS$ is a infinite set.

Unsolved problem 1. How many prime are there of the form

$$\underbrace{111\cdots111}_{k\ \ 1's}$$

where of course k is odd.

Unsolved problem 2.

$$\lim_{n\to\infty}\frac{SPDSN(n)}{\pi(n)}=0,$$

where $SPDSN(n)$ represent the number of elements of $SPDS$ that are less than or equal to n, and $\pi(n)$ denotes the number of all primes not exceeding n.

A short UBASIC program was run for all numbers up to $1,000,000$, and the counts were 78498 primes $< 1,000,000$; 587 members of $SPDS < 1,000,000$.

But at present, we still can not prove that $SPDS$ is a finite set, and we can not also solve Problem 1. In this paper, we will use the elementary method and analytic method to study Problem 2, and solved it completely. That is, we shall prove the following:

Theorem. Let $SPDSN(n)$ denotes the number of all elements of $SPDS$ that are less than or equal to n, and $\pi(n)$ denotes the number of all primes not exceeding n. Then we have the limit

$$\lim_{n\to\infty} \frac{SPDSN(n)}{\pi(n)} = 0.$$

It is clear that our Theorem solved the problem 2.

§2. Proof of the theorem

To complete the proof of our theorem, we need a simple Lemma which stated as follows:

Lemma. For every integer $n \geq 2$, we have the estimate

$$\frac{1}{6} \cdot \frac{n}{\ln n} < \pi(n) < 6 \cdot \frac{n}{\ln n}.$$

Proof. See reference [3].

Now we use this Lemma to prove our theorem. For any positive integer n, if digits of n in decimal notation are $A_{k-1},\ A_{k-2},\ \cdots,\ A_1,\ A_0$, then

$$n = A_{k-1}10^{k-1} + A_{k-2}10^{k-2} + \cdots + A_1 10^1 + A_0.$$

where $1 \leq A_i \leq 9$.

It is clear that

$$10^{k-1} \leq n \leq 10^k.$$

Therefore,

$$\lg n \leq k \leq \lg n + 1$$

or

$$k = \lg n + O(1).$$

Since

$$SPDNS(n) = \sum_{\substack{m\leq n \\ m\in SPDS}} 1 \leq 4^1 + 4^2 + 4^3 + \cdots + 4^k = \frac{4}{3}(4^k - 1) < 4^{k+1}$$

and

$$\frac{1}{6} \cdot \frac{n}{\ln n} < \pi(n) < 6 \cdot \frac{n}{\ln n}.$$

We have

$$0 \leq \frac{SPDSN(n)}{\pi(n)} \quad \leq \quad \frac{4^{k+1} \cdot 6 \cdot \ln n}{n} = \frac{4^{\lg n + O(1)} \cdot 6 \cdot \ln n}{n}.$$

Taking $x \to \infty$, we find that

$$\begin{aligned} 0 \leq \lim_{x\to\infty} \frac{4^{\lg x+O(1)} \cdot 6 \cdot \ln x}{x} &\ll \lim_{x\to\infty} \frac{4^{\lg x} \cdot 6 \cdot \ln x}{x} = \lim_{x\to\infty} \frac{e^{\ln 4 \cdot \lg x} \cdot 6 \cdot \ln x}{x} \\ &= \lim_{x\to\infty} \frac{e^{\ln x \cdot \frac{\ln 4}{\ln 10}} \cdot 6 \cdot \ln x}{x} = \lim_{x\to\infty} \frac{x^{\frac{\ln 4}{\ln 10}} \cdot 6 \cdot \ln x}{x} \\ &= \lim_{x\to\infty} \frac{6 \cdot \ln x}{x^{(1-\frac{\ln 4}{\ln 10})}} = \lim_{x\to\infty} \frac{6}{(1-\frac{\ln 4}{\ln 10}) \cdot x^{(1-\frac{\ln 4}{\ln 10})}} \\ &= 0. \end{aligned}$$

So from the properties of the limit we have

$$\lim_{n\to\infty} \frac{4^{\lg n+1+O(1)} \cdot 6 \cdot \ln n}{n} = 0.$$

Therefore,

$$\lim_{n\to\infty} \frac{SPDSN(n)}{\pi(n)} = 0.$$

This completes the proof of Theorem.

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On the mean value of $a^2(n)$ [1]

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Abstract Let $a(n)$ denote the number of nonisomorphic Abelian groups with n elements. In this paper we shall study the mean value of $a^2(n)$ by the convolution method.

Keywords Piltz divisor problem, mean value, convolution method.

§1. Introduction and main results

Let $a(n)$ denote the number of nonisomorphic Abelian groups with n elements. The mean value of $a(n)$ was first studied by P. Erdös and G. Szekeres[1], who proved that

$$\sum_{n\le x} a(n) = c_1 x + O(x^{1/2}). \tag{1}$$

Kendall and Rankin[2] proved that

$$\sum_{n\le x} a(n) = c_1 x + c_2 x^{1/2} + O(x^{1/3}\log x). \tag{2}$$

H. -E. Richert[7] proved

$$\sum_{n\le x} a(n) = c_1 x + c_2 x^{1/2} + c_3 x^{1/3} + O(x^{3/10}\log^{9/10} x). \tag{3}$$

Let $\Delta(x) := \sum_{n\le x} a(n) - c_1 x - c_2 x^{1/2} - c_3 x^{1/3}$.
The following is the list of the improvements to (3).
$\Delta(x) \ll x^{20/69}\log^{21/23} x$, [8]W. Schwarz; $\Delta(x) \ll x^{7/27}\log^2 x$, [9]P. G. Schmidt
$\Delta(x) \ll x^{97/381}\log^{35} x$, [4]G. Kolesnik; $\Delta(x) \ll x^{40/159+\epsilon}$, [5]H. Q. Liu
$\Delta(x) \ll x^{50/199+\epsilon}$, [5] H. Q. Liu; $\Delta(x) \ll x^{55/219\log^7 x}$, [10]Sargos and Wu;
$\Delta(x) \ll x^{1/4+\varepsilon}$, [6]Robert and Sargos.

In this paper, we shall prove a result about the mean value of $a^2(n)$. Our main result is the following

Theorem. We have

$$\begin{aligned}\sum_{n\le x} a^2(n) &= c_4 x + c_5 x^{\frac{1}{2}}\log^2 x + c_6 x^{\frac{1}{2}}\log x \\ &\quad + c_7 x^{1/2} + O(x^{96/245+\varepsilon}),\end{aligned} \tag{4}$$

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where $c_j (j = 4, 5, 6, 7)$ are computable constants.

Notations. Throughout this paper, $[t]$ denotes the integer part of t, $\psi(t) = t - [t] - 1/2$, $\varepsilon > 0$ denotes a small positive constant.

§2. Proof of the theorem

The proof of our theorem is closely related to the Piltz divisor problem. Let $\Delta_3(x)$ denotes the error term in the asymptotic formula for $\sum\limits_{n\le x} d_3(n)$, where $d_3(n)$ is the number of ways n can be written as a product of 3 factors. We know that

$$D_3(x) = \sum_{n\le x} d_3(n) = xH_3(\log x) + \Delta_3(x), \tag{5}$$

where $H_3(u)$ is a polynomial of degree 2 in u.

For the upper bound of $\Delta_3(x)$, Kolesnik[3] proved that

$$\Delta_3(x) \ll x^{\frac{43}{96}+\varepsilon}. \tag{6}$$

We begin with the function $d(1, 2, 2, 2; n) = \sum\limits_{n=n_1n_2^2n_3^2n_4^2} 1.$

By the hyperbolic summation method we have

$$\begin{aligned}\sum_{n\le x} d(1,2,2,2;n) &= \sum_{ml^2\le x} d_3(l) \\ &= \sum_{l\le y} d_3(l) \sum_{m\le \frac{x}{y^2}} 1 + \sum_{m\le \frac{x}{y^2}} \sum_{l\le (\frac{x}{m})^{\frac{1}{2}}} d_3(l) - [\frac{x}{y^2}]D_3(y),\end{aligned} \tag{7}$$

where $1 < y < x^{1/2}$ is a parameter to be determined.

Inserting (5) into (7) and by some calculations, we obtain

$$\sum_{n\le x} d(1,2,2,2;n) = c_0'x + c_1'x^{1/2}\log^2 x + c_2'x^{1/2}\log x + c_3'x^{1/2} + \Delta(1,2,2,2;x) \tag{8}$$

with

$$\Delta(1,2,2,2;x) = -\sum_{l\le y} d_3(l)\psi(\frac{x}{l^2}) + \sum_{m\le \frac{x}{y^2}} \Delta_3((\frac{x}{m})^{\frac{1}{2}}) + O(\frac{y^3}{x}(\log y)^2 + \frac{x}{y^{2-\varepsilon}}), \tag{9}$$

where $c_j' (j = 0, 1, 2, 3)$ are computable constants. Choosing $y = x^{96/245}$ we get

$$\Delta(1,2,2,2;x) \ll x^{96/245+\varepsilon}. \tag{10}$$

By the Euler product we get for $\Re s > 1$ that

$$\sum_{n=1}^{\infty} a^2(n)n^{-s} = (1+\sum_{\alpha=1}^{\infty}\frac{a^2(p^\alpha)}{p^{\alpha s}}) = \prod_p (1+\frac{1}{p^s}+\frac{4}{p^{2s}}+\cdots) = \zeta(s)\zeta^3(2s)G(s), \tag{11}$$

where $G(s) = \prod\limits_p (1-\frac{1}{ps})(1-\frac{3}{p^{2s}})\prod\limits_p (1+\frac{1}{p^s}+\frac{4}{p^{2s}}+\cdots).$

Write $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$ $(Rs > 1)$. It is easy to see that this infinite series is absolutely convergent in the range $\sigma > \frac{1}{3}$, which implies that

$$\sum_{n \le x} |g(n)| = O(x^{\frac{1}{3}+\varepsilon}). \tag{12}$$

From (11) we have the relation

$$a^2(n) = \sum_{n=ml} d(1,2,2,2;m)g(l). \tag{13}$$

By (10)-(13)we get

$$\begin{aligned}
&\sum_{n \le x} a^2(n) = \sum_{l \le x} g(l) \sum_{m \le x/l} d(1,2,2,2;m) = \sum_{l \le x} g(l)(c_0'(\frac{x}{l}) + c_1'(\frac{x}{l})^{1/2} \log^2(\frac{x}{l}) \\
&+c_2\prime(\frac{x}{l})^{1/2} \log(\frac{x}{l}) + c_3'(\frac{x}{l})^{1/2}) + O(\sum_{l \le x} |g(l)|(\frac{x}{l})^{96/245+\varepsilon}), \\
&= c_4 x + c_5 x^{\frac{1}{2}} \log^2 x + c_6 x^{\frac{1}{2}} \log x + c_7 x^{1/2} + O(x^{96/245+\varepsilon}).
\end{aligned}$$

This completes the proof of Theorem.

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Monotonicity properties for the Gamma and Psi functions

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Abstract We present some the monotonicity and complete monotonicity properties for the gamma and psi functions. This extends some known results.

Keywords Gamma function, psi function, polygamma function, monotonicity, complete monoto nicity, logarithmically complete monotonicity.

§1. Introduction and result

The classical gamma function is usually defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \tag{1}$$

which is one of the most important special functions and has much extensive applications in many branches, for example, statistics, physics, engineering, and other mathematical sciences. The history and the development of this function are described in detail in [1]. The psi or digamma function $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed for $x > 0$ and $k = 1, 2, \cdots$ as

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}} dt, \tag{2}$$

$$\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k e^{-xt}}{1 - e^{-t}} dt, \tag{3}$$

where $\gamma = 0.57721566490153286 \cdots$ is the Euler-Mascheroni constant.

There exists a very extensive literature on these functions. In particular, inequalities, monotonicity and complete monotonicity properties for these functions have been published. Please refer to the papers [2-4] and the references therein. Recall that a function f is said to be completely monotonic on an interval I if f has derivatives of all orders on I and

$$(-1)^n f^{(n)}(x) \geq 0 \tag{4}$$

for $x \in I$ and $n \geq 0$. Let $\mathcal{C}$ denote the set of completely monotonic functions. Some related references and a detailed collection of the most important properties of the completely monotonic functions can be found in [5] and [6, Chapter IV].

A positive function f is said to be logarithmically completely monotonic on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^k[\ln f(x)]^{(k)} \geq 0 \tag{5}$$

for $k = 1, 2, \cdots$ on I. Let $\mathcal{L}$ on $(0, \infty)$ stand for the set of logarithmically completely monotonic functions.

A function f on $(0, \infty)$ is called a Stieltjes transform if it can be written in the form

$$f(x) = a + \int_0^\infty \frac{d\mu(s)}{s + x}, \tag{6}$$

where a is a nonnegative number and μ a nonnegative measure on $[0, \infty)$ satisfying

$$\int_0^\infty \frac{1}{1 + s} d\mu(s) < \infty.$$

The set of Stieltjes transforms is denoted by $\mathcal{S}$.

The notion "logarithmically completely monotonic function" was posed in [7] and a much useful and meaningful relation $\mathcal{L} \subset \mathcal{C}$ between the completely monotonic functions and the logarithmically completely monotonic functions was proved in [7]. In fact, the relation $\mathcal{L} \subset \mathcal{C}$ is an old result and can be found in [8]. It is proved in [9] that $\mathcal{S} \setminus \{0\} \subset \mathcal{L} \subset \mathcal{C}$. The class of logarithmically completely monotonic functions can be characterized as the infinitely divisible completely monotonic functions which are established by Horn in [10, Theorem 4.4] and restated in [9, Theorem 1.1].

When studying a problem on upper bound for permanents of $(0, 1)$–matrices, H. Minc and L. Sathre [11] discovered several noteworthy inequalities involving $(n!)^{1/n}$. One of them is the following: If n is a positive integer, then

$$\frac{n}{n+1} < \frac{\sqrt[n]{n!}}{\sqrt[n+1]{(n+1)!}} < 1. \tag{7}$$

Motivated by the left-hand inequality of (7), it was shown in [7] that the function $\frac{[\Gamma(x+1)]^{1/x}}{x}$ is strictly logarithmically completely monotonic on $(0, \infty)$. This extends a result of D. Kershaw and A. Laforgia [12], who proved that the function $x[\Gamma(1 + \frac{1}{x})]^x$ is strictly increasing on $(0, \infty)$, which is equivalent to the function $\frac{[\Gamma(x+1)]^{1/x}}{x}$ being strictly decreasing on $(0, \infty)$.

Motivated by the right-hand inequality of (7), we establish the following result.

Theorem 1. Let $r \geq 0, s \geq 0$ be two real numbers. The function

$$f_{r,s}(x) = \frac{[\Gamma(x + s + 1)]^{1/(x+s)}}{[\Gamma(x + r + 1)]^{1/(x+r)}}$$

is strictly logarithmically completely monotonic on $(0, \infty)$ if and only if $s > r$.

In 1997, G. D. Anderson and S.-L.Qiu [13] proved that the function $\ln \Gamma(x + 1)/(x \ln x)$ is strictly increasing on $(1, \infty)$, and then, used this result to show that the sequence $\Omega^{1/(n \ln n)}(n = 2, 3, \cdots)$ is strictly decreasing. Here, $\Omega_n = \pi^{n/2}/\Gamma(n/2 + 1)$ denotes the n-dimensional volume of the unit ball in R^n. In order to show the monotonicity of the function $\ln \Gamma(x + 1)/(x \ln x)$, they investigated the function

$$f(x) = \sum_{n=1}^{\infty} \frac{n - x}{(n + x)^3}$$

and they found the representation

$$f(x) = \psi'(x+1) + x\psi''(x+1). \tag{8}$$

They proved, in a complicated way, that $f(x) > 0$ for $x \in [1, 4)$. A. Elbert and A. Laforgia [14] showed, in a simple way, that $f(x) > 0$ for $x \in (-1, \infty)$.

The following Theorem 2 consider the complete monotonicity property of the function f defined by (8).

Theorem 2. The function f, defined by (8), is strictly completely monotonic on $(0, \infty)$.

As direct consequence of Theorem 2, the following Theorem 3 is deduced immediately.

Theorem 3. The function $g(x) = x^2\psi'(x+1) - x\psi(x+1) + \ln\Gamma(x+1)$ is strictly increasing on $(0, \infty)$. The function $h(x) = x\psi'(x+1) - \ln\Gamma(x+1)$ is positive and strictly increasing on $(0, \infty)$. The function h'' is strictly completely monotonic on $(0, \infty)$.

As an application of Theorem 3, we provide an extension of the result given by Anderson and Qiu [13].

Theorem 4. The function $F(x) = \Gamma(x+1)/(x \ln x)$ is strictly increasing on $(0, \infty)$.

§2. Proofs of the theorems

Proof of Theorem 1. Firstly, we show that the function $x \mapsto f_{r,s}(x)$ is strictly logarithmically completely monotonic on $(0, \infty)$ for $s > r$. Define for $x > 0$,

$$g(x) = \frac{\ln[\Gamma(x+1)]}{x}.$$

Clearly,

$$(-1)^n (\ln f_{r,s}(x))^{(n)} = (-1)^n g^{(n)}(x+s) - g^{(n)} g(x+r). \tag{9}$$

For $n \geq 1$, we imply

$$\begin{aligned}
&\frac{x^{n+1}}{n!}(-1)^n g^{(n)}(x) = \frac{x^{n+1}}{n!}(-1)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{x}\right)^{(n-k)} [\ln\Gamma(x+1)]^{(k)} \\
&= \frac{x^{n+1}}{n!}(-1)^n \left[\frac{(-1)^n n!}{x^{n+1}} \ln\Gamma(x+1) + \sum_{k=1}^{n} \frac{n!}{k!} \frac{(-1)^{n-k}}{x^{n-k+1}} \psi^{(k-1)}(x+1) \right] \\
&= \ln\Gamma(x+1) + \sum_{k=1}^{n} \frac{(-1)^k}{k!} x^k \psi^{(k-1)}(x+1) \triangleq h(x),
\end{aligned}$$

and

$$\begin{aligned}
h'(x) &= \psi(x+1) + \sum_{k=1}^{n} \frac{(-1)^k}{(k-1)!} x^{k-1} \psi^{(k-1)}(x+1) + \sum_{k=1}^{n} \frac{(-1)^k}{k!} x^k \psi^{(k)}(x+1) \\
&= \frac{(-1)^n}{n!} x^n \psi^{(n)}(x+1) = -\frac{x^n}{n!} \int_0^\infty \frac{t^n}{e^t - 1} e^{-xt} dt < 0.
\end{aligned}$$

Hence, $h(x) < h(0) = 0$ and

$$(-1)^n g^{(n)}(x) < 0 \quad (x > 0; n = 1, 2, \cdots). \tag{10}$$

This implies $((-1)^n g^{(n)}(x))' > 0$, and then, the function $x \mapsto (-1)^n g^{(n)}(x)$ is strictly increasing on $(0, \infty)$, so that (9) implies $(-1)^n (\ln f_{r,s}(x))^{(n)} > 0$ for $x > 0$ and $n = 1, 2, \cdots$.

Next, we assume that the function $x \mapsto f_{r,s}(x)$ is strictly logarithmically completely monotonic on $(0, \infty)$. Then we have for all real $x > 0$,

$$(\ln f_{r,s}(x))' = g'(x + s) - g'(x + r) < 0. \tag{11}$$

By (10), it is easy to see that the function g' is strictly decreasing on $(0, \infty)$, and thus, we conclude from (11) that $s > r$. The proof of Theorem 1 is complete.

Proof of Theorem 2. Using the representation (3), we imply for $n \geq 0$,

$$\begin{aligned}
(-1)^n f^{(n)}(x) &= (-1)^n \left[\psi^{(n+1)}(x+1) + \sum_{k=0}^{n} \binom{n}{k} (x)^{(k)} \psi^{(n-k+2)}(x) \right] \\
&= (-1)^n \left[x\psi^{(n+2)}(x+1) + (n+1)\psi^{(n+1)}(x) \right] \\
&= -x \int_0^\infty \frac{t^{n+2}}{1 - e^{-t}} e^{-(x+1)t} dt + (n+1) \int_0^\infty \frac{t^{n+1}}{1 - e^{-t}} e^{-(x+1)t} dt \\
&= \int_0^\infty \delta(t) t^n e^{-xt} [(n+1) - xt] dt,
\end{aligned} \tag{12}$$

where

$$\delta(t) = \frac{t}{e^t - 1}.$$

It is easy to see that the function δ is strictly decreasing on $(0, \infty)$ with $\lim_{t \to 0} \phi(x) = 1$ and $\lim_{t \to \infty} \phi(x) = 0$. Hence, we get from (12),

$$\begin{aligned}
(-1)^n f^{(n)}(x) &= \int_0^{(n+1)/x} \delta(t) t^n e^{-xt} [(n+1) - xt] dt \\
&\qquad + \int_{(n+1)/x}^\infty \delta(t) t^n e^{-xt} [(n+1) - xt] dt \\
&> \delta\left(\frac{n+1}{x}\right) \int_0^{(n+1)/x} t^n e^{-xt} [(n+1) - xt] dt \\
&\qquad + \delta\left(\frac{n+1}{x}\right) \int_{(n+1)/x}^\infty t^n e^{-xt} [(n+1) - xt] dt \\
&= \delta\left(\frac{n+1}{x}\right) \int_0^\infty t^n e^{-xt} [(n+1) - xt] dt = 0.
\end{aligned}$$

At the last step, by applying the following representation

$$\frac{m!}{(x+a)^{m+1}} = \int_0^\infty t^m e^{-(x+a)t} dt \quad (x > 0; a \geq 0, m = 0, 1, 2, \cdots).$$

The proof of Theorem 2 is complete.

Proof of Theorem 3. Clearly,

$$g'(x) = x\psi'(x+1) + x^2\psi''(x+1) = xf(x) > 0 \quad (x > 0).$$

where f is defined by (8). Hence, the function g is strictly increasing on $(0, \infty)$.

It is easy to see that

$$h'(x) = x\psi'(x+1) > 0 \quad (x > 0).$$

Hence, the function h is strictly increasing and $h(x) > h(0) = 0$ on $(0, \infty)$. Easy computation reveals that

$$h''(x) = \psi'(x+1) + x\psi''(x+1) = f(x),$$

By Theorem 2,

$$(-1)^n (h''(x))^{(n)} = (-1)^n f^{(n)}(x) > 0,$$

for $x > 0$ and $n = 0, 1, 2, \cdots$. The proof of Theorem 3 is complete.

In order to prove Theorem 4, we need the following lemma given in [15].

Lemma 1. Let $u \in C^1(0, \infty)$ with $u(1) = 0$ and $v \in C^1(0, \infty)$ such that $v < 0$ on $(0, 1)$, $v > 0$ on $(1, \infty)$ and $v' > 0$ on $(0, \infty)$. If u'/v' is strictly increasing on $(0, \infty)$, then u/v is also strictly increasing on $(0, \infty)$.

Proof of Theorem 4. Define for $x > 0$

$$u(x) = \frac{1}{x} \ln \Gamma(x+1), \quad v(x) = \ln x.$$

Easy computation yields

$$x^2 \left(\frac{u'(x)}{v'(x)} \right)' = x^2\psi'(x+1) - x\psi(x+1) + \ln \Gamma(x+1) = g(x).$$

By Theorem 3, we have $g(x) > g(0) = 0$ for $x > 0$. This implies that u'/v' is strictly increasing on $(0, \infty)$. From Lemma 1 we conclude that the function $F = u/v$ is also strictly increasing on $(0, \infty)$. The proof of Theorem 4 is complete.

We remark that Theorem 4 was first proved by H. Alzer [4], who showed that the function g is strictly increasing on $(0, \infty)$ by using the convolution theorem for Laplace transformas.

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Resolvent dynamical systems for set-valued quasi variational inclusions in Banach spaces[1]

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Abstract In this paper, a new class of generalized set-valued quasi variational inclusions in Banach spaces are studied, which include many variational inclusions studied by others in recent years. Using the resolvent operator technique, we establish the equivalence between the generalized set-valued quasi variational inclusions in Banach spaces and the fixed point problems, suggest and analyze a class of resolvent dynamical systems for quasi variational inclusion in Banach spaces. We show that the trajectory of the solution of the dynamics system converges globally exponentially to the unique solution of quasi variational inclusions in Banach spaces. Our results can be considered as a significant extension of previously known results.

Keywords Set-valued quasi variational inclusion, dynamical systems, Banach space.

§1. Introductions

In recent years, variational inequalities have been extended and generalized in different directions by using novel and innovative techniques and ideas both for their own sake and for their applications. An important and useful generalization is called the variational inclusion, see [1-8] and references therein. But almost all discussions about variational inclusions are limited in Hilbert spaces. In this paper, we introduce a new class of set-valued quasi-variational inclusions in Banach spaces. In recent years, much attention has been given to consider and analyze the projected dynamics systems associated with variational inequalities and nonlinear programming problems, in which the righthand side of the ordinary differential equation is a projection operator. Such types of projected dynamical systems were introduced and studied by Dupuis and Nagurney [9]. Projected dynamical systems are characterized by a discontinuous right-hand side. The discontinuity arises from the constraint governing the question. The innovative and novel feature of a projected dynamical system is that its set of stationary points corresponds to the set of solutions of the corresponding variational inequality problems. It has been shown in [9-15] that the dynamical systems are useful in developing efficient and powerful

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numerical technique for solving variational inequalities and related optimization problems. Xia and Wang [13], Zhang and Nagurner [15] and Nagurner and Zhang [9] have studied the globally asymptotic stability of these projected dynamics systems. Noor [16,17] has also suggested and analyzed similar resolvent dynamical systems for mixed variational inequalities and quasi variational inclusions by extending and modifying there techniques. It is worth mentioning that there is no such type of dynamical systems for quasi variational inclusions in Banach spaces. In this paper, we suggest and analyze dynamical systems for quasi variational inclusions in Banach spaces. Using the resolvent operator method, we establishes the equivalence between the quasi variational inclusions in Banach spaces, resolvent equations and fixed-point problems. We use this alternative formation to suggest a class of resolvent dynamical systems associated with quasi variational inclusions in Banach spaces. We show that the trajectory of the solutions of these dynamical systems converges globally exponentially to the unique solution of the related quasi variational inclusions in Banach spaces. The analysis is in the spirit of Xia and Wang [14] and Noor [16,17]. Since the quasi variational inequalities and nonlinear programming problems as special cases, the results obtained in this paper continue to hold for these problems.

§2. Preliminary

Let E be a real Banach space, E^* is the topological dual space of E, $CB(E)$ is the family of all nonempty closed and bounded subsets of E, $< \cdot, \cdot >$ is the dual pair between E and E^*, $D(T)$ denotes the domain of T, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by

$$J(x) = \{x^* \in E^* :< x, x^* >= \|x\|^2 = \|x^*\|^2\}, x \in E.$$

We now give the following well-known concepts and notions.

Definition 2.1. Let $A : D(A) \subset E \times E \to 2^E$ be a set-value mapping. $P : E \times E \to E$ is a projection mapping, that is $P(x, y) = x$, for any $(x, y) \in E \times E$.

(i) A is said to be accretive (k-strongly accretive) with respect to the first argument, if for any $x, y \in D(A), u \in Ax, v \in Ay$, there exist $j(P(x - y)) \in J(P(x - y))$, such that

$$< u - v, j(P(x - y)) >\geq 0(\geq k\|u - v\|^2);$$

(ii) A is said to be an m-accretive mapping with respect to the first argument, if A is accretive with respect to the first argument and $(I + \rho A(u))(P(D(A)) = E$, for every $u \in E$ and $\rho > 0$ (equivalently , if A is accretive with respect to the first argument and $(I + A(u))(P(D(A))) = E$, for all $u \in E$), where $A(\cdot, u) \equiv A(u)$.

Remark 2.1. If $A(u, v) = A(u)$, definition 2.1 is the very definition2.1 proposed by S. S. Chang [2]. Furthermore, if $E = E^* = H$ is a Hilbert space, the definition of an accretive mapping with respect to the first argument is in fact the definition of a monotone mapping with respect to the first argument proposed by Noor [1].

Proposition 2.1. If $E = H$ is a Hilbert space, an m-accretive mapping A with respect to the first argument is a maximal monotone mapping with respect to the first argument.

Proof. If we use the technique given in S. S. Chang [2], we can prove this proposition immediately.

Definition 2.2. Let $A : D(A) \subset E \times E \to 2^E$ be an m-accretive mapping with respect to the first argument, for any $\rho > 0$, the mapping defined by:

$$R_{A(u)}(u) = (I + \rho A(u))^{-1}(u),$$

for any $u \in E$, which is called the resolvent operator, where $A(\cdot, u) = A(u)$.

Problem 2.1. Let E be a real Banach space, $T, V : E \to CB(E)$ set-valued mappings, $g : E \to E$ a single-valued mapping, $A(\cdot,\cdot) : E \times E \to 2^E$ be an m-accretive mapping with respect to the first argument, and $N(\cdot,\cdot) : E \times E \to E$ be a nonlinear mapping, now let us to consider the following problem of finding $u \in E, w \in T(u), y \in V(u)$ such that

$$0 \in N(w, y) + A(g(u), u). \tag{2.1}$$

Remark 2.2. For a suitable choice of the mappings N, T, V, A, g, and the space E, we can obtain a number of known and new classes of variational inequalities, variational inclusions and the corresponding optimization problems. Furthermore, these variational inclusions provide us with a general and unified framework for studying a wide class of problems arising in mathematics, physics and engineering science [1-4].

Definition 2.3. [3] Let $A : E \to CB(E)$ be a set-valued mapping and $H(\cdot,\cdot)$ be a hausdorff metric on $CB(E)$, T is said to be ξ-Lipschitz continuous if, for any $x, y \in E$,

$$H(Tx, Ty) \leq \xi\|x - y\|,$$

where $\xi > 0$ is a constant.

Definition 2.4. [17] The dynamical system is said to converge to the solution set K^* of (2.1), if, irrespective of the initial point, the trajectory of the dynamical system satisfies

$$\lim_{t\to\infty} dist(u(t), K^*) = 0,$$

where

$$dist(u, K^*) = inf_{v\in K^*}\|u - v\|.$$

Definition 2.5. [17] The dynamical system is said to be globally exponentially stable with degree η at u^*, if , irrespective of the initial point, the trajectory of the system satisfies

$$\|u(t) - u^*\| \leq \mu_1\|u(t_0) - u^*\|exp(-\eta(t - t_0)), \forall t \geq t_0,$$

where μ_1 and η are positive constants independent of the initial point. It is clear that globally exponentially stability is necessarily globally asymptotically stable and the dynamical system converges arbitrarily fast.

Lemma 2.1. [2,7] Let E is a real Banach space and $J : E \to 2^{E^*}$is a normalized duality mapping, then for any $x, y \in E$,

$$\|x + y\|^2 \leq \|x\|^2 + 2 < y, j(x + y) >, \forall j(x + y) \in J(x + y).$$

Lemma 2.2. [17] Let $\hat{u}$ and $\hat{v}$ be real-valued nonnegative continuous functions with domain $\{t : t \leq t_0\}$ and let $\alpha(t) = \alpha_0(|t - t_0|)$, where α_0 is a monotone increasing function. If, for $t \geq t_0$,

$$\hat{u} \leq \alpha(t) + \int_{t_0}^{t} \hat{u}(s)\hat{v}(s)ds,$$

then $\hat{u}(s) \leq \alpha(t)exp\{\int_{t_0}^{t} \hat{v}(s)ds\}$.

Assumption 2.1. For all $u, v, w \in E$, the resolvent operator $R_{A(u)}$ satisfying

$$\|R_{A(u)}w - R_{A(v)}w\| \leq \nu\|u - v\|,$$

where $\nu > 0$ is a constant.

§3. Main result

In this section, by using the technique of resolvent operator we establish the equivalence between the variational inclusion (2.1) and the fixed point problem. This equivalence is used to suggest a class of resolvent dynamical systems for the quasi variational inclusions (2.1). For this purpose, we need the following well-known result.

Lemma 3.1. The following statements are equivalent:

(1) (u, w, y), where $u \in E, w \in T(u), y \in V(u)$, is the solution of generalized set-valued quasi-variational inclusion (2.1);

(2) (u, w, y) is the solution of resolvent equation

$$g(u) = R_{A(u)}[g(u) - \rho N(w, y)], \tag{3.1}$$

where $\rho > 0$ is a constant, and $R_{A(u)}$ is a resolvent operator;

(3) (z, u, w, y) is the solution of implicit resolvent equation

$$z = g(u) - \rho N(w, y), g(u) = R_{A(u)}(z). \tag{3.2}$$

Proof. If we use the technique given in Noor [1],we can prove this lemma immediately.

From Lemma 3.1, we conclude that the set-valued quasi variational inclusion (2.1) is equivalent to the fixed point problem (3.1). We use this equivalence to suggest a class of resolvent dynamical system associated with quasi variational inclusion (2.1) as

$$\frac{du}{dt} = \lambda\{R_A(u)[g(u) - \rho N(w, y)] - g(u)\}, u(t_0) = u_0 \in E, \tag{3.3}$$

where λ is a parameter. The system of type (3.3) is called the resolvent dynamical system associated with quasi variational inclusion (2.1). Here the right-hand side is related to the resolvent operator and is discontinuous on the boundary. It is clear from the definition that the solution to (3.3) always stays in the constraint set. This implies that the qualitative results such that the existence, uniqueness and continuous dependence of the solution to (3.3) can be studied.

We now show that the trajectory of the solution of the resolvent dynamical system (3.3) converges to the unique solution of quasi variational inclusion (2.1) by using the technique of Xia and Wang [13,14] as extended by Noor [16,17].

Theorem 3.1. Let E be a real Banach space, $T, V : E \to 2^E$ be set-valued mappings, $g : E \to E$ be a single valued mapping, $A(\cdot,\cdot) : E \times E \to 2^E$ be an m-accretive with respect to the first argument, $N(\cdot,\cdot) : E \times E \to E$ be nonlinear mappings satisfying the following conditions :

(i) g is Lipschitz continuous with constants δ and k-strongly accretive, where k is a constant;

(ii) $A(\cdot,\cdot) : E \times E \to 2^E$ is m-accretive with respect to the first argument;

(iii) $T, V : E \to CB(E)$ are Lipschitz continuous with respect to constants μ, ξ;

(iv) for a given $y \in E$, the mapping $x \to N(x, y)$ is β -Lipschitz continuous with respect to the set-valued mapping T;

(v) for a given $x \in E$, the mapping $y \to N(x, y)$ is γ_1 -Lipschitz continuous with respect to the set-valued mapping B;

if the Assumption 2.1 holds, then, for each $u_0 \in E, w_0 \in Tu_0, y_0 \in Vu_0$, there exists a unique continuous solution $u(t)$ of dynamical system (3.3) with $u(t_0) = u_0$ over $[t_0, \infty)$.

Proof. Let

$$G(u) = \lambda\{R_{A(u)}[g(u) - \rho N(w, y)] - g(u)\},$$

where $\lambda > 0$ is a constant,$w \in Tu, y \in Vu$. For all $u, v \in E, w \in Tu, y \in Vu, w' \in Tv, y' \in Vv$, and using conditions (i),(iii),(iv),(v) and Assumption 2.1, we have

$$\begin{aligned}
\|G(u) - G(v)\| &\le \lambda\{\|R_{A(u)}[g(u) - \rho N(w, y)] - R_{A(v)}[g(v) - \rho N(w', y')]\| \\
&\quad + \|g(u) - g(v)\|\} \\
&\le \lambda\|g(u) - g(v)\| \\
&\quad + \lambda\|R_{A(u)}[g(u) - \rho N(w, y)] - R_{A(u)}[g(v) - \rho N(w', y')]\| \\
&\quad + \lambda\|R_{A(u)}[g(v) - \rho N(w', y')] - R_{A(v)}[g(v) - \rho N(w', y')]\| \\
&\le 2\lambda\|g(u) - g(v)\| + \nu\lambda\|u - v\| + \rho\lambda\|N(w, y) - N(w', y')\| \\
&\le \lambda\{(2\delta + \nu)\|u - v\| + \rho(\mu\beta + \gamma\xi)\|u - v\|\} \\
&= \lambda\{2\delta + \nu + \rho(\mu\beta + \gamma\xi)\}\|u - v\|.
\end{aligned}$$

This implies that operator $G(u)$ is Lipschitz continuous in E. So, for each $u_0 \in E$, there exists a unique and continuous solution $u(t)$ of the dynamical system (3.3), defined in an interval $t_0 \le t \le T_1$ with the initial condition $u(t_0) = u_0$. Let $[t_0, T_1]$ be its maximal interval of existence. Then we have to show that $T_1 = \infty$. Consider

$$\begin{aligned}
\|G(u)\| &= \lambda\|R_{A(u)}[g(u) - \rho N(w, y)] - g(u)\| \\
&\le \lambda\{\|R_{A(u)}[g(u) - \rho N(w, y)] - R_{A(u)}[g(u)]\| \\
&\quad + \|R_{A(u)}[g(u)] - R_{A(u^*)}[g(u)]\| \\
&\quad + \|R_{A(u^*)}[g(u)] - R_{A(u^*)}[g(u^*)]\| + \|R_{A(u^*)}[g(u^*)] - g(u)\|\} \\
&\le \lambda\rho\|N(w, y)\| + \lambda\nu\|u - u^*\| + \delta\|u - u^*\| + \lambda\|R_{A(u^*)}[g(u^*)] - g(0)\| + \lambda\delta\|u\| \\
&= \lambda(2\delta + \rho(\mu\beta + \gamma\xi) + \nu)\|u\| + \lambda\nu\|u^*\| + \lambda\|R_{A(u^*)}[g(u^*)] - g(0)\|,
\end{aligned}$$

for any $u \in E$, then

$$\|u(t)\| \leq \|u_0\| + \int_{t_0}^{t} \|Gu(s)\|ds \leq (\|u_0\| + k_1(t-t_0)) + k_2 \int_{t_0}^{t} \|u(s)\|ds,$$

where $k_1 = \nu\lambda\|u^*\| + \lambda\|R_{A(u^*)}[g(u^*)] - g(0)\|$ and $k_2 = \lambda(2\delta + \rho(\mu\beta + \gamma\xi) + \nu)$. Hence by invoking Lemma 2.2, we have

$$\|u(t)\| \leq \|u_0\| + k_1(t-t_0)\}e^{k_2(t-t_0)}, t \in [t_0, T_1).$$

This show that the solution is bounded on $[t_0, T_1)$. So $T_1 = \infty$.

Theorem 3.2. Let the operators N, T, V, A, g be as Theorem 3.1, if Assumption 2.1 holds, then the resolvent dynamical system (3.3) converges globally exponentially to the unique solution of the quasi variational inclusion (2.1).

