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Connectedness in topology of intuitionistic fuzzy rough sets

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Abstract In this paper Type-1 connectedness and Type-2 connectedness in topology of intuitionistic fuzzy rough sets are defined and properties of two types of connectedness are studied.

Keywords Fuzzy subsets, rough sets, intuitionistic fuzzy sets, intuitionistic fuzzy rough sets, fuzzy topology, Type-1 connectedness, Type-2 connectedness.

2000 Mathematics Subject Classification: 54A40

§1. Introduction

After introducing the idea of fuzzy subsets by Lotfi Zadeh [22] different variations/generalisations of fuzzy subsets were made by several authors. Pawlak [20] introduced the idea of rough sets. Nanda [19] and Çoker [7] gave the definition of fuzzy rough sets. Atanassov [2] introduced the idea of intuitionistic fuzzy sets. Combining the ideas of fuzzy rough sets and intuitionistic fuzzy sets T. K. Mandal and S. K. Samanta [16] introduced the concept of intuitionistic fuzzy rough sets (briefly call IFRS, definition 1.15). On the other hand fuzzy topology (we call it topology of fuzzy subsets) was first introduced by C. L. Chang [4]. Later many authors dealt with the idea of fuzzy topology of different kinds of fuzzy sets. M. K. Chakroborti and T. M. G. Ahsanullah [3] introduced the concept of fuzzy topology on fuzzy sets. T. K. Mondal and S. K. Samanta introduced the topology of interval valued fuzzy sets in [18] and the topology of interval valued intuitionistic fuzzy sets in [17]. In [11], we introduced the concept of topology of intuitionistic fuzzy rough sets and study its various properties. In defining topology on an IFRS from the parent space, we observed that two topologies are induced on the IFRS and accordingly two types of continuity are defined. In [12], we studied the connectedness in topology of intuitionistic fuzzy sets.

In this paper, we have defined Type-1 connectedness and Type-2 connectedness in topology of intuitionistic fuzzy rough sets and studied properties of two types of connectedness.
§2. Preliminaries

Unless otherwise stated, we shall consider \((V, \mathcal{B})\) to be a rough universe where \(V\) is a nonempty set and \(\mathcal{B}\) is a Boolean subalgebra of the Boolean algebra of all subsets of \(V\). Also consider a rough set \(X = (X_L, X_U) \in \mathbb{B}^2\) with \(X_L \subset X_U\).

Moreover we assume that \(\mathcal{C}_X\) be the collection of all IFRSs in \(X\).

**Definition 2.1.** A fuzzy rough set (briefly FRS) in \(X\) is an object of the form \(A = (A_L, A_U)\), where \(A_L\) and \(A_U\) are characterized by a pair of maps \(A_L : X_L \rightarrow \mathcal{L}\) and \(A_U : X_U \rightarrow \mathcal{L}\) with \(A_L(x) \leq A_U(x)\), \(\forall x \in X_L\), where \((\mathcal{L}, \leq)\) is a fuzzy lattice (i.e. complete and completely distributive lattice whose least and greatest elements are denoted by 0 and 1 respectively with an involutive order reversing operation \(':\mathcal{L} \rightarrow \mathcal{L}\)).

**Definition 2.2.** For any two fuzzy rough sets \(A = (A_L, A_U)\) and \(B = (B_L, B_U)\) in \(X\),
(i) \(A \subseteq B\) if \(A_L(x) \leq B_L(x)\), \(\forall x \in X_L\) and \(A_U(x) \leq B_U(x)\), \(\forall x \in X_U\);
(ii) \(A = B\) if \(A \subseteq B\) and \(B \subseteq A\).

If \(\{A_i : i \in I\}\) be any family of fuzzy rough sets in \(X\), where \(A_i = (A_{L_i}, A_{U_i})\), then
(iii) \(E = \bigcup_i A_i\), where \(E_L(x) = \vee A_{L_i}(x)\), \(\forall x \in X_L\) and \(E_U(x) = \vee A_{U_i}(x)\), \(\forall x \in X_U\);
(iv) \(F = \bigcap_i A_i\), where \(F_L(x) = \wedge A_{L_i}(x)\), \(\forall x \in X_L\) and \(F_U(x) = \wedge A_{U_i}(x)\), \(\forall x \in X_U\).

**Definition 2.3.** If \(A\) and \(B\) are fuzzy sets in \(X_L\) and \(X_U\) respectively where \(X_L \subset X_U\). Then the restriction of \(B\) on \(X_L\) and the extension of \(A\) on \(X_U\) (denoted by \(B_{\leq L}\) and \(A_{\geq U}\) respectively) are defined by
\[
A_{\leq U}(x) = \begin{cases}
A(x), & \forall x \in X_L; \\
\vee_{\xi \in X_L} \{A(\xi)\}, & \forall x \in X_U - X_L.
\end{cases}
\]

Complement of an FRS \(A = (A_L, A_U)\) in \(X\) are denoted by \(\bar{A} = ((\bar{A})_L, (\bar{A})_U)\) and is defined by

\[
(\bar{A})_L(x) = (A_{U \geq L})'(x), \forall x \in X_L\) and \((\bar{A})_U(x) = (A_{L \leq U})'(x), \forall x \in X_U.
\]

For simplicity we write \((\bar{A}_L, \bar{A}_U)\) instead of \(((\bar{A})_L, (\bar{A})_U)\).

**Theorem 2.1.** If \(A, B, C, D\) and \(B_i, i \in J\) are FRSs in \(X\), then
(i) \(A \subseteq B \text{ and } C \subseteq D\) implies \(A \cup C \subseteq B \cup D\) and \(A \cap C \subseteq B \cap D\),
(ii) \(A \subseteq B\) and \(B \subseteq C\) implies \(A \subseteq C\),
(iii) \(A \cap B \subseteq A\) and \(B \subseteq A \cup B\),
(iv) \(A \cup (\bigcap_i B_i) = \bigcap_i (A \cup B_i)\) and \(A \cap (\bigcup_i B_i) = \bigcup_i (A \cap B_i)\),
(v) \(A \subseteq B \Rightarrow \bar{A} \supseteq \bar{B}\),
(vi) \(\bigcap_i B_i = \bigcap_i \bar{B}_i\) and \(\bigcup_i B_i = \bigcup_i \bar{B}_i\).

**Theorem 2.2.** If \(A\) be any FRS in \(X\), \(0 = (0_L, 0_U)\) be the null FRS and \(1 = (\bar{1}_L, \bar{1}_U)\) be the whole FRS in \(X\), then
(i) \(0 \subseteq A \subseteq 1\),
(ii) \(0 = \bar{1}, 1 = \bar{0}\).

**Notation 2.1.** Let \((V, \mathcal{B})\) and \((V_1, \mathcal{B}_1)\) be two rough universes and \(f : V \rightarrow V_1\) be a mapping. If \(f(\lambda) \in B_1\), \(\forall \lambda \in \mathcal{B}\), then \(f\) maps \((V, \mathcal{B})\) to \((V_1, \mathcal{B}_1)\) and it is denoted by \(f : (V, \mathcal{B}) \rightarrow (V_1, \mathcal{B}_1)\). If \(f^{-1}(\mu) \in \mathcal{B}\), \(\forall \mu \in \mathcal{B}_1\), then it is denoted by \(f^{-1} : (V_1, \mathcal{B}_1) \rightarrow (V, \mathcal{B})\).
Definition 2.4. If $(V, B)$ and $(V_1, B_1)$ be two rough universes and $f : (V, B) \to (V_1, B_1)$.

Let $A = (A_L, A_U)$ be a FRS in $X$. Then $Y = f(X) \in B_1^2$ and $Y_L = f(X_L), Y_U = f(X_U)$.
The image of $A$ under $f$, denoted by $f(A) = (f(A_L), f(A_U))$ and is defined by
$$f(A_L)(y) = \forall \left\{ A_L(x) : x \in X_L \cap f^{-1}(y) \right\}, \forall y \in Y_L;$$

and
$$f(A_U)(y) = \forall \left\{ A_U(x) : x \in X_U \cap f^{-1}(y) \right\}, \forall y \in Y_U.$$

Next let $f : V \to V_1$ be such that $f^{-1} : (V_1, B_1) \to (V, B)$. Let $B = (B_L, B_U)$ be a FRS in $Y$, where $Y = (Y_L, Y_U) \in B_2^2$ is a rough set. Then $X = f^{-1}(Y) \in B^2$, where $X_L = f^{-1}(Y_L), X_U = f^{-1}(Y_U)$. Then the inverse image of $B$, under $f$, denoted by $f^{-1}(B) = (f^{-1}(B_L), f^{-1}(B_U))$ and is defined by
$$f^{-1}(B_L)(x) = B_L(f(x)), \forall x \in X_L \text{ and } f^{-1}(B_U)(x) = B_U(f(x)), \forall x \in X_U.$$

Theorem 2.3. If $f : (V, B) \to (V_1, B_1)$ be a mapping, then for all FRSs $A, A_1$ and $A_2$ in $X$, we have

(i) $f(A) \supset f(A)$,

(ii) $A_1 \subset A_2 \Rightarrow f(A_1) \subset f(A_2)$.

Theorem 2.4. If $f : V \to V_1$ be such that $f^{-1} : (V_1, B_1) \to (V, B)$. Then for all FRSs $B, B_i, i \in J$ in $Y$ we have

(i) $f^{-1}(B) = f^{-1}(B)$,

(ii) $B_1 \subset B_2 \Rightarrow f^{-1}(B_1) \subset f^{-1}(B_2)$.

(iii) If $g : V_1 \to V_2$ be a mapping such that $g^{-1} : (V_2, B_2) \to (V_1, B_1)$, then $(g \circ f)^{-1}(C) = f^{-1}(g^{-1}(C))$, for any FRS $C$ in $Z$ where $Z = (Z_L, Z_U) \in B_2^2$ is a rough set. $g \circ f$ is the composition of $g$ and $f$.

(iv) $f^{-1}(\bigcup_i B_i) = \bigcup_i f^{-1}(B_i),$

(v) $f^{-1}(\bigcap_i B_i) = \bigcap_i f^{-1}(B_i)$.

Theorem 2.5. If $f : (V, B) \to (V_1, B_1)$ be a mapping such that $f^{-1} : (V_1, B_1) \to (V, B)$. Then for all FRS $A$ in $X$ and $B$ in $Y$, we have

(i) $B = f(f^{-1}(B))$,

(ii) $A \subset f^{-1}(f(A))$.

Definition 2.5. If $A$ and $B$ are two FRSs in $X$ with $B \subset \bar{A}$ and $A \subset \bar{B}$, then the ordered pair $(A, B)$ is called an intuitionistic fuzzy rough set (briefly IFRS) in $X$.

The condition $A \subset B$ and $B \subset \bar{A}$ are called intuitionistic condition (briefly IC).

Definition 2.6. Let $P = (A, B)$ and $Q = (C, D)$ be two IFRSs in $X$. Then

(i) $P \subset Q$ if $A \subset C$ and $B \supset D$,

(ii) $P = Q$ if $P \subset Q$ and $Q \subset P$.

(iii) The complement of $P$ in $X$, denoted by $P'$, is defined by $P' = (B, A)$,

(iv) For IFRSs $P_i = (A_i, B_i)$ in $X, i \in J$, define $\bigcup_{i \in J} P_i = (\bigcup_{i \in J} A_i, \bigcap_{i \in J} B_i)$ and $\bigcap_{i \in J} P_i = (\bigcap_{i \in J} A_i, \bigcup_{i \in J} B_i)$.

Theorem 2.6. Let $P = (A, B), Q = (C, D), R = (E, F)$ and $P_i = (A_i, B_i), i \in J$ be IFRSs in $X$, then
(i) \( P \cap P = P = P \cup P \),
(ii) \( P \cap Q = Q \cap P, P \cup Q = Q \cup P \),
(iii) \( (P \cap Q) \cap R = P \cap (Q \cap R), (P \cup Q) \cup R = P \cup (Q \cup R) \),
(iv) \( P \cap Q \subset P, Q \subset P \cup \),
(v) \( P \subset Q \) and \( Q \subset R \Rightarrow P \subset R \),
(vi) \( P_i \subset Q_i, \forall i \in J \Rightarrow \bigcup_{i \in J} P_i \subset Q \),
(vii) \( Q \cap P_i, \forall i \in J \Rightarrow Q \subset \bigcap_{i \in J} P_i \),
(viii) \( Q \cap (\bigcup_{i \in J} P_i) = \bigcap_{i \in J} (Q \cap P_i) \),
(ix) \( Q \cap (\bigcup_{i \in J} P_i) = \bigcup_{i \in J} (Q \cap P_i) \),
(x) \( (P')' = P \),
(xi) \( P \subset Q \Leftrightarrow Q' \subset P' \),
(xii) \( (\bigcup_{i \in J} P_i)' = \bigcap_{i \in J} P'_i \) and \( (\bigcap_{i \in J} P_i)' = \bigcup_{i \in J} P'_i \).

**Definition 2.7.** [16] \( 0^* = (\bar{0}, 1) \) and \( 1^* = (\bar{1}, \bar{0}) \) are respectively called null IFRS and whole IFRS in \( X \). Clearly \( (0^*)' = 1^* \) and \( (1^*)' = 0^* \).

**Theorem 2.7.** [16] If \( P \) be any IFRS in \( X \), then \( 0^* \subset P \subset 1^* \).

Slightly changing the definition of the image of an IFRS under \( f \) given by Samanta and Mondal [16] we have given the following:

**Definition 2.8.** [11] Let \( (V, B) \) and \( (V_1, B_1) \) be two rough universes and \( f : (V, B) \rightarrow (V_1, B_1) \) be a mapping. Let \( P = (A, B) \) be an IFRS in \( X = (X_L, X_U) \) and \( Y = f(X) \in B_1^2 \), where \( Y_L = f(X_L) \) and \( Y_U = f(X_U) \).

Then we define image of \( P \) under \( f \) by \( f(P) = (\tilde{f}(A), \tilde{f}(B)) \), where \( \tilde{f}(A) = (f(A_L), f(A_U)) \), \( A = (A_L, A_U) \) and \( \tilde{f}(B) = (C_L, C_U) \) (where \( B = (B_L, B_U) \)) is defined by

\[
C_L(y) = \{ B_L(x) : x \in X_L \cap f^{-1}(y), \forall y \in Y_L \},
\]

\[
C_U(y) = \begin{cases} 
\{ B_U(x) : x \in X_L \cap f^{-1}(y), \forall y \in Y_L \}; \\
\{ B_U(x) : x \in X_U \cap f^{-1}(y), \forall y \in Y_U \}; 
\end{cases}
\]

**Definition 2.9.** [16] Let \( f : V \rightarrow V_1 \) be such that \( f^{-1} : (V_1, B_1) \rightarrow (V, B) \). Let \( Q = (C, D) \) be an IFRS in \( Y \), where \( Y = (Y_L, Y_U) \in B_1^2 \) be a rough set. Then \( X = f^{-1}(Y) \in B^2 \), where \( X_L = f^{-1}(Y_L) \) and \( X_U = f^{-1}(Y_U) \). Then the inverse image \( f^{-1}(Q) \) of \( Q \) under \( f \) is defined by \( f^{-1}(Q) = (f^{-1}(C), f^{-1}(D)) \), where \( f^{-1}(C) = (f^{-1}(C_L), f^{-1}(C_U)) \) and \( f^{-1}(D) = (f^{-1}(D_L), f^{-1}(D_U)) \).

The following three theorems of Samanta and Mondal [16] are also valid for this modified definition of functional image of an IFRS.

**Theorem 2.8.** Let \( f : (V, B) \rightarrow (V_1, B_1) \) be a mapping. Then for all IFRSs \( P \) and \( Q \), we have

(i) \( f(P') \supset (f(P))' \),
(ii) \( P \subset Q \Rightarrow f(P) \subset f(Q) \).

**Theorem 2.9.** Let \( f : V \rightarrow V_1 \) be such that \( f^{-1} : (V_1, B_1) \rightarrow (V, B) \). Then for all IFRSs \( R, S \) and \( R_i, i \in J \) in \( Y \),

(i) \( f^{-1}(R_i) = (f^{-1}(R))' \),
(ii) \( R \subset S \Rightarrow f^{-1}(R) \subset f^{-1}(S) \),
(iii) If \( g : V_1 \to V_2 \) be a mapping such that \( g^{-1} : (V_2, B_2) \to (V_1, B_1) \), then \((g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))\) for any IFRS \( W \) in \( Z \), where \( Z = (Z_L, Z_U) \) is a rough set, \( g \circ f \) is the composition of \( g \) and \( f \),

(iv) \( f^{-1}(\bigcup_{i \in J} R_i) = \bigcup_{i \in J} f^{-1}(R_i) \),

(v) \( f^{-1}(\bigcap_{i \in J} R_i) = \bigcap_{i \in J} f^{-1}(R_i) \).

**Theorem 2.10.** Let \( f : (V, B) \to (V_1, B_1) \) be a mapping such that \( f^{-1} : (V_1, B_1) \to (V, B) \). Then for all IFRS \( P \) in \( X \) and \( R \) in \( Y \), we have

(i) \( R = f(f^{-1}(R)) \),

(ii) \( P \subseteq f^{-1}(f(P)) \).

**Theorem 2.11.**[11] If \( P \) and \( Q \) be two IFRSs in \( X \) and \( f : (V, B) \to (V_1, B_1) \) be a mapping, then \( f(P \cup Q) = f(P) \cup f(Q) \).

**Theorem 2.12.**[11] If \( P, Q \) be two IFRSs in \( X \) and \( f : (V, B) \to (V_1, B_1) \) be a mapping, then \( f(P \cap Q) \subseteq f(P) \cap f(Q) \).

**Note 2.1.** If \( f : (V, B) \to (V_1, B_1) \) be one-one, then clearly \( f(P \cap Q) = f(P) \cap f(Q) \). But in general \( f(P \cap Q) \neq f(P) \cap f(Q) \).

**Definition 2.10.**[11] Let \( X = (X_L, X_U) \) be a rough set and \( \tau \) be a family of IFRSs in \( X \) such that

(i) \( 0^* , 1^* \in \tau \),

(ii) \( P \cap Q \in \tau, \forall P, Q \in \tau \),

(iii) \( P_i, i \in \Delta \Rightarrow \bigcup_{i \in \Delta} P_i \in \tau \).

Then \( \tau \) is called a topology of IFRSs in \( X \) and \((X, \tau)\) is called a topological space of IFRSs in \( X \).

Every member of \( \tau \) is called open IFRS. An IFRS \( C \) is called closed IFRS if \( C' \in \tau \). Let \( \mathcal{F} \) denote the collection of all closed IFRSs in \((X, \tau)\). If \( \tau_I = \{0^*, 1^*\} \), then \( \tau_I \) is a topology of IFRSs in \( X \). This topology is called the indiscrete topology. The discrete topology of IFRSs in \( X \) contains all the IFRSs in \( X \).

**Theorem 2.13.**[11] The collection \( \mathcal{F} \) of all closed IFRSs satisfies the following properties:

(i) \( 0^* , 1^* \in \mathcal{F} \),

(ii) \( P, Q \in \mathcal{F} \Rightarrow P \cup Q \in \mathcal{F} \),

(iii) \( P_i, i \in \Delta \Rightarrow \bigcap_{i \in \Delta} P_i \in \mathcal{F} \).

**Definition 2.11.**[11] Let \( P \) be an IFRS in \( X \). The closure of \( P \) in \((X, \tau)\), denoted by \( \text{cl}_P \), is defined by the intersection of all closed IFRSs in \((X, \tau)\) containing \( P \).

Clearly \( \text{cl}_P \) is the smallest closed IFRS containing \( P \) and \( P \) is closed iff \( P = \text{cl}_P \).

**Definition 2.12.**[11] Let \((X, \tau)\) and \((Y, u)\) be two topological spaces of IFRSs and \( f : (V, B) \to (V_1, B_1) \) be a mapping such that \( f^{-1} : (V_1, B_1) \to (V, B) \). Then \( f : (X, \tau) \to (Y, u) \) is said to be IFR continuous if \( f^{-1}(Q) \in \tau, \forall Q \in u \).

Unless otherwise stated we consider \((X, \tau)\) and \((Y, u)\) be topological spaces of IFRSs and \( f : (V, B) \to (V_1, B_1) \) be a mapping such that \( f^{-1} : (V_1, B_1) \to (V, B) \).

