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Generalized separation axioms

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Abstract In this paper we discuss new separation axioms using g-open sets.

Keywords G-closed sets, g-open sets, g_i spaces.

§1. Introduction

Norman Levine introduced generalized closed sets, K. Balachandaran and P. Sundaram studied generalized continuous functions and generalized homeomorphism. V. K. Sharma studied generalized separation axioms. Following V. K. Sharma we are going to further study the properties of generalized separation axioms. Throughout the paper a space $X$ means a topological space $(X, \tau)$. For any subset $A$ of $X$ its complement, interior, closure, g-interior, g-closure are denoted respectively by the symbols $A^c$, $A^0$, $A^-$, $g(A)^0$ and $g(A)^-$. $(Y/A)^-$ represents the closure in the subspace $Y$ in $X$ with relativized topology $\tau/Y$.

§2. Preliminaries

Definition 2.1. $A \subset X$ is called
(i) generalized closed [5] (briefly g-closed) if $(A)^- \subseteq U$ whenever $A \subseteq U$ and U is open.
(ii) generalized open if its compliment is g-closed.
(iii) regularly open if $A = ((A)^-)^0$.

Definition 2.2. Let $A \subset X$ and $x \in X$. Then $x$ is called g-limit point of $A$ if each g-open set containing $x$ intersects $A$. There exists a g-open set $U$ containing $x$ such that $x \in U \subseteq A$.

Note 1. The class of g-open sets and regularly open sets are denoted by $GO(X)$ and $RO(X)$ respectively. Clearly $RO(X) \subset \tau(X) \subset GO(X)$.

Definition 2.3. $X$ is said to be
(i) $GO$-compact [6] if every g-open cover has a finite subcover.
(ii) $r-T_0$ if for each pair of distinct points $x, y$ of $X$, there exists a regular-open set $G$ containing either $x$ or $y$.
(iii) $rT_1$ [resp : $rT_2$] if for each pair of distinct points $x, y$ of $X$ there exists [resp: disjoint] regular-open sets $G$ and $H$ such that $G$ containing $x$ but not $y$ and $H$ containing $y$ but not $x$.

Definition 2.4. A function $f: X \to Y$ is said to be
(i) nearly continuous [resp: nearly-irresolute] if inverse image of every open [resp: regular-open] set is regular-open.
Theorem 2.1.
(i) If is a g-limit point of any \( A \subset X \), then every g-neighbourhood of \( x \) contains infinitely many distinct points.
(ii) Let \( A \subseteq Y \subseteq X \) and \( Y \) is regularly open subspace of \( X \) then \( A \) is g-open in \( X \) if \( A \) is g-open in \( \tau_Y \).

Theorem 2.2. If \( f \) is g-continuous [resp: g-irresolute\{g-homeomorphism\}] and \( G \) is open [resp: g-open\{g-closed\}] set in \( Y \), then \( f^{-1}(G) \) is g-open [resp: g-open \{g-closed\}] in \( Y \).

Theorem 2.3. [5] Let \( Y \) and \( \{X_\alpha : \alpha \in I\} \) be Topological Spaces. Let \( f : Y \to \Pi X_\alpha \) be a function. If \( f \) is g-continuous, then \( \pi_\alpha \circ f : Y \to X_\alpha \) is g-continuous.

Theorem 2.4. [5] If \( Y = T_2 \) and \( \{X_\alpha : \alpha \in I\} \) be, then \( f \) is g-continuous iff \( \pi_\alpha \circ f : Y \to X_\alpha \) is g-continuous.

Corollary 2.1. [5] Let \( f_\alpha : X_\alpha \to Y_\alpha \) be a function and let \( f : \Pi X_\alpha \to \Pi Y_\alpha \) be defined by \( f ((x_\alpha)_{\alpha \in I}) = (f_\alpha (x_\alpha))_{\alpha \in I} \). If \( f \) is g-continuous then each \( f_\alpha \) is g-continuous.

Corollary 2.2. [5] For each \( \alpha \), \( X_\alpha \) be \( T_2 \) and let \( f_\alpha : X_\alpha \to Y_\alpha \) be a function and let \( f : \Pi X_\alpha \to \Pi Y_\alpha \) be defined by \( f ((x_\alpha)_{\alpha \in I}) = (f_\alpha (x_\alpha))_{\alpha \in I} \), then \( f \) is g-continuous if each \( f_\alpha \) is g-continuous.

§3. \( g_i \) spaces, \( i = 0, 1, 2 \).

Definition 3.1. \( X \) is said to be
(i) \( g_0 \) space if for each pair of distinct points \( x, y \) of \( X \), there exists a g-open set \( G \) containing either \( x \) or \( y \).
(ii) \( g_1 \) space if for each pair of distinct points \( x, y \) of \( X \), there exists a g-open set \( G \) containing \( x \) but not \( y \) and a g-open set \( H \) containing \( y \) but not \( x \).
(iii) \( g_2 \) space if for each pair of distinct points \( x, y \) of \( X \), there exists disjoint g-open sets \( G \) and \( H \) such that \( G \) containing \( x \) but not \( y \) and \( H \) containing \( y \) but not \( x \).

Note 2. By note 1. \( r - T_i \Rightarrow T_i \Rightarrow g_i \) for \( i = 0, 1, 2 \) but the converse is not true in general.

Example 3.1. Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{a, c\}, X\} \) then \( X \) is \( g_0 \) but not \( r-T_0 \) and \( T_0 \).

Example 3.2. Let \( X = \{a, b, c, d\} \).
(i) \( \tau = \phi(X) \) then \( X \) is \( g_i \) for \( i = 0, 1, 2 \).
(ii) \( \tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\} \) then \( X \) is not \( g_i \) for \( i = 0, 1, 2 \).

Remark 3.1. If \( X \) is \( T_2 \) then \( T_i \) and \( g_i \) are one and the same for \( i = 0, 1, 2 \).

Proof. Since \( X \) is \( T_2 \), every g-closed set is closed and hence proof follows from the definitions.

Theorem 3.1. Every [regular] open subspace of \( g_i \) space is \( g_i \), for \( i = 0, 1, 2 \).

Theorem 3.2. \( X \) is \( g_0 \) iff \( \forall x \in X, \exists U \in GO(X) \) containing \( x \) \( \supset \) the subspace \( U \) is \( g_0 \).
Theorem 3.3. If $f: X \rightarrow Y$ is injective, $g$-irresolute and $Y$ is $g_i$ then $X$ is $g_i$ for $i = 0, 1, 2$.

Theorem 3.4.
(i) The product of $g_i$ [7] spaces is again $g_i$ for $i = 0, 1, 2$.
(ii) $X$ is $g_0$ [7] iff distinct points of $X$ have disjoint $g$-closures.

Corollary 3.1. (i) If $f: X \rightarrow Y$ is injective, $g$-continuous and $Y$ is $T_i$ then $X$ is $g_i$ for $i = 0, 1, 2$.
(ii) If $f: X \rightarrow Y$ is injective, nearly-irresolute [nearly-continuous] and $Y$ is $r - T_i$ then $X$ is $g_i$ for $i = 0, 1, 2$.

Theorem 3.5. The following are equivalent:
(i) $X$ is $g_1$.
(ii) Each one point set is $g$-closed.
(iii) Each subset of $X$ is the intersection of all $g$-open sets containing it.
(iv) For any $x \in X$, intersection of all $g$-open sets containing the point is the set $\{x\}$.

Theorem 3.6. If a space $X$ is $g_1$ then distinct points of $X$ have disjoint $g$-closures.

Theorem 3.7. Suppose $x$ is a $g$-limit point of a subset $A$ of a $g_1$ space $X$. Then every $g$-neighborhood of $x$ contains infinitely many distinct points of $A$.

Theorem 3.8. Let $X$ be $T_1$ and $f: X \rightarrow Y$ be $g$-closed sujection. Then $X$ is $g_1$.

Remark 3.2. $X$ is $g_2$ $\Rightarrow$ $X$ is $g_1$ $\Rightarrow$ $X$ is $g_0$.

Theorem 3.9. A space $X$ is $g_2$ iff the intersection of all $g$-closed, g-neighbourhoods of each point of the space is reduced to that point.

Proof. Let $X$ be $g_2$ and $x \in X$, then for each $y \neq x$ in $X$, there exists $g$-open sets $U$ and $V$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Since $x \in U - V$, hence $X - V$ is a $g$-closed, g-neighbourhood of $x$ to which $y$ does not belong. Consequently, the intersection of all $g$-closed, g-neighbourhoods of $x$ is reduced to $\{x\}$.

Conversely let $x, y \in X$, such that $y \neq x$ in $X$, then by hypothesis there exists a $g$-closed, g-neighbourhood $U$ of $x$ such that $y \notin U$. Now there exists a $g$-open set $G$ such that $x \in G \subset U$. Thus $G$ and $X - U$ are disjoint g-open sets containing $x$ and $y$ respectively. Hence $X$ is $g_2$.

Theorem 3.10. If to each $x \in X$, there exists a $g$-closed, g-open subset of $X$ containing $x$ and which is also a $g_2$ subspace of $X$, then $X$ is $g_2$.

Proof. Let $x \in X$, $U$ a $g$-closed, g-open subset of $X$ containing $x$ and which is also a $g_2$ subspace of $X$, then the intersection of all $g$-closed, g-neighbourhood of $x$ in $U$ is reduced to $x$. $U$ being $g$-closed, g-open, these are $g$-closed, g-neighbourhood of $x$ in $X$. Thus the intersection of all $g$-closed, g-neighbourhoods of $x$ is reduced to $\{x\}$. Hence $X$ is $g_2$.

Theorem 3.11. If $X$ is $g_2$ then the diagonal $\Delta$ in $X \times X$ is $g$-closed.

Proof. Suppose $(x, y) \in X \times X - \Delta$. As $(x, y) \notin \Delta$ and $x \neq y$. Since $X$ is $g_2$, $\exists U, V \in GO(X) \ni x \in U$, $y \in V$ and $U \cap V = \phi$, $U \cap V = \phi$ $\Rightarrow$ $(U \times V) \cap \Delta = \phi$ and therefore $(U \times V) \subset X \times X - \Delta$. Further $(x, y) \in (U \times V)$ and $(U \times V)$ is $g$-open in $X \times X$ gives $X \times X - \Delta$ is $g$-open. Hence $\Delta$ is $g$-closed.

Corollary 3.2. (i) In an $T_1[rT_1]$ space, each singleton set is $g$-closed.
(ii) If $X$ is $T_2[r - T_2]$ then the diagonal $\Delta$ in $X \times X$ is g-closed.

**Theorem 3.12.** In $g_2$-space, g-limits of sequences, if exists, are unique.

**Proof.** Let $(x_n)$ be a sequence in $g_2$-space $X$ and if $(x_n) \to x; (x_n) \to y$ as $n \to \infty$. If $x \neq y$ then, for $X$ is $g_2$, $\exists$ disjoint $U; V \in GO(X) \ni x \in U, y \in V$ and $U \cap V = \phi$. Then $\exists N_1, N_2 \ni x_n \in U; \forall n \geq N_1$, and $x_n \in V; \forall n \geq N_2$. Let $m \in Z^+ \ni m > \{N_1; N_2\}$. Then $x_m \in U \cap V$ contradicting the fact $U \cap V = \phi$. So $x = y$ and thus the g-limits are unique.

**Theorem 3.13.** In a $g_2$ space, a point and disjoint $GO - compact$ subspace can be separated by disjoint g-open sets.

**Proof.** Let $X$ be a $g_2$ space, $x$ a point in $X$ and $C$ the disjoint $GO - compact$ subspace of $X$ not containing $x$. Let $y$ be a point in $C$ then for $x \neq y$, in $X$ and $X$ is $g_2$, there exist disjoint g-neighborhoods $G_x$ and $H_y$. Allowing this for each $y$ in $C$, we obtain a class $\{H_y\}$ whose union covers $C$; and since $C$ is $GO - compact$, some finite subclass, which we denote by $\{H_i, i = 1$ to $n\}$ covers $C$. If $G_i$ is the g-neighborhood of $x$ corresponding to $H_i$, we put $G = \bigcup_{i=1}^{n} G_i$ and $H = \bigcap_{i=1}^{n} H_i$, satisfying the required properties.

**Theorem 3.14.** Every $GO - compact$ subspace of a $g_2$ space $X$ is g-closed.

**Proof.** Let $C$ be $GO - compact$ subspace of a $g_2$ space. If $x$ be any point in $C^c$, by above theorem $x$ has a g-neighborhood $G$ such that $x \in G \subset C^c$. This shows that $C^c$ is the union of g-open sets and therefore $C^c$ is g-open. Thus $C$ is g-closed.

**Corollary 3.3.**

(i) Show that in a $T_2[r - T_2]$ space, a point and disjoint compact [r-compact] subspace can be separated by disjoint g-open sets.

(ii) Every compact [r-compact] subspace of a $T_2[r - T_2]$ space is g-closed.

**Theorem 3.15.** Every g-irresolute map from a $GO - compact$ space into a $g_2$ space is g-closed.

**Proof.** Suppose $f: X \to Y$ is g-irresolute where $X$ is $GO - compact$ and $Y$ is $g_2$. Let $C$ be any g-closed subset of $X$. Then $C$ is $GO - compact$ and $f(C)$ is $GO - compact$. But then $f(C)$ is g-closed in $Y$. Hence the image of any g-closed set in $X$ is g-closed set in $Y$. Thus $f$ is g-closed.

**Theorem 3.16.**

(i) Any g-irresolute bijection from a $GO - compact$ space onto a $g_2$ space is a gc-homeomop -hism.

(ii) Any g-continuous bijection from a $GO - compact$ space onto a $g_2$ space is a gc-homeomop -hism.

**Proof.** (i) Let $f: X \to Y$ be a g-irresolute bijection from a $GO - compact$ space onto a $g_2$ space. Let $G$ be an g-open subset of $X$. Then $X - G$ is g-closed and hence $f(X - G)$ is g-closed. Since $f$ is bijective $f(X - G) = Y - f(G)$. Therefore $f(G)$ is g-open in $Y$. This means that $f$ is g-open. Hence $f$ is bijective g-irresolute and g-open. Thus $f$ is gc-homeomorphism.

(ii) Similar to the previous and so omitted.

**Theorem 3.17.** The following are equivalent:

(i) $X$ is $g_2$.

(ii) For each pair $x \neq y \in X$, $\exists$ a g-open, g-closed set $V \ni x \in V; y \notin V$, and
(iii) For each pair \( x \neq y \in X \), \( \exists \) a \( g \)-continuous function \( f : X \to [0,1] \ni f(x) = 0 \) and \( f(C) = 1 \).

**Corollary 3.4.** Let \( Y \) be a \( g_2 \) space. If \( f : X \to Y \) be one-one and \( g \)-irresolute [nearly-irresolute]. Then \( X \) is \( g_2 \).

**Theorem 3.18.** If \( f : X \to Y \) is \( g \)-irresolute and \( Y \) is \( g_2 \) then
(i) the set \( A = \{(x_1, x_2) : f(x_1) = f(x_2)\} \) is \( g \)-closed in \( X \times X \).
(ii) \( G(f) \), the graph of \( f \) is \( g \)-closed in \( X \times Y \).

**Proof.** (i) Let \( A = \{(x_1, x_2) : f(x_1) = f(x_2)\} \). If \( (x_1, x_2) \in X \times X - A \), then \( f(x_1) \neq f(x_2) \) \( \Rightarrow \exists \) disjoint \( V_1 \), \( V_2 \in GO(Y) \nexists f(x_1) \in V_1 \) and \( f(x_2) \in V_2 \), then by \( g \)-irresoluteness of \( f \),
\[ f^{-1}(V_j) \subseteq GO(X, x_j) \text{ for each } j. \]
Thus \( (x_1, x_2) \in f^{-1}(V_1) \times f^{-1}(V_2) \subseteq GO(X \times X) \). Therefore \( f^{-1}(V_1) \times f^{-1}(V_2) \subseteq X \times X - A \Rightarrow X \times X - A \) is \( g \)-open. Hence \( A \) is \( g \)-closed.

(ii) Let \( (x, y) \notin G(f) \Rightarrow y \neq f(x) \Rightarrow \exists \) disjoint \( V; W \in GO(X) \ni f(x) \in V \) and \( y \in W \). Since \( f \) is \( g \)-irresolute, \( \exists U \in GO(X) \ni x \in U \) and \( f(U) \subseteq W \). Therefore we obtain \( (x, y) \in U \times V \subseteq X \times Y - G(f) \). Hence \( X \times Y - G(f) \) is \( g \)-open. Hence \( G(f) \) is \( g \)-closed in \( X \times Y \).

**Theorem 3.19.** If \( f : X \to Y \) is \( g \)-open and the set \( A = \{(x_1, x_2) : f(x_1) = f(x_2)\} \) is closed in \( X \times X \). Then \( Y \) is \( g_2 \).

**Theorem 3.20.** Let \( X \) be an arbitrary space, \( R \) an equivalence relation in \( X \) and \( p : X \to X/R \) the identification map. If \( R \subseteq X \times X \) is \( g \)-closed in \( X \times X \) and \( p \) is \( g \)-open map, then \( X/R \) is \( g_2 \).

**Proof.** Let \( p(x), p(y) \) be distinct members of \( X/R \). Since \( x \) and \( y \) are not related, \( R \subseteq X \times X \) is \( g \)-closed in \( X \times X \). There are \( U; V \in GO(X) \ni x \in U \), \( y \in V \) and \( U \times V \subseteq R^c \). Thus \( p(U), p(V) \) are disjoint and also \( g \)-open in \( X/R \) since \( p \) is \( g \)-open map.

**Theorem 3.21.** The following four properties are equivalent:
(i) \( X \) is \( g_2 \).
(ii) \( X \) is \( g_2 \).
(iii) For each \( x \in X \), \( \exists U \in GO(X) \ni x \in U \) and \( y \notin g(U)^c \).
(iv) For each \( x \in X \), \( \Delta = \{(x, x) : x \in X\} \) is \( g \)-closed in \( X \times X \).

**Proof.** (i) \( \Rightarrow \) (ii) Let \( x \in X \) and \( y \neq x \). Then \( \exists \) disjoint \( U; V \in GO(X) \ni x \in U \) and \( y \in V \). Clearly \( V^c \) is \( g \)-closed, \( g(U)^c \subseteq V^c \) and \( y \notin V^c \) and therefore \( y \notin g(U)^c \).

(iii) \( \Rightarrow \) (iv) If \( y \neq x \), then \( \exists U \in GO(X) \ni x \in U \) and \( y \notin g(U)^c \). So \( y \notin \cap\{g(U)^c : U \in GO(X) \text{ and } x \in U\} \).

(iv) \( \Rightarrow \) (i) \( y \neq x \), then \( (x, y) \notin \Delta \) and thus \( \exists U; V \in GO(X) \ni (x, y) \in U \times V \) and \( (U \times V) \cap \Delta = \phi \). Clearly, for \( U; V \in GO(X) \) we have: \( x \in U \), \( y \in V \) and \( U \cap V = \phi \).

**§4. \( g - R_i \) spaces \( i = 0, 1 \).**

**Definition 4.1.** Let \( x \in X \). Then
(i) \( g \)-kernel of \( x \) is denoted and defined by \( ker_g(x) = \cap \{U : U \in GO(X) \text{ and } x \in U\} \).
(ii) $\text{Ker}_g F = \cap \{U : U \in \text{GO}(X) \text{ and } F \subseteq U\}$.

**Lemma 4.1.** Let $A \subseteq X$, then $\text{ker}_g A = \{x \in X : g(x) - \cap A \neq \phi\}$

**Lemma 4.2.** Let $x \in X$. Then $y \in \text{Ker}_g \{x\}$ iff $x \in g\{y\}^{-}$.

**Proof.** Suppose that $y \notin \text{Ker}_g \{x\}$. Then $\exists V \in \text{GO}(X)$ containing $x \ni y \notin V$. Therefore we have $x \notin g\{y\}^-$. The proof of converse case can be done similarly.

**Lemma 4.3.** For any points $x \neq y \in X$, the following are equivalent:

(i) $\text{Ker}_g \{x\} \neq \text{Ker}_g \{y\}$;

(ii) $g\{x\}^- \neq g\{y\}^-$.  

**Proof.** (1) $\Rightarrow$ (2) Let $\text{Ker}_g \{x\} \neq \text{Ker}_g \{y\}$, then $\exists z \in X \ni z \in \text{Ker}_g \{x\}$ and $z \notin \text{Ker}_g \{y\}$. From $z \in \text{Ker}_g \{x\}$ it follows that $x \cap g\{z\}^- \neq \phi \Rightarrow x \in g\{z\}^-$. By $z \notin \text{Ker}_g \{y\}$, we have $\{y\} \cap g\{z\}^- = \phi$. Since $x \in g\{z\}^- \cap g\{z\}^-$ and $\{y\} \cap g\{x\}^- = \phi$. Therefore $g\{x\}^- \neq g\{y\}^-$. Now $\text{Ker}_g \{x\} \neq \text{Ker}_g \{y\} \Rightarrow g\{x\}^- \neq g\{y\}^-$.  

(2) $\Rightarrow$ (1) If $g\{x\}^- \neq g\{y\}^-$. Then $\exists z \in X \ni z \in g\{x\}^-$ and $z \notin g\{y\}^-$. Then $\exists$ a $g$-open set containing $z$ and therefore containing $x$ but not $y$, namely, $y \notin \text{Ker}_g \{x\}$. Hence $\text{Ker}_g \{x\} \neq \text{Ker}_g \{y\}$.

**Definition 4.2.** $X$ is said to be

(i) $g - R_0$ if and only if $g\{x\}^- \subseteq G$ whenever $x \in G \in g - O(\tau)$.

(ii) weakly $g - R_0$ iff $\cap g\{x\}^- = \phi$.

(iii) $g - R_1$ iff for $x, y \in X \ni g\{x\}^- \neq g\{y\}^- \cap$ disjoint $U; V \in \text{GO}(X) \ni g\{x\}^- \subseteq U$ and $g\{y\}^- \subseteq V$.

**Example 4.1.**

(i) Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{b, c\}, X\}$, then $X$ is weakly $g - R_0$ and $g - R_i$, $i = 0, 1$.

(ii) Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$, then $X$ is neither weakly $g - R_0$ nor $g - R_i$, $i = 0, 1$.

**Remark 4.1.** $rR_0 \Rightarrow R_i \Rightarrow gR_i$, $i = 0, 1$.

**Theorem 4.1.**

(i) Every weakly-$R_0$ space is weakly $g - R_0$.

(ii) Every subspace of $g - R_1$ space is again $g - R_1$.

(ii) Product of any two $g - R_1$ spaces is again $g - R_1$.

**Lemma 4.4.** Every $g - R_0$ space is weakly $g - R_0$.

Converse of the above lemma is not true in general by the following examples.

**Example 4.2.**

(i) Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, X\}$. Clearly, the space $(X, \tau)$ is weakly $g - R_0$, since $\cap g\{x\}^- = \phi$. But it is not $g - R_0$, for $\{b\} \subseteq X$ is $g$-open and $g\{b\}^- = \{b, c\} \not\subseteq \{b\}$.

(ii) Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{b, c\}, X\}$. Clearly, the space $(X, \tau)$ is $g - R_0$ and $R_0$.

**Theorem 4.2.** $X$ is $g - R_0$ iff given $x \neq y \in X; g\{x\}^- \neq g\{y\}^-$.  

**Proof.** Assume $X$ is $g - R_0$ and let $x, y$ be two distinct points of $X$. Suppose $U$ is a $g$-open set containing $x$ but not $y$, then $y \notin g\{y\}^- \subseteq X - U$ and so $x \notin g\{y\}^-$. Hence $g\{x\}^- \neq g\{y\}^-$.  

Conversely, let $x \neq y$ be any two points in $X \ni g\{x\}^- \neq g\{y\}^- \Rightarrow g\{x\}^- \subseteq X - g\{y\}^- = U$(say) a $g$-open set in $X$. This is true for every $g\{x\}^-$. Thus $\cap g\{x\}^- \subseteq U$ where
\( x \in g\{x\}^- \subseteq U \in GO(\tau) \), which in turn implies \( \cap g\{x\}^- \subseteq U \) where \( x \in U \in g - O(\tau) \). Hence \( X \) is \( g - R_0 \).

**Theorem 4.3.** \( X \) is weakly \( g - R_0 \) iff \( \ker_g \{x\} \neq X \) for any \( x \in X \).

**Proof.** Let \( x_0 \in X \ni \ker_g \{x_0\} = X \). This means that \( x_0 \) is not contained in any proper \( g \)-open subset of \( X \). Thus \( x_0 \) belongs to the \( g \)-closure of every singleton set. Hence \( x_0 \in \cap g\{x\}^- \), a contradiction.

Conversely assume that \( \ker_g \{x\} \neq X \) for any \( x \in X \). If there is a point \( x_0 \in X \ni x_0 \in \cap \{g\{x\}^-\} \), then every \( g \)-open set containing \( x_0 \) must contain every point of \( X \). Therefore, the unique \( g \)-open set containing \( x_0 \) is \( X \). Hence \( \ker_g \{x_0\} = X \), which is a contradiction. Thus \( X \) is weakly \( g - R_0 \).

**Theorem 4.4.** The following statements are equivalent:

(i) \( X \) is \( g - R_0 \) space.

(ii) For each \( x \in X \), \( g\{x\}^- \subseteq \ker_g \{x\} \).

(iii) For any \( g \)-closed set \( F \) and a point \( x \notin F \), \( \exists U \in GO(X) \ni x \notin U \) and \( F \subset U \).

(iv) Each \( g \)-closed set \( F \) can be expressed as \( F = \cap \{G : G \text{ is } g \text{-open and } F \subset G\} \).

(v) Each \( g \)-open set \( G \) can be expressed as the union of \( g \)-closed sets \( A \) contained in \( G \).

(vi) For each \( g \)-closed set \( F \), \( x \notin F \) implies \( g\{x\}^- \cap F = \phi \).

**Proof.** (i) \( \Rightarrow \) (ii) For any \( x \in X \), we have \( \ker_g \{x\} = \cap \{U : U \in GO(X) \text{ and } x \in U\} \). Since \( X \) is \( g - R_0 \), each \( g \)-open set containing \( x \) contains \( g\{x\}^- \). Hence \( g\{x\}^- \subseteq \ker_g \{x\} \).

(ii) \( \Rightarrow \) (iii) Let \( F \in GC(X) \& x \in X - F \). Then for any \( y \in F \), \( g\{y\}^- \subseteq F \) and so \( x \notin g\{y\}^- \Rightarrow y \notin g\{x\}^- \) that is \( \exists U_y \in GO(X) \ni y \notin U_y \) and \( x \notin U_y \) for all \( y \in F \). Let \( U = \cup \{U_y : U_y \text{ is } g \text{-open}, y \in U_y \text{ and } x \notin U_y\} \). Then \( U \) is \( g \)-open such that \( x \notin U \) and \( F \subset U \).

(iii) \( \Rightarrow \) (iv) Let \( F \in GC(X) \) and \( N = \cap \{G : G \text{ is } g \text{-open and } F \subset G\} \). Then \( F \subset N \Rightarrow (1) \).

Let \( x \notin F \), then by (iii), \( \exists G \in GO(X) \ni x \notin G \) and \( F \subset G \), hence \( x \notin N \) which implies \( x \in N \Rightarrow x \in F \). Hence \( N \subset F \Rightarrow (2) \).

Therefore from (1)&(2), each \( g \)-closed set \( F = \cap \{G : G \text{ is } g \text{-open and } F \subset G\} \).

(iv) \( \Rightarrow \) (v) obvious.

(v) \( \Rightarrow \) (vi) Let \( F \) be any \( g \)-closed set and \( x \notin F \). Then \( X - F = G \) is a \( g \)-open set containing \( x \). Then by (v), \( G \) can be expressed as the union of \( g \)-closed sets \( A \) contained in \( G \), and so there is a \( g \)-closed set \( M \) such that \( x \in M \subset G \); and hence \( g\{x\}^- \subset G \) which implies \( g\{x\}^- \cap F = \phi \).

(vi) \( \Rightarrow \) (i) Let \( G \) be any \( g \)-open set and \( x \in G \). Then \( x \notin X - G \), which is a \( g \)-closed set.

Therefore by (vi) \( g\{x\}^- \cap X - G = \phi \), which implies that \( g\{x\}^- \subseteq G \). Thus \((X, \tau)\) is \( g - R_0 \) space.

**Theorem 4.5.** Let \( f : X \rightarrow Y \) be a \( g \)-closed one-one function. If \( X \) is weakly \( g - R_0 \), then so is \( Y \).

**Theorem 4.6.** If \( X \) is weakly \( g - R_0 \), then for every space \( Y, X \times Y \) is weakly \( g - R_0 \).

**Proof.** \( \cap g\{(x, y)\}^- \subseteq \cap \{g\{x\}^- \times g\{y\}^-\} = \cap \{g\{x\}^-\} \times \{g\{y\}^-\} \subseteq \phi \times Y = \phi \). Hence \( X \times Y \) is \( g - R_0 \).

**Corollary 4.1.**

(i) If \( X \) and \( Y \) are weakly \( g - R_0 \), then \( X \times Y \) is weakly \( g - R_0 \).

(ii) If \( X \) and \( Y \) are (weakly-)\( R_0 \), then \( X \times Y \) is weakly \( g - R_0 \).

(iii) If \( X \) and \( Y \) are \( g - R_0 \), then \( X \times Y \) is weakly \( g - R_0 \).
Lemma 4.2, it follows that $y \subset \emptyset$. This implies that $\forall x \in X, g(x)^- \neq g(y)^- \Rightarrow g(x)^- \cap g(y)^- = \emptyset$.

**Theorem 4.7.** $X$ is $g - R_0$ iff for any $x, y \in X$, $g(x)^- \neq g(y)^- \Rightarrow g(x)^- \cap g(y)^- = \emptyset$.

**Proof.** Let $x$ be a point in $g - R_0$ and $y \in X \setminus g(x)^-$. Then, $\exists z \in g(x)^- \setminus g(y)^-$. There exists $V \in GO(X) \ni y \notin V$ and $z \in V$; hence $x \in V$. Therefore, we have $x \notin g(y)^-$. Thus $x \in [g(y)^-]^c \in GO(X)$, which implies $g(x)^- \subset [g(y)^-]^c$ and $g(x)^- \cap g(y)^- = \emptyset$. The proof for otherwise is similar.

**Sufficiency:** Let $V \in GO(X)$ and let $x \in V$. We show that $g(x)^- \subset V$. Let $y \notin V$, i.e., $y \in [V]^c$. Then $x \neq y$ and $x \notin g(y)^-$. Hence $g(x)^- \neq g(y)^-$. By assumption, $g(x)^- \cap g(y)^- = \emptyset$. Hence $y \notin g(x)^-$. Therefore $g(x)^- \subset V$.

**Theorem 4.8.** A space $X$ is $g - R_0$ iff for any points $x, y \in X$, $\forall x \in X, \forall y \in Y$, $g(x)^- \neq g(y)^- \Rightarrow Ker_g[x] \cap Ker_g[y] \neq \emptyset$.

**Proof.** Suppose $X$ is $g - R_0$. Thus by lemma 4.3, for any points $x, y \in X$ if $Ker_g[x] \neq Ker_g[y]$ then $g(x)^- \neq g(y)^-$. Assume that $z \in Ker_g[x] \cap Ker_g[y]$. By $z \in Ker_g[x]$ and lemma 4.2, it follows that $z \notin g(z)^-$. Since $x \in g(x)^-$, $g(x)^- = g(z)^-$. This is a contradiction. Therefore, we have $Ker_g[x] \cap Ker_g[y] = \emptyset$.

Conversely, let $x$ and $y$ be any two points in $X$, $\exists g(x)^- \neq g(y)^- \Rightarrow g(y)^- = g(x)^-$. This implies that $g(x)^- \subset Ker_g[x]$ and therefore $Ker_g[x] \cap Ker_g[z] \neq \emptyset$. Therefore $X$ is a $g - R_0$ space.

**Theorem 4.9.** The following properties are equivalent:

1. $X$ is a $g - R_0$ space;
2. For any $A \neq \emptyset$ and $G \in GO(X, \tau) \ni A \cap G \neq \emptyset$, $\exists F \in GC(X, \tau) \ni A \cap F \neq \emptyset$ and $F \subset G$.

**Proof.** (1)$\Rightarrow$(2) Let $A \neq \emptyset$ and $G \in GO(X) \ni A \cap G \neq \emptyset$. There exists $x \in A \cap G$. Since $x \in E \in GO(X)$, $g(x)^- \subset G$. Set $F = g(x)^-$, then $F \in GC(X), F \subset G$ and $A \cap F \neq \emptyset$.

(2)$\Rightarrow$(1) Let $G \in GO(X)$ and $x \in G$. Suppose $y \in Ker_g[x]$, then $x \in g(y)^- \mbox{ and } y \in G$. This implies that $g(x)^- \subset Ker_g[x] \subset G$. Hence $X$ is a $g - R_0$ space.

**Corollary 4.2.** The following properties are equivalent:

1. $X$ is a $g - R_0$ space;
2. $g(x)^- = Ker_g[x] \forall x \in X$.

**Proof.** (1)$\Rightarrow$(2) Suppose $X$ is $g - R_0$. By theorem 4.9, $g(x)^- \subset Ker_g[x] \forall x \in X$. Let $y \in Ker_g[x]$, then $x \in g(y)^- \mbox{ and } g(x)^- = g(y)^-$. Therefore, $y \notin g(x)^- \mbox{ and } g(x)^- = Ker_g[x]$. Hence $g(x)^- = Ker_g[x]$.

(2)$\Rightarrow$(1) This is obvious by theorem 4.9.

**Theorem 4.10.** The following properties are equivalent:

1. $X$ is a $g - R_0$ space;
2. $x \in g(y)^- \mbox{ if and only if } y \in g(x)^- \mbox{ for any points } x \mbox{ and } y \mbox{ in } X$.

**Proof.** (1)$\Rightarrow$(2) Assume that $X$ is $g - R_0$. Let $x \in g(y)^- \mbox{ and } D \mbox{ be any } g\mbox{-open set such that } y \in D$. Now by hypothesis, $x \in D$. Therefore, every $g\mbox{-open set which containy contains } x$.

Hence $y \in g(x)^-$. This implies that $g(x)^- \subset U$. Hence $X$ is a $g - R_0$.
Theorem 4.11. The following properties are equivalent:

(1) $X$ is a $g - R_0$ space.
(2) If $F$ is g-closed, then $F = Ker_g(F)$.
(3) If $F$ is g-closed and $x \in F$, then $Ker_g\{x\} \subseteq F$.
(4) If $x \in X$, then $Ker_g\{x\} \subseteq g\{x\}^\circ$.

Proof. (1)$\Rightarrow$(2) Let $x \notin F \in GC(X) \Rightarrow X - F \in GO(X)$ and contains $x$. For $X$ is $g - R_0$, $G(\{x\})^- \subseteq X - F$. Thus $G(\{x\})^- \cap F = \phi$ and $x \notin Ker_g(F)$. Hence $Ker_g(F) = F$.

(2)$\Rightarrow$(3) $A \subset B \Rightarrow Ker_g(A) \subset Ker_g(B)$. Therefore, by (2) $Ker_g\{x\} \subset Ker_g(F) = F$.

(3)$\Rightarrow$(4) Since $x \in g\{x\}^-$ and $g\{x\}^-$ is g-closed, by (3) $Ker_g\{x\} \subset g\{x\}^-$. 

(4)$\Rightarrow$(1) Let $x \in g\{y\}^-$. Then by lemma 4.2, $y \in Ker_g\{x\}$. Since $x \in g\{x\}^-$ and $g\{x\}^-$ is g-closed, by (4) we obtain $y \in Ker_g\{x\} \subseteq g\{x\}^-$. Therefore $x \in g\{y\}^-$ implies $y \in g\{x\}^-$. The converse is obvious and $X$ is $g - R_0$.

Recall that a filterbase $F$ is called g-convergent to a point $x$ in $X$, if for any g-open set $U$ of $X$ containing $x$, there exists $B \in F$ such that $B \subseteq U$.

Lemma 4.5. Let $x$ and $y$ be any two points in $X$ such that every net in $X$ g-converging to $y$ g-converges to $x$. Then $x \in g\{y\}^-$. 

Proof. Suppose that $x_n = y$ for each $n \in N$. Then $\{x_n\}_{n\in N}$ is a net in $g\{y\}^-$. Since $\{x_n\}_{n\in N}$ g-converges to $y$, then $\{x_n\}_{n\in N}$ g-converges to $x$ and this implies that $x \in g\{y\}^-$. 

Theorem 4.12. The following statements are equivalent:

(1) $X$ is a $g - R_0$ space.
(2) If $x, y \in X$, then $y \in g\{x\}^-$ if every net in $X$ g-converging to $y$ g-converges to $x$.

Proof. (1)$\Rightarrow$(2) Let $x, y \in X \ni y \in g\{x\}^-$. Suppose that $\{x_\alpha\}_{\alpha \in \Lambda}$ is a net in $X \ni \{x_\alpha\}_{\alpha \in \Lambda}$ g-converges to $y$. Since $y \in g\{x\}^-$, by theorem 4.7, we have $g\{x\}^- = g\{y\}^-$. Therefore $x \in g\{y\}^-$. This means that $\{x_\alpha\}_{\alpha \in \Lambda}$ g-converges to $x$.

Conversely, let $x, y \in X$ such that every net in $X$ g-converging to $y$ g-converges to $x$. Then $x \in g\{y\}^-$ by theorem 4.4. By theorem 4.7, we have $g\{x\}^- = g\{y\}^-$. Therefore $y \in g\{x\}^-$. 

(2)$\Rightarrow$(1) Let $x, y$ be any two points of $X \ni g\{x\}^- \ni g\{y\}^-$. Let $z \in g\{x\}^- \cap g\{y\}^-$. So $\exists$ a net $\{x_\alpha\}_{\alpha \in \Lambda}$ in $g\{x\}^- \ni \{x_\alpha\}_{\alpha \in \Lambda}$ g-converges to $z$. Since $z \in g\{y\}^-$, then $\{x_\alpha\}_{\alpha \in \Lambda}$ g-converges to $y$. It follows that $y \in g\{x\}^-$. Similarly we obtain $x \in g\{y\}^-$. Therefore $g\{x\}^- = g\{y\}^-$. Hence, $X$ is $g - R_0$.

Theorem 4.13. A space $X$ is $g - R_1$ iff given $x \neq y \in X$, $g\{x\}^- \neq g\{y\}^-$. 

Theorem 4.14. Every $g_2$ space is $g - R_1$.

The converse is not true. However, we have the following result.

Theorem 4.15. Every $g_1$ space is $g_2$.

Proof. Let $X$ be $g_1$ and $g - R_1$ space. Let $x \neq y \in X$. Since $X$ is $g_1$, $\{x\}$ is g-closed set and $\{y\}$ is g-closed set such that $g\{x\}^- \neq g\{y\}^-$. Since $X$ is $g - R_1$, there exists disjoint g-open sets $U$ and $V$ such that $x \in U$, $y \in V$. Hence $X$ is $g_2$.

Corollary. $X$ is $g_2$ iff it is $g - R_1$ and $g_1$.

Theorem 4.16. The following are equivalent:

(i) $X$ is $g - R_1$.
(ii) $\cap g\{x\}^- = \{x\}$.
(iii) For any $x \in X$, intersection of all g-neighborhoods of $x$ is $\{x\}$.
**Theorem 4.17.** The following are equivalent:
(i) \( X = g - R_1 \).
(ii) For each pair \( x, y \in X \) such that \( g\{x\}^- \neq g\{y\}^- \), there exists a g-open, g-closed set \( V \) such that \( x \in V \) and \( y \notin V \), and 
(iii) For each pair \( x, y \in X \) such that \( g\{x\}^- \neq g\{y\}^- \), there exists a g-continuous function \( f: (X, \tau) \to [0, 1] \) such that \( f(x) = 0 \) and \( f(C) = 1 \).

**Proof.** (i)⇒(ii) Let \( y \neq x \in X \), \( y \notin g\{x\}^- \). Since \( X = g - R_1 \), there exists a g-open set \( U \) such that \( y \in U \), \( x \notin U \), \( y \notin U \). In either case \( y \notin g\{x\}^- \). Hence \( \cap g\{x\}^- = \{x\} \).

(ii)⇒(iii) If \( x, y \in X \) where \( y \neq x \), then \( x \notin g\{y\}^- \), so there is a g-open set containing \( x \) but not \( y \). Therefore \( y \) does not belong to the intersection of all g-neighborhoods of \( x \). Hence intersection of all g-neighborhoods of \( x \) is \( \{x\} \).

(iii)⇒(i) Let \( x, y \in X \) where \( y \neq x \). By hypothesis, \( y \) does not belong to the intersection of all g-neighborhoods of \( x \) and \( x \) does not belong to the intersection of all g-neighborhoods of \( y \), which implies \( g\{x\}^- \neq g\{y\}^- \), therefore by theorem 4.13, \( X = g - R_1 \).

**Theorem 4.18.** If \( x \in X = g - R_1 \), then \( X = g - R_0 \).

**Proof.** Let \( x \in U \in GO(X) \). If \( y \notin U \), then \( g\{x\}^- \neq g\{y\}^- \). Hence, \( \exists \) a g-open \( V_y \ni g\{y\}^- \subset V_y \) and \( x \notin V_y \Rightarrow y \notin g\{x\}^- \). Thus \( g\{x\}^- \subset U \). Therefore \( X = g - R_0 \).

**Theorem 4.19.** \( X = g - R_1 \) iff for \( x, y \in X \), \( Ker_g\{x\} \neq Ker_g\{y\} \), \( \exists \) disjoint \( U; V \in GO(X) \ni g\{x\}^- \subset U \) and \( g\{y\}^- \subset V \).

Following diagram indicates the interrelation among the separation axioms.

\[ \text{§5. \( g - C_i \) and \( g - D_i \) spaces, \( i = 0, 1, 2 \).} \]

**Definition 5.1.** \( X \) is said to be a
(i) \( g - C_0 \) space if for each pair of distinct points \( x, y \) of \( X \) there exists a g-open set \( G \) whose closure contains either of the point \( x \) or \( y \).

(ii) \( g - C_1 \) space if for each pair of distinct points \( x, y \) of \( X \) there exists a g-open set \( G \) whose closure containing \( x \) but not \( y \) and a g-open set \( H \) whose closure containing \( y \) but not \( x \).

(iii) \( g - C_2 \) space if for each pair of distinct points \( x, y \) of \( X \) there exists disjoint g-open sets \( G \) and \( H \) such that \( G \) containing \( x \) but not \( y \) and \( H \) containing \( y \) but not \( x \).

**Note.** \( g - C_2 \Rightarrow g - C_1 \Rightarrow g - C_0 \) but converse need not be true in general as shown by the following example.
Example 5.1.
(i) Let \( X = \{a, b, c, d\} \) and \( \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\} \), then \( X \) is \( gC_t \), \( i = 0, 1, 2 \).
(ii) Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\} \) then \( X \) is not \( gC_t \), \( i = 0, 1, 2 \).

Theorem 5.1.
(i) Every subspace of \( g - C_t \) space is \( g - C_t \).
(ii) Every \( g_t \) space is \( g - C_t \).
(iii) Product of \( g - C_t \) spaces are \( g - C_t \).

Theorem 5.2. Let \( (X, \tau) \) be any \( g - C_t \) space and \( A \) be any non empty subset of \( X \) then \( A \) is \( g - C_t \) iff \( (A, \tau/A) \) is \( g - C_t \).

Theorem 5.3.
(i) If \( X \) is \( g - C_t \) then each singleton set is \( g \)-closed.
(ii) In an \( g - C_t \) space disjoint points of \( X \) has disjoint \( g \)-closures.

Definition 5.2. \( A \subset X \) is called a \( g \)Difference (shortly \( gD \)-set) set if there are two \( U, V \in GO(X) \) such that \( U \neq X \) and \( A = U - V \).

Clearly every \( g \)-open set \( U \) different from \( X \) is a \( g \)D-set if \( A = U \) and \( V = \phi \).

Definition 5.3. \( X \) is said to be a
(i) \( g - D_0 \) if for any pair of distinct points \( x \) and \( y \) of \( X \) there exist a \( gD \)-set in \( X \) containing \( x \) but not \( y \) or a \( gD \)-set in \( X \) containing \( y \) but not \( x \).
(ii) \( g - D_1 \) if for any pair of distinct points \( x \) and \( y \) in \( X \) there exist a \( gD \)-set of \( X \) containing \( x \) but not \( y \) and a \( gD \)-set in \( X \) containing \( y \) but not \( x \).
(iii) \( g - D_2 \) if for any pair of distinct points \( x \) and \( y \) of \( X \) there exists disjoint \( gD \) sets \( G \) and \( H \in X \) containing \( x \) and \( y \), respectively.