Proof. Since the operator N, T, V, A, g be as Theorem 3.1, it follows from Theorem 3.1 that the resolvent dynamical system (3.3) has a unique solution $u(t)$ over $[t_0, T_1)$ for any fixed $u_0 \in E$. Let $u(t) = u(t, t_0 : u_0)$ be the initial value problem (3.3). For a given $u^* \in E, w^* \in Tu^*, y^* \in Vu^*$, satisfying (2.1), consider the following Lyapunov function:

$$L(u) = \lambda\|u - u^*\|, u \in E. \tag{3.4}$$

From (3.3) and (3.4), we have

$$\begin{aligned}
\frac{dL}{dt} &= 2\lambda < j(u(t) - u^*), R_{A(u(t))}[g(u(t)) - \rho N(w(t), y(t))] - g(u(t)) > \\
&= -2\lambda < j(u(t) - u^*), g(u(t)) - g(u^*) > \\
&+2\lambda < j(u(t) - u^*), R_{A(u(t))}[g(u(t)) - \rho N(w(t), y(t))] - g(u^*) > \\
&\leq -2\lambda k\|u(t) - u^*\|^2 \\
&+2\lambda < j(u(t) - u^*), R_{A(u(t))}[g(u(t)) - \rho N(w(t), y(t))] - g(u^*) >,
\end{aligned} \tag{3.5}$$

where $u^*, w^* \in Tu^*, y^* \in Vu^*$ is the solution of the quasi variational inclusion (2.1), that is ,

$$g(u^*) = R_{A(u^*)}[g(u^*) - \rho N(w^*, y^*)].$$

Now using Assumption 2.1 and conditions (i), (iii),(iv)and (v), we have

$$\begin{aligned}
&\|R_{A(u)}[g(u) - \rho N(w, y)] - R_{A(u^*)}[g(u^*) - \rho N(w^*, y^*)]\| \\
&\leq \|R_{A(u)}[g(u) - \rho N(w, y)] - R_{A(u^*)}[g(u) - \rho N(w, y)]\| \\
&+\|R_{A(u^*)}[g(u) - \rho N(w, y)] - R_{A(u^*)}[g(u^*) - \rho N(w^*, y^*)]\| \\
&\leq \nu\|u - u^*\| + \|g(u) - g(u^*) - \rho(N(w, y) - N(w^*, y^*))\| \\
&\leq \nu\|u - u^*\| + \delta\|u - u^*\| + \rho(\mu\beta + \gamma\xi)\|u - u^*\| \\
&\leq (\nu + \delta + \rho(\mu\beta + \gamma\xi))\|u - u^*\|.
\end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we have $\frac{d}{dt}\|u(t) - u^*\| \leq 2\alpha\lambda\|u(t) - u^*\|$, where $\alpha = \nu + \delta + \rho(\mu\beta + \gamma\xi) - k$. Thus, for $\lambda = -\lambda_1$, where λ_1 is a positive constant, we have

$$\|u(t) - u^*\| \leq \|u(t_0) - u^*\|e^{-\alpha\lambda_1(t-t_0)},$$

which shows that the trajectory of the solution of resolvent dynamical system (3.3) converges globally exponentially to the unique solution of the quasi variational inclusion (2.1).

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A note on a theorem of Calderón [1]

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Abstract Let $a(n)$ denote the number of non-isomorphic finite abelian groups with n elements and $\omega(n) = (a * a * a)(n)$. In this paper we shall study the mean value $\sum_{n \leq x} \omega^2(n)$.

Keywords Divisor problem, finite abelian groups

§1. Introduction

Let $a(n)$ denote the number of non-isomorphic finite abelian groups with n elements. This is a well-known multiplicative function, such that $a(p^\alpha) = P(\alpha)$, where $P(\alpha)$ is the unrestricted partition function. Define $\omega(n) = (a * a * a)(n)$. H. Menzer [3] proved that

$$W(x) = \sum_{n \leq x} \omega(n) = xP_2(\log x) + x^{\frac{1}{2}} Q_2(\log x) + O(x^{\frac{76}{153}} \log^6 x), \tag{1}$$

where $P_2(u)$, $Q_2(u)$ are polynomials in u of degree 2 with explicit coefficients. W. Zhai [5] improved the error term in (1.1) to $O(x^{\frac{53}{116}+\varepsilon})$. J. Wu [4] improved Zhai's exponent $\frac{53}{116}$ to $\frac{4}{9}$.

C.Calderón[1] studied the sum $\sum_{n \leqq x} \omega^2(n)$ and proved that

$$\sum_{n \leqq x} \omega^2(n) = xP_8(\log x) + O(x^{\frac{2}{3}+\epsilon}), \tag{2}$$

where $P_8(u)$ is a polynomial in u of degree 8.

In this short note we first prove the following

Theorem 1. We have the asymptotic formula

$$\sum_{n \leqq x} \omega^2(n) = xP_8(\log x) + O(x^{\frac{35}{54}+\epsilon}). \tag{3}$$

Numerically , we have $2/3 = 0.666\cdots$, $35/54 = 0.648\cdots$.

The mean value of $\omega^2(n)$ is closely related to the general divisor problem. Let $k \geq 2$ be a fixed integer, $d_k(n)$ denote the number of ways n can be written as a product of k factors. Define $D_k(x) := \sum_{n \leq x} d_k(n)$. The study of $D_k(x)$ is an important problem in the analytic number theory.

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When $k = 2$, it is the famous Dirichlet divisor problem. Usually $D_k(x)$ has an asymptotic formula of the type

$$D_k(x) = xQ_{k-1}(\log x) + O(x^{\delta_k+\varepsilon}) \tag{4}$$

for some $\delta_k < 1$, where $Q_{k-1}(u)$ is a polynomial in u of degree $k-1$ and ε denotes a sufficiently small positive constant. It is believed that the following conjecture

$$\delta_k = (k-1)/2k \tag{5}$$

holds for any $k \geq 2$. For more details of $D_k(x)$, see Ivić[2].

Theorem 2. If the conjecture (5) is true, then we have the asymptotic formula

$$\sum_{n\leq x} \omega^2(n) = xP_8(\log x) + x^{\frac{1}{2}}P_{35}(\log x) + O(x^{\frac{508}{1053}+\epsilon}), \tag{6}$$

where $P_{35}(u)$ is a polynomial of degree 35 in u.

Numerically , we have $508/1053 = 0.482\cdots < 1/2$.

§2. Proof of Theorems

In order to prove our theorems, we need the following Lemma, whose proof is contained in Chapter 14 of Ivić[2].

Main Lemma. Suppose arithmetic functions $f(n)$ and $g(n)$ satisfy

$$\sum_{m\leq x} f(m) = \sum_{j=1}^{J} x^{\alpha_j}V_j(\log x) + O(x^{\alpha}), \sum_{l\leq x}|g(l)| = O(x^{\beta}), \tag{7}$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_J > \alpha > \beta > 0$, $V_j(u)(j = 1, \cdots, J)$ are polynomials in u. Let $h(n) = \sum\limits_{n=ml} f(m)g(l)$, then

$$\sum_{n\leq x} h(n) = \sum_{j=1}^{J} x^{\alpha_j}U_j(\log x) + O(x^{\alpha}),$$

where $U_j(u)(j = 1, \cdots, J)$ are polynomials in u.

Proof of Theorem 1. By the Euler product we have for $\Re s > 1$ that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\omega^2(n)}{n^s} &= \prod_p(1+\sum_{k=1}^{\infty}\frac{\omega^2(n)}{p^{ks}}) \\ &= \zeta^9(s)\prod_p(1-p^{-s})^9\{1+9p^{-s}+81p^{-2s}\cdots\} \\ &= \zeta^9(s)\zeta^{36}(2s)G(s),\end{aligned} \tag{8}$$

where $G(s)$ can be represented as a Dirichlet series, which converges absolutely for $\sigma > \dfrac{1}{3}$. From Theorem 13.2 of Ivić[2] we have that

$$\sum_{n\leq x} d_9(n) = xQ_8(\log x) + O(x^{\frac{35}{54}+\epsilon}),$$

where $Q_8(u)$ is a polynomial in u of degree 8. Write

$$B(s) = \zeta^{36}(2s)G(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s},$$

it is easy to see that

$$\sum_{n \le x} |b(n)| \ll x^{\frac{1}{2}+\epsilon}.$$

Now Theorem 1 follows from the Main Lemma by taking $f(n) = d_9(n), g(n) = b(n)$.

Proof of Theorem 2. For any $k \ge 2$, define $\Delta_k(x) := D_k(x) - xQ_{k-1}(\log x)$. If the conjecture (5) is true, we then have

$$\Delta_9(x) = O(x^{\frac{4}{9}+\epsilon}), \tag{9}$$

$$\Delta_{36}(x) = O(x^{\frac{35}{72}+\epsilon}). \tag{10}$$

Let $f(n) = \sum_{n=ml^2} d_9(m)d_{36}(l)$. By the hyperbolic summation method we can write

$$\begin{aligned} \sum_{n \le x} f(n) &= \sum_{nm^2 \le x} d_9(n)d_{36}(m) \\ &= \sum_{n \le y} d_9(n) \sum_{m \le (\frac{x}{n})^{\frac{1}{2}}} d_{36}(m) + \sum_{m \le z} d_{36}(m) \sum_{n \le \frac{x}{m^2}} d_9(n) \\ &\quad - \sum_{n \le y} d_9(n) \sum_{m \le z} d_{36}(m) \\ &= S_1 + S_2 - S_3, \end{aligned} \tag{11}$$

say, where y, z are parameters for which $yz^2 = x, 1 < y < x$. We first compute S_1. By (4) and (10) we get

$$\begin{aligned} S_1 &= \sum_{n \le y} d_9(n) \sum_{m \le (\frac{x}{n})^{\frac{1}{2}}} d_{36}(m) \\ &= \sum_{n \le y} d_9(n)\{x^{\frac{1}{2}} n^{-\frac{1}{2}} Q_{35}(\log(\frac{x}{n})^{\frac{1}{2}}) + O((\frac{x}{n})^{\frac{35}{144}+\epsilon})\} \\ &= \sum_{n \le y} d_9(n) x^{\frac{1}{2}} n^{-\frac{1}{2}} Q_{35}(\log(\frac{x}{n})^{\frac{1}{2}}) + O(\sum_{n \le y} |d_9(n)|(\frac{x}{n})^{\frac{35}{144}+\epsilon}) \\ &= S_{11} + O(x^{\frac{35}{144}+\epsilon} y^{\frac{109}{144}+\varepsilon}), \end{aligned} \tag{12}$$

where

$$S_{11} = \sum_{n \le y} d_9(n) x^{\frac{1}{2}} n^{-\frac{1}{2}} Q_{35}(\log(\frac{x}{n})^{\frac{1}{2}}).$$

Suppose $Q_{35}(u)=\sum_{j=0}^{35}c_ju^j$. Then

$$\begin{aligned}
S_{11} &= x^{\frac{1}{2}}\sum_{n\le y}d_9(n)n^{-\frac{1}{2}}\sum_{j=0}^{35}c_j(\log(\frac{x}{n})^{\frac{1}{2}})^j\\
&= x^{\frac{1}{2}}\sum_{n\le y}d_9(n)n^{-\frac{1}{2}}\sum_{j=0}^{35}c_j(\frac{1}{2})^j(\log x-\log n)^j\\
&= x^{\frac{1}{2}}\sum_{n\le y}d_9(n)n^{-\frac{1}{2}}\sum_{j=0}^{35}c_j(\frac{1}{2})^j\sum_{l=0}^{j}\binom{j}{l}(\log x)^{j-l}(-1)^l(\log n)^l \qquad (13)\\
&= x^{\frac{1}{2}}\sum_{j=0}^{35}c_j(\frac{1}{2})^j\sum_{l=0}^{j}\binom{j}{l}(\log x)^{j-l}(-1)^l\sum_{n\le y}d_9(n)n^{-\frac{1}{2}}(\log n)^l\\
&= x^{\frac{1}{2}}\sum_{j=0}^{35}c_j(\frac{1}{2})^j\sum_{l=0}^{j}\binom{j}{l}(\log x)^{j-l}(-1)^lS_{12,l},
\end{aligned}$$

where

$$S_{12,l}=\sum_{n\le y}d_9(n)n^{-\frac{1}{2}}(\log n)^l.$$

Now compute the $S_{12,l}$. We have

$$\begin{aligned}
S_{12,l} &= \textstyle\int_{1^-}^{y}t^{-\frac{1}{2}}(\log t)^l dD_9(t)\\
&= \textstyle\int_{1^-}^{y}t^{-\frac{1}{2}}(\log t)^l dtQ_8(\log t)+\int_{1^-}^{y}t^{-\frac{1}{2}}(\log t)^l d\Delta_9(t) \qquad (14)\\
&= \textstyle\int_{1^-}^{y}t^{-\frac{1}{2}}(\log t)^l\left(Q_8(\log t)+Q_8'(\log t)\right)dt+\int_{1^-}^{y}t^{-\frac{1}{2}}(\log t)^l d\Delta_9(t).
\end{aligned}$$

Suppose $Q_8(u)=\sum_{s=0}^{8}d_su^s$, then

$$(\log t)^l\left(Q_8(\log t)+Q_8'(\log t)\right)=\sum_{s=0}^{8}f_s(\log t)^{s+l},$$

where $f_s=d_s+(s+1)d_{s+1}$ for $s=0,1,\cdots,7$, and $f_8=d_8$. Thus we have

$$\begin{aligned}
&\int_{1^-}^{y}t^{-\frac{1}{2}}(\log t)^l\left(Q_8(\log t)+Q_8'(\log t)\right)dt \qquad (15)\\
&=\sum_{s=0}^{8}f_s\int_{1^-}^{y}t^{-1/2}(\log t)^{s+l}dt\\
&=\sum_{s=0}^{8}f_s\sum_{i=0}^{s+l}u_iy^{1/2}\log^i y \qquad (16)
\end{aligned}$$

with computable constants $u_i(i\ge 0)$.

By partial integration and (9) it is easy to show that

$$\int_{1^-}^{y}t^{-\frac{1}{2}}(\log t)^l d\Delta_9(t)=g_l+O(y^{-1/18+\varepsilon}), \qquad (17)$$

where g_l is a computable constant.

From (12)-(17) we get

$$\begin{aligned} S_1 &= x^{1/2}y^{1/2}\sum_{j=0}^{35} c_j(\frac{1}{2})^j \sum_{l=0}^{j}\binom{j}{l}(\log x)^{j-l}(-1)^l \sum_{s=0}^{8} f_s \sum_{i=0}^{s+l} u_i \log^i y \\ &+x^{1/2}\sum_{j=0}^{35} c_j(\frac{1}{2})^j \sum_{l=0}^{j} g_l\binom{j}{l}(\log x)^{j-l}(-1)^l \\ &+O(x^{1/2+\varepsilon}y^{-1/18}+x^{\frac{35}{144}+\varepsilon}y^{\frac{109}{144}}). \end{aligned} \tag{18}$$

By similar arguments we can show that

$$\begin{aligned} S_2 &= x\sum_{s=0}^{8} d_s \sum_{l=0}^{s}\binom{s}{l}2^l\zeta^{(l)}(2)(\log x)^{s-l} \\ &+xz^{-1}\sum_{s=0}^{8} d_s \sum_{l=0}^{s}\binom{s}{l}(-2)^l(\log x)^{s-l}\sum_{i=0}^{36} v_i \log^i z \\ &+O(x^{1/2+\varepsilon}y^{-1/18}+x^{\frac{35}{144}+\varepsilon}y^{\frac{109}{144}}). \end{aligned} \tag{19}$$

Inserting (18) and (19) into (11), choosing $y = x^{\frac{37}{117}}$ and then by some calculations we can get that

$$\sum_{n\le x} f(n) = xV_8(\log x) + x^{1/2}V_{25}(\log x) + O(x^{508/1053+\varepsilon}), \tag{20}$$

where $V_8(u)$ is a polynomial in u of degree 8 and $V_{35}(u)$ is a polynomial in u of degree 35, respectively.

Write $G(s) = \sum_{n=1}^{\infty} g(n)n^{-s}$. Then this infinite series is absolutely convergent for $\Re s > 1/3$. Thus we have

$$\sum_{n\le x} |g(n)| \ll x^{1/3+\varepsilon}. \tag{21}$$

Now Theorem 2 follows from (20) and (21) with the help of the Main Lemma.

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On the Smarandache sequences

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Abstract In this paper, we use the elementary method to study the convergence of the Smarandache alternate consecutive, reverse Fibonacci sequence and Smarandache multiple sequence.

Keywords Convergence, Smarandache sequences, elementary method.

§1. Introduction and results

For any positive integer n, the Smarandache alternate consecutive and reverse Fibonacci sequence $a(n)$ is defined as follows: $a(1) = 1$, $a(2) = 11$, $a(3) = 112$, $a(4) = 3211$, $a(5) = 11235$, $a(6) = 853211$, $a(7) = 11235813$, $a(8) = 2113853211$, $a(9) = 112358132134, \cdots$. The Smarandache multiple sequence $b(n)$ is defined as: $b(1) = 1$, $b(2) = 24$, $b(3) = 369$, $b(4) = 481216$, $b(5) = 510152025$, $b(6) = 61218243036$, $b(7) = 7142128354249$, $b(8) = 816243240485664$, $b(9) = 91827364554637281$, $\cdots$.

These two sequences were both proposed by professor F.Smarandache in reference [1], where he asked us to study the properties of these two sequences.

About these problems, it seems that none had studied it, at least we have not seen any related papers before. However, in reference [1] (See chapter III, problem 6 and problem 21), professor Felice Russo asked us to study the convergence of

$$\frac{a(n)}{a(n+1)}, \quad \lim_{k\to\infty}\sum_{n=1}^{k}\frac{b(n)}{b(n+1)}$$

and other properties.

The main purpose of this paper is using the elementary method to study these problems, and give some interesting conclusions. That is, we shall prove the following:

Theorem 1. For Smarandache alternate consecutive and reverse Fibonacci sequence $a(n)$, we have $\lim\limits_{n\to\infty}\dfrac{a(n)}{a(n+1)} = 0$.

Theorem 2. For Smarandache multiple sequence $b(n)$, the series $\sum\limits_{n=1}^{\infty}\dfrac{b(n)}{b(n+1)}$ is convergent.

§2. Proof of the theorems

In this section, we shall using elementary method to prove our Theorems. First we prove Theorem 1. If n is an odd number, then from the definition of $a(n)$ we know that $a(n)$ can be written in the form:

$$a(n) = F(1)F(2)\cdots F(n) \quad \text{and} \quad a(n+1) = F(n+1)F(n)\cdots F(1),$$

where $F(n)$ be the Fibonacci sequence.

Let α_n denote the number of the digits of $F(n)$ in base 10, then

$$a(n) = F(n) + F(n-1)\cdot 10^{\alpha_n} + F(n-2)\cdot 10^{\alpha_n+\alpha_{n-1}} + \cdots + F(1)\cdot 10^{\alpha_n+\alpha_{n-1}+\cdots+\alpha_2}$$

and

$$a(n+1) = F(1) + F(2)\cdot 10^{\alpha_1} + F(3)\cdot 10^{\alpha_1+\alpha_2} + \cdots + F(n+1)\cdot 10^{\alpha_1+\alpha_2+\cdots+\alpha_n}.$$

So

$$\begin{aligned}\frac{a(n)}{a(n+1)} &\leq \frac{F(n) + F(n-1)\cdot 10^{\alpha_n} + \cdots + F(1)\cdot 10^{\alpha_n+\alpha_{n-1}+\cdots+\alpha_2}}{F(n+1)\cdot 10^{\alpha_1+\alpha_2+\cdots\alpha_n}} \\ &= \frac{\dfrac{F(n)}{10^{\alpha_1+\cdots+\alpha_n}} + \dfrac{F(n-1)}{10^{\alpha_1+\cdots+\alpha_{n-1}}} + \cdots + \dfrac{F(1)}{10^{\alpha_1}}}{F(n+1)}.\end{aligned}$$

For $1 \leq k \leq n$, since the number of the digits of $F(k)$ in base 10 is α_k, we can suppose $F(k) = a_1\cdot 10^{\alpha_k-1} + a_2\cdot 10^{\alpha_k-2} + \cdots + a_{\alpha_k}$, $0 \leq a_i \leq 9$ and $1 \leq i \leq \alpha_k$. Therefore, we have

$$\begin{aligned}\frac{F(k)}{10^{\alpha_1+\alpha_2+\cdots+\alpha_k}} &= \frac{a_1\cdot 10^{\alpha_k-1} + a_2\cdot 10^{\alpha_k-2} + \cdots + a_{\alpha_k}}{10^{\alpha_1+\alpha_2+\cdots+\alpha_k}} \\ &\leq \frac{9\cdot(1 + 10^{-1} + 10^{-2} + \cdots + 10^{1-\alpha_k})}{10^{\alpha_1+\alpha_2+\cdots+\alpha_{k-1}+1}} \\ &= \frac{10\cdot(1 - 10^{-\alpha_k})}{10^{\alpha_1+\alpha_2+\cdots+\alpha_{k-1}+1}} \\ &\leq \frac{1}{10^{\alpha_1+\alpha_2+\cdots+\alpha_{k-1}}} \leq \frac{1}{10^{k-1}}.\end{aligned}$$

Thus,

$$0 \leq \frac{a(n)}{a(n+1)} \leq \frac{1 + 10^{-1} + 10^{-2} + \cdots + 10^{1-n}}{F(n+1)} \leq \frac{10}{9F(n+1)} \to 0, \text{ as } n \to \infty.$$

That is to say,

$$\lim_{n\to\infty} \frac{a(n)}{a(n+1)} = 0.$$

If n be an even number, then we also have

$$a(n) = F(1) + F(2)\cdot 10^{\alpha_1} + F(3)\cdot 10^{\alpha_1+\alpha_2} + \cdots + F(n)\cdot 10^{\alpha_1+\alpha_2+\cdots+\alpha_{n-1}}$$

and

$$a(n+1) = F(n+1) + F(n)\cdot 10^{\alpha_{n+1}} + F(n-1)\cdot 10^{\alpha_{n+1}+\alpha_n} + \cdots + F(1)\cdot 10^{\alpha_{n+1}+\alpha_n+\cdots+\alpha_2}.$$

We can use the similar methods to evaluate the value of $\dfrac{a(n)}{a(n+1)}$. And

$$\frac{a(n)}{a(n+1)} \leq \frac{F(1)+F(2)\cdot 10^{\alpha_1}+F(3)\cdot 10^{\alpha_1+\alpha_2}+\cdots+F(n)\cdot 10^{\alpha_1+\alpha_2+\cdots+\alpha_{n-1}}}{10^{\alpha_2+\alpha_3\cdots+\alpha_{n+1}}}.$$

For every $1 \leq k \leq n$, similarly, let $F(k) = a_1 \cdot 10^{\alpha_k-1} + a_2 \cdot 10^{\alpha_k-2} + \cdots + a_{\alpha_k}$, we have

$$\begin{aligned}
\frac{F(k)\cdot 10^{\alpha_1+\alpha_2+\cdots+\alpha_{k-1}}}{10^{\alpha_2+\alpha_3+\cdots+\alpha_{n+1}}} &= \frac{(a_1\cdot 10^{\alpha_k-1}+a_2\cdot 10^{\alpha_k-2}+\cdots+a_{\alpha_k})\cdot 10^{\alpha_1+\alpha_2+\cdots+\alpha_{k-1}}}{10^{\alpha_2+\alpha_3+\cdots+\alpha_{n+1}}} \\
&\leq \frac{9(1+10^{-1}+10^{-2}+\cdots+10^{1-\alpha_k})}{10^{\alpha_{k+1}+\alpha_{k+2}+\cdots+\alpha_{n+1}}} \\
&\leq \frac{1}{10^{\alpha_{k+1}+\alpha_{k+2}+\cdots+\alpha_{n+1}-1}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 \leq \frac{a(n)}{a(n+1)} &\leq \frac{1}{10^{\alpha_2+\alpha_3+\cdots+\alpha_{n+1}-1}}+\frac{1}{10^{\alpha_3+\alpha_4+\cdots+\alpha_{n+1}-1}}+\cdots+\frac{1}{10^{\alpha_{n+1}-1}} \\
&\leq \frac{1}{10^{\alpha_{n+1}-1}}(1+10^{-1}+10^{-2}+\cdots+10^{1-n)}) \\
&\leq \frac{100}{9\cdot 10^{\alpha_{n+1}}} \to 0, \quad \text{as } n\to\infty.
\end{aligned}$$

So $\lim\limits_{n\to\infty}\dfrac{a(n)}{a(n+1)} = 0$. This proves Theorem 1.

Now we prove Theorem 2. For the sequence $b(n) = n(2n)(3n)\cdots(n\cdot n)$, let $\gamma(n)$ denote the number of the digits of n^2 in base 10, then by observation, we can obtain $\gamma(2)=\gamma(3)=1$, $\gamma(4)=2, \gamma(10)=3, \gamma(40)=4, \gamma(100)=5, \gamma(400)=6, \gamma(1000)=7, \cdots$. When n ranges from $4\cdot 10^{\alpha}$ to $10^{\alpha+1}$, $\gamma(n)$ increases. That is to say, $\gamma(4\cdot 10^{\alpha}) = 2\alpha+2$, and $\gamma(10^{\alpha+1}) = 2\alpha+3$, where $\alpha = 0,1,2,\cdots$.

For every positive integer n, it is obvious that $k\cdot n \leq k\cdot(n+1)$, where $1\leq k\leq n$. So we can evaluate $\dfrac{b(n)}{b(n+1)}$ as

$$\begin{aligned}
\frac{b(n)}{b(n+1)} &= \frac{n(2n)(3n)\cdots(n\cdot n)}{(n+1)(2(n+1))\cdots(n\cdot(n+1))((n+1)\cdot(n+1))} \\
&\leq \frac{b(n)}{b(n)\cdot 10^{\gamma(n+1)}} = \frac{1}{10^{\gamma(n+1)}},
\end{aligned}$$

thus,

$$\sum_{n=1}^{\infty}\frac{b(n)}{b(n+1)} \leq \sum_{n=1}^{\infty}\frac{1}{10^{\gamma(n+1)}}.$$

If $4\cdot 10^{\alpha} \leq n \leq 10^{\alpha+1}-1$, then $2\alpha+2 \leq \gamma(n) \leq 2\alpha+3$, where $\alpha = 0,1,2,3,\cdots$. In addition, if $10^{\alpha+1} \leq n \leq 4\cdot 10^{\alpha+1}-1$, then $2\alpha+3 \leq \gamma(n) \leq 2\alpha+4$, where $\alpha = 0,1,2,3,\cdots$.

Therefore, we can get

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{10^{\gamma(n+1)}} &= \frac{2}{10}+\frac{1}{10^{\gamma(4)}}+\frac{1}{10^{\gamma(5)}}+\cdots+\frac{1}{10^{\gamma(9)}}+\frac{1}{10^{\gamma(10)}}+\cdots+\frac{1}{10^{\gamma(39)}} \\
&\quad +\cdots+\frac{1}{10^{\gamma(4\cdot 10^{\alpha})}}+\cdots+\frac{1}{10^{\gamma(10^{\alpha+1}-1)}} \\
&\quad +\frac{1}{10^{\gamma(10^{\alpha+1})}}+\cdots+\frac{1}{10^{\gamma(4\cdot 10^{\alpha+1}-1)}}+\cdots \\
&= \frac{2}{10}+\sum_{\alpha=0}^{\infty}\left(\frac{1}{10^{\gamma(4\cdot 10^{\alpha})}}+\cdots+\frac{1}{10^{\gamma(10^{\alpha+1}-1)}}\right) \\
&\quad +\sum_{\alpha=0}^{\infty}\left(\frac{1}{10^{\gamma(10^{\alpha+1})}}+\cdots+\frac{1}{10^{\gamma(4\cdot 10^{\alpha+1}-1)}}\right) \\
&\leq \frac{1}{5}+\sum_{\alpha=0}^{\infty}\frac{6\cdot 10^{\alpha}}{10^{2\alpha+2}}+\sum_{\alpha=0}^{\infty}\frac{3\cdot 10^{\alpha+1}}{10^{2\alpha+3}} \\
&= \frac{1}{5}+\sum_{\alpha=0}^{\infty}\frac{6}{10^{\alpha+2}}+\sum_{\alpha=0}^{\infty}\frac{3}{10^{\alpha+2}}=\frac{3}{10}.
\end{aligned}$$

So the series $\sum_{n=1}^{\infty}\frac{b(n)}{b(n+1)}$ is convergent. This completes the proof of Theorem 2.

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The study of $\sigma-$index on $Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs

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Abstract A $Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs be a graph abtained from P_k whose every vertex $v_i(i = 1, 2, \cdots, k)$ attached one cycle $C_i(i = 1, 2, \cdots, k)$. In this paper, we determine the lower and the higher bound for the Merrifield−simmons index in $Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs in terms of the order k, and characterize the $Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs with the smallest and the largest Merrifield-simmons index.

Keywords $Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs, σ-index or Merrifield-Simmons index.

§1. Introduction

Let $G = (V, E)$ be a simple connected graph with the vertex set $V(G)$ and the edge set $E(G)$. For any $v \in V$, $N_G(v) = \{u \mid uv \in E(G)\}$ denotes the neighbors of v, and $d_G(v) =| N_G(v) |$ is the degree of v in G; $N_G[v] = \{v\} \cup N_G(v)$. A leaf is a vertex of degree one and a stem is a vertex adjacent to at least one leaf. Let $E^{'} \subseteq E(G)$, we denote by $G - E^{'}$ the subgraph of G obtained by deleting the edges of $E^{'}$. $W \subseteq V(G)$, $G - W$ denotes the subgraph of G obtained by deleting the vertices of W and the edges incident with them. If a graph G has components $G_1, G_2, \cdots, G_k$, then G is denoted by $\bigcup\limits_{i=1}^{k} G_i$. P_n denotes the path on n vertices, C_n is the cycle on n vertices, and S_n is the star consisting of one center vertex adjacent to $n-1$ leaves and T_n is a tree on n vertices.

For a graph $G = (V, E)$, a subset $S \subseteq V$ is called independent if no two vertices of S are adjacent in G. The set of independent sets in G is denoted by $I(G)$. The empty set is an independent set. The number of independent sets in G, denoted by $\sigma-$index, is called the $Merrifield - Simmons$ index in theoratical chemistry. the $Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs is abtained from P_k whose every vertex $v_i(i = 1, 2, \cdots, k)$ attached one cycle $C_{s_i}(i = 1, \cdots, k)$.

The $Merrifield-Simmons$ index [1-3] is one of the topogical indices whose mathematical properties were studied in some detail [4-12] whereas its applicability for QSPR and QSAR was examined to a much lesser extent; in [2] it was shown that $\sigma-$index is correlated with the boiling points.

In this paper, we investigate the $Merrifield - Simmons$ index on $Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs . We characterize the $Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs with the smallest and the largest Merrifield-Simmons index.

§2. Some known results

We give with several important lemmas from [2-6] will be helpful to the proofs of our main resuts, and also give three lemmas which will increase the Merrifield-Simmons index.

Lemma2.1[2]**.** Let G be a graph with k components $G_1, G_2, \cdots, G_k$, then

$$\sigma(G) = \prod_{i=1}^{k} \sigma(G_i).$$

Lemma 2.2[4]**.** For any graph G with any $v \in V(G)$, we have

$$\sigma(G) = \sigma(G - v) + \sigma(G - [v]),$$

where $[v] = N_G(v) \bigcup v$.

Lemma 2.3[3]**.** Let T be a tree, then

$$F_{n+2} \leq \sigma(T) \leq 2^{n-1} + 1$$

and

$$\sigma(T) = F_{n+2}$$

if and only if $T \cong P_n$ and $\sigma(T) = 2^{n-1} + 1$ if and only if $T \cong S_n$.

Lemma 2.4[5]**.** Let $n = 4m + i (i \in \{1, 2, 3, 4\})$ and $m \geq 2$, then

$$\begin{aligned}&\sigma((P_n, v_2, T)) > \sigma((P_n, v_4, T)) > \cdots > \sigma((P_n, v_{2m+2\rho}, T))\\ &> \cdots > \sigma((P_n, v_{2m+1}, T)) > \sigma((P_n, v_3, T)) > \sigma((P_n, v_1, T)),\end{aligned}$$

where $\rho = 0$ if $i = 1, 2$ and $\rho = 1$ if $i = 3, 4$.

Lemma 2.5[6]**.** Let $\alpha = \dfrac{1+\sqrt{5}}{2}$ and $\beta = \dfrac{1-\sqrt{5}}{2}$ and by definition of *Fibonacci* number F_n and *Lucas* number L_n, we know

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}},\ L_n = \alpha^n + \beta^n,\ F_n \cdot F_m = \frac{1}{5}(L_{n+m} - (-1)^n \cdot L_{m-n}).$$

Lemma 2.6. Let G is $Q(P_3, C_s, C_l, C_{n-s-l})$ graphs with n vertices, then

$$\sigma(Q(P_3, C_s, C_l, C_{n-s-l})) = L_s L_{n-s-l} F_{l+1} + F_{s+1} F_{n-s-l+1} F_{l-1}.$$

Proof. From the lemmas 2.1, 2.2, we have
$\sigma(Q(P_3, C_s, C_l, C_h)) = L_s L_{n-s-l} F_{l+1} + F_{s+1} F_{n-s-l+1} F_{l-1}$.

Lemma 2.7. Let G are $Q(P_k, C_4, C_4, \cdots, C_4, C_{n-4(k-1)})$ and $Q(P_k, C_3, C_3, \cdots, C_3, C_{n-3(k-1)})$ graphs with n vertices, then

$$\begin{aligned}&\sigma(Q(P_k, C_4, C_4, \cdots, C_4, C_{n-4(k-1)}))\\ &= F_5 F_{n-4(k-1)+1} a_{k-2} + F_5 F_{n-4(k-1)-1} F_5 a_{k-3}\\ &+ F_3 F_5 (F_{n-4(k-1)+1} a_{k-3} + F_{n-4(k-1)-1} F_5 a_{k-4})\\ &\sigma(Q(P_k, C_3, C_3, \cdots, C_3, C_{n-3(k-1)}))\\ &= F_4 F_{n-3(k-1)+1} b_{k-2} + F_4 F_{n-3(k-1)-1} F_4 b_{k-3}\\ &+ F_4 F_{n-3(k-1)+1} b_{k-3} + F_4 F_{n-3(k-1)-1} F_4 b_{k-4}),\end{aligned}$$

where $a_i = \sigma(Q(P_i, C_4, C_4, \cdots, C_4))$ and $b_i = \sigma(Q(P_i, C_3, C_3, \cdots, C_3)), i = k-2, k-3, k-4$.

Proof. According to the lemmas 2.1, 2.2, it will be proved easily.

§3. The graph with the largest Merrifield-Simmons index in $Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs

In this section, we will find the $Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs with the largest σ-index in $Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs. and give some good results on orders of σ-index.

Definition 3.1. Let $Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs be a graph abtained from P_k whose every vertex $v_i(i = 1, 2, \cdots, k)$ attached one cycle $C_i(i = 1, 2, \cdots, k)$.

Theorem 3.1. Let G be $Q(P_3, C_s, C_l, C_{n-s-l})$ graphs with n vertices, then

$$\sigma(Q(P_3, C_4, C_4, C_{n-8})) \geq \sigma(Q(P_3, C_s, C_l, C_{n-s-l}))$$

and the equality is correct if only if $Q(P_3, C_4, C_4, C_{n-8}) \cong Q(P_3, C_s, C_l, C_{n-s-l})$.

Proof. From Lemma 2.6, we have

$$\sigma(Q(P_3, C_s, C_l, C_{n-s-l})) = L_s L_{n-s-l} F_{l+1} + F_{s+1} F_{n-s-l+1} F_{l-1}.$$

From Lemma 2.5, we have

$$\begin{aligned}
&\sigma(Q(S_3, C_s, C_l, C_{n-s-l-1})) \\
&= L_s L_{n-s-l} F_{l+1} + \frac{1}{5}(L_{n-l+2} + (-1)^s L_{n-2s-l}) F_{l-1} \\
&= (L_{n-l} + (-1)^s L_{n-2s-l}) F_{l+1} + \frac{1}{5}(L_{n-l+2} F_{l-1} + (-1)^s L_{n-2s-l} F_{l-1}) \\
&= L_{n-l} F_{l+1} + (-1)^s L_{n-2s-l} F_{l+1} + \frac{1}{5}(L_{n-l+2} F_{l-1} + (-1)^s L_{n-2s-l} F_{l-1}) \\
&= (F_{n+1} + (-1)^l F_{n-2l-1}) + (-1)^s (F_{n-2s+1} + (-1)^l F_{n-2s-2l-1}) \\
&+ \frac{1}{5}[(F_{n+1} + (-1)^l F_{n-2l+3}) + (-1)^s (F_{n-2s-1} + (-1)^l F_{n-2s-2l+1})] \\
&= \frac{6}{5} F_{n+1} + (-1)^l (F_{n-2l-1} + \frac{1}{5} F_{n-2l+3}) + (-1)^s (F_{n-2s+1} + \frac{1}{5} F_{n-2s-1}) \\
&+ (-1)^{s+l} (F_{n-2s-2l+1} + \frac{1}{5} F_{n-2s-2l+1}).
\end{aligned}$$

From above, we know that the result is correct.

Theorem 3.2. Let G be $Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs with n vertices, then

$$\sigma(Q(P_k, C_4, C_4, \cdots, C_4, C_{n-4(k-1)})) \geq \sigma(Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})),$$

and the equality is correct if only if $Q(P_k, C_4, C_4, \cdots, C_4, C_{n-4(k-1)}) \cong Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$.

Proof. If $K = 3$, we have proofed that the result is correct. We presume that the result is correct, if $K \leq k-1$, then if $K = k$ we have

$$\begin{aligned}
&\sigma(Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})) \\
&= \sigma(Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k}) - v) + \sigma(Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_{k+1}}) - [v]) \\
&\leq F_{s_k+1}F_{n-s_k-4(k-2)+1}a_{k-2} + F_{s_k+1}F_{n-s_k-4(k-2)-1}F_5a_{k-3} \\
&+F_{s_k-1}F_{s_{k-1}+1}F_{n-s_k-s_{k-1}-4(k-3)+1}a_{k-3} + F_{s_k-1}F_{s_{k-1}+1}F_{n-s_k-s_{k-1}-4(k-3)-1}F_5a_{k-4} \\
&= F_{s_k+1}F_{n-s_k-4(k-2)+1}a_{k-2} + F_{s_k+1}F_{n-s_k-4(k-2)-1}F_5a_{k-3} \\
&+\frac{1}{5}F_{s_k-1}(L_{n-s_k-4(k-3)+2} + (-1)^{s_{k-1}}L_{n-s_k-2s_{k-1}-4(k-3)})a_{k-3} \\
&+\frac{1}{5}F_{s_k-1}(L_{n-s_k-4(k-3)} + (-1)^{s_{k-1}}L_{n-s_k-2s_{k-1}-4(k-3)-2})F_5a_{k-4} \\
&= F_{s_k+1}F_{n-s_k-4(k-2)+1}a_{k-2} + F_{s_k+1}F_{n-s_k-4(k-2)-1}F_5a_{k-3} \\
&+\frac{1}{5}[F_{s_k-1}L_{n-s_k-4(k-3)+2}a_{k-3} + (-1)^{s_{k-1}}F_{s_k-1}L_{n-s_k-2s_{k-1}-4(k-3)}a_{k-3} \\
&+F_{s_k-1}L_{n-s_k-4(k-3)}F_5a_{k-4} + (-1)^{s_{k-1}}F_{s_k-1}L_{n-s_k-2s_{k-1}-4(k-3)-2}F_5a_{k-4}] \\
&= \frac{1}{5}[(L_{n-4(k-2)+2} + (-1)^{s_k}L_{n-2s_k-4(k-2)}a_{k-2}) \\
&+(L_{n-4(k-2)} + (-1)^{s_k}L_{n-2s_k-4(k-2)-2})F_5a_{k-3} \\
&+(F_{n-4(k-3)+1} + (-1)^{s_k}F_{n-2s_k-4(k-3)+3})a_{k-3} \\
&+(-1)^{s_{k-1}}(F_{n-2s_{k-1}-4(k-3)-1} + (-1)^{s_k}F_{n-2s_k-2s_{k-1}-4(k-3)+1})a_{k-3} \\
&+(F_{n-4(k-3)-1} + (-1)^{s_k}F_{n-2s_k-4(k-3)+1})F_5a_{k-4} \\
&+(-1)^{s_{k-1}}(F_{n-2s_{k-1}-4(k-3)-3} + (-1)^{s_k}F_{n-2s_k-2s_{k-1}-4(k-3)-1})F_5a_{k-4}] \\
&\leq F_5F_{n-4(k-1)+1}a_{k-2} + F_5F_{n-4(k-1)-1}F_5a_{k-3} \\
&+F_3F_5(F_{n-4(k-1)+1}a_{k-3} + F_{n-4(k-1)-1}F_5a_{k-4}) \\
&= \sigma(Q(P_k, C_4, C_4, \cdots, C_4, C_{n-4(k-1)})),
\end{aligned}$$

where $a_i = \sigma(Q(P_i, C_4, C_4, \cdots, C_4)), i = k-2, k-3, k-4$.
From above, we know that the result is correct.