**Theorem 2.14.**[11] The following statements are equivalent:

(i) \( f : (X, \tau) \to (Y, u) \) is IFR continuous,

(ii) \( f^{-1}(Q) \) is closed IFRS in \((X, \tau)\), for every closed IFRS \( Q \) in \((Y, u)\),

(iii) \( f(\text{cl}_P) \subseteq \text{cl}_u(f(P)) \), for every IFRS \( P \) in \( X \).
Definition 2.13. Let \( P \in \mathcal{C}_X \). Then a subfamily \( T \) of \( \mathcal{C}_X \) is said to be a topology on \( P \) if

(i) \( Q \in T \Rightarrow Q \subseteq P \),

(ii) \( 0^* \in T \),

(iii) \( P_1, P_2 \in T \Rightarrow P_1 \cap P_2 \in T \),

(iv) \( P_i \in T, i \in \Delta \Rightarrow \bigcup_{i \in \Delta} P_i \in T \).

Then \( (P, T) \) is called a subtopology of \((X, \tau)\).

Theorem 2.15. Let \( \tau \) be a topology of \( IFRSs \) in \( X \) and let \( P \in \mathcal{C}_X \). Then \( \tau_1 = \{ P \cap R : R \in \tau \} \) is a topology on \( P \). Every member of \( \tau_1 \) is called open \( IFRS \) in \((P, \tau_1)\). If \( Q \in \tau_1 \), then \( Q' \) is called a closed \( IFRS \) in \((P, \tau_1)\), where \( Q' = P \cap Q' \). We take \( 0^* \) as closed \( IFRS \) also in \((P, \tau_1)\). Let \( C_1 = \{ Q'_p = P \cap Q' : Q \in \tau_1 \} \cup \{ 0^* \} \).

Theorem 2.16. \( C_1 \) is closed under arbitrary intersection and finite union.

Remark 2.1. Clearly the collection \( \tau_2 = \{(S'_p)'_p = (P \cap P') \cup S : S \in \tau_1 \} \cup \{ 0^* \} \) forms a topology of \( IFRSs \) on \( P \) of which \( C_1 \) is a family of closed \( IFRSs \). But \( C_1 \) is also a family of closed \( IFRSs \) in \((P, \tau_1)\). Thus \( \exists \) two topologies of \( IFRSs \) \( \tau_1 \) and \( \tau_2 \) on \( P \). \( \tau_1 \) is called the first subspace topology of \((X, \tau)\) on \( P \) and \( \tau_2 \) is called the second subspace topology of \((X, \tau)\) on \( P \). We briefly write, \( \tau_1 \) and \( \tau_2 \) are first and second topologies respectively on \( P \), where there is no confusion about the topological space \((X, \tau)\) of \( IFRSs \).

Definition 2.14. Let \((X, \tau)\) and \((Y, u)\) be two topological spaces of \( IFRSs \) and \( P \in \mathcal{C}_X \). Let \( \tau_1 \) and \( u_1 \) be first topologies on \( P \) and \( f(P) \) respectively. Then \( f : (P, \tau_1) \rightarrow (f(P), u_1) \) is said to be \( IFR_1 \) continuous if \( P \cap f^{-1}(Q) \in \tau_1, \forall Q \in u_1 \).

Theorem 2.17. Let \((X, \tau)\) and \((Y, u)\) be two topological spaces of \( IFRSs \) in \( X \) and \( Y \) respectively and \( P \in \mathcal{C}_X \) and let \( \tau_1 \) and \( u_1 \) be first topologies on \( P \) and \( f(P) \) respectively. If \( f : (X, \tau) \rightarrow (Y, u) \) is \( IFR \) continuous, then \( f : (P, \tau_1) \rightarrow (f(P), u_1) \) is \( IFR_1 \) continuous.

Definition 2.15. Let \((X, \tau)\) and \((Y, u)\) be two topological spaces of \( IFRSs \) in \( X \) and \( Y \) respectively and \( P \in \mathcal{C}_X \) and let \( \tau_2 \), \( u_2 \) be second topologies on \( P \) and \( f(P) \) respectively. Then \( f : (P, \tau_2) \rightarrow (f(P), u_2) \) is said to be \( IFR_2 \) continuous if \( P \cap (P' \cup f^{-1}(Q)) \in \tau_2, \forall Q \in u_2 \).

Theorem 2.18. Let \((X, \tau)\) and \((Y, u)\) be two topological spaces of \( IFRSs \) in \( X \) and \( Y \) respectively and \( P \in \mathcal{C}_X \). Let \( \tau_1 \) and \( u_1 \) be first topologies on \( P \) and \( f(P) \) respectively and \( \tau_2 \), \( u_2 \) be second topologies on \( P \) and \( f(P) \) respectively. If \( f : (P, \tau_1) \rightarrow (f(P), u_1) \) is \( IFR_1 \) continuous, then \( f : (P, \tau_2) \rightarrow (f(P), u_2) \) is \( IFR_2 \) continuous.

Corollary 2.1. If \( f : (X, \tau) \rightarrow (Y, u) \) is \( IFR \) continuous, then \( f : (P, \tau_2) \rightarrow (f(P), u_2) \) is \( IFR_2 \) continuous, where the symbols have usual meaning.

§3. Connectedness of an \( IFRS \)

Connectedness in fuzzy topological space has been studied by several authors (for reference see [1], [5], [6], [8], [15], [21]). We studied connectedness in topology of intuitionistic fuzzy sets in [12]. In this section we study two types of connectedness in a topological space of \( IFRSs \).

Definition 3.1. Let \( \tau_1 \) be the first topology on \( P \). An \( IFRS \) \( Q \subseteq P \) is said to be nonempty in \((P, \tau_1)\) if \( Q \neq 0^* \).
Definition 3.2. Let \( \tau_1 \) be the first topology on \( P \). Two IFRSs \( Q_1 \) and \( Q_2 \) are said to be disjoint in \((P, \tau_1)\) if \( Q_1 \cap Q_2 = \emptyset \).

Definition 3.3. Let \((X, \tau)\) be a topological space of IFRSs in \( X \) and let \( P \in \mathcal{C}_X \). Then \( P \) is said to be Type-1 connected in \((X, \tau)\), if \( P \) can not be expressed as a union of two nonempty disjoint open sets in \((P, \tau_1)\).

Theorem 3.1. Let \( f : (P, \tau_1) \to (f(P), u_1) \) be IFR1 continuous and \( P \) be Type-1 connected in \((X, \tau)\), then \( f(P) \) is Type-1 connected in \((Y, u)\).

Proof. Let \( f : (P, \tau_1) \to (f(P), u_1) \) be IFR1 continuous and \( P \) be Type-1 connected in \((X, \tau)\). We shall show that \( f(P) \) is Type-1 connected in \((Y, u)\), where \( Y = f(X) \) and \( u \) is the topology of IFRSs in \( Y \). If possible, let \( f(P) \) be not Type-1 connected in \((Y, u)\).

Therefore \( f(P) = Q_1 \cup Q_2 \), where \( Q_1, Q_2 \in u_1 \) and \( Q_1, Q_2 \) are nonempty disjoint open sets in \((Y, u_1)\). Therefore

\[
P = P \cap f^{-1}(f(P)) = P \cap f^{-1}(Q_1 \cup Q_2) = P \cap (f^{-1}(Q_1) \cup f^{-1}(Q_2)) = (P \cap f^{-1}(Q_1)) \cup (P \cap f^{-1}(Q_2)).
\]

(1)

Since \( f : (P, \tau_1) \to (f(P), u_1) \) is IFR1 continuous and \( Q_1, Q_2 \in u_1 \), we have \( P \cap f^{-1}(Q_1), P \cap f^{-1}(Q_2) \in \tau_1 \). Now we shall show that \( P \cap f^{-1}(Q_1) \) and \( P \cap f^{-1}(Q_2) \) are nonempty and disjoint. If possible let \( P \cap f^{-1}(Q_1) = \emptyset \). Therefore from (1) we have \( P = P \cap f^{-1}(Q_2) \) and hence \( f^{-1}(Q_2) \supset P \) and consequently \( f(f^{-1}(Q_2)) \supset f(P) \), i.e., \( Q_2 \supset f(P) \), which contradicts \( f(P) = Q_1 \cup Q_2 \). Thus \( P \) can not be Type-1 connected in \((X, \tau)\), which is a contradiction. Hence \( f(P) \) is Type-1 connected in \((Y, u)\).

Definition 3.4. Let \((X, \tau)\) be a topological space of IFRSs in \( X \) and let \( P \in \mathcal{C}_X \) and let \( \tau_2 \) be the second topology on \( P \). Then an IFRS \( Q \subseteq P \) is said to be nonempty in \((P, \tau_2)\) if \( Q \supset P \cap P' \) and \( Q \neq P \cap P' \). Two IFRSs \( Q \) and \( R \) are said to be disjoint in \((P, \tau_2)\) if \( Q \cap R \subset P \cap P' \).

Definition 3.5. Let \((X, \tau)\) be a topological space of IFRSs in \( X \) and let \( P \in \mathcal{C}_X \). Then \( P \) is said to be Type-2 connected in \((X, \tau)\) if \( P \) can not be expressed as a union of two nonempty disjoint open sets in \((P, \tau_2)\).

Note 3.1. If \( f : (P, \tau_2) \to (f(P), u_2) \) is IFR2 continuous and \( P \) is Type-2 connected in \((X, \tau)\), then \( f(P) \) may not be Type-2 connected in \((Y, u)\). This is shown by the following example.

Example 3.1. Let \( X_L = X_U = V = \{x, y, z, u, v, w\} \), \( Y_L = Y_U = V_1 = \{a, b\} \). Define
\[ f : V \to V_1 \text{ by } f(x) = f(y) = f(z) = a, f(u) = f(v) = f(w) = b. \text{ Let} \]

\[ Q = \{(x/0.3, y/0.3, z/0.3, u/0.4, v/0.4, w/0.4), \{x/0.4, y/0.4, z/0.4, u/0.5, v/0.5, w/0.5\}\}, \]
\[ R = \{(x/0.4, y/0.4, z/0.4, u/0.3, v/0.3, w/0.3), \{x/0.5, y/0.5, z/0.5, u/0.4, v/0.4, w/0.4\}\}, \]
\[ T = Q \cup R \]
\[ = \{(x/0.4, y/0.4, z/0.4, u/0.4, v/0.4, w/0.4), \{x/0.5, y/0.5, z/0.5, u/0.5, v/0.5, w/0.5\}\}, \]
\[ S = Q \cap R \]
\[ = \{(x/0.3, y/0.3, z/0.3, u/0.3, v/0.3, w/0.3), \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.4, w/0.4\}\}, \]
\[ M = \{(x/0.35, y/0.4, z/0.4, u/0.3, v/0.3, w/0.3), \{x/0.45, y/0.5, z/0.5, u/0.4, v/0.4, w/0.4\}\}, \]
\[ N = \{(x/0.35, y/0.4, z/0.4, u/0.4, v/0.4, w/0.4), \{x/0.45, y/0.5, z/0.5, u/0.5, v/0.5, w/0.5\}\}, \]
\[ \text{Clearly } \tau = \{0^*, 1^*, Q, R, T, S, M, N\} \text{ forms a topology of IFRSs in } X = (X_L, X_U). \text{ Let} \]
\[ P = \{(x/0.4, y/0.3, z/0.1, u/0.4, v/0.3, w/0.3), \{x/0.5, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}\}, \]
\[ \{(x/0.3, y/0.3, z/0.3, u/0.4, v/0.3, w/0.3), \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}\}. \]

Therefore
\[ P \cap P' = \{(x/0.3, y/0.3, z/0.1, u/0.4, v/0.3, w/0.3), \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}\}, \]
\[ \{(x/0.4, y/0.3, z/0.3, u/0.4, v/0.3, w/0.3), \{x/0.5, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}\}. \]

Now
\[ Q = P \cap (P' \cup Q) \]
\[ = \{(x/0.3, y/0.3, z/0.1, u/0.4, v/0.3, w/0.3), \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}\}, \]
\[ \{(x/0.4, y/0.3, z/0.3, u/0.4, v/0.3, w/0.3), \{x/0.5, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}\}\}
\[ = P \cap P', \]
\[ R = P \cap (P' \cup R) \]
\[ = \{(x/0.4, y/0.3, z/0.1, u/0.4, v/0.3, w/0.3), \{x/0.5, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}\}, \]
\[ \{(x/0.3, y/0.3, z/0.3, u/0.4, v/0.3, w/0.3), \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}\}\}
\[ = P, \]
\[ T = P \cap (P' \cup T) = ((\{x/0.4, y/0.3, z/0.1, u/0.4, v/0.3, w/0.3\}, \{x/0.5, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}),
\{x/0.3, y/0.3, z/0.3, u/0.4, v/0.3, w/0.3\}, \{x/0.5, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\})) = P, \]
\[ S = P \cap (P' \cup S) = ((\{x/0.3, y/0.3, z/0.1, u/0.4, v/0.3, w/0.3\}, \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}),
\{x/0.4, y/0.3, z/0.3, u/0.4, v/0.3, w/0.3\}, \{x/0.5, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\})) = P \cap P', \]
\[ M = P \cap (P' \cup M) = ((\{x/0.35, y/0.3, z/0.1, u/0.4, v/0.3, w/0.3\}, \{x/0.45, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\}),
\{x/0.3, y/0.3, z/0.3, u/0.4, v/0.3, w/0.3\}, \{x/0.4, y/0.4, z/0.4, u/0.4, v/0.5, w/0.5\})) = M, \]
\[ P \cap (P' \cup 0^*) = P \cap P', \]
\[ P \cap (P' \cup 1^*) = P. \]

Thus \( \tau_2 = \{0^*, P, P \cap P', M\} \) forms a second topology on \( P \). Clearly \( P \) is Type-2 connected in \((X, \tau)\). Let
\[ \hat{Q} = ((\{a/0.3, b/0.4\}, \{a/0.4, b/0.5\}), (\{a/0.4, b/0.3\}, \{a/0.5, b/0.4\})), \]
\[ \hat{R} = ((\{a/0.4, b/0.3\}, \{a/0.5, b/0.4\}), (\{a/0.3, b/0.4\}, \{a/0.4, b/0.5\})), \]
\[ \hat{T} = \hat{Q} \cup \hat{R} = ((\{a/0.4, b/0.4\}, \{a/0.5, b/0.5\}), (\{a/0.3, b/0.3\}, \{a/0.4, b/0.4\})), \]
\[ \hat{S} = \hat{Q} \cap \hat{R} = ((\{a/0.3, b/0.3\}, \{a/0.4, b/0.4\}), (\{a/0.4, b/0.4\}, \{a/0.5, b/0.5\})). \]

Clearly \( u = \{0^*, 1^*, \hat{Q}, \hat{R}, \hat{T}, \hat{S}\} \) forms a topology of IFRSs in \( Y = (Y_L, Y_U) \). Now
\[ f(P) = ((\{a/0.4, b/0.4\}, \{a/0.5, b/0.5\}), (\{a/0.3, b/0.3\}, \{a/0.4, b/0.4\})) = \hat{T}. \]

Therefore
\[ f(P) \cap (f(P))^\prime = ((\{a/0.3, b/0.3\}, \{a/0.4, b/0.4\}), (\{a/0.4, b/0.4\}, \{a/0.5, b/0.5\})) = \hat{S}, \]
\[ \hat{Q} = f(P) \cap ((f(P))^\prime \cup \hat{Q}) = ((\{a/0.3, b/0.4\}, \{a/0.4, b/0.5\}), (\{a/0.4, b/0.3\}, \{a/0.5, b/0.4\})), \]
\[ \hat{R} = f(P) \cap ((f(P))^\prime \cup \hat{R}) = ((\{a/0.4, b/0.3\}, \{a/0.5, b/0.4\}), (\{a/0.3, b/0.4\}, \{a/0.4, b/0.5\})), \]
\[ \hat{T} = f(P) \cap ((f(P))^\prime \cup \hat{T}) = f(P). \]
Since \( \tilde{T} = f(P) \), \( \tilde{S} = f(P) \cap ((f(P))' \cup \tilde{\tilde{S}}) = f(P) \cap (f(P))' \), \( f(P) \cap ((f(P))' \cup 0^* = f(P) \cap (f(P))' \), \( f(P) \cap ((f(P))' \cup 1^*) = f(P) \). Thus \( u_2 = 0^*, f(P) \cap (f(P))', f(P), \tilde{Q}, \tilde{\tilde{R}} \) forms a second topology on \( f(P) \). Clearly \( f(P) = \tilde{\tilde{Q}} \cup \tilde{\tilde{R}} \), where \( f(P) \cap (f(P))' \subset \tilde{\tilde{Q}}, \tilde{\tilde{R}} \cap \tilde{\tilde{\tilde{R}}} \cap \tilde{\tilde{\tilde{R}}} \) and \( \tilde{\tilde{Q}}, \tilde{\tilde{\tilde{R}}} \neq f(P) \) and \( \tilde{\tilde{Q}}, \tilde{\tilde{\tilde{R}}} \neq f(P) \cap (f(P))' \). Also \( \tilde{\tilde{Q}} \cap \tilde{\tilde{R}} = f(P) \cap (f(P))' \). Therefore \( f(P) \) can not be Type-2 connected in \((Y, u)\). Since \( f^{-1}(0^*) = 0^* \), \( f^{-1}(1^*) = 1^* \), \( f^{-1}(\tilde{\tilde{Q}}) = Q \), \( f^{-1}(\tilde{\tilde{R}}) = R \), \( f^{-1}(\tilde{T}) = T \), \( f^{-1}(\tilde{S}) = S \), it follows that \( f : (X, \tau) \rightarrow (Y, u) \) is IFR continuous and hence \( f : (P, \tau_2) \rightarrow (f(P), u_2) \) is IFR\(_2\) continuous. Thus IFR\(_2\) continuous image of a Type-2 connected IFRS need not be Type-2 connected IFRS.

**Definition 3.6.** Let \( f : (V, \mathcal{B}) \rightarrow (V_1, \mathcal{B}_1) \) be a mapping and \( P \in \mathcal{C}_X \). Then \( f \) is said to satisfy disjoint condition with respect to \( P \), if \( \forall Q \in \mathcal{C}_X \), \( P \cap Q \subset P \cap P' \Rightarrow f(P) \cap f(Q) \subset f(P) \cap f(P') \).

**Theorem 3.2.** If \( P \in \mathcal{C}_X \) and \( f : (V, \mathcal{B}) \rightarrow (V_1, \mathcal{B}_1) \) be a mapping and is one-one, then \( f(P') = (f(P))' \).

The proof is straightforward.

**Remark 3.1.** The converse of the theorem 3.2 is not true. This can be shown by the following Example.

**Example 3.2.** Let \( V = \{x, y\} \), \( X_L = X_U = V \), \( X = (X_L, X_U) \), \( V_1 = \{a\} \), \( Y_L = Y_U = V_1 \), \( Y = (Y_L, Y_U) \). Let \( f : V \rightarrow V_1 \) be defined by \( f(x) = f(y) = a \). Let \( P = (\{x/0.4, y/0.4\}, \{x/0.4, y/0.4\}, \{x/0.4, y/0.4\}, \{x/0.4, y/0.4\}) \). Clearly \( P \in \mathcal{C}_X \) and \( f(P') = (f(P))' \), but \( f \) is not one-one.

**Theorem 3.3.** If \( P \in \mathcal{C}_X \) and \( f : (V, \mathcal{B}) \rightarrow (V_1, \mathcal{B}_1) \) be a mapping and is one-one, then \( f \) satisfies disjoint condition with respect to \( P \).

**Proof.** Let \( f : (V, \mathcal{B}) \rightarrow (V_1, \mathcal{B}_1) \) be one-one and \( P \in \mathcal{C}_X \). Then \( \forall Q \in \mathcal{C}_X \), \( P \cap Q \subset P \cap P' \Rightarrow f(P) \cap f(Q) \subset f(P) \cap f(P') \), since \( f \) is one-one. \( \Rightarrow f(P) \cap f(Q) \subset f(P) \cap (f(P))' \), since \( f \) is one-one, \( f(P') = (f(P))' \). Thus \( f \) satisfies disjoint condition with respect to \( P \).

But the converse of the above theorem is not true, which can be shown by the following Example.

**Example 3.3.** Let \( V = \{x, y\} \), \( X_L = X_U = V \), \( X = (X_L, X_U) \), \( V_1 = \{a\} \), \( Y_L = Y_U = V_1 \), \( Y = (Y_L, Y_U) \). Let \( f : V \rightarrow V_1 \) be defined by \( f(x) = f(y) = a \). Let \( P = (\{x/0.4, y/0.4\}, \{x/0.4, y/0.4\}, \{x/0.4, y/0.4\}) \). Then \( P \cap P' = P \). \( f(P) = (\{a/0.4\}, \{a/0.4\}, \{a/0.4\}) \). Therefore \( f(P) \cap (f(P))' = f(P) \). Clearly \( f(P) \cap f(Q) \subset f(P) \cap (f(P))' \), \( \forall Q \in \mathcal{C}_X \). Thus \( \forall Q \in \mathcal{C}_X \), \( P \cap Q \subset P \cap P' \Rightarrow f(P) \cap f(Q) \subset f(P) \cap (f(P))' \). Thus \( f \) satisfies disjoint condition with respect to \( P \). Clearly \( f \) is not one-one. Thus \( f \) satisfies disjoint condition with respect to \( P \), but \( f \) is not one-one.