Example 5.2. (iii) Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{a\}, \{b, c\}, X\} \) then \( X \) is \( gD_t, i = 0, 1, 2 \).

Remark 5.2.
(i) If \( X \) is \( r - T_t \), then it is \( g_t \), \( i = 0, 1, 2 \) and converse is false.
(ii) If \( X \) is \( g_t \), then it is \( g_{t-1}, i = 1, 2 \).
(iii) If \( X \) is \( g_t \), then it is \( g - D_t, i = 0, 1, 2 \).
(iv) If \( X \) is \( g - D_t \), then it is \( g - D_{t-1}, i = 1, 2 \).

Theorem 5.4. The following statements are true:
(i) \( X \) is \( g - D_0 \) if and only if it is \( g_0 \).
(ii) \( X \) is \( g - D_1 \) if and only if it is \( g - D_2 \).

Proof. (i) The sufficiency is stated in remark 5.1(iii).

To prove necessity, let \( X \) be \( g - D_0 \). Then for each distinct pair of points \( x, y \in X \), at least one of \( x, y \), say \( x \), belongs to a \( gD \)-set \( G \) but \( y \notin G \). Let \( G = U_1 - U_2 \) where \( U_1 \neq X \) and \( U_1, U_2 \in GO(X, \tau) \). Then \( x \in U_1 \) and for \( y \notin G \) we have two cases: (a) \( y \notin U_1 \); (b) \( y \in U_1 \) and \( y \notin U_2 \).

In case (a), \( x \in U_1 \) but \( y \notin U_1 \);
In case (b), \( y \in U_2 \) but \( x \notin U_2 \). Hence \( X \) is \( g_0 \).

(ii) Sufficiency. remark 5.1(iv).

Necessity. Suppose \( X \) is \( g - D_1 \). Then for each \( x \neq y \in X \), we have \( gD \) sets \( G_1, G_2 \ni x \in G_1; y \notin G_1; y \in G_2, x \notin G_2 \). Let \( G_1 = U_1 - U_2, G_2 = U_3 - U_4 \). From \( x \notin G_2 \), it follows that
either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. We discuss the two cases separately.

(1) $x \notin U_3$. By $y \notin G_1$ we have two subcases:

(a) $y \notin U_1$. From $x \in U_1 - U_2$, it follows that $x \in U_1 - (U_2 \cup U_3)$ and by $y \in U_3 - U_4$ we have $y \in U_3 - (U_1 \cup U_4)$. Therefore $(U_1 - (U_2 \cup U_3)) \cap (U_3 - (U_1 \cup U_4)) = \phi$.

(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1 - U_2, y \in U_2, (U_1 - U_2) \cap U_2 = \phi$.

(2) $x \in U_3$ and $x \in U_4$. We have $y \in U_3 - U_4, x \in U_4, (U_3 - U_4) \cap U_4 = \phi$. Therefore $X$ is $g - D_2$.

**Corollary 5.1.** If $X$ is $g - D_1$, then it is $g_0$.

**Proof.** remark 5.1(iv) and theorem 5.2(i).

**Definition 5.4.** A point $x \in X$ which has $X$ as the unique $g-$neighborhood is called gc.c point.

**Theorem 5.5.** For an $g_0$ space $X$ the following are equivalent:

(1) $X$ is $g - D_1$.

(2) $X$ has no gc.c point.

**Proof.** (1)$\Rightarrow$(2) Since $X$ is $g - D_1$, then each point $x$ of $X$ is contained in a $gD$-set $O = U - V$ and thus in $U$. By definition $U \neq X$. This implies that $x$ is not a gc.c point.

(2)$\Rightarrow$(1) If $X$ is $g_0$, then for each $x \neq y \in X$, at least one of them, $x$ (say) has a $g-$neighborhood $U$ containing $x$ and not $y$. Thus $U$ which is different from $X$ is a gD-set. If $X$ has no gc.c point, then $y$ is not a gc.c point. This means that there exists a $g-$neighborhood $V$ of $y$ such that $V \neq X$. Thus $y \in (V - U)$ but not $x$ and $V - U$ is a $gD$-set. Hence $X$ is $g - D_1$.

**Corollary 5.2.** A $g_0$ space $X$ is $g - D_1$ if and only if there is a unique gc.c point in $X$.

**Proof.** Only uniqueness is sufficient to prove. If $x_0$ and $y_0$ are two gc.c points in $X$ then since $X$ is $g_0$, at least one of them has a $g-$neighborhood $U$ containing $x$ but not $y$. This is a contradiction since $x \neq U$.

**Remark 5.2.** It is clear that an $g_0$ space $X$ is not $g - D_1$ if and only if there is a unique gc.c point in $X$. It is unique because if $x$ and $y$ are both gc.c point in $X$, then at least one of them say $x$ has a $g-$neighborhood $U$ containing $x$ but not $y$. But this is a contradiction since $U \neq X$.

**Definition 5.5.** $X$ is $g-$symmetric if for $x$ and $y$ in $X$, $x \in g\{y\}^-$ implies $y \in g\{x\}^-$. 

**Theorem 5.6.** $X$ is $g-$symmetric if and only if $\{x\}$ is $gg$-closed for each $x \in X$.

**Proof.** Assume that $x \in g\{y\}^-$ but $y \notin g\{x\}^-$. This means that $[g\{x\}]^c$ contains $y$. This implies that $g\{y\}^-$ is a subset of $[g\{x\}]^c$. Now $[g\{x\}]^c$ contains $x$ which is a contradiction.

Conversely, suppose that $\{x\} \subseteq E \in g(X, \tau)$ but $g\{x\}$ is not a subset of $E$. This means that $g\{x\}$ and $E^c$ are not disjoint. Let $y$ belongs to their intersection. Now we have $x \in g\{y\}^-$ which is a subset of $E^c$ and $x \notin E$. But this is a contradiction.

**Corollary 5.3.** If $X$ is a $g_1$, then it is $g-$symmetric.

**Proof.** In a $g_1$ space, singleton sets are $g-$closed (theorem 2.2(ii)) and therefore $gg-$closed (remark 5.3). By theorem 5.6, the space is $g-$symmetric.

**Corollary 5.4.** The following are equivalent:

(1) $X$ is $g-$symmetric and $g_0$.

(2) $X$ is $g_1$. 

Proof. By corollary 5.3 and remark 5.1 it suffices to prove only (1)⇒(2). Let \( x \neq y \) and by \( g_0 \), we may assume that \( x \in G_1 \subset \{ y \}^c \) for some \( G_1 \in GO(X, \tau) \). Then \( x \notin g\{ y \}^c \) and hence \( y \notin g\{ x \}^c \). There exists a \( G_2 \in GO(X, \tau) \) such that \( y \in G_2 \subset \{ x \}^c \) and \( X \) is a \( g_1 \) space.

Theorem 5.7. For an \( g \)-symmetric space \( X \) the following are equivalent:

(1) \( X \) is \( g_0 \).
(2) \( X \) is \( g - D_1 \).
(3) \( X \) is \( g_1 \).

Proof. (1)⇒(3) corollary 5.4 and (3)⇒(2)⇒(1) remark 5.1.

Theorem 5.8. If \( f : X \to Y \) is a \( g \)-irresolute surjective function and \( E \) is a \( gD \)-set in \( Y \), then \( f^{-1}(E) \) is a \( gD \)-set in \( X \).

Proof. Let \( E \) be a \( gD \)-set in \( Y \). Then there \( \exists U_1; U_2 \in GO(X) \ni E = U_1 - U_2 \) and \( U_1 \neq Y \). By the \( g \)-irresoluteness of \( f \), \( f^{-1}(U_1) \) and \( f^{-1}(U_2) \) are \( g \)-open in \( X \). Since \( U_1 \neq Y \), we have \( f^{-1}(U_1) \neq X \). Hence \( f^{-1}(E) = f^{-1}(U_1) - f^{-1}(U_2) \) is a \( g \)-D-set.

Theorem 5.9. If \( (Y, \sigma) \) is \( g - D_1 \) and \( f : (X, \tau) \to (Y, \sigma) \) is \( g \)-irresolute and bijective, then \( (X, \tau) \) is \( g - D_1 \).

Proof. Suppose that \( Y \) is a \( g - D_1 \) space. Let \( x \neq y \in X \) be any pair of points. Since \( f \) is injective and \( Y \) is \( g - D_1 \), there exist \( g \)-D-sets \( G_x \) and \( G_y \) of \( Y \) containing \( f(x) \) and \( f(y) \) respectively, such that \( f(y) \notin G_x \) and \( f(x) \notin G_y \). By theorem 5.8, \( f^{-1}(G_x) \) and \( f^{-1}(G_y) \) are \( g \)-D-sets in \( X \) containing \( x \) and \( y \) respectively. This implies that \( X \) is a \( g - D_1 \) space.

Theorem 5.10. \( X \) is \( g - D_1 \) if and only if for each pair of distinct points \( x, y \) in \( X \) there exists a \( g \)-irresolute surjective function \( f : (X, \tau) \to (Y, \sigma) \), where \( Y \) is a \( g - D_1 \) space such that \( f(x) \) and \( f(y) \) are distinct.

Proof. Necessity. For every \( x \neq y \in X \), it suffices to take the identity function on \( X \).

Sufficiency. Let \( x \) and \( y \) be any pair of distinct points in \( X \). By hypothesis, there exists a \( g \)-irresolute, surjective function \( f \) of a space \( X \) onto a \( g - D_1 \) space \( Y \) such that \( f(x) \neq f(y) \). Therefore, there exist disjoint \( g \)-D-sets \( G_x \) and \( G_y \) in \( Y \) such that \( f(x) \in G_x \) and \( f(y) \in G_y \). Since \( f \) is \( g \)-irresolute and surjective, by theorem 5.8, \( f^{-1}(G_x) \) and \( f^{-1}(G_y) \) are disjoint \( g \)-D-sets in \( X \) containing \( x \) and \( y \) respectively. Therefore \( X \) is \( g - D_1 \) space.

Corollary 5.5. Let \( \{ X_\alpha : \alpha \in I \} \) be any family of topological spaces. If \( X_\alpha \) is \( g - D_1 \) for each \( \alpha \in I \), then the product \( \Pi X_\alpha \) is \( g - D_1 \).

Proof. Let \((x_\alpha)\) and \((y_\alpha)\) be any pair of distinct points in \( \Pi X_\alpha \). Then there exists an index \( \beta \in I \) such that \( x_\beta \neq y_\beta \). The natural projection \( P_\beta : \Pi X_\alpha \to X_\beta \) almost continuous and almost open and \( P_\beta((x_\alpha)) = P_\beta((y_\alpha)) \). Since \( X_\beta \) is \( g - D_1 \), \( \Pi X_\alpha \) is \( g - D_1 \).

§4. Conclusion

In this paper we defined new separation axioms using \( g \)-open sets and studied their inter-relations with other separation axioms.

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References

On the bounds of the largest eigen value and the Laplacian energy of certain class of graphs

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Abstract The energy of a graph $G$ is defined as the sum of the eigen values of the adjacency matrix of the graph $G$. We report the upper Bounds for the Laplacian energy of the $L(S(K_n))$, $L(S(W_n))$, $L(S(T_{n,k}))$ and $L(S(L_n))$, where $L$ and $S$ stands for line graph and subdivision graph of $G$. The bounds for the largest eigen value of the line graph of the subdivision graph of same class of graphs are obtained.

Keywords Energy, subdivision graph, zagrab index, Laplacian matrix.

§1. Introduction

Let $G$ be a simple graph and $A(G)$ be the adjacency matrix of $G$. The eigenvalues of $G$ are just the eigenvalues of the matrix $A(G)$ [2]. The energy $E(G)$ of $G$ is defined as the sum of the absolute values of the eigenvalues of $G$. The maximum eigen value of the adjacency matrix is denoted by $\lambda_{\text{max}}$. This notion was proposed by Gutman [3] and found applications in the molecular orbital theory of conjugated $\pi$-electron systems [4,6].

Let $D(G)$ be the degree diagonal matrix of the graph $G$. Then $L(G) = D(G) - A(G)$ is the Laplacian matrix of $G$. Denote by $\mu_1(G)$, $\mu_2(G)$,..., $\mu_n(G)$ the Laplacian eigenvalues of $G$, arranged in a nonincreasing order, where $n$ is the number of vertices of $G$ (see [8]). The Laplacian energy $LE(G)$ [7,15] of $G$ is defined as the sum of the distance between Laplacian eigenvalues of $G$ and the average degree $d(G)$ of $G$, for which more results may be found in [15,16].

The first Zagreb index [5,12] of a graph $G$ is defined as

$$M(G) = \sum d^2(u),$$

where $d(u)$ is the degree of the vertex $u$ in $G$. The subdivision graph $S(G)$ is the graph obtained from $G$ by replacing each of its edge by a path of length 2, or equivalently by inserting an additional vertex into each edge of $G$ [11]. The line graph $L(G)$ is the graph whose vertices corresponds to the edges of $G$ with two vertices being adjacent if and only if the corresponding edges in $G$ have a vertex in common.

The tadpole graph $T_{n,k}$ [13] is the graph obtained by joining a cycle graph $C_n$ to a path of length $k$. The wheel graph $W_{n+1}$ is defined as the graph $K_1 + C_n$ where $K_1$ is the singleton...
graph and $C_n$ is the cycle graph $[9]$. The helm graph $H_{n+1}$ is the graph obtained from the wheel Graph $W_{n+1}$, by adjoining a pendent edge at each node of the cycle. The ladder graph $L_n = K_2 \square P_n$, where $P_n$ is a path graph. In [10], the authors obtained some results using the notion ladder graph.

§2. Preliminaries

The following results will be useful in our further investigation.

**Theorem 2.1.** [14] Let $G$ be a graph with $n$ vertices and $m \geq 1$ edges. Then the line graph $L(G)$ have $m$ vertices and $\frac{M(G)}{2} - m$ edges.

**Theorem 2.2.** [14] Let $G$ be a graph with $n$ vertices and $m \geq 1$ edges, then

1. $E(L(G)) \leq 4m - 2$.
2. $LE(L(G)) \leq (2M(G) - 4m)(1 - \frac{1}{m})$.

**Theorem 2.3.** [1] The maximal eigen value $\lambda_{\text{max}}(G)$ of the bipartite graph $G$ with $e$ number of edges is

$$\lambda_{\text{max}}(G) \leq \sqrt{e}.$$ 

§3. The maximal eigen value and energy

This section deals with the bounds for the laplacian energy and the largest eigen value of the line graph of the subdivision graph of the tadpole graph, wheel graph, complete graph and ladder graph.

**Theorem 3.1.** For the line graph of the subdivision graph of the tadpole graph $T_{n,k}$, the bounds for the laplacian energy and the maximal eigen value is given by theorem 2.2. Then

$$LE(L(S(T_{n,k}))) \leq \frac{2(2n + 2k + 1)(2n + 2k - 1)}{n + k} \quad \text{and} \quad \lambda_{\text{max}}(L(S(T_{n,k}))) \leq \sqrt{2n + 2k + 1}.$$ 

**Proof.** The cardinality of the subdivision graph of the tadpole graph $T_{n,k}$ is $2(n + k)$ of which one vertex of degree 3, one pendent vertex and $2n + 2k - 2$ vertices of degree 2. Hence the zagreb index of the subdivision graph of the tadpole graph $T_{n,k}$ is

$$M(S(T_{n,k})) = 8n + 8k + 2.$$ 

(1)

Hence from equation 1 and theorem 2.2 the Laplacian energy of the line graph of the subdivision graph of the tadpole graph $T_{n,k}$ written as

$$LE(L(S(T_{n,k}))) \leq \frac{2(2n + 2k + 1)(2n + 2k - 1)}{n + k}.$$ 

The graph $S(T_{n,k})$ is a bipartite graph having the cardinality of the edge set $2n + 2k$. Hence

$$\lambda_{\text{max}}(S(T_{n,k})) \leq \sqrt{2(n + k)}.$$ 

From equation (1) and theorem 2.1, the number of edges in the line graph of the subdivision graph of the tadpole graph

$$2n + 2k + 1.$$ 

(2)
So, the maximum eigen value of $L(S(T_n,k))$ written from equation (2) as

$$\lambda_{max}(L(S(T_n,k))) \leq \sqrt{2n + 2k + 1}.$$  

**Theorem 3.2.** For the line graph of the subdivision graph of the wheel graph $W_{n+1}$, the bounds for the Laplacian energy and the maximal eigen value is

$$LE(L(S(W_{n+1}))) \leq \frac{1}{2}(n + 9)(4n - 1) \quad \text{and} \quad \lambda_{max}(L(S(W_{n+1}))) \leq \sqrt{\frac{n(n + 9)}{2}}.$$  

**Proof.** The cardinality of the vertex set of the subdivision graph of the wheel graph $W_{n+1}$ is $3n + 1$ among which the hub of the wheel of degree $n$, the $2n$ vertices of degree 2 and $n$ vertices of degree 3. So the zagreb index of $S(W_{n+1})$ is

$$M(S(W_{n+1})) = n(n + 17). \quad (3)$$  

Hence from the equation (3) the laplacian energy of the line graph of the subdivision graph of the wheel graph is

$$LE(L(S(W_{n+1}))) \leq \frac{1}{2}(n + 9)(4n - 1).$$  

The graph $S(W_{n+1})$ contains $4n$ edges. So the cardinality of the edge set of $L(S(W_{n+1}))$ written using equation (3) and theorem 2.1 as

$$\frac{n(n + 9)}{2}. \quad (4)$$  

Using equation (4), then

$$\lambda_{max}(L(S(W_{n+1}))) \leq \sqrt{\frac{n(n + 9)}{2}}.$$  

**Theorem 3.3.** For the line graph of the subdivision graph of the complete graph $S_n$, the bounds for the Laplacian energy and the maximal eigen value is

$$LE(L(S(S_n)))) \leq 2(n - 1)(n^2 - n - 1) \quad \text{and} \quad \lambda_{max}(L(S(S_n)))) \leq \sqrt{\frac{n(n - 1)^2}{2}}.$$  

**Proof.** The subdivision graph of the complete graph $S_n$, the vertex set is of cardinality $\frac{n(n+1)}{2}$. Out of which the $n$ vertices of degree $n - 1$, and $\frac{n(n-1)}{2}$ vertices of degree 2. Hence the zagreb index of the subdivision graph of the complete graph.

$$M(S(S_n)) = n(n^2 - 1). \quad (5)$$  

For the line graph of the subdivision graph of the complete graph $S_n$, the bounds for the Laplacian energy is given from equation (5) and theorem 2.2

$$LE(L(S(S_n)))) \leq 2(n - 1)(n^2 - n - 1).$$  

The cardinality of the edge set of the subdivision graph of the complete graph $S_n$ is $4n$ so that

$$\lambda_{max}(S(S_n)) \leq 2\sqrt{n}.$$
The number of edges in the line graph of the subdivision graph of the complete graph $S_n$ can be written using equation (5) and theorem 2.1 as

$$\frac{n(n-1)^2}{2}. \quad (6)$$

Using equation (6) the maximal eigen value of $L(S(S_n))$ is

$$\lambda_{\text{max}}(L(S(S_n))) \leq \sqrt{\frac{n(n-1)^2}{2}}.$$

**Theorem 3.4.** For the line graph of the subdivision graph of the ladder graph $L_n$, the bounds for the Laplacian energy and the maximal eigen value is

$$LE(L(S(L_n))) \leq 2(n-1)(n^2 - n - 1) \quad \text{and} \quad \lambda_{\text{max}}(L(S(L_n))) \leq \sqrt{9n - 10}.$$

**Proof.** The cardinality of the vertex set of the subdivision graph of the ladder graph $L_n$ is $5n - 2$ among which $3n + 2$ vertices of degree 2 and $2n - 4$ vertices of degree 3. Hence the zagreb index of the subdivision graph of the ladder graph $L_n$ is

$$M(S(L_n)) = 30n - 28. \quad (7)$$

For the line graph of the subdivision graph of the ladder graph $L_n$, the bounds for the Laplacian energy is given from equation (7) and theorem 2.2 as

$$LE(L(S(L_n))) \leq 2 \left(\frac{(9n-10)(6n-5)}{3n-2}\right).$$

The cardinality of the edge set of $S(L_n)$ is $6n - 4$. Hence the bounds for the largest eigen value of $S(L_n)$ is

$$\lambda_{\text{max}}(S(L_n)) \leq \sqrt{6n - 4}.$$

The number of edges in the line graph of the subdivision graph of the ladder graph $L_n$ can be written using equation (7) and theorem 2.1 as

$$9n - 10. \quad (8)$$

From equation (8) the largest eigen value of the line graph of the subdivision graph of the ladder graph is

$$\lambda_{\text{max}}(L(S(L_n))) \leq \sqrt{9n - 10}.$$

**References**


On the hybrid mean value of the Smarandache $kn$-digital sequence and Smarandache function

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Abstract The main purpose of this paper is using the elementary method to study the hybrid
mean value properties of the Smarandache $kn$-digital sequence and Smarandache function, and
give an interesting asymptotic formula for it.

Keywords Smarandache $kn$-digital sequence, Smarandache function, hybrid mean value, as-
ymptotic formula, elementary method.

§1. Introduction

For any positive integer $k$, the famous Smarandache $kn$-digital sequence $a(k,n)$ is defined
as all positive integers which can be partitioned into two groups such that the second part is
$k$ times bigger than the first. For example, Smarandache $2n$ and $3n$ digital sequences $a(2,n)$
and $a(3,n)$ are defined as $\{a(2,n)\} = \{12, 24, 36, 48, 510, 612, 714, 816, \cdots \}$ and $\{a(3,n)\} =
\{13, 26, 39, 412, 515, 618, 721, 824, \cdots \}$.

Recently, Professor Gou Su told me that she studied the hybrid mean value properties of
the Smarandache $kn$-digital sequence and the divisor sum function $\sigma(n)$, and proved that the
asymptotic formula

$$\sum_{n \leq x} \frac{\sigma(n)}{a(k,n)} = \frac{3\pi^2}{k \cdot 20 \cdot \ln x} \cdot \ln \ln x + O(1)$$

holds for all integers $1 \leq k \leq 9$.

When I read professor Gou Su’s work, I found that the method is very new, and the results
are also interesting. This paper as a note of Gou Su’s work, we consider the hybrid mean value
properties of the Smarandache $kn$-digital sequence and Smarandache function $S(n)$, which is
defined as the smallest positive integer $m$ such that $n|m!$. That is, $S(n) = \min\{m : n|m!, \ m \in
N\}$. In this paper, we will use the elementary and analytic methods to study a similar problem,
and prove a new conclusion. That is, we shall prove the following:

Theorem. Let $1 \leq k \leq 9$, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \leq x} \frac{S(n)}{a(k,n)} = \frac{3\pi^2}{k \cdot 20} \cdot \ln \ln x + O(1).$$

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§2. Proof of the theorem

In this section, we shall use the elementary and combinational methods to complete the proof of our theorem. First we need following:

**Lemma.** For any real number $x > 1$, we have

$$
\sum_{n \leq x} \frac{S(n)}{n} = \frac{\pi^2}{6} \cdot \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).
$$

**Proof.** For any real number $x > 2$, from [4] we have the asymptotic formula

$$
\sum_{n \leq x} S(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right).
$$

Then from Euler summation formula (see theorem 3.1 of [3]) we can deduce that

$$
\sum_{1 < n \leq x} \frac{S(n)}{n} = \frac{1}{x} \left(\frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + O\left(\frac{x^2}{\ln^2 x}\right)\right) + \int_1^x \left(\frac{\pi^2}{12} \cdot \frac{t^2}{\ln t} + O\left(\frac{t^2}{\ln^2 t}\right)\right) \frac{1}{t} dt
$$

$$
= \frac{\pi^2}{12} \cdot \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right) + \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \frac{13\pi^2}{12} \int_1^x \frac{1}{\ln^2 t} dt
$$

$$
= \frac{\pi^2}{6} \cdot \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right).
$$

This proves our Lemma.

Now we take $k = 2$ (or $k = 4$), then for any real number $x > 1$, there exists a positive integer $M$ such that

$$
5 \cdot 10^M \leq x < 5 \cdot 10^{M+1},
$$

then we can deduce that

$$
M = \frac{1}{\ln 10} \cdot \ln x + O(1).
$$

So from the definition of $a(2, n)$ we have

$$
\sum_{1 \leq n \leq x} \frac{S(n)}{a(2, n)} = \sum_{n=1}^{4} \frac{S(n)}{a(2, n)} + \sum_{n=5}^{49} \frac{S(n)}{a(2, n)} + \sum_{n=50}^{499} \frac{S(n)}{a(2, n)} + \ldots + \sum_{n=5 \cdot 10^{M-1}}^{5 \cdot 10^M-1} \frac{S(n)}{a(2, n)}
$$

$$
= \sum_{n=1}^{4} \frac{S(n)}{n \cdot (10 + 2)} + \sum_{n=5}^{49} \frac{S(n)}{n \cdot (10^2 + 2)} + \sum_{n=50}^{499} \frac{S(n)}{n \cdot (10^3 + 2)} + \ldots + \sum_{n=5 \cdot 10^{M-1}}^{5 \cdot 10^M-1} \frac{S(n)}{n \cdot (10^{M+1} + 2)}
$$

$$
+ \sum_{n=5 \cdot 10^{M-1}}^{5 \cdot 10^M-1} \frac{S(n)}{n \cdot (10^{M+2} + 2)}
$$

$$
(3)
$$
and

\[
\sum_{1 \leq n \leq x} \frac{S(n)}{a(4,n)} = \sum_{n=1}^{2} \frac{S(n)}{a(4,n)} + \sum_{n=3}^{24} \frac{S(n)}{a(4,n)} + \sum_{n=25}^{249} \frac{S(n)}{a(4,n)} + \cdots + \sum_{n=\frac{1}{4} \cdot 10^{M-1}}^{\frac{1}{4} \cdot 10^{M-1}} \frac{S(n)}{a(4,n)}
\]

\[
= \frac{2}{n \cdot (10 + 4)} + \frac{24}{n \cdot (10^2 + 4)} + \frac{249}{n \cdot (10^3 + 4)} + \cdots + \frac{1}{n \cdot (10^{M+1} + 4)}.
\]

Then from (2), (3) and Lemma we may immediately deduce

\[
\sum_{n=\frac{5}{10^k-1}}^{\frac{5}{10^k}-1} \frac{S(n)}{n \cdot (10^k+1 + 2)} = \sum_{n=\frac{5}{10^k-1}}^{\frac{5}{10^k}-1} \frac{S(n)}{n \cdot (10^k+1 + 2)} - \sum_{n=\frac{5}{10^k-1}}^{\frac{5}{10^k}-1} \frac{S(n)}{n \cdot (10^k+1 + 2)}
\]

\[
= \frac{\pi^2}{6} \cdot \frac{5 \cdot 10^k - 5 \cdot 10^k - 1}{10^k + 1 + 2} \cdot \frac{1}{\ln(5 \cdot 10^k)} + O \left( \frac{1}{k^2} \right)
\]

\[
= \frac{3\pi^2}{40} \cdot \frac{1}{k} + O \left( \frac{1}{k^2} \right)
\]

Similarly,

\[
\sum_{n=\frac{1}{4} \cdot 10^k-1}^{\frac{1}{4} \cdot 10^k-1} \frac{S(n)}{n \cdot (10^k+4)} = \sum_{n=\frac{1}{4} \cdot 10^k-1}^{\frac{1}{4} \cdot 10^k-1} \frac{S(n)}{n \cdot (10^k+4)} - \sum_{n=\frac{1}{4} \cdot 10^k-1}^{\frac{1}{4} \cdot 10^k-1} \frac{S(n)}{n \cdot (10^k+4)}
\]

\[
= \frac{\pi^2}{6} \cdot \frac{\frac{1}{4} \cdot 10^k - \frac{1}{4} \cdot 10^k - 1}{10^k + 4} \cdot \frac{1}{\ln(\frac{1}{4} \cdot 10^k)} + O \left( \frac{1}{k^2} \right)
\]

\[
= \frac{3\pi^2}{80} \cdot \frac{1}{k} + O \left( \frac{1}{k^2} \right)
\]

Noting that the identity \( \sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6 \) and the asymptotic formula

\[
\sum_{1 \leq k \leq M} \frac{1}{k} = \ln M + \gamma + O \left( \frac{1}{M} \right),
\]

where \( \gamma \) is the Euler constant.
From (2), (3) and (5) we have
\[
\sum_{1 \leq n \leq x} \frac{S(n)}{a(2, n)} = \sum_{n=1}^{4} \frac{S(n)}{a(2, n)} + \sum_{n=5}^{49} \frac{S(n)}{a(2, n)} + \sum_{n=50}^{499} \frac{S(n)}{a(2, n)} + \cdots + \sum_{n=5 \cdot 10^{M-1}}^{5 \cdot 10^{M-1}} \frac{S(n)}{a(2, n)}
\]
\[
= \sum_{k=1}^{M} \frac{3 \pi^2}{40} \cdot \frac{1}{k} + O \left( \sum_{k=1}^{M} \frac{1}{k^2} \right)
\]
\[
= \frac{3 \pi^2}{40} \ln \ln x + O(1).
\]

Similarly,
\[
\sum_{1 \leq n \leq x} \frac{S(n)}{a(4, n)} = \sum_{n=1}^{2} \frac{S(n)}{a(4, n)} + \sum_{n=3}^{24} \frac{S(n)}{a(4, n)} + \sum_{n=25}^{249} \frac{S(n)}{a(4, n)} + \cdots + \sum_{n=\frac{1}{4} \cdot 10^{M-1}}^{\frac{1}{4} \cdot 10^{M-1}} \frac{S(n)}{a(4, n)}
\]
\[
= \sum_{k=1}^{M} \frac{3 \pi^2}{80} \cdot \frac{1}{k} + O \left( \sum_{k=1}^{M} \frac{1}{k^2} \right)
\]
\[
= \frac{3 \pi^2}{80} \ln \ln x + O(1).
\]

For using the same methods, we can also prove that the theorem holds for all integers \(k = 1, 3, 5, 6, 7, 8, 9\). This completes the proof of our theorem.

References

$M^*$-paranormal composition operators on
Weighted Hardy spaces

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Abstract If $T$ is an analytic function mapping the unit disk $D$ into itself, we define the
composition operator $C_T$ on the space $H^2(\beta)$ by $C_Tf = f \circ T$. In this paper, we investigate the
relationship between properties of the symbol $T$ and the $M^*$-paranormality of the operators
$C_T$ and $C^*_T$.

Keywords Weighted Hardy space, $M^*$-paranormal, hyponormal, partial isometry, composi-
tion operators.

Mathematics Subject Classification: Primary 47B38, Secondary 47B37, 47B35.

§1. Preliminaries

Let $f$ be an analytic map on the open unit disk $D$ given by the Taylor’s series

$$f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$$

Let $\beta = \{\beta_n\}_{n=0}^{\infty}$ be a sequence of positive numbers with $\beta_0 = 1$ and $\frac{\beta_{n+1}}{\beta_n} \to 1$ as $n \to \infty$.
The set $H^2(\beta)$ of formal complex power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that

$$\|f\|_{\beta}^2 = \sum_{n=0}^{\infty} |a_n|^2 \beta_n^2 < \infty$$

is a Hilbert Space of function analytic in the unit disk with the inner product $<f, g>_{\beta} = \sum_{n=0}^{\infty} a_n b_n \beta_n^2$ for $f$ as above and $g(z) = \sum_{n=0}^{\infty} b_n z^n$.

Let $D$ be the Open Unit disk in the complex plane and let $T : D \to D$ be an analytic
self-map of the unit disk and consider the corresponding composition operator $C_T$ acting on
$H^2(\beta)$ i.e., $C_T(f) = f \circ T$, $f \in H^2(\beta)$.

The operators $C_T$ are not necessarily defined on all of $H^2(\beta)$. They are everywhere defined
in some special cases on the classical Hardy Space $H^2$ (the case when $\beta_n = 1$ for all $n$). See
for example of [10] and on a general space $H^2(\beta)$ if the function $T$ is analytic on some open
set containing the closed unit disk having supremum norm strictly smaller than one (see [13]).
There are a lot of other known properties of composition operators, on the classical Hardy Space $H^2$ (see for example [3], [7] and [10]) and on more general space $H^2(\beta)$. (see [5], [6], [11], [12] and [13]).

In [4], Cowen’s and Kriete obtained a nice correlation between hyponormality of composition operators on $H^2$ and the Denjoy-Wolff point of the induced map.

In [14], Nina Zorboska obtained some results on the hyponormality of Composition Operators and their adjoints.

Let $\omega$ be a point on the open disk.

Define $k^n(\omega) = \sum_{n=0}^{\infty} \frac{|\omega^n|^n}{n!}$.

Then the function $k^n(\omega)$ is a point evaluation for $H^2(\beta)$.

Then $k^n(\omega)$ is in $H^2(\beta)$ and $\|k^n(\omega)\|^2 = \sum_{n=0}^{\infty} \frac{|\omega|^n}{n!}$. Thus, $\|k_0\|$ is an increasing function of $|\omega|$.

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ then $< f, k^n(\omega) > = \sum_{n=0}^{\infty} a_n \omega^n \bar{\omega^n} = f(\omega)$.

Therefore $< f, k^n(\omega) > = f(\omega)$ for all $f$ and $k^n(\omega)$ is known as the point evaluation kernel at $\omega$.

It can be easily shown that $C_T^*k^n(\omega) = k^n(\omega)$ and $k^n(\omega) = 1$ (the function identically equal to 1).

In this article, we are interested in the $M^*$-paranormal composition operators and their adjoints on $H^2(\beta)$.

§2. $M^*$-paranormal composition operators

An operator $T$ defined on a Hilbert Space $H$ is said to be $M^*$-paranormal [1] if $\|T^*x\|^2 \leq M \|T^2x\|$ for each vector $x \in H$ or equivalently $\|T^*x\|^2 \leq M \|T^2x\| |x|$ for all $x \in H$.

An operator $T$ defined on a Hilbert space $H$ is said to be $[2]$:

(i) Co-isometry if $TT^* = I$ or equivalently $< T^*(x), T^*(y) > = < TT^*(x), y > = < x, y >$ for all $x, y \in H$.

(ii) Partial isometry if $T^*T = T$ or equivalently $TT^*T = T$.

An operator $T$ on a complex Hilbert Space $H$ is of class $(M, k)$, $k \geq 2$ if $T^kT^k \geq (T^kT)^k$ [9].

Arora and Thukral [1] proved that $T$ is $M^*$-paranormal if and only if $M^2T^2 - 2\lambda TT^* + \lambda^2 \geq 0$, for each $\lambda > 0$.

**Theorem 2.1.** If composition operator $C_T$ on $H^2(\beta)$ is $M^*$-paranormal then $\|k_T(0)\|^2 \geq M$.

**Proof.** If $C_T$ is $M^*$-paranormal

$\Rightarrow M^2C_T^2C_T - 2\lambda C_TC_T + \lambda^2 \geq 0$ for all $\lambda > 0$.

$\Rightarrow < (M^2C_T^2C_T - 2\lambda C_TC_T + \lambda^2) f, f > \geq 0$ for all $f \in H^2(\beta)$.

$\Rightarrow M^2 < C_T^2C_T f, f > - 2 \lambda < C_TC_T f, f > + \lambda^2 < f, f > \geq 0$.

$\Rightarrow M^2 < C_T^2 f, C_T^2 f > - 2 \lambda < C_T f, C_T f > + \lambda^2 < f, f > \geq 0$.

$\Rightarrow M^2 \|C_T^2 f\|^2 - 2 \lambda \|C_T f\|^2 + \lambda^2 \|f\|^2 \geq 0$.

Let $f = k_0 \in H^2(\beta)$.

$M^2 \|C_Tk_0\|^2 - 2 \lambda \|C_Tk_0\|^2 + \lambda^2 \|k_0\|^2 \geq 0$. 

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Theorem 2.3. If partial isometry $C_T$ on $H^2(\beta)$ is $M^*$-paranormal and if co-isometry then $M^2 \geq 1$.

**Proof.** $C_T$ is $M^*$-paranormal

$M^2 C_T^2 C_T^* - 2\lambda C_T C_T^* + \lambda^2 \geq 0,$ for all $\lambda > 0$.

$(M^2 C_T^2 C_T^* - 2\lambda C_T C_T^* + \lambda^2 ) C_T \geq 0.$

$M^2 C_T^* C_T C_T^* C_T - 2 \lambda C_T C_T^* C_T + \lambda^2 C_T^* C_T \geq 0.$

Since $C_T$ is partial isometry.

$M^2 C_T^2 C_T C_T - 2 \lambda C_T C_T^* C_T + \lambda^2 C_T^* C_T \geq 0.$

Since $C_T$ is co-isometry.

$M^2 C_T^2 C_T^* - 2\lambda C_T C_T^* + \lambda^2 C_T^* C_T \geq 0.$

$M^2 \parallel C^2_T f \parallel^2 - 2\lambda \parallel C_T f \parallel^2 + \lambda^2 \parallel C_T f \parallel^2 \geq 0$ for all $f \in H^2(\beta)$.

Let $f = k_0^2 \in H^2(\beta)$,

$M^2 \parallel C_T C_T^* k_0^2 \parallel^2 - 2\lambda \parallel C_T k_0^2 \parallel^2 + \lambda^2 \parallel C_T k_0^2 \parallel^2 \geq 0.$

$M^2 \parallel C_T k_0^2 \parallel^2 - 2\lambda \parallel k_0^2 \parallel^2 + \lambda^2 \parallel k_0^2 \parallel^2 \geq 0 \ [14].$

$M^2 \parallel k_0^2 \parallel^2 - 2\lambda \parallel k_0^2 \parallel^2 + \lambda^2 \parallel k_0^2 \parallel^2 \geq 0.$

$M^2 - 2\lambda + \lambda^2 \geq 0 \ [14].$

By elementary properties of real quadratic form, we get $M^2 \geq 1$.

**Theorem 2.4.** If composition operator $C_T$ is on $H^2(\beta)$ and $C_T^*$ is $M^*$-paranormal then

$M^2 \parallel k_0^2 \parallel^2 \geq 1.$

**Proof.** $C_T^*$ is $M^*$-paranormal,

$M^2 C_T^2 - 2 \lambda C_T + \lambda^2 \geq 0,$ for all $\lambda > 0$.

$\Rightarrow \ < (M^2 C_T^2 - 2\lambda C_T + \lambda^2) f, f >= 0,$ for all $f \in H^2(\beta)$.

$\Rightarrow \ M^2 < C_T^2 f, f > - 2\lambda < C_T f, f > + \lambda^2 < f, f > \geq 0.$

$\Rightarrow \ M^2 < C_T^2 f, C_T^2 f > - 2\lambda < C_T f, C_T f > + \lambda^2 < f, f > \geq 0.$

$M^2 \parallel C_T^2 f \parallel^2 - 2\lambda \parallel C_T f \parallel^2 + \lambda^2 \parallel f \parallel^2 \geq 0.$

Let $f = k_0^2 \in H^2(\beta)$,

$M^2 \parallel C_T^2 k_0^2 \parallel^2 - 2\lambda \parallel C_T k_0^2 \parallel^2 + \lambda^2 \parallel k_0^2 \parallel^2 \geq 0.$

$M^2 \parallel C_T k_0^2 \parallel^2 - 2\lambda \parallel k_0^2 \parallel^2 + \lambda^2 \parallel k_0^2 \parallel^2 \geq 0.$

$M^2 \parallel k_0^2 \parallel^2 - 2\lambda \parallel k_0^2 \parallel^2 + \lambda^2 \parallel k_0^2 \parallel^2 \geq 0 \ [14].$

$M^2 \parallel k_0^2 \parallel^2 \geq 1 \ [14].$
\[ M^2 \left\| k_{T}^{\beta} \right\|_{\beta} \geq 1. \]

**Theorem 2.5.** If composition operator \( C_T \) on \( H^2(\beta) \) is of class \((M, 3)\), then \( \left\| C_T k_{T}^{\beta} \right\|_{\beta} \leq 1. \)

**Proof.** \( C_T \) is of class \((M, 3)\) if
\[ \langle C_T^3 C_T^3 f, f \rangle \geq \langle C_T^3 C_T C_T f, f \rangle \]
\[ \langle C_T^3 C_T C_T f, f \rangle \leq \langle C_T^3 f, f \rangle \geq 0. \]
\[ \langle C_T^3 C_T C_T C_T C_T f, f \rangle \leq \langle C_T C_T f, f \rangle \leq \langle C_T^3 f, f \rangle \]
\[ \| C_T^3 C_T C_T f \|^2 \leq \| C_T^3 f \|^2. \]

Let \( f = k_{0}^{\beta} \in H^2(\beta) \), we have
\[ \left\| C_T C_T C_T k_{0}^{\beta} \right\|_{\beta} \leq \left\| C_T^3 k_{0}^{\beta} \right\|_{\beta}^{2}, \]
\[ \left\| C_T C_T k_{0}^{\beta} \right\|_{\beta} \leq \left\| C_T^3 C_T k_{0}^{\beta} \right\|_{\beta}^{2} \]
\[ \left\| C_T k_{T}^{\beta} \right\|_{\beta} \leq \left\| C_T C_T k_{0}^{\beta} \right\|_{\beta}^{2} \]
\[ \left\| C_T k_{T}^{\beta} \right\|_{\beta} \leq \left\| C_T k_{0}^{\beta} \right\|_{\beta}^{2} \]
\[ \left\| C_T k_{T}^{\beta} \right\|_{\beta} \leq \left\| k_{0}^{\beta} \right\|_{\beta}^{2} \]
\[ \left\| C_T k_{T}^{\beta} \right\|_{\beta} \leq 1. \]

**References**


On the mean value of some new sequences

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Abstract The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of the Smarandache repetitional sequence, and give two asymptotic formulas for it.

Keywords Euler’s summation formula, Abel’s identity, Smarandache repetitional sequence.

§1. Introduction

Let \( k \) be a fixed positive integer. The famous Smarandache repetitional generalized sequence \( S(n, k) \) is defined as following: \( n1n, n22n, n333n, n444n, n5555n, n66666n, n7777777n, n88888888n, n999999999n, n101010101010101010101010101010101010101010101010101010101010101010\), \( \cdots \). In problem 3 of reference [1], Professor Mihaly Benze asked us to study the arithmetical properties about this sequence. It is interesting for us to study this problem. But it’s a pity none had studied it before. At least we haven’t seen such a paper yet. In this paper, we shall use the elementary and analytic methods to study the arithmetical properties of the special Smarandache repetitional generalized sequence, and give a sharper asymptotic formula for it. That is, we shall prove the following:

Theorem. For any real number \( x \geq 1 \), we have the asymptotic formula

\[
\sum_{k \leq x} S(n, k) = \sum_{i=0}^{m} di(x - i)^3 + fi(2i + 1)(x - i)^2 + h \cdot 10^x + O\left(\frac{100}{9} \cdot 10^x\right) + A.
\]

§2. Proof of the theorem

In this section, we shall complete the proof of our theorem. First we give three simple lemmas which are necessary in the proof of our theorem. The proofs of these lemmas can be found in reference [7].