§4. The graph with the smallest Merrifield-Simmons index in $Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs

In this section, we will find the $Q(S_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs with the smallest Merrifield-Simmons index.

Theorem 4.1. Let G be $Q(P_3, C_s, C_l, C_{n-s-l})$ graphs with n vertices, then

$$\sigma(Q(P_3, C_3, C_3, C_{n-6})) \leq \sigma(Q(P_3, C_s, C_l, C_{n-s-l}))$$

and the equality is correct if only if $Q(P_3, C_3, C_3, C_{n-6}) \cong Q(P_3, C_s, C_l, C_{n-s-l})$.

Proof. From Lemma 2.6, we have
$\sigma(Q(P_3, C_s, C_l, C_{n-s-l})) = L_sL_{n-s-l}F_{l+1} + F_{s+1}F_{n-s-l+1}F_{l-1}$.

From Lemma 2.5, we have

$$\begin{aligned}
&\sigma(Q(S_3, C_s, C_l, C_{n-s-l-1}))\\
&= L_s L_{n-s-l} F_{l+1} + \frac{1}{5}(L_{n-l+2} + (-1)^s L_{n-2s-l}) F_{l-1}\\
&= (L_{n-l} + (-1)^s L_{n-2s-l}) F_{l+1} + \frac{1}{5}(L_{n-l+2} F_{l-1} + (-1)^s L_{n-2s-l} F_{l-1})\\
&= L_{n-l} F_{l+1} + (-1)^s L_{n-2s-l} F_{l+1} + \frac{1}{5}(L_{n-l+2} F_{l-1} + (-1)^s L_{n-2s-l} F_{l-1})\\
&= (F_{n+1} + (-1)^l F_{n-2l-1}) + (-1)^s (F_{n-2s+1} + (-1)^l F_{n-2s-2l-1})\\
&+\frac{1}{5}[(F_{n+1} + (-1)^l F_{n-2l+3}) + (-1)^s (F_{n-2s-1} + (-1)^l F_{n-2s-2l+1})]\\
&= \frac{6}{5} F_{n+1} + (-1)^l (F_{n-2l-1} + \frac{1}{5} F_{n-2l+3}) + (-1)^s (F_{n-2s+1} + \frac{1}{5} F_{n-2s-1})\\
&+(-1)^{s+l} (F_{n-2s-2l+1} + \frac{1}{5} F_{n-2s-2l+1}).
\end{aligned}$$

From above, we know that the result is correct.

Theorem 4.2. Let G be $Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$ graphs with n vertices, then

$$\sigma(Q(P_k, C_3, C_3, \cdots, C_3, C_{n-3(k-1)})) \leq \sigma(Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k}))$$

and the equality is correct if only if $Q(P_k, C_3, C_3, \cdots, C_3, C_{n-3(k-1)}) \cong Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k})$.

Proof. If $K = 3$, we have proved that the result is correct. We presume that the result is correct, if $K \leq k-1$, then if $K = k$ we have

$$\begin{aligned}
&\sigma(Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k}))\\
&\quad= \sigma(Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_k}) - v) + \sigma(Q(P_k, C_{s_1}, C_{s_2}, \cdots, C_{s_{k+1}}) - [v])\\
&\quad\geq F_{s_k+1} F_{n-s_k-3(k-2)+1} b_{k-2} + F_{s_k+1} F_{n-s_k-3(k-2)-1} F_4 b_{k-3}\\
&\qquad +F_{s_k-1} F_{s_{k-1}+1} (F_{n-s_k-s_{k-1}-3(k-3)+1} b_{k-3} + F_{n-s_k-s_{k-1}-3(k-3)-1} F_3 b_{k-4})\\
&\quad= F_{s_k+1} F_{n-s_k-3(k-2)+1} b_{k-2} + F_{s_k+1} F_{n-s_k-3(k-2)-1} F_4 b_{k-3}\\
&\qquad +\frac{1}{5}[F_{s_k-1}(L_{n-s_k-3(k-3)+2} + (-1)^{s_{k-1}} L_{n-s_k-2s_{k-1}-3(k-3)}) b_{k-3}\\
&\qquad +F_{s_k-1}(L_{n-s_k-3(k-3)} + (-1)^{s_{k-1}} L_{n-s_k-2s_{k-1}-3(k-3)-2}) F_4 b_{k-4}]\\
&\quad= F_{s_k+1} F_{n-s_k-3(k-2)+1} b_{k-2} + F_{s_k+1} F_{n-s_k-3(k-2)-1} F_4 b_{k-3}\\
&\qquad +\tfrac{1}{5}[F_{s_k-1} L_{n-s_k-3(k-3)+2} b_{k-3} + (-1)^{s_{k-1}} F_{s_k-1} L_{n-s_k-2s_{k-1}-3(k-3)} b_{k-3}\\
&\qquad +F_{s_k-1} L_{n-s_k-3(k-3)} F_4 b_{k-4} + (-1)^{s_{k-1}} F_{s_k-1} L_{n-s_k-2s_{k-1}-3(k-3)-2} F_4 b_{k-4}]\\
&\quad= \frac{1}{5}[(L_{n-3(k-2)+2} + (-1)^{s_k} L_{n-2s_k-3(k-2)} b_{k-2})\\
&\qquad +(L_{n-3(k-2)} + (-1)^{s_k} L_{n-2s_k-3(k-2)-2}) F_3 b_{k-3}\\
&\qquad +(F_{n-3(k-3)+1} + (-1)^{s_k} F_{n-2s_k-3(k-3)+3}) b_{k-3}\\
&\qquad +(-1)^{s_{k-1}} (F_{n-2s_{k-1}-3(k-3)-1} + (-1)^{s_k} F_{n-2s_k-2s_{k-1}-3(k-3)+1}) b_{k-3}\\
&\qquad +(F_{n-3(k-3)-1} + (-1)^{s_k} F_{n-2s_k-3(k-3)+1}) F_4 b_{k-4}\\
&\qquad +(-1)^{s_{k-1}} (F_{n-2s_{k-1}-3(k-3)-3} + (-1)^{s_k} F_{n-2s_k-2s_{k-1}-3(k-3)-1}) F_4 b_{k-4}]\\
&\quad\geq F_4 F_{n-3(k-1)+1} b_{k-2} + F_4 F_{n-3(k-1)-1} F_4 b_{k-3}\\
&\qquad +F_4 F_{n-3(k-1)+1} b_{k-3} + F_4 F_{n-3(k-1)-1} F_4 b_{k-4})\\
&\quad= \sigma(Q(P_k, C_3, C_3, \cdots, C_3, C_{n-3(k-1)})),
\end{aligned}$$

where $b_i = \sigma(Q(P_i, C_3, C_3, \cdots, C_3)), i = k-2, k-3, k-4$.

From above, we know that the result is correct.

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The natural partial order on Semiabundant semigroups [1]

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Abstract In this paper, the natural partial order on semiabundant semigroups is firstly defined. And then, some properties and two examples of semiabundant semigroups which are not abundant semigroups are given. Finally one of the main resulsts is proved: if the natural partial order on semiabundant semigroup is compatible with the multiplication, then the semigroup is locally semiadequate.

Keywords Green's $\sim -$ relation, semiabundant semigroup, natural partial order.

§1. Introduction

As we know, constructions and many properties on regular and abundant semigroups have been described in terms of their natural partial orders (see [1]-[3]). The natural partial orders on these two classes of semigroups were firstly investigated by Nampooripad and Lawson in [1] and [2] respectively. Also, we know that the class of semiabundant semigroups contains regular semigroups and abundant semigroups ([4]), naturally we will consider to study the natural partial order on semiabundant semigroups in order to investigate some constructions and properties of this class of semigroups.

In this paper, we firstly discuss some properties of Green's $\sim -$relations, and then, define a natural partial order on semiabundant semigroups. Finally, we give some properties of them and prove that if the natural partial order on semiabundant semigroup S is compatible with respective to the multiplication, then the semigroup S is locally semiadequate.

For the notations not mentioned in this paper, readers are referred to [5]-[7].

§2. Green's $\sim$ -relations

The notation $E(S)$ denotes the set of all idempotent elements of a semigroup S.

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First, we recall Green's $\sim -$ relations introduced in [8]: let S be an arbitrary semigroup.

$$\tilde{\mathcal{L}} = \{(a,b) \in S \times S : (\forall e \in E(S))\ ae = a \Leftrightarrow be = b\},$$

$$\tilde{\mathcal{R}} = \{(a,b) \in S \times S : (\forall e \in E(S))\ ea = a \Leftrightarrow eb = b\},$$

$\tilde{\mathcal{H}} = \tilde{\mathcal{L}} \cap \tilde{\mathcal{R}}, \tilde{\mathcal{D}} = \tilde{\mathcal{L}} \vee \tilde{\mathcal{R}}$.

Since results about $\tilde{\mathcal{L}}$ there exists a dual result for $\tilde{\mathcal{R}}$, in the following, we only need to discuss the properties related to the $\tilde{\mathcal{L}}-$relations.

Definition 1[4]**.** A semigroup S is called a semiabundant semigroup if its each $\tilde{\mathcal{L}}-$class and each $\tilde{\mathcal{R}}-$class contains at least one idempotent.

If U is a subset of S, we write $E(S) \cap U$ as $E(U)$. Also, for $a \in S$, the equivalence relation $\tilde{\mathcal{L}}-$class ($\tilde{\mathcal{R}}-$class) containing the element a is denoted by $\tilde{L_a}$ ($\tilde{R_a}$).

Definition 2. Let S be a semigroup. I is said to be a left (right) $\sim -$ ideal of S if I is a left (right) ideal of S and $\tilde{L_a} \subseteq I$ ($\tilde{R_a} \subseteq I$) for any $a \in I$. We call I a $\sim -$ ideal of S if I is both a left $\sim -$ ideal and a right $\sim -$ ideal.

Proposition 1. If $\{I_\alpha : \alpha \in \Lambda\}$ is a set of $\sim -$ ideals (left $\sim -$ ideals, right $\sim -$ ideals) of a semigroup S, then

(1) $\cap\{I_\alpha : \alpha \in \Lambda\}$ is also a $\sim -$ ideal (left $\sim -$ ideal, right $\sim -$ ideal);

(2) $\cup\{I_\alpha : \alpha \in \Lambda\}$ is also a $\sim -$ ideal (left $\sim -$ ideal, right $\sim -$ ideal).

Proof. It is easy to verify.

Now, let $a \in S$. In view of (1) of Proposition 1 and the fact S is a $\sim -$ ideal of itself, there exists a smallest $\sim -$ ideal $\tilde{J}(a)$ containing a, a smallest left $\sim -$ideal $\tilde{L}(a)$ containing a and a smallest right $\sim -$ideal $\tilde{R}(a)$ containing a. We shall call $\tilde{J}(a)$ ($\tilde{L}(a)$, $\tilde{R}(a)$) the principal $\sim -$ ideal (principal left $\sim -$ideal, principal right $\sim -$ideal) generated by a. It is clear that $\tilde{L}(a) \subseteq \tilde{J}(a)$ and $\tilde{R}(a) \subseteq \tilde{J}(a)$.

Next, we shall give some characterizations of these $\sim -$ ideals.

Proposition 2. Let a be an element of a semigroup S. Then

(1) $b \in \tilde{L}(a)$ if and only if there are elements $a_0, a_1, ..., a_n \in S$, $x_1, x_2, ..., x_n \in S^1$ such that $a = a_0, b = a_n$ and $(a_i, x_i a_{i-1}) \in \tilde{\mathcal{L}}$ for $i = 1, 2, ..., n$.

(2) $b \in \tilde{R}(a)$ if and only if there are elements $a_0, a_1, ..., a_n \in S$, $x_1, x_2, ..., x_n \in S^1$ such that $a = a_0, b = a_n$, and $(a_i, a_{i-1} x_i) \in \tilde{\mathcal{R}}$ for $i = 1, 2, ..., n$.

(3) $b \in \tilde{J}(a)$ if and only if there are elements $a_0, a_1, ..., a_n \in S$, $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in S^1$ such that $a = a_0, b = a_n$, and $(a_i, x_i a_{i-1} y_i) \in \tilde{\mathcal{D}}$ for $i = 1, 2, ..., n$.

Proof. (1) Let $I = \{b \in S : (\exists a_0, a_1, ..., a_n \in S, x_1, x_2, ..., x_n \in S^1)\ a = a_0, b = a_n$ and $(a_i, x_i a_{i-1}) \in \tilde{\mathcal{L}}$, $i = 1, 2, ..., n\}$. We only need to prove that $I = \tilde{L}(a)$.

If $b \in I$, then there exists elements $a_0, a_1, ..., a_n \in S, x_1, x_2, ..., x_n \in S^1$ such that $a = a_0, b = a_n$, and $(a_i, x_i a_{i-1}) \in \tilde{\mathcal{L}}$ for $i = 1, 2, ..., n$. If $a_{i-1} \in \tilde{L}(a)$, we have $x_i a_{i-1} \in \tilde{L}(a)$ since $\tilde{L}(a)$ is a left ideal, and so $a_i \in \tilde{L}(a)$ by the fact that $\tilde{L}(a)$ is a left $\sim -$ ideal. Further, since $a_0 = a \in \tilde{L}(a)$, we have $a_i \in \tilde{L}(a)$ for $i = 0, 1, ..., n$. Therefore, $b = a_n \in \tilde{L}(a)$, and so $I \subseteq \tilde{L}(a)$.

On the other hand, if $b \in I$, we have elements $a_0, a_1, ..., a_n \in S$, $x_1, x_2, ..., x_n \in S^1$ such that $a = a_0, b = a_n$, and $(a_i, x_i a_{i-1}) \in \tilde{\mathcal{L}}$ for $i = 1, 2, ..., n$. Since $(sa_n, sa_n) \in \tilde{\mathcal{L}}$ and $b = a_n \in I$, we immediately have that $sb \in I$. Clearly, $\tilde{L_b} \subseteq I$. Hence, I is a left $\sim -$ ideal. By $a \in I$, we have $\tilde{L}(a) = I$.

Similarly, we can show that (2) and (3) hold.

Corollary 1. For elements a, b of a semigroup S, we have

(1) $(a, b) \in \tilde{\mathcal{L}}$ if and only if $\tilde{L}(a) = \tilde{L}(b)$;

(2) $(a, b) \in \tilde{\mathcal{R}}$ if and only if $\tilde{R}(a) = \tilde{R}(b)$.

Proof. We only need to prove (1), the proof of (2) is similar.

Firstly, if $(a, b) \in \tilde{\mathcal{L}}$, then clearly by Proposition 2, we have $\tilde{L}(a) = \tilde{L}(b)$. Now, suppose that $\tilde{L}(a) = \tilde{L}(b)$, then we have $b \in \tilde{L}(a)$ and so by Proposition 2, there are elements $a_0, a_1, ..., a_n \in S, x_1, x_2, ..., x_n \in S^1$ such that $a = a_0, b = a_n$, and $(a_i, x_i a_{i-1}) \in \tilde{\mathcal{L}}$ for $i = 1, 2, ..., n$. Let $e \in E(S)$ satisfying $ae = a$. If $a_{i-1}e = a_{i-1}$, then $x_i a_{i-1} e = x_i a_{i-1}$. Since $(a_i, x_i a_{i-1}) \in \tilde{\mathcal{L}}$, we have $a_i e = a_i$. Since $a = a_0$, it follows that $be = b$. Similarly, for any $e \in E(S)$, if $be = b$, then we have $ae = a$. Hence, we have $(a, b) \in \tilde{\mathcal{L}}$. Therefore, (1) holds.

Let e be an idempotent in a semigroup S. For $a \in Se$, we have $a = ae$. Then for any $b \in \tilde{L}_a$, $b = be \in Se$. It means that the left ideal Se is a left $\sim -$ ideal. Thus, if a is a regular element in S, then $(a, e) \in \mathcal{L}$ for some $e \in E(S)$, we have $Sa = Se = \tilde{L}(a) = \tilde{L}(e)$. So we immediately have the following corollary :

Corollary 2. A semigroup S is semiabundant if and only if for any $a \in S$, there are idempotents $e, f \in E(S)$ such that $\tilde{L}(a) = Se, \tilde{R}(a) = fS$.

Proposition 3. Let S be a semiabundant semigroup and $a \in S$. Then for any $e \in E(S)$, $(a, e) \in \tilde{\mathcal{L}}$ $(\tilde{\mathcal{R}})$ if and only if $a \in Se$ (eS) and Se (Se) is contained in every idempotent–generated left (right) ideal to which a belongs.

Proof. Let $e \in E(S)$ and $a \in S$. If $(a, e) \in \tilde{\mathcal{L}}$, then $\tilde{L}(a) = Se$, and so $a \in Se$. If $a \in Sf$ for any idempotent f, then $af = a$ and so $ef = e$. Hence, we have $Se \subseteq Sf$.

Conversely, suppose that $a \in Se$ and that for some idempotent f, $a \in Sf$ implies $Se \subseteq Sf$. Then since Se is a left $\sim -$ideal, we have $\tilde{L}(a) \subseteq Se$. Since S is semiabundant, by Corollary 2, we have $\tilde{L}(a) = Sf$. Hence $Se = Sf$, and so $(a, e) \in \tilde{\mathcal{L}}$.

Similarly, we can show its dual.

Finally, we define the Green's relation $\tilde{\mathcal{J}}$ by analogy with the characterizations of $\tilde{\mathcal{L}}$ and $\tilde{\mathcal{R}}$ in Corollary 1. That is : $(a, b) \in \tilde{\mathcal{J}}$ if and only if $\tilde{J}(a) = \tilde{J}(b)$

It is clear that $\tilde{\mathcal{L}} \subseteq \tilde{\mathcal{J}}$ and $\tilde{\mathcal{R}} \subseteq \tilde{\mathcal{J}}$, and so $\tilde{\mathcal{D}} \subseteq \tilde{\mathcal{J}}$.

§3. Examples of semiabundant semigroups

In the following,we will give two examples of semiabundant semigroups which are not abundant semigroups.

Example 1. Let $S = \{0, e, f, a, g\}$ be a semigroup, which multiplication is as follows:

$\cdot$	0	e	f	a	g
0	0	0	0	0	0
e	0	e	a	a	0
f	0	a	f	0	0
a	0	0	a	0	0
g	0	0	0	0	g

It is clear that $E(S) = \{0, e, f, g\}$. Also, we can check that the $\tilde{\mathcal{L}}$-classes of S are $\{0\}$,$\{e\}$, $\{a, f\}$, $\{g\}$, the $\tilde{\mathcal{R}}$-classes of S are $\{0\}$,$\{a, e\}$,$\{f\}$,$\{g\}$. Thus, S is semiabundant.

Alao, we can check that L_a^* contains no idempotents, hence, S is not abundant.

From the above example, we can see that the class of semiabundant semigroups properly contains the class of abundant semigroups. And also we can see that the semigroup in Example 1 is finite. Next, we will give another example which is infinite semiabundant semigroup but not abundant one.

Example 2. Let $a = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $a_n = 3^n a$ and $S = \{e, f, g, h, u, v, a, a_n\}$, where $n = 1, 2, 3, \cdots$. The multiplication of S is as follows:

$\cdot$	a	a_n	e	f	g	h	u	v
a	a	a_n	e	f	g	h	u	v
a_m	a_m	a_{m+n}	e	f	g	h	u	v
e	e	e	e	f	g	h	u	v
f	f	e	e	f	g	h	u	v
g	g	g	g	h	u	v	e	f
h	h	g	g	h	u	v	e	f
u	u	u	u	v	e	f	g	h
v	v	u	u	v	e	f	g	h

It is easy to check that the $\tilde{\mathcal{L}}$-classes of S are $\{e, g, u\}$,$\{f, h, v\}$, $\{a, a_n\}(n \in N)$, the $\tilde{\mathcal{R}}$-classes of S are $\{e, f, g, h, u, v\}$, $\{a, a_n\}$ $(n \in N)$. Hence, S is a semiabundant semigroup. But for every $n \in N$, the $\mathcal{R}^*-$class of a_n in S contains no idempotents, thus, S is not abundant.

§4. The natural partial order on semiabundant semigroups

In this section, we will firstly introduce the natural partial order on semiabundant semigroups, and then give some properties for the partial order. Finally, we will discuss a relation between the natural partial order of semiabundant semigroups and locally semiadequate semigroups.

Now, we introduce a partial ordering on the $\tilde{\mathcal{L}}-$ and $\tilde{\mathcal{R}}-$ classes of a semigroup S. For any $a, b \in S$, we say that $\tilde{L}_a \leq \tilde{L}_b$ if and only if $\tilde{L}(a) \subseteq \tilde{L}(b)$, where $\tilde{L}(a)$ is the principal left $\sim -$ ideal generated by a. Dually, we can define the partial ordering on the $\tilde{\mathcal{R}}-$ classes.

Lemma 1. For any elements a and x of S, we have $\tilde{L}_{xa} \leq \tilde{L}_a$.

Proof. We only need to prove that $\tilde{L}(xa) \subseteq \tilde{L}(a)$. Clearly, we have $xa \in S^1a$, where S^1a is the smallest left ideal containing a. Since $\tilde{L}(a)$ is a left ideal containing a, we have that $S^1a \subseteq \tilde{L}(a)$. And then we have $xa \in \tilde{L}(a)$. On the other hand, since $\tilde{L}(xa)$ is the smallest left $\sim -$ ideal containing xa and $\tilde{L}(a)$ is a left $\sim -$ ideal, we immediately have $\tilde{L}(xa) \subseteq \tilde{L}(a)$.

In particular, for the regular elements of S, the partial ordering is just the one which is given in [1].

Lemma 2. Let S be a semigroup and let a, b be regular element of S. Then $\tilde{L}_a \leq \tilde{L}_b$ if and only if $L_a \leq L_b$.

Proof. Suppose that $\tilde{L}_a \leq \tilde{L}_b$ and let a', b' be the inverses of a, b respectively. Then we have $(a, a'a) \in \mathcal{L}$ and $(b, b'b) \in \mathcal{L}$. And then by Corollary 2 we have $\tilde{L}(a) = Sa'a$ and $\tilde{L}(b) = Sb'b$. By the assumption, we have $Sa'a \subseteq Sb'b$, and so $a \in (Sb')b \subset Sb$. Hence $L_a \leq L_b$.

Conversely, if $L_a \leq L_b$, then we have $a \in S^1 b$. Thus, there exists some element $x \in S^1$ such that $a = xb$. By Lemma 1, we immediately have that $\tilde{L}_a = \tilde{L}_{xb} \leq \tilde{L}_b$.

Proposition 4. Let S be a semiabundant semigroup. Define two relations on S as follows: for all $x, y \in S$,

$$\begin{aligned} x \quad &\leq_l \quad y \Leftrightarrow \tilde{L}_x \leq \tilde{L}_y \text{ and } (\exists e \in \tilde{L}_x \cap E(S))\ x = ye; \\ x \quad &\leq_r \quad y \Leftrightarrow \tilde{R}_x \leq \tilde{R}_y \text{ and } (\exists f \in \tilde{R}_x \cap E(S))\ x = fy. \end{aligned}$$

Then $\leq_l$ ($\leq_r$) is a partial order on S which coincides with ω on $E(S)$, where $e\omega f$ if and only if $ef = fe = e$.

Proof. We only need to show that $\leq_l$ is a partial order. The proof can be done by the following steps :

(i) The reflexivity of $\leq_l$ is clear, it follows from that S is a semiabundant semigroup.

(ii) $\leq_l$ is anti-symmetric. In fact, if we let $x \leq_l y \leq_l x$, then $\tilde{L}_x = \tilde{L}_y$ and there exists $f \in \tilde{L}_x$ such that $x = yf$, and this implies that $x = y$.

(iii) $\leq_l$ is transitive. Suppose that $x \leq_l y \leq_l z$ for $x, y, z \in S$. Then $\tilde{L}_x \leq \tilde{L}_z$ and there exists $f \in \tilde{L}_x$ and $g \in \tilde{L}_y$ such that $x = yf$, $y = zg$. And it follows that $\tilde{L}_f \leq \tilde{L}_g$. By Lemma 2, we have $L_f \leq L_g$, and then $gf \in E(S)$, $gf\omega g$ and $(gf, f) \in \mathcal{L}$. But $x = z(gf)$ and $gf \in E(\tilde{L}_f) = E(\tilde{L}_x)$. This means that $\leq_l$ is transitive.

Similarly, we can show that $\leq_r$ is a partial order. The proof of which $\leq_l$ ($\leq_r$) coincides with ω on $E(S)$ is not hard to verify.

Now, we define the natural partial order to be $\leq = \leq_l \cap \leq_r$similar with the ones of the natural partial order of regular or abundant semigroups.

Proposition 5. Let S be a semiabundant semigroup. Then

(1) if $x \leq_l e$ ($x \leq_r e$) where $e \in E(S)$, then $x \in E(S)$;

(2) if $b \leq_l a$ ($b \leq_r a$) where a is regular, then b is regular;

(3) if $x, y \in S$ with $(x, y) \in \tilde{\mathcal{L}}$ ($(x, y) \in \tilde{\mathcal{R}}$) and $x \leq_l y$ ($x \leq_r y$), then $x = y$.

Proof. In the following, we only prove the results in the case of $\leq_l$, and the dual can be similarly verified.

(1) Assume that $x \leq_l e$. Then $\tilde{L}_x \leq \tilde{L}_e$, and also there exists $f \in \tilde{L}_x \cap E(S)$ such that $x = ef$. So we can get $\tilde{L}_f = \tilde{L}_x \leq \tilde{L}_e$. By Lemma 2, we have $L_f \leq L_e$, and then $fe = f$, so $x = ef$ is an idempotent.

(2) Assume that $b \leq_l a$ and $a \in Reg(S)$. Then $\tilde{L}_b \leq_l \tilde{L}_a$, and also there exists $f \in \tilde{L}_b$ such that $b = af$. Let $a' \in V(a)$. Then we have $ba'b = af \cdot a' \cdot af = a(f \cdot a'a)f$. However, $(a'a, a) \in \mathcal{L}$, and we have $\tilde{L}_f = \tilde{L}_b \leq \tilde{L}_a = \tilde{L}_{a'a}$. By Lemma 2, we have $L_f \leq L_{a'a}$. And then $f \cdot a'a = f$. Applying the above, $ba'b = a(f \cdot a'a)f = a \cdot f \cdot f = af = b$. Hence, b is regular.

(3) Assume that $(x, y) \in \mathcal{L}$ and $x \leq_l y$. Then $\tilde{L}_x \leq \tilde{L}_y$, and also there exists an idempotent $e \in \tilde{L}_x$ such that $x = ye$. By assumption, we have $(e, y) \in \tilde{\mathcal{L}}$, and then we have $y = ye = x$.

Recall that a subset A of a semigroup S is said to be an order ideal if for each element a of A and any x with $x \leq a$, then x belongs to A. Also, we say a partial order $\leq$ on a semigroup S is compatible with the multiplication if whenever $a \leq b$ and $c \leq d$, then $ac \leq bd$.

Remark. In Proposition 5, properties (1) and (2) may be paraphrased by saying that $(E(S), \leq_l)$ and $(Reg(S), \leq_l)$ are order ideals of $(S, \leq_l)$.

Proposition 6. Let x, y be elements of a semiabundant semigroup S. Then $x \leq_l y (x \leq_r y)$ if and only if for each idempotent $y^* \in \tilde{L}_y (y^+ \in \tilde{R}_y)$, there exists an idempotent $x^* \in \tilde{L}_x (x^+ \in \tilde{R}_x)$ such that $x^* \omega y^* (x^+ \omega y^+)$ and $x = yx^* (x = x^+ y)$.

Proof. We will prove the results in the case of $\leq_l$, and the dual can be similarly verified.

Suppose that $x \leq_l y$, then $\tilde{L}_x \leq \tilde{L}_y$ and there exists an idempotent $e \in \tilde{L}_x$ such that $x = ye$. Let $f \in \tilde{L}_y$. Then $\tilde{L}_e = \tilde{L}_x \leq \tilde{L}_y = \tilde{L}_f$, by Lemma 2, $L_e \leq L_f$. Then we have $e\tilde{\mathcal{L}}e_1 = fe\omega f$ and $ye_1 = yfe = ye = x$.

Conversely, suppose that $x = ye$, where e is an idempotent in $\tilde{L}_x$ and $e\omega f$ for some idempotent $f \in \tilde{L}_y$. Then $e = ef$. By Lemma 1, we have $\tilde{L}_e = \tilde{L}_{ef} \leq \tilde{L}_f$, and then $x \leq_l y$.

Proposition 7. The order $\leq_l$ and $\leq_r$ coincide on $Reg(S)$.

Proof. Let $a, b \in Reg(S)$ with $a \leq_r b$. Pick an idempotent f with $f\tilde{\mathcal{R}}b$. By Proposition 6, there exists an idempotent $e \in \tilde{R}_a$, $e\omega f$ and $a = eb$. Choose an idempotent g with $g\tilde{\mathcal{L}}b$. Since $b \in Reg(S)$, we have $g\mathcal{L}b$. But then $f\mathcal{D}b$ and D_g is a regular $\mathcal{D}-$ class, and so there exists $b' \in V(b)$ with $b'b = g$. Further, by $bb'\mathcal{R}b$ we have $bb'\tilde{\mathcal{R}}b$. By Proposition 6 again, there exists an idempotent e with $e\tilde{\mathcal{R}}a$, and such that $e\omega bb'$ and $a = eb$. Put $e_1 = b'eb$. Then $e_1\tilde{\mathcal{L}}a$. In fact, if $e_1 h = e_1$ for any $h \in E(S)$, then $b'ebh = b'eb, bb'ebh = bb'eb = eb$, i.e., $e_1 h = a$, also, we can easily check that $e_1 \omega b'b = g$ and $be_1 = (bb')eb = eb = a$. And so we have $a \leq_l b$.

In the following, we call a non-zero element of a semiabundant semigroup S primitive if it is minimal amongst the non-zero elements of S with respect to $\leq$. Since the restriction of $\leq$ to $E(S)$ is ω, the definition coincides with the usual definition when it is applied to the idempotents.

Proposition 8. A semiabundant semigroup is primitive with respect to ω if and only if it is primitive with respect to $\leq$.

Proof. $\Rightarrow$) Suppose that every non-zero idempotent is minimal in the set of non-zero idempotents, and let x, y be two non-zero elements of S with $x \leq y$. Then for each idempotent f, with $f \in \tilde{L}_y$, by Proposition 6, there exists an idempotent e with $e \in \tilde{L}_x$, $e\omega f$ and $x = ye$. But by the primitivity of the idempotents, $e\omega f$ implies that $e = f$. Hence $x = ye = yf = y$.

$\Leftarrow$) It is clear.

Now, by Proposition 8, we can immediately obtain the following corollary:

Corollary 3. A semiabundant semigroup without zero is primitive if and only if the natural partial order is the identity relation.

Lemma 3. Let U be a semiabundant subsemigroup of semigroup S such that the idempotents of U form an order ideal of S. Then

$$\tilde{\mathcal{L}}(U) = \tilde{\mathcal{L}}(S) \cap (U \times U), \tilde{\mathcal{R}}(U) = \tilde{\mathcal{R}}(S) \cap (U \times U).$$

Proof. We only prove that $\tilde{\mathcal{L}}(U) = \tilde{\mathcal{L}}(S) \cap (U \times U)$, the dual can be similarly verified.

Firstly, it is clear that $\tilde{\mathcal{L}}(U) \supseteq \tilde{\mathcal{L}}(S) \cap (U \times U)$. Next, we will show that $\tilde{\mathcal{L}}(U) \subseteq \tilde{\mathcal{L}}(S) \cap (U \times U)$.

Let $a \in U$. Since U and S are semiabundant, there are idempotents e, g in U and S respectively with $a\tilde{\mathcal{L}}(U)e$, $a\tilde{\mathcal{L}}(S)g$. Since $ae = a = ag$, we have $ge = g$. Hence eg is an idempotent and $eg \leq e$ so that $eg \in U$. But $eg\mathcal{L}g$, so that $eg\tilde{\mathcal{L}}(S)a$, and from above, this gives $eg\tilde{\mathcal{L}}(U)a$. And then,we have $eg\mathcal{L}e$. Now, if $a, b \in U$ and $a\tilde{\mathcal{L}}(U)b$, then for some idempotent $e \in U$, a, b are $\tilde{\mathcal{L}}-$ related in U and hence are $\tilde{\mathcal{L}}-$ related in S to e. Thus, $a\tilde{\mathcal{L}}(S)b$ as required.

Proposition 9. Let S be a semiabundant semigroup and let U be a semiabundant subsemigroup with $E(U)$ an order ideal of $E(S)$. Then

(i) for $x, y \in U$, if $x \leq y$ in U, then $x \leq y$ in S;

(ii) for $x, y \in U$, if $x \leq y$ in S, then $x \leq y$ in U.

Proof. (i) It is clear.

(ii) By Lemma 3, U has the property of that $\tilde{\mathcal{L}}(U) = \tilde{\mathcal{L}}(S) \cap (U \times U), \tilde{\mathcal{R}}(U) = \tilde{\mathcal{R}}(S) \cap (U \times U)$. Suppose that $x, y \in U$ with $x \leq y$ in S. Then by Proposition 6, for each idempotent $y^* \in \tilde{L}_y$, there exists an idempotent $x^* \in \tilde{L}_x$ such that $x^* \omega y^*$ and $x = yx^*$. Since U is semiabundant, there is an idempotent f in U with $f\tilde{\mathcal{L}}(U)y$. By the property above, this means that we may take $y^* = f$ in S with $x^* \omega f$ and $x = yx^*$ for some idempotent x^* in $\tilde{L}_x$. This means in particular that x^* actually belongs to U. Notice that $\tilde{\mathcal{L}}(U) = \tilde{\mathcal{L}}(S) \cap (U \times U)$, we immediately have that $x^* \tilde{\mathcal{L}}(U)x$. Hence, we have shown that if $x \leq_l y$ in S, then $x \leq_l y$ in U. Similarly, we can show that if $x \leq_r y$ in S, then $x \leq_r y$ in U. Hence, (ii) holds.

Proposition 10. Let S be a semiabundant semigroup and let U be a semiabundant subsemigroup. Then $E(U)$ is an order ideal of $E(S)$ if and only if U is an order ideal of S with respect to the natural partial order.

Proof. $\Leftarrow$) It is clear.

$\Rightarrow$) Suppose that $E(U)$ is an order ideal of $E(S)$. Let y be an element of U and let x be an element of S with $x \leq y$. Choose an idempotent f in U with $f\tilde{\mathcal{L}}(U)y$. By Lemma 3, we have $f\tilde{\mathcal{L}}(S)y$. Consequently, there exists an idempotent e with $e\tilde{\mathcal{L}}(S)x$, $e\omega f$ and $x = ye$. But $e \in E(U)$ and $E(U)$ is an order ideal of $E(S)$, we have that $x = ye \in U$. Therefore, U is an order ideal of S with respect to the natural partial order.

Proposition 11. Let S be semiabundant with $\tilde{\mathcal{L}}$ a right congruence and $\tilde{\mathcal{R}}$ a left congruence. Then for elements x and y of S, $x \leq y$ if and only if there are idempotents e and f of S such that $x = ey = yf$.

Proof. $\Rightarrow$) It is clear.

$\Leftarrow$) Suppose that $x = ey = yf$. By $x = ey$ and Lemma 1, we have $\tilde{L}_x \leq \tilde{L}_y$. Choose an idempotent $x^* \in \tilde{L}_x$ such that $x = xx^* = yfx^*$. By $x = ey$, we also have $xf = x$, and then we can get $x^*f = x^*$. This means that x^*f is an idempotent. Since $\tilde{\mathcal{L}}$ is a right congruence, we deduce that $x^*f \in \tilde{L}_x$. Hence, $x \leq_l y$. Similarly, we can show that $x \leq_r y$ by the fact of that $\tilde{\mathcal{R}}$ is a left congruence. And so we have $x \leq y$.

Finally, we shall establish a necessary condition for the natural partial order $\leq = \leq_l \cap \leq_r$ to be compatible with the multiplication on a semiabundant semigroup S. The following notations are useful.

Definition 3. A semiabundant semigroup S is called semiadequate if all of its idempotents $E(S)$ forms a semilattice.

In a semiadequate semigroup S, because $E(S)$ forms a semilattice, we can get each $\tilde{\mathcal{L}}-$

class and each $\tilde{\mathcal{R}}-$ class only contains a unique idempotent.

Recall that in a semigroup S, subsemigroups of the form eSe where e is an idempotent are called local submonoids. In particular, if each local submonoid is semiadequate, then S is said to be locally semiadequate.

Lemma 4[10]**.** If U is a regular subsemigroup of a semigroup S and $a, b \in U$, then $(a, b) \in \mathcal{L}(U)$ if and only if $(a, b) \in \mathcal{L}(S)$.

In the above Lemma, $\mathcal{L}$ may be replaced by $\tilde{\mathcal{L}}$ since a, b are regular elements.

Theorem 1. Let S be a semigroup and T the set of regular elements in S. Then the following conditions are equivalent :

(1) S is semiadequate;

(2) T is an inverse subsemigroup of S and $E(S)$ has non-empty intersection with each $\tilde{\mathcal{L}}-$ class and each $\tilde{\mathcal{R}}-$ class of S;

(3) T is an inverse subsemigroup of S and T has non-empty intersection with each $\tilde{\mathcal{L}}-$ class and each $\tilde{\mathcal{R}}-$ class of S;

(4) each $\tilde{\mathcal{L}}-$ class and each $\tilde{\mathcal{R}}-$ class of S contains a unique idempotent and the subsemigroup generated by $E(S)$ is regular.

Proof. (1) $\Rightarrow$ (2) If (1) holds, then by the definition of semiadequate semigroup, we have each $\tilde{\mathcal{L}}-$ class and each $\tilde{\mathcal{R}}-$ class of S contains an idempotent. Since $E(S)$ is a subsemilattice of T and by Section 7.1 Exercise 1 in [9], we obtain that T is an inverse subsemigroup of S.