**Theorem 3.4.** If \( P \) is Type-2 connected in \((X, \tau)\) and \( f : (P, \tau_2) \rightarrow (f(P), u_2) \) is IFR\(_2\) continuous satisfying disjoint condition with respect to \( P \), then \( f(P) \) is Type-2 connected in \((Y, u)\).

**Proof.** Let \( P \) be Type-2 connected in \((X, \tau)\) and \( f : (P, \tau_2) \rightarrow (f(P), u_2) \) be IFR\(_2\) continuous satisfying disjoint condition with respect to \( P \). If possible, let \( f(P) \) be not Type-2 connected in \((Y, u)\). Therefore \( f(P) = Q_1 \cup Q_2 \) for some \( Q_1, Q_2 \in u_2 \), where \( Q_1 \cap Q_2 \subset \)}
\[ f(P) \cap (f(P))' \] and \( Q_1, Q_2 \supseteq f(P) \cap (f(P))' \).

\[
P = P \cap f^{-1}(f(P)) = P \cap f^{-1}(Q_1 \cup Q_2) = P \cap (f^{-1}(Q_1) \cup f^{-1}(Q_2)) = (P \cap f^{-1}(Q_1)) \cup (P \cap f^{-1}(Q_2)) \subseteq (P \cap (P' \cup f^{-1}(Q_1))) \cup (P \cap (P' \cup f^{-1}(Q_2))) \subseteq P.
\]

Therefore \( P = (P \cap (P' \cup f^{-1}(Q_1))) \cup (P \cap (P' \cup f^{-1}(Q_2))) = P_1 \cup P_2 \) (say), where \( P_1 = P \cap (P' \cup f^{-1}(Q_1)), P_2 = P \cap (P' \cup f^{-1}(Q_2)) \in \tau_2 \), since \( f : (P, \tau_2) \to (f(P), u_2) \) is IFR\(_2\) continuous.

\[
P_1 \cap P_2 = P \cap (P' \cup f^{-1}(Q_1)) \cap (P' \cup f^{-1}(Q_2)) = P \cap (P' \cup (f^{-1}(Q_1) \cap f^{-1}(Q_2))) = P \cap (P' \cup (f^{-1}(Q_1 \cap Q_2))) \subseteq P \cap (P' \cup (f^{-1}(f(P))')) = P \cap (P' \cup ((f^{-1}(f(P))) \cap (f^{-1}(f(P)))')) = (P \cap P') \cup (P \cap (f^{-1}(f(P))))' = (P \cap P') \cup (P \cap (f^{-1}(f(P)))') = P \cap (P' \cup (f^{-1}(f(P)))'). = P \cap P'.
\]

Therefore \( P_1 \cap P_2 \subseteq P \cap P' \). We claim that \( P_1, P_2 \supseteq P \cap P' \). If possible, let \( P_1 = P \cap P' \). Therefore \( P \cap (P' \cup f^{-1}(Q_1)) \subseteq P \cap P' \), i.e., \( (P \cap P') \cup (P \cap f^{-1}(Q_1)) \subseteq P \cap P' \). Therefore \( P \cap f^{-1}(Q_1) \subseteq P \cap P' \). Therefore \( f(P) \cap f(f^{-1}(Q_1)) \subseteq f(P) \cap (f(P))' \), since \( f \) satisfies disjoint condition with respect to \( P \). Therefore \( f(P) \cap Q_1 \subseteq f(P) \cap (f(P))' \) and hence \( Q_1 \subseteq f(P) \cap (f(P))' \), since \( Q_1 \subseteq f(P) \). But this contradicts the fact that \( Q_1 \supseteq f(P) \cap (f(P))' \). Thus \( P_1 \supseteq P \cap P' \). Similarly we can prove that \( P_2 \supseteq P \cap P' \). Thus \( P \) can be expressed as the union of two nonempty disjoint open sets in \( (P, \tau_2) \), which contradicts that \( P \) is Type-2 connected in \( (X, \tau) \). Hence \( f(P) \) is Type-2 connected in \( (Y, u) \).

References


Some fixed point theorems in asymmetric metric spaces

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Abstract In this work, we define the concept of forward and backward contractions. Then, we prove the Banach’s contraction principle in asymmetric metric spaces. Also, we prove a fixed point theorem in partially ordered asymmetric metric spaces.

Keywords Asymmetric metric space, fixed point theorems.

§1. Introduction

Asymmetric metric spaces are defined as metric spaces, but without the requirement that the (asymmetric) metric $d$ has to satisfy $d(x,y) = d(y,x)$.

In the realms of applied mathematics and materials science we find many recent applications of asymmetric metric spaces, for example, in rate-independent models for plasticity [1], shape-memory alloys [2], and models for material failure [3].

There are other applications of asymmetric metrics both in pure and applied mathematics; for example, asymmetric metric spaces have recently been studied with questions of existence and uniqueness of Hamilton-Jacobi equations [4] in mind. The study of asymmetric metrics apparently goes back to Wilson [5]. Following his terminology, asymmetric metrics are often called quasi-metrics.

Author in [6] has completely discussed on asymmetric metric spaces. Also, In [7], Aminpour, Khorshidvandpour and Mousavi have proved some useful results in asymmetric metric spaces. In this work we prove some theorems in asymmetric metric spaces. We start with some elementary definitions from [6].

Definition 1.1. A function $d : X \times X \rightarrow \mathbb{R}$ is an asymmetric metric and $(X,d)$ is an asymmetric metric space if

(i) For every $x, y \in X$, $d(x,y) \geq 0$ and $d(x,y) = 0$ holds if and only if $x = y$,

(ii) For every $x, y, z \in X$, we have $d(x,z) \leq d(x,y) + d(y,z)$. Henceforth, $(X,d)$ shall be an asymmetric metric space.

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Example 1.1. Let $\alpha > 0$. Then $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+ \geq 0$ defined by

$$d(x, y) = \begin{cases} 
    x - y, & x \geq y; \\
    \alpha(y - x), & y > x.
\end{cases}$$

is obviously an asymmetric metric.

Definition 1.2. The forward topology $\tau^+$ induced by $d$ is the topology generated by the forward open balls $B^+ (x, \varepsilon) = \{ y \in X : d(x, y) < \varepsilon \}$ for $x \in X$, $\varepsilon > 0$.

Likewise, the backward topology $\tau^-$ induced by $d$ is the topology generated by the backward open balls $B^- (y, \varepsilon) = \{ x \in X : d(y, x) < \varepsilon \}$ for $x \in X$, $\varepsilon > 0$.

Definition 1.3. A sequence $\{x_k\}_{k \in \mathbb{N}}$ forward converges to $x_0 \in X$, respectively backward converges to $x_0 \in X$, if and only if

$$\lim_{k \to \infty} d(x_0, x_k) = 0,$$

respectively

$$\lim_{k \to \infty} d(x_k, x_0) = 0.$$ 

Then we write $x_k \xrightarrow{f} x_0$, $x_k \xrightarrow{b} x_0$ respectively.

Example 1.2. Let $(\mathbb{R}, d)$ be an asymmetric space, where $d$ is as in example 1.1. It is easy to show that the sequence $\{x + \frac{1}{n}\}_{n \in \mathbb{N}}$ is both forward and backward converges to $x_0 \in X$.

Definition 1.4. Suppose $(X, d_X)$ and $(Y, d_Y)$ are asymmetric metric spaces. Let $f : X \to Y$ be a function. We say $f$ is forward continuous at $x \in X$, respectively backward continuous, if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $y \in B^+(x, \delta)$ implies $f(y) \in B^+(f(x), \varepsilon)$, respectively $f(y) \in B^-(f(x), \varepsilon)$.

However, note that uniform forward continuity and uniform backward continuity are the same.

Definition 1.5. A set $S \subset X$ is forward compact if every open cover of $S$ in the forward topology has a finite subcover. We say $S$ is forward relatively compact if $\overline{S}$ is forward compact, where $\overline{S}$ denotes the closure in the forward topology. We say $S$ is forward sequentially compact if every sequence has a forward convergent subsequence with limit in $S$. Finally, $S \subset X$ is forward complete if every forward Cauchy sequence is forward convergent.

Note that there is a corresponding backward definition in each case, which is obtained by replacing forward with backward in each definition.

Lemma 1.1.\cite{6} Let $d : X \times X \to \mathbb{R}^+ \geq 0$ be an asymmetric metric. If $(X, d)$ is forward sequentially compact and $x_n \xrightarrow{b} x_0$ then $x_k \xrightarrow{f} x_0$.

Lemma 1.2. Let $(x_n)$ be a backward Cauchy sequence in $X$. If $(x_n)$ has a backward convergent subsequence, then $(x_n)$ converges to the limit of it’s subsequence.

As symmetric case.

§2. Banach’s contraction principle

In this section we prove Banach’s contraction principle in the sense of asymmetric metric spaces. First of all, we have the following definition. Throughout this section $(X, d)$ denotes an
asymmetric metric space, unless the contrary is specified.

**Definition 2.1.** A mapping \( T : X \to X \) is said forward (backward) contraction when there exists \( 0 < \alpha < 1 \) such that

\[
d(Tx, Ty) \leq \alpha d(x, y) (d(Tx, Ty) \leq \alpha d(y, x)),
\]

for each \( x, y \in X \).

**Example 2.1.** Let \( X = \mathbb{R} \geq 0 \) be a nonempty set. Consider the map \( d : X \times X \to \mathbb{R} \geq 0 \) defined by

\[
d(x, y) = \begin{cases} 
        y - x, & y \geq x; \\ 
        \frac{1}{4} (x - y), & x > y.
\end{cases}
\]

It is easy to show that \((X, d)\) is an asymmetric metric space. Consider \( T : X \to X \) by \( Tx = \frac{1}{2} x \). Clearly, \( T \) is a forward contraction, whereas it is not backward. For this, let \( \alpha \) be an arbitrary with \( 0 < \alpha < 1 \). Set \( x = 2^{1-\alpha} \) and \( y = 2^{-\alpha} \). Then we have

\[
d(Tx, Ty) = \frac{1}{2} (2^{1-\alpha} - 2^{-2\alpha}), \ d(y, x) = \frac{1}{4} (2^{1-\alpha} - 2^{-2\alpha}).
\]

Since \( 0 < \alpha < 1 \), \( d(Tx, Ty) > \alpha d(y, x) \). Therefore, \( T \) is not a backward contraction.

Next, we prove asymmetric version of Banach’s contraction principle.

**Theorem 2.1.** Let \((X, d)\) be a forward complete space. Let \( T : X \to X \) be a forward contraction. Also, suppose that forward convergence implies backward convergence in \( X \). Then \( T \) has an unique fixed point.

**Proof.** Choose \( x_0 \in X \) and construct the sequence \((x_n)\) as follow:

\[
x_0, x_1 = Tx_0, x_2 = Tx_1 = T^2 x_0, \ldots, x_n = T^m x_0, \ldots
\]

Since \( T \) is a forward contraction we have

\[
d(x_m, x_{m+1}) = d(Tx_{m-1}, Tx_m) \leq \alpha d(x_{m-1}, x_m) = \alpha d(Tx_{m-2}, Tx_{m-1}) \leq \alpha^2 d(x_{m-2}, x_{m-1}) \leq \cdots \leq \alpha^m d(x_0, x_1).
\]

If \( n > m \), then

\[
d(x_m, x_n) \leq d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{n-1}, x_n) \leq (\alpha^m + \alpha^{m+1} + \cdots + \alpha^{n-1}) d(x_0, x_1) = \alpha^m - \alpha^n d(x_0, x_1) \leq \frac{\alpha^m}{1 - \alpha} d(x_0, x_1).
\]

If \( m \) is sufficiently large, then the right side of the last inequality approaches to zero, since \( 0 < \alpha < 1 \). Hence \((x_n)\) is a forward Cauchy sequence in \( X \). Since \( X \) is forward complete, so \( x_n \to x \in X \) as \( n \to \infty \). By assumption \( x_n \xrightarrow{b} x \). Therefore we have

\[
d(x, Tx) \leq d(x, x_n) + d(x_n, Tx) \leq d(x, x_n) + \alpha d(x_{n-1}, x).
\]
Now, the right side of last inequality approaches to zero as \( n \to \infty \). Hence \( d(x, Tx) = 0 \) which means \( Tx = x \). Now, let \( y \in X \) be another fixed point of \( T \), i.e., \( Ty = y \). Then
\[
d(x, y) = d(Tx, Ty) \leq \alpha d(x, y).
\]
Since \( 0 < \alpha < 1 \), so \( d(x, y) = 0 \) which implies \( x = y \).

The following theorem is an another version of Banach’s contraction principle.

**Theorem 2.2.** Let \((X, d)\) be a forward sequentially compact space and \( T : X \to X \) backward contraction. Then \( T \) has an unique fixed point.

**Proof.** Choose \( x_0 \in X \) and construct the sequence \((x_n)\) as the proof of theorem 2.1. Then \((x_n)\) is a backward Cauchy sequence in \( X \). Since \( X \) is backward sequentially compact, so \((x_n)\) has a backward convergent subsequence, say \((x_{n_k})\), which \( x_{n_k} \downarrow x \in X \) as \( k \to \infty \). So, \( x_{n_k} \downarrow x \) by lemma 1.2, also by lemma 1.1, \( x_{n_k} \not\rightarrow x \). Now
\[
d(x, Tx) \leq d(x, x_{n_k}) + d(x_{n_k}, Tx) \leq d(x, x_{n_k}) + \alpha d(x_{n_k-1}, x).
\]
The right side of last inequality approaches to zero as \( n \to \infty \). Hence \( d(x, Tx) = 0 \) which implies \( Tx = x \). Now, Let \( y \in X \) be another fixed point of \( T \), i.e., \( Ty = y \). Then
\[
d(x, y) = d(Tx, Ty) \leq \alpha d(x, y).
\]
Since \( 0 < \alpha < 1 \), so \( d(x, y) = 0 \) which implies \( x = y \).

**Lemma 2.1.** Any forward (backward) contraction is forward (backward) continuous. The proof is clear.

Authors in [8] have completely discussed on the fixed point of a nondecreasing continuous mappings on a partially ordered cone metric spaces. We wish to study the fixed points of nondecreasing contraction on a partially ordered asymmetric metric spaces.

**Theorem 2.3.** Let \((X, \sqsubseteq)\) be a partially ordered set and there exists a metric \( d \) on \( X \) such that \((X, d)\) to be a backward complete asymmetric metric space. Consider \( T : X \to X \) as a forward contraction and nondecreasing w.r.t. \( \sqsubseteq \). If there exists \( x_0 \in X \) with \( x_0 \sqsubseteq Tx_0 \) then \( T \) has a fixed point.

**Proof.** If \( T(x_0) = x_0 \), then there is nothing for proof. Now, let \( T(x_0) \neq x_0 \). Since \( x_0 \sqsubseteq Tx_0 \), we can obtain the following sequence by induction
\[
x_0 \sqsubseteq T(x_0) \sqsubseteq T^2(x_0) \sqsubseteq T^3(x_0) \sqsubseteq \cdots \sqsubseteq T^n(x_0) \sqsubseteq T^{n+1}(x_0).
\]
Also, we have
\[
d(T^{n+1}(x_0), T^n(x_0)) \leq \alpha^n d(T(x_0), x_0).
\]
For all \( n \in \mathbb{N} \), since \( T \) is a forward contraction. By (1) and (2) we have
\[
d(T^{n+2}(x_0), T^{n+1}(x_0)) = d(T(T^{n+1}(x_0)), T(T^n(x_0))
\leq \alpha d(T^{n+1}(x_0), T^n(x_0))
\leq \alpha^{n+1} d(f(x_0), x_0) .
Now, let \( m > n \). Then
\[
d(T^m(x_0), T^n(x_0)) \leq d(T^m(x_0), T^{m-1}(x_0)) + \cdots + d(T^{n+1}(x_0), T^n(x_0)) \\
\leq (\alpha^{m-1} + \alpha^{m-2} + \cdots + \alpha^n)d(T(x_0), x_0) \\
= \frac{\alpha^n - \alpha^m}{1 - \alpha}d(T(x_0), x_0) \leq \frac{\alpha^n}{1 - \alpha}d(T(x_0), x_0).
\]

Now, if \( n \) is sufficiently large, then we deduce \( \{T^n(x_0)\} \) is a backward cauchy sequence in \( X \).
Since \( X \) is backward complete, so there exists \( y \in X \) such that \( T^n(x_0) \overset{b}{\to} y \) as \( n \to \infty \). Finally, we prove that \( y \) is a fixed point of \( T \). Fixed \( \varepsilon > 0 \). \( T \) is forward continuous by lemma 2.1.
Hence there exists \( \delta > 0 \) so that \( d(y, x) < \delta \) implies \( d(T(y), T(x)) < \frac{\varepsilon}{2} \). Set \( \gamma := \min \{ \frac{\varepsilon}{2}, \delta \} \).
Then there exists \( N \in \mathbb{N} \) such that
\[
d(T^n x_0, y) < \gamma, \text{ for all } n \geq N.
\]

Now, we have
\[
d(Ty, y) \leq d(Ty, T(T^n(x_0))) + d(T^{n+1}(x_0), y) < \frac{\varepsilon}{2} + \gamma \leq \varepsilon.
\]

Since \( \varepsilon \) was arbitrary, so \( d(Ty, y) = 0 \) which implies \( Ty = y \).

**Remark 2.1.** Taslim [9], introduced the concept of denseness in asymmetric metric spaces.
In particular, she proved that if \( X \) is forward and backward compact and \( Y \subset X \) both of backward and forward in \( X \), then \( \tau^+ = \tau^- \). Therefore, in the case, all of results in the literature go to symmetric case.

**References**

Schur harmonic convexity of Stolarsky extended mean values

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Abstract In this paper, the Schur harmonic convexity of Stolarsky extended mean values are discussed.

Keywords Stolarsky, harmonic, concavity, convexity.

2000 Mathematics Subject Classification: 26D15

§1. Introduction and preliminaries

In literature the importance and applications of means and its inequality to science and technology is explored by eminent researchers see [3]. In [20-28], we studied some results on contra harmonic mean. In [11,41], authors studied the different properties of the so-called Stolarsky (extended) two parameter mean values, is defined as follows:

\[
E_{r,s}(a, b) = \begin{cases} 
\left[ \frac{r}{s} \left( \frac{a^r - b^r}{a^s - b^s} \right) \right]^\frac{1}{s-r}, & rs(s - r) \neq 0; \\
\exp \left( \frac{1}{s} + \frac{(a^r \ln a - b^r \ln b)}{(a^s - b^s)} \right), & r = s \neq 0; \\
\left( \frac{(a^r \ln a - b^r \ln b)}{s(a^s - b^s)} \right)^\frac{1}{s}, & r = 0, s \neq 0; \\
\sqrt{ab}, & r = s = 0; \\
a, & a = b > 0.
\end{cases}
\] (1)

Some of the classical two parameter means are special cases.

Here we recall some of the known means which are essential for the paper, the arithmetic mean in weighted form,

\[ A_{p,q}(a, b) = pa + qb = A(a, b; p, q); \]

such that \( a, b > 0 \) and \( p + q = 1 \), where \( p \) and \( q \) are the weights.

The Stolarsky means \( S_{p,q}(a, b) \) are \( C^\infty \) function on the domain \( (p, q, a, b) \), \( p, q \in R, a, b > 0 \). Obviously, Stolarsky means \( S_{p,q}(a, b) \) are symmetric with respect to \( a, b \) and \( p, q \). Most of the classical two variable means are special cases of \( S_{p,q}(a, b) \), Stolarsky mean. For example:
Qi’s result and proved that Stolarsky means consider a new mean in the following form: in weighted forms in two and such that conditions from lemma 2, and then prove these conditions are sufficient.

harmonic convexity of Stolarsky means

for fixed concave on $[0, \infty)$ and inequalities are studied in papers [4,9,13,16,17,19,22,23,30-35,44,49,50,51,52,55].

convexity of generalized exponent mean $S$ and obtained a sufficient condition for the Schur convexity of

and declared an incorrect conclusion. Shi [40] Chu and Zhang [6] perfectly solved the Schur convexity of Stolarsky means with respect to $(a, b)$.

Qi [36] tried to obtain the Schur convexity of $S_{p,q}(a,b)$ with respect to $(a,b)$ for fixed $(p,q)$ and declared an incorrect conclusion. Shi [40] observed that the above conclusion is wrong and obtained a sufficient condition for the Schur convexity of $S_{p,q}(a,b)$ with respect to $(a,b)$. Chu and Zhang [6] improved Shi’s results and gave an necessary and sufficient condition. This perfectly solved the Schur convexity of Stolarsky means with respect to $(a,b)$.