Lemma 1. For any real number \( x \geq 1 \) and \( \alpha > 0 \), we have the asymptotic formula

\[
\sum_{n \leq x} n^\alpha = \frac{x^{1+\alpha}}{1+\alpha} + O\left(x^\alpha\right).
\]

Lemma 2. If \( f \) has a continuous derivative \( f' \) on the interval \([x, y]\), where \( 0 < y < x \),

\[
\sum_{y < k \leq x} f(n) = \int_y^x f(t)dt + \int_y^x (t - [t])f'(t)dt + f(x)([x] - x) - f(y)([y] - y).
\]
Lemma 3. For any arithmetical function \(a(n)\), let \(A(x) = \sum_{n \leq x} a(n)\). If \(f\) has a continuous derivative \(f'\) on the interval \([y, x]\), where \(0 < y < x\),

\[
\sum_{y < k \leq x} a(n) f(n) = A(x) f(x) - A(y) f(y) + \int_{y}^{x} A(t) f'(t) \, dt.
\]

Now we use these lemmas to prove our conclusion. First we use the elementary method to obtain an asymptotic formula. To accomplish our theorem easily, we can get the following equations by observing the classification of the Smarandache repetitional sequence.

\[
\begin{align*}
S(n, 1) &= n \cdot 10^2 + 1 \cdot 10^1 + n \cdot 10^0 \\
S(n, 2) &= n \cdot 10^3 + 2 \cdot 10^2 + 2 \cdot 10^1 + n \cdot 10^0 \\
S(n, 3) &= n \cdot 10^3 + 3 \cdot 10^3 + 3 \cdot 10^2 + 3 \cdot 10^1 + n \cdot 10^0 \\
S(n, 4) &= n \cdot 10^5 + 4 \cdot 10^4 + 4 \cdot 10^3 + 4 \cdot 10^2 + 4 \cdot 10^1 + n \cdot 10^0 \\
& \quad \ldots \ldots \\
S(n, a-2) &= n \cdot 10^{a-1} + (a-2) \cdot 10^{a-2} + \cdots + (a-2) \cdot 10^2 + (a-2) \cdot 10^1 + n \cdot 10^0 \\
S(n, a-1) &= n \cdot 10^a + (a-1) \cdot 10^{a-1} + \cdots + (a-1) \cdot 10^2 + (a-1) \cdot 10^1 + n \cdot 10^0 \\
S(n, a) &= n \cdot 10^{a+1} + a \cdot 10^a + \cdots + a \cdot 10^2 + a \cdot 10^1 + n \cdot 10^0.
\end{align*}
\]

Now we estimate the right hand side of the above equations, by lemma 2 we have

\[
\sum_{k \leq x} S(n, k) = \sum_{k \leq x} n \cdot \left[ \frac{100 \cdot (10^k - 1)}{9} \right] + \sum_{k \leq x} \frac{k(k + 1)}{2} \cdot 10^1 + \sum_{k \leq x} \frac{(k - 1)(k + 2)}{2} \cdot 10^2 \\
+ \sum_{k \leq x} \frac{(k - 2)(k + 3)}{2} \cdot 10^3 + \cdots + \sum_{k \leq x} \frac{k + (k - 2)}{2} \cdot 10^{k-2} \\
+ \sum_{k \leq x} \frac{k + (k - 3)}{2} \cdot 10^{k-1} + k \cdot 10^k.
\] (1)

Now we estimate the one part of the right hand side of the above equations, by lemma 2 and lemma 3 we have

\[
\sum_{k \leq x} k(k + 1) = \left[ \frac{x^2}{2} + O(x) \right] (x + 1) - 1 - O(2) - \int_{1}^{x} \left[ \frac{t^2}{2} + O(t) \right] \, dt
\]

\[
= \frac{x^3}{2} + \frac{x^2}{2} - 1 - \int_{1}^{x} \left[ \frac{t^2}{2} + O(t) \right] \, dt + O(x^2)
\]

\[
= \frac{x^3}{2} + \frac{x^2}{2} - 1 - \frac{t^2}{2} \, dt + O(\int_{1}^{x} t \, dt) + O(x^2)
\]

\[
= \frac{x^3}{3} + \frac{x^2}{2} + O(x^2) + C1,
\] (2)

where we have used the identity \(C1\).
Similarly, we also have the asymptotic formulae

\[
\sum_{k \leq x} (k - 1)(k + 2) = \frac{(x - 1)^2}{2} + O(x - 1)(x + 2) - \int_{1}^{x} \frac{(t - 1)^2}{2} + O(t - 1)\] \, dt
\[
= \frac{(x - 1)^2(x + 2)}{2} - \int_{1}^{x} \frac{(t - 1)^2}{2} \, dt + \int_{1}^{x} O(t - 1)\] \, dt + O(x - 1)^2
\[
= \frac{(x - 1)^2(x - 1 + 3)}{2} - \frac{(x - 1)^3}{6} + O(\int_{1}^{x} (t - 1)\] \, dt) + O(x^2)
\[
= \frac{(x - 1)^3}{3} + \frac{3(x - 1)^2}{2} + O((x - 1)^2) + C2,
\]

where we have used the identity C2. Similarly, we also have the asymptotic formulae

\[
\sum_{k \leq x} (k - a + 1)(k + a) = \frac{(x - a + 1)^2}{2} + O(x - a + 1)(x + a) - \int_{1}^{x} \frac{(t - a + 1)^2}{2} \] \, dt
\[
+ O(t - a + 1)\] \, dt
\[
= \frac{(x - a + 1)^2(x + a)}{2} - \int_{1}^{x} \frac{(t - a + 1)^2}{2} \, dt + \int_{1}^{x} O(t - a + 1)\] \, dt
\[
= \frac{(x - a + 1)^2(x - a + 1 + 2a - 1)}{2} - \frac{(x - a + 1)^3}{6} + C2.
\]

Finally, we can use the lemma 2 to get the following formulae

\[
\sum_{k \leq x} k \cdot 10^k = \frac{x^2}{2} \cdot 10^x + O(x \cdot 10^x) - \int_{1}^{x} \log(10)\] \, dt
\[
= \frac{x^2}{2} \cdot 10^x + O(x \cdot 10^x) - \frac{1}{2} \int_{1}^{x} t^2 \, dt - \int_{1}^{x} O(t \cdot 10^t)\] \, dt
\[
= \frac{x^2}{2} \cdot 10^x + O(x \cdot 10^x) - \frac{x^2}{2} \cdot 10^x + O(\int_{1}^{x} t \cdot 10^t)\] \, dt
\[
= \frac{x \cdot 10^x}{\log(10)} - \frac{10^x}{(\log(10))^2} + O(x \cdot 10^x) + A1.
\]
From (1) (2), (3), (4), (5) and (6) we deduce the asymptotic formula

$$\sum_{k \leq x} S(n, k) = \sum_{i=0}^{m} di(x - i)^3 + f i(2i + 1)(x - i)^2 + h \cdot 10^r + O\left(\frac{100 \cdot n}{9} \cdot 10^r\right) + A.$$

Now combining two methods we may immediately deduce our theorem.

References


Eigenvalues and eigenfunctions of Sturm-Liouville problem with two-point discontinuities containing eigenparameter-dependent boundary conditions

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Abstract In this work, we extend some spectral properties of regular Sturm-Liouville problems to the problems consisting of a Sturm-Liouville equation with piecewise continuous potential together with eigenparameter-dependent boundary conditions and four supplementary transmission conditions. We give an operator-theoretical formulation, construct fundamental solutions and investigate some properties of the eigenvalues and corresponding eigenfunctions of the discontinuous Sturm-Liouville problem.

Keywords Discontinuous Sturm-Liouville problems, the asymptotic of eigenvalues and eigenfunctions.


§1. Introduction

It is well known that many topics in mathematical physics lead to the Sturm-Liouville type boundary-value problems. The Sturmian theory is one of the most actual and extensively developing fields in theoretical and applied mathematics. Particularly, in recent years, there have been increasing interests in spectral analysis of boundary value problems with eigenvalue-dependent boundary conditions. There are quite substantial literatures on such problems. Here we mention the results of [1-7, 11 and 13, 15-17] and corresponding references cited therein. Basically, boundary-value problems with continuous coefficients at the highest derivative of the equation have been investigated. Note that, discontinuous Sturm-Liouville problems with eigen-dependent boundary conditions and with two supplementary transmission conditions at the point of discontinuity were investigated in [2, 8-11]. In this paper, we shall investigate following discontinuous eigenvalue problem

\[ \tau u := -a(x)u'' + q(x)u = \lambda u \] (1)
on \( x \in [a, \xi_1] \cup (\xi_1, \xi_2) \cup [\xi_2, b] \) with boundary conditions at \( x = a \)
\[
L_1 u := \alpha_1 u(a) + \alpha_2 u'(a) = 0
\]  
(2)

four transmission conditions at the points of discontinuities \( x = \xi_1 \) and \( x = \xi_2 \),
\[
L_2 u := \gamma_1 u(\xi_1 - 0) - \delta_1 u(\xi_1 + 0) = 0,
\]  
(3)
\[
L_3 u := \gamma_1' u'(\xi_1 - 0) - \delta_1' u'(\xi_1 + 0) = 0,
\]  
(4)
\[
L_4 u := \gamma_2 u(\xi_2 - 0) - \delta_2 u(\xi_2 + 0) = 0,
\]  
(5)
\[
L_5 u := \gamma_2' u'(\xi_2 - 0) - \delta_2' u'(\xi_2 + 0) = 0,
\]  
(6)
and eigen-dependent boundary conditions at \( x = b \)
\[
L_0(\lambda) u := \lambda [\beta_1 u(b) - \beta_2 u'(b)] + [\beta_1 u(b) - \beta_2 u'(b)] = 0.
\]  
(7)

Where \( a(x) = a_1^2 \) for \( x \in [a, \xi_1] \), \( a(x) = a_2^2 \) for \( x \in (\xi_1, \xi_2) \), \( a(x) = a_3^2 \) for \( x \in [\xi_2, b] \); \( a_1 > 0, a_2 > 0 \) and \( a_3 > 0 \) are given real numbers; \( q(x) \) is a given real-valued function continuous in \([a, \xi_1], [\xi_1, \xi_2] \) and \([\xi_2, b]\) (that is, continuous in \([a, \xi_1], (\xi_1, \xi_2) \) and \([\xi_2, b]\) and has finite limits \( q(\xi_i \pm) := \lim_{x \to \xi_i \pm} q(x) \), \( q(\xi_i \pm) := \lim_{x \to \xi_i \pm} q(x) \)); \( \lambda \) is a complex eigenvalue parameter; the coefficients of the boundary and transmission conditions are real numbers. We assume that \( |\alpha_1| + |\alpha_2| \neq 0 \), \( |\gamma_1| + |\delta_1| \neq 0 \), \( |\gamma_1'| + |\delta_1'| \neq 0 \) \((i = 1, 2)\) and \( \rho := \beta_1' \beta_2 - \beta_1 \beta_2' > 0 \).

\section{2. Preliminaries}

For convenience let us introduce the next notations:
\[
\Omega_1 = [a, \xi_1], \quad \Omega_2 = [\xi_1, \xi_2], \quad \Omega_3 = [\xi_2, b],
\]

\[
\begin{align*}
\begin{cases}
 u(1)_1(x) := \\
 u(1)_2(x) :=
\end{cases}
\begin{cases}
 u(x) & x \in [a, \xi_1] \\
 \lim_{x \to \xi_1^-} u(x) & x = \xi_1
\end{cases}
\begin{cases}
 u(x) & x \in (\xi_1, \xi_2) \\
 \lim_{x \to \xi_1^+} u(x) & x = \xi_1
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
 u(3)_1(x) := \\
 u(3)_2(x) :=
\end{cases}
\begin{cases}
 u(x) & x \in (\xi_1, \xi_2) \\
 \lim_{x \to \xi_2^-} u(x) & x = \xi_2
\end{cases}
\begin{cases}
 u(x) & x \in (\xi_2, b) \\
 \lim_{x \to \xi_2^+} u(x) & x = \xi_2
\end{cases}
\end{align*}
\]

\[
(u)_\beta := \lim_{x \to b} \left( \beta_1 u(x) - \beta_2 u'(x) \right), \quad (u)'_\beta := \lim_{x \to b} \left( \beta_1' u(x) - \beta_2' u'(x) \right), \quad \tilde{u}(x) := \begin{cases}
 u(x), & x \in [a, b] \\
 (u)'_\beta, & x = b.
\end{cases}
\]

Note that, everywhere in below we shall assume that \( \gamma_i', \delta_i' > 0 \) \((i = 1, 2)\) and for the Lebesque measurable subsets \( M \subset [a, \xi_1] \cup (\xi_1, \xi_2) \cup (\xi_2, b] \) with Lebesque measure \( \mu_L(M) \) we shall define a new positive measure \( \mu_\rho(M) \) by
\[
\mu_\rho(M) := \frac{1}{a_1^2} \frac{\gamma_1' \gamma_2'}{\delta_1' \delta_2'} \mu_L(M \cap [a, \xi_1)) + \frac{1}{a_2^2} \mu_L(M \cap (\xi_1, \xi_2)) + \frac{1}{a_3^2} \frac{\delta_2' \delta_3'}{\gamma_2' \gamma_3'} \mu_L(M \cap (\xi_2, b]) + \frac{\delta_2' \delta_3'}{\gamma_2' \gamma_3'} b(M) + \frac{\delta_2' \delta_3'}{\gamma_2' \gamma_3'} \rho.
\]
Finally, substituting (9)-(12) in (8) yield the required equality

Further, it is easy to verify that

where as usual,

whereas the Wronskian’s of the functions \( f \) and \( g \). Since \( f \) and \( \overline{\gamma} \) satisfy the boundary condition (2) it follows that

From the transmission conditions (3)-(6) we get

Further, it is easy to verify that

Finally, substituting (9)-(12) in (8) yield the required equality

**Corollary 2.1.** All eigenvalues of the considered problem (1-7) are real.

We can now assume that all eigenfunctions are real-valued.

**Theorem 2.1.** The operator \( A \) is symmetric.

**Proof.** Let \( f, g \in D(A) \). By two partial integrations we get

\[
\langle Af, g \rangle_{\mathcal{H}_{\mu}} - \langle f, Ag \rangle_{\mathcal{H}_{\mu}} = \frac{\gamma_1 \gamma_2}{\delta_1 \delta_2} [W(\overline{f}, \overline{g}; \xi_1 - 0) - W(\overline{f}, \overline{g}; a)] + W(\overline{f}, \overline{g}; \xi_2 - 0) - W(\overline{f}, \overline{g}; \xi_1 + 0) + \frac{\delta_2 \beta_2}{\gamma_2 \gamma_2} [W(\overline{f}, \overline{g}; b) - W(\overline{f}, \overline{g}; \xi_2 + 0)] - \frac{\delta_2 \beta_2}{\gamma_2 \gamma_2} \rho [(f_{\beta}^\prime(\overline{g})_{\beta} - (f)_{\beta}(\overline{g})_{\beta}] \\
\] (8)

\[
W(f, g; x) = f(x)g^\prime(x) - f^\prime(x)g(x) \\
\] (9)

\[
W(\overline{f}, \overline{g}; a) = 0. \\
\] (10)

\[
\gamma_1 \gamma_2 W(\overline{f}, \overline{g}; \xi_i - 0) = \delta_i \delta_i W(\overline{f}, \overline{g}; \xi_i + 0) \quad (i = 1, 2). \\
\] (11)

\[
(f)_{\beta}(\overline{g})_{\beta} - (f)_{\beta}(\overline{g})_{\beta} = \rho W(\overline{f}, \overline{g}; b). \\
\] (12)

\[
\langle Af, g \rangle_{\mathcal{H}_{\mu}} = \langle f, Ag \rangle_{\mathcal{H}_{\mu}} \quad (f, g \in \mathcal{H}_{\mu}). \\
\] (13)
Corollary 2.2. If $\lambda_1$ and $\lambda_2$ are two different eigenvalues of the problem (1-7) then corresponding eigenfunctions $u_1$ and $u_2$ of this problem satisfy the following equality:

$$\frac{1}{a_1^2} \frac{\gamma_1 \gamma_1'}{\delta_1 \delta_1'} \int_a^{\xi_1} u_1(x)u_2(x)dx + \frac{1}{a_2^2} \int_{\xi_1}^{\xi_2} u_1(x)u_2(x)dx + \frac{1}{a_3^2} \frac{\delta_2 \delta_2'}{\gamma_2 \gamma_2'} \int_{\xi_2}^{b} u_1(x)u_2(x)dx$$

$$+ \frac{1}{\rho^2} \frac{\delta_2 \delta_2'}{\gamma_2 \gamma_2'} (u_1)'_\beta (u_2)'_\beta = 0.$$  \hfill (14)

In fact, this formula means the orthogonality of eigenfunctions $u_1$ and $u_2$ in the Hilbert space $H_\rho$. We need the following Lemma, which can be proved similarly to theorem 2 in [2].

**Lemma 2.1.** Let the real-valued function $q(x)$ be continuous in $[a, b]$ and $f(\lambda), g(\lambda)$ be given entire functions. Then for any $\chi(x, \lambda_0)$ the equation $-u'' + q(x)u = \lambda u, x \in [a, b]$ has a unique solution $u = u(x, \lambda)$ satisfying the initial conditions

$$u(a) = f(\lambda), u'(a) = g(\lambda) \text{ or } (u(b) = f(\lambda), u'(b) = g(\lambda)).$$

For each $x \in [a, b], u(x, \lambda)$ is an entire function of $\lambda$.

We shall define two solutions

$$\phi_\lambda(x) = \begin{cases} \phi_{1\lambda}(x), & x \in [a, \xi_1) \\ \phi_{2\lambda}(x), & x \in (\xi_1, \xi_2) \\ \phi_{3\lambda}(x), & x \in [\xi_2, b) \end{cases}$$

and

$$\chi_\lambda(x) = \begin{cases} \chi_{1\lambda}(x), & x \in [a, \xi_1) \\ \chi_{2\lambda}(x), & x \in (\xi_1, \xi_2) \\ \chi_{3\lambda}(x), & x \in [\xi_2, b) \end{cases}$$

of the equation (1) as follows: Let $\phi_{1\lambda}(x) = \phi_1(x, \lambda)$ be the solution of equation (1) on $[a, \xi_1)$, which satisfies the initial conditions

$$u(a) = \alpha_2, \ u'(a) = -\alpha_1.$$  \hfill (15)

Using lemma 2.1, after defining this solution we may define the solution $\phi_2(x, \lambda)$ of equation (1) on $[\xi_1, \xi_2]$ by means of the solution $\phi_1(x, \lambda)$ by the nonstandard initial conditions

$$u(\xi_1 + 0) = \frac{\gamma_1}{\delta_1} \phi_{1\lambda}(\xi_1 - 0, \lambda), \ u'(\xi_1 + 0) = \frac{\gamma_1'}{\delta_1'} \phi'_{1\lambda}(\xi_1 - 0, \lambda).$$  \hfill (16)

After defining this solution, we may define the solution $\phi_3(x, \lambda)$ of equation (1) on $[\xi_2, b]$ by means of the solution $\phi_2(x, \lambda)$ by the nonstandard initial conditions

$$u(\xi_2 + 0) = \frac{\gamma_2}{\delta_2} \phi_{2\lambda}(\xi_2 - 0, \lambda), \ u'(\xi_2 + 0) = \frac{\gamma_2'}{\delta_2'} \phi'_{2\lambda}(\xi_2 - 0, \lambda).$$  \hfill (17)

Hence, $\phi(x, \lambda)$ satisfies the equation (1) on $[a, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, b)$, the boundary condition (2) and the transmission conditions (3-6). Analogically first we define the solution $\chi_{3\lambda} = \chi_3(x, \lambda)$ on $[\xi_2, b]$ by the initial conditions

$$u(b) = \beta_2 \lambda + \beta_2, \ u'(b) = \beta_1 \lambda + \beta_1.$$  \hfill (18)
Again, after defining this solution we define the solution \( \chi_{2\lambda} = \chi_2(x, \lambda) \) of the equation (1) on \([\xi_1, \xi_2]\) by the initial conditions

\[
u(\xi_2 - 0) = \frac{\delta_1}{\gamma_1} \chi_{3\lambda}(\xi_2 + 0, \lambda), \quad \nu'(\xi_2 - 0) = \frac{\delta_1}{\gamma_1} \chi_{3\lambda}(\xi_2 + 0, \lambda).
\]

Using this solution, we define the solution \( \chi_{1\lambda} = \chi_1(x, \lambda) \) of (1) on \([\xi_2, b]\) by conditions

\[
u(\xi_1 - 0) = \frac{\delta_2}{\gamma_2} \chi_{3\lambda}(\xi_1 + 0, \lambda), \quad \nu'(\xi_1 - 0) = \frac{\delta_2}{\gamma_2} \chi_{3\lambda}(\xi_1 + 0, \lambda).
\]

Hence, \( \chi(x, \lambda) \) satisfies the equality (1) on \([a, \xi_1) \cup (\xi_1, \xi_2) \cup (\xi_2, b]\), the boundary condition (7) and the transmission conditions (3-6). Further it follows from (1) that the Wronskians

\[\omega_i(\lambda) := W_\lambda(\phi_i, \chi_i; x) := \phi_i(x, \lambda)\chi_i'(x, \lambda) - \phi_i'(x, \lambda)\chi_i(x, \lambda), x \in \Omega, \quad (i = 1, 2, 3)\]

are independent of \( x \in \Omega \). Moreover, these functions are entire in \( \lambda \).

**Lemma 2.2.** For each \( \lambda \in \mathbb{C} \), \( \gamma_1\gamma_2\gamma_2'\omega_1(\lambda) = \delta_1\delta_1'\delta_2'\omega_2(\lambda) = \delta_1\delta_2'\delta_2'\omega_3(\lambda) \).

**Proof.** In view of (16-17) and (19-20), short calculation gives

\[\gamma_1\gamma_2\gamma_2'W(\phi_1, \chi_1; \xi_1 - 0) = \delta_1\delta_1'\delta_2'W(\phi_2, \chi_2; \xi_1 + 0) = \delta_1\delta_2'\delta_2'W(\phi_3, \chi_3; \xi_2 + 0),\]

so \( \gamma_1\gamma_2\gamma_2'\omega_1(\lambda) = \delta_1\delta_1'\delta_2'\omega_2(\lambda) = \delta_1\delta_2'\delta_2'\omega_3(\lambda) \) for each \( \lambda \in \mathbb{C} \).

Now we may introduce the characteristic function

\[\omega(\lambda) := \gamma_1\gamma_2\gamma_2' \omega_1(\lambda) = \delta_1\delta_1'\delta_2' \omega_2(\lambda) = \delta_1\delta_2'\delta_2' \omega_3(\lambda).
\]

**Theorem 2.2.** The eigenvalues of the problem (1-7) are the zeros of the function \( \omega(\lambda) \).

**Proof.** Let \( \omega(\lambda_0) = 0 \). Then \( W_\lambda(\phi_1, \chi_1; x) = 0 \) and therefore the functions \( \phi_{1\lambda_0}(x) \) and \( \chi_{1\lambda_0}(x) \) are linearly dependent i.e. \( \chi_{1\lambda_0}(x) = k_1\phi_{1\lambda_0}(x), x \in [a, \xi_1) \) for some \( k_1 \neq 0 \). From this it follows that \( \chi(x, \lambda_0) \) satisfies also the first boundary condition (2), so \( \chi(x, \lambda_0) \) is an eigenfunctions for the eigenvalue \( \lambda_0 \). Now let \( u_0(x) \) be any eigenfunction corresponding to eigenvalue \( \lambda_0 \), but \( \omega(\lambda_0) \neq 0 \). Then the functions \( \phi_1, \chi_1, \phi_2, \chi_2, \) and \( \phi_3, \chi_3 \) would be linearly independent on \([a, \xi_1), [\xi_1, \xi_2] \) and \([\xi_2, b]\) respectively. Therefore \( u_0(x) \) may be represented as the form

\[u_0(x) = \begin{cases} c_1\phi_1(x, \lambda_0) + c_2\chi_1(x, \lambda_0), & x \in [a, \xi_1) \\ c_3\phi_2(x, \lambda_0) + c_4\chi_2(x, \lambda_0), & x \in (\xi_1, \xi_2) \\ c_5\phi_3(x, \lambda_0) + c_6\chi_3(x, \lambda_0), & x \in [\xi_2, b) \end{cases} \]

where at least one of the constants \( c_1, c_2, c_3, c_4, c_5, c_6 \) is not zero. Considering the equations

\[L_\nu(u_0(x)) = 0, \quad \nu = 1, 6, \]

as a system of linear equations of the variables \( c_i, i = 1, 6, \) and taking (16-17) and (19-20) into account, it can be shown that the determinant of this system is different from zero. Therefore, the system (21) has only the trivial solution \( c_i = 0, i = 1, 6 \). Thus we get a contradiction, which completes the proof.

**Lemma 2.3.** If \( \lambda = \lambda_0 \) is an eigenvalue, then \( \phi(x, \lambda_0) \) and \( \chi(x, \lambda_0) \) are linearly dependent.
Proof. Let $\lambda = \lambda_0$ be eigenvalue. From theorem 2, $W(\phi_{i\lambda_0}; \chi_{i\lambda_0}; x) = \omega_i(\lambda_0) = 0$ and therefore

$$\chi_{i\lambda_0}(x) = k_i\phi_{i\lambda_0}(x) \quad i = (1, 2, 3)$$

(22)

for some $k_1 \neq 0, k_2 \neq 0$ and $k_3 \neq 0$. We must show that $k_1 = k_2 = k_3$. Suppose, if possible that $k_1 \neq k_2$. Using the definitions of $\phi_i(x, \lambda_0)$ and $\chi_i(x, \lambda_0)$ and the equalities (22) we have

$$\delta_1(k_1 - k_2)\phi_{2\lambda}(\xi_1 + 0) = \delta_1 k_1\phi_{2\lambda}(\xi_1 + 0) - \delta_1 k_2\phi_{2\lambda}(\xi_1 + 0)$$

$$= k_1\chi_{1\lambda}(\xi_1 - 0) - k_2\delta_1\phi_{2\lambda}(\xi_1 + 0)$$

$$= \gamma_1\chi_{1\lambda}(\xi_1 - 0) - \delta_1\chi_{2\lambda}(\xi_1 + 0) = 0.$$

Hence

$$\phi_{2\lambda}(\xi_1 + 0) = 0.$$  

(23)

Analogically, starting from $\delta'_1(k_1 - k_2)\phi'_{2\lambda}(\xi_1 + 0)$ and following the same procedure we can derive that

$$\phi'_{2\lambda}(\xi_1 + 0) = 0.$$  

(24)

From the fact that $\phi_{2\lambda_0}(x)$ is a solution of the differential equation (1) on $[\xi_1, \xi_2]$ and satisfies the initial conditions (23-24) it follows that $\phi_{2\lambda_0}(x) = 0$ identically on $[\xi_1, \xi_2]$. Making use of (16-17) and (23-24) we may also derive that

$$\phi_{1\lambda_0}(\xi_1 - 0) = \phi'_{1\lambda_0}(\xi_1 - 0) = 0 \quad and \quad \phi_{3\lambda_0}(\xi_2 - 0) = \phi'_{3\lambda_0}(\xi_2 - 0) = 0$$

respectively. From this by the same argument as for $\phi_{2\lambda_0}(x)$ it follows that $\phi_{1\lambda_0}(x) = 0$ identically on $[a, \xi_1]$ and $\phi_{3\lambda_0}(x) = 0$ identically on $[\xi_2, b]$. Hence $\phi(x, \lambda_0) = 0$ identically on $[a, \xi_1] \cup (\xi_1, \xi_2) \cup (\xi_2, b)$. However, this contradicts (15), since $|\alpha_1| + |\alpha_2| \neq 0$.

Corollary 2.3. If $\lambda = \lambda_0$ is an eigenvalue, then both $\phi(x, \lambda_0)$ and $\chi(x, \lambda_0)$ are eigenfunctions corresponding to this eigenvalue.

Lemma 2.4. All eigenvalues $\lambda_n$ are simple zeros of $\omega(\lambda)$.

Proof. Using the well-known Lagrange’s formula [12] it can be shown that

$$\frac{1}{a^1} \int_a^{\xi_1} \phi_1(x)\phi_{\lambda_n}(x)dx + \frac{1}{b^2} \int_{\xi_2}^{b^2} \phi_2(x)\phi_{\lambda_n}(x)dx + \frac{1}{a^3} \int_{\xi_3}^{b} \phi_3(x)\phi_{\lambda_n}(x)dx = \frac{W(\phi_1, \phi_{\lambda_n}; b)}{\lambda - \lambda_n}$$

(25)

for any $\lambda$. Since $\chi_{\lambda_n}(x) = k_n\phi_{\lambda_n}(x), \ x \in [a, \xi_1] \cup (\xi_1, \xi_2) \cup (\xi_2, b]$ for some $k_n \neq 0$ ($n = 1, 2, ...$).

Using this equality for the right side of (25) we have

$$W(\phi_1, \phi_{\lambda_n}; b) = \frac{1}{k_n} W(\phi_1, \chi_{\lambda_n}; b)$$

$$= \frac{1}{k_n} \left[ \lambda_n(\phi_1)' + (\phi_1)_b \right]$$

$$= \frac{1}{k_n} \left[ \omega(\lambda) + (\lambda - \lambda_n)(\phi_1)' \right]$$

$$= (\lambda - \lambda_n) \frac{1}{k_n} \left[ \frac{\omega(\lambda)}{\lambda - \lambda_n} - (\phi_1)' \right].$$

(26)
Substituting this formula in (25) and letting $\lambda \to \lambda_n$ we get

$$\frac{1}{a_1^2} \int_a^{\xi_1} |\phi_{\lambda_n}(x)|^2 \, dx + \frac{1}{a_2^2} \int_{\xi_1}^{\xi_2} |\phi_{\lambda_n}(x)|^2 \, dx + \frac{1}{a_3^2} \int_{\xi_2}^b |\phi_{\lambda_n}(x)|^2 \, dx = \frac{1}{k_n} \left[ \omega'(\lambda_n) - (\phi_{\lambda_n})_0 \right].$$

(27)

Now putting $(\phi_{\lambda_n})_0 = \frac{1}{k_n}(\phi_{\lambda_n})_0 = \frac{\lambda_n}{k_n}$ in (26) seems that $\omega'(\lambda_n) \neq 0$.

§3. Asymptotic approximate formulas of the characteristic function

**Lemma 3.1.** Let $\phi(x, \lambda)$ be the solutions of equation (1) defined in section 2 and let $\lambda = s^2$. Then the next integral equations are hold:

$$\phi^{(k)}_{1\lambda}(x) = \alpha_2 \left[ \cos \frac{s(x-a)}{a_1} \right]^{(k)} - \alpha_1 \frac{a_1}{s} \left[ \sin \frac{s(x-a)}{a_1} \right]^{(k)} + \frac{1}{a_1 s} \int_a^x \left[ \sin \frac{s(x-y)}{a_1} \right]^{(k)} q(y) \phi_{1\lambda}(y) \, dy,$$

(28)

$$\phi^{(k)}_{2\lambda}(x) = \frac{\gamma_1}{\delta_1} \phi_{1\lambda}(\xi_1 - 0) \left[ \cos \frac{s(x-\xi_1)}{a_2} \right]^{(k)} \frac{a_2}{s} \gamma_1' \phi_{1\lambda}(\xi_1 - 0) \left[ \sin \frac{s(x-\xi_1)}{a_2} \right]^{(k)} + \frac{1}{a_2 s} \int_{\xi_1}^x \left[ \sin \frac{s(x-y)}{a_2} \right]^{(k)} q(y) \phi_{2\lambda}(y) \, dy,$$

(29)

$$\phi^{(k)}_{3\lambda}(x) = \frac{\gamma_2}{\delta_2} \phi_{2\lambda}(\xi_2 - 0) \left[ \cos \frac{s(x-\xi_2)}{a_3} \right]^{(k)} \frac{a_3}{s} \gamma_2' \phi_{2\lambda}(\xi_2 - 0) \left[ \sin \frac{s(x-\xi_2)}{a_3} \right]^{(k)} + \frac{1}{a_3 s} \int_{\xi_2}^x \left[ \sin \frac{s(x-y)}{a_3} \right]^{(k)} q(y) \phi_{3\lambda}(y) \, dy,$$

(30)

where $(\bullet)^{(k)} = \frac{d^k}{dx^k}(\bullet)$.

**Proof.** It is enough to substitute $s^2\phi_{1\lambda}(y) + a_1^2\phi_{1\lambda}''(y)$, $s^2\phi_{2\lambda}(y) + a_2^2\phi_{2\lambda}''(y)$ and $s^2\phi_{3\lambda}(y) + a_3^2\phi_{3\lambda}''(y)$ instead of $q(y)\phi_{1\lambda}(y)$, $q(y)\phi_{2\lambda}(y)$ and $q(y)\phi_{3\lambda}(y)$ in the integral terms of the equations (27), (28) and (29) for $k = 0, 1, \ldots$, respectively and integrate by parts twice.

**Lemma 3.2.** Let $\lambda = s^2$, $Im s = t$. Then the functions $\phi_{\lambda}(x)$ have the following asymptotic representations for $|\lambda| \to \infty$, which hold uniformly for $x \in \Omega_i$ ($i = 1, 2, 3$) and $k = 0, 1$:

If $\alpha_2 \neq 0$,

$$\phi^{(k)}_{1\lambda}(x) = \alpha_2 \left[ \cos \frac{s(x-a)}{a_1} \right]^{(k)} + O \left( |s|^{-1} \exp \frac{|t|}{a_1} \right),$$

(31)

$$\phi^{(k)}_{2\lambda}(x) = \alpha_2 \frac{\gamma_1}{\delta_1} \left[ \cos \frac{s(x-\xi_1)}{a_2} \right]^{(k)} \cos \frac{s(\xi_1 - a)}{a_1} - \alpha_1 \frac{a_1}{\delta_1} \left[ \sin \frac{s(x-\xi_1)}{a_2} \right]^{(k)} \sin \frac{s(\xi_1 - a)}{a_1} + O \left( |s|^{-1} \exp \frac{|t|}{a_1a_2} \right),$$

(32)
\[ \phi^{(k)}_{\alpha}(x) = a_1 \frac{\gamma_1 \gamma_2}{\delta_1} \left[ \cos \left( \frac{s(x - \xi_2)}{a_3} \right) \cos \frac{s(\xi_2 - \xi_1)}{a_2} \sin \frac{s(\xi_1 - a)}{a_1} \right]^{(k)} \]

\[ - a_2 \frac{\gamma_1 \gamma_2}{\delta_1 \delta_2} \left[ \cos \left( \frac{s(x - \xi_2)}{a_3} \right) \sin \frac{s(\xi_2 - \xi_1)}{a_2} \sin \frac{s(\xi_1 - a)}{a_1} \right]^{(k)} \]

\[ - a_3 \frac{\gamma_1 \gamma_2'}{\delta_1 \delta_2} \left[ \sin \left( \frac{s(x - \xi_2)}{a_3} \right) \sin \frac{s(\xi_2 - \xi_1)}{a_2} \cos \frac{s(\xi_1 - a)}{a_1} \right]^{(k)} \]

\[ - a_4 \frac{\gamma_1 \gamma_2'}{\delta_1 \delta_2} \left[ \sin \left( \frac{s(x - \xi_2)}{a_3} \right) \cos \frac{s(\xi_2 - \xi_1)}{a_2} \sin \frac{s(\xi_1 - a)}{a_1} \right]^{(k)} \]

\[ + O \left( |s|^{-k} \exp \left[ \frac{|t|}{a_1 a_2 a_3} \right] \right). \] (33)

If \( \alpha_2 = 0 \),

\[ \phi^{(k)}_{1\alpha}(x) = - \frac{a_1}{a_1} \left[ \sin \left( \frac{s(x - a)}{a_1} \right) \right]^{(k)} + O \left( |s|^{-k} \exp \left[ \frac{|t|}{a_1 a_2 a_3} \right] \right), \] (34)

\[ \phi^{(k)}_{2\alpha}(x) = - \frac{a_1 a_1}{a_1} \frac{\gamma_1}{\delta_1} \left[ \cos \left( \frac{s(x - \xi_1)}{a_2} \right) \sin \frac{s(\xi_1 - a)}{a_1} \right]^{(k)} \]

\[ - a_2 \frac{a_2}{a_2} \frac{\gamma_1'}{\delta_1} \left[ \sin \left( \frac{s(x - \xi_1)}{a_2} \right) \cos \frac{s(\xi_1 - a)}{a_1} \right]^{(k)} \]

\[ + O \left( |s|^{-k} \exp \left[ \frac{|t|}{a_1 a_2 a_3} \right] \right). \] (35)

\[ \phi^{(k)}_{3\alpha}(x) = - \frac{a_1 a_1}{a_1} \frac{\gamma_1 \gamma_2}{\delta_1 \delta_2} \left[ \cos \left( \frac{s(x - \xi_2)}{a_3} \right) \cos \frac{s(\xi_2 - \xi_1)}{a_2} \sin \frac{s(\xi_1 - a)}{a_1} \right]^{(k)} \]

\[ - a_2 \frac{a_1 a_1}{a_1} \frac{\gamma_1 \gamma_2'}{\delta_1 \delta_2} \left[ \cos \left( \frac{s(x - \xi_2)}{a_3} \right) \sin \frac{s(\xi_2 - \xi_1)}{a_2} \cos \frac{s(\xi_1 - a)}{a_1} \right]^{(k)} \]

\[ + \frac{a_1 a_1}{a_1} \frac{\gamma_1 \gamma_2'}{\delta_1 \delta_2} \left[ \sin \left( \frac{s(x - \xi_2)}{a_3} \right) \sin \frac{s(\xi_2 - \xi_1)}{a_2} \sin \frac{s(\xi_1 - a)}{a_1} \right]^{(k)} \]

\[ - a_2 \frac{a_1 a_1}{a_1} \frac{\gamma_1 \gamma_2'}{\delta_1 \delta_2} \left[ \sin \left( \frac{s(x - \xi_2)}{a_3} \right) \cos \frac{s(\xi_2 - \xi_1)}{a_2} \sin \frac{s(\xi_1 - a)}{a_1} \right]^{(k)} \]

\[ + O \left( |s|^{-k} \exp \left[ \frac{|t|}{a_1 a_2 a_3} \right] \right). \] (36)

**Proof.** Since the proof of formulas for \( \phi_{1\alpha}(x) \) are identical to the Titchmarsh’s proof of similar results for \( \phi_{\lambda}(x) \) [14], we may formulate them without proving. But the similar formulas for \( \phi_{2\alpha}(x) \) and \( \phi_{3\alpha}(x) \) need individual consideration, since the last solutions are defined by initial conditions of special nonstandard forms. We shall only prove the formula (31) for \( k = 0 \). Let \( \alpha_2 \neq 0 \). Then according to (30),

\[ \phi_{1\alpha}(\xi_1 - 0) = \frac{a_2 \cos \frac{s(\xi_1 - a)}{a_1}}{a_1} + O \left( |s|^{-1} \exp \left[ \frac{|t|}{a_1 a_2 a_3} \right] \right) \]

and

\[ \phi_{1\alpha}'(\xi_1 - 0) = - \frac{a_2}{a_1} \sin \frac{s(\xi_1 - a)}{a_1} + O \left( \exp \left[ \frac{|t|}{a_1 a_2 a_3} \right] \right). \]
Substituting this asymptotic expressions into (28) for \( k = 0 \) we get

\[
\phi_{2\lambda}(x) = a_2 a_\lambda \left[ \frac{1}{a_2} \gamma_1 \cos \frac{s(x - \xi_1)}{a_2} \cos \frac{s(\xi_1 - a)}{a_1} - \frac{1}{a_1} \gamma_1 \sin \frac{s(x - \xi_1)}{a_2} \sin \frac{s(\xi_1 - a)}{a_1} \right] \\
+ \frac{1}{a_2^2} \int_{\xi_1}^{x} \sin \frac{s(x - y)}{a_2} q(y) \phi_{2\lambda}(y) dy \\
+ O \left( |s|^{-1} \exp \left( \frac{t}{a_1} [a_1 (x - \xi_1) + a_2 (\xi_1 - a)] \right) \right). \tag{37}
\]

Multiplying through by \( \exp \left( \frac{t}{a_1} [a_1 (x - \xi_1) + a_2 (\xi_1 - a)] \right) \) and denoting

\[
F_{2\lambda}(x) := \exp \left( \frac{t}{a_1} [a_1 (x - \xi_1) + a_2 (\xi_1 - a)] \right) \phi_{2\lambda}(x),
\]

we have

\[
F_{2\lambda}(x) := a_2 a_\lambda \exp \left( \frac{t}{a_1} [a_1 (x - \xi_1) + a_2 (\xi_1 - a)] \right) \left[ \frac{1}{a_2} \gamma_1 \cos \frac{s(x - \xi_1)}{a_2} \cos \frac{s(\xi_1 - a)}{a_1} \\
- \frac{1}{a_1} \gamma_1 \sin \frac{s(x - \xi_1)}{a_2} \sin \frac{s(\xi_1 - a)}{a_1} \right] \\
+ \frac{1}{a_2^2} \int_{\xi_1}^{x} \sin \frac{s(x - y)}{a_2} q(y) \exp \left( \frac{t}{a_1} [a_1 (x - \xi_1) + a_2 (\xi_1 - a)] \right) F_{2\lambda}(y) dy + O \left( |s|^{-1} \right).
\]

Denoting \( M(\lambda) := \max_{x \in [\xi_1, \xi_2]} |F_{2\lambda}(x)| \) from the last formula it follows that

\[
M(\lambda) \leq \left| \frac{a_2 \gamma_1}{\delta_1} \right| + \frac{a_2}{a_1} \gamma_1 \beta_2 |a_2| \int_{\xi_1}^{\xi_2} q(y) dy + \frac{M_0}{|s|}
\]

for some \( M_0 > 0 \). From this it follows that \( M(\lambda) = O(1) \) as \( \lambda \to \infty \), so

\[
\phi_{2\lambda}(x) = O \left( \exp \left( \frac{t}{a_1} [a_1 (x - \xi_1) + a_2 (\xi_1 - a)] \right) \right).
\]

Substituting back into the integral on the right of (36) yields (31) for \( k = 0 \). The other assertions can be proved similarly.

**Theorem 3.1.** Let \( \lambda = s^2 \), \( t = Ims \). Then the characteristic function \( \omega(\lambda) \) has the following asymptotic representations:

Case 1: If \( \beta_2 \neq 0, \alpha_2 \neq 0 \), then

\[
\omega(\lambda) = a_2 b_2 s^3 \gamma_2 \left[ \frac{1}{a_2} \gamma_1 \cos \frac{s(b - \xi_2)}{a_2} \cos \frac{s(\xi_2 - \xi_1)}{a_1} - \frac{1}{a_1} \gamma_1 \sin \frac{s(b - \xi_2)}{a_2} \sin \frac{s(\xi_2 - \xi_1)}{a_1} \right] \\
- a_2 b_2 s^3 \frac{a_2}{a_1} \gamma_2 \left[ \frac{1}{a_2} \gamma_1 \sin \frac{s(b - \xi_2)}{a_2} \sin \frac{s(\xi_2 - \xi_1)}{a_1} \right] \\
+ a_2 b_2 s^3 \left[ \frac{1}{a_2} \gamma_1 \cos \frac{s(b - \xi_2)}{a_2} \cos \frac{s(\xi_2 - \xi_1)}{a_1} \right] \\
+ a_2 b_2 s^3 \left[ \frac{1}{a_2} \gamma_1 \sin \frac{s(b - \xi_2)}{a_2} \sin \frac{s(\xi_2 - \xi_1)}{a_1} \right] \\
+ O \left( |s|^2 \exp \left( \frac{t}{a_1} [a_1 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)] \right) \right). \tag{38}
\]
Case 2: If $\beta_2 \neq 0, \alpha_2 = 0$, then

$$
\omega_3(\lambda) = -\alpha_1\beta'_2s^2a_1\alpha_3\sin(\beta_2 - \xi_2)\cos(\xi_2 - \xi_1)\sin(\beta_1 - \xi_1) + \alpha_2s^2a_2\beta_2\sin(\beta_2 - \xi_2)\cos(\xi_2 - \xi_1)\sin(\beta_1 - \xi_1)
- \alpha_1\beta'_2s^2a_1\alpha_3\sin(\beta_2 - \xi_2)\cos(\xi_2 - \xi_1)\sin(\beta_1 - \xi_1)
+ \alpha_1\beta'_2s^2a_2\beta_2\sin(\beta_2 - \xi_2)\cos(\xi_2 - \xi_1)\sin(\beta_1 - \xi_1)
+ O\left(|s| \exp\left[|a_1a_2(\beta - \xi_2) + a_1a_3(\xi_2 - \xi_1) + a_2a_3(\beta - \xi_1)|\right]\right).
$$

(39)

Case 3: If $\beta_2 = 0, \alpha_2 \neq 0$, then

$$
\omega_3(\lambda) = \alpha_2\beta'_1s^2\delta_1\delta_2\cos(\beta_2 - \xi_2)\cos(\xi_2 - \xi_1)\sin(\beta_1 - \xi_1)
- \alpha_2\beta'_1s^2\delta_2\delta_3\cos(\beta_2 - \xi_2)\cos(\xi_2 - \xi_1)\sin(\beta_1 - \xi_1)
- \alpha_2\beta'_1s^2\delta_2\delta_3\cos(\beta_2 - \xi_2)\cos(\xi_2 - \xi_1)\sin(\beta_1 - \xi_1)
+ \alpha_2\beta'_1s^2\delta_1\delta_2\cos(\beta_2 - \xi_2)\cos(\xi_2 - \xi_1)\sin(\beta_1 - \xi_1)
+ O\left(|s| \exp\left[|a_1a_2(\beta - \xi_2) + a_1a_3(\xi_2 - \xi_1) + a_2a_3(\beta - \xi_1)|\right]\right).
$$

(40)

Case 4: If $\beta_2 = 0, \alpha_2 = 0$, then

$$
\omega_3(\lambda) = -\alpha_1\beta'_1s^2\alpha_3\gamma_2\cos(\beta_2 - \xi_2)\cos(\xi_2 - \xi_1)\sin(\beta_1 - \xi_1)
- \alpha_1\beta'_1s^2\alpha_3\gamma_2\cos(\beta_2 - \xi_2)\cos(\xi_2 - \xi_1)\sin(\beta_1 - \xi_1)
- \alpha_1\beta'_1s^2\alpha_3\gamma_2\cos(\beta_2 - \xi_2)\cos(\xi_2 - \xi_1)\sin(\beta_1 - \xi_1)
+ \alpha_1\beta'_1s^2\alpha_3\gamma_2\cos(\beta_2 - \xi_2)\cos(\xi_2 - \xi_1)\sin(\beta_1 - \xi_1)
+ O\left(|s| \exp\left[|a_1a_2(\beta - \xi_2) + a_1a_3(\xi_2 - \xi_1) + a_2a_3(\beta - \xi_1)|\right]\right).
$$

(41)

Where $k = 0, 1$.