(2) $\Rightarrow$ (3) It is clear, since $E(S) \subseteq T$.

(3) $\Rightarrow$ (4) If (3) holds, then $E(S)$ is a semilattice.If L is an $\tilde{\mathcal{L}}-$ class of S and $a \in L \cap T$, then in T and by Lemma 4, in S also, a is $\mathcal{L}-$ related to $a^{-1}a$ where a^{-1} is the inverse of a. Thus, each $\tilde{\mathcal{L}}-$ class of S contains an idempotent. Similarly, we can show that each $\tilde{\mathcal{R}}-$ class of S contains an idempotent. The uniqueness follows from Lemma 4 and the fact that T is inverse.

(4) $\Rightarrow$ (1) If (4) holds, to prove (1) we only need to show that $E(S)$ is a semilattice. In fact, since the subsemigroup $< E(S) >$ generated by $E(S)$ is regular, by Lemma 4, we have each $\mathcal{L}-$ class and each $\mathcal{R}-$ class of $< E >$ contains a unique idempotent. Thus, $< E >$ is an inverse semigroup so that $E(< E(S) >) = E(S)$, and then $E(S)$ is a semilattice.

Proposition 12. Let S be semiabundant.Then each local submonoid of S is semiabundant.

Proof. Let a be an element of the local submonoid eSe and let f be an idempotent of S with $f\tilde{\mathcal{L}}(S)a$. Clearly, we have $ae = a$, and then $fe = f$. Hence, $ef \in E(S)$, $ef\omega e$ and $ef\mathcal{L}f$. At this time, it is not hard to check that $E(eSe)$ is an order ideal of $E(S)$ and the element ef belongs to $E(eSe)$. Since $ef\tilde{\mathcal{L}}(S)a$, we have $ef\tilde{\mathcal{L}}(eSe)a$. This implies that each element of eSe is $\tilde{\mathcal{L}}-$related to an idempotent in eSe likewise belonging to eSe. Similarly, we can show that each $\tilde{\mathcal{R}}-$ class of eSe contains an idempotent. Hence, eSe is semiabundant.

By Proposition 12, each local submonoid of the semiabundant semigroup S is semiabundant. Now we set about to give a necessary condition for the natural partial order $\leq = \leq_l \cap \leq_r$ to be compatible with the multiplication on a semiabundant semigroup S. In other words, we begin to give a sufficiency for the submonoid of a semiabundant semigroup S to be semiadequate.

Theorem 2. If the natural partial order on semiabundant semigroup S is compatible with

the multiplication, then the semigroup S is locally semiadequate.

Proof. Let $e, f, h \in E(S)$ with $f \leq e$ and $h \leq e$. By the assumption that $\leq$ is compatible with the multiplication, we have $fh \leq e^2 = e$. According to Proposition 5 (1), the element fh is therefore an idempotent and so $fh\omega e$. This shows that the idempotents of each local submonoid form a band.

Let u, v be any two idempotents in the local submonoid eSe. And suppose that in addition, $u\tilde{\mathcal{L}}v$ in eSe. From the fact that $u\omega e$ and $v\omega e$, we have both $u \leq e$ and $v \leq e$. Since idempotents are regular, $u\mathcal{L}v$, we can also get $uv = u, vu = v$. But then applying the compatibility of $\leq$, we have $u = uv \leq ev = v$ and $v = vu \leq eu = u$. Hence, $u = v$. And we have shown that each local submonoid is a semiabundant semigroup in which each $\tilde{\mathcal{L}}-$ class and each $\tilde{\mathcal{R}}-$ class contains a unique idempotent. Also, we can show that $< E(eSe) >= E(eSe)$. In fact, if we let $g = uv \in < E(eSe) >$, where $u, v \in E(eSe)$, then $u \leq e$ and $v \leq e$, applying the compatibility of $\leq$, we have $uv \leq e$, by Proposition 5, we immediately have that $g = uv \in E(eSe)$. Hence, by Theorem 1, the local submonoid is semiadequate. Our proof is completed.

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On the Smarandache ceil function and the Dirichlet divisor function [1]

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Abstract In this paper we shall study several mean values of the composite function involving Dirichlet divisor function and Smarandache ceil function by the convolution method.

Keywords Smarandache Ceil function, Dirichlet divisor function, asymptotic formula, convolution method.

§1. Introduction

For a fixed positive integer k and any positive integer n, the Smarandache ceil function $S_k(n)$ is defined as

$$\{S_k(n) = \min m \in N : n|m^k\}.$$

This function was introduced by Professor Smarandache. About this function, many scholars studied its properties. Ibstedt [2] presented the following property: $(\forall a, b \in N)(a, b) = 1 \Longrightarrow S_k(ab) = S_k(a)S_k(b)$. It is easy to see that if $(a, b) = 1$, then $(S_k(a), S_k(b)) = 1$.

In her thesis, Ren Dongmei [4] proved the asymptotic formular

$$\sum_{n\leq x} d(S_k(n)) = c_1 x \log x + c_2 x + O(x^{1/2+\epsilon}), \tag{1}$$

where c_1 and c_2 are computable constants, and ε is any fixed positive number.

The aim of this short note is to prove the following

Theorem. Let $d_3(n)$ denote the Piltz divisor function of dimensional 3, then for any real number $x \geq 2$, we have

$$\sum_{n\leq x} d_3(S_k(n)) = xP_{2,k}(\log x) + O(x^{1/2}e^{-c\delta(x)}), \tag{2}$$

where $P_{2,k}(\log x)$ is a polynomial of degree 2 in $\log x$, $\delta(x) := \log^{\frac{3}{5}} x(\log\log x)^{-\frac{1}{5}}$, $c > 0$ is an absolute constant.

Remark. The estimate $O(x^{1/2+\epsilon})$ in (1) can also be improved to $O(x^{1/2}e^{-c\delta(x)})$ by a similar approach.

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§2. Proof of the theorem

In order to prove our theorem, we need the following two lemmas .

Lemma 1. Let $f(n)$ be an arithmetical function for which

$$\sum_{n\le x} f(n) = \sum_{j=1}^{l} x^{a_j} P_j(\log x) + O(x^a), \sum_{n\le x} |f(n)| = O(x^{a_1}\log^r x),$$

where $a_1 \ge a_2 \ge \cdots \ge a_l > 1/c > a \ge 0$, $r \ge 0, P_1(t), \cdots, P_l(t)$ are polynomials in t of degrees not exceeding r, and $c \ge 1$ and $b \ge 1$ are fixed integers. Suppose for $\Re s > 1$ that

$$\sum_{n=1}^{\infty} \frac{\mu_b(n)}{n^s} = \frac{1}{\zeta^b(s)}.$$

If $h(n) = \sum_{d^c|n} \mu_b(d) f(n/d^c)$, then

$$\sum_{n\le x} h(n) = \sum_{j=1}^{l} x^{a_j} R_j(\log x) + E_c(x), \tag{3}$$

where $R_1(t)\cdots R_l(t)$ are polynomials in t of degrees not exceeding r, and for some $D > 0$

$$E_c(x) \ll x^{1/c} exp(-D(\log x)^{3/5}(\log\log x)^{-1/5}). \tag{4}$$

Proof. If $b = 1$, Lemma 1 is Theorem 14.2 of Ivić[3]. When $b \ge 2$, Lemma 1 can be proved by the same approach.

Lemma 2. Let $f(m), g(n)$ be arithmetical functions such that

$$\sum_{m\le x} f(m) = \sum_{j=1}^{J} x^{\alpha_j} P_j(\log x) + O(x^\alpha), \qquad \sum_{n\le x} |g(n)| = O(x^\beta),$$

where $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_J > \alpha > \beta > 0$, where $P_j(t)$ is polynomial in t. If $h(n) = \sum_{n=md} f(m)g(d)$, then

$$\sum_{n\le x} h(n) = \sum_{j=1}^{J} x^{\alpha_j} Q_j(\log x) + O(x^\alpha), \tag{5}$$

where $Q_j(t)(j = 1, \cdots, J)$ are polynomials in t.

Now we prove our theorem, which is closely related to the Piltz divisor problem. Let $\Delta_3(x)$ denotes the error term in the asymptotic formula for $\sum_{n\le x} d_3(n)$. We know that

$$D_3(x) = \sum_{n\le x} d_3(n) = xH_3(\log x) + \Delta_3(x), \tag{6}$$

where $H_3(u)$ is a polynomial of degree 2 in u. For the upper bound of $\Delta_3(x)$, Kolesnik[1] proved that

$$\Delta_3(x) \ll x^{\frac{43}{96}+\epsilon}. \tag{7}$$

Let $f(s)=\sum_{n=1}^{\infty}\frac{d_3(S_k(n))}{n^s}$ $(Res>1)$. By the Euler product formula we get for $\Re s>1$ that

$$\begin{aligned} f(s) &= \prod_p(1+\frac{d_3(S_k(p))}{p^s}+\frac{d_3(S_k(p^2))}{p^{2s}})\frac{d_3(S_k(p^3))}{p^{3s}}+\cdots \\ &= \prod_p(1+\frac{3}{p^s}+\frac{3}{p^{2s}}+\frac{3}{p^{3s}}+\cdots) \\ &= \prod_p(1+\frac{3}{p^s}+\frac{3}{p^{2s}}+\frac{9}{p^{3s}})G_{l,k}(s) \\ &= \prod_p(1+\frac{3}{p^s}+\frac{3}{p^{2s}}(1+\frac{3}{p^s}))G_k(s) \\ &= (1-p^{-s})^{-3}(1-p^{-2s})^6(1-p^{-2s})^{-3}G_k(s) \\ &= (1-p^{-s})^{-3}(1-p^{-2s})^3G_k(s) \\ &= \frac{\zeta^3(s)}{\zeta^3(2s)}G_k(s). \end{aligned} \tag{8}$$

It is easy to prove that $G_k(s)$ is absolutely convergent for $Res>1/3$.

Let $\zeta^3(s)G_k(s)=\sum_{n=1}^{\infty}\frac{f_k(n)}{n^s}$ $(Rs>1)$. By Lemma 2 and (2.4) we can get

$$\sum_{n\le x}f_k(n)=xM_3(\log x)+O(x^{\frac{43}{96}+\epsilon}), \tag{9}$$

where $M_3(u)$ is a polynomial of degree 2 in u. Then we can get

$$\sum_{n\le x}|f_k(n)|\ll x\log^2 x. \tag{10}$$

We konw $\frac{1}{\zeta^3(s)}=\sum_{n=1}^{\infty}\frac{\mu_3(d)}{d^s}$ $(Rs>1)$. From (8) we have the relation

$$d_3(S_k(n))=\sum_{n=md^2}f_k(m)\mu_3(d). \tag{11}$$

Now Theorem follows from (9)-(11) with the help of Lemma 1.

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On fuzzy number valued Choquet integral

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Abstract This paper first introduces the fuzzy number valued Choquet integral on crisp sets with respect to fuzzy number valued fuzzy measure, Its properties are obtained, Then it introduces the fuzzy number valued Choquet integral on fuzzy sets with respect to fuzzy number valued fuzzy measure, All these are generalizations of Choquet integral.

Keywords Fuzzy number, fuzzy measure, fuzzy integral.

§1. Introduction

After the formulation of fuzzy integral by Sugeno, various generalizations of fuzzy integral were introduced and investigated. Fuzzy number fuzzy integral (FNFI) were defined by various authors in [3], [5] and [6].

Zhang Guang-Quan [5] used the concept of Sugeno's fuzzy integral as λ-cuts to define the fuzzy number valued fuzzy integral. He defined it as

$$\int_{\bar{A}} f \mathrm{d}\mu = \bigcup_{\lambda\in[0,1]} \lambda[\sup_{\alpha\in[0,\infty]} (\alpha \wedge \mu(\chi_{F_\alpha} \cap \bar{A})^-_\lambda), \sup_{\alpha\in[0,\infty]} (\alpha \wedge \mu(\chi_{F_\alpha} \cap \bar{A})^+_\lambda)].$$

Leechay Jang et al [3], defined fuzzy number valued fuzzy Choquet integral as the Choquet integral of fuzzy number valued function. But the concepts in [3] are all based on the interval-valued Choquet integrals.

We in this paper define the fuzzy number valued Choquet integral that is neither based on interval-valued Choquet integrals nor fuzzy valued functions. The properties are then investigated. Fuzzy number valued Choquet integral has many applications as indicated in [3]. For the basic definitions that are relevant to fuzzy number, the reader may refer [2].

§2. Definition and properties

Definition 2.1. Let (X, Ω)be a measurable space where Ω is a non-empty class of subsets of X. A fuzzy number valued fuzzy measure (FNFM) μ on X is a set function μ: $\Omega \to F'_+$ where F'_+ is the class of all fuzzy numbers in R_0^+ with the following properties.

(1) $\mu\phi = \bar{0}$;

(2) $A, B \in \Omega, A \subseteq B \Rightarrow \mu A \leq \mu B$.

Definition 2.2. A FNFM μ is said to be continuous above if for E_i $(i = 1, 2, \cdots) \in \Omega$ with $E_1 \subset E_2 \cdots$ and $\bigcup_{n=1}^{\infty} E_n \in \Omega$, $lim\ \mu E_n = \mu(\bigcup_{n=1}^{\infty} E_n)$. Similarly A FNFM μ is said to be continuous below if for E_i $(i = 1, 2, \cdots) \in \Omega$ with $E_1 \supset E_2 \cdots$ and $\bigcap_{n=1}^{\infty} E_n \in \Omega$ with $\mu(E_1)$ is finite , $lim\ \mu E_n = \mu(\bigcap_{n=1}^{\infty} E_n)$.

A FNFM which is both continuous above and continuous below is called continuous.

Definition 2.3. (Let (X, Ω, μ) be a fuzzy number valued fuzzy measure space where μ is continuous. The fuzzy number valued Choquet integral of a measurable function f with respect to μ on a crisp subset A of X is defined as

$$(C)\int_A f\mathrm{d}\mu = \bigcup_{\lambda\in[0,1]} \lambda[\int_0^{\infty} \mu(F_\alpha \cap A)_\lambda^- \mathrm{d}\alpha, \int_0^{\infty} \mu(F_\alpha \cap A)_\lambda^+ \mathrm{d}\alpha]$$

where $F_\alpha = \{x : f(x) \geq \alpha\}$ and the integrals on the right side represent Lebesgue integrals.

Proposition 2.1.

(i) If $\mu A = \bar{0}$ then $(C)\int_A f\mathrm{d}\mu = \bar{0}$ for any f.

(ii) If $(C)\int_A f\mathrm{d}\mu = \bar{0}$ then $\mu(F_{0+} \cap A) = \bar{0}$.

(iii) If $f_1 \leq f_2$ then $(C)\int_A f_1\mathrm{d}\mu \leq (C)\int_A f_2\mathrm{d}\mu$.

(iv) $(C)\int_A f\mathrm{d}\mu = (C)\int_X f\chi_A\mathrm{d}\mu$ where χ_A is the characteristic function of A.

(v) $(C)\int_A a\mathrm{d}\mu = a\mu(A)$ for any constant $a \in [0, \infty)$.

(vi) If $A \subseteq B$ then $(C)\int_A f\mathrm{d}\mu \leq (C)\int_B f\mathrm{d}\mu$.

(vii) $(C)\int_A (f_1 \vee f_2)\mathrm{d}\mu \geq (C)\int_A f_1\mathrm{d}\mu \vee (C)\int_A f_2\mathrm{d}\mu$.

(viii) $(C)\int_A (f_1 \wedge f_2)\mathrm{d}\mu \leq (C)\int_A f_1\mathrm{d}\mu \wedge (C)\int_A f_2\mathrm{d}\mu$.

(ix) $(C)\int_{A\cup B} f\mathrm{d}\mu \geq (C)\int_A f\mathrm{d}\mu \vee (C)\int_B f\mathrm{d}\mu$.

(x) $(C)\int_{A\cap B} f\mathrm{d}\mu \leq (C)\int_A f\mathrm{d}\mu \wedge (C)\int_B f\mathrm{d}\mu$.

Proof.

(i) If $\mu A = \bar{0}$ then $\mu(F_\alpha \cap A) = \bar{0}$ because of monotonicity of μ.

Hence

$$(C)\int_A f\mathrm{d}\mu = \bigcup_{\lambda\in[0,1]} \lambda[\int_0^{\infty} 0\mathrm{d}\alpha, \int_0^{\infty} 0\mathrm{d}\alpha] = \bar{0}.$$

(ii) Let $(C)\int_A f\mathrm{d}\mu = \bar{0} \Rightarrow \int_0^{\infty} \mu(F_\alpha \cap A)_\lambda^- \mathrm{d}\alpha = \int_0^{\infty} \mu(F_\alpha \cap A)_\lambda^+ \mathrm{d}\alpha = 0$ for any $\lambda \in [0, 1]$

Suppose $\mu(F_{0+} \cap A) = \bar{c} > \bar{0}$. As $F_{\frac{1}{n}} \cap A \nearrow F_0 \cap A$, by continuity from below of μ, $(\bar{\rho})lim\ \mu(F_{\frac{1}{n}} \cap A) = \mu(F_0 \cap A) = \bar{c}$. Hence by theorem 2.2 of [5] there exists a $\lambda_0 \in [0, 1]$ such that $lim\ \mu(F_{\frac{1}{n}} \cap A) = c_{\lambda_0}^- > 0$. Hence there exists a n_0 such that $\mu(F_{\frac{1}{n_0}} \cap A)_{\lambda_0}^- > \frac{c_{\lambda_0}^-}{2}$

$$\Rightarrow \int_0^{\infty} \mu(F_\alpha \cap A)_\lambda^- \mathrm{d}\alpha, \int_0^{\infty} \mu(F_\alpha \cap A)_\lambda^+ \mathrm{d}\alpha \geq \int_0^{\infty} \mu(F_{\frac{1}{n_0}} \cap A)_\lambda^- \mathrm{d}\alpha > \int_0^{\infty} \frac{c_{\lambda_0}^-}{2}\mathrm{d}\alpha > 0$$

$\Rightarrow (C)\int\limits_A f\mathrm{d}\mu \neq \bar{0}$, a contradiction. Hence the result.

(iii) Let $F_\alpha = \{x : f_1(x) \geq \alpha\}$ and $F_\alpha^{'} = \{x : f_2(x) \geq \alpha\}$
As $f_1 \leq f_2, F_\alpha \subseteq F_\alpha^{'}$ for all $\alpha \in [0, \infty]$.
Hence $\mu(F_\alpha \cap A) \leq \mu(F_\alpha^{'} \cap A)$ and hence $\mu(F_\alpha \cap A)_\lambda^- \leq \mu(F_\alpha^{'} \cap A)_\lambda^-$ and
$\mu(F_\alpha \cap A)_\lambda^+ \leq \mu(F_\alpha^{'} \cap A)_\lambda^+$
$\Rightarrow \int\limits_0^\infty \mu(F_\alpha \cap A)_\lambda^- \mathrm{d}\alpha \leq \int\limits_0^\infty \mu(F_\alpha^{'} \cap A)_\lambda^- \mathrm{d}\alpha$ and $\int\limits_0^\infty \mu(F_\alpha \cap A)_\lambda^+ \mathrm{d}\alpha \leq \int\limits_0^\infty \mu(F_\alpha^{'} \cap A)_\lambda^+ \mathrm{d}\alpha$
from which the result follows.

(iv) Let $F_\alpha = \{x : f(x) \geq \alpha\}$ and $J_\alpha = \{x : f(x)\chi_A(x) \geq \alpha\}$
When $\alpha \in [0, \infty), x \in J_\alpha, f(x)\chi_A(x) \geq \alpha \Rightarrow f(x) \geq \alpha \Rightarrow x \in F_\alpha$
Therefore $F_\alpha \cap A = J_\alpha$ and hence $\mu(F_\alpha \cap A) = \mu(J_\alpha) = \mu(J_\alpha \cap X)$

$$(C)\int\limits_A f\mathrm{d}\mu = \bigcup_{\lambda\in[0,1]} \lambda[\int\limits_0^\infty \mu(F_\alpha \cap A)_\lambda^- \mathrm{d}\alpha, \int\limits_0^\infty \mu(F_\alpha \cap A)_\lambda^+ \mathrm{d}\alpha] =$$

$$\bigcup_{\lambda\in[0,1]} \lambda[\int\limits_0^\infty \mu(J_\alpha \cap X)_\lambda^- \mathrm{d}\alpha, \int\limits_0^\infty \mu(J_\alpha \cap X)_\lambda^+ \mathrm{d}\alpha] = (C)\int\limits_X f\chi_A \mathrm{d}\mu.$$

(v) As $F_\alpha = \{x : f(x) \geq \alpha\}, F_\alpha$ is X when $a \geq \alpha$ and is ϕ when $a \leq \alpha$,

$$(C)\int\limits_A a\mathrm{d}\mu = \bigcup_{\lambda\in[0,1]} \lambda[\int\limits_0^a \mu(A)_\lambda^- \mathrm{d}\alpha + \int\limits_a^\infty 0\mathrm{d}\alpha, \int\limits_0^a \mu(A)_\lambda^+ \mathrm{d}\alpha + \int\limits_a^\infty 0\mathrm{d}\alpha]$$

$$= \bigcup_{\lambda\in[0,1]} \lambda[a\mu(A)_\lambda^-, a\mu(A)_\lambda^+] = a\mu(A).$$

(vi) As $F_\alpha \cap A \subseteq F_\alpha \cap B$, we have

$$\mu(F_\alpha \cap A) \leq \mu(F_\alpha \cap B) \Rightarrow (C)\int\limits_A f\mathrm{d}\mu = \bigcup_{\lambda\in[0,1]} \lambda[\int\limits_0^\infty \mu(F_\alpha \cap A)_\lambda^- \mathrm{d}\alpha, \int\limits_0^\infty \mu(F_\alpha \cap A)_\lambda^+ \mathrm{d}\alpha]$$

$$\leq \bigcup_{\lambda\in[0,1]} \lambda[\int\limits_0^\infty \mu(F_\alpha \cap B)_\lambda^- \mathrm{d}\alpha, \int\limits_0^\infty \mu(F_\alpha \cap B)_\lambda^+ \mathrm{d}\alpha] = (C)\int\limits_B f\mathrm{d}\mu.$$

(vii) Let $M_\alpha = \{x : (f_1 \vee f_2)(x) \geq \alpha\}$
$F_\alpha^{'} = \{x : f_1(x) \geq \alpha\}$

$F_\alpha^2 = \{x : f_2(x) \geq \alpha\}$
Clearly $M_\alpha = F_\alpha^{'} \cup F_\alpha^2$
Therefore $\int\limits_0^\infty \mu(M_\alpha \cap A)_\lambda^- \mathrm{d}\alpha \geq \int\limits_0^\infty \mu(F_\alpha^{'} \cap A)_\lambda^- \mathrm{d}\alpha$ and $\int\limits_0^\infty \mu(M_\alpha \cap A)_\lambda^+ \mathrm{d}\alpha \geq \int\limits_0^\infty \mu(F_\alpha^{'} \cap A)_\lambda^+ \mathrm{d}\alpha$
$\Rightarrow (C)\int\limits_A (f_1 \vee f_2)\mathrm{d}\mu \geq (C)\int\limits_A f_1 \mathrm{d}\mu$
Similarly $(C)\int\limits_A (f_1 \vee f_2)\mathrm{d}\mu \geq (C)\int\limits_A f_2 \mathrm{d}\mu$.
Hence the result.

(viii) By setting $m_\alpha = \{x : (f_1 \wedge f_2)(x) \geq \alpha\}$,the proof of the result is made analogously to that of (vii).

(ix) As $\mu(F_\alpha \cap A), \mu(F_\alpha \cap B) \leq \mu(F_\alpha \cap (A \cup B))$, we have $(C)\int\limits_{A\cup B} f\mathrm{d}\mu \geq (C)\int\limits_A f\mathrm{d}\mu, (C)\int\limits_B f\mathrm{d}\mu$ and hence $(C)\int\limits_{A\cup B} f\mathrm{d}\mu \geq (C)\int\limits_A f\mathrm{d}\mu \vee (C)\int\limits_B f\mathrm{d}\mu$.

(x) The proof is analogous to that of (ix).

Definition 2.3. Let (X, Ω, μ) be a fuzzy number valued fuzzy measure space. Suppose that $f : X \to [0, \infty)$ and $h : X \to [0, 1]$ is a fuzzy set on X. If h is measurable, then the fuzzy number valued Choquet integral of f on the fuzzy set h is defined as

$$(C)\int_h f\mathrm{d}\mu = (C)\int_X (hf)\mathrm{d}\mu$$

The following results are immediate.

Proposition 2.2.

(i) $(C)\int_h a\mathrm{d}\mu = a(C)\int_X h\mathrm{d}\mu$ for $a \geq 0$.

(ii) If $f : X \to [0, 1]$ is measurable then $(C)\int_{\chi_A} f\mathrm{d}\mu = (C)\int_A f\mathrm{d}\mu$.

(iii) If $h_1 \leq h_2$ then $(C)\int_{h_1} f\mathrm{d}\mu \leq (C)\int_{h_2} f\mathrm{d}\mu$.

(iv) If $f_1 \leq f_2$ then (C)$\int_h f_1\mathrm{d}\mu \leq (C)\int_h f_2\mathrm{d}\mu$.

(v)$(C)\int_h (f_1 \vee f_2)\mathrm{d}\mu \geq (C)\int_h f_1\mathrm{d}\mu \vee (C)\int_h f_2\mathrm{d}\mu$.

(vi) $(C)\int_h (f_1 \wedge f_2)\mathrm{d}\mu \leq (C)\int_h f_1\mathrm{d}\mu \wedge (C)\int_h f_2\mathrm{d}\mu$

(vii) $(C)\int_{h_1 \vee h_2} f\mathrm{d}\mu \geq (C)\int_{h_1} f\mathrm{d}\mu \ \vee (C)\int_B f\mathrm{d}\mu$.

(viii) $(C)\int_{h_1 \wedge h_2} f\mathrm{d}\mu \leq (C)\int_{h_1} f\mathrm{d}\mu \ \wedge (C)\int_{h_2} f\mathrm{d}\mu$.

By (ii) of above proposition it is clear that the definition 2.3 is well defined.

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The Smarandache bisymmetric arithmetic determinant sequence

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Abstract Murthy [1] introduced the Smarandache bisymmetric arithmetic determinant sequence. In this paper, we derive the sum of the first n terms of the sequence.

Keywords The Smarandache bisymmetric arithmetic determinant sequence, the $n-th$ term, the sum of the first n terms.

§1. Introduction and results

The Smarandache bisymmetric arithmetic determinant sequence (SBADS), introduced by Murthy [1], is defined as follows.

Definition 1. The Smarandache bisymmetric arithmetic determinant sequence, {SBADS(n)}, is

$$\left\{a,\ \begin{vmatrix} a & a+d \\ a+d & a \end{vmatrix},\ \begin{vmatrix} a & a+d & a+2d \\ a+d & a+2d & a+d \\ a+2d & a+d & a \end{vmatrix},\ \ldots\right\}.$$

The following result is due to Majumdar [2].

Theorem 1. Let a_n be the $n-th$ term of the Smarandache bisymmetric arithmetic determinant sequence. Then,

$$a_n = \begin{vmatrix} a & a+d & a+2d & \cdots & a+(n-3)d & a+(n-2)d & a+(n-1)d \\ a+d & a+2d & a+3d & \cdots & a+(n-2)d & a+(n-1)d & a+(n-2)d \\ a+2d & a+3d & a+4d & \cdots & a+(n-1)d & a+(n-2)d & a+(n-3)d \\ \vdots & & & & & & \\ a+(n-3)d & a+(n-2)d & a+(n-1)d & \cdots & a+4d & a+3d & a+2d \\ a+(n-2)d & a+(n-1)d & a+(n-2)d & \cdots & a+3d & a+2d & a+d \\ a+(n-1)d & a+(n-2)d & a+(n-3)d & \cdots & a+2d & a+d & a \end{vmatrix}$$

$$= (-1)^{\left[\frac{n}{2}\right]}\left(a+\tfrac{n-1}{2}d\right)(2d)^{n-1}.$$

Let $\{S_n\}$ be the sequence of $n-th$ partial sums of the sequence $\{a_n\}$, so that

$$S_n = \sum_{k=1}^{n} a_k, n \geqslant 1.$$

This paper gives explicit expressions for the sequence $\{S_n\}$. This is given in Theorem 3.1 in Section 3. In Section 2, we give some preliminary results that would be necessary for the proof of the theorem. We conclude this paper with some remarks in the final section, Section 4.

§2. Some preliminary results

In this section, we derive some preliminary results that would be necessary in deriving the expressions of $\{S_n\}$ in the next section. These are given in the following two lemmas.

Lemma 1. For any integer $m \geq 1$,

$$\sum_{k=1,3,\cdots,(2m-1)} (2d)^{2(k-1)} = \frac{(2d)^{4m}-1}{(2d)^4-1}.$$

Proof. Since the series

$$\sum_{k=1,3,\cdots,(2m-1)} (2d)^{2(k-1)} = 1 + (2d)^4 + (2d)^8 + \cdots + (2d)^{4(m-1)}$$

is a geometric series with common ratio $(2d)^4$, the result follows.

Lemma 2. For any integer $m \geq 1$,

$$1 + 3y + 5y^2 + \cdots + (2m-1)y^{m-1} = \frac{2m}{y-1}y^m - \frac{(y+1)(y^m-1)}{(y-1)^2}.$$

Proof. Let

$$s = 1 + 3y + 5y^2 + \cdots + (2m-1)y^{m-1}. \tag{1}$$

Multiplying throughout by y, we get

$$ys = y + 3y^2 + \cdots + (2m-3)y^{m-1} + (2m-1)y^m. \tag{2}$$

Now, subtracting (1) from (2), we have

$$\begin{aligned}
(1-y)s &= 1 + 2(1 + y + y^2 + \cdots + y^{m-1}) - (2m-1)y^m \\
&= 2(1 + y + y^2 + \cdots + y^{m-1}) - 1 - (2m-1)y^m \\
&= 2\frac{y^m-1}{y-1} - 1 - (2m-1)y^m \\
&= \frac{(y+1)(y^m-1)}{y-1} - 2my^m
\end{aligned}$$

which now gives the desired result after dividing throughout by $y-1$.

§3. Main results

In this section, we derive the explicit expressions of the $n-th$ partial sums, S_n, of the Smarandache bisymmetric arithmetic determinant sequence.

From Theorem 1, we see that, for any integer $k \geq 1$,

$$a_{2k} + a_{2k+1} = (-1)^k \left[a(2d+1) - \frac{d}{2} + d(2d+1)k\right](2d)^{2k-1},$$

$$a_{2k+2} + a_{2k+3} = (-1)^{k+1} \left[a(2d+1) + \frac{d}{2}(4d+1) + d(2d+1)k\right](2d)^{2k+1},$$

so that

$$\begin{aligned} &a_{2k} + a_{2k+1} + a_{2k+3} + a_{2k+4} \\ &= (-1)^{k+1} d \left[2a(2d+1)(4d^2-1) + d(16d^3+4d^2+1) + 2d(2d+1)(4d^2-1)k\right](2d)^{2(k-1)}. \end{aligned} \tag{3}$$

Let S_n be the sum of the first n terms of the sequence, that is, let

$$S_n = a_1 + a_2 + \cdots + a_n.$$

We have the following result.

Theorem 2. For any integer $m \geq 0$,

(i)$S_{4m+1} = \dfrac{m(2d+1)}{4d^2+1}(2d)^{4m+2} + d\left[\dfrac{2a(2d+1)}{4d^2+1} - \dfrac{d(4d^2-4d-1)}{(4d^2+1)^2}\right](2d)^{4m}$
$+ \left[\dfrac{d^2(4d^2-4d-1)}{(4d^2+1)^2} - \dfrac{a(2d-1)}{4d^2+1}\right],$

(ii)$S_{4m+2} = -\dfrac{m(2d-1)}{4d^2+1}(2d)^{4m+3} - \left[\dfrac{a(2d-1)}{4d^2+1} + \dfrac{d(4d^3+3d-1)}{(4d^2+1)^2}\right](2d)^{4m+2}$
$+ \left[\dfrac{d^2(4d^2-4d-1)}{(4d^2+1)^2} - \dfrac{a(2d-1)}{4d^2+1}\right],$

(iii)$S_{4m+3} = -\dfrac{m(2d+1)}{4d^2+1}(2d)^{4(m+1)} - d\left[\dfrac{2a(2d+1)}{4d^2+1} + \dfrac{d(16d^3+4d^2+8d+3)}{(4d^2+1)^2}\right](2d)^{4m+2}$
$+ \left[\dfrac{d^2(4d^2-4d-1)}{(4d^2+1)^2} - \dfrac{a(2d-1)}{4d^2+1}\right],$

(iv)$S_{4m+4} = -\dfrac{m(2d-1)}{4d^2+1}(2d)^{4m+5} + \left[\dfrac{a(2d-1)}{4d^2+1} + \dfrac{d(12d^3-4d^2+5d-2)}{(4d^2+1)^2}\right](2d)^{4(m+1)}$
$+ \left[\dfrac{d^2(4d^2-4d-1)}{(4d^2+1)^2} - \dfrac{a(2d-1)}{4d^2+1}\right].$

Proof. To prove the theorem, we make use of Lemmas 1 and 2, as well as Theorem 1

(i) Since S_{4m+1} can be written as

$$S_{4m+1} = a_1 + a_2 + \cdots + a_{4m+1} = a_1 + \sum_{k=1,3,\cdots,(2m-1)} (a_{2k} + a_{2k+1} + a_{2k+2} + a_{2k+3}),$$

by virtue of (3), Lemma 1 and Lemma 2 (with $y = (2d)^4$), we get

$$\begin{aligned} S_{4m+1} = a + d\left[2a(2d+1)(4d^2-1) + d(16d^3+4d^2+1)\right] &\sum_{k=1,3,\cdots,(2m-1)} (-1)^{k+1}(2d)^{2(k-1)} \\ +2d^2(2d+1)(4d^2-1) &\sum_{k=1,3,\cdots,(2m-1)} (-1)^{k+1}k(2d)^{2(k-1)} \end{aligned}$$

$$= a + d\left[2a(2d+1)(4d^2-1) + d(16d^3+4d^2+1)\right]\ \frac{(2d)^{4m}-1}{(2d)^4-1}$$
$$+2d^2(2d+1)(4d^2-1)\left\{\frac{2m}{(2d)^4-1}(2d)^{4m} - \frac{(2d)^4+1}{[(2d)^4-1]^2}[(2d)^{4m}-1]\right\}$$

$$= a - d\left[\frac{2a(2d+1)}{4d^2+1} + \frac{d(16d^3+4d^2+1)}{(2d)^4-1}\right] + \frac{m(2d+1)}{4d^2+1}(2d)^{4m+2}$$
$$+d\left[\frac{2a(2d+1)}{4d^2+1} + \frac{d(16d^3+4d^2+1)}{(2d)^4-1}\right](2d)^{4m} - \frac{2d^2[(2d)^4+1]}{(2d-1)(4d^2+1)^2}[(2d)^{4m}-1].$$

In the above expression, collecting together the coefficients of $(2d)^{4m}$ as well as the constant terms, we see that coefficient of $(2d)^{4m}$ is

$$d\left[\frac{2a(2d+1)}{4d^2+1} + \frac{d(16d^3+4d^2+1)}{(2d)^4-1} - \frac{2d(16d^4+1)}{(2d-1)(4d^2+1)^2}\right]$$

$$= d\left[\frac{2a(2d+1)}{4d^2+1} - \frac{d(4d^2-4d-1)}{(4d^2+1)^2}\right],$$

constant term is

$$a - \frac{2ad(2d+1)}{4d^2+1} - \frac{d^2(16d^3+4d^2+1)}{(2d)^4-1} + \frac{2d^2(16d^4+1)}{(2d-1)(4d^2+1)^2}$$

$$= \frac{d^2(4d^2-4d-1)}{(4d^2+1)^2} - \frac{a(2d-1)}{4d^2+1}.$$

Hence, finally, we get the desired expression for S_{4m+1}.

(ii) Since $S_{4m+2} = S_{4m+1} + a_{4m+2}$, by virtue Theorem 1,

$$S_{4m+2} = S_{4m+1} + a_{4m+2} = S_{4m+1} - (a + \frac{4m+1}{2}d)(2d)^{4m+1}.$$

Now, using part (i), and noting that coefficient of $m(2d)^{4m+2}$ is

$$\frac{2d+1}{4d^2+1} - 1 = -2d\frac{2d-1}{4d^2+1},$$

coefficient of $(2d)^{4m}$ is

$$d\left[\frac{2a(2d+1)}{4d^2+1} - \frac{d(4d^2-4d-1)}{(4d^2+1)^2}\right] - 2d(a+\frac{d}{2})$$

$$= -4d^2\left[\frac{a(2d-1)}{4d^2+1} + \frac{d(4d^3+3d-1)}{(4d^2+1)^2}\right],$$

we get the desired expression for S_{4m+2}.

(iii) Since $S_{4m+3} = S_{4m+2} + a_{4m+3} = S_{4m+2} - (a + \frac{4m+2}{2}d)(2d)^{4m+2}$, using part (ii), and noting that coefficient of $m(2d)^{4m+3}$ is $-\frac{2d-1}{4d^2+1} - 1 = -2d\frac{2d+1}{4d^2+1}$, coefficient of $(2d)^{4m+2}$ is

$$-\left[\frac{a(2d-1)}{4d^2+1} + \frac{d(4d^3+3d-1)}{(4d^2+1)^2}\right] - (a+d)$$

$$= -d\left[\frac{2a(2d+1)}{4d^2+1} + \frac{d(16d^3+4d^2+8d+3)}{(4d^2+1)^2}\right],$$

we get the desired expression for S_{4m+3}.

(iv) Since $S_{4m+4} = S_{4m+3} + a_{4m+4} = S_{4m+3} + (a + \frac{4m+3}{2}d)(2d)^{4m+3}$, using part (iii), and noting that coefficient of $m(2d)^{4m+4}$ is

$$-\frac{2d+1}{4d^2+1} + 1 = \frac{2d(2d-1)}{4d^2+1},$$

coefficient of $(2d)^{4m+2}$ is

$$-d\left[\frac{2a(2d+1)}{4d^2+1} + \frac{d(16d^3+4d^2+8d+3)}{(4d^2+1)^2}\right] + (a+\frac{3}{2}d)2d$$

$$= 4d^2\left[\frac{a(2d-1)}{4\mathrm{d}^2+1} + \frac{d(12d^3-4d^2+5d-2)}{(4d^2+1)^2}\right],$$

we get the desired expression for S_{4m+4}.