For the Schur geometrically convexity, Chu and Zhang [5] proved that Stolarsky means $S_{p,q}(a,b)$ are Schur convex with respect to $(a,b)$, of which the idea is to find the necessary conditions from lemma 2, and then prove these conditions are sufficient.

In [3], the weighted contra harmonic mean is defined on the basis of proportions by,

$$ C_{p,q}(a,b) = \frac{pa^2 + qb^2}{pa + qb} = C(a,b;p,q), $$

such that $a, b > 0$ and $p + q = 1$, where $p$ and $q$ are the weights.

This work motivates us to introduce a new family of Stolarsky’s extended type mean values in weighted forms in two and $n$ variables.

For two variables $a, b > 0, p, q \in R$ and $p, q$ are the weights, such that $p + q = 1$, then consider a new mean in the following form:

$$ N_{r,s}(a,b;p,q) = \left[ \frac{r^2}{s^2} \frac{C(a^r, b^r; p, q) - A(a^r, b^r; p, q)}{C(a^s, b^s; p, q) - A(a^s, b^s; p, q)} \right]^{\frac{1}{r-s}}, $$
which is equivalently,

\[ N_{r,s}(a, b; p, q) = \left[ \frac{r^2}{s^2} \left( \frac{pa^r + qb^r}{pa^s + qb^s} \right) \left( \frac{pa^2s + qb^2s - (pa^s + qb^s)^2}{pa^2r + qb^2r - (pa^r + qb^r)^2} \right) \right]^{\frac{1}{r-s}}, \]

which is equivalently,

\[ N_{r,s}(a, b; p, q) = \left[ \frac{r^2}{s^2} \left( \frac{pa^r + qb^r}{pa^s + qb^s} \right) \left( \frac{a^s - b^s}{a^r - b^r} \right)^2 \right]^{\frac{1}{r-s}}. \]

In [50], the authors introduced the homogeneous function with two parameters \( r \) and \( s \) by,

\[ H_p(a, b; r, s) = \left[ \frac{f(a^s, b^s)}{f(a^r, b^r)} \right]^\frac{1}{r-s}, \]

and studied its monotonicity and deduces some inequalities involving means, where \( f \) is a homogeneous function for \( a \) and \( b \).

In particular, \( f = A \), is the arithmetic mean of \( a \) and \( b \).

\[ H_A(a, b; s, r) = \begin{cases} \left( \frac{(a^s + b^s)^{r}}{(a^s + b^s)} \right)^{\frac{1}{r-s}}, & r \neq s; \\ G_{A,p}(a, b) = a^{\frac{ps}{r+qs}}b^{\frac{qs}{r+ps}}, & r = s \neq 0; \\ \sqrt{ab}, & r = s = 0; \\ a, & a = b > 0. \end{cases} \tag{2} \]

Here, \( G_{A,s}(a, b) = Z_s(a, b) = Z_{A}^{\frac{1}{2}}(a^p, b^p) = Z_s \). \( Z(a, b) = a^{\frac{1}{r+s}}b^{-\frac{s}{r+s}} \) is named power-exponential mean between two positive numbers \( a \) and \( b \).

In weighted form,

\[ H_{A(a,b,p,q)}(a, b; s, r) = \begin{cases} \left( \frac{(pa^r + qb^r)^{s}}{(pa^s + qb^s)} \right)^{\frac{1}{r-s}}, & r \neq s; \\ G_{A(a,b,p,q)}(a, b) = a^{\frac{ps}{r+qs}}b^{\frac{qs}{r+ps}}, & r = s \neq 0; \\ a^p b^q, & r = s = 0; \\ a, & a = b > 0. \end{cases} \tag{3} \]

In [42], author introduced and studied the various properties and log-convexity results of the class \( W \) of weighted two parameter means is given by

\[ W_{r,s}(a, b; p, q) = \begin{cases} \left( \frac{r^2}{s^2} \left( \frac{pa^r + qb^r - a^s b^s}{pa^s + qb^s - a^r b^r} \right) \right)^{\frac{1}{r-s}}, & rs(s-r)(a-b) \neq 0; \\ \frac{2}{(r+1)^2} \left( \frac{pa^r + qb^r - a^s b^s}{pa^s + qb^s - a^r b^r} \right)^2, & s(a-b) \neq 0, r = 0; \\ e^{\exp \left( \frac{\pi}{a} + \frac{pa^r + qb^r - a^s b^s}{pa^s + qb^s - a^r b^r} \right)}, & r = s, s \neq 0; \\ a^{(r+1)/2}b^{(s+1)/2}, & a \neq b, r = s = 0; \\ a, & a = b > 0. \end{cases} \tag{4} \]

The above definitions leads to express the mean values \( N_{r,s}(a, b; p, q) \) in the following form:

\[ N_{r,s}(a, b; p, q) = \left[ \frac{pa^r + qb^r}{pa^s + qb^s} \right]^{\frac{1}{r-s}} \left[ \left( \frac{r}{s} \left( a^s - b^s \right) \right)^{\frac{1}{r-s}} \right]^2, \]
which is equivalently

\[ N_{r,s}(a,b;p,q) = \left[ \frac{f(a^r, b^r; p, q)}{f(a^s, b^s; p, q)} \right] \frac{1}{r-s} E_{r,s}(a,b), \]

or

\[ N_{r,s}(a,b;p,q) = H_{f=A(a,b;p,q)}(a,b;s,r)E_{r,s}^2(a,b). \] (5)

Here \( f = A(a,b;p,q) \) is arithmetic mean in weighted form.

The various properties and identities concerning to \( N_{r,s}(a,b;p,q) \) are also listed. The laborious calculations gives the following different cases of the mean value \( N_{r,s}(a,b;p,q) \).

\[ N_{r,s}(a,b;p,q) = \begin{cases} 
\frac{r^2 \left( \frac{pa^r+qb^r}{pa^r+qb^r} \right) \left( \frac{x^r-b^r}{x^r-b^r} \right)^2}{1}, & rs(s-r)(a-b) \neq 0; \\
\frac{2}{s} \exp \left( -\frac{2}{s} - \frac{pa^s ln_a+qb^s ln_b}{pa^s+qb^s} + 2 \frac{a^s ln_a-b^s ln_b}{a^r-b^r} \right), & s(a-b) \neq 0, r=0; \\
a^{1-p}b^{1-q}, & a \neq b, r=s=0; \\
a, & a=b>0.
\end{cases} \] (6)

§2. Definition and properties

Schur convexity was introduced by Schur in 1923 \cite{24}, and it has many important applications in analytic inequalities \cite{2,12,54}, linear regression \cite{43}, graphs and matrices \cite{8}, combinatorial optimization \cite{15}, information theoretic topics \cite{10}, Gamma functions \cite{25}, stochastic orderings \cite{39}, reliability \cite{14}, and other related fields. Recently, Anderson \cite{1} discussed an attractive class of inequalities, which arise from the notation of harmonic convexity. For convenience of readers, we recall some definitions as follows.

We recall the definitions which are essential to develop this paper.

**Definition 2.1.** \cite{24,46} Let \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \in R^n \),

1. \( x \) is majorized by \( y \) (in symbol \( x \prec y \)). If \( \sum_{i=1}^k x_{[i]} \leq \sum_{i=1}^k y_{[i]} \) and \( \sum_{i=1}^n x_{[i]} \leq \sum_{i=1}^n y_{[i]} \), where \( x_{[1]} \geq \cdots \geq x_{[n]} \) and \( y_{[1]} \geq \cdots \geq y_{[n]} \) are rearrangements of \( x \) and \( y \) in descending order.

2. \( x \succeq y \) means \( x_i \geq y_i \) for all \( i = 1, 2, \ldots, n \). Let \( \Omega \in R^n(n \geq 2) \). The function \( \varphi : \Omega \to R \) is said to be decreasing if and only if \( -\varphi \) is increasing.

3. \( \Omega \subseteq R^n \) is called a convex set if \( (\alpha x_1 + \beta y_1, \cdots, \alpha x_n + \beta y_n) \) for every \( x \) and \( y \in \Omega \) where \( \alpha, \beta \in [0,1] \) with \( \alpha + \beta = 1 \).

4. Let \( \Omega \subseteq R^n \) the function \( \varphi : \Omega \to R \) be said to be a schur convex function on \( \Omega \) if \( x \preceq y \) on \( \Omega \) implies \( \varphi(x) \leq \varphi(y) \). \( \varphi \) is said to be a schur concave function on \( \Omega \) if and only if \( -\varphi \) is schur convex.
Definition 2.2. [55] Let \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \in R^n_+ \), \( \Omega \subseteq R^n \) is called Harmonically convex set if \( (x_1^\alpha y_1^{\beta}, \ldots, x_n^\alpha y_n^{\beta}) \in \Omega \) for all \( x \) and \( y \in \Omega \), where \( \alpha, \beta \in [0, 1] \) with \( \alpha + \beta = 1 \).

Let \( \Omega \subseteq R^n_+ \), the function \( \varphi : \Omega \rightarrow R_+ \) is said to be schur Harmonically convex function on \( \Omega \), if \( (b x_1, \ldots, b x_n) < (b y_1, \ldots, b y_n) \) on \( \Omega \) implies \( \varphi(x) \leq \varphi(y) \). Then \( \varphi \) is said to be a schur Harmonically concave function on \( \Omega \) if and only if \( -\varphi \) is schur Harmonically convex.

Definition 2.3. [24, 46] Let \( \Omega \subseteq R^n \) is called symmetric set if \( x \in \Omega \) implies \( P x \in \Omega \) for every \( n \times n \) permutation matrix \( P \), the function \( \varphi : \Omega \rightarrow R \) is called symmetric if for every permutation matrix \( P \), \( \varphi(P x) = \varphi(x) \) for all \( x \in \Omega \).

Definition 2.4. [24, 46] Let \( \Omega \subseteq R^n \), \( \varphi : \Omega \rightarrow R \) is called symmetric and convex function. Then \( \varphi \) is schur convex on \( \Omega \).

Lemma 2.1. [55] Let \( \Omega \subseteq R^n \) be symmetric with non empty interior Harmonically convex set and let \( \varphi : \Omega \rightarrow R_+ \) be continuous on \( \Omega \) and differentiable in \( \Omega^0 \). If \( \varphi \) is symmetric on \( \Omega \) and
\[
(x_1 - x_2)(x_1^2 \frac{\partial \varphi}{\partial x_1} - x_2^2 \frac{\partial \varphi}{\partial x_2}) \geq 0(\leq 0),
\]
holds for any \( x = (x_1, x_2, \ldots, x_n) \in \Omega^0 \), then \( \varphi \) is a schur-Harmonically convex (Schur-Harmonically concave concave function).

Lemma 2.2. [46] Let \( a \leq b, u(t) = ta + (1 - t)b, v(t) = tb + (1 - t)a, \frac{1}{2} \leq t_2 \leq t_1 \leq 1, 0 \leq t_1 \leq t_2 \leq \frac{1}{2} \), then
\[
\frac{(a + b)}{2} \times (u(t_2), v(t_2)) \times (u(t_1), v(t_1)).
\]

§3. Main results

In this section, we shall prove some the lemmas required for proving main theorem.

Lemma 3.1. Stolarsky’s extended family type means \( N_{p,q}(a, b; r, s) \) are schur-Harmonic convex or Schur-Harmonic concave with respect to \( (a, b) \in (0, \infty) \times (0, \infty) \) if and only if \( g(t) \geq 0 \) or \( g(t) \leq 0 \) for all \( t > 0 \), where
\[
g(t) = g_{p,q}(t)
\]
and
\[
A = p + q + 1, \quad B = p - q + 1, \quad C = p - q - 1, \quad D = p + q, \quad E = p - q.
\]
\textbf{Proof.} Let Stolarsky’s extended family type mean \( N = N_{p,q}(a, b; r, s) \) defined for \( pq(p - q) \neq 0 \) as

\[ N = N_{p,q}(a, b; r, s) = \left[ \frac{q^2}{p^2} \left( \frac{ra^p + sb^p}{ra^q + sb^q} \right) \left( \frac{a^p - b^p}{a^q - b^q} \right)^2 \right]^{\frac{1}{p-q}}. \]  \hspace{1cm} (11)

Let \( r = s = \frac{1}{2} \), take log on both sides and differentiate partially with respect to \( a \) and multiply by \( a^2 \), gives

\[ a^2 \frac{\partial N}{\partial a} = \frac{N}{p-q} \left[ \frac{q(a^{q+1} - b^{q+1})}{a^q + b^q} - \frac{p(a^{p+1} - b^{p+1})}{a^p + b^p} + 2 \frac{ba^{p+1}}{a^p - b^p} - 2 \frac{qa^{q+1}}{a^q - b^q} \right]. \]  \hspace{1cm} (12)

Similarly,

\[ b^2 \frac{\partial N}{\partial a} = \frac{N}{p-q} \left[ \frac{q(b^{q+1} - a^{q+1})}{b^q + a^q} - \frac{p(b^{p+1} - a^{p+1})}{b^p + a^p} + 2 \frac{ba^{p+1}}{a^p - b^p} - 2 \frac{qa^{q+1}}{a^q - b^q} \right], \]  \hspace{1cm} (13)

then,

\[ (a - b) \left( a^2 \frac{\partial G}{\partial a} - b^2 \frac{\partial G}{\partial b} \right) = \frac{(a - b)N}{p-q} \left[ \Delta \right], \]  \hspace{1cm} (14)

where,

\[ \Delta = \left[ \frac{q(a^{q+1} - b^{q+1})}{a^q + b^q} - \frac{p(a^{p+1} - b^{p+1})}{a^p + b^p} + 2 \frac{ba^{p+1}}{a^p - b^p} - 2 \frac{qa^{q+1}}{a^q - b^q} \right]. \]

Substituting \( \ln \sqrt{a/b} = t \) and using \( \sinh x = \frac{1}{2} (e^x - e^{-x}) \), \( \cosh x = \frac{1}{2} (e^x + e^{-x}) \), we have

\[ \Delta = \sqrt{ab} \left[ \frac{\sinh(q+1)t}{\cosh qt} - p \frac{\sinh(p+1)t}{\cosh pt} + 2p \frac{\cosh(p+1)t}{\sinh pt} - 2q \frac{\cosh(q+1)t}{\sinh qt} \right]. \]

Using the product into sum formula for hyperbolic functions leads to: For \( pq(p - q) \neq 0 \)

\[ (a - b) \left( a^2 \frac{\partial G}{\partial a} - b^2 \frac{\partial G}{\partial b} \right) = \frac{(pq)N(a - b)\sqrt{ab}}{2 \sinh pt \sinh qt \cosh pt \cosh qt} [g_{p,q}(t)], \]  \hspace{1cm} (15)

where,

\[ g_{p,q}(t) = \frac{[(p - q) \sinh At \{ \cosh(p + q)t + 3 \cosh(p - q)t \}]}{pq(p - q)} \]

\[ - \frac{[(p \sinh Bt + q \sinh Ct) \{ 3 \cosh(p + q)t + \cosh(p - q)t \}]}{pq(p - q)}. \]  \hspace{1cm} (16)

In case of \( p \neq q = 0 \). Since \( N_{p,q} \in C^{\infty} \), we have

\[ \frac{\partial N_{p,0}}{\partial a} = \lim_{q \to 0} \frac{\partial N_{p,q}}{\partial a}, \quad \frac{\partial N_{p,0}}{\partial b} = \lim_{q \to 0} \frac{\partial N_{p,q}}{\partial b}, \]

\[ \frac{\partial N_{p,p}}{\partial a} = \lim_{q \to p} \frac{\partial N_{p,q}}{\partial a}, \quad \frac{\partial N_{p,p}}{\partial b} = \lim_{q \to p} \frac{\partial N_{p,q}}{\partial b}, \]

\[ \frac{\partial N_{0,0}}{\partial a} = \lim_{p \to 0} \frac{\partial N_{p,q}}{\partial a}, \quad \frac{\partial N_{0,0}}{\partial b} = \lim_{p \to 0} \frac{\partial N_{p,q}}{\partial b}. \]
Thus
\[
(a - b) \left( a^2 \frac{\partial N_{p,0}}{\partial a} - b^2 \frac{\partial N_{p,0}}{\partial b} \right) = \lim_{q \to 0} \left[ (a - b) \left( a^2 \frac{\partial N_{p,0}}{\partial a} - b^2 \frac{\partial N_{p,0}}{\partial b} \right) \right]
\]
\[
= \lim_{q \to 0} \left( \frac{(pq)N_{p,q}(a - b)\sqrt{ab}}{2 \sinh pt \sinh qt \cosh pt \cosh qt} \left[ g_{p,q}(t) \right] \right) = \frac{(p)N_{p,0}(a - b)\sqrt{ab}}{2t \sinh pt \cosh pt} \left[ g_{p,0}(t) \right].
\]
Likewise for \( q \neq p = 0 \),
\[
(a - b) \left( a^2 \frac{\partial N_{0,q}}{\partial a} - b^2 \frac{\partial N_{0,q}}{\partial b} \right) = \lim_{p \to 0} \left[ (a - b) \left( a^2 \frac{\partial N_{0,q}}{\partial a} - b^2 \frac{\partial N_{0,q}}{\partial b} \right) \right]
\]
\[
= \lim_{q \to p} \left( \frac{(pq)N_{p,q}(a - b)\sqrt{ab}}{2 \sinh pt \cosh qt \cosh pt} \left[ g_{p,q}(t) \right] \right) = \frac{(p^2)N_{p,0}(a - b)\sqrt{ab}}{2t \sinh pt \cosh pt} \left[ g_{p,0}(t) \right].
\]
For \( q = p \neq 0 \),
\[
(a - b) \left( a^2 \frac{\partial N_{0,0}}{\partial a} - b^2 \frac{\partial N_{0,0}}{\partial b} \right) = \lim_{p \to 0} \left[ (a - b) \left( a^2 \frac{\partial N_{0,0}}{\partial a} - b^2 \frac{\partial N_{0,0}}{\partial b} \right) \right]
\]
\[
= \lim_{q \to 0} \left( \frac{(pq)N_{p,p}(a - b)\sqrt{ab}}{2 \sinh^2 pt \cosh^2 pt} \left[ g_{p,p}(t) \right] \right) = \frac{N_{0,0}(a - b)\sqrt{ab}}{2t^2} \left[ g_{0,0}(t) \right].
\]
By summarizing all cases above yield
\[
(a - b) \left( a^2 \frac{\partial N}{\partial a} - b^2 \frac{\partial N}{\partial b} \right) = \begin{cases} 
\frac{(pq)N(a-b)\sqrt{ab}}{2 \sinh pt \sinh qt \cosh pt \cosh qt} \left[ g_{p,q}(t) \right], & pq(p - q) \neq 0; \\
\frac{(p)N_{0,q}(a-b)\sqrt{ab}}{2t \sinh qt \cosh pt} \left[ g_{0,q}(t) \right], & p = 0, q \neq 0; \\
\frac{(p^2)N_{p,0}(a-b)\sqrt{ab}}{2t \sinh pt \cosh qt} \left[ g_{p,0}(t) \right], & p \neq 0, q = 0; \\
\frac{(p^2)N_{p,p}(a-b)\sqrt{ab}}{2t^2} \left[ g_{p,p}(t) \right], & p = q \neq 0; \\
\frac{N_{0,0}(a-b)\sqrt{ab}}{2t^2} \left[ g_{0,0}(t) \right], & p = q = 0.
\end{cases}
\]

Since \((a - b) \left( a^2 \frac{\partial N}{\partial a} - b^2 \frac{\partial N}{\partial b} \right)\) is symmetric with respect to \( a \) and \( b \), without loss of generality we assume \( a > b \), then \( t = \ln \sqrt{ab} > 0 \). It is easy to verify that \( \frac{(a-b)\sqrt{ab}}{2t} > 0 \), if \( pq \neq 0 \) for \( t > 0 \). Thus by lemma 2 Stolarsky means \( S_{p,q}(a,b) \) are Schur harmonic convex (Schur harmonic concave) with respect to \((a,b) \in (0,\infty) \times (0,\infty)\), if and only if \((a - b) \left( a^2 \frac{\partial N}{\partial a} - b^2 \frac{\partial N}{\partial b} \right) \geq 0\), if and only if \( g(t) = g_{p,q}(t) \geq 0\) for all \( t > 0 \). This completes the proof of lemma 3.1.

**Lemma 3.2.** The function \( g(t) = g_{p,q}(t) \) defined by (3.1) and \( g'(t) = \frac{\partial g_{p,q}(t)}{\partial t} \) both are symmetric with respect to \( p \) and \( q \), and both continuous with respect to \( p \) and \( q \) on \( R \times R \).
Proof. It is easy to check that $g_{p,q}(t)$ and $\frac{\partial g_{p,q}(t)}{\partial t}$ are symmetric with respect to $p$ and $q$, then $\frac{\partial g_{p,q}(t)}{\partial t} = \frac{\partial g_{q,p}(t)}{\partial t}$.