**Proof.** The proof is immediate by substituting (32) and (35) for $k = 0$ in the representation

$$
\omega_3(\lambda) = \lambda \left[\beta_1\phi_3(\beta) - \beta'_2\phi_3(\beta)\right] + \left[\beta_1\phi_3(\beta) - \beta'_2\phi_3(\beta)\right]
- \lambda\beta_2\phi'_3(\beta) + \lambda\beta'_1\phi_3(\beta) + \beta_1\phi_3(\beta) - \beta'_2\phi_3(\beta).
$$

(42)

**Corollary 3.1.** The eigenvalues of the problem (1-7) are bounded below.

**Proof.** Putting $s = it$ ($t > 0$) in the above formulæ it follows that $\omega_3(-t^2) \to \infty$ as $t \to \infty$. Hence $\omega_3(\lambda) \neq 0$ for $\lambda$ negative and sufficiently large.
§4. Asymptotic formula for eigenvalues and eigenfunctions

Here we can obtain the asymptotic approximation formula for the eigenvalues of the problem (1-7). Since the eigenvalues coincide with the zeros of the entire function $\omega_1(\lambda)$, it follows that they have no finite limit. Moreover, we know from corollary 2.1 and 2.2 that all eigenvalues are real and bounded below. Therefore, we may renumber them as $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$, listed according to their multiplicity. In this section for the sake of simplicity, we shall assume that $a_i = a_{i+1} = a_i$ (i = 1, 2).

**Theorem 4.1.** The eigenvalues $\lambda_n = s_n^2$ (n = 0, 1, 2, ...) of the problem (1-7) have the following asymptotic representation for $n \rightarrow \infty$:

Case 1: If $\beta_2 \neq 0$, $\alpha_2 \neq 0$, then

$$s_n = \frac{a_1 a_2 a_3}{a_1 a_2 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)} \pi (n - 1) + O \left( \frac{1}{n} \right). \quad (43)$$

Case 2: If $\beta_2 \neq 0$, $\alpha_2 = 0$, then

$$s_n = \frac{a_1 a_2 a_3}{a_1 a_2 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)} \pi \left( n - \frac{1}{2} \right) + O \left( \frac{1}{n} \right). \quad (44)$$

Case 3: If $\beta_2 = 0$, $\alpha_2 \neq 0$, then

$$s_n = \frac{a_1 a_2 a_3}{a_1 a_2 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)} \pi \left( n - \frac{1}{2} \right) + O \left( \frac{1}{n} \right). \quad (45)$$

Case 4: If $\beta_2 = 0$, $\alpha_2 = 0$, then

$$s_n = \frac{a_1 a_2 a_3}{a_1 a_2 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)} \pi n + O \left( \frac{1}{n} \right). \quad (46)$$

**Proof.** We shall only consider the first case (the other cases may be considered analogically). Denoting $\omega_0(s) := \omega_1(s^2) = \omega_3(\lambda)$,

$$\omega_1(s) = \frac{a_2 \gamma_1 \gamma_2 s^3}{a_3 \delta_1 \delta_2} \sin \left( \frac{a_1 a_2 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)}{a_1 a_2 a_3} s \right)$$

and

$$\omega_2(s) := \omega_0(s) - \omega_1(\lambda).$$

We write $\omega(\lambda)$ as $\omega_0(s) = \omega_1(s) + \omega_2(\lambda)$. In view (37) from elementary considerations we have

$$\omega_2(s) = O \left( |s|^2 \exp \left( \frac{|a_1 a_2 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)|}{a_1 a_2 a_3} s \right) \right).$$

We shall apply the well-known Rouche theorem which asserts that if $f(s)$ and $g(s)$ analytic inside and on a closed contour $C$, and $|g(s)| < |f(s)|$ on $C$, then $f(s)$ and $f(s) + g(s)$ have the
same number zeros inside $C$, provided that each zero is counted according to their multiplicity. For sufficiently large $n$, it is readily shown that $|\omega_1(s)| > |\omega_2(s)|$ on the contours

$$C_n := \left\{ s \in \mathbb{C} \mid |s| = \frac{a_1a_2a_3}{a_1a_2(b - \xi_2) + a_1a_3(\xi_2 - \xi_1) + a_2a_3(\xi_1 - a)} \left( n + \frac{1}{2} \right) \pi \right\}$$

for sufficiently large $n$. Let $\lambda_0 = \lambda_1 \leq \ldots$ are zeros of $\omega(\lambda)$ and $\lambda_n = s_n^2$. Since inside the contour $C_n$, $\omega_1(s)$ has zeros at points $s = 0$ (with multiplicity 4) and

$$s = \frac{a_1a_2a_3}{a_1a_2(b - \xi_2) + a_1a_3(\xi_2 - \xi_1) + a_2a_3(\xi_1 - a)} \frac{k\pi}{2}, \quad k = \pm 1, \pm 2, \ldots, \pm n$$

(with multiplicity 1), and so the number of zeros is $2n + 4$, it follows that

$$s_n = \frac{a_1a_2a_3}{a_1a_2(b - \xi_2) + a_1a_3(\xi_2 - \xi_1) + a_2a_3(\xi_1 - a)} (n - 1)\pi + \delta_n, \quad (48)$$

where $\delta_n = O(1)$, more precisely

$$|\delta_n| < \frac{a_1a_2a_3}{a_1a_2(b - \xi_2) + a_1a_3(\xi_2 - \xi_1) + a_2a_3(\xi_1 - a)} \frac{\pi}{4}$$

for sufficiently large $n$. By substituting in (37) we derive that $\delta_n = O \left( \frac{1}{n} \right)$, which completes the proof. The next approximation for the eigenvalues may be obtained by following procedure. For this, we shall suppose that $q(y)$ is of bounded variation in $[a, b]$. We only consider the case $\beta_2 \neq 0, \alpha_2 \neq 0$ (since the other cases may be considered similarly). Putting $x = \xi_1$ in (27), $x = \xi_2$ in (28) and then substituting in (29), we derive that

$$\phi_3''(b) = -\frac{8\alpha_2\gamma_1\gamma_2}{a_2\delta_1\delta_2} \sin \left( \frac{a_1a_2(b - \xi_2) + a_1a_3(\xi_2 - \xi_1) + a_2a_3(\xi_1 - a)}{a_1a_2a_3} s \right)$$

$$- \frac{\alpha_1\gamma_2}{\delta_1\delta_2} \cos \left( \frac{a_1a_2(b - \xi_2) + a_1a_3(\xi_2 - \xi_1) + a_2a_3(\xi_1 - a)}{a_1a_2a_3} s \right) q(y)\phi_{1\lambda}(y)dy$$

$$+ \frac{\gamma_1\gamma_2}{a_1a_2a_3} \int_{\xi_1}^{\xi_2} \cos \left( \frac{a_1a_2(b - \xi_2) + a_1a_3(\xi_2 - \xi_1) + a_2a_3(\xi_1 - y)}{a_1a_2a_3} s \right) q(y)\phi_{2\lambda}(y)dy$$

$$+ \frac{\gamma_2}{a_2a_3} \int_{\xi_1}^{\xi_2} \cos \left( \frac{a_2(b - \xi_2) + a_3(\xi_2 - y)}{a_2a_3} s \right) q(y)\phi_{3\lambda}(y)dy$$

$$+ \frac{1}{a_3} \int_{\xi_1}^{b} \cos \left( \frac{(b - y)}{a_3} s \right) q(y)\phi_{3\lambda}(y)dy.$$
Substituting (30), (31) and (32) in the right side of the last integral equality then gives

\[
\phi_{3\lambda}'(b) = -\frac{s\alpha_2\gamma_2}{a_2\delta_2} \sin \left( \frac{a_1 a_3 (b + \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)}{a_1 a_2 a_3} \right)
- \frac{\alpha_1 \gamma_2}{\delta_1 \delta_2} \cos \left( \frac{a_1 a_3 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)}{a_1 a_2 a_3} \right) q(y) dy
+ \frac{\gamma_2}{\delta_2} \int_a^b \cos \left( \frac{a_1 a_3 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)}{a_1 a_2 a_3} \right) q(y) dy
+ O \left( \frac{1}{|s|^{1/2}} \right).
\]

On the other hand, from (32) it follows that

\[
\phi_{3\lambda}(b) = \frac{s^{1/2} \gamma_2}{\delta_1 \delta_2} \cos \left( \frac{a_1 a_3 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)}{a_1 a_2 a_3} \right) q(y) dy
+ O \left( \frac{1}{|s|^{1/2}} \right).
\]

Putting these formulas in (41) we have

\[
\omega_{3}\lambda = s^3 \beta_3 \frac{\gamma_2}{\delta_1 \delta_2} \sin \left( \frac{a_1 a_3 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)}{a_1 a_2 a_3} \right)
+ s^2 \left( \frac{\gamma_2}{\delta_1 \delta_2} \cos \left( \frac{a_1 a_3 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)}{a_1 a_2 a_3} \right) \right) q(y) dy
+ s^2 \left( \frac{\gamma_2}{\delta_2} \int_a^b \cos \left( \frac{a_1 a_3 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)}{a_1 a_2 a_3} \right) q(y) \phi_{3\lambda}(y) dy \right)
+ s^2 \left( \frac{\gamma_2}{\delta_2} \cos \left( \frac{a_1 a_3 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)}{a_1 a_2 a_3} \right) \right) q(y) \phi_{3\lambda}(y) dy
+ O \left( \frac{1}{|s|^{1/2}} \right).
\]

Putting (42) in the last equality we find that

\[
\sin \delta_n = -\frac{\cos \delta_n}{s_n} \left[ \frac{\beta_3}{\beta_2} + \frac{\alpha_1}{\alpha_2} \frac{Q}{2} + O \left( \frac{1}{n} \right) \right] + O \left( \frac{1}{|s_n|^{1/2}} \right),
\]

(49)
where

\[ Q = \frac{1}{a_1} \int_a^{\xi_1} q(y)dy + \frac{1}{a_2} \int_{\xi_1}^{\xi_2} q(y)dy + \frac{1}{a_3} \int_{\xi_2}^b q(y)dy. \]

Recalling that \( q(y) \) is of bounded variation in \([a, b]\) and applying the well-known Riemann-Lebesque Lemma [18] to the third integral on the right in (47), this term is \( O\left(\frac{1}{n^2}\right)\). Consequently, from (47) it follows that

\[ \delta_n = -\frac{a_1 a_2 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)}{a_1 a_2 a_3} \frac{1}{\pi (n - 1)} \left[ \frac{\beta'_1}{\beta_2} + \frac{\alpha_1}{\alpha_2} - \frac{Q}{2} \right] + O\left(\frac{1}{n^2}\right). \]

Substituting in (46), we have

\[ s_n = \frac{a_1 a_2 a_3}{a_1 a_2 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)} \pi (n - 1) \]

\[ - \frac{1}{\pi (n - \frac{1}{2})} \left[ \frac{\beta'_1}{\beta_2} + \frac{\alpha_1}{\alpha_2} - \frac{Q}{2} \right] + O\left(\frac{1}{n^2}\right). \]

Similar formulae in the other cases are as follows:

Case 2:

\[ s_n = \frac{a_1 a_2 a_3}{a_1 a_2 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)} \pi \left( n - \frac{1}{2} \right) \]

\[ + \frac{1}{\pi \left( n - \frac{1}{2} \right)} \left[ \frac{\alpha_1}{\alpha_2} - \frac{1}{a_3} \frac{\beta_2}{\beta'_1} - \frac{Q}{2} \right] + O\left(\frac{1}{n^2}\right). \]

Case 3:

\[ s_n = \frac{a_1 a_2 a_3}{a_1 a_2 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)} \pi \left( n - \frac{1}{2} \right) \]

\[ + \frac{1}{\pi \left( n - \frac{1}{2} \right)} \left[ \frac{1}{a_3} \frac{\beta_2}{\beta'_1} + \frac{Q}{2} \right] + O\left(\frac{1}{n^2}\right). \]

Case 4:

\[ s_n = \frac{a_1 a_2 a_3}{a_1 a_2 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)} \pi n \]

\[ + \frac{1}{\pi n} \left[ \frac{1}{a_3} \frac{\beta_2}{\beta'_1} + \frac{Q}{2} \right] + O\left(\frac{1}{n^2}\right). \]

Recalling that \( \phi(x, \lambda_n) \) is an eigenfunction according to eigenvalue \( \lambda_n \), by putting (42) in the (30), (31) and (32) for \( k = 0 \) we derive that

\[ \phi_{1\lambda_n} = a_2 \cos \left( \frac{a_2 a_3 (x - a)}{a_1 a_2 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)} + O\left(\frac{1}{n}\right) \right), \]

\[ \phi_{2\lambda_n} = a_2 \gamma_1 \cos \left( \frac{a_1 a_3 (x - \xi_1) + a_2 a_3 (\xi_1 - a)}{a_1 a_2 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)} + O\left(\frac{1}{n}\right) \right). \]
and

$$
\phi_{\lambda_n} = \alpha_2 \frac{\gamma_1 \gamma_2}{\delta_1 \delta_2} \cos \left( \frac{a_1 a_2 (x - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)}{a_1 a_2 (b - \xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)} \right) + O \left( \frac{1}{n} \right)
$$

in the first case. Hence if $\beta_2 \neq 0$ and $\alpha_2 \neq 0$ then the eigenfunction $\phi(x, \lambda_n)$ has the asymptotic representation

$$
\phi(x, \lambda_n) = \begin{cases} 
\alpha_2 \cos \left( \frac{a_2 a_3 (x-a)}{a_1 a_2 (x-\xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)} \right) & \text{if } x \in [a, \xi_1), \\
\alpha_2 \frac{\gamma_1}{\delta_1} \cos \left( \frac{a_2 a_3 (x-a)}{a_1 a_2 (x-\xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)} \right) + O \left( \frac{1}{n} \right) & \text{if } x \in (\xi_1, \xi_2), \\
\alpha_2 \frac{\gamma_2}{\delta_2} \cos \left( \frac{a_2 a_3 (x-a)}{a_1 a_2 (x-\xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)} \right) + O \left( \frac{1}{n} \right) & \text{if } x \in [\xi_2, b). 
\end{cases}
$$

Which hold uniformly for $x \in [a, \xi_1] \cup [\xi_1, \xi_2] \cup [\xi_2, b)$. Similar formulae in the other cases are as follows:

In case 2:

$$
\phi(x, \lambda_n) = \begin{cases} 
-\alpha_1 a_1 a_2 (b-\xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a) & \text{if } x \in [a, \xi_1), \\
-\alpha_1 \frac{\gamma_1}{\delta_1} a_1 a_2 (b-\xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a) & \text{if } x \in (\xi_1, \xi_2), \\
-\alpha_1 \frac{\gamma_2}{\delta_2} a_1 a_2 (b-\xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a) & \text{if } x \in [\xi_2, b). 
\end{cases}
$$

In case 3:

$$
\phi(x, \lambda_n) = \begin{cases} 
\alpha_2 \cos \left( \frac{a_2 a_3 (x-a)}{a_1 a_2 (x-\xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)} \right) & \text{if } x \in [a, \xi_1), \\
\alpha_2 \frac{\gamma_1}{\delta_1} \cos \left( \frac{a_2 a_3 (x-a)}{a_1 a_2 (x-\xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)} \right) + O \left( \frac{1}{n} \right) & \text{if } x \in (\xi_1, \xi_2), \\
\alpha_2 \frac{\gamma_2}{\delta_2} \cos \left( \frac{a_2 a_3 (x-a)}{a_1 a_2 (x-\xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a)} \right) + O \left( \frac{1}{n} \right) & \text{if } x \in [\xi_2, b). 
\end{cases}
$$

In case 4:

$$
\phi(x, \lambda_n) = \begin{cases} 
-\alpha_1 a_1 a_2 (b-\xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a) & \text{if } x \in [a, \xi_1), \\
-\alpha_1 \frac{\gamma_1}{\delta_1} a_1 a_2 (b-\xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a) & \text{if } x \in (\xi_1, \xi_2), \\
-\alpha_1 \frac{\gamma_2}{\delta_2} a_1 a_2 (b-\xi_2) + a_1 a_3 (\xi_2 - \xi_1) + a_2 a_3 (\xi_1 - a) & \text{if } x \in [\xi_2, b). 
\end{cases}
$$

All these asymptotic approximations hold uniformly for $x \in [a, \xi_1] \cup [\xi_1, \xi_2] \cup [\xi_2, b)$.

References


On the reduced residue system modulo $m$

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Abstract Let $m, k, b$ be integers, $k \neq 0$ and $m > 1$. It is well known that if $a_1, a_2, \ldots, a_m$ is a complete system of incongruent residues modulo $m$ and $(k, m) = 1$, then $ka_1 + b, ka_2 + b, \ldots, ka_m + b$ is also a complete system of incongruent residues modulo $m$ for any integer $b$. In this paper, we give a necessary and sufficient condition for both $a_1, a_2, \ldots, a_{\phi(m)}$ and $ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b$ to be reduced residue systems modulo $m$.

Keywords Complete system, reduced residue system, necessary and sufficient condition.

§1. Introduction

We all know the following theorem [1,3]: if $a_1, a_2, \ldots, a_m$ is a complete system of incongruent residues modulo $m$ and $(k, m) = 1$, then $ka_1 + b, ka_2 + b, \ldots, ka_m + b$ is also a complete system of incongruent residues modulo $m$. If we replace the complete system of incongruent residues modulo $m$ in this theorem by a reduced residue system modulo $m$, the result is not true. For example, 1, 5 modulo 6 is a reduced residue system modulo 6, but $1 \times 1 + 2, 1 \times 5 + 2$ modulo 6 is not a reduced residue system modulo 6. We can easily find that $(k, m) = 1$ and $m | b$ are sufficient for $ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b$ to be a reduced residue system modulo $m$, but are they necessary? We answer it by the following example. Both 1, 3, 7, 9 modulo 10 and $2 \times 1 + 5, 2 \times 3 + 5, 2 \times 7 + 5, 2 \times 9 + 5$ modulo 10 are reduced residue systems modulo 10, but neither $(k, m) = 1$ nor $m | b$. In this paper, we give a necessary and sufficient condition for both $a_1, a_2, \ldots, a_{\phi(m)}$ and $ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b$ to be reduced residue systems modulo $m$.

§2. Main results

Theorem 1. Let $m, k, b$ be integers, $k \neq 0$ and $m > 1$. Let $m = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ ($p_1 < p_2 < \ldots < p_s$) be the standard factorization of $m$. Suppose that $a_1, a_2, \ldots, a_{\phi(m)}$ is a reduced residue system modulo $m$, then $ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b$ is also a reduced residue system modulo $m$ if and only if one of the following two conditions hold:
   i) $(k, m) = 1, p_1 \ldots p_s | b$;
   ii) $(k, m) = 2, m = 4t + 2$ ($t = 0, 1, 2, \ldots$), $2 | b$ and $p_2 \ldots p_s | b$. 

Before the proof, we give the following lemma.

**Lemma 2.** Let $n, d$ be positive integers, $n > 1$ and $d | n$. Then every reduced residue system modulo $n$ can be divided into $\phi(n)/\phi(d)$ reduced residue systems modulo $d$.

**Proof.** It’s sufficient for us to prove that if $(r, d) = 1$, then the set

$$A = \{r + d\ell : \ell = 1, 2, \ldots, n/d\}$$

has $\phi(n)/\phi(d)$ numbers prime to $n$.

Let

$$n = n_1n_2,$$

where $n_1$ has the same prime factors with $d$ and $(n_2, d) = 1$. Since

$$(r + d\ell, d) = (r, d) = 1,$$

we have

$$(r + d\ell, n_1) = 1.$$ Hence $(r + d\ell, n) = 1$ is equivalent to $(r + d\ell, n_2) = 1$. There are $\phi(n_2)$ numbers prime to $n_2$ in the set $\{1, 2, \ldots, n_2\}$. Take a number from it arbitrarily and denote it by $s$. Since $(n_2, d) = 1$, the congruence

$$r + d\ell \equiv s \pmod{n_2}$$

has only one solution. Hence there are $(\phi(n)/\phi(d)) \cdot \phi(n_2)$, namely $\phi(n)/\phi(d)$ numbers prime to $n$ in the set $A$. This proves lemma 2.

Next we divide theorem 1 into the following two theorems and give the proof respectively.

**Theorem 3.** Let $m, b$ be integers and $m > 1$. Let $m = p_1^{a_1}p_2^{a_2} \cdots p_s^{a_s}$ ($p_1 < p_2 < \ldots < p_s$) be the standard factorization of $m$. If $a_1, a_2, \ldots, a_{\phi(m)}$ is a reduced residue system modulo $m$, then $a_1 + b, a_2 + b, \ldots, a_{\phi(m)} + b$ is also a reduced residue system modulo $m$ if and only if $p_1 \cdots p_s \nmid b$.

**Proof.** Sufficiency. It is obvious that $a_1 + b, a_2 + b, \ldots, a_{\phi(m)} + b$ is a set of incongruent residues modulo $m$, so it suffices to prove that $(a_i + b, m) = 1$ for $i = 1, 2, \ldots, \phi(m)$. Suppose that there exists an integer $j$ such that $1 \leq j \leq \phi(m)$ and $(a_j + b, m) = d > 1$, we choose a prime factor $p$ of $d$ and then $p|m$. Since $p_1 \cdots p_s | b$, we have $p|b$. But $p|a_j + b$, we obtain $p|a_j$ and then $p|(a_j, m)$ which contradicts with $(a_j, m) = 1$. Therefore $a_1 + b, a_2 + b, \ldots, a_{\phi(m)} + b$ is also a reduced residue system modulo $m$.

Necessity. If $p_1 \cdots p_s \nmid b$, then there exists an integer $i$ such that $1 \leq i \leq s$ and $p_i \nmid b$. Let

$$b \equiv j \pmod{p_i} \quad \text{and} \quad 1 \leq j \leq p_i - 1,$$

then

$$(p_i - j, p_i) = 1.$$ By lemma 2, there exists an integer $\ell$ such that $1 \leq \ell \leq \phi(m)$ and

$$a_\ell \equiv p_i - j \pmod{p_i}.$$
From (1) and (2), we have
\[ a_i + b \equiv p_i - j + j \equiv 0 \pmod{p_i}, \]
i.e.,
\[ (a_i + b, m) > 1. \]
Therefore
\[ a_1 + b, a_2 + b, \ldots, a_{\varphi(m)} + b \]
is not a reduced residue system modulo \( m \) which is a contradiction. Hence \( p_1 \ldots p_s | b \). This completes the proof.

**Corollary 4.** Let \( m, k, b \) be integers, \( k \neq 0 \) and \( m > 1 \). Let \( m = p_1^{a_1}p_2^{a_2} \cdots p_s^{a_s} \) (\( p_1 < p_2 < \ldots < p_s \)) be the standard factorization of \( m \). Suppose that \( a_1, a_2, \ldots, a_{\varphi(m)} \) is a reduced residue system modulo \( m \) and \((k, m) = 1\), then \( ka_1 + b, ka_2 + b, \ldots, ka_{\varphi(m)} + b \) is also a reduced residue system modulo \( m \) if and only if all the following conditions hold: \((k, m) = 2, m = 4t + 2 \) (\( t = 0, 1, 2, \ldots \)), \( 2 \nmid b \) and \( p_2 \cdots p_s | b \).

**Proof.** Since \((k, m) = 1, ka_1, ka_2, \ldots, ka_{\varphi(m)} \) is a reduced residue system modulo \( m \). By theorem 3, we can get this corollary immediately.

**Theorem 5.** Let \( m, k, b \) be integers, \( k \neq 0 \) and \( m > 1 \). Let \( m = p_1^{a_1}p_2^{a_2} \cdots p_s^{a_s} \) (\( p_1 < p_2 < \ldots < p_s \)) be the standard factorization of \( m \). Suppose that \( a_1, a_2, \ldots, a_{\varphi(m)} \) is a reduced residue system modulo \( m \) and \((k, m) > 1\), then \( ka_1 + b, ka_2 + b, \ldots, ka_{\varphi(m)} + b \) is also a reduced residue system modulo \( m \) if and only if all the following conditions hold: \((k, m) = 2, m = 4t + 2 \) (\( t = 0, 1, 2, \ldots \)), \( 2 \nmid b \) and \( p_2 \cdots p_s | b \).

**Proof.** Necessity. Since \( ka_1 + b, ka_2 + b, \ldots, ka_{\varphi(m)} + b \) is a reduced residue system modulo \( m \), we have \( ka_i + b \not\equiv ka_j + b \pmod{m} \) for \( 1 \leq i < j \leq \varphi(m) \). It follows that \( m \nmid k(a_i - a_j) \), i.e.,
\[ \frac{m}{(m, k)} \nmid a_i - a_j, 1 \leq i < j \leq \varphi(m). \]
If \( \varphi(m) > \varphi\left(\frac{m}{(m, k)}\right) \), by lemma 2, then there exist integers \( i_0, j_0 \) such that \( 1 \leq i_0 < j_0 \leq \varphi(m) \) and
\[ a_{i_0} \equiv a_{j_0} \pmod{\frac{m}{(m, k)}}. \]
It follows that
\[ \frac{m}{(m, k)} \nmid a_{j_0} - a_{i_0}, \]
which is a contradiction. Hence we have
\[ \varphi(m) = \varphi\left(\frac{m}{(m, k)}\right). \]
It follows that
\[ m = 4t + 2 \ (t = 0, 1, 2, \ldots) \quad \text{and} \quad (m, k) = 2. \] (3)

Since for any \( i \) with \( 1 \leq i \leq \varphi(m) \), we have that
\[ (ka_i + b, m) = 1. \]
From (3), \( 2 | k \) and \( 2 | m \), it follows that \( 2 \nmid b \). By lemma 2, we have that both \( a_1, a_2, \ldots, a_{\varphi(m)} \) and \( ka_1 + b, ka_2 + b, \ldots, ka_{\varphi(m)} + b \) are reduced residue systems modulo \( \frac{m}{2} \). Since \((k, \frac{m}{2}) = 1\), by corollary 4, we obtain \( p_2 \cdots p_s \nmid b \).
Sufficiency. If \( m = 4t+2 \) (\( t = 0, 1, 2, \ldots \)), then \( \phi(m) = \phi\left(\frac{m}{2}\right) \). By lemma 2, \( a_1, a_2, \ldots, a_{\phi(m)} \) is a reduced residue system modulo \( \frac{m}{2} \). Since
\[
(k, m) = 2 \quad \text{and} \quad p_2 \ldots p_s \mid b,
\]
by corollary 4,
\[
ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b,
\]
is also a reduced residue system modulo \( \frac{m}{2} \). Then \( ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b \) are all incongruent modulo \( \frac{m}{2} \), and so they are all incongruent modulo \( m \). Besides, we have
\[
(ka_i + b, \frac{m}{2} ) = 1
\]
for \( i = 1, 2, \ldots, \phi(m) \). Since \( 2 \nmid b \), we have \( 2 \nmid ka_i + b \), and then \( (ka_i + b, m) = 1 \) for \( i = 1, 2, \ldots, \phi(m) \). Hence \( ka_1 + b, ka_2 + b, \ldots, ka_{\phi(m)} + b \) is also a reduced residue system modulo \( m \).

By theorem 3 and theorem 5 above, we have proved theorem 1.

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References

Several identities involving the classical Catalan numbers

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Abstract This paper mainly study some summation problems involving the classical Catalan numbers. By using the elementary method and the properties of the Catalan numbers, we give some interesting identities for it.

Keywords Catalan numbers, elementary method, identities.

§1. Introduction

For any positive integer $n$, the classical Catalan numbers $b_n$ are defined as follows:

$$b_n = \binom{2n}{n}/(n+1), \quad n = 0, 1, 2, 3, \cdots$$

the first several values of $b_n$ are: $b_0=1$, $b_1=1$, $b_2=2$, $b_3=5$, $b_4=14$, $b_5=42$, $b_6=132$, ···. This sequence has some wide applications in combinational mathematics and graph theory, so it had been studied by many people, some related results can be found in references [1] and [2]. In this paper, we shall study the calculating problem of the summation

$$\sum_{a_1+a_2+\cdots+a_k=n} b_{2a_1}b_{2a_2}\cdots b_{2a_k}$$

and

$$\sum_{a_1+a_2+\cdots+a_k=n} b_{2a_1+1}b_{2a_2+1}\cdots b_{2a_k+1},$$

where $\sum$ denotes the summation over all $k$-tuples with no-negative integer coordinates $(a_1,a_2,\cdots,a_k)$ such that $a_1+a_2+\cdots+a_k = n$.

We shall use the elementary method to give two exact calculating formulas for (1) and (2). That is, we shall prove the following conclusions:

Theorem 1. For any positive integers $n$ and $k$ with $2 \leq k \leq n$, if $k$ is an even number,
then we have
\[
\sum_{a_1+a_2+\ldots+a_k=n} b_{2a_1} b_{2a_2} \cdots b_{2a_k} = \frac{4^{2n}}{\prod_{i=0}^{l} \frac{(2i+1)}{(2i+1)(2n+k-2l)} \cdot (2n+k-l-1)!} \cdot c(1,1),
\]
where \( q = k/2 \), and
\[
c(1, i) = \begin{cases} 
1, & \text{if } i = 0; \\
\prod_{s=0}^{i-1} (1-2s), & \text{if } i \geq 1.
\end{cases}
\]

If \( k \) is an odd number, then
\[
\sum_{a_1+a_2+\ldots+a_k=n} b_{2a_1} b_{2a_2} \cdots b_{2a_k} = \frac{4^{2n}}{\prod_{i=0}^{l} \frac{(2i+1)}{(2i+1)(2n+k-2l)} \cdot (2n+k-l-1)!} \cdot c(1,1),
\]
where \( q = k - \frac{1}{2} \).
Corollary 2. For any integer \( n \geq 2 \), we have
\[
\sum_{a_1 + a_2 = n} b_{2a_1 + 1} b_{2a_2 + 1} = \frac{-2^{2n+1}c(1, 2n + 4)}{(2n + 4)!} + \frac{2^{3n+1}(-1)^n \cdot c(1, n + 2)}{(n + 2)!}.
\]

§2. Proof of the theorem

In this section, we shall use the elementary methods and the properties of the Catalan numbers to prove our theorems directly. First we prove theorem 1 and corollary 1. From the properties of the Catalan numbers we know that
\[
2(1 - \sqrt{1 - x}) = x \sum_{n=0}^{\infty} \frac{b_n x^n}{4^n}.
\]

Let
\[
f(x) = 2(1 - \sqrt{1 - x}) = \sum_{n=0}^{\infty} \frac{b_n x^{n+1}}{4^n},
\]
then
\[
f(-x) = 2(1 - \sqrt{1 + x}) = \sum_{n=0}^{\infty} \frac{b_n (-x)^{n+1}}{4^n},
\]
so
\[
f(x) - f(-x) = 2(\sqrt{1 + x} - \sqrt{1 - x}) = 2x \sum_{n=0}^{\infty} \frac{b_{2n} x^{2n}}{4^{2n}}.
\]

Then from the properties of the power series we have
\[
(\sqrt{1 + x} - \sqrt{1 - x})^k = x^k \sum_{n=0}^{\infty} \left( \sum_{a_1 + a_2 + \cdots + a_k = n} \frac{b_{2a_1} b_{2a_2} \cdots b_{2a_k}}{4^{2n}} \right) x^{2n}.
\]

On the other hand,
\[
(\sqrt{1 + x} - \sqrt{1 - x})^k = \sum_{m=0}^{k} \binom{k}{m} (\sqrt{1 + x})^m (-\sqrt{1 - x})^{k-m}.
\]

(a) If \( k \) is an even number, we discuss \( m \) in the following two cases:

(1) If \( m \) is an even number, let \( m = 2l, k = 2q \), then
\[
\sum_{m=0, m=2l}^{k} \binom{k}{m} (\sqrt{1 + x})^m (-\sqrt{1 - x})^{k-m} = \sum_{l=0}^{q} \binom{2q}{2l} (1 + x)^l (1 - x)^{q-l},
\]
where the maximum number of times of \( x \) is \( q \).

(2) If \( m \) is an odd number, let \( m = 2l + 1, k = 2q \), then
\[
\sum_{m=0, m=2l+1}^{k} \binom{k}{m} (\sqrt{1 + x})^m (-\sqrt{1 - x})^{k-m} = -\sqrt{1 - x}^2 \sum_{l=0}^{q-1} \binom{2q}{2l+1} (1 + x)^l (1 - x)^{q-l-1}.
\]
Note that the power series expansion of \((1 - x^2)^{\frac{1}{2}}\)
\[
(1 - x^2)^{\frac{1}{2}} = \sum_{n=0}^{\infty} (-1)^n \frac{c(1, n)}{2^n n!} x^{2n}.
\]
(8)

Applying (8) and comparing the coefficients of \(x^{2n+k}\) in (7) we may immediately deduce the identity (3).

(b) If \(k\) is an odd number, we discuss \(m\) in the following two cases:

(1) If \(m\) is an even number, let \(m = 2l\), \(k = 2q + 1\), then
\[
\sum_{m=0}^{k} \binom{k}{m} (\sqrt{1+x})^m (-\sqrt{1-x})^{k-m} = -\sqrt{1-x} \sum_{l=0}^{q} \binom{2q+1}{2l} (1+x)^l (1-x)^{q-l}.
\]

(2) If \(m\) is an odd number, let \(m = 2l + 1\), \(k = 2q + 1\), then
\[
\sum_{m=0}^{k} \binom{k}{m} (\sqrt{1+x})^m (-\sqrt{1-x})^{k-m} = \sqrt{1+x} \sum_{l=0}^{q} \binom{2q+1}{2l+1} (1+x)^l (1-x)^{q-l}.
\]

Note that the power series expansion of \((1 - x)^{\frac{1}{2}}\) and \((1 + x)^{\frac{1}{2}}\),
\[
(1 - x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot c(1, n)}{2^n n!} x^n,
\]
(9)
\[
(1 + x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \frac{c(1, n)}{2^n n!} x^n.
\]
(10)

Applying (9) and (10) and comparing the coefficients of \(x^{2n+k}\) in (7) we may immediately deduce the identity (4).

Taking \(k=2\) in (7), we have
\[
4^{2n} (2 - 2\sqrt{1 - x^2}) = \sum_{n=0}^{\infty} \left( \sum_{a_1+a_2=n} b_{2a_1} b_{2a_2} \right) x^{2n+2}.
\]
(11)

Applying (8) in (11) we can deduce (5).

Taking \(k=3\) in (7), we have
\[
4^{2n} [\sqrt{1-x}(2x-4) + \sqrt{1+x}(4-2x)] = \sum_{n=0}^{\infty} \left( \sum_{a_1+a_2+a_3=n} b_{2a_1} b_{2a_2} b_{2a_3} \right) x^{2n+3}.
\]
(12)

Applying (9) and (10) in (12) we can deduce (6).

This proves theorem 1 and corollary 1.

Now we prove theorem 2 and corollary 2. It is clear that
\[
f(x) + f(-x) = 2(2 - \sqrt{1 - x} - \sqrt{1 + x}) = 2x^2 \sum_{n=0}^{\infty} \frac{b_{2n+1} x^{2n}}{4^{2n+1}}.
\]
Then from the properties of the power series we have
\[
(2 - \sqrt{1 - x} - \sqrt{1 + x})^k = x^{2k} \sum_{n=0}^{\infty} \left( \sum_{a_1 + a_2 + \cdots + a_k = n} \frac{b_{2a_1+1}b_{2a_2+1} \cdots b_{2a_k+1}}{4^{2n+1}} \right) x^{2n}. \tag{13}
\]

On the other hand,
\[
(2 - \sqrt{1 - x} - \sqrt{1 + x})^k = 2^k \sum_{m=0}^{k} \sum_{l=0}^{m} \binom{k}{m} \binom{m}{l} (-1)^{m-k} 2^{m-l} (\sqrt{1 - x})^l (\sqrt{1 + x})^{m-l}.
\]

Note that the power series expansion of \((1 - x)^{\frac{l}{2}}\) and \((1 + x)^{\frac{m-l}{2}}\)
\[
(1 - x)^{\frac{l}{2}} = \sum_{n=0}^{\infty} (-1)^n \frac{c(l, n)}{2^n n!} x^n, \tag{14}
\]
\[
(1 + x)^{\frac{m-l}{2}} = \sum_{n=0}^{\infty} \frac{c(m-l, n)}{2^n n!} x^n, \tag{15}
\]
where
\[
c(l, n) = \begin{cases} 
1, & \text{if } n = 0; \\
\frac{l-2s}{n-1} \prod_{s=0}^{l-2s}, & \text{if } n \geq 1.
\end{cases}
\]

Applying (14) and (15) and comparing the coefficients of \(x^{2n+2k}\) in (13) we may immediately deduce theorem 2.

Taking \(k=2\) in (13), we have
\[
4^{2n+1}(-4\sqrt{1 - x} - 4\sqrt{1 + x} + 2\sqrt{1-x^2} + 6) = x^4 \sum_{n=0}^{\infty} \left( \sum_{a_1 + a_2 = n} b_{2a_1+1}b_{2a_2+1} \right) x^{2n}. \tag{16}
\]

Applying (8), (9) and (10) in (16), we can deduce corollary 2.
This completes the proof of theorem 2 and corollary 2.

References

A note on ambiguous probabilistic finite state automata

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Abstract In this paper we discuss the probability of strings generated by various forms of probabilistic finite state automata in terms of extended probability transition function. On the same line, the ambiguity of probabilistic deterministic finite state automata is also discussed.

Keywords Probabilistic finite state automata, extended probability transition function, ambiguous probabilistic finite state automata.

§1. Introduction

As in the classical automata theory, there are deterministic and non-deterministic finite state automata deals with probabilistic processes [5]. Over the years, there are papers analysing the parsing issues of strings generated by a probabilistic finite state automata [1,3,8]. Furthermore the general objects of probabilistic finite state automata and study on their definitions and properties have reached the communities [7]. Probabilistic finite state automata have been introduced to describe distributions over strings and are successfully used in several fields, including pattern recognition, computational biology, speech recognition etc. In this paper, we discuss the ambiguity of probabilistic finite state automata through parsing the strings. This basic idea may have larger influence over the general issues related to probabilities of strings generated by various types of probabilistic finite state automata. The presentation of some of the extensions of probabilistic finite state automata is also dealt through extended probability transition function.

§2. Probabilistic finite state automata

Probabilistic finite state automata (PA) are one of the most widespread backgrounds that knowledges to researchers in computer science [2,6]. An overview of definitions for Probabilistic finite state automata is first established for various generative processes. As in formal language theory there is a key difference between deterministic and nondeterministic finite state machines in probabilistic case. Probabilistic finite state automata admit determinism; the parsing challenges of probabilistic deterministic finite state automata [4] are discussed through extended
probability transition function. It should be noted that results dealing with parsing are also been studied [7]. The ambiguity of deterministic and nondeterministic probabilistic finite state automata is studied and some extensions of the same is recalled using extended probability transition function.

**Definition 1.** A PA is a 5 tuple $A = (Q, \Sigma, \phi, i, \tau)$, where $Q$ is a set of finite states, $\Sigma$ is the alphabet, $\phi, i, \tau$ are functions such that

- $\phi : Q \times \Sigma \times Q \rightarrow [0, 1]$ (Transition probability function).
- $i : Q \rightarrow R^+$ (Initial state probability function).
- $\tau : Q \rightarrow R^+$ (Final state probability function).

such that $\sum_{q \in Q} i(q) = 1; \forall q \in Q, \tau(q) + \sum_{a \in \Sigma, q' \in Q} \phi(q, a, q') = 1$.

It is assumed that $\phi(q, a, q') = 0$ for all $(q, a, q') \notin \phi$.

**Example 1.** As in the formal language theory, the PA can also be represented graphically. Figure 1. is a graphical representation of a PA, in which the functions $i, \tau$ and $\phi$ are defined such that

![Diagram](image)

\begin{align*}
i(q_0) &= 1, i(q) = 0, \forall q \in Q. \\
\tau(q_0) &= 1/16, \tau(q_1) = 0, \tau(q_2) = 2/3, \tau(q_3) = 4/5. \\
\phi(q_0, a, q_1) &= 1/8; \phi(q_0, c, q_0) = 1/4; \ldots
\end{align*}

**Definition 2.** Let $A$ be a PA and $w = x_1x_2 \ldots x_n \in \Sigma^*$ and $Q = \{q_0, q_1, \ldots, q_m\}$ with $i(q_i) = 1$. A sequence of the form $q_0x_1q_1x_2q_2 \ldots q_{n-1}x_nq_n$ such that $\phi(q_i, x_{i+1}, q_{i+1}) > 0$ for all $i = 0, 1, 2, \ldots, n - 1$ (not necessary all $q_i$’s are different in the sequence) is called a path of $w$ in $PA$, and it is denoted by the symbol $\theta$.

A string $w$ is said to be generated by the given $PA$ if there is a path $\theta = q_0x_1q_1x_2q_2 \ldots q_{n-1}x_nq_n$ such that $\tau(q_n) \neq 0$.

In the above example, for the string $w = aabd$ there is a path $q_0a q_1 a q_1 b q_2 d q_3$ with $\tau(q_3) = 4/5 \neq 0$. But for $w = c c a a a$, there is a path $\theta = q_0c q_0 c q_1 a q_1 a q_1$ with $\tau(q_1) = 0$. Therefore it is considered that the $PA$ generates the string $aabd$, but not $c c a a a$.

**Definition 3.** Let $w = xa \in \Sigma^*$ such that $|w| = n$ and $|x| = n - 1$. The function $\hat{\phi}$ defined from $Q \times \Sigma^* \times Q$ to $[0, 1]$ such that $\hat{\phi}(q_0, \epsilon, q_i) = 1$ for all $i$ and $\hat{\phi}(q_0, w, q_n) =$
\( \hat{\phi}(q_0, x, q_{n-1})\phi(q_{n-1}, a, q_n) \) is called extended probability transition function.

**Definition 4.** Let \( w = x_1x_2 \ldots x_n \in \Sigma^* \) and \( \theta = q_0x_1q_1x_2q_2 \ldots q_{n-1}x_nq_n \) be a path in a PA \( A \) such that \( i(q_0) = 1 \) and \( \tau(q_n) \neq 0 \) then we say that \( A \) generates \( w \) with the probability

\[
P_A(w) = i(q_0)\hat{\phi}(q_0, w, q_n)\tau(q_n).
\]

In the above example, the string \( w = aabd \) is generated by the PA through \( \theta \) with the probability

\[
P_A(\theta) = i(q_0)\phi(q_0, a, q_1)\phi(q_1, a, q_1)\phi(q_1, b, q_2)\phi(q_2, d, q_3)\tau(q_3)
= 1 \times \frac{1}{8} \times \frac{1}{2} \times \frac{1}{4} \times \frac{1}{3} \times \frac{1}{5} = \frac{1}{240}.
\]

**Theorem 1.** If there are two paths \( q_0x_1q_1x_2q_2 \ldots q_{n-1}x_nq_n \) and \( q_0x_1q_1'x_2q_2' \ldots q_{n-1}'x_{n}'q_n' \), then

\[
P_A(w) = i(q_0)(\hat{\phi}(q_0, x_1, x_2, \ldots, x_n, q_n)\tau(q_n) + \phi(q_0, x_1, q_1') (\hat{\phi}(q_1', x_1, x_2, \ldots, x_n, q_n')\tau(q_n'))).
\]

This result can also be extended to any finite number of paths.

**Definition 5.** A PA \( A = (Q, \Sigma, \phi, i, \tau) \) with \( \sum_{i=1}^{s} i(p_i) = 1 \), \( s > 1 \) is called a PA with multiple initial states, if for every such \( p_i \), \( i = 1 \ldots s \), \( \phi(p_i, x, p_j) = 0 \) and there exist \( q_i \in Q \) such that \( \phi(p_i, x, q_j) > 0 \), for some \( x \in \Sigma \).