To complete the proof, note that, corresponding to $m = 0$, the expressions for S_1, S_2, S_3 and S_4 are given as follows :

$$S_1 = d\left[\frac{2a(2d+1)}{4d^2+1} - \frac{d(4d^2-4d-1)}{(4d^2+1)^2}\right] + \left[\frac{d^2(4d^2-4d-1)}{(4d^2+1)^2} - \frac{a(2d-1)}{4d^2+1}\right] = a,$$

$$\begin{aligned} S_2 &= -\left[\frac{a(2d-1)}{4d^2+1} + \frac{d(4d^3+3\mathrm{d}-1)}{(4d^2+1)^2}\right] + \left[\frac{d^2(4d^2-4d-1)}{(4d^2+1)^2} - \frac{a(2d-1)}{4d^2+1}\right] \\ &= -a\frac{8d^3-4d^2+2d-1}{4d^2+1} - d^2\frac{16d^4+8d^2+1}{(4d^2+1)^2} \\ &= -a(2d-1) - d^2, \end{aligned}$$

$$\begin{aligned} S_3 &= -d\left[\frac{2a(2d+1)}{4d^2+1} + \frac{d(16d^3+4d^2+8d+3)}{(4d^2+1)^2}\right] + \left[\frac{d^2(4d^2-4\mathrm{d}-1)}{(4d^2+1)^2} - \frac{a(2d-1)}{4d^2+1}\right] \\ &= -a\frac{16d^4+8d^3+2d-1}{4d^2+1} - d^2\frac{64d^5+16d^4+32d^3+8d^2+4d+1}{(4\mathrm{d}^2+1)^2} \\ &= -a(4d^2+2d-1) - d^2(4d+1), \end{aligned}$$

$$\begin{aligned} S_4 &= \left[\frac{a(2d-1)}{4d^2+1} + \frac{d(12d^3-4d^2+5d-2)}{(4d^2+1)^2}\right] + \left[\frac{d^2(4d^2-4d-1)}{(4d^2+1)^2} - \frac{a(2d-1)}{4d^2+1}\right] \\ &= \frac{a(2d-1)}{4d^2+1}(64d^4-1) + \frac{d^2}{(4d^2+1)^2}(192d^6-64d^5+80d^4-32d^3+4d^2-4d-1) \\ &= a(8d^3-4d^2-2d+1) + d^2(12d^2-4d-1). \end{aligned}$$

All these complete the proof of the theorem.

§4. Remarks

A particular case of the Smarandache bisymmetric arithmetic determinant sequence (SBADS) is the Smarandache bisymmetric determinant natural sequence (SBDNS), and can be obtained from the former by setting $a = 1 = d$. In an earlier paper, Majumdar [3] has derived the expression for the first n terms of the SBDNS.

The Smarandache cyclic determinant natural sequence is

$$\left\{ 1, \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix}, \ldots \right\}.$$

The first few terms of the sequence are

$$1, -3, -18, 160, 1875, \cdots$$

and in general, the $n-th$ term of the sequence is (see Majumdar [2])

$$(-1)^{\left[\frac{n}{2}\right]} \left(\frac{n+1}{2}\right) n^{n-1}.$$

Open Problem To find a formula for the sum of the first n terms of the Smarandache cyclic determinant natural sequence.

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On the divisor function and the number of finite abelian groups [1]

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Abstract Let $d(n)$ denote the Dirichlet divisor function and $a(n)$ denote the number of nonisomorphic abelian groups with n elements , $l \geq 1$ is a fixed integer. In this paper we shall study the hybrid mean value $\sum\limits_{\substack{n \leqq x \\ a(n)=l}} d(n)$ and obtain its asymptotic formula.

Keywords Divisor problem, finite abelian groups.

§1. Introduction

Let $a(n)$ denote the number of non-isomorphic abelian groups with n elements. This is a well-known multiplicative function such that for any prime p we have $a(p^\alpha) = P(\alpha)$, where $P(\alpha)$ is the unrestricted partition function. Let $l \geq 1$ be a fixed integer. The asymptotic behavior of $A_l(x) = \sum\limits_{\substack{n \leqq x \\ a(n)=l}} 1$ was first investigated by Ivić[3] who obtained the result

$$A_l(x) = C_{l,a}x + O(x^{\frac{1}{2}} \log x), \tag{1}$$

which was improved to

$$A_l(x) = C_{l,a}x + O(x^{\frac{1}{2}} \log^{-1} x \log\log x) \tag{2}$$

by Krätzel[4], where $C_{l.a}$ is a constant. Note that when $l = 1$, $A_1(x)$ is just the counting function of the square-free numbers.

The well-known Dirichlet divisor problem is to study the asymptotic behavior of the sum $\sum\limits_{n \leqq x} d(n)$, where $d(n)$ is the Dirichlet divisor function. Dirichlet first proved

$$\sum_{n \leqq x} d(n) = x(\log x + 2\gamma - 1) + O(x^{\frac{1}{2}}). \tag{3}$$

The error term $O(x^{1/2})$ was improved by many authors. The latest result reads

$$\sum_{n \leqq x} d(n) = x(\log x + 2\gamma - 1) + O(x^{131/416} \log^{29647/8320} x), \tag{4}$$

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which was proved by Huxley [1].

In this paper we shall study the asymptotic behavior of the hybrid mean value $\sum_{\substack{n\leqq x\\ a(n)=l}} d(n)$. We shall prove the following

Theorem. Let $l \geq 1$ be any fixed integer. We then have the asymptotic formula

$$\sum_{\substack{n\leqq x\\ a(n)=l}} d(n) = A_{1,l}x\log x + A_{2,l}x + O(x^{\frac{1}{2}+\epsilon}), \tag{5}$$

where $A_{1,l}$, $A_{2,l}$ are computable constants .

§2. Some lemmas

Lemma 2.1. [2] (Eular product) If $f(n)$ is a multiplicative function of n , and the Dirichlet series $\sum_{n=1}^{\infty} f(n)n^{-s}$ is absolutely convergent on some half plane $\Re s > \sigma_0$ for some real σ_0, then we have

$$\sum_{n=1}^{\infty}\frac{f(n)}{n^s} = \prod_p(1+\sum_{\alpha=1}^{\infty}\frac{f(p^\alpha)}{p^{\alpha s}}), \quad \Re s > \sigma_0. \tag{6}$$

Lemma 2.2. Suppose $\mathcal{A}$ is an infinite subset of $\mathbb{N}$, $c(m)$ satisfies $c(m) \ll m^\varepsilon$ and $F(t) \ll t^{1/2}$, where $F(t) = \sum_{\substack{m\leq t\\ m\in\mathcal{A}}} c(m)$, then the infinite series $\sum_{\substack{m=1\\ m\in\mathcal{A}}}^{\infty}\frac{c(m)}{m}$ converges, and

$$\sum_{\substack{m\leq x\\ m\in\mathcal{A}}}\frac{c(m)}{m} = \sum_{\substack{m=1\\ m\in\mathcal{A}}}^{\infty}\frac{c(m)}{m} + O(x^{-\frac{1}{2}}). \tag{7}$$

Proof. Suppose $T \geq 2$ is any real number. Then

$$\begin{aligned}\sum_{\substack{T<m\leq 2T\\ m\in\mathcal{A}}}\frac{c(m)}{m} &= \int_T^{2T}\frac{dF(t)}{t}\\ &= \frac{F(t)}{t}|_T^{2T} + \int_T^{2T}\frac{F(t)}{t^2}dt \ll T^{-1/2},\end{aligned}$$

which implies Lemma 2.2 immediately.

§3. Proof of theorem

Consider the equation

$$a(n) = l, n \in \mathbb{N}. \tag{8}$$

If $l = 1$, then the solutions of (8) are just all square-free numbers. So later we always suppose $l \geq 2$. If $n > 1$ has a factorization $n = n_0p_1^{\beta_1}p_2^{\beta_2}\cdots p_s^{\beta_s}$, where n_0 is square-free , $p_1, \cdots, p_s$ are

distinct primes such that $(n_0, p_j) = 1 (j = 1, \cdots, s)$ and $\beta_j \geq 2 (j = 1, \cdots, s)$, then the equation (8) becomes

$$a(n) = \prod_{j=1}^{s} a(p_j^{\beta_j}) = \prod_{j=1}^{s} P(\beta_j) = l. \tag{9}$$

Namely l must have a factorization of the form $l = P(\beta_1) \cdots P(\beta_s)$. Conversely, if $\alpha_1 \geq 2, \cdots, \alpha_d \geq 2$ are a group of fixed integers such that $l = P(\alpha_1)P(\alpha_2) \cdots P(\alpha_d)$, then all numbers in the set

$$N(\alpha_1, \cdots \alpha_d) := \{n | n = n_0 p_1^{\alpha_1} \cdots p_d^{\alpha_d}, p_i \neq p_j,\ 1 \leq i \neq j \leq d, p_j \nmid n_0 (j = 1, \cdots, d), \mu(n_0) \neq 0\}$$

are solutions of the equation (8). Furthermore if l has two different factorizations

$$\begin{gathered} l = P(\alpha_1)P(\alpha_2) \cdots P(\alpha_d) = P(\beta_1)P(\beta_2) \cdots P(\beta_s), \\ \alpha_j \geq 2 (j = 1, \cdots, d), \beta_j \geq 2 (j = 1, \cdots, s), \end{gathered} \tag{10}$$

then we must have

$$N(\alpha_1, \cdots \alpha_d) \bigcap N(\beta_1, \cdots \beta_s) = \emptyset. \tag{11}$$

From the above analysis we see that all solutions of (8) are just all elements of the set

$$\bigcup_{l = P(\alpha_1) \cdots P(\alpha_d)} N(\alpha_1, \cdots \alpha_d).$$

Thus we have

$$\sum_{\substack{n \leq x \\ a(n) = l}} d(n) = \sum_{\substack{l = P(\alpha_1) \cdots P(\alpha_d) \\ \alpha_j \geq 2, j = 1, \cdots, d}} \sum_{\substack{n \leq x \\ n \in N(\alpha_1, \cdots \alpha_d)}} d(n). \tag{12}$$

If $n = n_0 p_1^{\alpha_1} \cdots p_d^{\alpha_d} \in N(\alpha_1, \cdots \alpha_d)$, then

$$d(n) = d(n_0 p_1^{\alpha_1} \cdots p_d^{\alpha_d}) = d(n_0)|\mu(n_0)|(1 + \alpha_1) \cdots (1 + \alpha_d). \tag{13}$$

Thus we get

$$\sum_{\substack{n \leq x \\ n \in N(\alpha_1, \cdots \alpha_d)}} d(n) = (1 + \alpha_1) \cdots (1 + \alpha_d) \sum_{\substack{p_1^{\alpha_1} \cdots p_d^{\alpha_d} \leq x \\ p_i \neq p_j}} \sum_{\substack{n_0 \leq x / p_1^{\alpha_1} \cdots p_d^{\alpha_d} \\ (n_0, p_1^{\alpha_1} \cdots p_d^{\alpha_d}) = 1}} d(n_0)|\mu(n_0)|. \tag{14}$$

The problem becomes to evaluate sums of the type($m \in \mathbb{N}$)

$$D_m(x) := \sum_{\substack{n \leq x \\ (n, m) = 1}} d(n)|\mu(n)|.$$

By Lemma 1, we have(for $\Re s > 1$)

$$\begin{aligned}
&\sum_{\substack{n=1\\(n,m)=1}}^{\infty} \frac{d(n)|\mu(n)|}{n^s} \\
= &\prod_{(p,m)=1} (1+\sum_{\alpha=1}^{\infty} \frac{d(p^\alpha)|\mu(p^\alpha)|}{p^{\alpha s}}) \\
= &\prod_{(p,m)=1} (1+\frac{2}{p^s}) \\
= &\prod_{p}(1+\frac{2}{p^s})\prod_{p|m}(1+\frac{2}{p^s})^{-1} \\
= &\zeta^2(s)G(s)\prod_{p|m}(1+\frac{2}{p^s})^{-1},
\end{aligned}$$

where $G(s)$ can be written as an infinite Dirichlet series, which is absolutely convergent for $\sigma > \frac{1}{2}$.

From the Perron's formula, we have

$$D_m(x) = c_1(m)x\log x + c_2(m)x + O(x^{\frac{1}{2}+\varepsilon}m^{\varepsilon}), \tag{15}$$

where $c_1(m) \ll m^{\varepsilon}$, $c_2(m) \ll m^{\varepsilon}$.

Now the Theorem follows from (12), (14) and (15) with the help of Lemma 2.

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The quintic supported spline wavelets with numerical integration [1]

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Abstract This paper gives the expression of quintic supported spline wavelets and then discusses the expansion of the integrand with the quintic supported spline-wavelet. As the main result a new useful numerical integration formula is obtained by our analyzing and proving. Furthermore, a numerical example is given at the end of this paper.

Keywords supported spline-wavelet, numerical integration, interpolation with wavelets.

§1. Introduction of the quintic supported spline wavelets

In 1992, C. K. Chui and J. Z. Wang have constructed the supported spline wavelets with B-spline as scaling function, and this kind of wavelets can be used in many areas(the reference[1]). Because of the spline wavelets have much advantages, such as the quintic supported spline wavelets, which use the quintic B-spline $N_6(x)$ as the scaling function, let $\psi_6(x)$ be the quintic supported spline wavelets. The properties of the quintic supported spline wavelets

$$\psi_6(x) = \frac{1}{2^5}\sum_{j=0}^{10}(-1)^j N_{12}(j+1)N_{12}^{(6)}(2x-j) = \sum_{n=0}^{10} q_n \cdot N_6(2x-n). \tag{1}$$

are as follows:

(1)Symmetry: $\psi_6(11/2+x) = \psi_6(11/2-x)$;

(2)Simple of orthogonality: $< N_6(x-p),\ \psi_6(x-p) >= 0$, where $p, q \in Z$;

(3)Vanishing: $\int_{-\infty}^{+\infty} x^j \psi_6(x)dx = 0,\ 0 \le j \le 5$;

(4)Continuous differentiability: the fourth derivative of $\psi_6(x)$ is continuous.

Where $q_n = \dfrac{(-1)^n}{2^5}\sum_{j=0}^{6}\binom{n}{j} N_{12}(n-j+1)$, $N_6(x)$ is quintic spline polynomial, $\psi_6(x)$is piecewise polynomial, the interval of $\psi_6(x)$ is half of $N_6(x)$'s , and supp$\psi_5(x) = [0, 11]$. Then, let $\psi_6(x)$ is the basis function, the figures of $\psi_6^{(i)}(i = 0, \cdots, 4)$ are following (Figure 1, 2).

§2. Interpolation of the quintic supported spline wavelets

The translation of $\{\psi_6(x)\}_{n\in Z}$ can compose a basis of wavelet space. let $\widetilde{\psi_6(x)} = \psi_6(x + 11/2)$, thus the quintic supported spline wavelet $\psi_6(x)$ translate on the left 11/2 unites, we can

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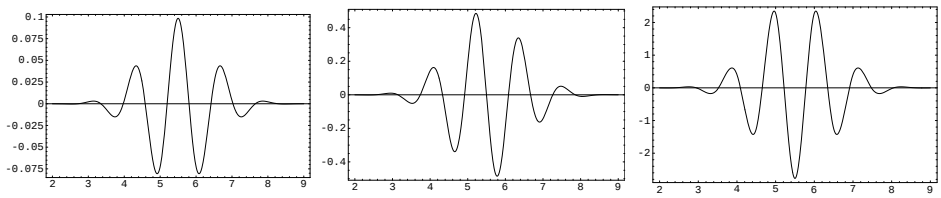

Figure 1: figure of ψ_6, ψ_6', ψ_6''

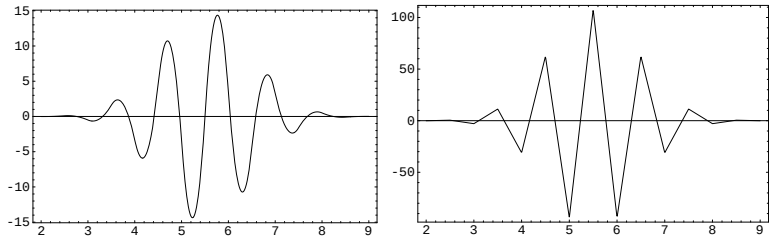

Figure 2: figure of ψ_6''', $\psi_6^{(4)}$

obtain $\mathtt{supp}\widetilde{\psi_6(x)} = [-11/2, 11/2]$, when $\mathtt{supp}\psi_6(x) = [0, 11]$. Then, we take partition on $[a, b]$, we have

$$x_{-1/2} < a = x_0 < x_{1/2} < \cdots < x_{N-1/2} < x_N = b < x_{N+1/2},$$

where $x_{-1/2}$ and $x_{N+1/2}$ are prolongation points out of $[a, b]$,

$$x_{j/2} = a + jh/2, j = 0, 1, \cdots, 2N, h = \frac{b-a}{N}. \tag{2}$$

Then $f(x)$ is expanded by the linear of supported spline wavelet $\psi_6(x)$ on $[a, b]$,

$$\widetilde{S_6}(x) = \sum_{j=0}^{2N+4} C_{j-2}\widetilde{\psi_6}(\frac{x - x_{(j-2)/2}}{h}), \tag{3}$$

where $\widetilde{\psi_6(x)} = \psi_6(x+11/2)$, from [1] we know that $\psi_6(x)$ is symmetric with respect to $x = 11/2$, that is $\mathtt{supp}\widetilde{\psi_6(x)} = [-11/2, 11/2]$, and $\widetilde{\psi_6(x)}$ is even function.

Thus

$$y_{i/2} = f(x_{i/2}), y_0' = f'(a), y_N' = f'(b), y_0'' = f''(a), y_N'' = f''(b), i = 0, 1, 2, \cdots, 2N.$$

are satisfy $2N + 5$ interpolation conditions,

$$\begin{cases} \widetilde{S_6'}(x_i) = y_i', & i = 0, N. \\ \widetilde{S_6}(x_{j/2}) = y_{j/2}, & j = 0, 1, 2, \cdots, 2N. \\ \widetilde{S_6''}(x_k) = y_k'', & k = 0, N. \end{cases} \tag{4}$$

Theorem 1.[1] Interpolation problem (4) has a unique solution.

§3. Numerical integral formula

With the quintic supported spline-wavelets, we give the interpolation function $\widetilde{S_6}(x)$ of $f(x)$, which is a integrand function in $\int_a^b f(x)\mathtt{d}x$,

$$f(x) \approx \widetilde{S_6}(x) = \sum_{j=0}^{2N+4} C_{j-2}\widetilde{\psi_6}(\frac{x - x_{(j-2)/2}}{h}), \tag{5}$$

where c_{-2}, c_{-1}, $\cdots$, c_{2N+2} can be obtained from (4). Therefore

$$\int_a^b f(x)\mathtt{d}x \approx \int_a^b \widetilde{S_6}(x)\mathtt{d}x = \sum_{j=0}^{2N+4} C_{j-2}\int_a^b \widetilde{\psi_6}(\frac{x - x_{(j-2)/2}}{h})\mathtt{d}x. \tag{6}$$

Let

$$\frac{x - x_{(j-2)/2}}{h} = t,$$

then we obtain with (6)

$$\int_a^b f(x)\mathtt{d}x \approx \sum_{j=0}^{2N+4} C_{j-2}h\int_{(2-j)/2}^{(3-j)/2} \widetilde{\psi_6(t)}\mathtt{d}t, \tag{7}$$

due to $\mathtt{supp}\widetilde{\psi_6(x)} = [-11/2, 11/2]$,we have

$$\begin{aligned}
&\int_1^{2N+2} \widetilde{\psi_6}(t)\mathtt{d}t = \frac{3323523607}{306561024000}, && \int_{1/2}^{(2N+1)/2} \widetilde{\psi_6}(t)\mathtt{d}t = -\frac{51590351}{4790016000},\\
&\int_0^{N} \widetilde{\psi_6(t)}\mathtt{d}t = 0, && \int_{-1/2}^{(2N-1/2} \widetilde{\psi_6(t)}\mathtt{d}t = \frac{51590351}{4790016000},\\
&\int_{-1}^{N-1} \widetilde{\psi_6(t)}\mathtt{d}t = -\frac{3323523607}{306561024000}, && \int_{-3/2}^{(2N-3)/2} \widetilde{\psi_6(t)}\mathtt{d}t = \frac{10985063}{2395008000},\\
&\int_{-2}^{N-2} \widetilde{\psi_6(t)}\mathtt{d}t = -\frac{21579329}{38320128000}, && \int_{-5/2}^{(2N-5)/2} \widetilde{\psi_6(t)}\mathtt{d}t = -\frac{429967}{2395008000},\\
&\int_{-3}^{N-3} \widetilde{\psi_6(t)}\mathtt{d}t = \frac{11044637}{204374016000}, && \int_{-7/2}^{(2N-7)/2} \widetilde{\psi_6(t)}\mathtt{d}t = -\frac{29573}{9580032000},\\
&\int_{-4}^{N-4} \widetilde{\psi_6(t)}\mathtt{d}t = -\frac{5849}{229920768000}, && \int_{-9/2}^{(2N-9)/2} \widetilde{\psi_6(t)}\mathtt{d}t = \frac{31}{28740096000},\\
&\int_{-5}^{N-5} \widetilde{\psi_6(t)}\mathtt{d}t = -\frac{1}{1839366144000}, && \int_{-11/2}^{(2N-11)/2} \widetilde{\psi_6(t)}\mathtt{d}t = 0.
\end{aligned}$$

For $14 \le j \le 2N-9$,

$$C_{j-2}h\int_{(2-j)/2}^{(2N-j+2)/2} \widetilde{\psi_6(t)}\mathtt{d}t \equiv 0.$$

For $2N-8 \le j \le 2N+4$,

$$\int_{5-N}^{5} \widetilde{\psi_6(t)}\mathtt{d}t = -\frac{1}{1839366144000}, \quad \int_{(9-2N)/2}^{9/2} \widetilde{\psi_6(t)}\mathtt{d}t = \frac{31}{28740096000},$$
$$\int_{4-N}^{4} \widetilde{\psi_6(t)}\mathtt{d}t = -\frac{5849}{229920768000}, \quad \int_{(7-2N)/2}^{7/2} \widetilde{\psi_6(t)}\mathtt{d}t = -\frac{29573}{9580032000},$$
$$\int_{3-N}^{3} \widetilde{\psi_6(t)}\mathtt{d}t = \frac{11044637}{204374016000}, \quad \int_{(5-2N)/2}^{5/2} \widetilde{\psi_6(t)}\mathtt{d}t = -\frac{429967}{2395008000},$$
$$\int_{2-N}^{2} \widetilde{\psi_6(t)}\mathtt{d}t = -\frac{21579329}{38320128000}, \quad \int_{(3-2N)/2}^{3/2} \widetilde{\psi_6(t)}\mathtt{d}t = \frac{10985063}{2395008000},$$
$$\int_{1-N}^{1} \widetilde{\psi_6(t)}\mathtt{d}t = -\frac{3323523607}{306561024000}, \quad \int_{(1-2N)/2}^{1/2} \widetilde{\psi_6(t)}\mathtt{d}t = \frac{51590351}{4790016000},$$
$$\int_{-N}^{0} \widetilde{\psi_6(t)}\mathtt{d}t = 0, \quad \int_{(-1-2N)/2}^{-1/2} \widetilde{\psi_6(t)}\mathtt{d}t = -\frac{51590351}{4790016000},$$
$$\int_{-1-N}^{-1} \widetilde{\psi_6(t)}\mathtt{d}t = \frac{3323523607}{306561024000}.$$

Then formula (7) can be rewritten as

$$\begin{aligned}
\int_a^b f(x)\mathtt{d}x \quad \approx \quad & \sum_{j=0}^{2N+4} C_{j-2} \int_{x_{(2-j)/2}}^{x_{(3-j)/2}} \widetilde{\psi_6}(\frac{x-x_{(j-2)/2}}{h})\mathtt{d}x \\
= \quad & (C_{-2}+C_{2N+2})h\frac{3323523607}{306561024000} - (C_{-1}+C_{2N+1})h\frac{51590351}{4790016000} \\
& +(C_1+C_{2N-1})h\frac{51590351}{4790016000} - (C_2+C_{2N-2})h\frac{3323523607}{306561024000} \\
& +(C_3+C_{2N-3})h\frac{10985063}{2395008000} - (C_4+C_{2N-4})h\frac{21579329}{38320128000} \\
& -(C_5+C_{2N-5})h\frac{429967}{2395008000} + (C_6+C_{2N-6})h\frac{11044637}{204374016000} \\
& -(C_7+C_{2N-7})h\frac{29573}{9580032000} - (C_8+C_{2N-8})h\frac{5849}{229920768000} \\
& +(C_9+C_{2N-9})h\frac{31}{28740096000} - (C_{10}+C_{2N-10})h\frac{1}{1839366144000}.
\end{aligned} \tag{8}$$

The above formula is a new numerical formula, where the coefficients c_{-2}, c_{-1}, $\cdots$, c_{2N+2} can be obtained uniquely from (4). Hence applying its to (8) we can get the approximate value.

§4. Numerical example

Example. The exact value of $\int_0^1 e^x\mathtt{d}x$ is $e-1$, Now we use the formula (8) to get approximate value of $\int_0^1 e^x\mathtt{d}x$. Let $N=20$, from Theorem 1, we know that the interpolation problem (4) has a unique solution.

$$\begin{cases} \widetilde{S_6'}(x_i) = y_i', & i=0,20. \\ \widetilde{S_6}(x_{j/2}) = y_{j/2}, & i=0,1,2,\cdots,40. \\ \widetilde{S_6''}(x_k) = y_k'', & k=0,20. \end{cases} \tag{9}$$

then, we obtain an algebraic system(see[6]), the approximate value 1.5974 can be easily obtained by solving this system. So the formula is efficient, since the error is 0.120879,

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Approximation in Hilbert algebras

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Abstract In this note the basic notion of rough set theory will be given. We shall introduce the notion of rough ideal and rough subalgebras of Hilbert algebra, which is a generalization of ideal and subalgebra in Hilbert algebra and we shall give some properties of the lower and upper approximation in Hilbert algebra. Finally we study fuzzy rough sets in Hilbert algebras.

Keywords Rough sets, lower approximation, upper approximation, Hilbert algebras, rough ideals, rough subalgebras.

§1. Introduction

The notion of a Hilbert algebra was introduced in early 50-ties by L. Henkin and T. Skolem for some investigations of implicative in intuitionistic and other non-classical logics. In 60-ties, these algebras were studied especially by A. Horn and A. Diego from algebraic point of view. I. Chajda and R. Halas [2] and W. A. Dudek [3] introduce and study deductive systems (ideals) and congruence relations in Hilbert algebra.

The concept of rough set was originally proposed by Pawlak [8] as a formal tool for modeling and processing incomplete information in formation system. Since then the subject has been investigated in many papers (see[9, 10]) the theory of rough set is an extension of set theory, in which a subset a universe is described by a pair of ordinary sets called the lower and upper approximation. A key notion in Pawlak rough set model is an equivalence relation. The equivalence classes are the building blocks for the construction of the lower and upper approximation. The lower approximation of a given set is the union of the equivalence classes which are subsets of the set, and upper approximation is the union of all equivalence classes which have a non-empty intersection with the sets. Some authors, for example, Iwinski [5] and Pomykala [10] studied algebraic properties of rough sets. The lattice theoretical approach has been suggested by Iwinski [5]. Pomykala and Pomykala [10] showed that the set of rough sets forms a Stone algebra. Kuroki in [7], introduced the notion of a rough ideal in a semigroups. Jun applied the rough set theory to BCK-algebras [6]. Dubois and Prade [4] began to investigate the problem of fuzzification of a rough set. Fuzzy rough sets are a notion introduced as a further extension of the idea of rough sets. Now, we propose a more general approach to this issue. In this paper basic notion of the Hilbert algebras and rough set theory will be given. In fact, the paper concerns a relationship between rough sets and Hilbert algebra theory. We shall

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introduce the notion of rough sub Hilbert algebra (resp. ideal) which is an extended notion of a sub Hilbert algebra (resp. ideal) in a Hilbert algebra, and we shall give some properties of the upper approximation in a Hilbert algebra.

§2. Preliminaries

Definition 2.1. [3] A Hilbert algebra is an algebra $(H, *, 1)$ where H is a nonempty set, $*$ is a binary operation and 1 is a constant such that the following axioms hold for each $x, y, z \in H$: To complete the proof of the theorem, we need the following :

(H1) $x * (y * x) = 1$,

(H2) $(x * (y * z)) * ((x * y) * (x * z)) = 1$,

(H3) $x * y = 1$ and $y * x = 1$ imply $x = y$.

Lemma 2.2. [3] In each Hilbert algebra H, the following relations hold for all $x, y, z \in H$:

(1) $x * x = 1$,

(3) $x * 1 = 1$,

(4) $x * (y * z) = y * (x * z)$.

It is easily checked that in a Hilbert algebra H the relation $\leq$ defined by

$$x \leq y \Leftrightarrow x * y = 1$$

is a partial order on H with 1 as the largest element.

Definition 2.3. [2] A nonempty subset I of a Hilbert algebra H is called an ideal of H if

(1) $1 \in I$,

(2) $x * y \in I$ for all $x \in H, y \in I$,

(3) $(y_2 * (y_1 * x)) * x \in I$ for all $x \in H, y_1, y_2 \in I$.

Definition 2.4. [3] A deductive system of a Hilbert algebra H is a nonempty set $D \subseteq H$ such that

(1) $1 \in D$,

(2) $x \in D$ and $x * y \in D$ imply $y \in D$.

Theorem 2.5. [3] A nonempty subset A of a Hilbert algebra H is an ideal if and only if it is a deductive system of H.

Theorem 2.6. [3] If D is a deductive system of a Hilbert algebra H, then the relation Θ_D defined by

$$(a, b) \in \Theta_D \Leftrightarrow a * b \in D \text{ and } b * a \in D$$

is a congruence relation on H.

Suppose that U is a non-empty set. A classification of U is a family $\mathcal{P}$ of non-empty subsets of U such that each element of U is contained in exactly one element of $\mathcal{P}$.

The following notation will be used.

By $\mathcal{P}(U)$ we will denote the power-set of U. If θ is an equivalence relation on U then for every $x \in U$, $[x]_\theta$ denotes the equivalence class of θ determined by x. For any $X \subseteq U$, we write X^c to denote the complementation of X in U.

Definition 2.7. A pair $(U, \theta\ )$ where $U \neq \emptyset$ and θ is an equivalence relation on U, is called an approximation space.

Definition 2.8. For an approximation space (U, θ), by a rough approximation in (U, θ) we mean a mapping $Apr : \mathcal{P}\ (U) \longrightarrow \mathcal{P}\ (U) \times \mathcal{P}\ (U)$ defined by $Apr(X) = (\underline{Apr}(X), \overline{Apr}(X))$, where $\underline{Apr}(X) = \{x \in X \mid [x]_\theta \subseteq X\}$, $\overline{Apr}(X) = \{x \in X \mid [x]_\theta \cap X \neq \emptyset\}$. $\underline{Apr}(X)$ is called a lower rough approximation of X in (U, θ), and $\overline{Apr}(X)$ is called upper rough approximation of X in (U, θ).

Denote H-positive region of X by $POS(X) = \underline{Apr}(X)$, H-negative region of X by $NEG(X) = U \setminus \overline{Apr}(X)$ and H-borderline region of X by $BN(X) = \overline{Apr}(X) \setminus \underline{Apr}(X)$. The positive region $POS(X)$ or $\underline{Apr}(X)$ is the collection of those objects which can be classified with full certainty as member of the set X, using the knowledge H. The negative region $NEG(X)$ is the collection of object which can be determined without any ambiguity, employing knowledge H, that they do not belong to the set X; that is, they belong to the complement $\neg X$ of X. The borderline region is, in a sense, the undecidable area of the universe, i.e., none of the objects belonging to the borderline region can be classified with certainty into X or it's complement $\neg X$ as far as knowledge H is concerned.

Definition 2.9. A subset X of U is called definable if $\underline{Apr}(X) = \overline{Apr}(X)$. if $X \subseteq U$ is given by a predicate P and $x \in U$, then

1. $x \in \underline{Apr}(X)$ mean that x certainly has property P,
2. $x \in \overline{Apr}(X)$ mean that possibly P,
3. $x \in \ Apr(X)$ mean that x definitely does not have property P.

Definition 2.10. Let $Apr(A) = (\underline{Apr}(A), \overline{Apr}(A))$ and $Apr(B) = (\underline{Apr}(B), \overline{Apr}(B))$ be any two rough sets in the approximation space (U, θ). Then

(i) $Apr(A) \sqcup Apr(B) = (\underline{Apr}(A) \cup \underline{Apr}(B), \overline{Apr}(A) \cup \overline{Apr}(B))$,

(ii) $Apr(A) \sqcap Apr(B) = (\underline{Apr}(A) \cap \underline{Apr}(B), \overline{Apr}(A) \cap \overline{Apr}(B))$,

(iii) $Apr(A) \sqsubseteq Apr(B) \Leftrightarrow Apr(A) \sqcap Apr(B) = Apr(A)$. when $Apr(A) \sqsubseteq Apr(B)$, we say that $Apr(A)$ is a rough subset of $Apr(B)$. Thus in the case of rough sets $Apr(A)$ and $Apr(B)$,

$Apr(A) \sqsubseteq Apr(B)$ if and only if $\underline{Apr}(A) \subseteq \underline{Apr}(B)$ and $\overline{Apr}(A) \subseteq \overline{Apr}(B)$.

This property of rough inclusion has all the properties of set inclusion. The rough complement of $Apr(A)$ we denote it by $Apr^c(A)$ is defined by $Apr^c(A) = (U \setminus \overline{Apr}(A), U \setminus \underline{Apr}(A))$.

Also, we can define $Apr(A) \setminus Apr(B)$ as follows:

$$Apr(A) \setminus Apr(B) = Apr(A) \sqcap Apr^c(B) = (\underline{Apr}(A) \setminus \overline{Apr}(B), \overline{Apr}(A) \setminus \underline{Apr}(B)).$$

§3. Rough ideals

From now on let H be a Hilbert algebra, I be an ideal of H and X be a nonempty subset of H.

For $a, b \in H$ we say a is congruent to b mod I, written as $a \equiv b(modI)$ if and only if $a * b \in I$ and $b * a \in I$. It easy to see that the relation $a \equiv b(modI)$ is an equivalence relation. Therefore, when $U = H$ and θ is the above equivalence relation, then we use the pair (H, I) instead of approximation space (U, θ), and denote lower rough approximation and upper rough

approximation of X in (H, I), by $\underline{Apr}_I(X)$ and $\overline{Apr}_I(X)$ respectively. Sometimes we remove I for shortness.

Proposition 3.1. For every approximation space (H, I) and every subset $A, B \subseteq H$, we have:

(1) $\underline{Apr}(A) \subseteq A \subseteq \overline{Apr}(A)$;
(2) $\underline{Apr}(\emptyset) = \emptyset = \overline{Apr}(\emptyset)$;
(3) $\underline{Apr}(H) = H = \overline{Apr}(H)$;
(4) If $A \subseteq B$ then $\underline{Apr}(A) \subseteq \underline{Apr}(B)$ and $\overline{Apr}(A) \subseteq \overline{Apr}_I B)$;
(5) $\underline{Apr}(\underline{Apr}(A)) = \underline{Apr}(A)$;
(6) $\overline{Apr}(\overline{Apr}(A)) = \overline{Apr}(A)$;
(7) $\overline{Apr}(\underline{Apr}(A)) = \underline{Apr}(A)$;
(8) $\underline{Apr}(\overline{Apr}(A)) = \overline{Apr}(A)$;
(9) $\underline{Apr}(A) = (\overline{Apr}(A^c))^c$;
(10) $\overline{Apr}(A) = (\underline{Apr}(A^c))^c$;
(11) $\underline{Apr}(A \cap B) = \underline{Apr}(A) \cap \underline{Apr}(B)$;
(12) $\overline{Apr}(A \cap B) \subseteq \overline{Apr}(A) \cap \overline{Apr}(B)$;
(13) $\underline{Apr}(A \cup B) \supseteq \underline{Apr}(A) \cup \underline{Apr}(B)$;
(14) $\overline{Apr}(A \cup B) = \overline{Apr}(A) \cup \overline{Apr}(B)$;
(15) $\underline{Apr}([x]) = \overline{Apr}([x])$.

Proof. We prove some of them and the others are similar.

(5). By (1) we have $\underline{Apr}(A) \subseteq A$, then by (4) $\underline{Apr}(\underline{Apr}(A)) \subseteq \underline{Apr}(A)$. Now, if $x \in \underline{Apr}(A)$ then $[x]_\theta \subseteq A$. If $z \in [x]_\theta$ then $[x]_\theta = [z]_\theta$ thus $[z]_\theta \subseteq A$ hence $z \in \underline{Apr}(A)$, therefore $[x]_\theta \subseteq \underline{Apr}(A)$ so $\underline{Apr}(A) \subseteq \underline{Apr}(\underline{Apr}(A))$.

(7). By part (4), we have $\underline{Apr}(A) \subseteq \overline{Apr}(\underline{Apr}(A))$. Let $x \in \overline{Apr}(\underline{Apr}(A))$ then $[x] \cap \underline{Apr}(A) \neq \emptyset$ thus there exists $z \in [x] \cap \underline{Apr}(A)$. So $[z] = [x]$ and $[z] \subseteq A$ which means that $[x] \subseteq A$, hence $x \in \underline{Apr}(A)$.

(10). Let $x \in \overline{Apr}(A)$ then $[x] \cap A \neq \emptyset$ thus $[x] \subseteq A^c$ so $x \in \underline{Apr}(A^c)$. Which means that $x \in (\underline{Apr}(A^c)^c$.

Conversely, $x \in (\underline{Apr}(A^c))^c$ so $[x] \subseteq A^c$, hence $[x] \cap A \neq \emptyset$ therefore $x \in \overline{Apr}(A)$.

Corollary 3.2. For every approximation space (H, D),

(i) for every $A \subseteq H$, $\underline{Apr}(A)$ and $\overline{Apr}(A)$ are definable sets,

(ii) for every $x \in H$, $[x]$ is definable set.

Proposition 3.3. Let A, B non-empty subsets of H. Then $\overline{Apr}(A) * \overline{Apr}(B) \subseteq \overline{Apr}(A * B)$.

Proof. $x \in \overline{Apr}(A) * \overline{Apr}(B)$ then $x = z * y$ where $z \in \overline{Apr}(A)$ and $y \in \overline{Apr}(B)$ we must show $x \in \overline{Apr}(A * B)$. By hypothesis we have $[y] \cap B \neq \emptyset$ and $[z] \cap A \neq \emptyset$ then there exist $z_1 \in [z] \cap A$ and $y_1 \in [y] \cap B$, so $[z] = [z_1]$, $[y] = [y_1]$ and $z_1 * y_1 \in A * B$. Consider $[x] = [z * y] = [z_1 * y_1] = [z_1] * [y_1]$, on other hand we have $[z_1] \cap A \neq \emptyset$ and $[y_1] \cap B \neq \emptyset$, thus $([z_1] * [y_1]) \cap A * B \neq \emptyset$ therefore $[x] \cap A * B \neq \emptyset$ hence $x \in \overline{Apr}(A * B)$.

Proposition 3.4. Let A, B are non-empty subsets of H. Then $\underline{Apr}(A) * \underline{Apr}(B) \subseteq \underline{Apr}(A * B)$.

Proof. $x \in \underline{Apr}(A) * \underline{Apr}(B)$ then $x = z * y$ where $z \in \underline{Apr}(A)$ and $y \in \underline{Apr}(B)$ we must show $x \in \underline{Apr}(A * B)$. By hypothesis we have $[y] \subseteq B$ and $[z] \subseteq A$. Consider $[x] = [z * y] =$

$[z] * [y] \subseteq A * B$ hence $x \in \underline{Apr}(A * B)$.