By lemma 3.1, we note that $g(t) = g_{p,q}(t)$ is continuous with respect to $p$ and $q$ on $R \times R$.

Finally, we prove that $g'(t) = \frac{\partial g_{p,q}(t)}{\partial t}$ is also continuous with respect to $p$ and $q$ on $R \times R$.

A simple calculations yield,

Case i: For $pq(p - q) \neq 0$,

$$g'(t) = \frac{\partial g_{p,q}(t)}{\partial t} = \frac{(p - q) \cosh At \{\cosh(p + q)t + 3\cosh(p - q)t\}}{pq(p - q)}$$

$$+ \frac{(p - q) \sinh At \{(p + q) \sinh(p + q)t + 3(p - q) \sinh(p - q)t\}}{pq(p - q)}$$

$$- \frac{(pB \cosh Bt + qC \cosh Ct)\{3\cosh(p + q)t + \cosh(p - q)t\}}{pq(p - q)}$$

$$- \frac{pq(p - q)}{pq(p - q)}.$$

Case ii: For $q = 0, p \neq 0$,

$$g'(t) = \frac{\partial g_{p,q}(t)}{\partial t} = \frac{2p \cosh(1 + 2p)t + 2pt(1 + 2p) \sinh(2p + 1)t - 6p \cosh t}{-p^2}$$

$$- \frac{6pt \sinh t - 2(2p + 1) \cosh(2p + 1)t - (2p - 1) \cosh(2p - 1)t}{-p^2}.$$

Case iii: For $p = 0, q \neq 0$,

$$g'(t) = \frac{\partial g_{p,q}(t)}{\partial t} = \frac{2q \cosh(1 + 2q)t + 2qt(1 + 2q) \sinh(2q + 1)t - 6q \cosh t}{-p^2}$$

$$- \frac{6qt \sinh t - 2(2q + 1) \cosh(2q + 1)t - (2q - 1) \cosh(2q - 1)t}{-p^2}.$$

Case iv: For $p = q \neq 0$,

$$g'(t) = \frac{\partial g_{p,q}(t)}{\partial t} = \frac{(1 + 2q)(3 + \cosh 2qt) \cosh(2q + 1)t - 6q(\sinh 2qt)(2qt \cosh t + \sinh t)}{q^2}$$

$$- \frac{(1 + 3 \cosh 2qt)((1 + 2q) \cosh t + 2qt \sinh t) + 2q(\sinh 2qt) \sinh(2q + 1)t}{q^2}.$$

Case v: For $p = q = 0$,

$$g'(t) = \frac{\partial g_{p,q}(t)}{\partial t} = 4t^2 \cosh t + 8t \sinh t.$$
It is obvious that \( \frac{\partial g_{p,q}(t)}{\partial t} \) is continuous with respect to \( p \) and \( q \) on \( R \times R \), again in view of

\[
\lim_{q \to 0} \frac{\partial g_{p,q}(t)}{\partial t} = \frac{\partial g_{p,0}(t)}{\partial t}; \quad \lim_{p \to 0} \frac{\partial g_{p,q}(t)}{\partial t} = \frac{\partial g_{0,q}(t)}{\partial t};
\]

\[
\lim_{q \to p} \frac{\partial g_{p,q}(t)}{\partial t} = \frac{\partial g_{p,p}(t)}{\partial t}; \quad \lim_{p \to 0} \frac{\partial g_{p,p}(t)}{\partial t} = \frac{\partial g_{0,0}(t)}{\partial t}.
\]

These arguments leads that the above five cases are continuous for all values of \( p \) and \( q \). This completes the proof of lemma 3.2.

**Lemma 3.3.** \( \lim_{t \to 0, t>0} t^{-3} g(t) = \frac{1}{3} (p + q - 3) \).

**Proof.**

It is easy to check that first and second derivatives of \( g(t) = 0 \), at \( t = 0 \). In the case of \( pq(p - q) \neq 0 \). Applying L-Hospital’s rule (three times) yields

\[
\lim_{t \to 0, t>0} \frac{g_{p,q}(t)}{t^3} = \lim_{t \to 0, t>0} \frac{g'_{p,q}(t)}{3t^2} = \cdots = \lim_{t \to 0, t>0} \frac{g''_{p,q}(t)}{6} = \frac{-4}{3} (p + q - 3).
\]

Similarly, for \( p = 0 \), \( q \neq 0 \),

\[
\lim_{t \to 0, t>0} \frac{g_{p,0}(t)}{t^3} = \lim_{t \to 0, t>0} \frac{g'_{p,0}(t)}{3t^2} = \cdots = \lim_{t \to 0, t>0} \frac{g''_{p,0}(t)}{6} = \frac{-4}{3} (q - 3);
\]

for \( q = 0 \), \( p \neq 0 \),

\[
\lim_{t \to 0, t>0} \frac{g_{p,p}(t)}{t^3} = \lim_{t \to 0, t>0} \frac{g'_{p,p}(t)}{3t^2} = \cdots = \lim_{t \to 0, t>0} \frac{g''_{p,p}(t)}{6} = \frac{-4}{3} (p - 3);
\]

for \( p = q \neq 0 \),

\[
\lim_{t \to 0, t>0} \frac{g_{p,p}(t)}{t^3} = \lim_{t \to 0, t>0} \frac{g'_{p,p}(t)}{3t^2} = \cdots = \lim_{t \to 0, t>0} \frac{g''_{p,p}(t)}{6} = \frac{-4}{3} (2p - 3);
\]

for \( p = q = 0 \),

\[
\lim_{t \to 0, t>0} \frac{g_{0,0}(t)}{t^3} = \lim_{t \to 0, t>0} \frac{g'_{0,0}(t)}{3t^2} = \cdots = \lim_{t \to 0, t>0} \frac{g''_{0,0}(t)}{6} = 4.
\]

This completes the proof.

The proof of our main result stated below follows from the above lemmas:

**Theorem 3.1.** For fixed \( (p, q) \in R \times R \),

(1) Stolarsky’s extended family type mean means \( N_{r,s}(a, b; p, q) \) are Schur harmonic convex with respect to \( (a, b) \) if \( p + q - 3 \geq 0 \).

(2) Stolarsky’s extended family type mean means \( N_{r,s}(a, b; p, q) \) are Schur harmonic concave if \( p + q - 3 \leq 0 \).

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Differential sandwich-type results for starlike functions

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Abstract In this paper, we present a generalized criterion for starlike univalent functions. We show that our result unifies some known results of starlike functions. Using the dual concept of differential subordination and superordination, we find some sandwich-type results regarding starlike univalent functions. Mathematica 7.0 is used to plot the images of the unit disk under certain functions.

Keywords differential subordination, differential superordination, starlike function.

§1. Introduction

Let $H$ be the class of functions analytic in the open unit disk $\mathbb{E} = \{ z : |z| < 1 \}$ and for $a \in \mathbb{C}$ (complex plane) and $n \in \mathbb{N}$ (set of natural numbers), let $H[a,n]$ be the subclass of $H$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $A$ be the class of functions $f$, analytic in $\mathbb{E}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Let $S$ denote the class of all analytic univalent functions $f$ defined in the unit disk $\mathbb{E}$ which are normalized by the conditions $f(0) = f'(0) - 1 = 0$.

A function $f \in A$ is said to be starlike in the open unit disk $\mathbb{E}$ if it is univalent in $\mathbb{E}$ and $f(\mathbb{E})$ is a starlike domain. Denote by $S^*(\alpha)$, the class of starlike functions of order $\alpha$ which is analytically defined as follows:

$$S^*(\alpha) = \left\{ f \in A : \Re \left( \frac{zf'(z)}{f(z)} \right) > \alpha, z \in \mathbb{E} \right\},$$

where $\alpha (0 \leq \alpha < 1)$ is a real number. Write $S^* = S^*(0)$, the class of univalent starlike w.r.t. the origin.

If $f$ is analytic and $g$ is analytic univalent in the open unit disk $\mathbb{E}$, we say that $f$ is subordinate to $g$ in $\mathbb{E}$ and write as $f(z) \prec g(z)$ if $f(0) = g(0)$ and $f(\mathbb{E}) \subset g(\mathbb{E})$. To derive certain sandwich-type results, we use the dual concept of differential subordination and superordination.

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, $p$ be an analytic function in $\mathbb{E}$ such that $(p(z),zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and $h$ be univalent in $\mathbb{E}$. Then the function $p$ is said to satisfy first order differential subordination if

$$\Phi(p(z),zp'(z); z) \prec h(z), \Phi(p(0),0;0) = h(0). \quad (1)$$
A univalent function $q$ is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p(z) < q(z)$ for all $p$ satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} < q$ for all dominants $q$ of (1), is said to be the best dominant of (1).

Let $\Psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be analytic and univalent in domain $\mathbb{C}^2 \times \mathbb{E}$, $h$ be analytic in $\mathbb{E}$, $p$ be analytic univalent in $\mathbb{E}$, with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$. Then $p$ is called a solution of the first order differential superordination if

$$h(z) \prec \Psi(p(z), zp'(z); z), \; h(0) = \Psi(p(0), 0; 0). \tag{2}$$

An analytic function $q$ is called a subordinant of the differential superordination (2), if $q(z) \prec p(z)$ for all $p$ satisfying (2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants $q$ of (2), is said to be the best subordinant of (2).

The expressions $zf'(z)/f(z)$ and $1 + zf''(z)/f'(z)$ play an important role in the theory of univalent functions. Several new classes have been introduced and studied by various researchers by combining these expressions in different manners. For example, in 1976, Lewandowski et al. [2] proved the following result.

**Theorem 1.1.** If $f \in A$ satisfies

$$\Re \left[ \frac{zf'(z)}{f(z)} \left( \frac{zf''(z)}{f'(z)} + 1 \right) \right] > 0, \; z \in \mathbb{E},$$

then $f \in S^*$. In 2002, Li and Owa [3] generalized and improved the above result by proving the next two results.

**Theorem 1.2.** If $f \in A$ satisfies

$$\Re \left[ \frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right] > -\frac{\alpha}{2}, \; z \in \mathbb{E},$$

for some $\alpha(\alpha \geq 0)$, then $f \in S^*$. 

**Theorem 1.3.** If $f \in A$ satisfies

$$\Re \left[ \frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right] > -(1 - \alpha)\frac{\alpha^2}{4}, \; z \in \mathbb{E},$$

for some $\alpha(0 < \alpha \leq 2)$, then $f \in S^*(\alpha/2)$.

Ravichandran et al. [7] improved the above results further and gave the following result.

**Theorem 1.4.** If $f \in A$ satisfies

$$\Re \left[ \frac{zf'(z)}{f(z)} \left( \alpha \frac{zf''(z)}{f'(z)} + 1 \right) \right] > \alpha\beta \left( \beta - \frac{1}{2} \right) + \left( \beta - \frac{\alpha}{2} \right), \; z \in \mathbb{E},$$

for some $\alpha(0 \leq \alpha, \beta \leq 1)$, then $f \in S^*(\beta)$.

The main objective of this paper is to generalize and improve the results of above nature and obtain certain sandwich-type results for starlike functions. Mathematica 7.0 is used to plot the images of the open unit disk $\mathbb{E}$ under certain functions.
§2. Preliminaries

We shall use the following definition and lemmas to prove our main results.

Definition 2.1. ([5], p.21, definition 2.2b) We denote by $Q$ the set of functions $p$ that are analytic and injective on $\mathbb{E} \setminus \mathbb{B}(p)$, where

$$\mathbb{B}(p) = \left\{ \zeta \in \partial \mathbb{E} : \lim_{z \to \zeta} p(z) = \infty \right\},$$

and are such that $p'(\zeta) \neq 0$ for $\zeta \in \partial \mathbb{E} \setminus \mathbb{B}(p)$.

Lemma 2.1. ([5], p.132, theorem 3.4 h) Let $q$ be univalent in $\mathbb{E}$ and let $\theta$ and $\phi$ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q_1(z)$ and suppose that one of the following conditions satisfies:

1. $h$ is convex,
2. $Q_1$ is starlike.

In addition, assume that $\Re \frac{zh'(z)}{Q_1(z)} > 0$, $z \in \mathbb{E}$. If $p$ is analytic in $\mathbb{E}$, with $p(0) = q(0), p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)],$$

then $p(z) \prec q(z)$ and $q$ is the best dominant.

Lemma 2.2. ([1]) Let $q$ be univalent in $\mathbb{E}$ and let $\theta$ and $\phi$ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\phi[q(z)]$, $h(z) = \theta[q(z)] + Q_1(z)$ and suppose that $Q_1$ is starlike in $\mathbb{E}$ and $\Re \frac{\theta'(q(z))}{\phi[q(z)]} > 0$, $z \in \mathbb{E}$. If $p \in \mathcal{H}(q(0),1) \cap Q$, with $p(\mathbb{E}) \subset \mathbb{D}$ and $\theta[p(z)] + zp'(z)\phi[p(z)]$ is univalent in $\mathbb{E}$ and

$$\theta[q(z)] + zq'(z)\phi[q(z)] \prec \theta[p(z)] + zp'(z)\phi[p(z)],$$

then $q(z) \prec p(z)$ and $q$ is the best subordinant.

§3. Main results

In what follows, the value of any complex power taken is the principal one.

Theorem 3.1. Let $q$, $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that

1. $\Re \left( 1 + zq''(z) \frac{q'(z)}{q(z)} + (\gamma - 1)zq'(z) \frac{q'(z)}{q(z)} \right) > 0,$
2. $\Re \left( 1 + zq''(z) \frac{q'(z)}{q(z)} + (\gamma - 1)zq'(z) \frac{q'(z)}{q(z)} + (\gamma + 1)q(z) + \frac{(1 - \alpha)\gamma}{\alpha} \right) > 0$ for all $z \in \mathbb{E}.$

If $f \in \mathcal{A}$, $\frac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfies the differential subordination

$$\left( \frac{zf'(z)}{f(z)} \right)^\gamma \left( 1 + \alpha \frac{zf''(z)}{f'(z)} \right) \prec \alpha zq'(z)q^{-1}(z) + \alpha q^\gamma(z) + (1 - \alpha)q^\gamma(z),$$

where $\alpha, \gamma$ are complex numbers with $\alpha \neq 0$, then $\frac{zf'(z)}{f(z)} \prec q(z)$ and $q$ is the best dominant.
**Proof.** On writing $p(z) = \frac{zf'(z)}{f(z)}$, the subordination (3) becomes:

$$\alpha q'(z)p^{\gamma-1}(z) + \alpha q^{\gamma+1}(z) + (1 - \alpha)p^{\gamma}(z) \prec \alpha q'(z)q^{\gamma-1}(z) + \alpha q^{\gamma+1}(z) + (1 - \alpha)q^{\gamma}(z).$$  

(4)

Define the functions $\theta$ and $\phi$ as under:

$$\theta(w) = \alpha w^{\gamma+1} + (1 - \alpha)w^{\gamma} \quad \text{and} \quad \phi(w) = \alpha w^{\gamma-1}.$$

Obviously, the functions $\theta$ and $\phi$ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\phi(w) \neq 0$, $w \in \mathbb{D}$.

Setting the functions $Q_1$ and $h$ as follows:

$$Q_1(z) = q'(z)\phi(q(z)) = \alpha q'(z)q^{\gamma-1}(z),$$

and

$$h(z) = \theta(q(z)) + Q_1(z) = \alpha q'(z)q^{\gamma-1}(z) + \alpha q^{\gamma+1}(z) + (1 - \alpha)q^{\gamma}(z).$$

A little calculation yields

$$\frac{zQ'_1(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1)\frac{zq'(z)}{q(z)},$$

and

$$\frac{zh'(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1)\frac{zq'(z)}{q(z)} + (\gamma + 1)q(z) + \frac{(1 - \alpha)\gamma}{\alpha}.$$

In view of the given conditions 1 and 2, we have that $Q_1$ is starlike in $\mathbb{E}$ and $\Re \left( \frac{zh'(z)}{Q_1(z)} \right) > 0$, $z \in \mathbb{E}$. Thus conditions 2 of lemma 2.1, is satisfied. In view of (4), we have

$$\theta[p(z)] + zq'(z)\phi[p(z)] \prec \theta[q(z)] + zq'(z)\phi[q(z)].$$

Therefore, the proof follows from lemma 2.1.

**Theorem 3.2.** Let $q$, $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that

1. $\Re \left( 1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1)\frac{zq'(z)}{q(z)} \right) > 0,$

2. $\Re \left( (\gamma + 1)q(z) + \frac{(1 - \alpha)\gamma}{\alpha} \right) > 0$ for all $z \in \mathbb{E}$.

If $f \in \mathcal{A}$, $\frac{zf'(z)}{f(z)} \in \mathcal{H}[q(0), 1] \cap Q$ with $\frac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfies the differential superordination

$$\alpha q'(z)q^{\gamma-1}(z) + \alpha q^{\gamma+1}(z) + (1 - \alpha)q^{\gamma}(z) \prec \left( \frac{zf'(z)}{f(z)} \right)^{\gamma} \left( \frac{zf''(z)}{f'(z)} \right) = h(z),$$

(5)

where $\alpha$, $\gamma$ are complex numbers with $\alpha \neq 0$ and $h$ is univalent in $\mathbb{E}$, then $q(z) \prec \frac{zf'(z)}{f(z)}$ and $q$ is the best subordinant.

**Proof.** Setting $p(z) = \frac{zf'(z)}{f(z)}$, the superordination (5) becomes:

$$\alpha q'(z)q^{\gamma-1}(z) + \alpha q^{\gamma+1}(z) + (1 - \alpha)q^{\gamma}(z) \prec \alpha zp'(z)p^{\gamma-1}(z) + \alpha p^{\gamma+1}(z) + (1 - \alpha)p^{\gamma}(z).$$

(6)
By defining the functions $\theta$, $\phi$ and $Q_1$ same as in case of theorem 3.1 and observing that
\[
\frac{\theta'(q(z))}{\phi(q(z))} = (γ + 1)q(z) + \frac{(1 - α)γ}{α}.
\]
The use of lemma 2.2 along with (6) completes the proof on the same lines as in case of theorem 3.1.

On combining theorem 3.1 and theorem 3.2, we obtain the following sandwich-type theorem.

**Theorem 3.3.** Suppose $α$, $γ$ are complex numbers with $α \neq 0$ and suppose that $q_1, q_2$ ($q_1(z) \neq 0, q_2(z) \neq 0, z \in \mathbb{E}$) are univalent functions in $E$ such that $q_1$ satisfies the conditions 1 and 2 of theorem 3.2 and $q_2$ follows the conditions 1 and 2 of theorem 3.1. If $f \in A$, $\frac{zf'(z)}{f(z)} \in \mathcal{H}[q_1(0), 1] \cap Q$ with $\frac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfies the differential sandwich-type condition
\[
αq_1'(z)q_1^{-1}(z) + αq_1^{γ+1}(z) + (1 - α)q_1'(z) \prec h(z) = \left( \frac{zf'(z)}{f(z)} \right)^γ \frac{zf''(z)}{f'(z)} \prec αq_2'(z)q_2^{-1}(z) + αq_2^{γ+1}(z) + (1 - α)q_2'(z),
\]
where $h$ is univalent in $\mathbb{E}$, then $q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z)$. Moreover $q_1$ and $q_2$ are, respectively, the best subordinant and the best dominant.

§4. Deductions

If we consider the dominant $q(z) = \frac{1 + (1 - 2λ)z}{1 - z}, 0 \leq λ < 1$, a little calculation yields that this dominant satisfies the conditions of theorem 3.1 in following particular cases. Select $γ = 1$ in theorem 3.1, we get the following result.

**Corollary 4.1.** Suppose that $α$ ($0 < α \leq 1$) is a real number and if $f \in A$, $\frac{zf'(z)}{f(z)} \neq 0$ in $\mathbb{E}$, satisfies
\[
\frac{zf'(z)}{f(z)} \left( 1 + α \frac{zf''(z)}{f'(z)} \right) < \frac{2α(1 - λ)z}{(1 - z)^2} + α \left( \frac{1 + (1 - 2λ)z}{1 - z} \right)^2 + (1 - α) \frac{1 + (1 - 2λ)z}{1 - z},
\]
then
\[
\frac{zf'(z)}{f(z)} \prec \frac{1 + (1 - 2λ)z}{1 - z}, \text{ i.e. } f \in S^*(λ), 0 \leq λ < 1.
\]

Note that the above corollary gives the result of Kwon \[6\] for $α = 1$.

**Example 4.1.** We compare the result of above corollary with the result of Ravichandran et al. \[7\] by considering the following particular cases. We see that the above corollary extends the result of Ravichandran et al. \[7\], stated in theorem 1.4. Write $α = 1/2$, $β = 0$ in theorem 1.4, we obtain:

If $f \in A$ satisfies
\[
\Re \left[ \frac{zf'(z)}{f(z)} \left( 1 + \frac{1}{2} \frac{zf''(z)}{f'(z)} \right) \right] > -\frac{1}{4}, \ z \in \mathbb{E}, \tag{7}
\]
then \( f \in \mathcal{S}^* \).