**Theorem 2.** Let \( A \) be a PA and also let \( w \in \Sigma^* \) accepted by \( A \) through \( m \) paths \( \theta_{A_1}, \theta_{A_2}, \ldots, \theta_{A_m} \) then

\[
P_A(w) = \sum_{i=1}^{m} i(q_0)\hat{\phi}(q_0, x_1x_2 \ldots x_n, q_n)\tau(q_n).
\]

Provided \( w = x_1x_2 \ldots x_n \),
\( \theta_{A_1} = q_0x_1q_1x_2q_2 \ldots x_nq_n, \)
\( \theta_{A_2} = q_0x_1q_2x_2q_2 \ldots x_nq_n, \)
\( \ldots \)

Suppose if \( A \) has only one start state \( q_0 \) then \( P_A(w) = i(q_0)\sum_{i=1}^{m} \Theta_{A_i}\tau(q_n) \), where \( \Theta_{A_i} = \hat{\phi}(q_0, x_1x_2 \ldots x_n, q_n). \)

**Proof.** Let \( w = x_1x_2 \ldots x_n \), \( \theta_{A_1} = q_0x_1q_1x_2q_12 \ldots x_nq_1n, \) and \( \theta_{A_2} = q_0x_1q_2x_2q_22 \ldots x_nq_2n. \) Then

\[
P_A(w) = i(q_{10})\hat{\phi}(q_{10}, w, q_{11})\tau(q_{11}) + i(q_{20})\hat{\phi}(q_{20}, w, q_{22})\tau(q_{22}).
\]

In general, if we denote \( \hat{\phi}(q_{i0}, w, q_{ir}) = \Theta_{A_i}. \)

Then (1) becomes

\[
P_A(w) = i(q_{10})\Theta_{A_1}\tau(q_{11}) + i(q_{20})\Theta_{A_2}\tau(q_{22}) + \ldots + i(q_{mn})\Theta_{A_m}\tau(q_{mn}).
\]

Suppose if there are \( m \) paths for \( w \),

\[
P_A(w) = i(q_{10})\Theta_{A_1}\tau(q_{11}) + i(q_{20})\Theta_{A_2}\tau(q_{22}) + \ldots + i(q_{mn})\Theta_{A_m}\tau(q_{mn}).
\]
Suppose if there exist \( q, q' \in Q \) such that \( i(q) = 1 \), \( i(p) = 0 \) and \( \tau(q') \neq 0 \), \( \tau(p) = 0 \) for all states \( p \) other than \( q \), then \( q_1 = q_2 = \cdots = q_m = q' \) and also

\[
P_A(w) = \sum_{i=1}^{m} \Theta_A^i \tau(q').
\]

**Theorem 3.** Let \( w = x_1x_2 \ldots x_m \in \Sigma^* \), then

\[
P_A(w) = i(q_0)\hat{\phi}(q_0, x_1x_2 \ldots x_{r-1}, q_{r-1}) m\phi(q_{r-1}, x_r, q_r) \hat{\phi}(q_r, x_{r+1}x_{r+2} \ldots, x_n, q_n) \tau(q_n).
\]

In the above theorem if \( m \to \infty \), \( P_A(w) = 0 \).

§3. Ambiguous probabilistic finite state automata

Let \( w \) be a string accepted by \( A \) through \( m \) valid paths. Let \( \theta_A(w) \) denote the set of all such \( m \) valid paths. Then \( P_A(w) = \sum_{i=1}^{m} \Theta_A^i \tau(q') \).

If \( \sum_{w} P_A(w) = 1 \), then \( A \) defines a distribution \( D \) on \( \Sigma^* \). Before discussing conditions for which a \( PA \) defines a probability distribution, we discuss an example in \( \Sigma^3 \). Assume that a coin is tossed 3 times then \( \Sigma = \{H, T\} \) and \( \Sigma^3 = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\} \). The \( PFA \) which generates \( \Sigma^3 \) may be given as follows.

\[
\begin{align*}
q_0 & \xrightarrow{H(1/2)} q_1 & q_1 & \xrightarrow{H(1/2)} q_2 & q_2 & \xrightarrow{H(1/2)} q_3 \\
& \xrightarrow{T(1/2)} q_0 & \xrightarrow{T(1/2)} q_1 & \xrightarrow{T(1/2)} q_2
\end{align*}
\]

Figure 2.

\[
i(q_i) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i = 1, 2, 3. \end{cases} \quad \tau(q_i) = \begin{cases} 1 & \text{if } i = 3 \\ 0 & \text{if } i = 0, 1, 2. \end{cases}
\]

Then \( A \) defines the probability distribution.

<table>
<thead>
<tr>
<th>( w )</th>
<th>( HHH )</th>
<th>( HHT )</th>
<th>( HTH )</th>
<th>( THH )</th>
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</tr>
</thead>
<tbody>
<tr>
<td>( P_A(w) )</td>
<td>( 1/8 )</td>
<td>( 1/8 )</td>
<td>( 1/8 )</td>
<td>( 1/8 )</td>
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</tr>
</tbody>
</table>

\[
\sum_{w} P_A(w) = 1.
\]

**Definition 6.** A \( PA \) is said to be ambiguous if there is at least one string which can be generated by \( A \) through more than one path. i.e., \( |\theta_A(w)| > 1 \) for at least one \( w \) in an ambiguous \( PA \).

**Definition 7.** A \( PA \) is called probabilistic deterministic finite state automaton, if
i. there exist $q_0 \in Q$ such that $i(q_0) = 1$.

ii. $\forall q \in Q, \forall a \in \Sigma, |\{q' : (q, a, q')\}| \leq 1$.

**Theorem 4.** No Probabilistic Deterministic finite state automaton (PDFA) is ambiguous.

**Proof.** Let the PA $A = (Q, \Sigma, \phi, i, \tau)$ be a PDFA. Therefore $\forall q \in Q, \forall a \in \Sigma, |\{q' : (q, a, q')\}| \leq 1$. We have to show that $|\theta_A(w)| \leq 1$.

For a string $w$ generated by the PDFA, $\theta_A(w)$ denote the set of all valid paths for $w$ in $A$.

Suppose assume that $w = x_1 x_2 \ldots x_n$ and $q_0 x_1 q_1 x_2 q_2 \ldots q_{n-1} x_n q_n$ and $p_0 x_1 p_1 x_2 p_2 \ldots x_n p_n$ are two valid paths for $w$ in $A$ with $p_0 = q_0$ and $i(p_0) = i(q_0) = 1$, $i(p) = 0$ for all other $p$.

Consider now the triplets $(q_{i-1}, x, q_i)$ and $(p_{i-1}, x, p_i)$. Let $q_{i-1}$ be same as $p_{i-1}$. Then if $q_i$ and $p_i$ are different, then we have $|\{q' : (q, a, q')\}| = |\{q_i\}| = 2$. This contradicts the fact that $|\{q' : (q, a, q')\}| \leq 1$. Thus $q_i$ is same as $p_i$ whenever $q_{i-1}$ is same as $p_{i-1}$.

Hence the paths $q_0 x_1 q_1 x_2 q_2 \ldots q_{n-1} x_n q_n$ and $p_0 x_1 p_1 x_2 p_2 \ldots x_n p_n$ are the same valid paths. Therefore the PDFA is unambiguous.

**Theorem 5.** Let $M$ be a PA with multiple initial states generating a sub probability space $D$ over the set of finite strings $\Sigma^*$, then there is a PA $A$ with single initial state generating the same sub probability space $D$.

**Proof.** Let $p_0, p_1, p_2, \ldots, p_k$ are initial states of $M = (Q_M, \Sigma, \phi_M, i_M, \tau_M)$ with initial probabilities $n_1, n_2, \ldots, n_k$ respectively such that $\sum_{i=1}^k n_i = 1$. Let $q_1, q_2, \ldots, q_n$ be the other states of $M$, i.e. $Q_M = \{q_1, q_2, \ldots, q_n\} \cup \{p_0, p_1, \ldots, p_k\}$.

Now we define $A = (Q, \Sigma, \phi, i, \tau)$ as follows. $Q = Q_M \cup \{p\}$. We have, in general $P_A(\theta) = i(p_0) \phi(p_0, w', q_k) \tau(q_k)$. Where $w' = x'_1 x'_2 x'_3 \ldots x'_k$.

Suppose if $PA$ is ambiguous, then

$$P_A(\theta) = \sum_{j=1}^n i(s_j) \phi(s_j, w', s_k) \tau(s_k).$$

We define $i$ such that $i(p) = 1$ and $i(q) = 0$ for all other $q \in Q$ and $\phi$ includes $(p, \epsilon, p_i)$, $i = 1, 2, \ldots, k$ with $\phi_M(p, \epsilon, p_i) = n_i$, $i = 1, 2, \ldots, k$. $P_A(q, a, q') = P_M(q, a, q')$, $\forall q, q' \in Q_M$ and define $P_A(\theta) = i(p) \sum_{j=1}^k \phi(p, \epsilon, p_{j-1}) \phi(p_{j-1}, w', q_j) \tau(q_j) \tau_A = \tau_M$ and $\tau_A(p) = 0$.

Then $P_A$ generates exactly the strings generated by $P_M$ with $P_A(\theta) = P_M(\theta), \forall \theta \in \theta_A$ and all $\theta \in \theta_M$.

**Conclusion**

In this paper, we have discussed that the PDFA is not ambiguous. The result will play a significant role in the study of parsing the strings and searching for optimal path for a string in a given $PA$. 
References


More about functional Alexandroff topological spaces

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Abstract In the following text we study the product of functional Alexandroff spaces and obtain a theorem on functional Alexandroff topological groups which recognize all functional Alexandroff topologies on a group which made it a topological group, this theorem is parallel to a well-known theorem on Alexandroff topological groups.

Keywords Alexandroff topology, functional Alexandroff topology.

§1. Introduction

A topological space $X$ is called Alexandroff if for any $x \in X$ there exists a smallest open neighborhood of $x$, namely $V(x)$ (in other words the intersection of any nonempty collection of open sets is open). For more details on Alexandroff spaces we refer the interested reader to [2], we should mention here Alexandroff’s well-known paper “Diskrete Räume” [1] in this area. If $\alpha : X \to X$ is an arbitrary map then $\downarrow x = \bigcup \{\alpha^{-n}(x) : n \geq 0\}$ $(x \in X)$, is a topological basis on $X$, the topology generated by this basis is denoted by $\tau_{\alpha}$ and topological space $(X, \tau_{\alpha})$ is called a functional Alexandroff topological space (induced by $\alpha$), moreover in this case $V(x) = \downarrow x$ $(x \in X)$. Functional Alexandroff spaces occurs naturally in dynamical systems, for more details on functional Alexandroff spaces we refer the interested reader to [3].

The following remark show the main characters of functional Alexandroff space:

Remark 1.1.[3] (theorem 3.5, main theorem) Let $X$ be an Alexandroff space. $X$ is a functional Alexandroff space if and only if the following statements hold:

$(C_1) \forall x, y \in X$, $V(x) \subseteq V(y) \lor V(y) \subseteq V(x) \lor V(x) \cap V(y) = \emptyset$;

$(C_2)$ For $x \in X$ if there exists $y \in X$ with $V(x) \subset V(y)$, then for all $z \in X \setminus \{x\}$ we have $V(x) \neq V(z)$;

$(C_3)$ For all $x, y \in X$, $\{z \in X : V(x) \subseteq V(z) \subseteq V(y)\}$ is finite.

The following note will be useful:

Note 1.2. Let $X$ be an Alexandroff space. The following statements are equivalent:

- $X$ is a functional Alexandroff space induced by a pointwise periodic function.
• $X$ is a functional Alexandroff space and

\[ \forall x, y \in X, \ (V(x) = V(y) \lor V(x) \cap V(y) = \emptyset). \]

• $\forall x, y \in X, \ ((V(x) = V(y) \lor V(x) \cap V(y) = \emptyset) \land |V(x)| < \aleph_0).$

• $X$ is a disjoint union of finite trivial spaces.

In the above case we name $X$ a **pointwise periodic functional Alexandroff space**.

**Remark 1.3.** Let $\alpha : X \to X$ be an arbitrary map, then $(X, \tau_{\alpha})$ is:

i. $T_0$ if and only if $\alpha$ does not have any periodic point except fixed points;

ii. $T_1$ if and only if $\alpha = \text{id}_X$.

In the following you will see:

• Products of functional Alexandroff topological spaces.

• A theorem on functional Alexandroff topological groups.

• A table on Alexandroff topological groups.

§2. Products of functional Alexandroff topological spaces

It is well known that finite product of Alexandroff spaces are Alexandroff ([2], theorem 2.2), but this is not true for arbitrary products, although with box topology product of Alexandroff spaces are Alexandroff. In this section we will deal with product of functional Alexandroff spaces. The question of this section may be indicated as: If the product of topological spaces is Alexandroff, when will it be functional Alexandroff?

**Note 2.1.** Let $\Gamma \neq \emptyset$ and for each $t \in \Gamma$, $X_t \neq \emptyset$. The following statements are equivalent:

1. $\prod_{t \in \Gamma} X_t$ is Alexandroff (with product topology).

2. Product and box topologies on $\prod_{t \in \Gamma} X_t$ are the same and for each $t \in \Gamma$, $X_t$ is an Alexandroff space.

3. There exist $t_1, \ldots, t_n \in \Gamma$ such that for each $t \in \Gamma \setminus \{t_1, \ldots, t_n\}$, $X_t$ has trivial topology, moreover for each $t \in \{t_1, \ldots, t_n\}$, $X_t$ is Alexandroff.

Moreover in the above case for all $(x_t)_{t \in \Gamma} \in \prod_{t \in \Gamma} X_t$, we have $V((x_t)_{t \in \Gamma}) = \prod_{t \in \Gamma} V(x_t)$.

The following theorem is the answer of the main question of this section.

**Theorem 2.2.** Let $\Gamma \neq \emptyset$ and for each $t \in \Gamma$, $|X_t| \geq 2$. The Alexandroff space $\prod_{t \in \Gamma} X_t$ is a functional Alexandroff space if and only if the following statements hold:

1. For all $t \in \Gamma$, $X_t$ is a functional Alexandroff space.

2. One of the following conditions hold:

   (S_1) There exists $s \in \Gamma$, such that for all $t \in \Gamma \setminus \{s\}$, $X_t$ is discrete.

   (S_2) For all $t \in \Gamma$, $X_t$ is a poitwise periodic functional Alexandroff space.

3. $\Gamma$ is finite.

**Proof.** First let $\prod_{t \in \Gamma} X_t$ be a functional Alexandroff space. For all $t \in \Gamma$, $X_t$ satisfies all three conditions $(C_1)$, $(C_2)$ and $(C_3)$ in remark 1.1, so it is functional Alexandroff. Now we distinguish the following cases:
• First case: There exist \( s \in \Gamma \) and \( x_s, y_s \in X_s \) such that \( V(x_s) \subset V(y_s) \). We claim that for all \( t \in \Gamma \setminus \{s\} \), \( X_t \) is discrete. Let \( r \in \Gamma \setminus \{s\} \) and \( x_r \in X_r \) be such that \( V(x_r) \neq \{x_r\} \). There exists \( y_t \in V(x_r) \setminus \{x_r\} \) and therefore \( V(y_t) \subset V(x_r) \) or \( V(y_t) = V(x_r) \). If \( V(y_t) \subset V(x_r) \) for each \( t \in \Gamma \setminus \{s, r\} \) choose \( x_t = y_t \in X_t \); Thus \( V((x_t)_{t \in \Gamma}) \cap V((y_t)_{t \in \Gamma}) \neq \emptyset \), \( V((x_t)_{t \in \Gamma}) \not\subset V((y_t)_{t \in \Gamma}) \) and \( V((y_t)_{t \in \Gamma}) \not\subset V((x_t)_{t \in \Gamma}) \) which is a contradiction (use \((C_1)\) in Remark 1.1). Therefore \( V(x_r) = V(y_t) \), for all \( t \in \Gamma \setminus \{r, s\} \) choose \( x_t = y_t \in X_t \), let

\[
Z_t = \begin{cases} x_t & t \neq r \\ y_t & t = r \end{cases}
\]

we have \( V((z_t)_{t \in \Gamma}) = V((x_t)_{t \in \Gamma}) \subset V((y_t)_{t \in \Gamma}) \) and \( (z_t)_{t \in \Gamma} \neq (x_t)_{t \in \Gamma} \) which is a contradiction according to \((C_2)\) in the main theorem. The above discussion leads us to the fact that for each \( x_r \in X_r \), \( V(x_r) = \{x_r\} \) and \( X_r \) is discrete. In this case \((S_1)\) is valid.

• Second case: For all \( s \in \Gamma \) and for all \( x_s, y_s \in X_s \) we have \( V(x_s) = V(y_s) \) or \( V(x_s) \cap V(y_s) = \emptyset \). By note 1.2, \( X_s \) is a pointwise periodic functional Alexandroff (for all \( t \in \Gamma \)). Let \( \Lambda = \{t \in \Gamma : X_t \) is not discrete\}. For each \( t \in \Lambda \) there exist \( x_t, y_t \in X_t \) such that \( x_t \neq y_t \) and \( V(x_t) = V(y_t) \), for all \( t \in \Gamma \setminus \Lambda \) choose \( x_t = y_t \in X_t \). Set \( B = \{(z_t)_{t \in \Gamma} \subseteq \Lambda \} \in \prod_{t \in \Gamma} X_t : V((z_t)_{t \in \Gamma}) = V((x_t)_{t \in \Gamma}) \}. \) Since \( B \supseteq \{(z_t)_{t \in \Gamma} : z_t \in \{x_t, y_t\} \}, \) thus \( |B| \geq 2^{|\Lambda}| \), by \((C_3)\) in the main theorem, \( B \) is finite, therefore \( \Lambda \) is finite. In this case \((S_2)\) is valid.

Using the above cases \( \{t \in \Gamma : X_t \) is not discrete\} is finite, since by note 2.1, box and product topologies on \( \prod_{t \in \Gamma} X_t \) are same thus \( \Gamma \) is finite.

Conversely let \( \prod_{t \in \Gamma} X_t \) be an Alexandroff space, which satisfies (1), (2) and (3). Consider \( \Gamma = \{t_1, \ldots, t_n\} \) with distinct \( t_s \)s. First suppose \((S_1)\) hold, \( X_{t_1} \) be a functional Alexandroff space induced by \( f_1 : X_{t_1} \to X_{t_1} \) and \( X_{t_2}, \ldots, X_{t_n} \) be discrete, for \( i \in \{2, \ldots, n\} \) set \( f_i := \text{id}_{X_{t_i}} \), for \( f = (f_1, f_2, \ldots, f_n) \) the functional Alexandroff topology \( \tau_f \) on \( X_{t_1} \times \cdots \times X_{t_n} \) is the same as the original topology on \( X_{t_1} \times \cdots \times X_{t_n} \), and \( X_{t_1} \times \cdots \times X_{t_n} \) is a functional Alexandroff space. Now suppose \((S_2)\) hold, by note 1.2, \( X_{t_1} \times \cdots \times X_{t_n} \) is a functional Alexandroff space.

The following theorem closes this section by answering an other question arisen from theorem 2.2: Let for each \( t \in \Gamma \), \( X_t \) be a functional Alexandroff space induced by map \( f_t : X_t \to X_t \). The following statements are equivalent:

**Theorem 2.3.** Let \( \Gamma \neq \emptyset \) and for each \( t \in \Gamma \), \( |X_t| \geq 2 \) and \( X_t \) be a functional Alexandroff space induced by map \( f_t : X_t \to X_t \). The following statements are equivalent:

1. \( \tau_{\prod_{t \in \Gamma} f_t} \) and product topology on \( \prod_{t \in \Gamma} X_t \) are the same.

2. \( \tau_{\prod_{t \in \Gamma} f_t} \) and box topology on \( \prod_{t \in \Gamma} X_t \) are the same.

3. \( \Gamma \) is finite and one of the following statements hold:
a. There exists \( s \in \Gamma \) such that for all \( t \in \Gamma - \{s\} \), \( f_t = \text{id}_{X_t} \).

b. For all \( t \in \Gamma \), \( X_t \) is a pointwise periodic functional Alexandroff space and for all distinct \( s, t \in \Gamma \) we have \( \gcd(|V(x_s)|, |V(x_t)|) = 1 \) (\( \forall x_s \in X_s, \forall x_t \in X_t \)).

**Proof.** By theorem 2.2, (1) and (2) are equivalent. Now suppose (1) and (2) hold. By theorem 2.2, \( \Gamma \) is finite. Moreover by theorem 2.2, \( (S_1) \) or \( (S_2) \), indicated in theorem 2.2, hold. If \((S_1)\) holds, then there exists \( s \in \Gamma \) such that for all \( t \in \Gamma - \{s\} \), \( X_t \) is discrete, i.e., for all \( t \in \Gamma - \{s\} \), \( f_t = \text{id}_{X_t} \), by remark 1.3 and (a) holds. If \((S_2)\) holds, then for all \( t \in \Gamma \), \( X_t \) is a pointwise periodic functional Alexandroff space, in addition if \( \Gamma = \{t_1, \ldots, t_n\} \) with distinct \( t_1, \ldots, x_n \in X_{t_1} \times \cdots \times X_{t_n} \) then period of \((x_1, \ldots, x_n)\) under \((f_{t_1}, \ldots, f_{t_n})\) equals to \(|V(x_1)\times \cdots \times V(x_n)| = |V(x_1)| \cdots |V(x_n)| = (\text{period of } x_1 \text{ under } f_{t_1}) \cdots \) (period of \( x_n \text{ under } f_{t_n} \)); Which leads to \( \gcd((\text{period of } x_i \text{ under } f_{t_i}), \text{ (period of } x_j \text{ under } f_{t_j})) = 1 \) for \( i \neq j \), i.e., \( \gcd(|V(x_i)|, |V(x_j)|) = 1 \) (for \( i \neq j \)) and (b) holds.

Conversely if (3) holds, then by theorem 2.2, \( \prod_{t \in \Gamma} X_t \) is a functional Alexandroff space.

Clearly (a) leads to (2), suppose (b) and let \( \Gamma = \{t_1, \ldots, t_n\} \) with distinct \( t_1, \ldots, x_n \in X_{t_1} \times \cdots \times X_{t_n} \). By \( \gcd(|V(x_1)|, |V(x_j)|) = 1 \) (1 \( \leq i < j \leq n \)) we have \( \{f_{t_1}, \ldots, f_{t_n}\} \)
\((x_1, \ldots, x_n)\) = \( |V(x_1)| \cdots |V(x_n)| < \kappa_0 \). Since we have \( \{f_{t_1}, \ldots, f_{t_n}\} (x_1, \ldots, x_n) \subseteq V(x_1) \times \cdots \times V(x_n) \), thus \( \{f_{t_1}, \ldots, f_{t_n}\} (x_1, \ldots, x_n) = V(x_1) \times \cdots \times V(x_n) \), which leads to (2), since \( V(x_1) \times \cdots \times V(x_n) \) is the smallest open neighborhood of \((x_1, \ldots, x_n)\) in \( X_{t_1} \times \cdots \times X_{t_n} \) in box topology and \( \{f_{t_1}, \ldots, f_{t_n}\} (x_1, \ldots, x_n) \) is the smallest open neighborhood of \((x_1, \ldots, x_n)\) in \( \tau(f_{t_1}, \ldots, f_{t_n}) \).

§3. A theorem on functional Alexandroff topological groups

It is well known that there exists a one to one correspondence between Alexandroff topologies on group \( G \) which made \( G \) a topological group and its normal subgroups ([4], theorem 4), moreover if for each normal subgroup \( N \) of \( G \), \( T_N \) denotes the Alexandroff topology on \( G \) obtained from \( N \), then \( T_N \) is the topology with minimal basis \( \{gN : g \in G\} \). We have the following parallel theorem:

**Theorem 3.1.** There exists a one to one correspondence between functional Alexandroff topologies on group \( G \) which made \( G \) a topological group and its finite normal subgroups. Moreover in this case \( G \) is a pointwise periodic functional Alexandroff space.

**Proof.** Let the topological group \( G \) be a functional Alexandroff space. Since in any topological group, the intersection of all open neighborhoods of \( e \) is a normal subgroup, thus \( V(e) \) is a normal subgroup of \( G \) such that \( V(g) = gV(e) \) (\( g \in G \)), moreover for each \( g \in V(e) \) we have \( V(g) = V(e) \). By remark 1.1, \( \{g \in G : V(g) = V(e)\} \) is finite, therefore \( V(e) \) is finite. Since \( V(e) \) is finite and for all \( g \in G \) we have \( |V(g)| = |V(e)| \), thus \( G \) is a pointwise periodic functional Alexandroff space, by note 1.2.

In fact \( \varphi : \{(G, \tau) : (G, \tau) \text{ is a functional Alexandroff space and a topological group} \} \rightarrow \{N : N \text{ is a finite normal subgroup of } G\} \) with \( \varphi(G, \tau) = V(e) \), is a bijection.

**Corollary 3.2.** Let \( G \) be a torsion free topological group. \( G \) is functional Alexandroff if
and only if $G$ is discrete (since in this case $G$ does not have any finite normal subgroup but \{e\}).

§4. A table on Alexandroff topological groups

Using theorem 3.1, we find out that any finite Alexandroff topological group is functional Alexandroff. The following table shows interaction between topological spaces with a group structure which are Alexandroff. Between subcategories of Alexandroff spaces with a group structure categories of finite topological spaces (with a group structure), functional Alexandroff topological spaces (with a group structure), and Alexandroff topological groups has been considered:

In the above table number (i) indicated in each area states that there exists a counterexample belongs to that area in the following counterexamples in item (i). In the following counterexamples $\mathbb{Z}$ and $\mathbb{Z}_3$ are considered under usual addition, with topological basis \{\(V(x) : x \in G\)}.

1. $G = \mathbb{Z}$, $V(0) = \{0\}$, $V(n) = \{m : m \geq |n| \vee m \leq -|n|\} \cup \{0\}$ ($n \in \mathbb{Z} - \{0\}$).
2. $G = \mathbb{Z}$, $V(n) = \{m \in \mathbb{Z} : m < n\} \cup \{n\}$ ($n \in \mathbb{Z}$).
3. $G = \mathbb{Z}$, $V(n) = n + 3\mathbb{Z}$ ($n \in \mathbb{Z}$).
4. $G = \mathbb{Z}_3$, $V(1) = V(2) = \{1, 2\}$, $V(0) = \mathbb{Z}_3$.
5. $G = \mathbb{Z}$, $V(n) = \{n\}$ ($n \in \mathbb{Z}$).
6. $G = \mathbb{Z}_3$, $V(0) = \{0\}$, $V(1) = \{0, 1\}$, $V(2) = \mathbb{Z}_3$.
7. $G = \mathbb{Z}_3$, $V(n) = \{n\}$ ($n \in \mathbb{Z}_3$).
References

Weak consistency of wavelet estimator for ρ-mixing errors

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Abstract This paper is concerned with the estimation of a semi-parametric regression model that is frequently used in statistical modeling. The wavelet technique is developed to estimate the unknown parameter and non-parameter. The weak consistencies of the estimators are obtained under the ρ-mixing condition.

Keywords Semi-parametric regression model, wavelet estimator, ρ-mixing, consistency.

§1. Introduction

Consider a semi-parametric regression model

\[ y_i = x_i \beta + g(t_i) + e_i, \quad i = 1, 2, \cdots, n \]

where \( x_i \in \mathbb{R}^1 \), \( t_i \in [0, 1] \), \( \{(x_i, t_i), 1 \leq i \leq n\} \) is deterministic design point sequence, \( \beta \) is an unknown regression parameter, \( g(\cdot) \) is an unknown Borel function; The unobserved process \( \{e_i, 1 \leq i \leq n\} \) is a stationary ρ-mixing sequence and satisfies \( Ee_i = 0, \quad i = 1, 2, \cdots, n \).

In recent years, the parametric or non-parametric estimators in the semi-parametric or non-parametric regression model have been widely studied in the literature when errors are a stationary mixing sequence (example [1]-[4]). But up to now, the discussion of the wavelet estimators in the semi-parametric regression model, whose errors are a stationary ρ-mixing sequence, has been scarcely seen.

Wavelets techniques, due to their ability to adapt to local features of curves, have recently received much attention from mathematicians, engineers and statisticians. Many authors have applied wavelet procedures to estimate nonparametric and semi-parametric models ([5]-[8]).

In this article, we establish weak consistencies of the wavelet estimators in the semi-parametric regression model with ρ-mixing errors, which enrich existing estimation theories and methods for semi-parametric regression models.

Suppose that \( \varphi(\cdot) \) is given scaling function in Schwartz space with order \( q \). A multiresolution analysis of \( L^2(\mathbb{R}) \) consists of an increasing sequence of closed subspace \( \{V_m\} \), where \( L^2(\mathbb{R}) \)
is the set of square integral functions over real line. The associated integral kernel of $V_m$ is given by

$$E_m(t, s) = 2^m E_0(2^m t, 2^m s) = 2^m \sum_{k \in \mathbb{Z}} \varphi(2^m t - k) \varphi(2^m s - k).$$

When $\beta$ is known, we define the estimator of $g(\cdot)$

$$\hat{g}_0(t, \beta) = \frac{1}{n} \sum_{i=1}^{n} (y_i - x_i \beta) \int_{A_i} E_m(t, s) ds,$$

where $A_i = [s_{i-1}, s_i]$ is a partition of interval $[0, 1]$ with $t_i \in A_i$, $1 \leq i \leq n$. Then, we minimize the following equation

$$\min_{\beta \in \mathbb{R}^n} \sum_{i=1}^{n} (y_i - x_i \beta - \hat{g}_0(t, \beta))^2.$$

Let its resolution be $\hat{\beta}_n$, we have

$$\hat{\beta}_n = S^{-2} n \sum_{i=1}^{n} \tilde{x}_i \tilde{y}_i,$$

where $\tilde{x}_i = x_i - \frac{1}{n} \sum_{j=1}^{n} x_j \int_{A_j} E_m(t_i, s) ds$, $\tilde{y}_i = y_i - \frac{1}{n} \sum_{j=1}^{n} y_j \int_{A_j} E_m(t_i, s) ds$, $\tilde{S}^2 n = \sum_{i=1}^{n} \tilde{x}_i^2$.

Finally, we can define the estimator of $g(\cdot)$

$$\hat{g}(t) = \int_{A_i} E_m(t, s) ds.$$

§2. Assumption and lemmas

In what follows, we will use $C$ and $C_i$, $i \geq 1$ to denote positive constants whose values are unimportant and may vary. We begin the following assumptions required to derive the large sample property of the estimators in section 1.

§2.1. Basic assumption

(A1) $g(\cdot)$ satisfies the Lipschitz condition of order $\gamma$ and $g(\cdot) \in H^v(v > 1/2)$;
(A2) $\varphi$ has compactly supported set and is a $q$-regular function;
(A3) $\varphi$ satisfies $|\hat{\varphi}(\xi) - 1| = O(\xi)$ as $\xi \rightarrow \infty$, where $\hat{\varphi}$ denotes the Fourier transformation of $\varphi$;
(A4) $\max_{1 \leq i \leq n} (s_i - s_{i-1}) = O(n^{-1})$, $\max_{1 \leq i \leq n} |\tilde{x}_i| = O(2^m)$;
(A5) $C_1 \leq \tilde{S}^2 n / n \leq C_2$, where $n$ is large enough;
(A6) $\left| \sum_{i=1}^{n} x_i \int_{A_i} E_m(t, s) ds \right| = \lambda$, $t \in [0, 1]$, where $\lambda$ is a constant that depends only on $t$.

The above conditions are mild and easily satisfied. In order to prove the main results, we present the following several lemmas.

§2.2. Lemmas

Lemma 1. Let $\varphi(\cdot) \in S_q$. Under basic assumptions of (A1)-(A3), if for each integer $k \geq 1$, $\exists c_k > 0$, such that
\( (i) \, |E_0(t, s)| \leq \frac{c_k}{1 + |t - s|}, \quad |E_m(t, s)| \leq \frac{2^m c_k}{1 + 2^m |t - s|}, \quad (ii) \sup_{t, m} \int_0^1 |E_m(t, s)| ds \leq C. \)

**Lemma 2.**\(^{[3]}\) Under basic assumption (A1)-(A4), we have

\[
(i) \left| \int_{A_i} E_m(t, s) ds \right| = O\left(\frac{2^m}{n}\right); \quad (ii) \sum_i \left( \int_{A_i} E_m(t, s) ds \right)^2 = O\left(\frac{2^m}{n}\right).
\]

**Lemma 3.**\(^{[3]}\) Under basic assumption (A1)-(A5), we have

\[
\sup_t \left| \sum_{i=1}^n g(t_i) \int_{A_i} E_m(t, s) ds - g(t) \right| = O(n^{-\gamma} + O(\tau_m), n \to \infty,
\]

where \(\tau_m = \begin{cases} 
(2^m)^{-\alpha + 1/2}, & (1/2 < \alpha < 3/2), \\
\sqrt{m}/2^m, & (\alpha = 3/2), \\
1/2^m, & (\alpha > 3/2).
\end{cases} \)

### §3. Main results and proofs

Now we establish the large sample property of the estimators described in section 1.

**Theorem 1.** Under basic assumption (A1)-(A6), and \(\sum_{i=1}^\infty \rho(i) < \infty, Ee_i = 0, Ee_i^2 = \sigma^2, \frac{2^m}{n} \to 0,\) we have

\[
\hat{\beta}_n - \beta = o_p(1),
\]

\[
\sup_t |\hat{g}(t) - g(t)| = o_p(1).
\]

**Proof.** It is obvious that

\[
\hat{\beta}_n - \beta = \hat{S}_n^{-2} \sum_{i=1}^n \bar{x}_i \hat{e}_i + \sum_{i=1}^n \bar{x}_i \hat{g}_i
\]

\[
= \hat{S}_n^{-2} \left[ \sum_{i=1}^n \bar{x}_i e_i - \sum_{i=1}^n \bar{x}_i \sum_{j=1}^n \int_{A_j} E_m(t_i, s) ds + \sum_{i=1}^n \bar{x}_i \hat{g}_i \right]
\]

\[
\Delta = B_{1n} + B_{2n} + B_{3n}.
\]

where \(\hat{e}_i = e_i - \sum_{j=1}^n e_j \int_{A_j} E_m(t_i, s) ds, \quad \hat{g}_i = g_i - \sum_{j=1}^n g_j \int_{A_j} E_m(t_i, s) ds.\)
Since
\[ EB_{1n}^2 = \sum_{i=1}^{n} (\tilde{S}_n^2 x_i e_i)^2 + 2 \sum_{1 \leq i < j \leq n} E[(\tilde{S}_n^2 x_i e_i) \cdot (\tilde{S}_n^2 x_j e_j)] \]
\[ \leq \sigma^2 \sum_{i=1}^{n} (\tilde{S}_n^2 x_i)^2 + 4 \sum_{i=1}^{n} \rho(j-i)E[(\tilde{S}_n^2 x_i e_i)^2] \cdot E[(\tilde{S}_n^2 x_j e_j)^2] \]
\[ \leq \sigma^2 \sum_{i=1}^{n} (\tilde{S}_n^2 x_i)^2 + 2\sigma^2 \sum_{i=1}^{n-1} \sum_{k=1}^{n-k} [ (\tilde{S}_n^2 x_i)^2 + (\tilde{S}_n^2 x_{i+k})^2] \]
\[ \leq \sigma^2 \sum_{i=1}^{n} (\tilde{S}_n^2 x_i)^2 + 2\sigma^2 \sum_{i=1}^{n-1} \sum_{k=1}^{n-k} (\tilde{S}_n^2 x_i)^2 \]
\[ = [1 + 4 \sum_{i=1}^{n} \rho(i)] \cdot \sigma^2 \sum_{i=1}^{n} (\tilde{S}_n^2 x_i)^2 \]
\[ \leq C \cdot \max(\tilde{S}_n^2 x_i) \cdot \sum_{i=1}^{n} (n^{-1} x_i) \cdot (n\tilde{S}_n^2) \leq C \cdot \frac{2m}{n} \rightarrow 0. \]

Hence, by Chebyshev inequality, we have for \( \forall \varepsilon > 0, P(|B_{1n}| \geq \varepsilon) \leq \frac{EB_{1n}^2}{\varepsilon^2} \rightarrow 0. \) Thus
\[ B_{1n} = o_p(1). \]  
(7)

Similarly to the proof of (7), and note that
\[ \sum_{i=1}^{n} \tilde{S}_n^2 x_i \int_{A_j} E_m(t_i, s) ds \leq \sum_{i=1}^{n} \tilde{S}_n^2 x_i \cdot \max_{i} \int_{A_j} E_m(t_i, s) ds \leq C \cdot \max_{i} \int_{A_j} E_m(t_i, s) ds. \]
We obtain
\[ B_{2n} = o_p(1). \]  
(8)

We terminate the proof of (4) with (6), (7), (8) and
\[ |B_{3n}| \leq \tilde{S}_n^2 \sum_{i=1}^{n} |x_i| \cdot \max_{1 \leq i \leq n} |\tilde{g}_i| = O(n^{-\gamma}) + O(\tau_m). \]

Now, consider (5). It is easy to see the following decompositions
\[ \sup_t |\tilde{g}(t) - g(t)| \leq \sup_t |\tilde{g}_0(t, \beta) - g(t)| + \sup_t \left| \sum_{i=1}^{n} x_i (\beta - \hat{\beta}_n) \int_{A_i} E_m(t, s) ds \right| \]
\[ \leq \sup_t \left| \sum_{i=1}^{n} g(t_i) \int_{A_i} E_m(t, s) ds - g(t) \right| + \sup_t \left| \sum_{i=1}^{n} c_i \int_{A_i} E_m(t, s) ds \right| \]
\[ + \left| \beta - \hat{\beta}_n \right| \cdot \sup_t \left| \sum_{i=1}^{n} x_i \int_{A_i} E_m(t, s) ds \right| \]
\[ \Delta = T_1 + T_2 + T_3. \]

By lemma 3, we have
\[ T_1 = O(n^{-\gamma}) + O(\tau_m). \]  
(10)

The proof for \( T_2 = o_p(1) \) is quite similar to that of (7) and hence it is omitted for the sake of brevity. And because of assumption (A6), we have \( \sum_{i=1}^{n} x_i \int_{A_i} E_m(t, s) ds = O_p(1) \). Hence, following from (4), we have
\[ T_3 = o_p(1). \]  
(11)

Thus, we have shown (5).
References


Involute-evolute curves in Galilean space $G_3$

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Abstract In this paper, definition of involute-evolute curve couple in Galilean space is given and some well-known theorems for the involute-evolute curves are obtained in 3-dimensional Galilean space.

Keywords Galilean space, involute-evolute curve.

§1. Introduction and preliminaries

C. Boyer discovered involutes while trying to build a more accurate clock $^{[1]}$. Later, H. H. Hacısalihoğlu $^{[2]}$ gave the relations Frenet apparatus of involute-evolute curve couple in the space $E^3$. A. Turgut $^{[3]}$ examined involute-evolute curve couple in $R^n$. At the beginning of the twentieth century, Cayley-Klein discussed Galilean geometry which is one of the nine geometries of projective space. After that, the studies with related to the curvature theory were maintained $^{[4,5,6]}$ and A. O. Öğrenmiş $^{[7]}$ and et al. studied the properties of the curves in the Galilean space were studied. In this paper, we define involute-evolute curves couple and give some theorems and conclusions, which are known from the classical differential geometry, in the three dimensional Galilean space $G_3$. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

The Galilean space $G_3$ is a Cayley-Klein space equipped with the projective metric of signature $(0,0,+,+)$, as in E. Molnar’s paper $^{[8]}$. The absolute figure of the Galilean Geometry consist of an ordered triple $\{w, f, I\}$, where $w$ is the ideal (absolute) plane, $f$ is the line (absolute line) in $w$ and $I$ is the fixed elliptic involution of points of $f$ $^{[8]}$. In the non-homogeneous coordinates the similarity group $H_8$ has the form

$$
\begin{align*}
\overline{x} &= a_{11} + a_{12}x, \\
\overline{y} &= a_{21} + a_{22}x + a_{23}y \cos \varphi + a_{24}z \sin \varphi, \\
\overline{z} &= a_{31} + a_{32}x - a_{33}y \sin \varphi + a_{34}z \cos \varphi.
\end{align*}
$$

(1)

Where $a_{ij}$ and $\varphi$ are real numbers $^{[5]}$.

In what follows the coefficients $a_{12}$ and $a_{23}$ will play the special role. In particular, for $a_{12} = a_{23} = 1$, (1) defines the group $B_6 \subset H_8$ of isometries of Galilean space $G_3$.

In $G_3$ there are four classes of lines:

i) (proper) non-isotropic lines- they don’t meet the absolute line $f$. 

\[\]
ii) (proper) isotropic lines—lines that don’t belong to the plane \( w \) but meet the absolute line \( f \).

iii) unproper non-isotropic lines—all lines of \( w \) but \( f \).

iv) the absolute line \( f \).

Planes \( x = \text{constant} \) are Euclidean and so is the plane \( w \). Other planes are isotropic.

If a curve \( C \) of the class \( C^r \) (\( r \geq 3 \)) is given by the parametrization 
\[
    r = r(x, y(x), z(x))
\]
then \( x \) is a Galilean invariant the arc length on \( C \).

The curvature is 
\[
    \kappa = \sqrt{y''(x)^2 + z''(x)^2}
\]
and torsion is 
\[
    \tau = \frac{1}{\kappa^2} \det(r'(x), r''(x), r'''(x)).
\]

The orthonormal trihedron is defined 
\[
    T(s) = \alpha'(s) = (1, y'(s), z'(s)) \\
    N(s) = \frac{1}{\kappa(s)}(0, y''(s), z''(s)) \\
    B(s) = \frac{1}{\kappa(s)}(0, -z''(s), y''(s)).
\]

The vectors \( T, N, B \) are called the vectors of tangent, principal normal and binormal line of \( \alpha \), respectively. For their derivatives the following Frenet formulas hold [9]
\[
    T'(s) = \kappa(s)N(s) \\
    N'(s) = \tau(s)B(s) \\
    B'(s) = -\tau(s)N(s).
\]

Galilean scalar product can be written as 
\[
    \langle u_1, u_2 \rangle = \begin{cases} 
    x_1x_2, & \text{if } x_1 \neq 0 \lor x_2 \neq 0, \\
    y_1y_2 + z_1z_2, & \text{if } x_1 = 0 \land x_2 = 0.
\end{cases}
\]

Where \( u_1 = (x_1, y_1, z_1) \) and \( u_2 = (x_2, y_2, z_2) \). It leaves invariant the Galilean norm of the vector \( u = (x, y, z) \) defined by 
\[
    \|u\| = \begin{cases} 
    x, & x \neq 0, \\
    \sqrt{y^2 + z^2}, & x = 0.
\end{cases}
\]

§2. Involute-evolute curves in Galilean space

In this section, we give a definition of involute-evolute curve and obtain some theorems about these curves in \( G_3 \).

Definition 2.1. Let \( \alpha \) and \( \alpha^* \) be two curves in the Galilean space \( G_3 \). The curve \( \alpha^* \) is called involute of the curve \( \alpha \) if the tangent vector of the curve \( \alpha \) at the point \( \alpha(s) \) passes through the tangent vector of the curve \( \alpha^* \) at the point \( \alpha^*(s) \) and 
\[
    \langle T, T^* \rangle = 0,
\]
where \(\{T,N,B\}\) and \(\{T^*,N^*,B^*\}\) are Frenet frames of \(\alpha\) and \(\alpha^*\), respectively. Also, the curve \(\alpha\) is called the evolute of the curve \(\alpha^*\).

This definition suffices to define this curve mate as (see Figure 1.)

\[
\alpha^* = \alpha + \lambda T.
\]

**Figure 1. Involute-evolute curves**

**Theorem 2.2.** Let \(\alpha\) and \(\alpha^*\) be two curves in the Galilean space \(G_3\). If the curve \(\alpha^*\) is an involute of the curve \(\alpha\), then the distance between the curves \(\alpha\) and \(\alpha^*\) is \(\lambda(s)\), where \(\lambda(s) = |c - s|\).

**Proof.** From definition of involute-evolute curve couple, we know

\[
\alpha^*(s) = \alpha(s) + \lambda(s)T(s).
\]  

(9)

Differentiating both sides of the equation (9) with respect to \(s\) and use the Frenet formulas, we obtain

\[
T^*(s) = T(s) + \frac{d\lambda}{ds}T(s) + \lambda(s)\kappa(s)N(s).
\]

Since the curve \(\alpha^*\) is involute of \(\alpha\), \(\langle T, T^* \rangle = 0\).