The following example shows that the converse of above propositions is not correct

Example 3.5. Let $B = \{0, a, b, c, 1\}$. Define $*$ as follow:

$*$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	a	1	c	1
c	0	a	b	1	1
1	0	a	b	c	1

Than H is a Hilbert algebra. Consider $I = \{1, b\}$, we get that

$[1] = \{y \in H \mid y * 1, 1 * y \in I\} = \{1\}$

$[a] = \{y \in H \mid y * a, a * y \in I\} = \{a\}$

$[b] = \{y \in H \mid y * b, b * y \in I\} = \{1, b\}$

$[c] = \{y \in H \mid y * c, c * y \in I\} = \{c\}$

$[0] = \{y \in H \mid y * 0, 0 * y \in I\} = \{0\}$

Let $A = \{a, b\}$ and $B = \{1, c\}$. Then $A * B = \{1, c\}$ so $\underline{Apr}(A * B) = \{1, c\}$, $\underline{Apr}(A) = \{a\}$, $\underline{Apr}(B) = \{1, c\}$ and $\underline{Apr}(A) * \underline{Apr}(B) = \{1\}$.

Similarly, we have $\overline{Apr}(A) = \{a, b\}$, $\overline{Apr}(B) = \{1, b, c\}$ thus $\overline{Apr}(A) * \overline{Apr}(B) = \{1, c\}$ but $\overline{Apr}(A * B) = \{1, b, c\}$.

Proposition 3.6. Let D, E are two ideals of H such that $D \subseteq E$ and let A be a non-empty subset of H. Then

(i) $\underline{Apr}_D(A) \supseteq \underline{Apr}_E(A)$,

(ii)$\overline{Apr}_E(A) \supseteq \overline{Apr}_D(A)$.

Proof. (i) Let $x \in \underline{Apr}_E(A)$, then $[x]_E \subseteq A$. Let $y \in [x]_D$ so $y * x \in D$ and $x * y \in D$ by hypothesis $y * x \in E$ and $x * y \in E$. Therefore $y \in [x]_E$ so $y \in A$, hence $x \in \underline{Apr}_D(A)$.

(ii) Let $x \in \overline{Apr}_D(A)$, then $[x]_D \cap A \neq \emptyset$, then there exists $y \in [x]_D \cap A$ so $y * x \in D$ and $x * y \in D$ by hypothesis $y * x \in E$ and $x * y \in E$. Therefore $y \in [x]_E \cap A$, hence $x \in \overline{Apr}_E(A)$.

The following corollary follows from above proposition

Corollary 3.7. Let D, E are two ideals of H and A a non-empty subset of H, then

(i) $\underline{Apr}_D(A) \cap \underline{Apr}_E(A) \subseteq \underline{Apr}_{(D \cap E)}(A)$,

(ii) $\overline{Apr}_{(D \cap E)} \subseteq \overline{Apr}_D(A) \cap \overline{Apr}_E(A)$.

Proposition 3.8. Let D, E are two ideals of H such that $D \subseteq E$, then:

(i) $\underline{Apr}_D(E)$

(ii) $\overline{Apr}_D(E)$

are ideals of H.

Proof. (i) Suppose $y \in [1]_D$ then $1 = 1 * y \in D$ and $y * 1 \in D$, since D is an ideal of H then $y \in D$ we get that $y \in E$ so $[1]_D \subseteq E$ therefore $1 \in \underline{Apr}_D(E)$.

Let x and $x * y \in \underline{Apr}_D(E)$, we must show that $y \in \underline{Apr}_D(E)$. Since $\underline{Apr}_D(E) \subseteq E$ then we get that x and $x * y \in E$ so $y \in E$. Now, let $z \in [y]_D$ thus $z * y \in D$ and $y * z \in D$. By

hypothesis we get that $y * z \in E$ and since $y \in E$ hence we conclude that $z \in E$, therefore $y \in \underline{Apr}_D(E)$.

(ii) Let $y \in [1]_D$, then $y * 1 \in D$ and $y = 1 * y \in D$ we get that $y \in D$, so $y \in E$, therefore $1 \in \overline{Apr}_D(E)$.

Let x and $x * y \in \overline{Apr}_D(E)$, we must show that $y \in \overline{Apr}_D(E)$. So $[x]_D \cap E \neq \emptyset$ and $[x*y]_D \cap E \neq \emptyset$. Now, let $z \in [x]_d \cap E$ then $z*x \in D$ and $z \in E$ thus $z*x \in E$ and $z \in E$ imply that $x \in E$. There exists $t \in [x*y]_D \cap E$ then $t*(x*y) \in D$ and $t \in E$ thus $t*(x*y) \in E$ and $t \in E$ since D is an ideal we get that $x * y \in E$. From $x * y \in E$ and $x \in E$ we can conclude that $y \in E$, thus $[y]_D \cap E \neq \emptyset$ hence $y \in \overline{Apr}_D(E)$.

Similarly, if D is an ideal and E is a sub-algebra of H, then $\underline{Apr}_D(E)$ and $\overline{Apr}_D(E)$ are sub-algebras of H.

Definition 3.9. Let I be an ideal of H and $Apr(A) = (\underline{Apr}, \overline{Apr})$ a rough set in the approximation space (H, I). If $\underline{Apr}$ and $\overline{Apr}$ are ideals (resp. sub-Hilbert algebra) of H, then we call $Apr(A)$ a rough ideal (resp. sub-Hilbert algebra). Note that a rough sub-Hilbert algebra also is called a rough Hilbert algebra.

Corollary 3.10. Let I, J are two ideals of H such that $I \subseteq J$, then $Apr_I(J)$ and $Apr_J(I)$ are rough ideals.

Theorem 3.11. Let H and $H^{'}$ be two Hilbert algebra and f a homomorphism from H to $H^{'}$. If A is a non-empty subset of H, then

(i)$f(\overline{Apr}_{kerf}(A)) = f(A)$,

(ii) Let f be an injective then $f(\underline{Apr}_{kerf}(A)) = f(A)$.

Proof. (i) It is clear that $A \subseteq \overline{Apr}_{kerf}(A)$, therefore $f(A) \subseteq f(\overline{Apr}_{kerf}(A))$.

Conversely, let $y \in f(\overline{Apr}_{kerf}(A))$. Then there exists an element $x \in \overline{Apr}_{kerf}(A)$ such that $f(x) = y$, so $[x]_{kerf} \cap A \neq \emptyset$, then there exists $a \in [x]_{kerf} \cap A$ so $a * x, x * a \in kerf$, thus $f(a) * f(x) = 1$ and $f(x) * f(a) = 1$, by (H3) we get that $f(a) = f(x)$. Since $y = f(x) = f(a)$ then $y \in f(A)$ hence $f(\overline{Apr}_{kerf}(A)) \subseteq f(A)$.

(ii) We show that in this case $\underline{Apr}_{kerf}(A) = A$. By Proposition 3.1, we have $\underline{Apr}_{kerf}(A) \subseteq A$. Now, let $z \in [a]_{kerf}$ then $z*x, x*z \in kerf$ so $f(a)*f(z) = 1$ and $f(z)*f(a) = 1$, by (H3) we get that $f(a) = f(z)$. By hypothesis we have $a = z$, hence $[a] = \{a\} \subseteq A$ then $a \in \underline{Apr}_{kerf}(A)$.

§4. Fuzzy Rough sets

Dubois and Prade [4] introduced the notion of fuzzy rough sets, here we review some of them

Let (U, Θ) be an approximation space and $Apr(X)$ a rough set in (U, Θ). A fuzzy rough set $Apr(X) = (\underline{Apr}(A), \overline{Apr}(A))$ in $Apr(X)$ is characterized by a pair of maps:

$$\mu_{\underline{Apr}(A)} : \underline{Apr}(X) \to [0, 1]$$

and

$$\mu_{\overline{Apr}(A)} : \overline{Apr}(X) \to [0, 1]$$

where $\mu_{\underline{Apr}(A)}(x) \leq \mu_{\overline{Apr}(A)}(x)$ for all $x \in \underline{Apr}(X)$.

For any two fuzzy rough sets $Apr(A) = (\underline{Apr}(A), \overline{Apr}(A))$ and $Apr(B) = (\underline{Apr}(B), \overline{Apr}(B))$ in $Apr(X)$, we define:

(1) $Apr(A) = Apr(B)$ if and only if:
$\mu_{\underline{Apr}(A)}(x) = \mu_{\underline{Apr}(B)}(x)$ for all $x \in \underline{Apr}(X)$.
$\mu_{\overline{Apr}(A)}(x) = \mu_{\overline{Apr}(B)}(x)$ for all $x \in \overline{Apr}(X)$.

(2) $Apr(A) \subseteq Apr(B)$ if and only if:
$\mu_{\underline{Apr}(A)}(x) \subseteq \mu_{\underline{Apr}(B)}(x)$ for all $x \in \underline{Apr}(X)$.
$\mu_{\overline{Apr}(A)}(x) \subseteq \mu_{\overline{Apr}(B)}(x)$ for all $x \in \overline{Apr}(X)$.

(3) $Apr(C) = Apr(A) \sqcup Apr(B)$ if and only if:
$\mu_{\underline{Apr}(C)}(x) = \max\{\mu_{\underline{Apr}(A)}(x), \mu\underline{Apr}(B)(x)\}$ for all $x \in \underline{Apr}(X)$.
$\mu_{\overline{Apr}(C)}(x) = \max\{\mu_{\overline{Apr}(A)}(x), \mu_{\overline{Apr}(B)}(x)\}$ for all $x \in \overline{Apr}(X)$.

(4) $Apr(D) = Apr(A) \sqcap Apr(B)$ if and only if:
$\mu_{\underline{Apr}(D)}(x) = \min\{\mu_{\underline{Apr}(A)}(x), \mu\underline{Apr}(B)(x)\}$ for all $x \in \underline{Apr}(X)$.
$\mu_{\overline{Apr}(D)}(x) = \min\{\mu_{\overline{Apr}(A)}(x), \mu_{\overline{Apr}(B)}(x)\}$ for all $x \in \overline{Apr}(X)$.

(5) The completion of $Apr(A)$ is $Apr^c(A) = (\underline{Apr}^c(A), \overline{Apr}^c(A))$ with the membership function:
$\mu_{\underline{Apr}^c(A)}(x) = 1 - \mu_{\underline{Apr}(A)}(x)$ for all $x \in \underline{Apr}(X)$.
$\mu_{\overline{Apr}^c(A)}(x) = 1 - \mu_{\overline{Apr}(A)}(x)$ for all $x \in \overline{Apr}(X)$.

Let $Apr(A) = (\underline{Apr}(A), \overline{Apr}(A))$ be a fuzzy rough set of $Apr(X)$, we define $\overline{\mu}_{\underline{Apr}(A)} : \overline{Apr}(X) \to [0,1]$ as follows:

$$\overline{\mu}_{\underline{Apr}(A)}(x) = \begin{cases} \mu_{\underline{Apr}(A)}(x) & \text{if } x \in \underline{Apr}(X), \\ 0 & \text{if } x \in BN(X). \end{cases}$$

Definition 4.1. Let $Apr(X)$ be a rough Hilbert algebra. An interval-valued fuzzy subset $\mathcal{A}$ is given by:

$$\mathcal{A} = \{(x, [\overline{\mu}_{\underline{Apr}(A)}(x), \mu_{\overline{Apr}(A)}(x)] \mid x \in \overline{Apr}(X)\}$$

where $(\underline{Apr}(A), \overline{Apr}(A))$ is a fuzzy rough set of $Apr(X)$.

Denote $\widetilde{\mu_{\mathcal{A}}}(x) = [\overline{\mu}_{\underline{Apr}(A)}(x), \mu_{\overline{Apr}(A)}(x)]$ for all $x \in \overline{Apr}(X)$, and denote $D([0,1])$ the family of all closed sub-intervals of $[0,1]$.

If $\overline{\mu}_{\underline{Apr}(A)}(x) = \mu_{\overline{Apr}(A)}(x) = c$ where $0 \leq c \leq 1$, then we have $\widetilde{\mu_{\mathcal{A}}}(x) = [c, c]$. Thus $\widetilde{\mu_{\mathcal{A}}}(x) \in D([0,1])$ for all $x \in \overline{Apr}(X)$.

Definition 4.2. Let $D_1 = [a_1, b_1], D_2 = [a_2, b_2]$ be elements of $D([0,1])$ then we define

$$rmax(D_1, D_2) = [a_1 \vee a_2, b_1 \vee b_2]$$

$$rmin(D_1, D_2) = [a_1 \wedge a_2, b_1 \wedge b_2]$$

and $D_1 \leq D_2$ if and only if $a_1 \leq a_2$ and $b_1 \leq b_2$.

Definition 4.3. Let $Apr(X)$ be a rough Hilbert algebra and $Apr(A) = (\underline{Apr}(A), \overline{Apr}(A))$ a fuzzy rough set of $Apr(X)$. Define:

$$\underline{\mathcal{A}}_t = \{x \in \underline{Apr}(X) \mid \mu_{\underline{Apr}(A)}(x) \geq t\}$$

$$\overline{\mathcal{A}}_t = \{x \in \overline{Apr}(X) \mid \mu_{\overline{Apr}(A)}(x) \geq t\}$$

$(\underline{\mathcal{A}}_t, \overline{\mathcal{A}}_t)$ is called a level rough set.

Recall that a fuzzy set A in H is called a fuzzy subalgebra of H, if

$$\mu_A(x * y) \geq \min\{\mu_A(x), \mu_A(y)\}$$

for all $x, y \in R$.

Definition 4.4. Let $Apr(X)$ be a rough Hilbert algebra. A fuzzy rough set $Apr(A) = (\underline{Apr}(A), \overline{Apr}(A))$ in $\overline{Apr}(X)$ is called a fuzzy rough sub algebra if for all $x, y \in \overline{Apr}(X)$, we have

$$\widetilde{\mu_{\mathcal{A}}}(x * y) \geq rmin\{\widetilde{\mu_{\mathcal{A}}}(x), \widetilde{\mu_{\mathcal{A}}}(y)\}.$$

In the definition above, if X be a definable set, i.e., $\underline{Apr}_I(X) = \overline{Apr}_I(X) = X$, then X is a Hilbert algebra. Suppose $\widetilde{\mu_{\mathcal{A}}} = [\mu_{\underline{Apr}(A)}(x), \mu_{\overline{Apr}(A)}(x)]$. If $\mu_{\underline{Apr}(A)}(x) = \mu_{\overline{Apr}(A)}(x) = c$, then $\widetilde{\mu_{\mathcal{A}}} = [c, c] = c$. Therefore if we have $\underline{Apr}(A) = \overline{Apr}(A)$, then above definition is definition of fuzzy sub algebra of Hilbert algebra, and so a non empty subset A of X is a sub algebra of X if and only if the characteristic function of A is a fuzzy sub algebra of X.

Lemma 4.5. Let $Apr(X)$ be a rough Hilbert algebra. If $Apr(A) = (\underline{Apr}(A), \overline{Apr}(A))$ and $Apr(B) = (\underline{Apr}(B), \overline{Apr}(B))$ are two fuzzy rough sub algebra of $Apr(X)$ then $A \cap B$ is a fuzzy rough sub algebra of $Apr(X)$.

Proof. The proof is straightforward.

Theorem 4.6. Let $Apr(X)$ be a rough Hilbert algebra and $Apr(A) = (\underline{Apr}(A), \overline{Apr}(A))$ a fuzzy rough set of $Apr(X)$. Then $Apr(A)$ is a fuzzy rough sub algebra of $Apr(X)$ if and only if for every $t \in Im\mu_{\underline{Apr}(A)} \cap Im\mu_{\overline{Apr}(A)}$, $(\underline{\mathcal{A}}_t, \overline{\mathcal{A}}_t)$ is a rough sub algebra of $Apr(X)$.

Proof. Suppose for every $0 \leq t \leq 1$, $(\underline{\mathcal{A}}_t, \overline{\mathcal{A}}_t)$ is a rough sub algebra of $Apr(X)$. Let $rmin\{\widetilde{\mu_{\mathcal{A}}}(x), \widetilde{\mu_{\mathcal{A}}}(y)\} = [t_0, t_1]$, then

$$\min\{\overline{\mu}_{\underline{Apr}(A)}(x), \overline{\mu}_{\underline{Apr}(A)}(y)\} = t_0, \qquad \min\{\mu_{\overline{Apr}(A)}(x), \mu_{\overline{Apr}(A)}(y)\} = t_1.$$

We have $x, y \in \overline{\mathcal{A}}_{t_1}$. then $x * y \in \overline{\mathcal{A}}_{t_1}$. On the other hand if $x \in BN(X)$ or $y \in BN(X)$, then $t_0 = 0$ and so $\overline{\mu}_{\underline{Apr}(A)}(x * y) \geq 0 = t_0$. If $x \notin BN(X)$ or $y \notin BN(X)$ then $\overline{\mu}_{\underline{Apr}(A)}(x) = \mu_{\underline{Apr}(A)}(x)$ and $\overline{\mu}_{\underline{Apr}(A)}(y) = \mu_{\underline{Apr}(A)}(y)$. So $t_0 = \min\{\mu_{\underline{Apr}(A)}(x), \mu_{\underline{Apr}(A)}(y)\}$. Therefore $x \in \underline{\mathcal{A}}_{t_0}$, $y \in \underline{\mathcal{A}}_{t_0}$ hence $x * y \in \underline{\mathcal{A}}_{t_0}$ which implies $\overline{\mu}_{\underline{Apr}(A)}(x * y) \geq t_0$. Thus

$$\widetilde{\mu}_{\mathcal{A}}(x*y) = [\overline{\mu}_{\underline{Apr}(A)}(x*y), \mu_{\overline{Apr}(A)}(x*y)] \geq [t_0, t_1].$$

Therefore

$\widetilde{\mu}_{\mathcal{A}}(x*y) \geq rmin\{\widetilde{\mu}_{\mathcal{A}}(x), \widetilde{\mu}_{\mathcal{A}}(y)\}$.

Hence $Apr(A)$ is a fuzzy rough sub algebra of $Apr(X)$.

Conversely, assume that $Apr(A) = (\underline{Apr}(A), \overline{Apr}(A))$ is a fuzzy rough sub algebra of $Apr(X)$. We prove that for every $0 \leq t \leq 1$, $\underline{\mathcal{A}}_t$ and $\overline{\mathcal{A}}_t$ are sub algebras. For every $x, y \in \underline{\mathcal{A}}_t$, we conclude that $\mu_{\underline{Apr}(A)}(x) \geq t$ and $\mu_{\underline{Apr}(A)}(y) \geq t$, so

$$rmin\{\widetilde{\mu}_{\mathcal{A}}(x), \widetilde{\mu}_{\mathcal{A}}(y)\} \geq [t, \min\{\mu_{\overline{Apr}(A)}(x), \mu_{\overline{Apr}(A)}(y)\}] \geq [t, \min\{\mu_{\underline{Apr}(A)}(x), \mu_{\underline{Apr}(A)}(y)\}].$$

Then $\widetilde{\mu}_{\mathcal{A}}(x*y) \geq [t, \min\{\mu_{\underline{Apr}(A)}(x), \mu_{\underline{Apr}(A)}(y)\}]$. Since $x, y \in \underline{Apr}(X)$ we have $x*y \in \underline{Apr}(X)$ and so $\mu_{\underline{Apr}(A)}(x*y) \geq t$. Hence $x*y \in \underline{\mathcal{A}}_t$. Let $x, y \in \overline{\mathcal{A}}_t$, then we have $\mu_{\overline{Apr}(A)}(x) \geq t$ and $\mu_{\overline{Apr}(A)}(y) \geq t$, then $rmin\{\widetilde{\mu}_{\mathcal{A}}(x), \widetilde{\mu}_{\mathcal{A}}(y)\} \geq [0, t]$, therefore $\widetilde{\mu}_{\mathcal{A}}(x*y) \geq [0, t]$, hence $x*y \in \overline{\mathcal{A}}_t$.

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On $\nu - T_i-, \nu - R_i-$ and $\nu - C_i-$ axioms

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Abstract In this paper we discuss new separation axioms using $\nu-$ open sets.

Keywords $\nu - T_i-, \nu - R_i-$ and $\nu - C_i-$ spaces

§1. Introduction

Norman Levine [6] introduced the concept of semi open sets in topological spaces. After the introduction of semi open sets by Norman Levine [6] various authors have turned their attentions to this concept and it becomes the primary aim of many mathematicians to examine and explore how far the basic concepts and theorem remain true if one replaces open sets by semi open set. Replacing open sets by regularly open sets in the definition of semi open sets, due to Kuratowski, Cameron defined regularly semi open sets and V. K. Sharma renamed these sets as $\nu-$open sets and studied the basic properties, characterizations and of $\nu-$open sets. Also V. K. Sharma studied $\nu - T_i$ axioms for $i = 0, 1, 2.$ in his papers [7-8].

As a generalization of regular closed sets, $\nu-$closed sets are introduced and studied by V. K. Sharma. He also defined and studied basic properties of $\nu-$continuous and $\nu-$irresolute functions. In the present paper we tried further to study separation axioms, its properties and characterizations using $\nu-$open sets. Also we studied the interrelation with other separation axioms.

Throughout the paper a space X means a topological space (x, τ). The class of $\nu-$open sets, regularly open sets and semi-open sets are denoted by $\nu - O(X)$,$RO(X)$ and $SO(X)$ respectively. For any subset A of X its complement, interior, closure, $\nu-$interior, $\nu-$closure are denoted respectively by the symbols A^c, int(A), cl(A), sint(A), scl(A), $\nu-$int(A) and $\nu-$cl(A). $cl(Y_{/A})$ represents the closure in the subspace Y in X with relativized topology $\tau_{/Y}$.

§2. Preliminaries

Definition 2.1. A subset A of a topological space (X, τ) is said to be

(i) regularly open if $A = int(cl(A))$

(ii) semi open(regularly semi open or $\nu-$open) if there exists an open(regularly open) set O such that $O \subset A \subset cl(O)$

(iii) $\nu-$closed if its complement is $\nu-$open

(iv)$\nu-$closed if $A = \nu - cl(A)$.

Note 1. Clearly every regularly open set is $\nu-$open and every $\nu-$open set is semi-open but the reverse implications do not holds good. that is, $RO(X) \subset \nu-$O(X)$\subset$SO(X).

Definition 2.2. The intersection of all $\nu-$closed sets containing A is called $\nu-$closure of A and is denoted by $\nu - cl(A)$. The class of all $\nu-$closed sets are denoted by $\nu - CL(X, \tau)$. The union of all $\nu-$open sets contained in A is called the $\nu-$interior of A, denoted by $\nu-$int(A)

Definition 2.3. A function f: $(X, \tau) \to (Y, \sigma)$ is said to be

(i) $\nu-$continuous if the inverse image of any open[closed]set in Y is a $\nu-$open[$\nu-$closed]set in X.

(ii)$\nu-$irresolute if the inverse image of any $\nu-$open[$\nu-$closed]set in Y is a $\nu-$open[$\nu-$closed] set in X.

(iii)$\nu-$open[$\nu-$closed] if the image of every $\nu-$open[$\nu-$closed]set is $\nu-$open[$\nu-$closed].

(iv)$\nu-$homeomorphism if f is bijective, $\nu-$irresolute and $\nu-$open.

Definition 2.4. Let x be a point of (X, τ) and V be a subset of X, then V is said to be $\nu-$neighbourhood of x if there exists a $\nu-$open set U of X such that $x \in U \subset V$.

Definition 2.5. Let A be a subset of (X, τ). A point $x \in A$ is said to be $\nu-$limit point of A iff for each $\nu-$open set U containing x, $U \cap (A - \{x\}) \neq \phi$.

Definition 2.6. The set of all $\nu-$limit points of A is called $\nu-$derived set of A and is denoted by $D_{\nu-}(A)$.

Theorem 2.1. If x is a $\nu-$limit point of any subset A of the topological space (X, τ), then every $\nu-$neighbourhood of x contains infinitely many distinct points.

Proof. Let x by any $\nu-$limit point of a subset A of (X, τ). Let U be any $\nu-$neighbourhood of x and let $V = A \cap (U - \{x\})$. If V is finite, then $X - V$ is a $\nu-$neighborhood of x. Evidently this neighborhood contains no point of A, except possibly x, contradicting that x is an $\nu-$accumulation point of A. So V is infinite, showing that every $\nu-$neighborhood of x contains infinitely many points of A.

Theorem 2.2. Let f:$(X, \tau)\to$(Y,σ)is $\nu-$irresolute.
(i) If x is $\nu-$limit point of the subset A of X then f(x) is the $\nu-$limit point of f(A).
(ii)If x is $\nu-$limit point of $A \subset X$ then there exists (x_n) in A such that $(x_n) \to x$ iff $f(x_n) \to f$(x).

Theorem 2.3. $A \subset X$ is $\nu-$open iff there exists a regularly open set F such that $int(F) \subset A \subset F$ iff int(cl(A))$\subset$ cl(int(A). iff A is semi open and semi closed in X. iff $A =$ scl(sint(A)) and $A =$ sint(scl(A)).

Theorem 2.4. union and intersection of any two $\nu-$open sets is not $\nu-$open.
Intersection of a regular open set and a $\nu-$open set is $\nu-$open.
If $B \subset X$ such that $A \subset$B$\subset$cl(A) then B is $\nu-$open iff A is $\nu-$open.
If A and R are regularly open and S is $\nu-$open such that $R \subset$S$\subset cl(R)$. Then $A\cap$R $= \phi \Rightarrow A \cap S = \phi$.

Theorem 2.5. If A is $\nu-$open then
(i) int(cl(A) = int(A)
(ii) clR = clA where A is r-open such that $R \subset A \subset cl(R)$.

Theorem 2.6.
(i) $\nu -$ open set is $r -$ open if $A \subset int(cl(A))$.
(ii)$s -$ open set is $\nu -$ open if $int(cl(A) \subset A$.
(iii)$s -$ closed set is $\nu -$ open if $A \subset cl(int(A))$.

Lemma 2.1. $RO(X, \tau) = int\nu - O(X, \tau)$.

Theorem 2.7. In a semi regular space, int$\nu - O(X, \tau)$ generates topology.

Theorem 2.8. (i) Let $A \subseteq Y \subseteq X$ and Y is regularly open subspace of X then A is $\nu-$open in X iff A is $\nu-$open in $\tau_{/Y}$.

(ii) Let $Y \subseteq X$ and $A \in \nu - O(Y, \tau_{/Y})$ then $A \in \nu - O(X, \tau)$ iff Y is $\nu-$open in X.

(iii) Let $Y \subseteq X$ and A is a $\nu-$neighborhood of x in Y. Then A is a $\nu-$neighborhood of x in Y iff Y is $\nu-$open in X.

Theorem 2.9. An almost continuous and almost open map is $\nu-$irresolute.

Example 1. Identity map is $\nu-$irresolute

§3. $\nu - T_0$ space

In this section we define new separation axioms using $\nu-$open sets, their properties and characterizations are verified.

Before doing this let we start with an example:

Example 3.1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X\}$ then
$SO(X) = \{\phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, X\}$
$\nu - O(X) = \{\phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{b, c, d\}, X\}$ and
$RO(X) = \{\phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, X\}$.

From the above example one can see that $\{c, d\}$ and $\{a, b, d\}$ are semi open but neither r-open nor $\nu-$open. $\{a, c\}$ is semi-open and $\nu-$open but not r-open. $\{b, c\}$ is $\nu-$open but not r-open.

Definition 3.1. A space (X, τ) is said to be $\nu - T_0$ space if for each pair of distinct points x, y of X there exists a $\nu-$open set G containing either of the point x or y.

Example 3.2.

1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. then X is $\nu - T_0$
2. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ then X is $\nu - T_0$.
3. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$. then X is not $\nu - T_0$.

Note. Since every $\nu-$open set is semi-open, $\nu - T_0$ space is semi-T_0 but the converse is not true in general, given by the following example.

Example 3.3. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$ then X is semi-T_0 but not $\nu - T_0$.

Note. Since every r-open set is $\nu-$open, $r - T_0$ space is $\nu - T_0$. but the converse is not true in general, given by the following example.

Example 3.4. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ then X is $\nu - T_0$ but not $r - T_0$.

Theorem 3.1.

(i) Every regular open subspace of $\nu - T_0$ space is $\nu - T_0$.

(ii) The product of $\nu - T_0$ spaces is again $\nu - T_0$

Proof.

(i)Let Y be an regular open subspace of a $\nu - T_0$ space X and let x, y be two distinct points of Y, which in turn are the distinct points X, then there exists a $\nu-$open set U containing x or y say, x but not y. By theorem (2.8), $U \cap Y$ is $\nu-$open containing x but not y. Hence Y is $\nu - T_0$.

(ii) Let X and Y be $\nu - T_0$ spaces and let $x, y \in X \times Y$, $x \neq$y. Let $x = (a, b), y = (c, d)$. Without loss of generality suppose that $a \neq c$ and $b \neq d$. Since a and c are distinct points of X, there exist $\nu-$open set U in X containing a or c, say $a \in U$, $c \notin U$. Similarly for b and d are distinct points of Y, there exist $\nu-$open set G in Y containing b or d, say $b \in G$, $d \notin G$. Then$(U \times G)$ is $\nu-$open in $X \times Y$ containing x but not y. Hence $X \times Y$ is $\nu - T_0$.

Theorem 3.2. A space is $\nu - T_0$ iff disjoint points of X has disjoint $\nu-$closures.

Proof. Assume X is $\nu - T_0$ and let x, y be two distinct points of X. suppose U is a $\nu-$open set containing x but not y, then $y \in \nu - cl\{y\} \subset X - U$ and so $x \notin \nu - cl\{y\}$. Hence $\nu-$cl{x}$\neq \nu - cl\{y\}$.

Conversely, let x, y be any two points in X such that $\nu - cl\{x\} \neq \nu - cl\{y\}$. Now for $\nu - cl\{x\} \neq \nu - cl\{y\}$ implies $\nu - cl\{x\} \subset X - \nu - cl\{y\} = U$(say) a $\nu-$open set in X. This is true for every $\nu - cl\{x\}$. Thus $\cap\nu - cl\{x\} \subseteq U$ where $x \in \cap\nu - cl\{x\} \subseteq U \in \nu - O(\tau)$ which in turn implies $\cap\nu - cl\{x\} \subseteq U$ where $x \in U \in \nu - O(\tau)$. Hence X is $\nu - T_0$.

Theorem 3.3. A space is $\nu-T_0$ iff For every $x \in X$, there exists a $\nu-$open set U containing x such that the subspace U is $\nu - T_0$.

Theorem 3.5.

If f: (X, τ) $\rightarrow (Y, \sigma)$ is $\nu-$irresolute and Y is $\nu - T_0$ then X is $\nu - T_0$.

Proof. omitted.

Remark 3.1. $\nu - T_0$ and T_0 are mutually independent.

Example 3.5.

1. Let $X = \{a, b, c\}$ and $\tau = \{\phi$, {a}, {a, b}, X} then X is T_0 but not $\nu - T_0$.

2. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$ then X is $\nu - T_0$ but not T_0.

Remark 3.2. $\nu - T_0$ and T_1 are mutually independent.

Example 3.6.

1. Let X be any countable set and $\tau =$ cofinite topology then X is T_1 but not $\nu - T_0$.

2. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X\}$ then X is $\nu - T_0$ but not T_1.

§4. $\nu - T_1$ space

Definition 4.1. A space (X, τ) is said to be $\nu - T_1$ space if for each pair of distinct points x, y of X there exists a $\nu-$open set G containing x but not y and a $\nu-$open set H containing y but not x.

Example 4.1.

1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$. then X is not $\nu - T_1$.
2. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$ then X is not $\nu - T_1$.
3. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ then X is $\nu - T_1$.

Note. Every $\nu - T_1$ space is $\nu - T_0$, but converse is not true by the following example.

Example 4.2. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. then X is $\nu - T_0$ but not $\nu - T_1$.

Theorem 4.1. A topological space is $\nu - T_1$ iff each singleton set is $\nu-$closed.

Proof. X is $\nu - T_1$
$\Leftrightarrow$ for each pair of points $x \neq y$ there exists $\nu-$open sets G and H such that $x \in G - H$ and $y \in H - G$
$\Leftrightarrow$ the point $x \in X - H$
$\Leftrightarrow$ the point $x \in \{x\} \subseteq X - H$
$\Leftrightarrow$ x is in the complement of $\nu-$open set H
$\Leftrightarrow$ $\{x\}$ is $\nu-$closed
$\Leftrightarrow$ each singleton set is $\nu-$closed in X.

Theorem 4.2.

(i) Every regular open subspace of $\nu - T_1$ space is $\nu - T_1$;

(ii)The product of $\nu - T_1$ spaces is again $\nu - T_1$.

Proof.

(i)Let Y be an regular open subspace of a $\nu - T_1$ space X and let $x \in Y$. Since X is $\nu - T_1, X - \{x\}$ is $\nu-$open in X. Now Y being regular open, By theorem (2.8) $[X - \{x\}] \cap Y = Y - \{x\}$ is $\nu-$open in Y, consequently $\{x\}$ is $\nu-$closed in Y. Hence by theorem (4.1), Y is $\nu - T_1$.

(ii)Let X and Y be $\nu - T_1$ spaces and let $x, y \in X \times Y$, $x \neq y$. Let $x = (a, b), y = (c, d)$. Without loss of generality suppose that $a \neq c$ and $b \neq d$. Since a and c are distinct points of X, there exist $\nu-$open sets U and V in X such that $a \in U$, $c \notin U$ and $c \in V$, $a \notin V$. Similarly for b and d are distinct points of Y, there exist $\nu-$open sets G and H in Y such that $b \in G$, $d \notin G$ and $d \in H$, $b \notin H$. Then $(U \times G)$ and $(V \times$H) are $\nu-$open in $X \times Y$ containing x and y respectively. Also $(U \times G) \cap (V \times H) = (U \cap V) \times (G \cap H) = \phi$. Hence $X \times Y$ is $\nu - T_1$.

Theorem 4.3. A space is $\nu - T_1$ iff disjoint points of X has disjoint $\nu-$closures.

Proof. Obvious from theorems (4.1), (4.2) and (3.2).

Theorem 4.4. Suppose x is a $\nu-$limit point of a subset of A of a $\nu - T_1$ space X. Then every neighborhood of x contains infinitely many points of A.

Proof. Let U be a $\nu-$neighborhood of x and let $V = A \cap (U - \{x\})$. If V is finite then $X - V$ is a $\nu-$open set containing x and so $U \cap (X - V)$ is a $\nu-$neighborhood of x, which is a

contradiction for x is a $\nu-$limit point of A. Hence V is infinite. Thus every neighborhood of x contains infinitely many points of A.

§5. $\nu - T_2$ space

Definition 5.1. A space (X, τ) is said to be $\nu - T_2$ space if for each pair of distinct points x, y of X there exists disjoint $\nu-$open sets G and H such that G containing x but not y and H containing y but not x.

Example 5.1.

1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$. Then X is not $\nu - T_2$.
2. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$ then X is not $\nu - T_2$.
3. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ then X is $\nu - T_2$.

Note. Every $\nu - T_2$ space is $\nu - T_1$, but converse is not true in general.

Theorem 5.1.

(i) Every regular open subspace of $\nu - T_2$ space is $\nu - T_2$.

(ii) The product of $\nu - T_2$ spaces is again $\nu - T_2$.

Proof. (i)Let Y be an regular open subspace of a $\nu - T_2$ space X and let $x \neq y \in Y$. Since X is $\nu - T_2$, there exist disjoint $\nu-$open sets U and V in X such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Now Y being regular open, By theorem (2.8), $U \cap Y$ is $\nu-$open containing x but not y and $V \cap Y$ is $\nu-$open containing y but not x in Y. Hence by theorem (4.1) Y is $\nu - T_1$.

(ii)Let X and Y be $\nu - T_2$ spaces and let $x, y \in X \times Y, x \neq y$. Let $x = (a, b)$, $y = (c, d)$. Without loss of generality suppose that $a \neq c$ and $b \neq d$. Since a and c are distinct points of X, there exist disjoint $\nu-$open sets U_4 and V in X such that $a \in U$, $c \in V$. Similarly for b and d are distinct points of Y, there exist disjoint $\nu-$open sets G and H in Y such that $b \in G$, $d \in H$, then $(U \times G)$ and $(V \times H)$ are $\nu-$open in $X \times Y$ containing x and y respectively. Also $(U \times G) \cap (V \times \mathrm{H}) = (\mathrm{U} \cap \mathrm{V}) \times (G \cap H) = \phi$. Hence $X \times Y$ is $\nu - T_2$.

Theorem 5.2. For any topological space,
(i) (X, τ) is $\nu - T_2 \Rightarrow$ (ii) (X, τ) is $\nu - T_1 \Rightarrow$ (iii) (X, τ) is $\nu - T_0$.

Proof. Obvious from the definitions and hence omitted.

Theorem 5.3. A space X is $\nu - T_2$ iff the intersection of all $\nu-$closed, $\nu-$neighbourhoods of each point of the space is reduced to that point.

Proof. Let X be $\nu - T_2$ and $x \in X$, then for each $y \neq$x in X, there exists $\nu-$open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Since $x \in U - V$, hence $X - V$ is a $\nu-$closed, $\nu-$neighbourhood of x to which y does not belong. Consequently, the intersection of all $\nu-$closed, $\nu-$neighbourhoods of x is reduced to $\{x\}$.

Conversely let $x, y \in X$, such that $y \neq x \mathrm{in} X$, then by hypothesis there exists a $\nu-$closed, $\nu-$neighbourhood U of x such that $y \notin U$. Now there exists a $\nu-$open set G such that $x \in G \subset U$. Thus G and $X - U$ are disjoint $\nu-$open sets containing x and y respectively. Hence X is $\nu - T_2$.

Theorem 5.4. If to each point x of a topological space X, there exists a $\nu-$closed, $\nu-$open subset of X containing x and which is also a $\nu - T_2$ subspace of X, then X is $\nu - T_2$.

Proof. Let $x \in X$, U a $\nu-$closed, $\nu-$open subset of X containing x and which is also a $\nu - T_2$ subspace of X, then the intersection of all $\nu-$closed, $\nu-$neighbourhoods of $x \in U$ is reduced to x. U being $\nu-$closed, $\nu-$open, these are the $\nu-$closed, $\nu-$neighbourhoods of x in X. Thus the intersection of all $\nu-$closed, $\nu-$neighbourhoods of x is reduced to $\{x\}$. Hence, by theorem (5.3) X is $\nu - T_2$.

Theorem 5.5. If X is $\nu - T_2$ then the diagonal $\Delta in X \times X$ is $\nu-$closed.

Proof. Suppose $(x, y) \in X \times X - \Delta$. As $(x, y) \notin \Delta$ and $x \neq y$. Since X is $\nu - T_2$ there exists $\nu-$open sets U and V in X such that $x \in U$, $y \in V$ and $U \cap V = \phi$. $U \cap V = \phi$ implies that $(U \times V) \cap \Delta = \phi$ and therefore $(U \times V) \subset X \times X - \Delta$. Further $(x, y) \in (U \times V)$ and $(U \times V)$ is $\nu-$open in $X \times X$(theorem 9.13,[2]). Hence $X \times X$ is $\nu-$open. Thus Δ is $\nu-$closed.