For \( \alpha = 1/2, \lambda = 0 \) in above corollary, we get:

If \( f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \neq 0 \) in \( \mathbb{E} \), satisfies

\[
\frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{2f'(z)} \right) < \frac{1 + 2z}{(1 - z)^2} = F(z),
\]

then \( f \in \mathcal{S}^* \).

The image of the open unit disk \( \mathbb{E} \) under \( F \) (given by (8)) is the shaded region in Figure 4.1. Therefore, in view of (8), the differential operator \( \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{2f'(z)} \right) \) takes the values in entire shaded region in Figure 4.1 to ensure the starlikeness of \( f \) whereas from (7), we see that \( f \) is starlike in \( \mathbb{E} \), when the operator \( \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{2f'(z)} \right) \) takes values in the portion right to the vertical dashing line at \( \Re(w) = -\frac{1}{4} \). Therefore, the result in (8) is an improvement over the result given in (7).

![Figure 4.1](image)

On writing \( \gamma = -1 \) in theorem 3.1, we get:

**Corollary 4.2.** Let \( \alpha \) be a real number such that \( \alpha \in (-\infty, 0) \cup [1, \infty) \) and let \( f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \neq 0 \) in \( \mathbb{E} \), satisfy

\[
1 + \alpha \frac{zf''(z)}{f'(z)} < \frac{1 + (1 - \alpha)(1 - z)}{1 + (1 - 2\lambda)z} + \frac{2\alpha(1 - \lambda)z}{(1 + (1 - 2\lambda)z)^2},
\]

then

\[
\frac{zf'(z)}{f(z)} < \frac{1 + (1 - 2\lambda)z}{1 - z}, \quad \text{i.e.} \quad f \in \mathcal{S}^*(\lambda), \quad 0 \leq \lambda < 1.
\]

Taking \( \gamma = 0 \) in theorem 3.1, we obtain:

**Corollary 4.3.** If \( f \in \mathcal{A}, \frac{zf'(z)}{f(z)} \neq 0 \) in \( \mathbb{E} \), satisfies

\[
1 + \alpha \frac{zf''(z)}{f'(z)} < 1 - \alpha + \frac{\alpha(1 + (1 - 2\lambda)z)}{1 - z} + \frac{2\alpha(1 - \lambda)z}{(1 - z)(1 + (1 - 2\lambda)z)},
\]

\[
\frac{zf'(z)}{f(z)} < \frac{1 + (1 - 2\lambda)z}{1 - z}, \quad \text{i.e.} \quad f \in \mathcal{S}^*(\lambda), \quad 0 \leq \lambda < 1.
\]
where $\alpha$ is a non-zero complex number. Then
\[
\frac{zf'(z)}{f(z)} < \frac{1 + (1 - 2\lambda)z}{1 - z},
\]
i.e. $f \in \mathcal{S}^*(\lambda)$, $0 \leq \lambda < 1$.

For $\alpha = 1$, $\lambda = 1/2$ in above corollary, we find below the result justifying the well-known result Marx [4] and Strohacker [9] that $\mathcal{K} \subset \mathcal{S}^*(1/2)$.

**Corollary 4.4.** If $f \in \mathcal{A}$, $zf'(z)/f(z) \neq 0$ in $E$, satisfies
\[
1 + \frac{zf''(z)}{f'(z)} f'(z) \prec 1 + (1 - 2\lambda)z,
\]
then
\[
\frac{zf'(z)}{f(z)} < \frac{1}{1 - z}, \quad z \in E.
\]

When we consider the dominant $q(z) = \frac{1 + az}{1 - z}$, $-1 < a \leq 1$, a little calculation yields that it satisfies the conditions of theorem 3.1 in following special cases and consequently we obtain the next results. Setting $\gamma = 1$ in theorem 3.1, we have the following result.

**Corollary 4.5.** Suppose that $\alpha$ ($0 < \alpha \leq 1$) is a real number and if $f \in \mathcal{A}$, $zf'(z)/f(z) \neq 0$ in $E$, satisfies
\[
1 + \frac{zf''(z)}{f'(z)} f'(z) \prec \alpha + (1 - 2\lambda)z
\]
then
\[
\frac{zf'(z)}{f(z)} < \frac{1 + az}{1 - z}, \quad z \in E, \quad -1 < a \leq 1.
\]

Writing $\gamma = -1$ in theorem 3.1, we have the following result.

**Corollary 4.6.** Let $\alpha$ be a real number such that $\alpha \in (-\infty, 0) \cup [1, \infty)$ and let $f \in \mathcal{A}$, $zf'(z)/f(z) \neq 0$ in $E$, satisfy
\[
1 + \frac{zf''(z)}{f'(z)} f'(z) \prec \alpha + (1 - \alpha)(1 - z) + \alpha(1 + az) \frac{1}{1 - z},
\]
then
\[
\frac{zf'(z)}{f(z)} < \frac{1 + az}{1 - z}, \quad z \in E, \quad -1 < a \leq 1.
\]

Setting $\gamma = 0$ in theorem 3.1, we get:

**Corollary 4.7.** Suppose that $\alpha$ is a non-zero complex number and if $f \in \mathcal{A}$, $zf'(z)/f(z) \neq 0$ in $E$, satisfies
\[
1 + \alpha \frac{zf''(z)}{f'(z)} f'(z) \prec 1 - \alpha + \alpha(1 + az) \frac{1}{1 - z} + \alpha(1 + az) \frac{1}{1 - z},
\]
then
\[
\frac{zf'(z)}{f(z)} < \frac{1 + az}{1 - z}, \quad z \in E, \quad -1 < a \leq 1.
\]

Note that for $\alpha = 1$, the above three corollaries reduce to the results of Singh and Gupta. [8]
§5. Sandwich-type results

In this section, we apply theorem 3.3 to find certain sandwich-type results which give the best subordinant and the best dominant for \( \frac{zf'(z)}{f(z)} \). By selecting the subordinant \( q_1(z) = 1 + az \) and the dominant \( q_2(z) = 1 + bz \), with \( 0 < a < b \), in theorem 3.3, we deduce, below, some criteria for starlike functions. Keeping \( \gamma = 1 \) in theorem 3.3, we obtain:

**Corollary 5.1.** Suppose \( \alpha, a, b \) are real numbers such that \( 0 < \alpha \leq 1 \), \( 0 < a < b < 1 \). If \( f \in \mathcal{A} \) is such that \( \frac{zf'(z)}{f(z)} \in \mathcal{H}[1, 1] \cap \mathcal{Q} \), with \( \frac{zf'(z)}{f(z)} \neq 0 \) and satisfies the condition

\[
1 + (1 + 2\alpha)az + \alpha a^2 z^2 < \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < 1 + (1 + 2\alpha)bz + \alpha b^2 z^2, \quad z \in \mathbb{E},
\]

where \( \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \) is univalent in \( \mathbb{E} \), then

\[
1 + az \prec \frac{zf'(z)}{f(z)} \prec 1 + bz.
\]

**Example 5.1.** For \( \alpha = 1, a = 1/10, b = 9/10 \) and \( f \) same as in above corollary, we obtain:

\[
1 + \frac{3}{10}z + \frac{1}{100}z^2 \prec \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) < 1 + \frac{27}{10}z + \frac{81}{100}z^2,
\]

then

\[
1 + \frac{1}{10}z \prec \frac{zf'(z)}{f(z)} \prec 1 + \frac{9}{10}z.
\]

We show the above results pictorially. In view of (9) and (10), when the operator \( \frac{zf'(z)}{f(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \) takes values in the light shaded portion of Figure 5.1, then \( \frac{zf'(z)}{f(z)} \) takes values in the light shaded portion of Figure 5.2 and hence \( f \) is starlike in \( \mathbb{E} \).

![Figure 5.1](image-url)
Writing $\gamma = -1$ in theorem 3.3, we obtain:

**Corollary 5.2.** Let $\alpha$, $a$, $b$ be real numbers such that $\alpha \in (-\infty, 0) \cup (1, \infty)$, $0 < a < b < 1$. If $f \in \mathcal{A}$ is such that $\frac{zf'(z)}{f(z)} \in \mathcal{H}[1, 1] \cap Q$, with $\frac{zf'(z)}{f(z)} \neq 0$ and $\frac{1 + \alpha zf''(z)/f'(z)}{zf'(z)/f(z)}$ is univalent in $E$. Then

$$\alpha + \frac{1 - \alpha}{1 + az} + \frac{\alpha az}{(1 + az)^2} \prec \frac{1 + \alpha zf''(z)/f'(z)}{zf'(z)/f(z)} \prec \alpha + \frac{1 - \alpha}{1 + bz} + \frac{abz}{(1 + bz)^2}, \quad z \in E,$$

implies

$$1 + az \prec \frac{zf'(z)}{f(z)} \prec 1 + bz, \quad z \in E.$$

Writing $\gamma = 0$ in theorem 3.3, we get:

**Corollary 5.3.** Suppose $\alpha$ is a non-zero complex number and $a$, $b$ are real numbers such that $0 < a < b < 1$. If $f \in \mathcal{A}$ is such that $\frac{zf'(z)}{f(z)} \in \mathcal{H}[1, 1] \cap Q$, with $\frac{zf'(z)}{f(z)} \neq 0$ and $1 + \alpha \frac{zf''(z)}{f'(z)}$ is univalent in $E$. Then

$$1 + \alpha az + \frac{\alpha az}{1 + az} \prec 1 + \alpha \frac{zf''(z)}{f'(z)} \prec 1 + abz + \frac{abz}{1 + bz}, \quad z \in E,$$

implies

$$1 + az \prec \frac{zf'(z)}{f(z)} \prec 1 + bz, \quad z \in E.$$

**Example 5.2.** For $\alpha = 1$, $a = 1/4$, $b = 3/4$ and $f$ same as in above corollary, we obtain:

$$1 + \frac{1}{4} z + \frac{z}{4 + z} \prec 1 + \frac{zf''(z)}{f'(z)} \prec 1 + \frac{3}{4} z + \frac{3z}{4 + 3z}, \quad (11)$$

then

$$1 + \frac{1}{4} z \prec \frac{zf'(z)}{f(z)} \prec 1 + \frac{3}{4} z. \quad (12)$$
In view of (11) and (12), we see that whenever the operator \( \frac{zf''(z)}{f'(z)} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \) takes values in the light shaded portion of Figure 5.3, then \( \frac{zf'(z)}{f(z)} \) takes values in the light shaded portion of Figure 5.4 and hence \( f \) is starlike in \( \mathbb{E} \).

![Figure 5.3](image1)

![Figure 5.4](image2)

**References**


On the convergence of some right circulant matrices

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Abstract In this paper, the conditions for the convergence of the matrices $RCIRC_n(\vec{d})$ and $RCIRC_n(\vec{g})$ were established.

Keywords Convergent matrix, right circulant matrix, spectral norm.

§1. Introduction

In [1] and [2] the followings were established about the spectral norm of $RCIRC_n(\vec{d})$ and $RCIRC_n(\vec{g})$:

1. $||RCIRC_n(\vec{d})||_2 = \max \left\{ |na + \frac{nd(n-1)}{2}|, \frac{|nd|}{2\sin \frac{\pi}{n}} \right\}$,
2. $||RCIRC_n(\vec{g})||_2 = \max \left\{ |\frac{a(1-r^n)}{1-r}|, \frac{|a(r^n-1)|}{\sqrt{r^2-2r\cos \frac{2\pi m}{n}+1}} \right\}$,

where

$$RCIRC_n(\vec{d}) = \begin{pmatrix}
a & a+d & a+2d & \cdots & a+(n-2)d & a+(n-1)d \\
a+(n-1)d & ad & a+d & \cdots & a+(n-3)d & a+(n-2)d \\
a+(n-2)d & a+(n-1)d & a & \cdots & a+(n-4)d & a+(n-3)d \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a+2d & a+3d & a+4d & \cdots & a & a+d \\
a+d & a+2d & a+3d & \cdots & a+(n-1)d & a
\end{pmatrix},$$

and

$$RCIRC_n(\vec{g}) = \begin{pmatrix}
a & ar & ar^2 & \cdots & ar^{n-2} & ar^{n-1} \\
ar^{n-1} & a & ar & \cdots & ar^{n-3} & ar^{n-2} \\
ar^{n-2} & ar^{n-1} & a & \cdots & ar^{n-4} & ar^{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
ar^2 & ar^3 & ar^4 & \cdots & a & ar \\
ar & ar^2 & ar^3 & \cdots & ar^{n-1} & a
\end{pmatrix}.$$

Definition 1.1. A matrix $C$ is said to be convergent if its spectral norm is less than 1, that is $||C||_2 < 1$. 
§2. Main results

**Theorem 2.1.** \( R_{\text{CIRC}_n} (\vec{d}) \) is convergent provided that \( |2an + nd(n-1)| < 2 \) or \( d^2 < \frac{2\sin^2(\frac{\pi}{n})}{n^2} \).

**Proof.**

Case 1: \( \| R_{\text{CIRC}_n} (\vec{d}) \|_2 = \left| na + \frac{nd(n-1)}{2} \right| < 1 \), then,

\[ |2an + nd(n-1)| < 2. \]

Case 2: \( \| R_{\text{CIRC}_n} (\vec{d}) \|_2 = \frac{|nd|}{2\sin \frac{\pi}{n}} < 1 \), then,

\[ |nd| < 2 \sin \left( \frac{\pi}{n} \right) \text{ and } d^2 < \frac{2\sin^2(\frac{\pi}{n})}{n^2}. \]

**Theorem 2.2.** \( R_{\text{CIRC}_n} (\vec{g}) \) is convergent provided that \( a^2 < \left( \frac{r-1}{r^n-1} \right)^2 \) or \( a^2 < \frac{r^2 - 2r \cos(\frac{2\pi m}{n}) + 1}{(r^n-1)^2} \).

**Proof.**

Case 1: \( \| R_{\text{CIRC}_n} (\vec{g}) \|_2 = \frac{|a(1-r^n)|}{1-r} < 1 \), then,

\[ a^2(1-r^n)^2 < (1-r)^2 \text{ and } a^2 < \left( \frac{1-r}{1-r^n} \right)^2. \]

Case 2: \( \| R_{\text{CIRC}_n} (\vec{g}) \|_2 = \frac{|a(1-r^n)|}{\sqrt{r^2 - 2r \cos \left( \frac{2\pi m}{n} \right) + 1}} < 1 \), then,

\[ a^2(1-r^n)^2 < r^2 - 2r \cos \frac{2\pi m}{n} + 1 \text{ and } a^2 < \frac{r^2 - 2r \cos \frac{2\pi m}{n} + 1}{(1-r^n)^2}. \]

**References**


Two summation formulae of half argument linked with contiguous relation

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Abstract The main object of present paper is the establishment of two summation formulae associated with the contiguous relation and hypergeometric function.

Keywords Contiguous relation, recurrence relation, Legendre's duplication formula.

2000 Mathematics Subject Classification: 33C05, 33C20, 33D15, 33D50, 33D60

§1. Introduction

Generalized Gaussian hypergeometric function of one variable:

\[ _{A}F_{B} \left[ \begin{array}{l} a_{1}, a_{2}, \cdots , a_{A} \\ b_{1}, b_{2}, \cdots , b_{B} \end{array} ; z \right] = \sum_{k=0}^{\infty} \frac{(a_{1})_{k}(a_{2})_{k} \cdots (a_{A})_{k} z^{k}}{(b_{1})_{k}(b_{2})_{k} \cdots (b_{B})_{k} k!}, \] (1)

or

\[ _{A}F_{B} \left[ \begin{array}{l} (a_{A}) \\ (b_{B}) \end{array} ; z \right] \equiv _{A}F_{B} \left[ \begin{array}{l} (a_{j})_{j=1}^{A} \\ (b_{j})_{j=1}^{B} \end{array} ; z \right] = \sum_{k=0}^{\infty} \frac{((a_{A})_{k} z^{k}}{((b_{B})_{k} k!}, \] (2)

where the parameters \( b_{1}, b_{2}, \cdots , b_{B} \) are neither zero nor negative integers and \( A, B \) are non-negative integers.

Contiguous relation: [1]

\[ (a-b) (1-z) \ _{2}F_{1} \left[ \begin{array}{l} a, b; \\ c; \end{array} z \right] = (c-b) \ _{2}F_{1} \left[ \begin{array}{l} a, b-1; \\ c; \end{array} z \right] + (a-c) \ _{2}F_{1} \left[ \begin{array}{l} a-1, b; \\ c; \end{array} z \right]. \] (3)

Recurrence relation :

\[ \Gamma(z+1) = z \Gamma(z). \] (4)

Legendre's duplication formula :

\[ \sqrt{\pi} \Gamma(2z) = 2^{(2z-1)} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right). \] (5)
\[ \Gamma \left( \frac{1}{2} \right) = \sqrt{\pi} = \frac{2^{(b-1)} \Gamma \left( \frac{b}{2} \right) \Gamma \left( \frac{b+1}{2} \right)}{\Gamma (b)}, \quad (6) \]

In the monograph of Prudnikov et al., a summation formula is given in the form: \[ (7) \]

\[ _2F_1 \left[ \begin{array}{c} a, \ b \\ \frac{a+b-1}{2} \end{array} ; \ 1 \right] = \sqrt{\pi} \left[ \frac{\Gamma \left( \frac{a+b+1}{2} \right) \Gamma \left( \frac{a+1}{2} \right)}{\Gamma (a) \Gamma (b)} + \frac{2 \Gamma \left( \frac{a+b-1}{2} \right)}{\Gamma (a) \Gamma (b)} \right]. \quad (8) \]

Now using Legendre’s duplication formula and recurrence relation for Gamma function, the above formula can be written in the form

\[ _2F_1 \left[ \begin{array}{c} a, \ b \\ \frac{a+b-1}{2} \end{array} ; \ 1 \right] = \frac{2^{(b-1)} \Gamma \left( \frac{a+b-1}{2} \right)}{\Gamma (b)} \left[ \frac{\Gamma \left( \frac{b}{2} \right)}{\Gamma \left( \frac{a+1}{2} \right)} \right] \left\{ \frac{(b+1)}{(a+1)} \right\} + \frac{2 \Gamma \left( \frac{a+b-1}{2} \right)}{\Gamma (a) \Gamma (b)} \right]. \quad (9) \]

It is noted that the above formula \[ (7) \], i.e. equation (8) or (9) is not correct. The correct form of equation (8) or (9) is obtained by \[ (2) \].

\[ _2F_1 \left[ \begin{array}{c} a, \ b \\ \frac{a+b-1}{2} \end{array} ; \ 1 \right] = \frac{2^{(b-1)} \Gamma \left( \frac{a+b-1}{2} \right)}{\Gamma (b)} \left[ \frac{\Gamma \left( \frac{b}{2} \right)}{\Gamma \left( \frac{a+1}{2} \right)} \right] \left\{ \frac{(b+1)}{(a+1)} \right\} + \frac{2 \Gamma \left( \frac{a+b-1}{2} \right)}{\Gamma (a) \Gamma (b)} \right]. \quad (10) \]

Involving the formula obtained by \[ (2) \], we establish the main formulae.