Then we have

\[
\frac{d\lambda}{ds} + 1 = 0.
\]

(10)

From the last equation, we easily get

\[
\lambda(s) = c - s.
\]

(11)

Where \(c\) is constant. Thus, the equation (9) can be written as

\[
\alpha^*(s) - \alpha(s) = (c - s)T(s).
\]

(12)

Taking the norm of the equation (12), we reach

\[
\|\alpha^*(s) - \alpha(s)\| = |c - s|.
\]

(13)

This completes the proof.

**Theorem 2.3.** Let \(\alpha\) and \(\alpha^*\) be two curves in Galilean space \(G_3\). \(\kappa, \tau\) and \(\kappa^*, \tau^*\) be the curvature functions of \(\alpha\) and \(\alpha^*\), respectively. If \(\alpha\) is evolute of \(\alpha^*\) then there is a relationship

\[
\kappa^* = \frac{\tau}{(c - s)\kappa},
\]
where $c$ is constant and $s$ is arc length parameter of $\alpha$.

**Proof.** Let Frenet frames be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$ at the points $\alpha(s)$ and $\alpha^*(s)$, respectively. Differentiating both sides of equation (12) with respect to $s$ and using Frenet formulas, we have following equation

$$T^*(s) \frac{ds^*}{ds} = (c - s)\kappa(s)N(s), \quad (14)$$

where $s$ and $s^*$ are the arc length parameter of the curves $\alpha$ and $\alpha^*$, respectively. Taking the norm of the equation (14), we reach

$$\frac{ds^*}{ds} = (c - s)\kappa(s) \quad (15)$$

and

$$T^* = N. \quad (16)$$

By taking the derivative of equation (16) and using the Frenet formulas and equation (15), we obtain

$$\kappa^* N^* = \frac{\tau}{(c - s)\kappa} B. \quad (17)$$

From the last equation, we get

$$\kappa^* = \frac{\tau}{(c - s)\kappa}. \quad (18)$$

**Theorem 2.4.** Let $\alpha$ be the non-planar evolute of curve $\alpha^*$, then $\alpha$ is a helix.

**Proof.** Under assumption $s$ and $s^*$ are arc length parameter of the curves $\alpha$ and $\alpha^*$, respectively. We take the derivative of the following equation with respect to $s$

$$\alpha^*(s) = \alpha(s) + \lambda(s)T(s).$$

We obtain that

$$T_s \frac{ds^*}{ds} = \lambda \kappa N.$$

{$\{T^*, N\}$} are linearly dependent. We may define function as

$$f = \langle T, T^* \wedge N^* \rangle$$

and take the derivative of the function $f$ with respect to $s$, we obtain

$$f' = -\tau \langle T, N^* \rangle. \quad (18)$$

From the equation (18) and the scalar product in Galilean space, we have

$$f' = 0.$$

That is,

$$f = const.$$

The velocity vector of the curve $\alpha$ always composes a constant angle with the normal of the plane which consist of $\alpha^*$. Then the non-planar evolute of the curve $\alpha^*$ is a helix.
**Theorem 2.5.** Let the curves $\beta$ and $\gamma$ be two evolutes of $\alpha$ in the Galilean space $G_3$. If the points $P_1$ and $P_2$ correspond to the point $P$ of $\alpha$, then the angle $P_1 P P_2$ is constant.

**Proof.** Let’s assume that the curves $\beta$ and $\gamma$ be two evolutes of $\alpha$ (see Figure 2.). And let the Frenet vectors of the curves $\alpha$, $\beta$ and $\gamma$ be $\{T, N, B\}$, $\{T^*, N^*, B^*\}$ and $\{\tilde{T}, \tilde{N}, \tilde{B}\}$, respectively.

![Figure 2. Evolutes of $\alpha$ curve](image)

Following the same way in the proof of the theorem 2.4, it is easily seen that $\{T, N^*\}$ and $\{T, \tilde{N}\}$ are linearly dependent.

Thus,
\[
\langle T, T^* \rangle = 0 \tag{19}
\]
and
\[
\langle T, \tilde{T} \rangle = 0. \tag{20}
\]

When $\theta$ is an angle between tangent vector $T^*$ and $\tilde{T}$, we define a function $f$. That is,
\[
f(s) = \langle T^*, \tilde{T} \rangle. \tag{21}
\]

Then, differentiating equation (21) with respect to $s$, we have
\[
f'(s) = \kappa \frac{d s^*}{d s} \langle N^*, \tilde{T} \rangle + \tilde{\kappa} \frac{d \tilde{s}}{d s} \langle T^*, \tilde{N} \rangle, \tag{22}
\]
where $s$, $s^*$ and $\tilde{s}$ are arc length parameter of the curves $\alpha$, $\beta$ and $\gamma$, respectively. Also, $\kappa$, $\kappa^*$ and $\tilde{\kappa}$ are the curvatures of the curves $\alpha$, $\beta$ and $\gamma$, respectively. Considering the equations (19), (20) and (22), we get
\[
f'(s) = 0.
\]
This means that
\[
f = \text{const}.
\]
So, $\theta$ is constant. The proof is completed.
References


S₂ near-rings

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Abstract A near-ring \( N \) is said to be \( S_1 \) if for every \( a \in N \), there exists \( x \in N^* = N - \{0\} \) such that \( axa = xa \). Closely following this, we introduce in this paper the concept of \( S_2 \) near-rings. A near-ring \( N \) is said to be an \( S_2 \) near-ring if, for every \( a \in N \), there exists \( x \in N^* \) such that \( axa = ax \). Further by generalizing this, we introduce strong \( S_2 \) near-rings i.e., near-rings in which \( aba = ab \) for all \( a, b \in N \). We discuss some of their properties, obtain a characterisation and also a structure theorem.

Keywords \( S_2 \) near-rings, strong \( S_2 \) near-rings, zero divisors.

§1. Introduction

Throughout this paper \( N \) stands for a right near-ring \((N, +, \cdot)\) and ‘0’ denotes the identity element of \((N, +)\). A subgroup \((M, +)\) of \((N, +)\) is called an \( N \)-subgroup if \( NM \subset M \). \( N \) is called weak commutative if \( abc = acb \) for all \( a, b, c \in N \) \([1]\). \( N \) is said to be regular if for every \( a \in N \), there exists \( b \in N \) such that \( a = aba \). \( N \) is called an \( S \) near-ring or an \( S' \) near-ring according as \( x \in Nx \) or \( x \in xN \) for all \( x \in N \). \( N \) is called a strong \( S_1 \) near-ring \([2]\) if \( N^* = N_{S_1}(a) \) for all \( a \in N \) where \( N^* = N - \{0\} \) and \( N_{S_1}(a) = \{x \in N^*/axa = xa\} \). For basic concepts and terms used but left undefined in this paper, we refer to Pilz \([1]\).

\( N_0 = \{a \in N/ax0 = 0\} \) is called the zero symmetric part of \( N \) and \( N \) is called zero symmetric if \( N = N_0 \).
\( N_c = \{a \in N/ad' = a \text{ for all } a' \in N\} \) is called the constant part of \( N \) and \( N \) is called constant if \( N = N_c \).
\( N_d = \{n \in N/n(a + a') = na + na' \text{ for all } a, a' \in N\} \) is the set of all distributive elements of \( N \) and \( N \) is called distributive if \( N = N_d \). For \( a \in N \), \( (0 : a) = \{n \in N/na = 0\} \) is the annihilator of \( a \). \( N \) is said to have IFP (i.e. Insertion of Factors Property) if for \( x, y \in N \), \( xy = 0 \) implies \( xny = 0 \) for all \( n \in N \). \( N \) has Property(P₂) if for all ideals \( I \) of \( N \), \( xy \in I \) implies \( yx \in I \) for all \( x, y \in N \) \([1]\).

We freely make use of the following results.

Theorem 1.1.\([1]\) Every near-ring is isomorphic to a subdirect product of subdirectly irreducible near-rings.
Theorem 1.2. \([4]\) If a near-ring \(N\) has IFP and if \(xy = 0\) implies \(yx = 0\) for \(x, y \in N\) then we say that \(N\) has \((\ast, \text{IFP})\).

Theorem 1.3. \([1]\) A near-ring \(N\) has strong IFP if for all ideals \(I\) of \(N\), \(ab \in I\) implies \(anb \in I\) for all \(a, b, n \in N\).

Theorem 1.4. \([1]\) A zero symmetric near-ring \(N\) has IFP if and only if \((0 : n)\) is an ideal for all \(n \in N\).

Theorem 1.5. \([2]\) A near-ring \(N\) is a strong \(S_1\) near-ring if and only if \(aba = ba\) for all \(a, b \in N\).

§2. \(S_2\) near-rings

In this section, we define \(S_2\) near-rings, give examples and prove certain elementary properties of such near-rings.

Definition 2.1. A near-ring \(N\) is said to be an \(S_2\) near-ring if for every \(a \in N\), there exists \(x \in N^*\) such that \(axa = ax\).

Examples 2.2.
(a) Every Boolean near-ring is an \(S_2\) near-ring.
(b) Consider the near-ring \((\mathbb{Z}_6, +, \cdot)\) where \((\mathbb{Z}_6, +)\) is the group of integers modulo ‘6’ and \(\cdot\) is defined as follows \([1]\). The multiplication table for \((N, \cdot)\) is given below:

<table>
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<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</tbody>
</table>

One can check that \((N, +, \cdot)\) is an \(S_2\) near-ring. It is worth noting that it is neither zero symmetric nor constant.

We furnish below a characterisation of \(S_2\) near-rings.

Theorem 2.3. Let \(N\) be a weak commutative near-ring without non-zero zero divisors. Then \(N\) is an \(S_2\) near-ring if and only if \(N\) is Boolean.

Proof. For the ‘only if’ part, let \(a \in N\). Since \(N\) is an \(S_2\) near-ring, there exists \(x \in N^*\) such that \(axa = ax\). Since \(N\) is weak commutative, \(aax = ax\). This implies \((a^2 - a)x = 0\). Since \(N\) has no non-zero zero divisors, we get \(a^2 - a = 0 \Rightarrow a^2 = a\). Thus \(N\) is Boolean. The proof of ‘if’ part is obvious.

Proposition 2.4. Let \(N\) be an \(S_2\) near-ring. If \(N\) is regular, then, for every \(a \in N\), there exist \(x \in N^*\) and \(y \in N\) such that \(axy = axa\).

Proof. The result is obvious when \(a = 0\). Now, let \(a \in N^*\). Since \(N\) is an \(S_2\) near-ring, there exists \(x \in N^*\) such that \(axa = ax\). Since \(N\) is regular, there exists \(y \in N\) such that \(a = aya\). Now \(axya = (ax)ya = (axa)ya = ax(aya) = axa\).
We furnish below a characterisation of regular near-rings.

**Corollary 2.5.** Let \( N \) be a distributive \( S_2 \) near-ring without non-zero zero divisors. Then \( N \) is regular if and only if \( N \) is Boolean.

**Proof.** For the ‘only if’ part, we observe that when \( a = 0 \), clearly \( a^2 = a \). Now, let \( a \in N^* \).

By proposition 2.4, there exist \( x \in N^* \) and \( y \in N \) such that \( aya = a \) and \( axya = axa \). Since \( N \) is distributive, this implies that \( ax(ya - a) = 0 \). Since \( N \) has no non-zero zero divisors, we get \( ya - a = 0 \). That is \( ya = a \). Now \( a^2 = a(ya) = a \) and so \( N \) is Boolean. The proof of ‘if’ part is obvious.

### §3. Strong \( S_2 \)-near-rings

In this section, we define strong \( S_2 \) near-rings and obtain its properties.

**Definition 3.1.** A near-ring \( N \) is called a strong \( S_2 \) near-ring, if \( aba = ab \) for all \( a, b \in N \).

**Examples 3.2.**

(a) Every constant near-ring is a strong \( S_2 \) near-ring.

(b) We consider the near-ring \( (N, +, \cdot) \) where \( (N, +) \) is the Klein’s four group \( \{0, a, b, c\} \) and ‘\( . \)’ is defined as follows [1]:

\[
\begin{array}{c|cccc}
. & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & a & a & a & a \\
b & 0 & 0 & 0 & 0 \\
c & a & a & a & a \\
\end{array}
\]

This is a strong \( S_2 \) near-ring.

**Remark 3.3.** Clearly every strong \( S_2 \) near-ring is an \( S_2 \) near-ring, but converse is not true. For, consider the near-ring \( (N, +, \cdot) \) where \( (N, +) \) is the Klein’s four group \( \{0, a, b, c\} \) and ‘\( . \)’ is defined as follows [1]:

\[
\begin{array}{c|cccc}
. & 0 & a & b & c \\
\hline
0 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & 0 \\
b & 0 & 0 & 0 & a \\
c & 0 & 0 & 0 & a \\
\end{array}
\]

This is an \( S_2 \) near-ring, but not a strong \( S_2 \) near-ring as \( bcb \neq bc \).

**Remark 3.4.** We observe that, when \( N \) is a commutative near-ring, the concepts of strong \( S_1 \) and strong \( S_2 \) near-rings coincide. It immediately follows from the definition 3.1 that when \( N \) is a strong \( S_2 \) near-ring, every ideal and every \( N-\)subgroup of \( N \) is also so.

One can easily verify the following.

**Theorem 3.5.** Homomorphic image of a strong \( S_2 \) near-ring is also a strong \( S_2 \) near-ring.

**Theorem 3.6.** Every strong \( S_2 \) near-ring is isomorphic to a subdirect product of subdirectly irreducible strong \( S_2 \) near-rings.
Proof. By theorem 1.1, $N$ is isomorphic to a subdirect product of subdirectly irreducible near-rings $N_i$'s, say, and each $N_i$ is a homomorphic image of $N$ under the usual projection map $\pi_i$. The desired result now follows from theorem 3.5.

**Theorem 3.7.** Let $N$ be a strong $S_2$ near-ring. Then the following are equivalent.

(i) $N$ is an $S$ near-ring.
(ii) $N$ is an $S'$ near-ring.
(iii) $N$ is Boolean.
(iv) $N$ is regular.

Proof. (i) $\Rightarrow$ (ii) Let $x \in N$. Since $N$ is an $S$ near-ring, there exists $y \in N$ such that $x = yx$. This implies that $x = yx = (yx)y = xy \in xN$. Thus $N$ is an $S'$ near-ring.

(ii) $\Rightarrow$ (iii) Let $x \in N$. Since $N$ is an $S'$ near-ring, $x \in xN$, there exists $y \in N$ such that $x = xy$. Now $x^2 = (xy)x = xyx = xy = x$. That is $x^2 = x$ for all $x \in N$ and so $N$ is Boolean.

(iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i) are obvious.

**Theorem 3.8.** Let $N$ be a zero symmetric strong $S_2$ near-ring. Then the following are true.

(i) $N$ has ($^*$, IFP).
(ii) $N$ has strong IFP.
(iii) $N$ has property $(P_1)$.
(iv) For every $a \in N$, $(0 : a)$ is an ideal of $N$.

Proof. (i) Let $x, y \in N$. Suppose $xy = 0$. Now $yx = yxy = y(0) = 0$. Now, for any $n \in N$, $xny = (xn)y = (xnx)y = x(xn)y = x(x(0)) = 0$. That is $xny = 0$. From these $N$ has ($^*$, IFP).

(ii) Let $x, y, n \in N$ and let $I$ be an ideal of $N$. Since $N$ is zero symmetric, $NI \subset I$. Suppose $xy \in I$. Now $xny = (xn)y = (xnx)y = x(xn)y = x(xn)(0) = 0$. Consequently $xny \in I$ and (ii) follows.

(iii) Let $x, y \in N$ and let $I$ be an ideal of $N$. Suppose $xy \in I$. As in (ii), $NI \subset I$. Now $yx = yxy = y(xy) \in NI$. Therefore $yx \in I$ and (iii) follows.

(iv) Follows from (i) and theorem 1.5.

References

Some clopen sets in uniform topology on $BE$-algebras

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Abstract In this paper, we consider a collection of filters of a $BE$-algebra $A$. We use the concept of congruence relation with respect to filters to construct a uniformity which induces a topology on $A$ and prove it is natural for $BE$-algebras to be topological $BE$-algebras with respect to this topology. We study the properties of this topology regarding different filters.

Keywords Uniformity, $BE$-algebra, filter, topological $BE$-algebra.

§1. Introduction

The study of $BCK/BCI$-algebras was initiated by K. Iseki as a generalization of the concept of set-theoretic difference and propositional calculus ([1],[2]). In [6], J. Neggeres and H. S. Kim introduced the notion of $d$-algebra which is a generalization of $BCK$-algebras. Moreover, Y. B. Jun, E. H. Roh and H. S. Kim [3] introduced a new notion, called $BH$-algebra, which is a generalization of $BCK/BCI$-algebras. Recently, as another generalization of $BCK$-algebras, the notion of $BE$-algebras was introduced by H. S. Kim and Y. H. Kim [5]. In this paper we consider a collection of filters and use congruence relation with respect to filters to define a uniformity and make the $BE$-algebra into a uniform topological space with the desired subset as the open sets.

Towards our goal, we renew some needed algebraic notions in §2. Then consider the uniformity based on congruence relations with respect to given collection of filters and construct the induced topology by this uniformity in §3. In the last sections we study the properties of these topologies such as compactness regarding different filters.

§2. Preliminaries

Definition 2.1.[5] A $BE$-algebra is an algebra $(A, *, 1)$ with a binary operation $*$, and a constant 1 such that:

(BE1) $a * a = 1$, $\forall a \in A$;
(BE2) $a * 1 = 1$, $\forall a \in A$;
(BE3) $1 * a = a$, $\forall a \in A$;
(BE4) $a * (b * c) = b * (a * c)$, $\forall a, b, c \in A$. 

We introduce the relation $\leq$ by setting $a \leq b$ if and only if $a * b = 1$, for any $a, b \in A$.

**Definition 2.2.** A filter of a BE-algebra $A$ is a nonempty subset $F$ of $A$ such that for all $a, b \in A$, we have

1. $1 \in F$;
2. $a * b \in F, a \in F$ then $b \in F$.

Let $A$ be a BE-algebra and $K$ a non-empty subset of $A$. Denote by $\sim_k$ the relation on $A$ given by

$$a \sim_k b \text{ if and only if } a * b \in K \text{ and } b * a \in K.$$  

**Theorem 3.3.** If $K$ be a filter of a BE-algebra $A$. Then relation $\sim_k$ is an congruence relation on $A$.

§3. Uniformity in BE-algebra

From now on $(A, *, 1)$ is a BE-algebra.

Let $X$ be a nonempty set and $U$, $V$ be any subset of $X \times X$. Define $U \circ V = \{(x, y) \in X \times X \mid (z, y) \in U \text{ and } (x, z) \in V, \text{ for some } z \in X\}$, $U^{-1} = \{(x, y) \in X \times X \mid (y, x) \in U\}$, $\Delta = \{(x, x) \in X \times X \mid x \in X\}$.

**Definition 3.1.** By a uniformity on $X$ we shall mean a nonempty collection $K$ of subsets of $X \times X$ which satisfies the following conditions:

1. $(U_1)$ $\Delta \subseteq U$ for any $U \in K$;
2. $(U_2)$ if $U \in K$, then $U^{-1} \in K$;
3. $(U_3)$ if $U \in K$, then there exist a $V \in K$ such that $V \circ V \subseteq U$;
4. $(U_4)$ if $U, V \in K$, then $U \cap V \in K$;
5. $(U_5)$ if $U \in K$, and $U \subseteq V \subseteq X \times X$ then $V \in K$.

The pair $(X, K)$ is called a uniform structure (uniform space).

**Theorem 3.2.** Let $\Lambda$ be an arbitrary family of filters of $A$ which is closed under intersection. If $F = \{(x, y) \in A \times A \mid x \sim_F y\}$ and $\mathcal{K} = \{F \mid F \in \Lambda\}$, then $\mathcal{K}$ satisfies the conditions $(U_1)$-$(U_4)$.

**Proof.** $(U_1)$: Since $F$ is a filter of $A$ then we have $x \sim_F x$ for any $x \in A$, hence $\Delta \subseteq U_F$, for all $U_F \in \mathcal{K}$.

$(U_2)$: For any $U_F \in \mathcal{K}$, we have $(x, y) \in (U_F)^{-1} \iff (y, x) \in U_F \iff y \sim_F x \iff x \sim_F y \iff (x, y) \in U_F$.

$(U_3)$: For any $U_F \in \mathcal{K}$, the transitivity of $\sim_F$ implies that $U_F \circ U_F \subseteq U_F$.

$(U_4)$: For any $U_F, U_J \in \mathcal{K}$, we claim that $U_F \cap U_J = U_{F \cap J}$. Let $(x, y) \in U_F \cap U_J$. Then $x \sim_F y$ and $x \sim_J y$. Hence $x * y \in F$, $y * x \in F$ and $x * y \in J$, $y * x \in J$. Then $x \sim_{F \cap J} y$ and hence $(x, y) \in U_{F \cap J}$. Conversely, let $(x, y) \in U_{F \cap J}$. Then $x \sim_{F \cap J} y$, hence $x * y \in F \cap J$ and $y * x \in F \cap J$. Then $x * y \in F$, $y * x \in F$, $x * y \in J$ and $y * x \in J$. Therefore $x \sim_F y$ and $x \sim_J y$.

Then $(x, y) \in U_F \cap U_J$, So $U_F \cap U_J = U_{F \cap J}$. Since $F, J \in \Lambda$, then $F \cap J \in \Lambda$, $U_F \cap U_J \in \mathcal{K}$.

**Theorem 3.3.** Let $\mathcal{K} = \{U \subseteq A \times A \mid U_F \subseteq U \text{ for some } U_F \in \mathcal{K}\}$. Then $\mathcal{K}$ satisfies a uniformity on $A$ and the pair $(A, \mathcal{K})$ is a uniform structure.
Proof. By theorem 3.2, the collection $\mathcal{K}$ satisfies the conditions $(U_1) - (U_4)$. It suffices to show that $\mathcal{K}$ satisfies $(U_5)$. Let $U \in \mathcal{K}$ and $U \subseteq V \subseteq A \times A$. Then there exists a $U_F \subseteq U \subseteq V$, which means that $V \in \mathcal{K}$. This proves the theorem.

Let $x \in A$ and $U \in \mathcal{K}$. Define

$$U[x] := \{y \in A \mid (x, y) \in U\}.$$ 

Theorem 3.4. Given a $BE$-algebra $A$, then

$$T = \{G \subseteq A \mid \forall x \in G, \exists U \in \mathcal{K}, U[x] \subseteq G\}$$

is a topology on $A$.

Proof. It is clear that $\emptyset$ and the set $A$ belong to $T$. Also from the very nature of that definition, it is clear that $T$ is closed under arbitrary union. Finally to show that $T$ is closed under finite intersection, let $G, H \in T$ and suppose $x \in G \cap H$. Then there exist $U$ and $V \in \mathcal{K}$ such that $U[x] \subseteq G$ and $V[x] \subseteq H$. Let $W = U \cap V$, then $W \in \mathcal{K}$. Also $W[x] \subseteq U[x] \cap V[x]$ and so $W[x] \subseteq G \cap H$ and so $G \cap H \in T$. Thus $T$ is topology on $A$.

Note that for any $x$ in $A$, $U[x]$ is an open neighborhood of $x$.

Lemma 3.5. Let $F$ be a filter of $A$. Then $F = \{1\}$ if and only if $U_F = U_{\{1\}}$.

Proof. Since $F \neq \{1\}$, there exist $z \in F$ such that $z \neq 1$. By (BE3), $1 \ast z = z \in F$ and by (BE2), $z \ast 1 = 1 \in F$. Hence $1 \in U_{F[z]}$ and then $(z, 1) \in U_F$. On other hand since $z \neq 1$, $(z, 1) \notin U_{\{1\}}$. Therefore if $U_F = U_{\{1\}}$ then $F = \{1\}$. Conversely is clear.

Definition 3.6. Let $(A, \mathcal{K})$ be a uniform structure. Then the topology $T$ is called the uniform topology on $A$ induced by $\mathcal{K}$.

Example 3.7. Let $A = \{a, b, c, 1\}$. Define $\ast$ as follow:

$$\begin{array}{c|cccc}
  \ast & 1 & a & b & c & d \\
  \hline
  1 & 1 & a & b & c & d \\
  a & 1 & 1 & b & c & d \\
  b & 1 & a & 1 & c & c \\
  c & 1 & 1 & b & 1 & b \\
  d & 1 & 1 & 1 & 1 & 1 \\
\end{array}$$

Easily we can check that $(A, \ast, 1)$ is a $BE$-algebra. It is easy to see that $F = \{c, a, 1\}$ is a filter and let $\Lambda = \{F\}$. Therefore as theorem 3.2, we construct

$$K^* = \{U_F\} = \{(x, y) \mid x \sim_F y\} = \{(1, 1), (a, 1), (1, a), (1, c), (c, 1), (a, a), (a, c), (c, a), (b, b), (b, d), (d, b), (c, c), (d, d)\}$.$$

We can check that $(A, \mathcal{K})$ is a uniform space, where $\mathcal{K} = \{U \mid U_F \subseteq U\}$. Open neighborhoods are:
\[ \begin{align*}
U_F[a] &= \{1, a, c\}, \\
U_F[b] &= \{b, d\}, \\
U_F[c] &= \{1, a, c\}, \\
U_F[d] &= \{b, d\}, \\
U_F[1] &= \{1, a, c\}.
\end{align*} \]

From above we get that
\[ T = \{\{1, a, c\}, \{b, d\}, \{1, a, b, c, d\}, \emptyset\}. \]

Thus \((A, T)\) is a uniform topological space.

\[\textbf{Proposition 3.8.} \text{ Topological space } (A, T) \text{ is completely regular.} \]

\[\textbf{Proof.} \text{ See theorem 14.2.9 of [4].} \]

\[\textbf{§4. Topological property of space } (A, T) \]

Let \(A\) be a \(BE\)-algebra and \(C, D\) subsets of \(A\). Then we define \(C \ast D\) as follows:
\[ C \ast D = \{x \ast y \mid x \in C, y \in D\}. \]

Let \(A\) be a \(BE\)-algebra and \(T\) a topology defined on the set \(A\). Then we say that the pair \((A, T)\) is a topological \(BE\)-algebra if the \(BE\)-algebraic operation \(\ast\) is continuous with respect to \(T\). The continuity of the \(BE\)-algebraic operation \(\ast\) is equivalent to having the following properties satisfied:

Let \(O\) be an open set and \(a, b \in A\) such that \(a \ast b \in O\). Then there are \(O_1\) and \(O_2\) such that \(a \in O_1, b \in O_2\) and \(O_1 \ast O_2 \subseteq O\).

\[\textbf{Theorem 4.1.} \text{ The pair } (A, T) \text{ is a topological } BE\text{-algebra.} \]

\[\textbf{Proof.} \text{ Assume that } x \ast y \in G, \text{ with } x, y \in A \text{ and } G \text{ an open subset of } A. \text{ Then there exist } U \in K, U[x \ast y] \subseteq G \text{ and a filter } F \text{ of } A \text{ such that } U_F \subseteq U. \text{ We claim that the following relation holds:} \]
\[ U_F[x] \ast U_F[y] \subseteq U[x \ast y]. \]

Indeed for \(h \in U_F[x]\) and \(k \in U_F[y]\) we get \(x \sim_F h\) and \(y \sim_F k\). Hence by theorem 2.3, \(x \ast y \sim_F h \ast k\). From that \((x \ast y, h \ast k) \in U_F \subseteq U\). Hence \(h \ast k \in U_F[x \ast y] \subseteq U[x \ast y]\). Then \(h \ast k \in G\). Thus the condition is verified.

\[\textbf{Theorem 4.2.}^{[5]} \text{ Let } X \text{ be a set and } S \subseteq P(X \times X) \text{ be a family such that for every } U \in S \text{ the following conditions hold:} \]

(a) \(\Delta \subseteq U;\)

(b) \(U^{-1}\) contains a member of \(S\), and

(c) there exists a \(V \in S\), such that \(V \circ V \subseteq U\). Then there exists a unique uniformity \(U\), for which \(S\) is a subbase.

\[\textbf{Theorem 4.3.} \text{ If we let } B = \{U_F \mid F \text{ is a filter of } A\}. \text{ Then } B \text{ is a subbase for a uniformity of } A, \text{ we denote this topology by } S. \]
Proof. Since \( \sim_F \) is an equivalence relation, then it is clear that \( B \) satisfies the axioms of theorem 4.2.

We say that topology \( \sigma \) is finer than \( \tau \) if \( \tau \subseteq \sigma \) as subsets of the power set. Then we have:

Corollary 4.4. Topology \( S \) is finer than \( T \).

Theorem 4.5. Any filter in the collection \( \Lambda \) is a clopen subset of \( A \).

Proof. Let \( F \) be a filter of \( A \) in \( \Lambda \) and \( y \in F^c \). Then \( y \in U_F[y] \) and we get \( F^c \subseteq \bigcup \{U_F[y] \mid y \in F^c\} \). We claim that, \( U_F[y] \subseteq F^c \), for all \( y \in F^c \). Let \( z \in U_F[y] \), then \( z \sim_F y \). Hence \( z \ast y \in F \). If \( z \in F \) then, \( y \in F \), that is a contradiction. So \( z \in F^c \) and we get \( \bigcup \{U_F[y] \mid y \in F^c\} \subseteq F^c \). Hence \( F^c = \bigcup \{U_F[y] \mid y \in F^c\} \) and since \( U_F[y] \) is open for all \( y \in A \), \( F \) is a closed subset. We show that \( F = \bigcup \{U_F[y] \mid y \in F\} \). If \( y \in F \) then \( y \in U_F[y] \) and we get \( F \subseteq \bigcup \{U_F[y] \mid y \in F\} \). Let \( y \in F \), if \( z \in U_F[y] \) then \( z \sim_F y \) and so \( z \ast y \in F \). Since \( y \in F \) hence \( z \in F \) and we get \( \bigcup \{U_F[y] \mid y \in F\} \subseteq F \). So \( F \) is also an open subset of \( A \).

Theorem 4.6. For any \( x \in A \) and \( F \in \Lambda \), \( U_F[x] \) is a clopen subset of \( A \).

Proof. We show that \( (U_F[x])^c \) is open. Let \( y \in (U_F[x])^c \), then \( x \ast y \in F^c \) or \( y \ast x \in F^c \). Let \( y \ast x \in F^c \). Hence by theorem 4.1 and the proof of theorem 4.5, \( (U_F[y] \ast U_F[x]) \subseteq U_F[y \ast x] \subseteq F^c \). We claim that: \( U_F[y] \subseteq (U_F[x])^c \). Let \( z \in U_F[y] \), then \( z \ast x \in (U_F[y] \ast U_F[x]) \). So \( z \ast x \in F^c \) then we get \( z \in (U_F[x])^c \). Hence \( U_F[x] \) is closed. It is clear that \( U_F[x] \) is open. So \( U_F[x] \) is clopen subset of \( A \).

A topological space \( X \) is connected if and only if has only \( X \) and \( \emptyset \) as clopen subsets. Therefore we have

Corollary 4.7. The space \( (A,T) \) is not a connected space.

§5. Some connection between uniform topology and filters

We denote the uniform topology obtained by an arbitrary family \( \Lambda \), by \( T_\Lambda \) and if \( \Lambda = \{F\} \), we denote it by \( T_F \).

Theorem 5.1. \( T_\Lambda = T_J \), where \( J = \bigcap \{F \mid F \in \Lambda\} \).

Proof. Let \( \mathcal{K} \) and \( \mathcal{K}^* \) be as in theorem 3.2 and 3.3. Now consider \( \Lambda_0 = \{J\} \), define: \( (\mathcal{K}_0)^* = \{U_J\} \) and \( \mathcal{K}_0 = \{U \mid U_J \subseteq U\} \).

Let \( G \in T_\Lambda \). So for all \( x \in G \), there exist \( U \in \mathcal{K} \) such that \( U[x] \subseteq G \). From \( J \subseteq F \) we get that \( U_J \subseteq U_F \), for all filter \( F \) of \( A \). Since \( U \in \mathcal{K} \), there exist \( F \in \Lambda \) such that \( U_F \subseteq U \). Hence \( U_J[x] \subseteq U_F[x] \subseteq G \). Since \( U_J \in \mathcal{K}_0 \), \( G \in T_J \). So \( T_\Lambda \subseteq T_J \).

Conversely, let \( H \in T_J \) then for all \( x \in H \), there exist \( U \in \mathcal{K}_0 \) such that \( U[x] \subseteq H \). So \( U_J[x] \subseteq H \) and since \( \Lambda \) is closed under intersection, \( J \subseteq \Lambda \). Then we get \( U_J \in \mathcal{K} \) and so \( H \in T_\Lambda \). Thus \( T_J \subseteq T_\Lambda \).

Corollary 5.2. Let \( F \) and \( J \) be filters of \( A \) and \( F \subseteq J \). Then \( J \) is clopen in topological space \( (A,T_F) \).

Proof. Consider \( \Lambda = \{F,J\} \). Then by theorem 5.1, \( T_\Lambda = T_F \) and therefore \( J \) is clopen in topological space \( (A,T_F) \).

Theorem 5.3. Let \( F \) and \( J \) be filters of \( A \). Then \( T_F \subseteq T_J \) if and only if \( J \subseteq F \).

Proof. Let \( J \subseteq F \). Consider:

\( \Lambda_1 = \{F\}, \mathcal{K}_1^* = \{U_F\}, \mathcal{K}_1 = \{U \mid U_F \subseteq U\} \) and
$A_2 = \{J\}, \mathcal{K}_2^* = \{U_J\}, \mathcal{K}_2 = \{U \mid U_J \subseteq U\}$.

Let $G \in T_F$. Then for all $x \in G$, there exist $U \in \mathcal{K}_1$ such that $U[x] \subseteq G$. Since $J \subseteq F$, then $U_J \subseteq U_F$ and since $U_F[x] \subseteq G$, we get $U_J[x] \subseteq G$. $U_J \in \mathcal{K}_2$ and so $G \in T_J$.

Conversely, let $T_F \subseteq T_J$. In contrary let $a \in J \setminus F$, since $F \in T_F$, by hypothesis we get that $F \in T_J$. Then for all $x \in F$, there exist $U \in \mathcal{K}_2$ such that $U[x] \subseteq F$, and so $U_J[x] \subseteq F$. Then $U_J[1] \subseteq F$. We have $a \ast 1 = 1 \in J$ and $1 \ast a = a \in J$. Then $a \sim_J 1$, so $a \in U_J[1]$, thus $a \in F$. Which is a contradiction.

**Corollary 5.4.** Let $F$ and $J$ be filters of $A$. Then $F = J$ if and only if $T_F = T_J$.

**Theorem 5.5.** Let $F$ be a filter of $A$. Then the following conditions are equivalent:

(1) Topological space $(A, T_F)$ is compact;

(2) Topological space $(A, T_F)$ is totally bounded;

(3) there exists $P = \{x_1, x_2, \ldots, x_n\} \subseteq A$ such that for all $a \in A$ there exists $x_i \in P$ where $a \ast x_i \in F$ and $x_i \ast a \in F$.

**Proof.** (1 $\Rightarrow$ 2): it is clear by theorem 14.3.8. of [5].

(2 $\Rightarrow$ 3): Let $U_F \in \mathcal{K}$ since $(A, T_F)$ is totally bounded, then there exists $x_1, x_2, \ldots, x_n \in F$ such that $A = \bigcup_{i=1}^n U_F[x_i]$. Now let $a \in A$ then there exist $x_i$ such that $a \in U_F[x_i]$, therefore $a \ast x_i \in F$ and $x_i \ast a \in F$.

(3 $\Rightarrow$ 1): For all $a \in A$ by hypothesis there exists $x_i \in P$ where $a \ast x_i \in F$ and $x_i \ast a \in F$.

Hence $a \in U_F[x_i]$, thus $A = \bigcup_{i=1}^n U_F[x_i]$. Now let $A = \bigcup_{\alpha \in I} O_\alpha$, where each $O_\alpha$ is an open set of $A$, then for any $x_i \in A$ there exists $\alpha_i \in I$ such that $x_i \in O_{\alpha_i}$, since $O_{\alpha_i}$ is an open set then $U_F[x_i] \subseteq O_{\alpha_i}$, so $A = \bigcup_{i=1}^n U_F[x_i] \subseteq \bigcup_{\alpha \in I} O_\alpha$, therefore $A = \bigcup_{\alpha \in I} O_\alpha$, which means that $(A, T_F)$ is compact.

**Theorem 5.6.** If $F$ is a filter of $A$ such that $F^c$ is a finite set. Then topological space $(A, T_F)$ is compact.

**Proof.** Let $A = \bigcup_{\alpha \in I} O_\alpha$, where each $O_\alpha$ is an open set of $A$. Let $F^c = \{x_1, x_2, \ldots, x_n\}$.

Then there exists $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_n \in I$ such that $1 \in O_\alpha, x_1 \in O_{\alpha_1}, x_2 \in O_{\alpha_2}, \ldots, x_n \in O_{\alpha_n}$. Then $U_F[1] \subseteq O_{\alpha_1}$, but $U_F[1] = F$. Hence $A = O_{\alpha_1} \cup O_{\alpha_2} \cup \ldots \cup O_{\alpha_n}$.

**Theorem 5.7.** If $F$ is a filter of $A$. Then $F$ is a compact set in topological space $(A, T_F)$.

**Proof.** Let $F \subseteq \bigcup_{\alpha \in I} O_\alpha$, where each $O_\alpha$ is an open set of $A$. Since $1 \in F$, then there exists $\alpha \in I$ such that $1 \in O_\alpha$. Then $F = U_F[1] \subseteq O_\alpha$. Hence $F$ is compact.

**Theorem 5.8.** If $F$ is a filter of $A$. Then $U_F[x]$ is a compact set in topological space $(A, T_F)$, for all $x \in A$.

**Proof.** Let $U_F[x] \subseteq \bigcup_{\alpha \in I} O_\alpha$, where each $O_\alpha$ is an open set of $A$. Since $x \in U_F[x]$, then there exists $\alpha \in I$ such that $x \in O_\alpha$. Then $U_F[x] \subseteq O_\alpha$. Hence $U_F[x]$ is compact.

**References**

Testing for comparison of two expectation functions of non-parametric regression

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Abstract In this study, we introduce a procedure for testing the equality of two regression functions, the hypothesis:

\[ H_0 : m_1 = m_2 \text{ vs } H_1 : m_1 \neq m_2. \]

The test statistic is based on the comparison of two estimators of the distribution of the error in each population, and the estimator of error in each population is

\[ \bar{Y}_j - \hat{m}_j(X_{ij}) \]

\[ \frac{\sigma_j(X_{ij})}{\sqrt{n_j}}. \]

Kuiper type statistic are considered.

In the simulation study, a bootstrap mechanism is used to approximate the critical value in practice.

Keywords Nonparametric regression, bootstrap procedure, comparison expectation function.

§1. Introduction and preliminaries

Regression analysis can be divided into two parts: 1) parametric regression and 2) nonparametric regression. The comparison of two or more groups is an important problem in statistical inference. This comparison can be performed through the regression curves in non-parametric regression context. Let \((X_{ij}, Y_{ij}), i = 1, \ldots, n_j, j = 1, 2\) be two independent random vectors, and assume the following non-parametric regression model

\[ Y_{ij} = m_j(X_{ij}) + \sigma_j(X_{ij})\varepsilon_{ij}, i = 1, \ldots, n_j, j = 1, 2, \] (1)

where \(m_j(X_{ij}) = E(Y_{ij}|(X_{ij}))\) is the unknown regression function, \(\sigma_j(X_{ij})\) is the conditional variance function \(\sigma_j^2(X_{ij}) = \text{Var}(Y_{ij}|(X_{ij}))\), \(\varepsilon_{ij}\) is the error variable with distribution \(F_{\varepsilon_j}\), and let \(n = n_1 + n_2\). In this paper, we are interested in comparison the equality of regression functions, the hypothesis \(H_0 : m_1 = m_2 \text{ vs } H_1 : m_1 \neq m_2\), and the test statistic is the Kuiper type statistics: \(U_{KU} = \sum_{j=1}^{2} \| \sup_y |U_j(y)| - \inf_y |U_j(y)| \|. \)

The problem of testing for comparison expectation functions of non-parametric regression or the problem of testing for the equality of non-parametric regression curves has been treated in the literature during the last eighteen years. The first reference about comparison of regression curves assume some restriction on the model. Fixed design and homoscedastic errors are common assumption in several references. W. Hardle and J. S. Marron \([4]\) analyzed semi-parametric models by comparing nonparametric regression functions under the assumption of fixed equal designs (\(i.e., n_1 = n_2, \ldots, = n_k\)). P. Hall and J. D. Hart \([3]\) designed two-sided tests for \(H_0 : m_1 = m_2\), and discussed the nonparametric homoscedastic (\(i.e., \sigma_i^2(t) = \sigma_j^2, i = 1, \ldots, k\)) models in the case of equal design...
points. E. C. King, J. D. Hart and T. E. Wehrly [5] also presented a test based on the difference between two curve estimators from kernel smoothers when the design points are fixed and equal. M. A. Delgado [2] investigated the test statistics for testing the equality of k regression function when the design point are fixed and equal. S. Young and A. W. Bowman [3] motivated the test statistic by the classical one-way analysis of variance. K. B. Kulasekera [7] presented three tests with common fixed design points. The first two tests are based on estimators Quasi-Residuals technique while the last test based on estimators of the variance of the error distributions. L. H. Koul and A. Schick [6] discussed the problem of testing the equality of non-parametric regression curves one-side alternatives when design points are common and distinct design points. A. Munk and H. Dette [8] considered the problem of testing for the equality of two curves with heteroscedastic errors and fixed design. The testing procedure is based on the estimation of $L^2$-distance of difference between two curves. N. Neumeyer and H. Dette [9] considered an empirical process approach in complete non-parametric, heteroscedastic and random designed setup to compare two regression curves. Their test can detect local alternatives which converge to the null hypothesis at the rate of order $n^{-1/2}$. The idea of our testing procedure come from [10]. Namely, if two empirical distributions of the errors are equal, there are evidence for the equality of the regression curves. On the contrary, if these two empirical distributions are different, there are evidence for the inequality of the regression curves. In this paper, the estimator of the error in $j$ population ($\varepsilon_{ij}$) for $j = 1, 2$ is 
$$ Y_j - \hat{m}_j(X_{ij}) $$

is the estimator of the error under the null hypothesis ($H_0$), where $Y_j$ is the average of $Y$ in $j$ population, $\hat{m}_j(X_{ij})$ is the estimator of regression function $m_j(X_{ij})$, $\tilde{m}(X_{ij})$ is the estimator of regression function $m(X_{ij})$ under the null hypothesis, $\sigma_j(X_{ij})$ is the estimator of variance $\sigma_j(X_{ij})$, and $n_j$ is sample size in population $j$ respectively. In addition, in this paper the test statistic can detect local alternatives which converge to the null hypothesis at the rate of order $n^{-1/2}$.

§2. Definition and assumptions

In this section, we introduced the definition and assumptions for proofs of the main results. Consider $(X_{ij}, Y_{ij}), i = 1, \ldots, n_j, j = 1, 2$ be an independent and identically distribution, and satisfying the non-parametric regression model (1). Let $\hat{m}_j(X_{ij})$ is the estimator of regression function $m_j(X_{ij})$, $\tilde{m}_j(X_{ij}) = \sum_{i=1}^{n_j} W_{ij}^{(j)}(x, h) Y_{ij}$, $W_{ij}^{(j)}(x, h) = \frac{K((x - X_{ij})/h)}{\sum_{i=1}^{n_j} K((x - X_{ij})/h)}$ is Nadaraya - Watson - type weight, $K$ is a known kernel, $h$ is an appropriate bandwidth, $\hat{\sigma}_j^2(x) = \sum_{i=1}^{n_j} W_{ij}^{(j)}(x, h) Y_{ij}^2 - \tilde{m}_j^2(x)$, and $\hat{m}(x) = \sum_{j=1}^{k} \sum_{i=1}^{n_j} W_{ij}^{(j)}(x, h) Y_{ij}$ be an estimator of the common regression function $m(x) = m_1(x) = m_2(x)$ under the null hypothesis $H_0$. For $j = 1, 2$, let $\hat{\varepsilon}_{ij} = \frac{Y_j - \hat{m}_j(X_{ij})}{\sigma_j(X_{ij})/\sqrt{n_j}}$ is the estimator of the error in $j$ population, $\hat{F}_{\varepsilon_j}(y)$ is the estimator of the error distribution function in $j$ population $F_{\varepsilon_j}(y) = P(\varepsilon_{ij} \leq y) = \frac{1}{n_j} \sum_{i=1}^{n_j} I(\frac{Y_j - \hat{m}_j(X_{ij})}{\sigma_j(X_{ij})/\sqrt{n_j}} \leq y)$, $I(.)$ is an indicator function, and probability density function of
$F_{ij}(y)$ is $f_{ij}(y)$. We used the notation $f_{ij}(y) = F'_{ij}(y)$. Let $\hat{\varepsilon}_{ij0} = \frac{\hat{Y}_j - \hat{m}(X_{ij})}{\hat{\sigma}_j(X_{ij})/\sqrt{n_j}}$ is an estimator of the error under the null hypothesis $H_0$, $\hat{F}_{00}(y)$ is the estimator of the error distribution function in $j$ population $F_{00}(y)$: $\hat{F}_{00}(y) = P(\varepsilon_{ij0} \leq y) = \frac{1}{n_j} \sum_{i=1}^{n_j} I[\frac{\hat{Y}_j - \hat{m}(X_{ij})}{\hat{\sigma}_j(X_{ij})/\sqrt{n_j}} \leq y]$, $I(.)$ is an indicator function, and probability density function of $F_{00}(y)$ is $f_{00}(y)$. We used the notation $f_{00}(y) = F'_{00}(y)$. For simplicity, we work with the same bandwidth $h$ to estimate $\hat{m}$, $\hat{\sigma}_j$, $\hat{\sigma}_j$. In addition, in this paper we perform the comparison between these two estimators of distribution of error in each population using the 2-dimensional process, $\hat{U}(y) = (\hat{U}_1(y), \hat{U}_2(y))$, $-\infty < y < \infty$, for $j = 1, 2$, $\hat{U}_j(y) = n_j^{1/2}(\hat{F}_{x0}(y) - \hat{F}_{ej}(y))$. More precisely, we will use the Kuiper test statistic $U_{KU} = \sum_{j=1}^{2}[\sup_{y}[\hat{U}_j(y)] - \inf_{y}[\hat{U}_j(y)]$. The assumptions we need for proofs of the main results are listed below for convenient reference.