Theorem 5.6. In $\nu - T_2$-space, $\nu-$limits of sequences, if exists, are unique.

Proof. Let (x_n) be a sequence in $\nu - T_2$-space X and if $(x_n) \to x$; $(x_n) \to y$ as n $\to \infty$. If $x \neq y$ then, for X is $\nu - T_2$ there exists disjoint $\nu-$open sets $U; V$ in X such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Then there exists N_1, $N_2 \in N$ such that $x_n \in U$ for all $n \geq N_1$, and $x_n \in V$ for all n $\geq N_2$. Let m be an integer greater than both N_1 and N_2. Then $x_m \in U \cap V$ contradicting the fact $U \cap V = \phi$. So $x = y$ and thus the $\nu-$limits are unique.

Theorem 5.7. Show that in a $\nu - T_2$ space, a point and disjoint $\nu-$compact subspace can be separated by disjoint $\nu-$open sets.

Proof. Let X be a $\nu - T_2$ space, x a point in X and C the disjoint $\nu-$compact subspace of X not containing x. Let y be a point in C then for $x \neq y$ in X and X is $\nu - T_2$, there exist disjoint $\nu-$neighborhoods G_x and H_y. Allowing this for each y in C, we obtain a class of $H_y^{'s}$ whose union covers C; and since C is $\nu-$compact, some finite subclass, which we denote by $H_i, i = 1$ to n covers C, say $C \subset \bigcup_{i=1}^{N} H_i,$. If $G_i, i = 1$ to n are the $\nu-$neighborhoods of x corresponding to the $H_i^{'s}$, we put $G = \bigcup_{i=1}^{N} G_i$ and $H = \bigcap_{i=1}^{N} H_i$, satisfying the required properties.

Corollary 5.1.

(i) Show that in an $r - T_2$ space, a point and disjoint r-compact subspace can be separated by disjoint $\nu-$open sets.

(ii) Show that in an $r - T_2$ space, a point and disjoint r-compact subspace can be separated by disjoint semi-open sets.

(iii)Show that in a $\nu - T_2$ space, a point and disjoint $\nu-$compact subspace can be separated by disjoint semi-open sets.

(iv) Show that in a $\nu - T_2$ space, a point and disjoint semi-compact subspace can be separated by disjoint semi-open sets.

Proof. Omitted.

Theorem 5.8. Every $\nu-$compact subspace of a $\nu - T_2$ space is $\nu-$closed.

Proof. Let C be $\nu-$compact subspace of $\nu - T_2$ space. If x by any point in C^c, by above theorem x has a $\nu-$neighborhood G such that $x \in G \subset C^c$. This shows that C^c is the union of $\nu-$open sets and therefore C^c is $\nu-$open. Thus C is $\nu-$ closed.

Corollary 5.2.

(i) Every $\nu-$compact subspace of a $\nu - T_2$ space is semi-closed.

(ii) Every r-compact subspace of a $r - T_2$ space is $\nu-$closed.

(iii) Every r-compact subspace of a $r - T_2$ space is semi-closed.

Proof. Obvious from note 1.

Theorem 5.9. Every $\nu-$irresolute map from a $\nu-$compact space into a $\nu - T_2$ space is $\nu-$closed.

Proof. Suppose $f\colon X \to Y$ is $\nu-$irresolute where X is $\nu-$compact and Y is $\nu - T_2$. Let C be any $\nu-$closed subset of X. Then C is $\nu-$compact and so $f(C)$ is $\nu-$compact. But then$f(C)$is $\nu-$closed in Y. Hence the image of any $\nu-$closed set in X is $\nu-$closed set in Y.Thus $f(C)$ is $\nu-$closed.

Theorem 5.10. Any $\nu-$continuous bijection from a $\nu-$compact space onto a $\nu - T_2$ space is a $\nu-$homeomorphism.

Proof. Let $f\colon X \to Y$ be a $\nu-$continuous bijection from a $\nu-$compact space onto a $\nu - T_2$ space. Let G be an $\nu-$open subset of X. Then $X - G$ is $\nu-$closed and hence $f(X - G)$is $\nu-$closed. Since f is bijective $f(X - G) = Y - f(G)$. Therefore $f(G)$ is $\nu-$open in Y implies f is $\nu-$open. Hence f is bijective $\nu-$irresolute and $\nu-$open. Thus f is $\nu-$homeomorphism.

Theorem 5.11. Let (X, τ) be a topological space. Then the following are equivalent:

(i) (X, τ) is $\nu - T_2$.

(ii) For each pair $x, y \in X$ such that $x \neq y$, there exists a $\nu-$open, $\nu-$closed set V such that $x \in V$ and $y \notin V$, and

(iii) For each pair $x, y \in X$ such that $x \neq y$, there exists a $\nu-$continuous function $f : (X, \tau) \to (0, 1]$ such that $f(x) = 0$ and $f(C) = 1$.

Theorem 5.12. Let X be a topological space and Y be a T_2 space. If $f\colon (X, \tau) \to (Y, \sigma)$ be one-one and r-continuous. Then X is $\nu - T_2$.

Proof. Let x and y be any two distinct points of X. since f is one-one $x \neq y$ implies $f(x) \neq f(y)$. Therefore there exists disjoint open sets U and V in Y such that $f(x) \in U$, $f(y) \in V$, then $f^{-1}(U)$ and $f^{-1}(V)$ are $r-$open sets in X, which in turn $\nu-$open sets in X, such that $x \in f^{-1}(U)$, $y \in f^{-1}(V)$ and $f^{-1}(U) \cap f^{-1}(V) = \phi$. Hence X is $\nu - T_2$.

Corollary 5.3. Let X be a topological space and Y be a $r - T_2$ space. If f: $(X, \tau) \to (Y, \sigma)$ be one-one and r-continuous. Then X is $\nu - T_2$.

Proof. Since every regular open set is $\nu-$open, the proof follows from the above theorem.

Corollary 5.4. Let X be a topological space and Y be a $\nu - T_2$ space. If f: $(X, \tau) \to (Y, \sigma)$ be one-one and r-irresolute. Then X is $\nu - T_2$.

Corollary 5.5. Let X be a topological space and Y be a $\nu - T_2$ space. If f: $(X, \tau) \to (Y, \sigma)$ be one-one and $\nu-$irresolute. Then X is $\nu - T_2$.

Proof. Straight forward and so omitted.

Problem. If X be any topological space, Y a $\nu - T_2$ space and A be any subspace of X. If f:$(A, \tau_{/A}) \to (Y, \sigma)$ is $\nu-$ continuous. Is there exists any extension F:$(X, \tau) \to (Y, \sigma)$.

Remark 5.1. $\nu - T_2 \Rightarrow \nu - T_1 \Rightarrow \nu - T_0$ none is reversible.

Remark 5.2.

$$\begin{array}{ccccc} T_0 \Leftarrow & & T_1 \Leftarrow & & T_2 \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{semi-}T_0 & \Leftarrow & \text{semi-}T_1 & \Leftarrow & \text{semi-}T_2 \\ \Uparrow & & \Uparrow & & \Uparrow \\ \nu - T_0 \Leftarrow & & \nu - T_1 \Leftarrow & & \nu - T_2 \\ \Uparrow & & \Uparrow & & \Uparrow \\ r - T_0 \Leftarrow & & r - T_1 \Leftarrow & & r - T_2 \end{array} \quad \text{ll1}$$

§6. $\nu - T_3$ space

Definition 6.1. A space (X, τ) is said to be

(i) $\nu-$regular ($\nu-T_3$)space at a point $x \in X$ if for every closed subset F of X not containing x, there exists disjoint $\nu-$open sets G and H such that $F \subset G$ and $x \in H$.

(ii)$\nu - T_3$ space if for a closed set F and a point $x \notin F$, there exists disjoint $\nu-$open sets G and H such that $F \subset G$ and $x \in H$.

Theorem 6.1. (i)Every regular open subspace of $\nu - T_3$ space is $\nu - T_3$.
(ii)The product of $\nu - T_3$ spaces is again $\nu - T_3$.

Theorem 6.2. For a topological space X the following conditions are equivalent
(i) X is $\nu-$regular
(ii) For any $x \in X$ and any $\nu-$open set G containing x there exists a $\nu-$open set H containing x such that $\nu-$cl(H)$\subset$G.
(iii) The family of all $\nu-$closed neighborhoods of any point of X forms a local base at that point.

Proof. (i) $\Rightarrow$ (ii)X is $\nu-$regular. Let G be a $\nu-$open set containing $x \in X$, then $X - G$ is $\nu-$closed set not containing x. So by $\nu-$regularity, there exists disjoint $\nu-$open sets U and V such that $x \in U, X - G \subset V$. Then $U \subset X - G$ and hence $\nu - cl(U) \subset X - V$, since $X - V$ is $\nu-$closed. But $X - V \subset G$ and thus $\nu - cl(U) \subset G$ which implies $\nu - cl(H) \subset G$ for $U = H$.
(ii) $\Rightarrow$ (iii)Let N be a $\nu-$neighborhood of $x \in X$. Let $G = \nu - int(N) \subset X$ then G be a $\nu-$open set containing x and so by (ii), there exist a $\nu-$open set H containing x such that $\nu-$cl(H)$\subset G$. Then $\nu-$cl(H) is a $\nu-$closed neighborhood of x contained in N. Hence the family of all $\nu-$closed neighborhoods of x is a local base at x.
(iii) $\Rightarrow$ (i) Let B be a $\nu-$closed subset of X not containing $x \in X$, then $X - B$ is a neighborhood of x. So by (iii) there exists a $\nu-$closed neighborhood M of x such that $M \subset X - B$. Let $U = \nu - int(M)$ and $V = X - M$, then U and V are mutually disjoint $\nu-$open sets such that $x \in U$ and $B \subset V$. Hence X is $\nu-$regular.

Theorem 6.3. Every $\nu-$compact, $\nu - T_2$ space is a $\nu - T_3$ space.

Proof. Let X be a $\nu-$compact, $\nu - T_2$ space. Then every $\nu-$closed subset of X is $\nu-$compact and so any point and disjoint $\nu-$compact subspace of X can be separated by disjoint $\nu-$open sets (by Theorem 5.7). Hence the space X is $\nu - T_3$.

Theorem 6.4. Let X be $\nu - T_3$, C a $\nu-$closed subset of X and F a $\nu-$compact subset of X disjoint from C, then there exists disjoint $\nu-$open sets U and V such that $C \subseteq U$ and F

$\subseteq V$.

Proof. Analogous to Theorem (5.7) and so omitted.

§7. Almost $\nu-$ regular space

Definition 7.1. A space (X, τ) is said to be Almost-regular space if for an $x \in X$ and for each regular open set U containing x there exists an open set V such that $x \in V \subset cl(V) \subset$U.

Definition 7.2. A space (X, τ) is said to be Almost $\nu-$regular space if for an $x \in X$ and for each $\nu-$open set U containing x there exists an open set V such that $x \in V \subset cl(V) \subset U$.

Theorem. Every almost regular space is almost $\nu-$regular.

Proof. Since every regular open set is $\nu-$open, the result follows from the definitions.

§8. $\nu - T_4$ space

Definition 8.1. A space (X, τ) is said to be $\nu - T_4$ space if for a pair of disjoint closed sets F_1 and F_2, there exists disjoint $\nu-$open sets G and H such that $F_1 \subset G$ and $F_2 \subset H$.

Lemma.

(i) Show that the $\nu-$closed subspace of a $\nu - T_4$ space is $\nu - T_4$.

(ii) Show that the regular closed subspace of a $\nu - T_4$ space is $\nu - T_4$.

(iii) Product of $\nu - T_4$ spaces is not a $\nu - T_4$ space.

Theorem 8.1. For a topological space X the following conditions are equivalent

(i) X is $\nu-$Normal.

(ii) For any $\nu-$closed set C and any $\nu-$open set G containing C, there exists an $\nu-$open set H such that C$\subset$H and cl(H)$\subset$G.

(iii) For any $\nu-$closed set C and any $\nu-$open set G containing C, there exists an $\nu-$open set H and a $\nu-$closed set K such that $C \subset$H$\subset$K$\subset G$.

Theorem 8.2. Every $\nu-$compact, $\nu - T_2$-space is $\nu-$Normal.

Proof. Let X be $\nu-$compact and $\nu - T_2$ space. Let A and B be any disjoint closed subsets of X. Since X is $\nu-$compact, A and B are disjoint $\nu-$compact subspaces of X. Let x be a point of A. By theorem (5.7) and X is $\nu - T_2$, x and B have disjoint $\nu-$neighborhoods G_x and H_B. Allowing x to vary over A, we obtain a class of $G_x^{'s}$ whose union contains A; and since A is $\nu-$compact, some finite subclass G_i $i = 1$ to n covers A. If $H_i^{'s}$ $i = 1$ to n are the $\nu-$neighborhoods of B corresponding to $G_i^{'s}$ it is clear that $G = \bigcup_{i=1}^{N} G_i$, and $H = \bigcap_{i=1}^{N} H_i$, are disjoint $\nu-$neighborhoods of A and B. Thus X is $\nu-$Normal.

Theorem 8.3. Every $\nu-$compact, $\nu - T_3$ space is a $\nu - T_4$ space. Weakening finite cover to countable cover we have the following results.

Corollary. (i) Every $\nu-$ Lindeloff, $\nu - T_3$ space is a $\nu - T_4$ space.
(ii)Every $\nu-$regular, second countable space is $\nu-$Normal.

Theorem 8.4. Every $\nu-$closed subspace of a $\nu-$Normal space is $\nu-$ Normal.

Theorem 8.5. Let X be $\nu - T_1$. Show that X is $\nu-$Normal iff each $\nu-$neighborhood of a $\nu-$closed set F contains the closure of some $\nu-$neighborhood of F.

Proof. same as that of Theorem(8.1).

Theorem 8.6. Let (X, τ) be a topological space. Then

(i) $(X, \tau)$is $\nu-$regular iff for each $\nu-$closed set C and each $x \notin C$, there exists a $\nu-$continuous function f: $(X, \tau) \to [0, 1]$ such that $f(\mathrm{x}) = 0$ and $f(\mathrm{C}) = 1$.

(ii) $(X, \tau)$is $\nu-$normal iff for each pair of disjoint $\nu-$ closed sets A and B, there exists a $\nu-$continuous function f: $(X, \tau) \to [0, 1]$ such that $f(A) = 0$ and $f(\mathrm{B}) = 1$.

§9. Lightly $\nu-$ normal

Definition 9.1. A subset A of a topological space (X, τ) is said to be $\nu-$zero set of X if there exists a $\nu-$continuous function $f: X \to R$ such that $A = x \in X : f(x) = 0$.

Its complement is called co$-\nu-$zero set of X.

Definition 9.2. A space (X, τ) is said to be lightly r-Normal if for a pair of disjoint closed set A and a $r-$zero set B, there exists disjoint $r-$open sets G and H such that $A \subset G$ and $B \subset H$.

Definition 9.3. A space (X, τ) is said to be lightly $\nu-$Normal if for a pair of disjoint closed set A and a $\nu-$zero set B, there exists disjoint $\nu-$open sets G and H such that $A \subset G$ and $B \subset H$.

The following theorem characterizes lightly $\nu-$normal spaces.

Theorem 9.1. For a topological space X the following conditions are equivalent:

(i) X is lightly $\nu-$Normal.

(ii) For every $\nu-$closed set A and every co-$\nu-$zero set G containing A, there exists an $\nu-$open set H such that $A \subset H \subset \nu-\mathrm{cl(H)} \subset G$.

(iii) For every $\nu-$zero set A and every $\nu-$open set G containing A, there exists an $\nu-$open set H such that $A \subset V \subset \nu-\mathrm{cl(V)} \subset G$.

(iv) For each pair of disjoint closed set A and a zero set B, there exists disjoint $\nu-$open sets U and V such that $A \subset U$ and $B \subset V$ and $\nu-\mathrm{clU} \cap \nu-\mathrm{clV} = \phi$.

Theorem 9.2. Every lightly r-Normal space is lightly $\nu-$Normal.

Theorem 9.3. Every r-closed subspace of a lightly $\nu-$Normal is lightly $\nu-$Normal. The following interrelations exists

Remark.

$$\begin{array}{ccc} \text{Lightly Normal} & \Rightarrow & \text{Lightly semi-Normal} \\ \Downarrow & & \Uparrow \\ \text{Lightly r-Normal} & \Rightarrow & \text{Lightly } \nu-\text{Normal} \end{array}$$

§10. $\nu - C_0$ space

Definition 10.1. A space $(X, \tau)$is said to be $\nu - C_0$ space if for each pair of distinct points x, y of X there exists a $\nu-$open set G whose closure contains either of the point x or y.

Example.

1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$.then X is not $\nu - C_0$.
2. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$ then X is $\nu - C_0$.

3. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ then X is $\nu - C_0$.

Theorem 10.1.
(i) Every subspace of $\nu - C_0$ space is $\nu - C_0$.
(ii) Every $\nu - T_0$ spaces is $\nu - C_0$.
(iii) Product of $\nu - C_0$ spaces are $\nu - C_0$.

Theorem 10.2. Let (X, τ) be any $\nu - C_0$ space and A be any non empty subset of X then A is $\nu - C_0$ iff $(A, \tau_{/A})$ is $\nu - C_0$.

§11. $\nu - C_1$ space

Definition 11.1. A space (X, τ) is said to be $\nu - C_1$ space if for each pair of distinct points x, y of X there exists a $\nu-$open set G whose closure containing x but not y and a $\nu-$open set H whose closure containing y but not x.

Example 11.1. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ then X is $\nu - C_1$.

Note. Every $\nu - C_1$ space is $\nu - C_0$, but converse need not be true in general as shown by the following example.

Example 11.2. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$ then X is $\nu - C_0$ but not $\nu - C_1$.

Theorem 11.1.
(i) Every subspace of $\nu - C_1$ space is $\nu - C_1$.
(ii) Every $\nu - T_1$ spaces is $\nu - C_1$.
(iii) Product of $\nu - C_1$ spaces are $\nu - C_1$.

Theorem 11.2. Let (X, τ) be any $\nu - C_1$ space and A be any non empty subset of X then A is $\nu - C_1$ iff $(A, \tau_{/A})$ is $\nu - C_1$.

Theorem 11.3. Every $\nu - C_1$ space is $\nu - C_0$.

Theorem 11.4.
(i) If (X, τ) is $\nu - C_1$ then each singleton set is $\nu-$closed.
(ii)In a $\nu - C_1$ space disjoint points of X has disjoint $\nu-$ closures.

§12. $\nu - C_2$ space

Definition 12.1. A space (X, τ) is said to be $\nu - C_2$ space if for each pair of distinct points x, y of X there exists disjoint $\nu-$open sets G and H such that G containing x but not y and H containing y but not x.

Example 12.1. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ then X is $\nu - C_2$.

Note. Every $\nu - C_2$ space is $\nu - C_0$, but converse need not be true in general as shown by the following example.

Example 12.2. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$ then X is $\nu - C_0$ but not $\nu - C_2$.

Theorem 12.1. (i) Every subspace of $\nu - C_2$ space is $\nu - C_2$.
(ii) Every $\nu - T_2$ spaces is $\nu - C_2$.
(iii) Product of $\nu - C_2$ spaces are $\nu - C_2$.

Theorem 12.2. Let (X, τ) be any $\nu - C_2$ space and A be any non empty subset of X then A is $\nu - C_2$ iff $(A, \tau_{/A})$ is $\nu - C_2$.

Theorem 12.3. Every $\nu - C_2$ space is $\nu - C_0$.

Remark 12.1. $\nu - C_2 \Rightarrow \nu - C_1 \Rightarrow \nu - C_0$ none is reversible

Remark 12.2.

$$\begin{array}{lll} C_2 \Rightarrow & C_1 \Rightarrow & C_0 \\ \Downarrow & \Downarrow & \Downarrow \\ \text{semi-}C_2 \Rightarrow & \text{semi-}C_1 \Rightarrow & \text{semi-}C_0 \\ \Uparrow & \Uparrow & \Uparrow \\ \nu - C_2 \Rightarrow & \nu - C_1 \Rightarrow & \nu - C_0 \\ \Uparrow & \Uparrow & \Uparrow \\ \text{r-}C_2 \Rightarrow & \text{r-}C_1 \Rightarrow & \text{r-}C_0 \end{array}$$

§13. $\nu - R_0$ space

Definition 13.1. A topological space X is said to be

(i) $\nu - R_0$ if and only if $\nu - cl\{x\} \subseteq G$ whenever $x \in G \in \nu - O(\tau)$.

(ii) weakly $\nu - R_0$ iff $\cap \nu - cl\{x\} = \phi$.

Example 13.1.

1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$.
2. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$.
3. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$.
4. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$.

All the above examples are both $\nu - R_0$ and weakly $\nu - R_0$.

Definition 13.2. Let X be a topological space and $x \in X$. Then

(i) $\nu-$kernel of x is denoted and defined by $\nu - ker\{x\} = \cap\{U : U \in \nu - O(X)\}$ and $x \in U$.

(ii) $\nu-$Kernel of $F = \cap\{U : U \in \nu - O(X)$ and $F \subset$ U$\}$

Lemma. Let A be any subset of a topological space X, Then $\nu - ker\{x\} = \{x \in X : \nu - cl\{x\} \cap A = \phi.\}$

Proof. $x \notin \nu{-}ker\{A\}$ implies $x \notin \cap\{U : U \in \nu - O(X)$ and $A \subset U\}$, so there is a $\nu-$open set U such that $A \subset U$ and $x \notin U$. Therefore, $\nu - cl\{x\} \cap U = \phi$ and $\nu - cl\{x\} \cap A = \phi$. Now, $\nu - cl\{x\} \cap A = \phi$, so $G = X - [\nu - cl\{x\}]$ is a $\nu-$open set such that $A \subset G$. Also, x does not belong to the intersection of all $\nu-$open neighborhoods of A, so $x \notin \nu - ker\{A\}$.

Theorem 13.1.

(i)Every R_0 space is $\nu - R_0$ and every $r - R_0$ space is $\nu - R_0$.

(ii) Every weakly-R_0 space is weakly $\nu - R_0$ and every weakly-R_0 space is weakly $\nu - R_0$.

(iii) Every $\nu - R_0$ space is weakly $\nu - R_0$.

Proof. Straight forward from the definitions and so omitted.

Converse of the above theorem is not true in general by the following examples.

Example 13.2.

1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$. Clearly, the space (X, τ) is weakly $\nu - R_0$. Since $\nu - cl\{x\} = \phi$. But it is not $\nu - R_0$, for $\{b\} \in X$ is $\nu-$open and $\nu - cl\{b\} = X \not\subset \{a, b\}$.

2. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{b, c\}, X\}$. Clearly, the space (X, τ) is not $\nu - R_0$ and not R_0.

3. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$. Clearly $\cap cl\{x\} = b, c \neq \phi$ and $\cap \nu - cl\{x\} = \phi$. Thus the space is weakly $\nu - R_0$, but not weakly-R_0.

Theorem 13.2. A topological space is $\nu - R_0$ iff given two distinct points of X their $\nu-$closures are distinct.

Proof. Assume X is $\nu - R_0$ and let x, y be two distinct points of X. suppose U is a $\nu-$open set containing x but not y, then $y \in \nu - cl\{y\} \subset X - U$ and so $x \notin \nu - cl\{y\}$. Hence $\nu - cl\{x\} \neq \nu - cl\{y\}$.

Conversely, let x, y be any two points in X such that $\nu - cl\{x\} \neq \nu - cl\{y\}$. Now for $\nu - cl\{x\} \neq \nu - cl\{y\}$ implies $\nu - cl\{x\} \subset X - \nu - cl\{y\} = U$(say) a $\nu-$open set in X. This is true for every $\nu - cl\{x\}$. Thus $\cap \nu - cl\{x\} \subseteq U$ where $x \in \nu - cl\{x\} \subseteq U \in \nu - O(\tau)$, which in turn implies $\cap \nu - cl\{x\} \subseteq U$ where $x \in U \in \nu - O(\tau)$. Hence X is $\nu - R_0$.

Theorem 13.3. A topological space X is weakly $\nu - R_0$ iff $\nu-$ker{x}$\neq X$ for any $x \in X$.

Proof. Let $x_0 \in X$ be such that $\nu-$ker$\{x_0\} = X$. This means that x_0 is not contained in any proper $\nu-$open subset of X. Thus x_0 belongs to the $\nu-$closure of every singleton set. Hence $x_0 \in \cap\ \nu - cl\{x\}$, a contradiction.

Conversely assume that $\nu-$ker{x}$\neq X$ for any $x \in X$. If there is a point $x_0 \in$ X such that $x_0 \in \cap\ \nu - cl\{x\}$, then every $\nu-$open set containing x_0 must contain every point of X. Therefore, the unique $\nu-$open set containing x_0 is X. Hence $\nu-$ker$\{x_0\} = X$, which is a contradiction. Thus X is weakly $\nu - R_0$.

Theorem 13.4. The following statements are equivalent:

(i) (X, τ) is $\nu - R_0$ space.

(ii) For each $x \in X$, $\nu-$cl{x}$\subset \nu - ker\{x\}$.

(iii) For any $\nu-$closed set F and a point $x \notin F$, there exists a $\nu-$open set U such that $x \notin U$ and $F \subset U$.

(iv) Each $\nu-$closed set F can be expressed as $F = \cap\{G : G is \nu-$open and $F \subset G\}$.

(v) Each $\nu-$open set G can be expressed as the union of $\nu-$closed sets A contained in G.

(vi) For each $\nu-$closed set F, $x \notin F$ implies $\nu - cl\{x\} \cap F = \phi$.

Proof. (i) $\Rightarrow$ (ii) For any $x \in X$, we have $\nu - ker\{x\} = \cap\{U : U \in \nu - O(X)$ and $x \in U\}$. Since $X$is $\nu - R_0$, each $\nu-$open set containing x contains $\nu - cl\{x\}$. Hence $\nu - cl\{x\} \subset \nu - ker\{x\}$.

(ii) $\Rightarrow$ (iii) Let F be any $\nu-$closed set and x a point in X such that $x \notin F$. Then for any $y \in F$, $\nu - cl\{y\} \subset F$ and so $x \notin \nu - cl\{y\} \Rightarrow y \notin \nu - cl\{x\}^{(13.02)}$ that is there exists a $\nu-$open set U_y such that $y \in U_y$ and $x \notin U_y$ for all $y \in F$. Let $U = \cup\{U_y : U_y$ is $\nu-$open,$y \in U_y$ and$x \notin$

$U_y\}$. Then U is $\nu-$open such that $x \notin U$ and $F \subset U$.

(iii) $\Rightarrow$(iv) Let F be any $\nu-$closed set and $N = \cap\{G : G is \nu - \text{open and } F \subset G\}$. Then $F \subset N \longrightarrow$ (1) Let $x \notin F$, then by (iii) there exists a $\nu-$open set G such that $x \notin G$ and $F \subset G$, hence $x \notin N$ which implies $x \in N \Rightarrow x \in F$. Hence $N \subset F \longrightarrow$ (2). Therefore from (1)&(2), each $\nu-$closed set F can be expressed as $F = \cap\{G : G is \nu - \text{open and } F \subset G\}$

(iv) $\Rightarrow$ (v) obvious.

(v) $\Rightarrow$ (vi) Let F be any $\nu-$closed set and $x \notin F$. Then $X - F = G$ is a $\nu-$open set containing x. Then by (v), G can be expressed as the union of $\nu-$closed sets A contained in G, and so there is a $\nu-$closed set M such that $x \in M \subset G$; and hence $\nu - cl\{x\} \subset G$ which implies $\nu - cl\{x\} \cap F = \phi$.

(vi) $\Rightarrow$ (i) Let G be any $\nu-$open set and $x \in G$. Then $x \notin X - G$, which is a $\nu-$closed set. Therefore by (vi) $\nu - cl\{x\} \cap X - G = \phi$, which implies that $\nu - cl\{x\} \subseteq G$. Thus (X, τ) is $\nu - R_0$ space.

Theorem 13.5. Let $f{:}X \to Y$ be a $\nu-$closed one-one function. If X is weakly $\nu - R_0$, then so is Y.

Theorem 13.6. If a space X is weakly $\nu - R_0$, then for every space $Y, X \times Y$ is also weakly $\nu - R_0$.

Proof. we have $\cap\nu - cl\{(x, y)\} \subseteq \cap\{\nu - cl\{x\} \times \nu - cl\{y\}\}$
$= \cap[\nu - cl\{x\}] \times [\nu - cl\{y\}]$
$\subseteq \phi \times Y$
$= \phi$.
Hence $X \times Y$ is $\nu - R_0$.

Corollary.

(i)If the spaces X and Y are weakly $\nu - R_0$, then $X \times Y$ is also weakly $\nu - R_0$.

(ii) If the spaces X and Y are (weakly-)R_0, then $X \times Y$ is also weakly $\nu - R_0$.

(iii) If the spaces $X$and Y are $\nu - R_0$, then $X \times Y$ is also weakly $\nu - R_0$.

(iv) If X is $\nu - R_0$ and Y are weakly R_0, then $X \times Y$ is also weakly $\nu - R_0$.

Proof. Obvious from theorem (13.1).

Following diagram indicates the interrelation among the separations axioms studied in this section:

Remark.

$$\begin{array}{ll} R_0 \Rightarrow & weakly - R_0 \\ \Downarrow & \Downarrow \\ \nu - R_0 \Rightarrow & weakly - \nu - R_0 \end{array}$$

§14. $\nu - R_1$ space

Definition14.1. A topological space X is said to be $\nu - R_1$ if and only if for $x, y \in X$

such that $\nu - cl\{x\} \neq \nu - cl\{y\}$ there exists $\nu-$open sets U and V such that $\nu - cl\{x\} \subseteq U$ and $\nu - cl\{y\} \subseteq V$.

Example 14.1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, then X is $\nu - R_1$.

Example 14.2.

Note. Every $\nu - R_1$ space is $\nu - R_0$, but converse is not true by the following examples.

Example 14.3.

1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$.then X is $\nu - R_0$ and but not $\nu - R_1$.
2. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, X\}$ then X is $\nu - R_0$ but not $\nu - R_1$.
3. Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$ then X is $\nu - R_0$ but not $\nu - R_1$.

Theorem 14.1.

(i) Every R_1 space is $\nu - R_1$.

(ii) Every $\nu - R_1$ space is $\nu - R_0$.

(iii) Every subspace of $\nu - R_1$ space is again $\nu - R_1$.

(iv) Product of any two $\nu - R_1$ spaces is again $\nu - R_1$.

Proof. straight forward from the definitions.

Converse of the above theorem need not be true in general.

Theorem 14.2. A topological space is $\nu - R_1$ iff given two distinct points of X their $\nu-$closures are distinct.

Proof. omitted.

Theorem 14.3. (X, τ) is $\nu - R_1$ iff for every pair of points x, y of X such that $\nu - cl\{x\} \neq \nu - cl\{y\}$ there exists $\nu-$open sets U and V such that $\nu - cl\{x\} \subseteq U$ and $\nu - cl\{y\} \subseteq V$.

Proof. Let (X, τ) is $\nu - R_1$ space. Let x, y be points of X such that $\nu - cl\{x\} \neq \nu - cl\{y\}$. Then there exists $\nu-$open sets U and V such that $x \in U,\ y \in V$. Since $\nu - R_1$ is $\nu - R_0$, therefore $x \in U$ implies $\nu - cl\{x\} \subset U$ and $y \in V$ implies $\nu - cl\{y\} \subset V$. Hence the part. The converse part is obvious.

Theorem 14.4. Every $\nu - T_2$ space is $\nu - R_1$.

Proof. obvious.

The converse is not true. However, we have the following result.

Theorem 14.5. Every $\nu - T_1,\ \nu - R_1$ space is $\nu - T_2$.

Proof. Let (X, τ) is $\nu - T_1$ and $\nu - R_1$ space. Let x, y be disjoint points of X. Since $(X, \tau)$is $\nu - T_1,\ \{x\}$ is $\nu-$closed set and $\{y\}$ is $\nu-$closed set such that $\nu - cl\{x\} \neq \nu - cl\{y\}$. Since (X, τ) is $\nu - R_1$, there exists disjoint $\nu-$open sets U and V such that $x \in U,\ y \in V$. Hence (X, τ) is $\nu - T_2$.

Corollary. (X, τ) is $\nu - T_2$ iff it is $\nu - R_1$ and $\nu - T_1$.

Theorem 14.6. In any topological space the following are equivalent

(i)$(X, \tau)$is $\nu - R_1$.

(ii) $\cap \nu - cl\{x\} = \{x\}$.

(iii) For every $x \in X$, intersection of all $\nu-$neighborhoods of x is $\{x\}$.

Proof. (i) $\Rightarrow$ (ii) Let $x \in X$ and $y \in \nu - cl\{x\}$, where $y \neq x$. since X is $\nu - R_1$, therefore there is a $\nu-$open set U such that $y \in U$, $x \notin U$ or $x \in U$, $y \notin U$. In either case $y \notin \nu - cl\{x\}$.

Hence $\cap\nu - cl\{x\} = \{x\}$.

(ii) $\Rightarrow$ (iii) If $x, y \in X$ where $y \neq x$, then x$\notin \cap\, \nu - cl\{y\}$, so there is a $\nu-$open set containing x but not y. Therefore y does not belong to the intersection of all $\nu-$neighborhoods of x. Hence intersection of all $\nu-$neighborhoods of x is $\{x\}$.

(iii) $\Rightarrow$ (i) Let $x, y \in X$ where $y \neq x$. by hypothesis, y does not belong to the intersection of all $\nu-$neighborhoods of x. therefore there exists a $\nu-$open set containing x but not y. Therefore y does not belong to the intersection of all $\nu-$neighborhoods of x. Hence intersection of all $\nu-$neighborhoods of x is $\{x\}$.

Theorem 14.7. Let (X, τ) be a topological space. Then the following are equivalent:

(i) (X, τ) is $\nu - R_1$.

(ii) For each pair $x, y \in X$ such that $\nu-$cl{x}$\neq \nu - cl\{y\}$, there exists a $\nu-$open, $\nu-$closed set V such that $x \in V$ and $y \notin V$, and

(iii) For each pair $x, y \in X$ such that $\nu-$cl{x}$\neq \nu-$cl{y}, there exists a $\nu-$continuous function
$f\colon (X, \tau) \to [0, 1]$ such that $f(x) = 0 and f(C) = 1$.

Proof. (i) $\Rightarrow$ (ii) Let $x, y \in X$ such that $\nu - cl\{x\} \neq \nu - cl\{y\}$, then there exists disjoint $\nu-$open sets U and W such that $\nu - cl\{x\} \subset U$ and $\nu - cl\{y\} \subset W$and $V = \nu - cl\{U\}$ is $\nu-$open and $\nu-$closed such that $x \in V$ and $y \notin V$.

(ii) $\Rightarrow$ (iii) Let $x, y \in X$ such that $\nu - cl\{x\} \neq \nu - cl\{y\}$, and let V be $\nu-$open and $\nu-$closed such that $x \in V$ and $y \notin V$. Then $f\colon (X, \tau) \to [0, 1]$ defined by $f(z) = 0$if $z \in V$ and $f(z) = 1$if $z \notin V$ satisfied the desired properties.

(iii) $\Rightarrow$ (i) Let $x, y \in X$ such that $\nu - cl\{x\} \neq \nu - cl\{y\}$, let $f\colon (X, \tau) \Rightarrow [0, 1]$ such that f is $\nu-$continuous, $f(x) = 0$ and $f(y) = 1$. Then $U = f^{-1}([0, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, 1])$ are disjoint $\nu-$open and $\nu-$closed sets in X, such that $\nu - cl\{x\} \subset U$ and $\nu - cl\{y\} \subset V$.

Following diagram indicates the interrelation among the separations axioms studied in this section

Remark.

$$\begin{array}{llll}
R_0 \Leftarrow & R_1 \Rightarrow & weakly - R_1 \Rightarrow & weakly - R_0 \\
\Downarrow & \Downarrow & \Downarrow & \Downarrow \\
\nu - R_0 \Rightarrow & \nu - R_1 \Rightarrow & weakly - \nu - R_1 \Rightarrow & weakly - \nu - R_0 \\
\Uparrow & \Uparrow & \Uparrow & \Uparrow \\
r - R_0 \Rightarrow & r - R_1 \Rightarrow & weakly - r - R_1 \Rightarrow & weakly - r - R_0 \\
\Uparrow & \Uparrow & \Uparrow & \Uparrow \\
semi - R_0 \Rightarrow & semi - R_1 \Rightarrow & weakly - semi - R_1 \Rightarrow & weakly - semi - R_0
\end{array}$$

Conclusion. In this paper we defined new separation axioms using a new variety of open sets called $\nu-$open sets and studied their interrelations with other separation axioms.

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A note on the near pseudo Smarandache function

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Abstract Vyawahare and Purohit [1] introduced the near pseudo Smarandache function, $K(n)$. In this paper, we derive some more recurrence formulas satisfied by $K(n)$. We also derive some new series, and give an expression for the sum of the first n terms of the sequence $\{K(n)\}$.

Keywords The near pseudo Smarandache function, recurrence formulas, the sum of the first n terms.

§1. Introduction

Vyawahare and Purohit [1] introduced a new function, called the near pseudo Smarandache function, and denoted by $K(n)$, is defined as follows.

Definition 1.1. The near pseudo Smarandache function, $K : N \longrightarrow N$, is

$$K(n) = \sum_{i=1}^{n} i + k(n),$$

where $k(n) = \min\left\{k : k \in N, n| \sum_{i=1}^{n} i + k\right\}$.

The following theorem, due to Vyawahare and Purohit [1], gives explicit expressions for $k(n)$ and $K(n)$.

Theorem 1.1. For any $n \in N$,

$$k(n) = \begin{cases} n, & \text{if } n \text{ is odd,} \\ \dfrac{n}{2}, & \text{if } n \text{ is even.} \end{cases}$$

with

$$K(n) = \begin{cases} \dfrac{n(n+3)}{2}, & \text{if } n \text{ is odd,} \\ \dfrac{n(n+2)}{2}, & \text{if } n \text{ is even.} \end{cases}$$

In [1], Vyawahare and Purohit give a wide range of results related to the near pseudo Smarandache function. Some of them are given in the following lemmas.

Lemma 1.1. $K(2n+1) - K(2n) = 3n + 2$, for any integer $n \in N$.

Lemma 1.2. $K(2n+1) - K(2n-1) = 4n+3$, for any integer $n \in N$.

This paper gives some recurrence relations satisfied by the near pseudo Smarandache function $K(n)$. These are given in Section 2. In Section 3, we give some series involving the functions $K(n)$ and $k(n)$. We also give an explicit expression for the sum of the first n terms of the sequences $\{K(n)\}_{n=1}^{\infty}$ and$\{k(n)\}_{n=1}^{\infty}$. We conclude this paper with some remarks in the final Section 4.

§2. More recurrence relations

In this section, we derive some new recurrence relations that are satisfied by the near pseudo Smarandache function $K(n)$.