### §2. Main results of summation formulae

For all the results \( a \neq b \),

\[ _2F_1 \left[ \begin{array}{c} a, \ b \\ \frac{a+b-23}{2} \end{array} ; \ 1 \right] = \frac{2^{(b-1)} \Gamma \left( \frac{a+b-23}{2} \right)}{(a-b) \Gamma (b)} \left[ \frac{\Gamma \left( \frac{b}{2} \right)}{\Gamma \left( \frac{a+1}{2} \right)} \right] \left\{ \frac{(b+1)}{(a+1)} \right\} + \frac{2 \Gamma \left( \frac{a+b-1}{2} \right)}{\Gamma (a) \Gamma (b)} \right]. \]

\[ \times \left\{ \begin{array}{c} (316234143225a - 703416314160a^2 + 590546123298a^3 - 264300628944a^4 + 72578259391a^5) \\ \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \} \\ + (-13137458400a^6 + 1628301884a^7 - 140529312a^8 + 8439783a^9 - 345840a^{10} + 9218a^{11}) \\ \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \} \\ + (-144a^{12} + a^{13} - 316234143225b + 987903828090a^2b - 1002491094240a^3b) \\ \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \} \\ + (523055123685a^4b - 13556289600a^5b + 27768329500a^6b - 2975972160a^7b + 297768865a^8b) \\ \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \} \right\} \]

\[ + (523055123685a^4b - 13556289600a^5b + 27768329500a^6b - 2975972160a^7b + 297768865a^8b) \\ \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \} \]
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\[
\begin{align*}
&+ \left( -14044800a^9b + 718410a^{10}b - 12000a^{11}b + 275a^{12}b + 703416314160b^2 - 987903828090ab^2 \right) \\
&\quad \prod_{\delta = 1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+ \left( 521997079830a^3b^2 - 287175957600a^4b^2 + 104854420780a^5b^2 - 15893250240a^6b^2 \right) \\
&\quad \prod_{\delta = 1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+ \left( 2529523900a^7b^2 - 149067600a^8b^2 + 12283150a^9b^2 - 242880a^{10}b^2 + 10350a^{11}b^2 \right) \\
&\quad \prod_{\delta = 1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+ \left( -590546123298b^3 + 100249194240ab^3 - 521997079830a^2b^3 + 82963937100a^4b^3 \right) \\
&\quad \prod_{\delta = 1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+ \left( -25551704640a^5b^3 + 6975270260a^6b^3 - 573638400a^7b^3 + 73053750a^8b^3 - 1821600a^9b^3 \right) \\
&\quad \prod_{\delta = 1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+ \left( 123970a^4b^4 + 264300628944b^4 - 523055123685ab^4 + 287175957600a^2b^4 - 82963937100a^4b^4 \right) \\
&\quad \prod_{\delta = 1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+ \left( 4841754610a^5b^4 - 785551200a^6b^4 + 167502100a^7b^4 - 5768400a^8b^4 + 600875a^9b^4 \right) \\
&\quad \prod_{\delta = 1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+ \left( -72578259391b^5 + 135562896000ab^5 - 104854420780a^2b^5 + 25551704640ab^5 \right) \\
&\quad \prod_{\delta = 1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+ \left( -4841754610a^5b^5 + 107126180a^6b^5 - 7131840a^7b^5 + 1225785a^8b^5 + 13137458400b^6 \right) \\
&\quad \prod_{\delta = 1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+ \left( -27768329500ab^6 + 15893250240a^2b^6 - 6975270260a^3b^6 + 785551200a^4b^6 - 107126180a^5b^6 \right) \\
&\quad \prod_{\delta = 1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+ \left( 742900a^7b^6 - 1628301884b^7 + 2975972160ab^7 - 2529523900a^2b^7 + 573638400a^3b^7 \right) \\
&\quad \prod_{\delta = 1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+ \left( -167502100a^4b^7 + 7131840a^5b^7 - 742900a^6b^7 + 140529312b^8 - 299768865ab^8 \right) \\
&\quad \prod_{\delta = 1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+ \left( 149067600a^2b^8 - 73053750a^3b^8 + 5768400a^4b^8 - 1225785a^5b^8 - 84397834b^9 + 14044800ab^9 \right) \\
&\quad \prod_{\delta = 1}^{12} \left\{ a - (2\delta - 1) \right\} 
\end{align*}
\]
\begin{align*}
&+ \frac{(-12283150a^2b^9 + 1821600a^3b^9 - 600875a^4b^9 + 345840b^{10} - 718410ab^{10} + 242880a^2b^{10})}{\prod_{\delta=1}^{12} \{a - (2\delta - 1)\}} \\
&+ \frac{(-123970a^3b^{10} - 9218b^{11} + 12000ab^{11} - 10350a^2b^{11} + 144b^{12} - 275ab^{12} - b^{13})}{\prod_{\delta=1}^{12} \{a - (2\delta - 1)\}} \\
&+ \frac{\Gamma\left(\frac{23+1}{2}\right)}{\Gamma\left(\frac{23-22}{2}\right)} \left\{ \frac{-483585689160a + 789891354792a^2 - 460744729944a^3 + 184587960184a^4}{\prod_{\zeta=1}^{11} \{a - 2\zeta\}} \right\} \\
&+ \frac{(-36115854800a^5 + 6465269136a^6 - 547131312a^7 + 50602992a^8 - 1870440a^9 + 89672a^{10})}{\prod_{\zeta=1}^{11} \{a - 2\zeta\}} \\
&+ \frac{(-1144a^{11} + 24a^{12} + 483585689160b - 628259352120a^2b + 596428543440a^3b)}{\prod_{\zeta=1}^{11} \{a - 2\zeta\}} \\
&+ \frac{(-195420950000a^4b + 56438379200a^5b - 6566394800a^6b + 92032160a^7b - 43386200a^8b)}{\prod_{\zeta=1}^{11} \{a - 2\zeta\}} \\
&+ \frac{(3256000a^9b - 51480a^{10}b + 2000a^{11}b - 789891354792b^2 + 628259352120ab^2)}{\prod_{\zeta=1}^{11} \{a - 2\zeta\}} \\
&+ \frac{(-173665216800a^3b^2 + 110863118800a^4b^2 - 20734777840a^5b^2 + 4558206240a^6b^2)}{\prod_{\zeta=1}^{11} \{a - 2\zeta\}} \\
&+ \frac{(-290628000a^7b^2 + 32738200a^8b^2 - 657800a^9b^2 + 40480a^{10}b^2 + 460744729944b^3)}{\prod_{\zeta=1}^{11} \{a - 2\zeta\}} \\
&+ \frac{(-596428543440a^3b^3 + 173665216800a^2b^3 - 15168969200a^4b^3 + 7064396640a^5b^3)}{\prod_{\zeta=1}^{11} \{a - 2\zeta\}} \\
&+ \frac{(-704664800a^6b^3 + 122396800a^7b^3 - 3289000a^8b^3 + 303600a^9b^3 - 184587960184b^4)}{\prod_{\zeta=1}^{11} \{a - 2\zeta\}} \\
&+ \frac{(195420950000a^4b^4 - 110863118800a^2b^4 + 15168969200a^3b^4 - 461297200a^5b^4)}{\prod_{\zeta=1}^{11} \{a - 2\zeta\}} \\
&+ \frac{(165186000a^6b^4 - 6817200a^7b^4 + 961400a^8b^4 + 36115854800b^5 - 56438379200a^5b^5)}{\prod_{\zeta=1}^{11} \{a - 2\zeta\}}
\end{align*}
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\[ + \frac{20734777840a^2b^5 - 7064396640a^3b^5 + 461297200a^4b^5 - 4160240a^6b^5 + 1188640a^7b^5}{\prod_{\zeta=1}^{11} \{a - 2\zeta\}} \]

\[ + \frac{(-6465269136b^6 + 6566394800ab^6 - 4558206240a^2b^6 + 704664800a^3b^6 - 165186000a^4b^6)}{\prod_{\zeta=1}^{11} \{a - 2\zeta\}} \]

\[ + \frac{(4160240a^5b^6 + 547131312b^7 - 920232160ab^7 + 290628000a^2b^7 - 122396800a^3b^7)}{\prod_{\zeta=1}^{11} \{a - 2\zeta\}} \]

\[ + \frac{(6817200a^4b^7 - 1188640a^5b^7 - 50602992b^8 + 43386200ab^8 - 32738200a^2b^8 + 3289000a^3b^8)}{\prod_{\zeta=1}^{11} \{a - 2\zeta\}} \]

\[ + \frac{(-961400a^4b^8 + 1870440b^9 - 3256000ab^9 + 657800a^2b^9 - 303600a^3b^9 - 89672b^{10} + 51480ab^{10})}{\prod_{\zeta=1}^{11} \{a - 2\zeta\}} \]

\[ + \frac{(-40480a^2b^{10} + 1144b^{11} - 2000ab^{11} - 24b^{12})}{\prod_{\zeta=1}^{11} \{a - 2\zeta\}} \]}

\[ _2F_1 \left[ \begin{array}{c} a, b \\ a + b - 2\delta \\ \frac{1}{2} \end{array} \right] = \frac{2^{(b-1)} \Gamma\left(\frac{a+b-2\delta}{2}\right)}{(a-b)\Gamma(b)} \]

\[ \times \frac{\Gamma\left(\frac{b+1}{2}\right)}{\Gamma\left(\frac{a-2\delta}{2}\right)} \left\{ \begin{array}{c} (1961990553600a - 4325828198400a^2 + 3874205859840a^3 - 1634441932800a^4) \\ \prod_{\delta=1}^{12} \{a - (2\delta - 1)\} \end{array} \right\} \]

\[ + \frac{(512039040000a^5 - 83602361600a^6 + 12742301120a^7 - 946046400a^8 + 77391600a^9)}{\prod_{\delta=1}^{12} \{a - (2\delta - 1)\}} \]

\[ + \frac{(-2574000a^{10} + 111540a^{11} - 1300a^{12} + 25a^{13} - 1961990553600b + 1281593180160ab)}{\prod_{\delta=1}^{12} \{a - (2\delta - 1)\}} \]

\[ + \frac{(3094576496640a^2b - 3825569710080a^3b + 2245742361600a^4b - 568140505600a^5b)}{\prod_{\delta=1}^{12} \{a - (2\delta - 1)\}} \]

\[ + \frac{(132653041600a^6b - 13220253440a^7b + 1600835600a^8b - 67152800a^9b + 4493060a^{10}b)}{\prod_{\delta=1}^{12} \{a - (2\delta - 1)\}} \]

\[ + \frac{(-64480a^{11}b + 2275a^{12}b + 3044235018240b^2 - 4990865375232ab^2 + 1232718520320a^2b^2)}{\prod_{\delta=1}^{12} \{a - (2\delta - 1)\}} \]
\begin{align*}
&+ (1115851130880a^3b^2 - 814203974400a^4b^2 + 341856934080a^5b^2 - 51313920000a^6b^2) \\
&\quad \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \} \\
&+ (9319592640a^7b^2 - 518226800a^8b^2 + 50979500a^9b^2 - 920920a^{10}b^2 + 50830a^{11}b^2) \\
&\quad \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \} \\
&+ (-1977916981248b^3 + 3503317499904ab^3 - 2219737221120a^2b^3 + 272783385600a^3b^3) \\
&\quad \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \} \\
&+ (136393348800a^4b^3 - 58313839360a^5b^3 + 18533032000a^6b^3 - 1521532800a^7b^3) \\
&\quad \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \} \\
&+ (221648700a^8b^3 - 5262400a^9b^3 + 427570a^{10}b^3 + 72397562266b^4 - 1485650361344ab^4) \\
&\quad \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \} \\
&+ (849188691200a^2b^4 - 304571115200a^3b^4 + 19959363200a^4b^4 + 6518851360a^5b^4) \\
&\quad \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \} \\
&+ (-1509922400a^6b^4 + 378005000a^7b^4 - 13066300a^8b^4 + 1562275a^9b^4 - 16824950016b^5) \\
&\quad \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \} \\
&+ (317597785600ab^5 - 259454322880a^2b^5 + 65660179200a^3b^5 - 15636448800a^4b^5) \\
&\quad \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \} \\
&+ (523700800a^5b^5 + 122727080a^6b^5 - 11886400a^7b^5 + 2414425a^8b^5 + 26376979200b^6) \\
&\quad \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \} \\
&+ (-56740812736a^6b^6 + 32671797760a^2b^6 - 15299636800a^3b^6 + 1786327200a^4b^6) \\
&\quad \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \} \\
&+ (-308714280a^5b^6 + 4160240a^6b^6 + 742900a^7b^6 - 287937478ab^7 + 5281066752ab^7) \\
&\quad \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \} \\
&+ (-4623802560a^2b^7 + 1051523200a^2b^7 - 329797000a^4b^7 + 145433600a^5b^7 - 1931540a^6b^7) \\
&\quad \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \} \\
&+ (2216522888b^8 - 477823632ab^8 + 237437200a^2b^8 - 120706300a^3b^8 + 9538100a^4b^8) \\
&\quad \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \}
\end{align*}
\[+ \left( -2187185a^5b^8 - 11991408b^9 + 19974240ab^9 - 17803500a^2b^9 + 2631200a^3b^9 - 904475a^4b^9 \right) \prod_{\delta=1}^{12} \{a - (2\delta - 1)\} \]
\[+ \left( 446160b^{10} - 932932ab^{10} + 314600a^2b^{10} - 164450a^3b^{10} - 10868b^{11} + 14144ab^{11} \right) \prod_{\delta=1}^{12} \{a - (2\delta - 1)\} \]
\[+ \left( -12350a^2b^{11} + 156b^{12} - 299ab^{12} - b^{13} \right) \prod_{\delta=1}^{12} \{a - (2\delta - 1)\} \]
\[+ \Gamma\left( \frac{3}{2} \right) \left\{ \Gamma\left( \frac{a-24}{2} \right) \prod_{\vartheta=1}^{12} \{a - 2\vartheta\} \right\} \]
\[+ \left( 1961990553600a - 3044235018240a^2 + 1977916981248a^3 - 723975622656a^4 \right) \prod_{\vartheta=1}^{12} \{a - 2\vartheta\} \]
\[+ (168244950016a^5 - 26376979200a^6 + 2879374784a^7 - 221652288a^8 + 11991408a^9) \prod_{\vartheta=1}^{12} \{a - 2\vartheta\} \]
\[+ \left( -446160a^{10} + 10868a^{11} - 156a^{12} + a^{13} - 1961990553600b - 1281593180160ab \right) \prod_{\vartheta=1}^{12} \{a - 2\vartheta\} \]
\[+ (4990865375232a^2b - 3503317499904a^3b + 1485650361344a^4b - 317597785600a^5b) \prod_{\vartheta=1}^{12} \{a - 2\vartheta\} \]
\[+ \left( -2219737221120a^3b^2 - 849188691200a^4b^2 + 259454322880a^5b^2 - 32671797760a^6b^2 \right) \prod_{\vartheta=1}^{12} \{a - 2\vartheta\} \]
\[+ \left( 4623802560a^7b^2 - 237437200a^8b^2 + 17803500a^9b^2 - 314600a^{10}b^2 + 12350a^{11}b^2 \right) \prod_{\vartheta=1}^{12} \{a - 2\vartheta\} \]
\[+ \left( -3874205859840b^3 + 3825569710080ab^3 - 1115851130880a^2b^3 - 272783385600a^3b^3 \right) \prod_{\vartheta=1}^{12} \{a - 2\vartheta\} \]
\[+ \left( 304571115200a^4b^3 - 65660179200a^5b^3 + 15299636800a^6b^3 - 1051523200a^7b^3 \right) \prod_{\vartheta=1}^{12} \{a - 2\vartheta\} \]
\[\begin{align*}
+ (120706300a^8b^3 - 2631200a^9b^3 + 164450a^6b^3 + 1634441932800b^4 - 2245742361600ab^4) \\
\quad \prod_{\vartheta=1}^{12} \{ a - 2\vartheta \} \\
+ (814203974400a^2b^4 - 136393348800a^2b^4 - 19959363200a^4b^4 + 15636448800a^5b^4) \\
\quad \prod_{\vartheta=1}^{12} \{ a - 2\vartheta \} \\
+ (568140505600ab^5 - 341856934080a^2b^5 + 58313839360a^3b^5 - 6518851360a^4b^5) \\
\quad \prod_{\vartheta=1}^{12} \{ a - 2\vartheta \} \\
+ (-523700800a^5b^5 + 308714280a^6b^5 - 14543360a^7b^5 + 2187185a^8b^5 + 83602361600b^6) \\
\quad \prod_{\vartheta=1}^{12} \{ a - 2\vartheta \} \\
+ (-132653041600ab^6 - 193154000a^2b^6 - 1509922400a^3b^6 + 13220253440a^4b^6) \\
\quad \prod_{\vartheta=1}^{12} \{ a - 2\vartheta \} \\
+ (-122727080a^5b^6 - 4160240a^6b^6 + 1995140a^7b^6 - 12742301120b^7 + 13220253440a^4b^7) \\
\quad \prod_{\vartheta=1}^{12} \{ a - 2\vartheta \} \\
+ (-9319592640a^2b^7 + 1521532800a^3b^7 - 378005000a^4b^7 + 11886400a^5b^7 - 742900a^6b^7) \\
\quad \prod_{\vartheta=1}^{12} \{ a - 2\vartheta \} \\
+ (946046400b^8 - 1600835600a^2b^8 + 518226800a^3b^8 - 221648700a^4b^8 + 13066300a^5b^8) \\
\quad \prod_{\vartheta=1}^{12} \{ a - 2\vartheta \} \\
+ (-2414425a^5b^9 - 77391600ab^9 + 67152800a^2b^9 - 50979500a^3b^9 + 5262400a^4b^9 - 1562275a^5b^9) \\
\quad \prod_{\vartheta=1}^{12} \{ a - 2\vartheta \} \\
+ (2574000b^{10} - 4493060ab^{10} + 9209920a^2b^{10} - 427570a^3b^{10} - 111540b^{11} + 64480ab^{11}) \\
\quad \prod_{\vartheta=1}^{12} \{ a - 2\vartheta \} \\
\quad \left\{ -\frac{50830a^2b^{11} + 1300b^{12} - 2275ab^{12} - 25b^{13}}{2} \right\}.
\end{align*}\]

\[\text{§3. Derivation of summation formulae}\]

Derivation of (11): substituting \( c = \frac{a+b-23}{2} \) and \( z = \frac{1}{2} \) in equation (3), we get

\[\left( \frac{a-b}{2} \right) _2F_1 \left[ \frac{a,b;}{a+b-23;}; \frac{1}{2} \right] = \left( \frac{a-b-23}{2} \right) _2F_1 \left[ \frac{a,b-1;}{a+b-23;}; \frac{1}{2} \right] \]
or

\[
(a-b)_{2F1}\left[\frac{a,b;}{a+b-23}, \frac{1}{2}\right] = (a-b-23)_{2F1}\left[\frac{a,b-1;}{a+b-23}, \frac{1}{2}\right] + (a-b+23)_{2F1}\left[\frac{a-1,b;}{a+b-23}, \frac{1}{2}\right].
\]

Now involving (10), we get

\[
L.H.S = \frac{\Gamma\left(\frac{a+b+23}{2}\right)}{\Gamma(b)}\left[\frac{(a - b - 23)(b - 1)}{(a - b + 1)}\right] \frac{\Gamma\left(\frac{b}{2}\right)}{\Gamma\left(\frac{a}{2}\right)} \left\{a - (2\delta - 1)\right\}^{12}\delta = 1
\]

\[
+ \frac{(-1085127120a^7 + 84910089a^8 - 4402200a^9 + 145222a^{10} - 2760a^{11} + 23a^{12})}{\prod_{\delta = 1}^{12}\left\{a - (2\delta - 1)\right\}}
\]

\[
+ \frac{(703416314160b - 1139061201588ab + 396896330784a^2b + 15200003492a^3b)}{\prod_{\delta = 1}^{12}\left\{a - (2\delta - 1)\right\}}
\]

\[
+ \frac{(-147359650912a^4b + 49578268792a^5b - 9037988736a^6b + 1074415560a^7b - 78535248a^8b)}{\prod_{\delta = 1}^{12}\left\{a - (2\delta - 1)\right\}}
\]

\[
+ \frac{(392956a^8b - 105248a^9b + 1748a^{10}b - 590546123298b^2 + 106982510354ab^2)}{\prod_{\delta = 1}^{12}\left\{a - (2\delta - 1)\right\}}
\]

\[
+ \frac{(-576865073266a^2b^2 + 81954887264a^3b^2 + 36378600348a^4b^2 - 16834239856a^5b^2)}{\prod_{\delta = 1}^{12}\left\{a - (2\delta - 1)\right\}}
\]

\[
+ \frac{(361693772a^6b^2 - 3916336966a^7b^2 + 30456646a^8b^2 - 1111176a^9b^2 + 31878a^{10}b^2)}{\prod_{\delta = 1}^{12}\left\{a - (2\delta - 1)\right\}}
\]

\[
+ \frac{(26430062894a^3b^3 - 50345114014a^4b^3 + 312991442432a^5b^3 - 79652150448a^6b^3)}{\prod_{\delta = 1}^{12}\left\{a - (2\delta - 1)\right\}}
\]

\[
+ \frac{(4715754400a^6b^3 + 2713871384a^7b^3 - 611995776a^8b^3 + 85728912a^9b^3 - 438308a^{10}b^3)}{\prod_{\delta = 1}^{12}\left\{a - (2\delta - 1)\right\}}
\]