A1 : $F_j(x) = P(X_j \leq x)$ and $F_j(y|x) = P(Y_j \leq y|X_j = x)$ be a distribution of independent variable X and the conditional distribution of response given X respectively. The probability density functions of $F_j(x)$ and $F_j(y|x)$ will be denoted respectively by $f_j(x)$ and $f_j(y|x)$. We used the notation $F'_j(x) = f_j(x)$ and $F'_j(y|x) = f_j(y|x)$.

A2 : For $j = 1, 2$,
(i) $F_{ij}(y)$ is an empirical distribution of error of $j$ population,
$$F_{ij}(y) = P(\varepsilon_{ij} \leq y) = \frac{1}{n_j} \sum_{i=1}^{n_j} I[\frac{\hat{Y}_j - m_j(X_{ij})}{\hat{\sigma}_j(X_{ij})/\sqrt{n_j}} \leq y]$$ and $f_{ij}(y) = F'_{ij}(y)$. (ii) $F_{00}(y)$ is an empirical distribution of error under the null hypothesis ($H_0$).
$$F_{00}(y) = P(\varepsilon_{ij0} \leq y) = \frac{1}{n_j} \sum_{i=1}^{n_j} I[\frac{\hat{Y}_j - m_j(X_{ij})}{\hat{\sigma}_j(X_{ij})/\sqrt{n_j}} \leq y],$$ and $f_{00}(y) = F'_{00}(y)$.

A3 : For $j = 1, 2$,
(i) $X_j$ is a continuous variable with compact support $x \in \mathbb{R}_x$, and density $f_j(x)$.
(ii) $f_j(x)$, $m_j(x)$, and $\sigma_j(x)$ are twice continuously differentiable.
(iii) $\inf_{x \in \mathbb{R}_x} f_j(x) > 0$ and $\inf_{x \in \mathbb{R}_x} \sigma_j(x) > 0$.

A4 : For $j = 1, 2$,
(i) $n_j \to p_j > 0$, as $n \to \infty$,
(ii) $nh^4 \to 0$, and $n_j h^{3+2\delta} \to \infty$ for some $\delta > 0$, as $h \to 0$.

A5 : $K$ is a symmetric density function.
(i) $\int u^2 K(u)du < \infty$.
(ii) $\int K(u)du = 1$.
(iii) $\int u K(u)du = 0$.

A6 : For $j = 1, 2$,
(i) $F_j(y|x)$ is the continuous in $(x, y)$ and differentiable with respect to $y$;
(ii) $F_j(y|x)$ is the continuous in $(x, y)$ and differentiable with respect to $x$.

§3. Main results

In this section, we derive the test statistic for finding the distribution of the test statistic.
Theorem 3.1. \( F_{0j}(y) = F_{\epsilon j}(y) \), \(-\infty < y < \infty\), for all \( j = 1, 2 \) if and only if \( m(x) = m_j(x) = m_j(x) \) for all \( x \in \mathbb{R}_x \).

Proof. Assume \( F_{0j}(y) = F_{\epsilon j}(y) \). This implies that two empirical distributions of the errors are equal, there are evidence for the equality of the 1st moment and the 2nd moment of the distribution. Consider the 1st moment : \( E(\frac{\bar{Y}_j - m_j(X_{ij})}{\sqrt{\sigma_j(X_{ij})^2}}) = E(\frac{\bar{Y}_j - m_j(X_{ij})}{\sqrt{\sigma_j(X_{ij})^2}}) = 0 \), then \( m(x) = m_j(x) \). In the same way, the 2nd moment have originate from the 1st moment, then \( m(x) = m_j(x) \). Namely, \( m(x) = m_1(x) = m_2(x) \). Conversely, assume \( m(x) = m_1(x) = m_2(x) \). Claim that \( F_{0j}(y) = F_{\epsilon j}(y) \). Consider the 1st moment : From \( m(x) = m_j(x) \), then

\[
\left( \frac{\bar{Y}_j - m_j(X_{ij})}{\sigma_j(X_{ij})/\sqrt{\tau_j}} \right) = \left( \frac{\bar{Y}_j - m_j(X_{ij})}{\sigma_j(X_{ij})/\sqrt{\tau_j}} \right), \quad E\left( \frac{\bar{Y}_j - m_j(X_{ij})}{\sigma_j(X_{ij})/\sqrt{\tau_j}} \right) = E\left( \frac{\bar{Y}_j - m_j(X_{ij})}{\sigma_j(X_{ij})/\sqrt{\tau_j}} \right) = 0.
\]

Consider the 2nd moment : From \( m(x) = m_j(x) \), then

\[
\left( \frac{\bar{Y}_j - m_j(X_{ij})}{\sigma_j(X_{ij})/\sqrt{\tau_j}} \right)^2 = \left( \frac{\bar{Y}_j - m_j(X_{ij})}{\sigma_j(X_{ij})/\sqrt{\tau_j}} \right)^2 \quad E\left( \frac{\bar{Y}_j - m_j(X_{ij})}{\sigma_j(X_{ij})/\sqrt{\tau_j}} \right)^2 = E\left( \frac{\bar{Y}_j - m_j(X_{ij})}{\sigma_j(X_{ij})/\sqrt{\tau_j}} \right)^2.
\]

From the 1st moment and the 2nd moment, if \( m(x) = m_1(x) = m_2(x) \) then \( F_{0j}(y) = F_{\epsilon j}(y) \).

Theorem 3.2. Assume (A1) - (A6). Under the null hypothesis \( (H_0) \), for \( j = 1, 2 \),

\[
\hat{F}_{\epsilon 0j}(y) - \hat{F}_{\epsilon j}(y) = f_{\epsilon j}(y) \sum_{r=1}^2 \sum_{i=1}^{n_r} \left( \frac{Y_{ir} - m(X_{ir})}{\sigma_j(X_{ir})/\sqrt{\tau_j}} \right) \left( \frac{f_{\epsilon j}(X_{ir})}{f_{mix}(X_{ir})} - \frac{I(r = j)}{p_j} \right) + o_p(n)^{-1/2}.
\]

Proof. From the theorem 1 in [1], we get the following representation:

\[
\hat{F}_{\epsilon 0j}(y) - \hat{F}_{\epsilon j}(y) = f_{\epsilon j}(y) \sum_{r=1}^2 \sum_{i=1}^{n_r} \left( \frac{Y_{ir} - m(X_{ir})}{\sigma_j(X_{ir})/\sqrt{\tau_j}} \right) \left( \frac{f_{\epsilon j}(X_{ir})}{f_{mix}(X_{ir})} - \frac{I(r = j)}{p_j} \right) + o_p(n)^{-1/2},
\]

uniformly in \( y \).

Theorem 3.3. Assume (A1) - (A6). Under the null hypothesis \( (H_0) \), 2 - dimensions of \( \hat{U}(y) = (\hat{U}_1(y), \hat{U}_2(y)) \) converges weakly to \( U(y) = (U_1(y), U_2(y)) \), where

\[
U_r(y) = f_{\epsilon j}(y) \left( \sum_{r=1}^{n_r} \frac{Y_{ir} - m(X_{ir})}{\sigma_j(X_{ir})/\sqrt{\tau_j}} \right) \left( \frac{f_{\epsilon j}(X_{ir})}{f_{mix}(X_{ir})} - \frac{I(r = j)}{p_j} \right) \quad \text{are normal random variables with mean zero and covariance structure :}
\]

\[
\text{Cov}(U_r(y), U_{r'}(y)) = f_{\epsilon j}(y)f_{\epsilon j'}(y)p_jp_{r'} \sum_{r=1}^2 \left( \frac{Y_{ir} - m(X_{ir})}{\sigma_j(X_{ir})/\sqrt{\tau_j}} \right)^2 \left( \frac{f_{\epsilon j}(X_{ir})}{f_{mix}(X_{ir})} - \frac{I(r = j)}{p_j} \right) \left( \frac{f_{\epsilon j'}(X_{ir})}{f_{mix}(X_{ir})} - \frac{I(r = j')}{p_j} \right)
\]

\[
\text{Proof. By Cramer - Wold device in R. J. Serfling [11], and theorem 1 in J. C. Pardo-Fernandez [10] showed that the weak convergence of multidimensional process is weak convergence of linear combination in each component. Assume \( b_j \) is any real number, \( \hat{U}(y) = (\hat{U}_1(y), \hat{U}_2(y))' \), where \( \hat{U}_j(y) = n_j^{1/2}(\hat{F}_{\epsilon 0j}(y) - \hat{F}_{\epsilon j}(y)), \) \(-\infty < y < \infty\) for \( j = 1, 2 \). Linear combination of \( \hat{U}(y) \) is easy to prove that \( \hat{Z}(y) = \sum_{j=1}^2 b_j n_j^{1/2}(\hat{F}_{\epsilon 0j}(y) - \hat{F}_{\epsilon j}(y)) \), from theorem 2,
\]

\[
\hat{Z}(y) = \sum_{r=1}^2 \sum_{i=1}^{n_r} \frac{Y_{ir} - m(X_{ir})}{\sigma_j(X_{ir})} \sum_{j=1}^2 b_j p_j f_{\epsilon j}(y) \frac{f_{\epsilon j}(X_{ir})}{f_{mix}(X_{ir})} - b_r f_{\epsilon r}(y) + o_p(n)^{-1/2},
\]

\[
\hat{Z}(y) = \sum_{r=1}^2 \sum_{i=1}^{n_r} \Psi_r(X_{ir}, Y_{ir}, y) + o_p(n)^{-1/2}, \quad \text{then} \quad \hat{Z}(y) = \sum_{i=1}^{n_r} \Psi_r(X_{ir}, Y_{ir}, y),
\]
where
\[ \Psi_r(u, v, y) = \frac{v - m(u)}{\sigma_r(u)} \left[ \sum_{j=1}^{2} b_j p_j f_{xj}(y) \frac{\sigma_r(u)}{\sigma_j(u)} \frac{f_{xj}(u)}{f_{mix}(u)} - b_r f_{r}(y) \right], r = 1, 2. \] (2)

From (2), given a collection \( \mathcal{F}_r \) of measurable functions, the empirical measure induces a map from \( \mathcal{F}_r \) to \( \mathbb{R} \) given by: \( \mathcal{F}_r = \{(u, v) \to \Psi_r(u, v, y), -\infty < y < \infty \} \), where \( \mathcal{F}_r \) is an indexed process. A. W. Van der Vaart and J. A. Wellner \cite{12} and J. C. Pardo-Fernandez \cite{10} introduced for any class of function \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \), define \( \mathcal{G}_1 + \mathcal{G}_2 = \{g_1 + g_2, g_1 \in \mathcal{G}_1, g_2 \in \mathcal{G}_2 \} \). The notation of class \( \mathcal{F}_r \) can be written as \( \mathcal{F}_r = \sum_{j=1}^{k+1} \mathcal{F}_{rj} \), for \( j = 1, 2 \). The empirical measure induces a map form \( \mathcal{F}_{rj} \) to \( \mathbb{R} \) given by: \( \mathcal{F}_{rj} = \{(u, v) \to b_j p_j f_{xj}(y) \frac{\sigma_r(u)}{\sigma_j(u)} \frac{f_{xj}(u)}{f_{mix}(u)} - b_r f_{r}(y) \}, -\infty < y < \infty \), and the empirical measure induces a map form \( \mathcal{F}_{r,(k+1)} \) to \( \mathbb{R} \) given by: \( \mathcal{F}_{r,(k+1)} = \{(u, v) \to -b_j f_{r}(y) \frac{v - m(u)}{\sigma_r(u)}, -\infty < y < \infty \} \). Let \( M \) be such that \( \sup_{y,j=1,2} |f_{xj}(y)| < M \). Then \( N_{11}(\delta, \mathcal{F}_{rj}, L_2(P)) \leq 2M \delta^{-1} \) if \( \delta < 2M \), and \( N_{11}(\delta, \mathcal{F}_{rj}, L_2(P)) = 1 \) if \( \delta > 2M \). Where \( N_{11} \) is a bracketing number, and the bracketing number \( N_{11}(\delta, \mathcal{F}_{rj}, L_2(P)) \) is the minimum number of \( \delta \)-brackets needed to cover \( \mathcal{F}_{rj} \). \( P \) is the measure of probability corresponding to the joint distribution of \( (X_r, Y_r) \), and \( L_2(P) \) is \( L_2 \) norm. Therefore, \( N_{11}(\delta, \mathcal{F}_{rj}, L_2(P)) \leq \Pi_{j=1}^{k+1} N_{11}(\delta, \mathcal{F}_{rj}, L_2(P)) \), (3) and, by taking logarithms, \( \log N_{11}(\delta, \mathcal{F}_{rj}, L_2(P)) \leq \sum_{j=1}^{k} \log N_{11}(\delta, \mathcal{F}_{rj}, L_2(P)) \). Now,
\[ \int_0^{\infty} \sqrt{\log N_{11}(\delta, \mathcal{F}_{rj}, L_2(P))} \delta d\delta \leq \sum_{j=1}^{k+1} \int_0^{2M} \sqrt{\log N_{11}(\delta, \mathcal{F}_{rj}, L_2(P))} \delta d\delta. \] (4)

From (4), can be conclude that \( \int_0^{\infty} \sqrt{\log N_{11}(\delta, \mathcal{F}_{rj}, L_2(P))} \delta d\delta \) is finite, therefore, the class of function \( \mathcal{F}_r \) is Donsker class or \( P \)-Donsker class, from Donsker-Theorem, if \( \mathcal{F}_r \) is Donsker class, it follows that the limit process \( \{Gf : f \in \mathcal{F}_r\} \) must be a zero-mean Gaussian process with covariance function \( E(Gf_1)(Gf_2) \), from this study \( \mathcal{F}_r \)- indexed process is \( \hat{Z}_r(y) \), then weak convergence of process \( \mathcal{F}_r \) must be a zero-mean process with covariance function \( \text{Cov}(Z_r(y), Z_r(y')) = E[\Psi(X_r, Y_r, y)\Psi(X_r, Y_r, y')] \). From Cramer - Wald device \( U_r(y) \) is \( \hat{Z}_r(y) \) where \( b_r = 1 \), then
\[ \hat{U}_r(y) = f_{xj}(y)p_j \sum_{r=1}^{2} \frac{Y_r - m(X_r)}{\sigma_j(X_r)} \left[ \frac{f_{xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r = j)}{p_j} \right] + a_r(n)^{-1/2}. \] (5)

From (5), \( \hat{U}(y) = (\hat{U}_1(y), \hat{U}_2(y)) \) converge to \( U(y) = (U_1(y), U_2(y)) \), where
\[ U_j(y) = f_{xj}(y)p_j \sum_{r=1}^{2} \frac{Y_r - m(X_r)}{\sigma_j(X_r)} \left( \frac{f_{xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r = j)}{p_j} \right) \]
are normal random variables with mean zero and covariance structure:
\[ \text{Cov}(U_j(y), U_{j'}(y)) = f_{xj}(y)f_{xj'}(y)p_j p_{j'} \sum_{r=1}^{2} E[\frac{(Y_r - m(X_r))^2}{\sigma_j(X_r)\sigma_{j'}(X_r)} \left( \frac{f_{xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r = j)}{p_j} \right) \left( \frac{f_{xj'}(X_r)}{f_{mix}(X_r)} - \frac{I(r = j')}{p_{j'}} \right)]. \]
Corollary 3.4. Assume (A1)-(A6). Then, under the null hypothesis ($H_0$),

$$U_{KU} \overline{d} \sum_{j=1}^{2} [\sup_{y}[U_j(y)] - \inf_{y}[U_j(y)]]$$

where $U_{KU} = \sum_{j=1}^{2} [\sup_{y}[\hat{U}_j(y)] - \inf_{y}[\hat{U}_j(y)]]$.

Proof. From theorem 3 and Continuous Mapping Theorem, $U_{KU} \overline{d} \sum_{j=1}^{2} [\sup_{y}[U_j(y)] - \inf_{y}[U_j(y)]]$, where $U_j(y)$ are the normal random variables with mean zero and covariance structure:

$$Cov(U_j(y), U_{j'}(y)) = f_{xj}(y)f_{xj'}(y)p_jp_{j'} \sum_{r=1}^{2} E\left[\left(\frac{Y_r - m(X_r)}{\sigma_j(X_r)\sigma_{j'}(X_r)} \left(\frac{f_{xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r = j)}{p_j}\right) - \frac{f_{xj'}(X_r)}{f_{mix}(X_r)} - \frac{I(r = j')}{p_{j'}}\right)\right] \sqrt{2/\pi},$$

and covariance structure:

$$Cov(U_j(y), U_{j'}(y)) = f_{xj}(y)f_{xj'}(y)p_jp_{j'} \sum_{r=1}^{2} E\left[\left(\frac{Y_r - m(X_r)}{\sigma_j(X_r)\sigma_{j'}(X_r)} \left(\frac{f_{xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r = j)}{p_j}\right) - \frac{f_{xj'}(X_r)}{f_{mix}(X_r)} - \frac{I(r = j')}{p_{j'}}\right)\right] \sqrt{2/\pi}.$$

§4. Bootstrap and simulations

§4.1 Bootstrap procedure

Asymptotic distribution of test statistics $U_{KU}$ given in corollary 3.4 are complicated. In practical applications, the critical values of the test statistic $U_{KU}$ can be approximated by bootstrap procedure. The bootstrap procedure can be described in the following step:

4.1.1. Assume bootstrap replication $b = 1, \ldots, B$ ($B = 200$), for $j = 1, 2, i = 1, \ldots, n_j$, let $\varepsilon_{ij,b}$ be an independent and identically distribution sample from the distribution of $(1 - h_j^2)^{1/2}V_j + h_jZ$, where $V_j$ is the distribution of $\varepsilon_{ij}$, $Z \sim N(0, 1)$, $h_j = cn_{j}^{-3/10}$ and $c=1^{10}$.

4.1.2. For $j = 1, 2, i = 1, \ldots, n_j$, the new response under the null hypothesis $Y_{ij,b}$, $b = 1, \ldots, B$, defined as follow : $Y_{ij,b} = \hat{m}_j(X_{ij}) + \hat{\sigma}_j(X_{ij})\varepsilon_{ij,b}$.

4.1.3. For $j = 1, 2, i = 1, \ldots, n_j$, calculate the test statistic $U_{KU}$ from bootstrap sample $X_{ij}, Y_{ij,b}$.

4.1.4. In step 4.1.3, let $U_{KU,b}^{*}$ be order statistic of $U_{KU,(1)}, \ldots, U_{KU,(B)}$ form 200 bootstrap replications respectively, then $U_{KU,(1-\alpha)}$ approximates the $(1 - \alpha)$ - quantile of distribution of the test statistic $U_{KU}$ under the null hypothesis.

4.1.5. The test statistic $U_{KU}$ iterates 500 trails and show the proportion of rejections.

§4.2 Simulations

4.2.1. Testing the equality of two regression functions, they are assumed in three types:
1) linear function \( Y = ax + b; \)
2) exponential function \( Y = \exp(ax) + b \) and
3) trigonometric function \( Y = \sin(2\pi x) + b \), where \( a \) and \( b \) are constants. The models for regression function are six models:

(1) linear function \( m_1(x) = m_2(x) \), where \( a_1 = a_2 = 2, b_1 = b_2 = 0; \)
(2) exponential function \( m_1(x) = m_2(x) \), where \( a_1 = a_2 = 2, b_1 = b_2 = 0; \)
(3) trigonometric function \( m_1(x) = m_2(x) \), where \( a_1 = a_2 = 2, b_1 = b_2 = 0; \)
(4) linear function \( m_1(x) = 2x, m_2(x) = 2x + 2; \)
(5) exponential function \( m_1(x) = \exp(2x), m_2(x) = \exp(2x) + 2; \)
(6) trigonometric function \( m_1(x) = \sin(2\pi x), m_2(x) = \sin(2\pi x) + 2. \)

4.2.2. The variance of two regression function are \( \sigma^2_1(x) = 0.25, \sigma^2_2(x) = 0.50. \)

4.2.3. Assume the error distribution are \( \varepsilon_{1n1} \sim N(0, 1) \) and \( \varepsilon_{2n2} \sim N(0, 1). \)

4.2.4. In all cases, the covariates \( X_1, X_2 \) are uniform distribution on the interval \([0,1]\).

4.2.5. The Kernel Function is Epanechnikov \( K(u) = 0.75(1 - U^2)I(|u| < 1). \)

4.2.6. In this paper, we consider bandwidth of the form \( h_j = cn_j^{-3/10}, \) where \( c = 1. \)

4.2.7. In each population, the samples sizes are determined \((n_1, n_2) = (20, 20), (50, 50) \) and \((100, 100). \)

§5. Results

From simulation study, we show six simulated examples with two regression functions. The model for regression function are three types: linear function, exponential function, and trigonometric function. Model (1)-(3) correspond to the null hypothesis, and model (4)-(6) correspond to the alternative hypothesis. Table 1 shows the proportion of rejections under the null hypothesis in 500 trails for sample sizes \((n_1, n_2) = (20, 20), (50, 50) \) and \((100, 100), \) \( B=200 \) bootstrap replications. The significance level are 0.05 and 0.10. Table 2 shows the proportion of rejections under the alternative hypothesis in 500 trails for sample sizes \((n_1, n_2) = (20, 20), (50, 50) \) and \((100, 100), \) \( B=200 \) bootstrap replications. The significance level are 0.05 and 0.10. In the simulation study we find out that the level is well - approximated in \((n_1, n_2) = (100,100). \) The power of the test based on \( U_{KU} \) is well - approximated in \((n_1, n_2) = (100,100). \) The behavior of the power of the test based on \( U_{KU} \) is good for models in linear form, exponential form and trigonometric form respectively.
Table 1. Rejection probabilities under the null hypothesis-model (1) to (3).

<table>
<thead>
<tr>
<th>$(n_1, n_2)$</th>
<th>regression function</th>
<th>$U_{KU}^1: \alpha = 0.05$</th>
<th>$U_{KU}^1: \alpha = 0.10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(20,20)</td>
<td>1</td>
<td>0.042</td>
<td>0.080</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.030</td>
<td>0.074</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.034</td>
<td>0.076</td>
</tr>
<tr>
<td>(50,50)</td>
<td>1</td>
<td>0.052</td>
<td>0.090</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.054</td>
<td>0.084</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.060</td>
<td>0.080</td>
</tr>
<tr>
<td>(100,100)</td>
<td>1</td>
<td>0.050</td>
<td>0.100</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>0.052</td>
<td>0.102</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>0.054</td>
<td>0.106</td>
</tr>
</tbody>
</table>

Table 2. Rejection probabilities under the alternative hypothesis-models (4) to (6).

<table>
<thead>
<tr>
<th>$(n_1, n_2)$</th>
<th>regression function</th>
<th>$U_{KU}^1: \alpha = 0.05$</th>
<th>$U_{KU}^1: \alpha = 0.10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(20,20)</td>
<td>4</td>
<td>0.910</td>
<td>0.930</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.890</td>
<td>0.930</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.384</td>
<td>0.450</td>
</tr>
<tr>
<td>(50,50)</td>
<td>4</td>
<td>0.930</td>
<td>0.965</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.926</td>
<td>0.960</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.400</td>
<td>0.520</td>
</tr>
<tr>
<td>(100,100)</td>
<td>4</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>0.970</td>
<td>0.980</td>
</tr>
<tr>
<td></td>
<td>6</td>
<td>0.546</td>
<td>0.644</td>
</tr>
</tbody>
</table>

§6. Conclusion and discussion

In this study, we introduce a procedure for testing the equality of two expectation function of non-parametric regression, the hypothesis $H_0 : m_1 = m_2$ vs $H_1 : m_1 \neq m_2$. The result was
found that, under the null hypothesis \( H_0 \), for \( j = 1, 2, \hat{U}(y) = (\hat{U}_1(y), \hat{U}_2(y)) \) converges weakly to \( U(y) = (U_1(y), U_2(y)) \), where \( U_j(y) = f_{xj}(y)p_j \sum_{r=1}^{2} \frac{Y_r - m(X_r)}{\sigma_j(X_r)} \left( \frac{f_{xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r = j)}{p_j} \right) \) are normal random variables with mean zero and covariance structure: 
\[
\text{Cov}(U_j(y), U_{j'}(y)) = f_{xj}(y)f_{xj'}(y)p_jp_{j'} \sum_{r=1}^{2} E \left[ \frac{(Y_r - m(X_r))^2}{\sigma_j(X_r)\sigma_{j'}(X_r)} \left( \frac{f_{xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r = j)}{p_j} \right) \left( \frac{f_{xj'}(X_r)}{f_{mix}(X_r)} - \frac{I(r = j')}{p_{j'}^2} \right) \right] \sqrt{2/\pi},
\]
and covariance structure:
\[
\text{Cov}(U_j(y), U_{j'}(y)) = f_{xj}(y)f_{xj'}(y)p_jp_{j'} \sum_{r=1}^{2} E \left[ \frac{(Y_r - m(X_r))^2}{\sigma_j(X_r)\sigma_{j'}(X_r)} \left( \frac{f_{xj}(X_r)}{f_{mix}(X_r)} - \frac{I(r = j)}{p_j} \right) \left( \frac{f_{xj'}(X_r)}{f_{mix}(X_r)} - \frac{I(r = j')}{p_{j'}^2} \right) \right] (1 - \frac{2}{\pi}).
\]

The result from the simulation study showed that under the null hypothesis, the regression function in linear form are well approximated in all situation, especially for the large sample sizes. The power of the test is well approximated for the large sample sizes, and the regression function in linear form provided the highest power of the test.

References


The existence of solution for $p(x)$-Laplace equation with no flux boundary 

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Abstract Using the principle least action, we investigate the existence solution for $p(x)$-Laplacian equation with no flux boundary condition in a bounded domain $\Omega \subset \mathbb{R}^N$, where no flux boundary condition is given in the following:

\[
\begin{aligned}
&u = \text{constant}, \quad x \in \partial \Omega, \\
&\int_{\partial \Omega} |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} \, ds = 0.
\end{aligned}
\]

Keywords Variable exponent Sobolev spaces, $p(x)$-Laplacian, no flux boundary, the principle least action.

§1. Introduction

In recent years there has been an increasing interest in the study of various mathematics problem with variable exponent, see the papers [1, 2, 3, 5]. The existence of solutions of $p(x)$-Laplace Dirichlet problems has been studied in many papers (see e.g. [7, 8, 9, 12, 14, 15]). The aim of the present paper is to study the existence of solutions of $p(x)$-Laplace equation with no flux boundary. Where no flux boundary condition is given in the following:

\[
\begin{aligned}
&u = \text{constant}, \quad x \in \partial \Omega, \\
&\int_{\partial \Omega} |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} \, ds = 0.
\end{aligned}
\]

Throughout the paper, $\Omega$ will be bounded domain in $\mathbb{R}^N$, $p \in C(\bar{\Omega})$ and

\[1 < p^- := \inf_{x \in \Omega} p(x) \leq p^+ := \sup_{x \in \Omega} p(x) < \infty.\]

Consider the existence of solution of the following problem

\[
\begin{aligned}
&-\Delta_{p(x)} u + f(x, u) = 0, \quad x \in \Omega, \\
&u(x) = \text{constant}, \quad x \in \Omega, \\
&\int_{\partial \Omega} |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \nu} \, ds = 0.
\end{aligned}
\]

1This paper is supported by the N. S. F. of P.R.China.
When a term of $|u|^{p(x)-2}u$ is involved in $p(x)$–Laplacian equation, we can apply variation method to obtain the existence of solution and multiplicity easily. In this paper, we may use the principle least action to get the existence of solution when $f$ satisfies some appropriate conditions.

§2. Preliminaries

Let $\Omega$ be an open subset of $\mathbb{R}^N$. On the basic properties of the space $W^{1,p(x)}(\Omega)$ we refer to [4, 13]. In the following we display some facts which we will use later.

Denote by $S(\Omega)$ the set of all measurable real functions defined on $\Omega$, and elements in $S(\Omega)$ that equal to each other almost everywhere are considered as one element. Denote $L^\infty_+(\Omega) = \{ p \in L^\infty(\Omega) : \inf_{\Omega} p(x) := p_- \geq 1 \}$.

For $p \in L^\infty_+(\Omega)$, define
\[
L^{p(x)}(\Omega) = \{ u \in S(\Omega) : \int_{\Omega} |u|^{p(x)}dx < \infty \},
\]
with the norm
\[
|u|_{L^{p(x)}(\Omega)} = |u|_{p(x)} = \inf \{ \lambda > 0 : \int_{\Omega} |u/\lambda|^{p(x)}dx \leq 1 \};
\]
and define
\[
W^{1,p(x)}(\Omega) = \{ u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega) \},
\]
with the norm
\[
\| u \|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}.
\]
Define $W^{1,p(x)}_0(\Omega)$ as the closure of $C^\infty_0(\Omega)$ in $W^{1,p(x)}(\Omega)$.

Define
\[
p^* = \begin{cases} \frac{Np(x)}{N-p(x)}, & \text{if } p(x) < N, \\ \infty, & \text{if } p(x) \geq N. \end{cases}
\]
Hereafter, we always assume that $p(x)$ is continuous and $p_- > 1$.

**Proposition 2.1.** [4,13] The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$, all are separable and reflexive Banach spaces.

**Proposition 2.2.** [4,10,13] The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p^*(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p^*(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p^*(x)}(\Omega)$, the Hölder inequality holds:
\[
\int_{\Omega} |uv|dx \leq 2\|u\|_{p(x)}\|v\|_{p^*(x)}. \tag{2}
\]

**Remark 2.3.** In the right of (2), the constants 2 is suitable, but not the best. The best constant is given in [10] denoted by $d_{(p_-p_+)}$ which only depends on $p_-$ and $p_+$ when $p(x)$ is given and $d_{(p_-p_+)}$ is smaller than $\frac{1}{p_-} + \frac{1}{p_+}$.

**Proposition 2.4.** [13] (Theorem 1.3.) Set $\rho(u) = \int_{\Omega} |u(x)|^{p(x)}dx$. For $u, u_k \in L^{p(x)}(\Omega)$, we have:
(i) $|u|_{p(x)} < 1$ ($= 1; > 1$) $\iff \rho(u) < 1$ ($= 1; > 1$);
(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p^-(x)} \leq \rho(u) \leq |u|_{p^+(x)}$; $|u|_{p^+(x)} < 1 \Rightarrow |u|_{p^+(x)} \leq \rho(u) \leq |u|_{p^-(x)}$.

(iii) $\lim_{k \to \infty} |u_k|_{p(x)} = 0$ ($= \infty$) $\iff \lim_{k \to \infty} \rho(u_k) = 0$ ($= \infty$).

**Proposition 2.5.**(11) (Theorem 1.1.) If $p: \Omega \to \mathbb{R}$ is Lipschitz continuous and $p_+ < N$, then for $q \in L^\infty(\Omega)$ with $p(x) \leq q(x) \leq p^*(x)$, there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

**Proposition 2.6.**(6) (Proposition 2.4.) $\Omega$ is bounded. Assume that the boundary of $\Omega$ possesses the cone property and $p \in C(\bar{\Omega})$. If $q \in C(\bar{\Omega})$ and $1 \leq q(x) < p^*(x)$ for $x \in \Omega$, then there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.

In this paper we use space $X := \{u \in W^{1,p(x)}(\Omega) : u|_{\partial \Omega} = \text{constant} \} = W^{1,p(x)}_0(\Omega) \oplus \mathbb{R}$. $X$ is a subspace of $W^{1,p(x)}_0(\Omega)$. The space $X$ is also separable and reflexive Banach space and with the equivalent norm

$$
\|u\|_X = \|u\|_X = \|u\| + \|\nabla u\|_{p(x)}, \quad u \in X.
$$

where $u = \bar{u} + \tilde{u}, \bar{u} \in R, \tilde{u} \in W^{1,p(x)}_0(\Omega)$.

**Definition 2.7.** We call $u \in X$ is a weak solution of (1), if $u$ satisfies

$$
\int_\Omega |\nabla u|^{p(x)} - 2\nabla u \cdot \nabla v dx + \int_\Omega |u|^{p(x)} - 2uv dx = \int_\Omega f(x, u)vdx, \quad \forall v \in X.
$$

In this paper, we always suppose $f$ satisfies the following basic assumption:

**Basic assumption.** Suppose $f$ satisfies Carathéodory, and

$$(F_0) \ |f(x, t)| \leq \alpha + \beta |t|^{q(x)-1}, \quad 1 < q(x) < p^*(x).$$

Consider the following function:

$$
I(u) = \int_\Omega \frac{1}{p(x)}|\nabla u|^{p(x)}dx + \int_\Omega F(x, u)dx, \quad u \in X,
$$

where $F(x, t) = \int_0^t f(x, s)ds$. We obtain $I \in C^1(X, R)$.

The main result of this paper is given by the following theorem.

**Theorem 2.8.** Suppose $(F_0)$ holds, $I$ has a bounded minimizing sequence. Then $I$ has a minimizer.

**Proof.** Set $\{u_n\}$ is a minimizing sequence of $I$, namely, $I(u_n) \to \inf_{n \to \infty} I(u_n), \|u_n\|$ is bounded. Suppose $u_n \to u_0$ in $X$, obviously, $I$ is weakly lower semicontinuous, thus we have $\inf_{u \in X} I(u_n) = \lim_{n \to \infty} I(u_n) \geq I(u_0)$, it follows that $I(u_0) = \inf_{u \in X} I(u)$.

§3. Main result

**Theorem 3.1.** Suppose $\alpha \in R$, and $0 \leq \alpha < p^* - 1$, for any $x \in \Omega, t \in R$. And the following assumptions hold:

$$
|f(x, t)| \leq C_1 + C_2|t|^\alpha. \quad (3)
$$

$$
|t|^{-\frac{\alpha}{p^*}} \int_\Omega F(x, t)dx \to +\infty \quad \text{if} \ |t| \to \infty. \quad (4)
$$

Then $I$ has a minimizer.
Proof. Clearly, $I$ is weakly lower semicontinuous. We shall prove $I$ is coercive.

$$I(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_\Omega (F(x, u(x)) - F(x, \bar{u})) dx + \int_\Omega F(x, \bar{u}) dx.$$ 

where

$$\int_\Omega (F(x, u(x)) - F(x, \bar{u})) dx = \int_\Omega \int_0^1 f(x, \bar{u} + s\bar{u}(x)) \bar{u}(x) ds dx$$

$$\leq \int_\Omega (C_1 + C_3|\bar{u}|^\alpha + C_4|\bar{u}(x)|^\alpha) |\bar{u}(x)| dx$$

$$\leq \int_\Omega C_1|\bar{u}(x)| dx + \int_\Omega C_3|\bar{u}|^\alpha |\bar{u}(x)| dx + \int_\Omega C_4|\bar{u}(x)|^{\alpha+1} dx$$

$$\leq I + II + III.$$

We next estimate forms $I$, $II$, $III$. Where

$$I = C_4|\bar{u}|_{L^p(\Omega)} \leq C_5|\bar{u}|_{L^{p(x)}(\Omega)} \leq C_6|\nabla \bar{u}|_{L^{p(x)}(\Omega)}.$$  (5)

$$III = C_3|\bar{u}|_{L^{p+1}} \leq C_7|\bar{u}|_{L^{p(x)}(\Omega)} \leq C_8|\nabla \bar{u}|_{L^{p(x)}(\Omega)}.$$  (6)

According to Young inequality, we deduce

$$II = C_3 \int_\Omega |\bar{u}|^\alpha |\bar{u}(x)| dx$$

$$\leq C_3 \int_\Omega (\varepsilon |\bar{u}(x)|^p + C_9(\varepsilon)|\bar{u}|^{\frac{p^*}{p-1}}) dx$$

$$\leq \varepsilon C_{10}|\nabla \bar{u}|_{L^{p(x)}(\Omega)} + C_3 C_9(\varepsilon)|\bar{u}|^{\frac{p^*}{p-1}}.$$

Fix $\varepsilon > 0$ so small that $\varepsilon C_{10} \leq \frac{1}{2p^*}$. we conclude that

$$II \leq \frac{1}{2p^*} |\nabla \bar{u}|_{L^{p(x)}(\Omega)}^{p^*} + C_{11}|\bar{u}|^{\frac{p^*}{p-1}}.$$  (7)

By (3), (4) and (5), we have

$$I(u) \geq \frac{1}{p^*} |\nabla \bar{u}|_{L^{p(x)}(\Omega)}^{p^*} - \frac{1}{p^*} - C_6|\nabla \bar{u}|_{L^{p(x)}(\Omega)} - \frac{1}{2p^*} |\nabla \bar{u}|_{L^{p(x)}(\Omega)}^{p^*} - C_{11}|\bar{u}|^{\frac{p^*}{p-1}}$$

$$- C_8|\nabla \bar{u}|_{L^{p(x)}(\Omega)}^{\alpha+1} + \int_\Omega F(x, \bar{u}) dx$$

$$\geq C_{10}|\nabla \bar{u}|_{L^{p(x)}(\Omega)}^{p^*} - C_{12} + \int_\Omega F(x, \bar{u}) dx - C_{11}|\bar{u}|^{\frac{p^*}{p-1}}$$

$$\geq C_{10}|\nabla \bar{u}|_{L^{p(x)}(\Omega)}^{p^*} - C_{12} + |\bar{u}|^{\frac{p^*}{p-1}}(|\bar{u}|^{\frac{p^*}{p-1}}(\int_\Omega F(x, \bar{u}) dx - C_{11})).$$  (8)

Using (1) and (6) we get $I$ is coercive, thus $I$ has a minimizer.

Given $\alpha = 0$ in theorem 3.1, we have

**Corollary 3.2.** Suppose $f$ satisfies

$$|f(x, t)| \leq C_1.$$  (9)
\[ \int_{\Omega} F(x, t) dx \to \infty, \text{ if } |t| \to \infty. \]  

(10)

Then \( I \) has a minimizer.

**Theorem 3.3.** Suppose \( F(x, t) \) is convex and \( \int_{\Omega} F(x, t) dx \to \infty \) when \( |t| \to \infty \). Then \( I \) has a minimizer.

**Proof.** Set \( g(t) = \int_{\Omega} F(x, t) dx \), for any \( t \in \mathbb{R} \). Obviously \( g : \mathbb{R} \to \mathbb{R} \) is a differentiable and convex function, and when \( |t| \to \infty \), \( g(t) \to +\infty \). We deduce \( g \) has a minimizer \( t_* \), thus \( g'(t_*) = 0 \). Namely \( \int_{\Omega} f(x, t_*) dx = 0 \). Suppose \( \{u_n\} \) is a minimizing sequence of \( I \), according to the proposition of \( F(x, t) \), we have

\[
I(u_n) \geq \frac{1}{p^+} \int_{\Omega} |\nabla \tilde{u}_n|^{p(x)} dx + \int_{\Omega} (F(x, t_*) + f(x, t_*))(u_n(x) - t_*) dx \\
\geq \frac{1}{p^+} \int_{\Omega} |\nabla \tilde{u}_n|^{p(x)} dx - C_1 + \int_{\Omega} f(x, t_*) \tilde{u}_n(x) dx \\
\geq \frac{1}{p^+} \int_{\Omega} |\nabla \tilde{u}_n|^{p(x)} dx - C_2 |\nabla \tilde{u}_n|_{L^{p(x)}(\Omega)}. 
\]

(11)

We derive that \( |\nabla \tilde{u}_n|_{L^{p(x)}(\Omega)} \) is bounded from (9). Next we proof \( |\tilde{u}_n|_{L^{p(x)}(\Omega)} \) is also bounded. Owing now to the convexity of \( F \), we deduce that

\[
F(x, \frac{\tilde{u}_n}{2}) = F(x, \frac{u_n(x) - \bar{u}_n(x)}{2}) \leq \frac{1}{2} (F(x, u_n(x)) + F(x, -\bar{u}_n(x))). 
\]

(12)

From (10), we have

\[
I(u_n) \geq \int_{\Omega} F(x, u_n) dx \geq 2 \int_{\Omega} F(x, \frac{\tilde{u}_n}{2}) dx - \int_{\Omega} F(X, -\bar{u}_n(x)) dx \\
\geq 2 \int_{\Omega} F(x, \frac{\tilde{u}_n}{2}) dx - \int_{\Omega} (C_{12} + C_{13} |\tilde{u}_n(x)|^{q(x)}) dx \\
\geq 2 \int_{\Omega} F(x, \frac{\tilde{u}_n}{2}) dx - C_{12}|\Omega| - C_{14} |\nabla \tilde{u}_n|^{p(x)}_{L^{p(x)}(\Omega)} - C_{15} \\
\geq 2 \int_{\Omega} F(x, \frac{\tilde{u}_n}{2}) dx - C_{16}. 
\]

(13)

We get that \( |\tilde{u}_n|_{L^{p(x)}(\Omega)} \) is bounded, thus \( \|u_n\| \) bounded. Consequently, \( I \) has a minimizer.

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**References**


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The rainbow $k$-connectivity of random graphs

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Abstract A path in an edge-colored graph (not necessarily a proper coloring) is a rainbow path if no two edges of it are colored the same. For an $l$-connected graph $G$ and $1 \leq k \leq l$, the rainbow $k$-connectivity $rc_k(G)$ of $G$ is the minimum integer $j$ for which there exists a $j$-edge-coloring of $G$ such that every two different vertices in $G$ are connected by $k$ independent rainbow paths. We prove that, in a random graph $G(n, p)$, $p = \sqrt{\ln n/n}$ is a sharp threshold function for the property $rc_k(G(n, p)) \leq 2$ when $k = O(\ln \ln n)$.

Keywords Rainbow connectivity, rainbow connection, graph coloring, random graph.

§1. Introduction

In this note, we use [1] for graph theory terminology not described here and consider only undirected simple graphs. Let $G$ be a nontrivial connected graph on which is defined an edge-coloring $f : E(G) \rightarrow \{1, 2, \ldots, k\}$, $k \in \mathbb{N}$, where adjacent edges may be colored the same. A path in $G$ is a rainbow path if no two edges of it are colored the same. The graph $G$ is rainbow-connected (with respect to $f$) if every two different vertices in $G$ are connected by a rainbow path [3]. The minimum integer $k$ for which there exists a $k$-coloring of the edges of $G$ that results in a rainbow-connected graph is called the rainbow connection number $rc(G)$ of $G$.

For two different vertices in a connected graph $G$, a rainbow geodesic in $G$ is a rainbow path which has the length of a shortest path between them in $G$ [3]. The graph $G$ is strongly rainbow-connected if every two different vertices in $G$ are connected by a rainbow geodesic. The minimum $k$ for which there exists a $k$-coloring of the edges of $G$ such that $G$ is strongly rainbow-connected is the strong rainbow connection number $src(G)$ of $G$. Clearly, $diam(G) \leq rc(G) \leq src(G)$ for every connected graph $G$, where $diam(G)$ denotes the diameter of $G$. Also note that $rc(G) = 1$ if and only if $G$ is a complete graph.