Lemma 2.1. For any integer $n \in N$,

$$K(2n) - K(2n-1) = n+1$$

Proof. Writing $K(2n) - K(2n-1)$ in the following form, and then using Lemma 1.2 and Lemma 1.1 (in this order), we get

$$\begin{aligned} K(2n) - K(2n-1) &= [K(2n+1) - K(2n-1)] - [K(2n+1) - K(2n)] \\ &= (4n+3) - (3n+2), \end{aligned}$$

which now gives the desired result.

We now have the following result.

Corollary 2.1. $K(n)$ is strictly increasing in n.

Proof. From Theorem 1.1, we see that both the subsequences $\{K(2n-1)\}_{n=1}^{\infty}$ and $\{K(2n)\}_{n=1}^{\infty}$ are strictly increasing. This, together with Lemma 1.1 and Lemma 2.1, shows that$\{K(n)\}_{n=1}^{\infty}$ is strictly increasing.

Lemma 2.2. For any integer $n \in N$,

$$K(2n+2) - K(2n) = 4(n+1).$$

Proof. Using Theorem 1.1, we get

$$K(2n+2) - K(2n) = \frac{(2n+2)(2n+4)}{2} - \frac{(2n)(2n+2)}{2} = 4(n+1),$$

after some algebraic simplifications.

Lemma 1.2 shows that the subsequence $\{K(2n-1)\}_{n=1}^{\infty}$ is strictly convex in the sense that

$$K(2n+3) - K(2n+1) = 4n+7 > 4n+3 = K(2n+1) - K(2n-1).$$

From Lemma 2.2, we see that the subsequence $\{K(2n)\}_{n=1}^{\infty}$ is also strictly convex, since

$$K(2n+4) - K(2n+2) = 4(n+2) > 4(n+1) = K(2n+2) - K(2n).$$

However, the sequence $\{K(n)\}_{n=1}^{\infty}$ is not convex, since

$$K(2n+2) - K(2n+1) = n+2 < 3n+2 = K(2n+1) - K(2n).$$

Note that

$$K(2n+1) - K(2n) = 3n+2 > n+1 = K(2n) - K(2n-1).$$

Corollary 2.2. $K(n+1) = K(n)$ has no solution.

Proof. If n is odd, say, $n = 2m-1$ for some integer $m > 1$, then from Lemma 2.1,

$$K(2m) - K(2m-1) = m+1 > 0,$$

and if n is even, say, $n = 2m$ for some integer $m > 1$, then from Lemma 1.1,

$$K(2m+1) - K(2m) = 3m+2 > 0.$$

These two inequalities establish the result.

Corollary 2.3. $K(n+2) = K(n)$ has no solution.

Proof. If n is odd, say, $n = 2m-1$ for some integer $m > 1$, then from Lemma 1.2,

$$K(2m+1) - K(2m-1) = 4m+3 > 0,$$

and if n is even, say, $n = 2m$ for some integer $m > 1$, then from Lemma 2.2,

$$K(2m+2) - K(2m) = 4(m+1) > 0.$$

Thus, the result is established.

Lemma 2.3. For any integers $m, n \in N$ with $m > n$,

(1) $K(2m-1) - K(2n-1) = (m-n)(2m+2n+1)$,

(2) $K(2m) - K(2n) = 2(m-n)(m+n+1)$.

Proof. For any integers m and n with $m \geq n \geq 1$, from Theorem 1.1,

(1)$K(2m-1) - K(2n-1) = (2m-1)(m+1) - (2n-1)(n+2)$

$= 2(m^2 - n^2) + (m-n)$

$= (m-n)(2m+2n+1)$,

(2) $K(2m) - K(2n) = 2m(m+1) - 2n(n+2)$

$= 2(m^2 - n^2) + 2(m-n)$

$= 2(m-n)(m+n+1)$,

which we intended to prove.

Corollary 2.4. For any integers $m, n \in N$ with $m > n$,

(1) $K(2m-1) - K(2n-1) = \dfrac{m-n}{m+n+2} K(2m+2n+1)$,

(2) $K(2m) - K(2n) = \dfrac{m-n}{m+n} K(2m+2n)$.

Proof. Let n and m be any two integers with $n \geq m \geq 1$.

(1) Since

$$K(2m+2n+1) = (m+n+2)(2m+2n+1),$$

we get the result by virtue of part (1) of Lemma 2.3 above.

(2) Note that

$$K(2m+2n) = 2(m+n)(m+n+1).$$

This, together with part (2) of Lemma 2.3, gives the desired result.

It may be mentioned here that, Lemma 1.2 is a particular case of part (1) of Corollary 2.4(when $m = n+1$) and Lemma 2.2 is a particular case of part (2) of Corollary 2.4 (when $m = n+1$).

§3. Series involving the functions $K(n)$ and $k(n)$

In this section, we derive some results in series involving $K(n)$ and $k(n)$. We also give explicit expressions of the $n-th$ partial sums in both the cases.

Lemma 3.1. For any integer $n \in N$,

(1) $\sum\limits_{m=1}^{n} [K(2m) - K(2m-1)] = \dfrac{n(n+3)}{2}$,

(2) $\sum\limits_{m=1}^{n} [K(2m+1) - K(2m)] = 3\dfrac{n(n+3)}{2} - n = \dfrac{n(3n+7)}{2}$.

Proof. (1) From Lemma 2.1,

$K(2m) - K(2m-1) = m+1$ for any integer $m \geq 1$.

Now, summing over m from 1 to n, we get

$$\sum_{m=1}^{n} [K(2m) - K(2m-1)] = \sum_{m=1}^{n} (m+1) = \frac{n(n+1)}{2} + n = \frac{n(n+3)}{2},$$

which is the result desired.

(2) From Lemma 1.1,

$K(2m+1) - K(2m) + 1 = 3(m+1)$ for any integer $m \geq 1$.

Therefore, summing over m from 1 to n, we get

$$\sum_{m=1}^{n} [K(2m+1) - K(2m) + 1] = 3\sum_{m=1}^{n} (m+1) = 3\left\{\frac{n(n+1)}{2} + n\right\} = 3\frac{n(n+3)}{2},$$

that is,

$$\sum_{m=1}^{n} [K(2m+1) - K(2m)] + n = 3\frac{n(n+3)}{2}$$

from which the desired result follows immediately.

Corollary 3.1. If n is an odd integer, then

(1) $\sum\limits_{m=1}^{n} [K(2m) - K(2m-1)] = K(n)$,

(2) $\sum\limits_{m=1}^{n} [K(2m+1) - K(2m)] = 3K(n) - n$.

Proof. Both the results follow immediately by virtue of Theorem 1.1 and Lemma 3.1.

Let $\{S_n\}$ be the sequence of $n-th$ partial sums of the sequence $\{K(n)\}_{n=1}^{\infty}$, so that

$$S_n = \sum_{m=1}^{n} K(m), n \geqslant 1$$

and likewise, let $\{s_n\}$ be the sequence of $n-th$ partial sums of $\{k(n)\}_{n=1}^{\infty}$.
Then, we have the following result.

Lemma 3.2. For any integer $n \geq 1$,
(1) $S_{2n} = \frac{n}{6}(8n^2 + 21n + 7)$,
(2) $S_{2n+1} = \frac{1}{6}(8n^3 + 33n^2 + 37n + 12)$.

Proof. From Theorem 1.1, for any integer $m \geq 1$,

$$K(2m-1) + K(2m) = (2m-1)(m+1) + 2m(m+1) = 4m^2 + 3m - 1.$$

(1) Since S_{2n} can be written as

$$S_{2n} = K(1) + K(2) + \cdots + K(2n) = \sum_{m=1}^{n} [K(2m-1) + K(2m)],$$

we get,

$$\begin{aligned} S_{2n} &= \sum_{m=1}^{n} (4m^2 + 3m - 1) = 4\sum_{m=1}^{n} m^2 + 3\sum_{m=1}^{n} m - n \\ &= 4\left\{\frac{n(n+1)(2n+1)}{6}\right\} + 3\left\{\frac{n(n+1)}{2}\right\} - n \end{aligned}$$

which now gives the desired result after some algebraic simplifications.

(2) Since $S_{2n+1} = S_{2n} + K(2n+1)$
from part (1) above, together with Theorem 1.1, we get

$$S_{2n+1} = \frac{n}{6}(8n^2 + 21n + 7) + (2n+1)(2n+3),$$

which gives the desired expression for S_{2n+1} after algebraic manipulations.

From Definition 1.1, we see that

$$k(2n-1) = 2n - 1 = k(2(2n-1)) \text{ for any integer } n \geq 1.$$

It then follows that the $n-th$ term of the subsequence $\{k(2n-1)\}_{n=1}^{\infty}$ is $2n-1$, while the $n-th$ term of the subsequence $\{k(2n)\}_{n=1}^{\infty}$ is n.

Lemma 3.3. For any integer $n \geq 1$,
(1) $s_{2n} = \frac{n}{2}(3n+1)$,
(2) $s_{2n+1} = \frac{1}{2}(3n^2 + 5n + 2)$.

Proof. We first note that, for any integer $m \geq 1$,

$$k(2m-1) + k(2m) = 2m - 1 + m = 3m - 1.$$

(1) We get the result from the following expression for s_{2n} :

$$s_{2n} = k(1) + k(2) + \cdots + k(2n) = \sum_{m=1}^{n} [k(2m-1) + k(2m)] = \sum_{m=1}^{n} (3m-1).$$

(2) Using part (1) above, we get

$$s_{2n+1} = s_{2n} + k(2n+1) = \frac{n}{2}(3n+1) + (2n+1),$$

which gives the result after some algebraic simplifications.

Lemma 3.4. For any integer $n \geq 0$,

$$\sum_{m=0}^{n} K(a^m) = \begin{cases} \frac{1}{2}\,\frac{a^{n+1}-1}{a^2-1}(a^{n+1}+3a+4), & \text{if } a \text{ is odd} \\ \frac{1}{2}\left[\frac{a^{n+1}-1}{a^2-1}(a^{n+1}+2a+3)+1\right], & \text{if } a \text{ is even} \end{cases}$$

Proof. We consider the two cases separately.

(1) When a is odd.

In this case,

$$\sum_{m=0}^{n} K(a^m) = \sum_{m=0}^{n} \frac{a^m(a^m+3)}{2} = \frac{1}{2}\left(\sum_{m=0}^{n} a^{2m} + 3\sum_{m=0}^{n} a^m\right).$$

Now, the first series on the right is a geometric series with common ratio a^2, while the second one is geometric with common ratio a. Therefore,

$$\sum_{m=0}^{n} K(a^m) = \frac{1}{2}\left(\frac{a^{2(n+1)}-1}{a^2-1} + 3\frac{a^{n+1}-1}{a-1}\right),$$

which gives the desired result after some algebraic simplifications.

(2) When a is even.

In this case,

$$\begin{aligned}\sum_{m=0}^{n} K(a^m) &= 2 + \sum_{m=1}^{n} \frac{a^m(a^m+2)}{2} = \frac{1}{2}\left(1+\sum_{m=0}^{n} a^{2m}\right) + \sum_{m=0}^{n} a^m \\ &= \frac{1}{2}\left(1+\frac{a^{2(n+1)}-1}{a^2-1}\right) + \frac{a^{n+1}-1}{a-1}.\end{aligned}$$

Now, simplifying the above, we get the result desired.

It can be shown that (see Yongfeng Zhang [2]) the series $\sum\limits_{n=1}^{\infty} \frac{1}{[K(n)]^s}$ is convergent for any real number $s > \frac{1}{2}$, and $\sum\limits_{n=1}^{\infty} \frac{1}{[K(n)]^s}$ is convergent for any real number $s > 1$ (Yu Wang [3]) with

$$\sum_{n=1}^{n} \frac{1}{[k(n)]^s} = \zeta(s)(2-\frac{1}{2^s}),$$

where $\zeta(s)$ is the Riemann zeta function.

§4. Some remarks

Since

$$\frac{n(n+3)}{2} > n, \quad \frac{n(n+2)}{2} > n \text{ for any integer } n \geqslant 1,$$

it follows that $K(n) > n$ for any integer $n \geq 1$. A consequence of this is that, the equation $K(n) = n$ has no solution.

Let T_m be the $m-th$ triangular number, that is

$$T_m = \frac{m(m+1)}{2}, m \geq 1.$$

Then, T_m satisfies the following recurrence relation :

$$T_{m+1} = T_m + m + 1, m \geq 1.$$

Now, by Definition 1.1,

$$K(m) - k(m) = T_m, m \geq 1,$$

so that

$$K(m) - [k(m) - m] + 1 = T_m + (m+1) = T_{m+1}.$$

Now, if m is odd, say, $m = 2n - 1$ (for some integer $n \geq 1$), then $k(m) - m = 0$, so that

$$K(2n-1) + 1 = T_{2n}$$

is a triangular number.
Again, since

$$K(m) - \left[k(m) - \frac{m}{2}\right] + \frac{m}{2} + 1 = T_{m+1},$$

it follows that, when m is even, say, $m = 2n$ (for some integer $n \geq 1$), then

$$K(2n-1) + n + 1 = T_{2n+1}$$

is a triangular number.

In a recent paper, [2] introduced a new function, which may be called the near pseudo Smarandache function of order t, where $t \geq 1$ is a fixed integer, and is defined as

$$K_t(n) = \sum_{i=1}^{n} i^t + k_t(n), \text{for any } n \in N$$

where

$$k_t(n) = \min\left\{k : k \in N, n| \sum_{i=1}^{n} i^t + k\right\}.$$

Then, the function introduced by Vyawahare and Purohit [1] is the near pseudo Smarandache function of order 1, that is,

$$K(n) = K_1(n) \quad \text{with } k(n) = k_1(n), n \in N.$$

In [3], Yu Wang has given the explicit expressions for $k_2(n)$ and $k_3(n)$, including the convergence of two infinite series involving these two functions. But the properties of $K_t(n)$ and $k_t(n)$ still remain to be investigated. The following two lemmas give the expressions for $k_2(n)$ and $k_3(n)$, due to Yu Wang [3].

Lemma 4.1. For any integer $n \geq 1$,

$$k_2(n) = \begin{cases} \frac{5n}{6}, & \text{if } n = 6m \\ \frac{n}{3}, & \text{if } n = 3(2m+1) \\ n, & \text{if } n = 6m+1 \text{ or } n = 6m+5 \\ \frac{n}{2}, & \text{if } n = 2(3m+1) \text{ or } n = 2(3m+2) \end{cases}$$

Lemma 4.2. For any integer $n \geq 1$,

$$k_3(n) = \begin{cases} \frac{n}{2}, \text{if } n = 2(2m+1) \\ n, \text{ otherwise} \end{cases}$$

Using the above two lemmas, we get the expressions for $K_2(n)$ and $K_3(n)$, given below.

Lemma 4.3. For any integer $n \geq 1$,

$$K_2(n) = \begin{cases} \frac{n}{6}(n^2+3n+6), & \text{if } n = 6m \\ \frac{n}{6}(n^2+3n+3), & \text{if } n = 3(2m+1) \\ \frac{n}{6}(n^2+3n+7), & \text{if } n = 6m+1 \text{ or } n = 6m+5 \\ \frac{n}{6}(n^2+3n+4), & \text{if } n = 2(3m+1) \text{ or } n = 2(3m+2) \end{cases}$$

Lemma 4.4. For any integer $n \geq 1$,

$$K_3(n) = \begin{cases} \frac{1}{4}n(n+2)(n^2+1), & \text{if } n = 2(2m+1) \\ \frac{1}{4}n(n^3+2n^2+n+4), & \text{otherwise} \end{cases}$$

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Research on the scheduling decision in fuzzy multi-resource emergency systems [1]

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Abstract The demand of resources in emergency systems often is uncertain and multiple. It has an important practical significance for research on the scheduling decision in multi-resource emergency systems. For the certain resource demand, the rich research results have achieved. The uncertain demand of single resource is discussed [7]. In this paper, for the uncertain demand of multi-resource, a weight coefficient is defined and a corresponding membership function is established for each resource by the policy-maker. A fuzzy mathematical programming model is established and an algorithm is proposed, which we find the maximum satisfactory degree of time restraint under the lowest satisfactory degree of resource demand. An example illustrates that the model is reasonable and the obtained algorithm is effective.

Keywords Emergency systems,fuzzy number,satisfactory degree,scheduling decision in multi-resource

§1. Introduction

The scheduling problem of resources in emergency systems has become a hot topic. For the emergency-time is a certain number or interval number and the different type of the resource consumption are discussed [4-5]. The fewest of retrieval depots as the objective function in the multi-resource problem is proposed [6]. The concept of the connection number is used to study scheduling problem in continuous consumption multi-resource [8]. The two-layer optimization model which make the fewest of retrieval depots is established [9-10]. Either the single resource or the multi-resource, the number of retrieval depots should be as few as possible on the base of the earliest emergency time. But there are many uncertain factors in the actual emergency systems.The emergency resources often are uncertain and multiple. The scheduling problem of the single resource in fuzzy emergency systems is studied [7]. In this paper, the scheduling decision of fuzzy multi-resource in emergency systems is presented on the base of [7].

On the one hand, from the point of the resources demand, a satisfactory degree of resource demand restraint is defined for any decision of each resource, and the scheduling decision should make the satisfactory degree as great as possible. On the other hand, the emergency-start-time should be as early as possible. Thus, a fuzzy optimization model which compute the maximum

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satisfactory degree of time restraint under the lowest satisfactory degree of resource demand restraint is established. An algorithm is proposed and an optimal decision is obtained.

§2. Establishment and solution to the fuzzy optimization model

2.1. Formulation of the problem

Consider an emergency systems having n retrieval depots $A_1, A_2, \cdots, A_n$, m kinds of demanded emergency resources, a emergency depot A. The supply quantity of emergency resources is fuzzy number $\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_m$. Let $x_{ij}^{'}$ be the stores quantity of the jth emergency resource of A_i. The $t_i (t_i \geq 0)$ represents the time from A_i to A. Assume $t_1 \leq t_2 \leq \cdots \leq t_n$. It is requested a scheduling decision that should determine the type and the quantity of each resource. Base on the lowest satisfactory degree of resource demand restraint, the emergency-start-time should be as early as possible. What's more, from the point of expense, the number of retrieval depots should be the fewest.

We describe an emergency scheduling decision f in matrix form:

$$f = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1m} \\ & & \vdots & \\ x_{i1} & x_{i2} & \cdots & x_{im} \\ & & \vdots & \\ x_{n1} & x_{n2} & \cdots & x_{nm} \end{pmatrix}$$

Where, $0 \leq x_{ij} \leq x_{ij}^{'}$, $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, m$. Let x_{ij} be the quantity of the jth resource supplied from A_i. Row i is all quantity of resources from A_i and column j is the quantity of the jth resource from each retrieval depot.

The jth column of f is the emergency decision of the jth resource, noted as ϕ_j:

$$\phi_j = \{(A_1, x_{1j}), (A_2, x_{2j}), \cdots, (A_n, x_{nj})\}.$$

Let $x(\phi_j)$ be the quantity of resource of the ϕ_j, $x(\phi_j) = \sum_{i=1}^{n} x_{ij}$, $j = 1, 2, \cdots, m$.

$T(\phi_j)$ represents the emergency-start-time of the jth resource for ϕ_j. Because of the disposable consumption emergency systems, it is obviously that $T(\phi_j) = \max_{1 \leq i \leq n} t_j^i$,$(j = 1, 2, \cdots, m)$. t_j^i stands for the emergency time from A_i to A when select the decision ϕ_j.

2.2. Establishment of model

Before the emergency activity starts, the policy-maker usually gives the value a_j, b_j, c_j, d, e of fuzzy set according to the actual situation of the emergency depot.

Suppose $\tilde{x}_j$ be the supply quantity of the jth resource, $\tilde{x}_j$ is the triangle fuzzy number. Its membership function is as following:

$$\mu_{\tilde{x_j}}(x_j) = \begin{cases} \dfrac{x_j - a_j}{b_j - a_j} & a_j \le x_j \le b_j, \\ 1 & x_j = b_j, \\ \dfrac{x_j - c_j}{b_j - c_j} & b_j \le x_j \le c_j, \\ 0, & \text{other}. \end{cases}$$

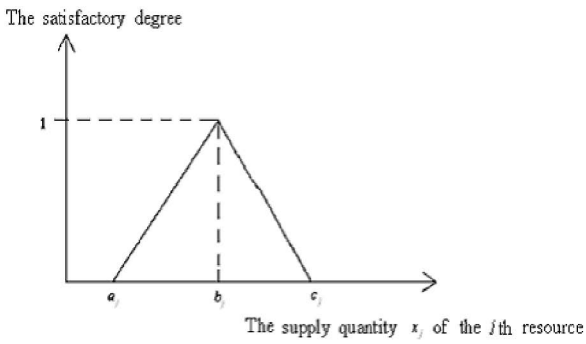

Fig.1 The satisfactory degree of the jth resource demand restraint

In Fig.1, we know that $\dfrac{x_j - a_j}{b_j - a_j}$ is an increasing function and $\dfrac{x_j - c_j}{b_j - c_j}$ decreasing function. When the supply quantity of the jth resource is equal to b_j, the satisfactory degree is 1. However, when the supply quantity is too few or excessively more, it will cause the resources insufficiency or waste, the degree will reduce. The $\mu_{\tilde{x_j}}(x(\phi_j))$ represents the satisfactory degree of the jth resource demand restraint.

Suppose that the emergency time of the jth resource is t_j which is the triangle fuzzy number. Its membership function is defined as following:

$$\mu_{\tilde{t_j}}(t_j) = \begin{cases} 1 & 0 \le t_j \le d, \\ \dfrac{e - t_j}{e - d} & d \le t_j \le e, \\ 0 & t_j \ge e. \end{cases}$$

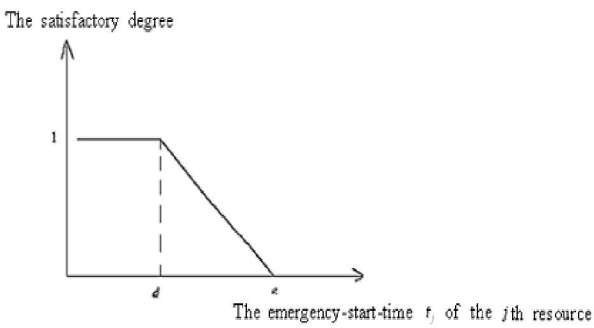

Fig.2 The satisfactory degree of the jth resource time restraint

In Fig.2, $\dfrac{e-t_j}{e-d}$ is a decreasing function. When the emergency time of the jth resource is less than or equal to d, the satisfactory degree is 1. However, when it is greater than d, the satisfactory degree reduces. It is showed that the earlier the emergency-start-time, the greater the satisfactory degree. The $\mu_{\tilde{t}_j}(T(\phi_j))$ stands for the satisfactory degree of the jth resource time restraint.

According to the importance of each resource for the emergency depot, the policy-maker defines a weight coefficient $\varepsilon_j(\sum_{j=1}^{m}\varepsilon_j = 1)$ for each resource. The $x(f)$ stands for the quantity of all resources for scheduling decision f. Thus, the satisfactory degree of the demand restraint in all resources $\mu_{\tilde{x}}(x(f))$ is defined as following:

$$\mu_{\tilde{x}}(x(f)) = \sum_{j=1}^{m}\varepsilon_j \cdot \mu_{\tilde{x}_j}(x(\phi_j)).$$

Meanwhile, $\mu_{\tilde{t}}(T(f))$ stands for the satisfactory degree of the time restraint for scheduling decision f.

$$\mu_{\tilde{t}}(T(f)) = \sum_{j=1}^{m}\varepsilon_j \cdot \mu_{\tilde{t}_j}(T(\phi_j)).$$

Considering the actual situation of the emergency depot, the demand restraint and the time restraint, the policy-maker gives the lowest satisfactory degree for demand restraint α. When $\mu_{\tilde{x}}(x(f)) \geq \alpha$, we find the greatest satisfactory degree of the time restraint, which is the optimal scheduling decision.

Therefore, based on the condition of the satisfactory degree of resource demand restraint $\mu_{\tilde{x}}(x(f))$ is greater than or be equal to α, the emergency-time should start as early as possible, and the number of retrieval depots should be fewer, the fuzzy optimization model is established:

$$\max \mu_{\tilde{t}}(T(f))$$
$$s.t. \begin{cases} u_{\tilde{x}}(x(f)) \geq \alpha, \\ \sum_{i=1}^{n} x_{ij} \leq b_j, j = 1, 2, \cdots, m. \end{cases}$$

§3. Solution to model

First of all, by the method of single resource [7], we can find all possible decisions of each resource. Then decisions are selected from each resource to form a matrix f, which is a feasible decision for all resources. The satisfactory degree of resource demand restraint $\mu_{\tilde{x}}(x(f))$ which greater than or be equal to α are selected. The greatest satisfactory degree of time restraint is an optimal decision. Especially, if the quantity of the jth$(j = 1, 2, \cdots, m)$ emergency resource from the ith retrieval depot of matrix f, is less than the sum of the no emergency resource before ith, and the all emergency resources on the ith retrieval depot satisfy such condition, the quantity of emergency resources on the ith retrieval depot will be distributed to the other retrieval depots before ith. Therefore, one retrieval depot can be saved.

The fuzzy algorithm based on the satisfactory degree:

Step 1. For the jth resource, let $q = 1$;

Step 2. If $q > n$, turn to Step4; otherwise, arrange $x'_{1j}, x'_{2j}, \cdots, x'_{qj}$ the order, then we get$x'_{k_1 j} \geq x'_{k_2 j} \geq \cdots \geq x'_{k_q j}$. If $\sum_{q=1}^{q} x'_{k_q j} \leq a_j$, $q = q + 1$, turn to Step2; otherwise, turn to Step3;

Step 3. The critical subscript p_j of sequence $x'_{k_1 j}, x'_{k_2 j}, \cdots, x'_{k_q j}$ for b_j is solved. If exist p_j, then

$$\phi_j = \{(A_{k_1}, x'_{k_1 j}), (A_{k_2}, x'_{k_2 j}), \cdots, (A_{k_q}, b_j - \sum_{q=1}^{p_j - 1} x'_{k_q j}),$$

turn to Step4, otherwise, let $\phi_j = \{(A_{k_1}, x'_{k_1 j}), (A_{k_2}, x'_{k_2 j}), \cdots, (A_{k_q}, x'_{k_q j})$, $q = q+1$, turn to Step2;

Step 4. A decision ϕ_j from the decisions of the jth resource is selected. These selected decisions constitute a matrix f. Finding all combinations that form matrix f and calculating the satisfactory degree of resource demand restraint. Then the decisions satisfied $\mu_{\tilde{x}}(x(f)) \geq \alpha$ are obtained;

Step 5. Calculating and comparing the satisfactory degree of time restraint $\mu_{\tilde{t}}(T(f))$ that obtained from Step4. Then, the greatest satisfactory degree of time restraint is optimal decision;

Step 6. To each column of the matrix f, the row which the last retrieval depot represents r_j. Calculating Q_j which is the quantity of all no emergency resources corresponding to the zero element which before the r_jth row on the jth column. Marking the non-zero element which less than Q_j on the jth column. If all non-zero elements on certain row are marked, the quantity of the resource corresponding to these non-zero elements is distributed to the location of the zero element, a retrieval depot will be saved. Turn to Step5, until there is no row r_j. Thus, the optimal decision is obtained.

§4. Numerical example

Now, we applied the above algorithm to compute an example.

There are 8 retrieval depots, 1 emergency depot and 3 emergency resources X_1, X_2, X_3 in an emergency system. The weight coefficient of importance for each resource is

$\varepsilon_1 = 0.2, \varepsilon_2 = 0.3, \varepsilon_3 = 0.5$, respectively.

The quantity of each resource as follows:

Table 1: The quantity of resources on retrieval depots

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
t_i	2	4	8	14	18	20	22	28
x'_{i1}	3	2	5	4	3	12	11	6
x'_{i2}	14	12	8	20	16	38	30	8
x'_{i3}	6	8	4	10	16	10	12	18

The policy-maker gives the value of a_j, b_j, c_j, d, e, which corresponding to each resource and time. The membership functions are defined as follows, respectively.

$$\mu_{\tilde{x}_1}(x(\phi_1)) = \begin{cases} \dfrac{x_1 - 2}{18} & 2 \le x_1 \le 20, \\ 1 & x_1 = 20, \\ \dfrac{x_1 - 35}{-15} & 20 \le x_1 \le 35, \\ 0 & \text{other}. \end{cases}$$

$$\mu_{\tilde{x}_2}(x(\phi_2)) = \begin{cases} \dfrac{x_2 - 8}{82} & 8 \le x_2 \le 90, \\ 1 & x_2 = 90, \\ \dfrac{x_2 - 150}{-60} & 90 \le x_2 \le 150, \\ 0 & \text{other}. \end{cases}$$

$$\mu_{\tilde{x}_3}(x(\phi_3)) = \begin{cases} \dfrac{x_3 - 3}{27} & 3 \le x_3 \le 30, \\ 1 & x_3 = 30, \\ \dfrac{x_3 - 50}{-20} & 30 \le x_3 \le 50, \\ 0 & \text{other}. \end{cases}$$

$$u_{\tilde{t}_j}(T(\phi_j)) = \begin{cases} 1 & 0 \le t_j \le 16, \\ \dfrac{32 - t_j}{16} & 16 \le t_j \le 32, \\ 0 & t_j \ge 32. \end{cases}$$

1) When the policy-maker requests the satisfactory degree of resource demand restraint $\alpha = 0.85$.

Applying the obtained algorithm based on MATLAB 7.0 platform, the optimal decision is solved.

$$f = \begin{pmatrix} 3 & 2 & 5 & 4 & 3 & 0 & 0 & 0 \\ 14 & 12 & 8 & 20 & 16 & 0 & 0 & 0 \\ 6 & 8 & 4 & 10 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

By calculating $r_1 = 5, r_2 = 5, r_3 = 4, Q_1 = Q_2 = Q_3 = 0$, so this decision is optimal. The satisfactory degree of resource demand restraint $\mu_{\tilde{x}}(x(f)) = 0.859$ and the satisfactory degree of time restraint $\mu_{\tilde{t}}(T(f)) = 0.938$. The optimal decision needs 5 retrieval depots.

2) When the policy-maker requests the satisfactory degree of resource demand restraint $\alpha = 0.95$.

Applying the obtained algorithm based on MATLAB 7.0 platform, the optimal decision is solved.

$$f = \begin{pmatrix} 0 & 0 & 5 & 3 & 0 & 12 & 0 & 0 \\ 14 & 2 & 0 & 20 & 16 & 38 & 0 & 0 \\ 6 & 8 & 4 & 10 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

By calculating $r_1 = 6, r_2 = 6, r_3 = 4, Q_1 = 8, Q_2 = 18, Q_3 = 0$, Marking the non-zero elements which is less than Q_j on the jth column, then it is stated as follows:

$$f = \begin{pmatrix} 0 & 0 & 5 & 3 & 0 & 12 & 0 & 0 \\ 14 & 2 & 0 & 20 & 16 & 38 & 0 & 0 \\ 6 & 8 & 4 & 10 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

We can see all of the non-zero elements are marked on the 5th row. So the quantity of resource 16 on the 5th row can be distributed to the 2ed row and the 3rd row. The Step6 is carried and a retrieval depot is saved.

$$f = \begin{pmatrix} 0 & 0 & 5 & 3 & 0 & 12 & 0 & 0 \\ 14 & 12 & 6 & 20 & 0 & 38 & 0 & 0 \\ 6 & 8 & 4 & 10 & 0 & 0 & 0 & 0 \end{pmatrix}^T$$

The optimal decision is obtained. The satisfactory degree of resource demand restraint $\mu_{\tilde{x}}(x(f)) = 0.965$ and the satisfactory degree of time restraint $\mu_{\tilde{t}}(T(f)) = 0.875$. Only 5 retrieval depots are needed.

§5. Conclusions

The fuzzy multi-resource scheduling problem in emergency systems is discussed in this paper. The fuzzy optimization model based on the satisfactory degree of resource demand and time restraint is established. It has a high value in actual problems. The policy-maker gives the suitable satisfactory degree and presents the reasonable scheduling decision according to the actual situation of emergency systems. This enables the emergency systems to have a very strong flexibility. However, there is an uncertainty of the vehicle routing in emergency systems. Therefore, the scheduling problem of the uncertain demand for the vehicle routing in emergency systems is waiting for the further research.

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On the Smarandache $3n$-digital sequence and the Zhang Wenpeng's conjecture

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Abstract The sequence $\{a_n\} = \{13,\ 26,\ 39, 412,\ 515,\ 618,\ 721,\ 824, \cdots\}$ is called the Smarandache $3n$-digital sequence. That is, the numbers that can be partitioned into two groups such that the second is three times bigger than the first. The main purpose of this paper is using the elementary method to study the properties of the Smarandache $3n$-digital sequence, and partly solved a conjecture proposed by Professor Zhang Wenpeng.

Keywords The Smarandache $3n$-digital sequence, elementary method, conjecture.

§1. Introduction and results

For any positive integer n, the famous Smarandache $3n$-digital sequence is defined as $\{a_n\} = \{13,\ 26,\ 39, 412,\ 515,\ 618,\ 721,\ 824, \cdots, \}$. That is, the numbers that can be partitioned into two groups such that the second is three times bigger than the first. For example, $a_{10} = 1030,\ a_{21} = 2163,\ a_{32} = 3296,\ a_{100} = 100300, \cdots$. This sequence is proposed by professor F.Smarandache, he also asked us to study its properties. About this problem, it seems that none had studied it yet, at least we have not seen any related papers before. Recently, Professor Zhang Wenpeng proposed the following:

Conjecture. There does not exist any complete square number in the Smarandache $3n$-digital sequence $\{a_n\}$. That is, the equation $a_n = m^2$ has no positive integer solution.

I think that this conjecture is interesting, because if it is true, then we shall obtain a deeply properties of the Smarandache $3n$-digital sequence. In this paper, we using the elementary method to study this problem, and prove that the Zhang's conjecture is correct for some special positive integers. That is, we shall prove the following three conclusions:

Theorem 1. If positive integer n is a square-free number (That is, for any prime p, if $p \mid n$, then $p^2 \nmid n$), then a_n is not a complete square number.

Theorem 2. If positive integer n is a complete square number, then a_n is not a complete square number.

Theorem 3. If a_n be a complete square number, then we must have

$$n = 2^{2\alpha_1} \cdot 3^{2\alpha_2} \cdot 5^{2\alpha_3} \cdot 11^{2\alpha_4} \cdot n_1,$$

where $(n_1,\ 330) = 1$.

From our theorems we know that a_n is not a complete square number for some special positive integers n, such as complete square numbers and square-free numbers. For general positive integer n, whether Zhang's conjecture is correct is an open problem.

§2. Proof of the theorems

In this section, we shall use the elementary method to complete the proof of our theorems. First we prove Theorem 1. For any square-free number n, let $3n = a_k a_{k-1} \cdots a_2 a_1$, where $1 \le a_k \le 9$, $0 \le a_i \le 9$, $i = 1, 2, \cdots, k-1$. Then from the definition of a_n we know that $a_n = n \cdot (10^k + 3)$. If n is a square-free number, and there exists a positive integer m such that

$$a_n = n \cdot (10^k + 3) = m^2. \tag{1}$$

Then from (1) and the definition of square-free number we know that $n \mid m$. Let $m = u \cdot n$, then (1) become

$$10^k + 3 = u^2 \cdot n. \tag{2}$$

In formula (2), it is not possible if $u = 1$, since $10^k + 3 > \underbrace{99 \cdots\cdots 9}_{k} \ge a_k a_{k-1} \cdots a_2 a_1 = 3n > n$.

If $u = 2$, then (2) is impossible. In fact, if (2) holds, then $10^k + 3 = 4 \cdot n$, since $10^k + 3$ is an odd number, and $4 \cdot n$ is an even number, this contradiction with $10^k + 3 = 4 \cdot n$.

If $u = 3$, then (2) does not hold. Because this time, we have the congruence $10^k + 3 \equiv 1 \mod 3$, but $u^2 \cdot n = 9 \cdot n \equiv 0 \mod 3$, so (2) is not possible.

If $u = 4$, then $10^k + 3$ is an odd number, but $u^2 \cdot n = 4^2 \cdot n$ is an even number, so (2) is not also possible.

If $u = 5$, then we have the congruence $10^k + 3 \equiv 3 \mod 5$, $u^2 \cdot n = 5^2 \cdot n \equiv 0 \mod 5$, so (2) is impossible.

If $u = 6$, then $10^k + 3$ is an odd number, and $u^2 \cdot n = 6^2 \cdot n$ is an even number, so (2) is not correct.

If $u \ge 7$, then note that $3n = a_k a_{k-1} \cdots a_2 a_1 \ge 10^{k-1}$, we have the inequality

$$u^2 \cdot n \ge 7^2 \cdot n = 49n \ge 10 \cdot 4n + 9 > 10 \cdot 10^{k-1} + 9 > 10^k + 3,$$

so formula (2) does not hold.

From the above we know that there does not exist any positive integer u such that formula (2). This proves Theorem 1.

Now we prove Theorem 2. Let $n = u^2$ be a complete square number, if there exists a positive integer m such that

$$n \cdot (10^k + 3) = u^2 \cdot (10^k + 3) = m^2, \tag{3}$$

then from (3) we deduce that $u \mid m$, let $m = u \cdot r$, then formula (3) become

$$10^k + 3 = r^2. \tag{4}$$

It is clear that (4) is not possible, since $10^k + 3 \equiv 1 \mod 3$, but $r^2 \equiv 0$ or $1 \mod 3$. This proves Theorem 2.

To prove Theorem 3, we note that for any positive integer k, $(10^k + 3, 2 \cdot 3 \cdot 5 \cdot 11) = (10^k + 3, 330) = 1$. In fact we have $10^k + 3 \equiv (-1)^k + 3 \equiv 2$ or $4 \mod 11$, $10^k + 3 \equiv 1 \mod 2$, $10^k + 3 \equiv 1 \mod 3$, $10^k + 3 \equiv 3 \mod 5$, so we have $(10^k + 3, 2 \cdot 3 \cdot 5 \cdot 11) = 1$. Thus, if $n \cdot (10^k + 3) = m^2$ and $p \mid n$ (where $p = 2, 3, 5, 11.$), then $p^2 \mid n$. That is, the power of p in the factorization of n is an even number. This completes the proof of Theorem 3.

§3. Some notes

Similar to the definition of the Smarandache $3n$-digital sequence, now we defined the Smarandache $4n$-digital sequence as $\{b_n\} = \{14,\ 28,\ 312, 416,\ 520,\ 624,\ 728,\ 832, \cdots, \}$. That is, the numbers that can be partitioned into two groups such that the second is four times bigger than the first. The Smarandache $5n$-digital sequence as $\{c_n\} = \{\ 15,\ 210,\ 315, 420,\ 525,\ 630,\ 735,\ 840, \cdots, \}$. The numbers that can be partitioned into two groups such that the second is five times bigger than the first. For these sequence, Professor Zhang Wenpeng also proposed the following:

Conjecture. There does not exist any complete square number in the Smarandache $4n$-digital sequence $\{b_n\}$ and the Smarandache $5n$-digital sequence $\{c_n\}$.

It is clear that using our method we can also deal with the Smarandache $4n$-digital sequence and the Smarandache $5n$-digital sequence, and obtain some similar conclusions.

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