The authors of [4] introduce a concept of rainbow connectivity recently. Suppose that $G$ is an $l$-connected graph for $l \geq 1$. For $1 \leq k \leq l$, the rainbow $k$-connectivity $rc_k(G)$ of $G$ is the minimum integer $j$ for which there exists an $i$-coloring of the edges of $G$ such that for every two different vertices of $G$, there is at least $k$ independent rainbow paths between them. Note that Menger’s Theorem justifies the definition (see e.g. [1] pp.52) and $rc(G) = rc_1(G)$. Moreover, $rc_k(G) \leq rc_j(G)$ for $1 \leq k \leq j \leq l$. 

Observe that \( rc_k(G) \) is a monotonic property in the sense that if we add an edge to \( G \) we cannot increase its rainbow connectivity. Hence, it is natural to explore the random graph setting \([2,7,8]\). Motivating this idea, in this paper we mainly consider the rainbow connectivity in Erdős–Rényi random graph model \( G(n, p) \) with \( n \) vertices and edge probability \( p \in [0, 1] \). A function \( f(n) \) is called a sharp threshold function for a graph property \( \mathcal{A} \) if there are two positive constants \( C \) and \( c \) such that \( G(n, p) \) satisfies \( \mathcal{A} \) almost surely for \( p \geq Cf(n) \) and \( G(n, p) \) almost surely does not satisfy \( \mathcal{A} \) for \( p \leq cf(n) \). A remarkable feature of random graphs is that all monotone graph properties have sharp thresholds (see e.g. \([5, 6]\)).

All the logarithms in this paper have natural base. Let \( \delta(G) \) be the minimum degree of graph \( G \). We establish the following results:

**Theorem 1.1.** Let \( k \in \mathbb{N} \) and \( k = O(\ln \ln n) \). Then, any non-complete graph \( G \) on \( n \) vertices with \( \delta(G) \geq n/2 + (3/2 \ln 2) \ln n + O(\ln \ln n) \) has \( rc_k(G) = 2 \).

**Theorem 1.2.** Let \( k \in \mathbb{N} \) and \( k = O(\ln \ln n) \). Then, \( p = \sqrt{\ln n/n} \) is a sharp threshold function for the property \( rc_k(G(n, p)) \leq 2 \).

The rest of the paper is organized as follows. In section 2, we provide the proof of theorem 1.1 and 1.2. In section 3, we give a remark on the strong rainbow connection number \( src(G) \) in a random graph.

## §2. Proofs

In this section, we show theorem 1.1 and theorem 1.2 with similar reasoning of \([2]\). Theorem 1.1 is a deterministic result which is proved in a probabilistic argument.

**Proof of Theorem 1.1.** For a graph \( G \) on \( n \) vertices and \( \delta(G) \geq n/2 + (3/2 \ln 2) \ln n + O(\ln \ln n) \), we randomly color the edges with two colors, red and blue. We will show that with positive probability, such a random coloring makes \( G \) rainbow \( k \)-connected for all \( k = O(\ln \ln n) \).

Given two vertices \( x \) and \( y \) in \( G \), the minimum degree requirement forces \( x \) and \( y \) to have more than \((3/\ln 2) \ln n + O(\ln \ln n)\) common neighbors whether \( x \) and \( y \) are adjacent or not. Hence, the graph \( G \) is \([(3/\ln 2) \ln n + O(\ln \ln n)]\)-connected by Menger’s Theorem \([1]\). Thus, we can well define rainbow connectivity \( rc_k(G) \) when \( k = O(\ln \ln n) \).

Now, for each such common neighbor \( z \), the probability that the path \( x, z, y \) is not a rainbow path is \( 1/2 \). Since the paths corresponding to distinct common neighbors are independent, the probability that there are at least \((3/\ln 2) \ln n\) paths of them being not rainbow path is bounded above by

\[
\left( \frac{1}{2} \right)^{(3/\ln 2) \ln n} \cdot \left( \frac{3/\ln 2 + O(\ln \ln n)}{O(\ln \ln n)} \right) \leq \frac{1}{n^2}, \quad n = n^{-2}
\]

(1)

involving the useful bound \( \binom{n}{2} \leq (en/m)^m \), valid for all positive \( n \) and \( m \).

Since there are \( \binom{n}{2} \) pairs \( x, y \) to consider, it follows from the union bound and (1) that with probability more than \( 1/2 \), each pair of vertices in \( G \) are connected by \( O(\ln \ln n) \) independent rainbow paths.

**Proof of Theorem 1.2.** Recall the definition of sharp threshold function in section 1. The proof consists of two parts.
For the first part of the theorem, we want to prove that for a sufficiently large constant \( C \), the random graph \( G(n, p) \) with \( p = C \sqrt{\ln n / n} \) almost surely has \( r_{ck}(G(n, p)) \leq 2 \). From the proof of theorem 1.1, we only need to show that almost surely any two vertices in \( G(n, p) \) have at least \( (3 / \ln 2) \ln n + O(\ln \ln n) \) common neighbors.

Fix a pair of vertices \( x, y \), and the probability that \( z \) is a common neighbor of them is \( C^2 \ln n / n \). Let random variable \( X \) represents the number of common neighbors of \( x \) and \( y \). Consequently, we obtain \( EX = (n - 2)(C^2 \ln n / n) \). By using the Chernoff bound (e.g. [7] pp.26), for large enough \( C \), we have

\[
P(X < (3 / \ln 2) \ln n + O(\ln \ln n)) \leq P(X < EX - C^2 \ln n / 4) = e^{-C^2 \ln n / 32} = o(n^{-2}).
\]

Since there are \( \binom{n}{2} \) pairs of vertices in \( G(n, p) \), the union bound readily yields the result.

For the other direction, it suffices to show that for a sufficiently small constant \( c \), the random graph \( G(n, p) \) with \( p = c \sqrt{\ln n / n} \) almost surely has \( \text{diam}(G(n, p)) \geq 3 \).

We give a remark on the strong rainbow connection number \( src(G(n, p)) \) of random graph \( G(n, p) \). Observe that \( src(G) \) is also a monotonic property in the sense that if we add an edge to \( G \) we cannot increase its strong rainbow connection number. We may establish the following result regarding the threshold function of \( src(G(n, p)) \).
Theorem 3.1. \( p = \sqrt{\ln n/n} \) is a sharp threshold function for the property \( src(G(n,p)) \leq 2 \).

Proof. Recall that \( diam(G) \leq rc_1(G) \leq src(G) \). The proof of theorem 1.1 and 1.2 in conjunction with proposition 1.1 in [3] yields the result directly.

References

The power-serieswise Armendariz rings with nilpotent subsets

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Abstract In this note, a new kind of power-serieswise Armendariz ring with nilpotent subsets was obtained. Necessary and sufficient condition for some rings without identity to be power-serieswise Armendariz are also showed.

Keywords Armendariz ring, power-serieswise Armendariz ring, ring with nilpotent subsets.

§1. Introduction

Throughout this paper $R$ denotes an associative ring. M. B. Rege and S. Chhawchharia introduced the notion of an Armendariz ring. They defined a ring $R$ to be an Armendariz ring if whenever polynomials $f(x) = a_0 + a_1 x + \cdots + a_n x^n$, $g(x) = b_0 + b_1 x + \cdots + b_m x^m \in R[x]$ satisfy $f(x)g(x) = 0$, then $a_i b_j = 0$ for each $i, j$. The name “Armendariz ring” was chosen because E. Armendariz had noted that a reduced ring satisfies this condition. After that, more comprehensive study of this kind of ring was carried out (see Anderson and Camillo, 1998; Hirano, 2002; Huh et al., 2002; Kim and Lee, 2000; Lee and Wong, 2003; Rege and Chhawchharia, 1997). Various known works on Armendariz rings deal with the Armendariz property of some extensions of an Armendariz ring $R$, such as $R[x]$, $R[x]/(x^n)$, the trivial extensions of $R \times M$ where $M$ is a bimodule over $R$, and the classical quotient ring $Q(R)$ (where $R$ satisfies the right Ore condition). Some Armendariz subrings of the matrix ring were studied in Lee and Zhou (2004), Wang (2006) and Yan (2003). Recently, the concept of Armendariz ring was connected with power series ring by Kim et al. (2006). They defined a ring $R$ to be a power-serieswise Armendariz ring if $a_i b_j = 0$ for all $i, j$ whenever power series $f(x) = \sum_{i=0}^{\infty} a_i x^i$, $g(x) = \sum_{j=0}^{\infty} b_j x^j$ in $R[[x]]$ satisfy $f(x)g(x) = 0$. They gave two kinds of power-serieswise Armendariz rings, one is the reduced ring and the other is not. The structure of the set of nilpotent elements in Armendariz rings and the concept of nil-Armendariz as a generalization were introduced by Antoine (2008). In this note we will give a new kind of power-serieswise Armendariz ring which is not reduced. The connection and difference between the three kinds of power-serieswise Armendariz rings are also showed. Some rings which are not Armendariz when they have identity are also considered. Necessary and sufficient condition for this ring without identity to be power-serieswise Armendariz are obtained.
§2. A new kind of power-serieswise Armendariz ring

Reduced ring is one of the earliest and most important Armendariz ring. With the development of study, the Armendariz rings with nilpotent subsets caught more and more one’s attention. The connection between power-serieswise Armendariz ring and ring with nilpotent subsets is showed below:

**Proposition 2.1.** Let $R$ be a ring. If $R^2 = 0$, then $R$ is power-serieswise Armendariz.

**Proposition 2.2.** Let $R$ be a power-serieswise Armendariz ring, then one of the following statements holds:
1) $R$ is reduced;
2) $R$ is not reduced and has a non-zero ideal $I$ with $I^2 = 0$.

**Proof.** If $R$ is not reduced, let $I = \{a \in R| a^2 = 0\}$, Anderson and Camillo (1998) showed $I$ is a non-zero ideal of $R$ and $I^2 = 0$.

Kim et al. (2006) proposition 3.7 showed that: $R$ be a ring and $J$ be an ideal of $R$ such that every element in $R \setminus J$ is regular and $J^2 = 0$, then $R$ is a power-serieswise Armendariz ring with nilpotent subsets. We give an equivalent characterization of this ring.

**Proposition 2.3.** Let $R$ be a ring. Then $R$ satisfies Kim et al. (2006) proposition 3.7 if and only if there is an ideal $I$ of $R$ satisfies:
1) $I^2 = 0$;
2) For non-zero elements $x_1$, $x_2$, $x_3$, $x_4$ in $R$, if $x_1x_2 = 0$, $x_1x_4 + x_3x_2 = 0$ and $x_3x_4 \in I$, then $x_1$, $x_2$, $x_3$, $x_4$ are all in $I$.

**Proof.** Suppose $R$ satisfies Kim et al. (2006) proposition 3.7, let $I = J$, where $J$ is the ideal in Kim et al. (2006) proposition 3.7. Then (1) holds. For non-zero elements $x_1$, $x_2$, $x_3$, $x_4$ in $R$, if $x_1x_2 = 0$, then $x_1$ and $x_2$ are in $I$. If $x_3x_4 = 0$, then $x_3$ and $x_4$ are in $I$. Assume to the contrary that $x_3x_4 \neq 0$. If $x_3x_4x_3 = 0$, then $x_3 \in I$ and $x_1x_4 + x_3x_2 = x_1x_4 = 0$, so $x_4 \in I$, which is a contradiction. If $x_3x_4x_3 \neq 0$, then $x_3x_4x_3x_4 = 0$ implies $x_4 \in I$. Then $x_1x_4 + x_3x_2 = x_3x_2 = 0$ implies $x_3 \in I$, which is a contradiction.

Conversely, let $J = I$, then $J^2 = 0$. We show that $R \setminus J$ is regular. For $a$ in $R \setminus J$, assume to the contrary there is $0 \neq b$ in $R$ such that $ab = 0$. Let $x_1 = a$, $x_2 = b$, $x_3 = -a$, $x_4 = b$, then (2) implies that $a$ and $b$ are in $J$, which is a contradiction. By the same method we obtain $a$ is not a right divisor of zero.

We now give another kind of power-serieswise Armendariz ring:

**Proposition 2.4.** If a non-empty subset $A$ of $R$ satisfies:
1) $A^2 = 0$;
2) If $ab = 0$ for $a$, $b$ in $R$, then at least one of the following statements holds:
   i) $a \in A$ and $aR = 0$;
   ii) $b \in A$ and $Rb = 0$.

then $R$ is a power-serieswise Armendariz ring.

**Proof.** Let $f(x) = \sum_{i=0}^{\infty} a_ix^i$, $g(x) = \sum_{j=0}^{\infty} b_jx^j$ be in $R[[x]]$ and $fg = 0$, then $a_0b_0 = 0$. By hypothesis,
if $a_0 \in A$, $b_0 \notin A$, then $a_0b_j = 0$ and
\[
0 = f(x)g(x) = \left( \sum_{i=1}^{\infty} a_i x^i \right) \left( \sum_{j=0}^{\infty} b_j x^j \right).
\]

if $a_0 \notin A$, $b_0 \in A$, then $a_i b_0 = 0$ and
\[
0 = f(x)g(x) = \left( \sum_{i=0}^{\infty} a_i x^i \right) \left( \sum_{j=1}^{\infty} b_j x^j \right).
\]

if $a_0 \in A$, $b_0 \in A$, then $a_0b_j = 0$, $a_i b_0 = 0$ and
\[
0 = f(x)g(x) = \left( \sum_{i=1}^{\infty} a_i x^i \right) \left( \sum_{j=1}^{\infty} b_j x^j \right).
\]

By the same method we obtain $a_i b_j = 0$ for all $i, j$.

We write $M_n(R)$ and $T_n(R)$ for the $n \times n$ matrix ring and the $n \times n$ upper triangular matrix ring over $R$. The $n \times n$ identity matrix is denoted by $I_n$.

**Example 2.5.** Let $R = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\} \subseteq T_2(\mathbb{Z}_2)$.

Then $R$ is a subring of $T_2(\mathbb{Z}_2)$. Let $A = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}$, then $A$ is the nonempty subset of $R$ in proposition 2.4, so $R$ is power-serieswise Armendariz. $R$ is not reduced and the only non-zero ideal of $R$ is $A$. $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in R \setminus A$, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$ imply that $R$ doesn’t satisfy Kim et al. (2006) proposition 3.7.

**Example 2.6.** $\mathbb{Z}_4$ is not reduced and the nonempty nilpotent subsets are $\{[0]\}$ and $I = \{[0], [2]\}$. $[2][2]=0$ and $[2][3] \neq 0$ imply that $\mathbb{Z}_4$ doesn’t satisfy proposition 2.4. But $I$ is the non-zero ideal of $R$ in Kim et al. (2006) proposition 3.7.

The two examples above can be applied to show the difference between reduced ring, the ring in Proposition 2.4 and the ring in Kim et al (2006) proposition 3.7. We now consider the connection between them. It is clear that the only nonempty nilpotent subset in a reduced ring is $\{0\}$. A reduced ring satisfied proposition 2.4 is the ring without divisors; A reduced ring $R$ satisfied Kim et al. (2006) proposition 3.7 is also the ring without divisors; A ring $R$ satisfied both proposition 2.3 and proposition 2.4 is either the ring without zero divisors or the ring with $R^2 = 0$.

§3. **Power-serieswise Armendariz extension rings**

In this section, we study the power-serieswise Armendariz property of some extensions of the ring $R$ with $R^2 = 0$. Kim and Lee (2000) example 1 showed that if $R$ has identity, then $T_2(R)$ are not Armendariz (hence not power-serieswise Armendariz). But for a ring without identity, we have:
Proposition 3.1. Let $R$ be a ring without identity, then $T_2(R)$ is power-serieswise Armendariz if and only if $R^2 = 0$.

**Proof.** Suppose that $T_2(R)$ is power-serieswise Armendariz. Assume to the contrary that $R^2 \neq 0$. Then there exists $a, b$ in $R$ such that $ab \neq 0$.

Let $f(x) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -a \\ 0 & 0 \end{pmatrix} x$, $g(x) = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} x$. Then $f(x)$ and $g(x)$ are in $T_2(R)[x]$ and $f(x)g(x) = 0$. But $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \neq 0$, which is a contradiction, therefore $R^2 = 0$.

Conversely, suppose that $R^2 = 0$. Then $(T_2(R))^2 = 0$ and the result follows from proposition 2.1.

Some notation we will use below can be found in Lee and Zhou (2004). They showed that for a ring $R$ with identity, $B_n^o(R)$ is not an Armendariz ring for $n = 2k \geq 2$, $B_n^o(R)$ is not an Armendariz ring for $n = 2k + 1 \geq 3$, $B_n(R)$ is not an Armendariz ring for $n \geq 2$. We have:

Proposition 3.2. Let $R$ be a ring without identity. The following hold:

1. For $n = 2k \geq 2$, $B_n^o(R)$ is a power-serieswise Armendariz ring if and only if $R^2 = 0$.
2. For $n = 2k + 1 \geq 3$, $B_n^o(R)$ is a power-serieswise Armendariz ring if and only if $R^2 = 0$.
3. For $n \geq 2$, $B_n(R)$ is a power-serieswise Armendariz ring if and only if $R^2 = 0$.

**Proof.** Suppose $R^2 = 0$ then the result is clear.

Conversely, assume to the contrary that $R^2 \neq 0$, then there exists $a, b$ in $R$ such that $ab \neq 0$ and

$$[aE_{1,k} + (aE_{1,k} - aE_{1,k+1})x][bE_{k+1,n} + (bE_{k,n} + bE_{k+1,n})x] = 0,$$

$$[aE_{1,k+1} + (aE_{1,k+1} - aE_{1,k+2})x][bE_{k+2,n} + (bE_{k+1,n} + bE_{k+2,n})x] = 0.$$ 

So (1) and (2) hold. (3) can be implied by (1) and (2).

Lee and Zhou (2004) theorem 1.4 showed that if $R$ is a reduced ring, then $A_n(R)$ is an Armendariz ring for every $n = 2k + 1 \geq 3$. We have:

**Theorem 3.3.** Let $R$ be a ring without identity. The following hold:

1. For $n = 2k \geq 4$ and positive integer $i, j$ satisfied $k \geq i \geq 1, k \geq j \geq i$ and $i \neq 1$ when $j = k$, any subring of $M_n(R)$ containing $A_n(R) + RE_{ij}$ is power-serieswise Armendariz if and only if $R^2 = 0$.
2. For $n = 2k + 1 \geq 3$ and positive integer $i, j$ satisfied $k + 1 \geq i \geq 1, k + 1 \geq j \geq i$ and $i \neq 1$ when $j = k + 1$, any subring of $M_n(R)$ containing $A_n(R) + RE_{ij}$ is power-serieswise Armendariz if and only if $R^2 = 0$.
3. If $n \geq 3$ is an even number, let $n = 2k$: If $n \geq 3$ is an odd number, let $n = 2k + 1$. For positive integer $i, j$ satisfied $n \geq i \geq k + 1, n \geq j \geq i$ and $i \neq k + 1$ when $j = n$, any subring of $M_n(R)$ containing $A_n(R) + RE_{ij}$ is power-serieswise Armendariz if and only if $R^2 = 0$.

**Proof.** (1) Let $a, b$ be in $R$. When $i = 1, n = 2k \geq 4$, we have

$$[aE_{ij} + (aE_{ij} - aV^{k-1})x][bE_{kn} + (bE_{kn} + bE_{jn})x] = 0.$$
When $i = 1$, $n = 2k + 1 \geq 3$, we have
\[ [aE_{1j} + (aE_{1j} - aE_{1,k+1})x][bE_{k+1,n} + (bE_{k+1,n} + bE_{jn})x] = 0. \]

When $i > 1$, we have
\[ [(aE_{ij})] + (aE_{ij} - aE_{j+1}x)[bE_{j+1,n} + (bE_{j+1,n} + bE_{jn})x] = 0. \]

Then the result is clear.

(2) When $j = n$, we have
\[ [aE_{1,k+1} + (aE_{1,k+1} - aE_{1i})x][bE_{i,n} + (bE_{i,n} + bE_{jn})x] = 0. \]

When $j < n$, we have
\[ [aE_{1i} + (aE_{1i} - aE_{1,n-j+i})x][(bV^{j-i} - bE_{ij}) + (bV^{j-i} - bE_{ij} + bV^{n-i})x] = 0. \]

Then the result is clear.

**Corollary 3.4.** For every $n \geq 2$, $T_n(R)$ is power-serieswise Armendariz if and only if $R^2 = 0$.

**Corollary 3.5.** For every $n \geq 2$, $M_n(R)$ is power-serieswise Armendariz if and only if $R^2 = 0$.

Let $R$ be a ring, $R_n = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}$.

Kim and Lee (2000) showed that $R_1, R_2$ and $R_3$ are Armendariz when $R$ is reduced. If $n \geq 4$, $|R| \geq 2$ and $R$ have identity, then $R_4$ is not Armendariz. We have:

**Corollary 3.6.** Let $R$ be a ring without identity. For every $n \geq 4$, $R_n$ is a power-serieswise Armendariz ring if and only if $R^2 = 0$.

Let $R$ be a ring, $S = R \oplus R$. Define $\alpha : S \to S$ by $\alpha((a, b)) = (b, a)$, then $\alpha$ is an automorphism of $S$. Kim and Lee (2000) showed that the skew polynomial ring $S[x; \alpha]$ is not Armendariz when $R$ has identity, we have:

**Proposition 3.7.** Let $R$ be a ring without identity. $S[x; \alpha]$ is power-serieswise Armendariz if and only if $R^2 = 0$.

**Proof.** Suppose $R^2 = 0$, then $(S[x; \alpha])^2 = 0$. The result is clear by proposition 2.1.

Conversely, suppose that $S[x; \alpha]$ is power-serieswise Armendariz. Assume to the contrary that $ab \neq 0$ for some $a, b$ in $R$. Let $f(y) = (a, 0) + [(a, 0)x]y$, $g(y) = (0, -b) + [(b, 0)x]y$. Then $f(y), g(y)$ are in $S[x; \alpha][y]$ and $f(y)g(y) = 0$, but $[(a, 0)x](0, -b) \neq 0$, which is a contradiction.

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References


ν—open mappings

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Abstract The aim of the paper is to study basic properties of ν—open mappings and inter-relations with other mappings.

Keywords ν—open mappings.

§1. Introduction

Mappings play an important role in the study of modern mathematics, especially in Topology and Functional Analysis. Open mappings are one such mappings which are studied for different types of open sets by various mathematicians for the past many years. In this paper we tried to study a new variety of open maps called ν—open maps. Throughout the paper X, Y means a topological spaces (X, τ) and (Y, σ) unless otherwise mentioned without any separation axioms defined in it.

§2. Preliminaries

Definition 2.1. A ⊂ X is called
(i) pre-open if A ⊆ (A) o and pre-closed if (A) o ⊆ A;
(ii) semi-open if A ⊆ (A) o and semi-closed if (A) o ⊆ A;
(iii) semipre-open [β-open] if A ⊆ ((A) o) and semipre-closed [β-closed] if ((A) o) ⊆ A;
(iv) α-open if A ⊆ ((A) o) and α-closed if ((A) o) ⊆ A;
(v) regular open if A = (A) o and regular closed if A = (A) o;
(vi) ν-open if there exists a regular open set U ⊃ U ⊆ A ⊆ U;
(vii) regular α—closed [α-closed] if there exists a regular closed set U ⊃ α(U) ⊆ A ⊆ U;
(viii) generalized closed [regular generalized closed] if A ⊆ U whenever A ⊆ U and U is open [resp: regular open].

Note 1. From the above definition we have the following implication diagram.
\[ ra\text{-open} \rightarrow \nu\text{-open} \]

Regular open \rightarrow \text{semi open} \rightarrow \beta\text{-open}

\[ \text{open} \rightarrow \alpha\text{-open} \]

\[ \text{pre-open} \]

**Definition 2.2.** A function \( f : X \rightarrow Y \) is said to be

1. continuous [resp: pre-continuous; semi-continuous; \( \beta \)-continuous; \( \alpha \)-continuous; nearly-continuous; \( \nu \)-continuous; \( ra \)-continuous] if the inverse image of every open set is open [resp: pre-open; semi-open; \( \beta \)-open; \( \alpha \)-open; regular-open; \( \nu \)-open; \( ra \)-open];

2. irresolute [resp: pre-irresolute; \( \beta \)-irresolute; \( \alpha \)-irresolute; nearly-irresolute; \( \nu \)-irresolute; \( ra \)-irresolute] if the inverse image of every semi-open [resp: pre-open; \( \beta \)-open; \( \alpha \)-open; regular-open; \( \nu \)-open; \( ra \)-open] set is semi-open [resp: pre-open; \( \beta \)-open; \( \alpha \)-open; regular-open; \( \nu \)-open; \( ra \)-open];

3. open [resp: pre-open; semi-open; \( \beta \)-open; \( \alpha \)-open; nearly-open; \( ra \)-open] if the image of every open set is open [resp: pre-open; semi-open; \( \beta \)-open; \( \alpha \)-open; regular-open; \( ra \)-open];

4. g-continuous [resp: rg-continuous] if the inverse image of every closed set is g-closed [resp: rg-closed].

**Definition 2.3.** \( X \) is said to be \( T_{1}^{2} \) [\( r - T_{1}^{2} \)] if every [regular-] generalized closed set is [regular-] closed.

§3. \( \nu \)-open mappings

**Definition 3.1.** A function \( f : X \rightarrow Y \) is said to be \( \nu \)-open if image of every open set in \( X \) is \( \nu \)-open in \( Y \).

**Theorem 3.1.**

(i) Every r-open map is \( \nu \)-open but not conversely.

(ii) Every r-open map is \( ra \)-open but not conversely.

(iii) Every \( ra \)-open map is \( \nu \)-open but not conversely.

(iv) Every \( \nu \)-open map is semi-open but not conversely.

(v) Every \( \nu \)-open map is \( \beta \)-open but not conversely.

(vi) Every r-open map is open but not conversely.

(vii) Every r-open map is semi-open but not conversely.

**Proof.** (i) \( f \) is r-open \( \Rightarrow \) image of every open set is r-open \( \Rightarrow \) image of every open set is \( \nu \)-open [since every r-open set is \( \nu \)-open] \( \Rightarrow f \) is \( \nu \)-open.

Similarly we can prove the remaining parts using definition 2.1 and note 1.

**Example 1.** Let \( X = Y = \{a, b, c\} ; \tau = \{\phi, \{a\}, X\} \) and \( \sigma = \{\phi, \{a\}, \{a, b\}, X\} \) and let \( f : X \rightarrow Y \) is identity map. Then \( f \) is open, semi-open and \( \nu \)-open but not r-open and \( ra \)-open.

**Example 2.** Let \( X = Y = \{a, b, c\} ; \tau = \{\phi, \{a\}, \{a, b\}, X\} \) and \( \sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, X\} \). Let \( f : X \rightarrow Y \) is identity map. Then \( f \) is semi open but not \( \nu \)-open.

**Example 3.** In example 2 above, \( f \) is \( \beta \)-open but not \( \nu \)-open.
Example 4. Let $X = Y = \{a, b, c, d\}; \tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Let $f: X \to Y$ be identity map. Then $f$ is open but not $r$-open and $\nu$-open.

Example 5. Let $X = Y = \{a, b, c\}; \tau = \{\phi, \{a\}, \{a, b\}, X\}$ and $\sigma = \{\phi, \{a\}, X\}$. Let $f: X \to Y$ be identity map. Then $f$ is not open, semi-open, $r$-open and $\nu$-open.

Theorem 3.2.
(i) If $R\alpha O(Y) = RO(Y)$, then $f$ is $r\alpha$-open iff $f$ is $r$-open.
(ii) If $R\alpha O(Y) = \nu O(Y)$, then $f$ is $r\alpha$-open iff $f$ is $\nu$-open.
(iii) If $\nu O(Y) = RO(Y)$, then $f$ is $r$-open iff $f$ is $\nu$-open.
(iv) If $\nu O(Y) = \alpha O(Y)$, then $f$ is $\alpha$-open iff $f$ is $\nu$-open.

Corollary 3.1.
(i) Every $r\alpha$-open map is semi-open and hence $\beta$-open but not conversely.
(ii) Every $r$-open map is $\beta$-open but not conversely.

Note 2.
(i) open map and $\nu$-open map are independent to each other.
(ii) $\alpha$-open map and $\nu$-open map are independent to each other.
(iii) pre-open map and $\nu$-open map are independent to each other.

Example 6. In example 2 above, $f$ is open; pre-open and $\alpha$-open but not $\nu$-open.

Example 7. Let $X = Y = \{a, b, c\}; \tau = \{\phi, \{a, c\}, X\}$ and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$. Let $f: X \to Y$ be identity map. Then $f$ is $\nu$-open but not open; pre-open and $\alpha$-open.

Example 8. Let $X = Y = \{a, b, c\}; \tau = \{\phi, \{a, b\}, X\}$, and $\sigma = \{\phi, \{a\}, \{b\}, \{a, b\}, Y\}$. $f: (X, \tau) \to (Y, \sigma)$ be defined by $f(a) = c; f(b) = b; f(c) = a$; is not $\nu$-open and $r$-open.

Example 9. Let $X = Y = \{a, b, c\}; \tau = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}$, and $\sigma = \{\phi, \{a\}, \{a, b\}, Y\}$. $f: (X, \tau) \to (Y, \sigma)$ be the identity map then $f$ is not $\nu$-open and $r$-open.

Note 3. We have the following implication diagram among the open maps.

```
\text{ro-map} \rightarrow \text{po-map} \quad \rightarrow \text{so-map} \rightarrow \beta \text{po-map} \quad \rightarrow \text{oo-map} \quad \rightarrow \text{oo-map}
```

Theorem 3.3.
(i) If $f$ is open and $g$ is $\nu$-open then $g \circ f$ is $\nu$-open.
(ii) If $f$ is open and $g$ is $r$-open then $g \circ f$ is $\nu$-open.
(iii) If $f$ and $g$ are $r$-open then $g \circ f$ is $\nu$-open.
(iv) If $f$ is $r$-open and $g$ is $\nu$-open then $g \circ f$ is $\nu$-open.

Proof. (i) Let $A$ be open set in $X \Rightarrow f(A)$ is open in $Y \Rightarrow g(f(A))$ is $\nu$-open in $Z \Rightarrow g \circ f(A)$ is $\nu$-open in $Z \Rightarrow g \circ f$ is $\nu$-open.

Similarly we can prove the remaining parts and so omitted.

Corollary 3.2.
(i) If $f$ is open and $g$ is $\nu$-open [r-open] then $g \circ f$ is semi-open and hence $\beta$-open.
(ii) If $f$ and $g$ are $r$-open then $g \circ f$ is semi-open and hence $\beta$-open.
(iii) If $f$ is $r$-open and $g$ is $\nu$-open then $g \circ f$ is semi-open and hence $\beta$-open.

**Theorem 3.4.** If $f: X \rightarrow Y$ is $\nu$-open, then $f(A^o) \subset \nu(f(A))^o$.

**Proof.** Let $A \subset X$ and $f: X \rightarrow Y$ is $\nu$-open gives $f(A^o)$ is $\nu$-open in $Y$ and $f(A^o) \subset f(A)$ which in turn gives

$$\nu(f(A^o))^o \subset \nu(f(A))^o. \quad (1)$$

Since $f(A^o)$ is $\nu$-open in $Y$,

$$\nu(f(A^o))^o = f(A^o). \quad (2)$$

Combaining (1) and (2) we have $f(A^o) \subset \nu(f(A))^o$ for every subset $A$ of $X$.

**Remark.** Converse is not true in general, as shown by the following:

**Example 10.** Let $X = Y = \{a, b, c\}$; $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$, and

$$\sigma = \{\emptyset, \{a\}, \{b\}, \{a, b\}, Y\}.$$

$f: (X, \tau) \rightarrow (Y, \sigma)$ be the identity map then $f(A^o) \subset \nu(f(A))^o$ for every subset $A$ of $X$ but $f$ is not $\nu$-open. Since $f(\{a, b\}) = \{a, b\}$ is not $\nu$-open.

**Corollary 3.3.** If $f: X \rightarrow Y$ is $r$-open, then $f(A^o) \subset \nu(f(A))^o$.

**Theorem 3.5.** If $f: X \rightarrow Y$ is $\nu$-open and $A \subset X$ is open, then $f(A)$ is $\tau_{\nu}$-open in $Y$.

**Proof.** Let $A \subset X$ and $f: X \rightarrow Y$ is $\nu$-open implies $f(A^o) \subset \nu(f(A))^o$ which in turn implies $\nu(f(A))^o \supset f(A)$, since $f(A) = f(A^o)$. But $\nu(f(A))^o \subset f(A)$. Combaining we get $f(A) = \nu(f(A))^o$. Therefore $f(A)$ is $\tau_{\nu}$-open in $Y$.

**Corollary 3.4.** If $f: X \rightarrow Y$ is $r$-open, then $f(A)$ is $\tau_{r}$-open in $Y$ if $A$ is $r$-open set in $X$.

**Theorem 3.6.** If $\nu(A)^o = r(A)^o$ for every $A \subset Y$, then the following are equivalent:

(i) $f: X \rightarrow Y$ is $\nu$-open map.

(ii) $f(A^o) \subset \nu(f(A))^o$.

**Proof.** (i) $\Rightarrow$ (ii) follows from theorem 3.4.

(ii) $\Rightarrow$ (i) Let $A$ be any open set in $X$, then $f(A) = f(A^o) \supset \nu(f(A))^o$ by hypothesis. We have $f(A) \subset \nu(f(A))^o$. Combaining we get $f(A) = \nu(f(A))^o = \nu(f(A))^o [\text{by given condition}]$ which implies $f(A)$ is $r$-open and hence $\nu$-open. Thus $f$ is $\nu$-open.

**Theorem 3.7.** $f: X \rightarrow Y$ is $\nu$-open iff for each subset $S$ of $Y$ and each open set $U$ containing $f^{-1}(S)$, there is a $\nu$-open set $V$ of $Y$ such that $S \subset V$ and $f^{-1}(V) \subset U$.

**Proof.** Assume $f$ is $\nu$-open, $S \subset Y$ and $U$ an open set of $X$ containing $f^{-1}(S)$, then $f(X-U)$ is $\nu$-open in $Y$ and $Y - f(X-U)$ is $\nu$-open in $Y$. $f^{-1}(S) \subset U$ implies $S \subset V$ and $f^{-1}(V) = X - f^{-1}(f(X-U)) \subset X - (X-U) = U$.

Conversely let $F$ be open in $X$, then $f^{-1}(f(F^c)) \subset F^c$. By hypothesis, $\exists V \in \nu O(Y) \ni f(F^c) \subset V$ and $f^{-1}(V) \subset F^c$ and so $F \subset (f^{-1}(V))^c$. Hence $V^c \subset f(F) \subset f([f^{-1}(V)]^c) \subset V^c$ implies $f(F) \subset V^c$, which implies $f(F) = V^c$. Thus $f(F)$ is $\nu$-open in $Y$ and therefore $f$ is $\nu$-open.

**Remark.** Composition of two $\nu$-open maps is not $\nu$-open in general.

**Theorem 3.8.** Let $X, Y, Z$ be topological spaces and every $\nu$-open set is open [r-open] in $Y$, then the composition of two $\nu$-open maps is $\nu$-open.

**Proof.** Let $A$ be open in $X$ $\Rightarrow f(A)$ is $\nu$-open in $Y$ $\Rightarrow f(A)$ is open in $Y$ [by assumption] $\Rightarrow g(f(A))$ is $\nu$-open in $Z$ $\Rightarrow g \circ f(A)$ is $\nu$-open in $Z$ $\Rightarrow g \circ f$ is $\nu$-open.
Theorem 3.9. If \( f : X \rightarrow Y \) is g-open; \( g : Y \rightarrow Z \) is \( \nu \)-open [r-open] and \( Y \) is \( T_{\frac{1}{2}}[r-T_{\frac{1}{2}}] \), then \( g \circ f \) is \( \nu \)-open.

Proof. (i) Let \( A \) be open in \( X \Rightarrow f(A) \) is g-open in \( Y \Rightarrow f(A) \) is open in \( Y \) [since \( Y \) is \( T_{\frac{1}{2}} \)] \( \Rightarrow g(f(A)) \) is \( \nu \)-open in \( Z \Rightarrow g \circ f(A) \) is \( \nu \)-open in \( Z \Rightarrow g \circ f \) is \( \nu \)-open.

(ii) Since every g-open set is rg-open, this part follows from the above case.

Corollary 3.5. If \( f : X \rightarrow Y \) is g-open; \( g : Y \rightarrow Z \) is \( \nu \)-open [r-open] and \( Y \) is \( T_{\frac{1}{2}}[r-T_{\frac{1}{2}}] \), then \( g \circ f \) is semi-open and hence \( \beta \)-open.

Theorem 3.10. If \( f : X \rightarrow Y \) is rg-open; \( g : Y \rightarrow Z \) is \( \nu \)-open [r-open] and \( Y \) is \( r-T_{\frac{1}{2}} \), then \( g \circ f \) is \( \nu \)-open.

Proof. Let \( A \) be open in \( X \Rightarrow f(A) \) is rg-open in \( Y \Rightarrow f(A) \) is r-open in \( Y \) [since \( Y \) is \( r-T_{\frac{1}{2}} \)] \( \Rightarrow f(A) \) is open in \( Y \) [since every r-open set is open] \( \Rightarrow g(f(A)) \) is \( \nu \)-open in \( Z \Rightarrow g \circ f(A) \) is \( \nu \)-open in \( Z \Rightarrow g \circ f \) is \( \nu \)-open.

Corollary 3.6. If \( f : X \rightarrow Y \) is rg-open; \( g : Y \rightarrow Z \) is \( \nu \)-open [r-open] and \( Y \) is \( r-T_{\frac{1}{2}} \), then \( g \circ f \) is semi-open and hence \( \beta \)-open.

Theorem 3.11. If \( f : X \rightarrow Y \); \( g : Y \rightarrow Z \) be two mappings such that \( g \circ f \) is \( \nu \)-open [r-open]. Then the following are true.

(i) If \( f \) is continuous [r-continuous] and surjective, then \( g \) is \( \nu \)-open.

(ii) If \( f \) is g-continuous, surjective and \( X \) is \( T_{\frac{1}{2}} \), then \( g \) is \( \nu \)-open.

(iii) If \( f \) is g-continuous [rg-continuous], surjective and \( X \) is \( r-T_{\frac{1}{2}} \), then \( g \) is \( \nu \)-open.

Proof. (i) Let \( A \) be open in \( Y \Rightarrow f^{-1}(A) \) is open in \( X \Rightarrow g \circ f(f^{-1}(A)) \) is \( \nu \)-open in \( Z \Rightarrow g(A) \) is \( \nu \)-open in \( Z \Rightarrow g \) is \( \nu \)-open.

Similarly we can prove the remaining parts and so omitted.

Corollary 3.7. If \( f : X \rightarrow Y \); \( g : Y \rightarrow Z \) be two mappings such that \( g \circ f \) is \( \nu \)-open [r-open]. Then the following are true.

(i) If \( f \) is continuous [r-continuous] and surjective, then \( g \) is semi-open and hence \( \beta \)-open.

(ii) If \( f \) is g-continuous, surjective and \( X \) is \( T_{\frac{1}{2}} \), then \( g \) is semi-open and hence \( \beta \)-open.

(iii) If \( f \) is g-continuous [rg-continuous], surjective and \( X \) is \( r-T_{\frac{1}{2}} \), then \( g \) is semi-open and hence \( \beta \)-open.

Theorem 3.12. If \( X \) is \( \nu \)-regular, \( f : X \rightarrow Y \) is r-closed, nearly-continuous, \( \nu \)-open surjection and \( A^o = A \) for every \( \nu \)-open set in \( Y \), then \( Y \) is \( \nu \)-regular.

Proof. Let \( p \in U \in \nu O(Y) \), \( \exists x \in X \ni f(x) = p \) by surjection. Since \( X \) is \( \nu \)-regular and \( f \) is nearly-continuous, \( \exists V \in RC(X) \ni x \in V^o \subset V \subset f^{-1}(U) \) which implies
\[
p \in f(V^o) \subset f(V) \subset U.
\] (3)

Since \( f \) is \( \nu \)-open, \( f(V^o) \subset U \) is \( \nu \)-open. By hypothesis
\[
\{f(V^o)\}^o = f(V^o) \text{ and } \{f(V^o)\}^o = \{f(V)\}^o.
\] (4)

Combining (3) and (4) \( p \in f(V^o) \subset f(V) \subset U \) and \( f(V) \) is r-closed. Hence \( Y \) is \( \nu \)-regular.

Corollary 3.8. If \( X \) is \( \nu \)-regular, \( f : X \rightarrow Y \) is r-closed, nearly-continuous, \( \nu \)-open surjection and \( A^o = A \) for every r-open set in \( Y \), then \( Y \) is \( \nu \)-regular.

Theorem 3.13. If \( f : X \rightarrow Y \) is \( \nu \)-open [r-open] and \( A \) is open set of \( X \), then \( f_A : (X, \tau(A)) \rightarrow (Y, \sigma) \) is \( \nu \)-open.


Proof. Let \( F \) be open set in \( A \). Then \( F = A \cap E \) for some open set \( E \) of \( X \) and so \( F \) is open in \( X \) which implies \( f(A) \) is \( \nu \)-open in \( Y \). But \( f(F) = f_A(F) \) and therefore \( f_A \) is \( \nu \)-open.

**Corollary 3.9.** If \( f : X \to Y \) is \( \nu \)-open [r-open] and \( A \) is open set of \( X \), then \( f_A : (X, \tau(A)) \to (Y, \sigma) \) is semi-open and hence \( \beta \)-open.

**Theorem 3.14.** If \( f : X \to Y \) is \( \nu \)-open [r-open], \( X \) is \( T_2 \) and \( A \) is g-open set of \( X \), then \( f_A : (X, \tau(A)) \to (Y, \sigma) \) is \( \nu \)-open.

**Corollary 3.10.** If \( f : X \to Y \) is \( \nu \)-open [r-open], \( X \) is \( T_2 \) and \( A \) is g-open set of \( X \), then \( f_A : (X, \tau(A)) \to (Y, \sigma) \) is semi-open and hence \( \beta \)-open.

**Theorem 3.15.** If \( f_i : X_i \to Y_i \) be \( \nu \)-open [r-open] for \( i = 1, 2 \). Let \( f : X_1 \times X_2 \to Y_1 \times Y_2 \) be defined as follows: \( f(x_1, x_2) = (f_1(x_1), f_2(x_2)) \). Then \( f : X_1 \times X_2 \to Y_1 \times Y_2 \) is \( \nu \)-open.

**Proof.** Let \( U_1 \times U_2 \subset X_1 \times X_2 \) where \( U_i \) is open in \( X_i \) for \( i = 1, 2 \). Then \( f(U_1 \times U_2) = f_1(U_1) \times f_2(U_2) \) a \( \nu \)-open set in \( Y_1 \times Y_2 \). Thus \( f(U_1 \times U_2) \) is \( \nu \)-open and hence \( f \) is \( \nu \)-open.

**Corollary 3.11.** If \( f_i : X_i \to Y_i \) be \( \nu \)-open [r-open] for \( i = 1, 2 \). Let \( f : X_1 \times X_2 \to Y_1 \times Y_2 \) be defined as follows: \( f(x_1, x_2) = (f_1(x_1), f_2(x_2)) \). Then \( f : X_1 \times X_2 \to Y_1 \times Y_2 \) is semi-open and hence \( \beta \)-open.

**Theorem 3.16.** Let \( h : X \to X_1 \times X_2 \) be \( \nu \)-open [r-open]. Let \( f_i : X \to X_i \) be defined as: \( h(x) = (x_1, x_2) \) and \( f_i(x) = x_i \). Then \( f_i : X \to X_i \) is \( \nu \)-open for \( i = 1, 2 \).

**Proof.** Let \( U_1 \) is open in \( X_1 \), then \( U_1 \times X_2 \) is open in \( X_1 \times X_2 \), and \( h(U_1 \times X_2) \) is \( \nu \)-open in \( X \). But \( f_1(U_1) = h(U_1 \times X_2) \), therefore \( f_1 \) is \( \nu \)-open. Similarly we can show that \( f_2 \) is also \( \nu \)-open and thus \( f_i : X \to X_i \) is \( \nu \)-open for \( i = 1, 2 \).

**Corollary 3.12.** Let \( h : X \to X_1 \times X_2 \) be \( \nu \)-open [r-open]. Let \( f_i : X \to X_i \) be defined as: \( h(x) = (x_1, x_2) \) and \( f_i(x) = x_i \). Then \( f_i : X \to X_i \) is semi-open and hence \( \beta \)-open for \( i = 1, 2 \).

**Conclusion.** We studied some properties and interrelations of \( \nu \)-open mappings.

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**References**