\[
+ \frac{(211508a^8b^3 - 72578259391b^4 + 140570255696a^9b^3 - 93342239796a^2b^4 + 2779323976a^3b^4)}{\prod_{\delta = 1}^{12}\left\{a - (2\delta - 1)\right\}}
\]
\[
\frac{(-3690175706a^4b^4 + 707240805b^4 + 70100044a^6b^4 - 6299792a^7b^4 + 572033a^8b^4)}{12 \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \}} \\
+ \frac{(13137458400b^5 - 25911980616ab^5 + 16865721152a^2b^5 - 5516113736a^3b^5 + 84236240a^4b^5)}{12 \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \}} \\
+ \frac{(-5695288a^5b^5 + 534888a^7b^5 - 1628301884b^6 + 3107909616ab^6 - 2154337780a^2b^6)}{12 \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \}} \\
+ \frac{(618811872a^3b^6 - 112314244ab^6 + 7488432a^5b^6 - 208012a^6b^6 + 140529312b^7)}{12 \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \}} \\
+ \frac{(-275591448ab^7 + 160818944a^2b^7 - 55228664a^3b^7 + 6306784a^4b^7 - 653752a^5b^7 - 8439783b^8)}{12 \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \}} \\
+ \frac{(14821224ab^8 - 10248018a^2b^8 + 2021976a^3b^8 - 389367a^4b^8 + 345840b^9 - 65556ab^9)}{12 \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \}} \\
+ \frac{(267168a^2b^9 - 92092a^3b^9 + 9218b^10 + 12760a^6b^{10} - 8602a^2b^{10} + 144b^{11} - 252ab^{11} - b^{12})}{12 \prod_{\delta=1}^{12} \{ a - (2\delta - 1) \}} \\
+ \frac{(a - b - 23) \Gamma \left( \frac{b+1}{2} \right) \Gamma \left( \frac{a-2b}{2} \right)}{(a - b + 1) \Gamma \left( \frac{a-2b}{2} \right)} \left\{ \frac{-497334999735 + 240349776192a + 283337378706a^2}{11 \prod_{\zeta=1}^{11} \{ a - 2\zeta \}} \right\} \\
+ \frac{(-301005003392a^3 - 120738023135a^4 - 2812787636a^5 + 4053197084a^6 - 397208064a^7)}{11 \prod_{\zeta=1}^{11} \{ a - 2\zeta \}} \\
+ \frac{(24985719a^8 - 1076416a^9 + 26290a^{10} - 384a^{11} + a^{12} + 125724083263b - 1100102048556ab)}{11 \prod_{\zeta=1}^{11} \{ a - 2\zeta \}} \\
+ \frac{(888092648a^2b + 29687486258a^3b - 147610629200a^4b + 35668641544a^5b - 5306483504a^6b)}{11 \prod_{\zeta=1}^{11} \{ a - 2\zeta \}} \\
+ \frac{(497150104a^7b - 31051944a^8b + 1180388a^9b - 27192a^{10}b + 252a^{11}b - 1245022762902b^2)}{11 \prod_{\zeta=1}^{11} \{ a - 2\zeta \}} \\
+ \frac{(134350139086ab^2 - 389126369998a^2b^2 - 55506401216a^3b^2 + 62934811220a^4b^2)}{11 \prod_{\zeta=1}^{11} \{ a - 2\zeta \}}
\]
\begin{align*}
&+ \left( -1726109784a^5b^2 + 2644880308a^6b^2 - 240208320a^7b^2 + 14146242a^8b^2 - 457424a^9b^2 \right) \\
&\prod_{\zeta=1}^{\eta} \{ a - 2\zeta \} \\
&+ \left( 8602a^{10}b^2 + 659273219368b^3 - 778752070180a^3b^3 + 316618638112a^2b^3 - 3627768352a^3b^3 \right) \\
&\prod_{\zeta=1}^{\eta} \{ a - 2\zeta \} \\
&+ \left( -8824358480a^4b^3 + 4191946696a^5b^3 - 630661472a^6b^3 + 61671280a^7b^3 - 2750616a^8b^3 \right) \\
&\prod_{\zeta=1}^{\eta} \{ a - 2\zeta \} \\
&+ \left( 92092a^9b^3 - 212974169753b^4 + 263201438400a^4b^4 - 119279262332a^2b^4 + 23952997824a^3b^4 \right) \\
&\prod_{\zeta=1}^{\eta} \{ a - 2\zeta \} \\
&+ \left( -825887174a^2b^4 - 370603968a^3b^4 + 96750052a^4b^4 - 6460608a^7b^4 + 389367a^8b^4 \right) \\
&\prod_{\zeta=1}^{\eta} \{ a - 2\zeta \} \\
&+ \left( 44829435056b^5 - 57198911416ab^5 + 26664410416a^2b^5 - 6095706008a^3b^5 + 634852624a^4b^5 \right) \\
&\prod_{\zeta=1}^{\eta} \{ a - 2\zeta \} \\
&+ \left( 7072408a^5b^5 - 4160240a^6b^5 + 653752a^7b^5 - 669754996b^6 + 804540784a^6b^6 \right) \\
&\prod_{\zeta=1}^{\eta} \{ a - 2\zeta \} \\
&+ \left( -3994074476a^2b^6 + 873356736a^3b^6 - 108374252a^4b^6 + 4992288a^5b^6 + 208012a^6b^6 \right) \\
&\prod_{\zeta=1}^{\eta} \{ a - 2\zeta \} \\
&+ \left( 644913744b^7 - 843555912ab^7 + 35377600a^2b^7 - 90259728a^3b^7 + 8201616a^4b^7 - 534888a^5b^7 \right) \\
&\prod_{\zeta=1}^{\eta} \{ a - 2\zeta \} \\
&+ \left( -48781689b^8 + 51717248ab^8 - 26461270a^2b^8 + 3999424a^3b^8 - 572033a^4b^8 + 2151512b^9 \right) \\
&\prod_{\zeta=1}^{\eta} \{ a - 2\zeta \} \\
&+ \left( -2852828a^9 + 795432a^2b^9 - 211508a^3b^9 - 84502b^{10} + 60720ab^{10} - 31878a^2b^{10} \right) \\
&\prod_{\zeta=1}^{\eta} \{ a - 2\zeta \} \\
&+ \left( \frac{1288b^{11} - 1748ab^{11} - 23b^{12}}{\prod_{\zeta=1}^{\eta} \{ a - 2\zeta \}} \right) + \frac{2^{(b-1)} \Gamma \left( \frac{a+b-23}{2} \right)}{\Gamma(b)} \left[ \frac{(a-b+23) \Gamma \left( \frac{b+1}{2} \right)}{(a-b-1) \Gamma \left( \frac{b-22}{2} \right)} \right] \\
&\times \left\{ \frac{497334999735 - 1257240832632a + 1245022762902a^2 - 659273219368a^3}{\prod_{\zeta=1}^{\eta} \{ a - 2\zeta \}} \right\}
\end{align*}
\[
+ (212974169753a^4 - 44829435056a^5 + 6609754996a^6 - 644913744a^7 + 48781689a^8)
\prod_{\zeta=1}^{11} \{ a - 2\zeta \} \\
+ (-2151512a^9 + 84502a^{10} - 1288a^{11} + 23a^{12} - 240349776192b + 1100102048556ab) \\
\prod_{\zeta=1}^{11} \{ a - 2\zeta \} \\
+ (-1343501390896a^2b + 778752070180a^3b - 263201438400a^4b + 57198911416a^5b) \\
\prod_{\zeta=1}^{11} \{ a - 2\zeta \} \\
+ (-804540784a^6b + 84355912a^7b - 51717248a^8b + 2852828a^9b - 60720a^{10}b + 1748a^{11}b) \\
\prod_{\zeta=1}^{11} \{ a - 2\zeta \} \\
+ (-28333738706b^2 - 8888092648ab^2 + 389126369998a^2b^2 - 31661863812a^3b^2) \\
\prod_{\zeta=1}^{11} \{ a - 2\zeta \} \\
+ (11927926332a^4b^2 - 26664410416a^5b^2 + 3994074476a^6b^2 - 353776800a^7b^2 + 26461270a^8b^2) \\
\prod_{\zeta=1}^{11} \{ a - 2\zeta \} \\
+ (-795432a^2b^2 + 31878a^1b^2 + 301005003392b^3 - 296874862508ab^3 + 55506401216a^2b^3) \\
\prod_{\zeta=1}^{11} \{ a - 2\zeta \} \\
+ (36277638352a^3b^3 - 23952997824a^4b^3 + 6095706008a^5b^3 - 873356736a^6b^3 + 90259728a^7b^3) \\
\prod_{\zeta=1}^{11} \{ a - 2\zeta \} \\
+ (-3999424a^8b^3 + 211508a^9b^3 - 120738023135b^4 + 147610629200ab^4 - 62934811220a^2b^4) \\
\prod_{\zeta=1}^{11} \{ a - 2\zeta \} \\
+ (8824358480a^3b^4 + 825887174a^4b^4 - 634852624a^5b^4 + 108374252a^6b^4 - 8201616a^7b^4) \\
\prod_{\zeta=1}^{11} \{ a - 2\zeta \} \\
+ (572033a^8b^4 + 28127876736b^5 - 35668641544ab^5 + 17261090784a^2b^5 - 4191946696a^3b^5) \\
\prod_{\zeta=1}^{11} \{ a - 2\zeta \} \\
+ (370603968a^4b^5 - 7072408a^5b^5 - 4992288a^6b^5 + 534888a^7b^5 - 405319708ab^6) \\
\prod_{\zeta=1}^{11} \{ a - 2\zeta \} \\
+ (5306483504a^6b^6 - 2644880308a^2b^6 + 630661472a^3b^6 - 96750052a^4b^6 + 4160240a^5b^6) \\
\prod_{\zeta=1}^{11} \{ a - 2\zeta \} 
\]
\[
\begin{align*}
&+ \left( -208012a^6b^7 + 397208064b^7 - 497150104ab^7 + 240208320a^2b^7 - 61671280a^3b^7 \right) \\
&\quad \prod_{\zeta=1}^{11} \{a - 2\zeta\} \\
&+ \left( 6460608a^4b^7 - 653752a^7b^7 - 24985719b^8 + 31051944ab^6 - 14146242a^2b^8 + 2750616a^3b^8 \right) \\
&\quad \prod_{\zeta=1}^{11} \{a - 2\zeta\} \\
&+ \left( -389367a^6b^8 + 1076416b^9 - 1180388ab^9 + 457424a^2b^9 - 92092a^3b^9 - 26290b^{10} \right) \\
&\quad \prod_{\zeta=1}^{11} \{a - 2\zeta\} \\
&+ \left( 27192a^{10} - 8602a^2b^{10} + 384b^{11} - 252ab^{11} - b^{12} \right) \\
&\quad \prod_{\zeta=1}^{11} \{a - 2\zeta\} \\
&\quad \left( a - b + 23 \right) \Gamma\left( \frac{b}{2} \right) \\
&\quad \left( a - b - 1 \right) \Gamma\left( \frac{a - b + 1}{2} \right) \\
&\times \left\{ \left( 31623443225 - 703416314160a + 590546123298a^2 - 264300628944a^3 + 72578259391a^4 \right) \\
&\quad \prod_{\delta=1}^{12} \{a - (2\delta - 1)\} \\
&+ \left( -1317458400a^5 + 1628308184a^6 - 140529312a^7 + 8439783a^8 - 345840a^9 + 9218a^{10} \right) \\
&\quad \prod_{\delta=1}^{12} \{a - (2\delta - 1)\} \\
&+ \left( -144a^{11} + a^{12} - 373432860360b + 1139061201588ab - 1069825103544a^2b \right) \\
&\quad \prod_{\delta=1}^{12} \{a - (2\delta - 1)\} \\
&+ \left( 50345140140a^3b - 140579255696a^3b + 25911980616a^5b - 3107909616a^6b + 2755914489^7b \right) \\
&\quad \prod_{\delta=1}^{12} \{a - (2\delta - 1)\} \\
&+ \left( -14821224a^8b + 655556a^9b - 12760a^{10}b + 252a^{11}b - 129106402182b^2 - 396896330784ab^2 \right) \\
&\quad \prod_{\delta=1}^{12} \{a - (2\delta - 1)\} \\
&+ \left( 57686507366a^2b^2 - 312991442332a^3b^2 + 93342239796a^4b^2 - 16865721152a^5b^2 \right) \\
&\quad \prod_{\delta=1}^{12} \{a - (2\delta - 1)\} \\
&+ \left( 2154337780a^6b^2 - 160818944a^7b^2 + 10248018a^8b^2 - 267168a^9b^2 + 8602a^{10}b^2 \right) \\
&\quad \prod_{\delta=1}^{12} \{a - (2\delta - 1)\} \\
&+ \left( 320632172520b^3 - 15200003492ab^3 - 81954887264a^2b^3 + 79652150448a^3b^3 \right) \\
&\quad \prod_{\delta=1}^{12} \{a - (2\delta - 1)\} \\
&+ \left( -2779329376a^4b^3 + 5516113736a^5b^3 - 618811872a^6b^3 + 55226864a^7b^3 - 2021976a^8b^3 \right) \\
&\quad \prod_{\delta=1}^{12} \{a - (2\delta - 1)\} \\
&\left( a^3b - 20a^2b^2 + \delta^2 - 1 \right)
\end{align*}
\]
\[\begin{align*}
&+\left(92092ab^3 - 17778184031b^4 + 147359650012ab^4 - 36378600348a^2b^4 - 471573400a^3b^4\right)
&\times \prod_{\delta=1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+\left(369017506a^4b^4 - 842326240a^5b^4 + 112314244a^6b^4 - 6306784a^7b^4 + 389367a^8b^4\right)
&\times \prod_{\delta=1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+\left(51711655760b^5 - 49578268792ab^5 + 16834239856a^2b^5 - 2713871384a^3b^5 - 70724080a^4b^5\right)
&\times \prod_{\delta=1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+\left(56995288a^5b^5 - 7488432a^6b^5 + 653752a^7b^5 - 9260845396b^6 + 9037988736ab^6\right)
&\times \prod_{\delta=1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+\left(-361693772ab^6 + 61995776a^3b^6 - 70100044a^4b^6 + 208012a^6b^6 + 1085127120b^7\right)
&\times \prod_{\delta=1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+\left(-1074415560ab^7 + 391633696a^2b^7 - 85728912a^3b^7 + 6299792a^4b^7 - 534888a^5b^7\right)
&\times \prod_{\delta=1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+\left(-84910089b^8 + 78535248ab^8 - 30456646a^2b^8 + 4383984a^3b^8 - 572033a^4b^8 + 4402200b^9\right)
&\times \prod_{\delta=1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+\left(-3929596ab^9 + 1111176a^2b^9 - 211508a^3b^9 - 145222b^{10} + 105248a^4b^{10} - 31878a^5b^{10}\right)
&\times \prod_{\delta=1}^{12} \left\{ a - (2\delta - 1) \right\} \\
&+\left(2760b^{11} - 1748ab^{11} - 23b^{12}\right) \\
&\times \prod_{\delta=1}^{12} \left\{ a - (2\delta - 1) \right\} \\
\end{align*}\]

On simplification, we get the result (11). By applying same method we can prove the result (12).

References


\[\nu g\]-separation axioms

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Abstract In this paper we discuss new separation axioms using \(\nu g\)-open sets.

Keywords \(\nu g\), spaces.

2000 Mathematics Subject Classification: 54D10, 54D15

§1. Introduction

Norman Levine introduced generalized closed sets, K. Balachandaran and P. Sundaram studied generalized continuous functions and generalized homeomorphism. V. K. Sharma studied generalized separation axioms. Following V. K. Sharma we are going to define a new variety of generalized axioms called \(\nu g\)-separation axioms and study their basic properties and inter-relation with other type of generalized separation axioms. Throughout the paper a space \(X\) means a topological space \((X, \tau)\). For any subset \(A\) of \(X\) its complement, interior, closure, \(\nu g\)-interior, \(\nu g\)-closure are denoted respectively by the symbols \(A^c\), \(A^0\), \(\nu g(A)^0\) and \(\nu g(\overline{A})\).

§2. Preliminaries

**Definition 2.1.** \(A \subset X\) is said to be
(i) Regular closed \([46]\) [resp: \(\alpha\)-closed \([29]\); \(\beta\)-closed \([1]\)] if \(A = \overline{A^\alpha}\) [resp: \((\overline{A^\alpha})^\alpha \subset A; \overline{(A^\alpha)} \subset A; \overline{(\overline{A^\alpha})} \subset A\)] and Regular open \([46]\) [resp: \(\alpha\)-open \([34]\); \(\beta\)-open \([28]\); \(\beta\)-open \([1]\)] if \(A = (A^\alpha)^\alpha\) [resp: \(A \subset ((A^\alpha)^\beta); A \subset (A^\beta); A \subset \overline{(A^\beta)}\)].

(ii) Semi open \([35]\) [resp: \(\nu\)-open] if there exists an open [resp: regular open] set \(U \ni U \subset A \subset U\) and semi closed \([13]\) [resp: \(\nu\)-closed] if its complement if semi open [\(\nu\)-open].

(iii) \(\varphi\)-closed \([36]\) [resp: \(\varphi\)-closed \([41]\)] if \(\overline{A} \subset U\) whenever \(A \subset U\) and \(U\) is open in \(X\).

(iv) \(sg\)-closed \([11, 18, 25]\) [resp: \(gs\)-closed \([32]\)] if \(s(\overline{A}) \subset U\) whenever \(A \subset U\) and \(U\) is semi-open[open] in \(X\).

(v) \(pg\)-closed [resp: \(gp\)-closed; \(gpr\)-closed \([23]\)] if \(p(\overline{A}) \subset U\) whenever \(A \subset U\) and \(U\) is pre-open[open; regular-open] in \(X\).

(vi) \(ag\)-closed [resp: \(go\)-closed \([25]\); \(rg\alpha\)-closed \([47]\)] if \(\alpha(\overline{A}) \subset U\) whenever \(A \subset U\) and \(U\) is \(\{\alpha\text{-open; } \alpha\text{-open}\} \) open in \(X\).

(vii) \(\nu g\)-closed \([6]\) if \(\nu(\overline{A}) \subset U\) whenever \(A \subset U\) and \(U\) is \(\nu\)-open in \(X\).
(viii) clopen [resp: nearly-clopen; \(\nu\)-clopen; semi-clopen; \(g\)-clopen; \(\nu g\)-clopen; \(\nu g\)-open] if it is both open [resp: regular-open; \(\nu\)-open; semi-open; \(g\)-open; \(rg\)-open; \(sg\)-open; \(\nu g\)-open] and closed [resp: regular-closed; \(\nu\)-closed; semi-closed; \(g\)-closed; \(rg\)-closed; \(sg\)-closed; \(\nu g\)-closed].

**Note 2.1.** From definition 2.1 we have the following interrelations among the closed sets.

\[
\begin{array}{c|c|c|c|c}
& r\alpha\text{-closed} & \rho g\text{-closed} & g\alpha\text{-closed} & \beta g\text{-closed} \\
\hline
r\alpha\text{-closed} & \downarrow & \downarrow & \downarrow & \downarrow \\
\hline
\rho g\text{-closed} & \downarrow & \downarrow & \downarrow & \downarrow \\
\hline
g\alpha\text{-closed} & \downarrow & \downarrow & \downarrow & \downarrow \\
\hline
\beta g\text{-closed} & \downarrow & \downarrow & \downarrow & \downarrow \\
\hline
\end{array}
\]

**Definition 2.2.** A function \(f: X \rightarrow Y\) is said to be \([4,7]\)

(i) \(\nu\)-continuous [resp: \(\nu g\)-continuous; \(\nu\)-irresolute; \(\nu g\)-irresolute] if the inverse image of every open [resp: closed; \(\nu\)-open; \(\nu g\)-closed] set is \(\nu\)-open [resp: \(\nu g\)-closed; \(\nu\)-closed; \(\nu g\)-open].

(ii) \(\nu\)-open [resp: \(\nu g\)-open] if the image of open set is \(\nu\)-open [resp: \(\nu g\)-open].

(iii) \(\nu\)-homeomorphism [resp: \(\nu\)-homeomorphism; \(\nu g\)-homeomorphism; \(\nu g\)-homeomorphism] if \(f\) and \(f^{-1}\) are bijective \(\nu\)-continuous [resp: \(\nu g\)-continuous; \(\nu g\)-irresolute].

**Definition 2.3.** \(X\) is said to be \(T_1\) [resp: \(T_2\), \(T_3\), \(sT_1\)] if every generalized [resp: regular generalized, \(\nu\)-generalized, semi-generalized] closed set is closed [resp: regular-closed, \(\nu\)-closed, semi-closed].

**Note 2.2.** The class of regular open sets, \(\nu\)-open sets, open sets, \(\nu g\)-open sets and \(\nu g\)-open sets are denoted by \(RO(X), \nu O(X), \tau(X), GO(X), RGO(X)\) and \(\nu GO(X)\) respectively. Clearly \(RO(X) \subseteq \tau(X) \subseteq GO(X) \subseteq RGO(X) \subseteq \nu GO(X)\).

**Note 2.3.** For any subset \(A\) in \(X\), \(A \in \nu GO(X, x)\) means \(A\) is a \(\nu g\)-neighborhood of \(x\).

**Definition 2.4.** \(X\) is said to be

(i) compact [resp: nearly compact, \(\nu\)-compact, semi-compact, \(g\)-compact, \(rg\)-compact, \(sg\)-compact, \(\nu g\)-compact] if every open [resp: regular-open, \(\nu\)-open, semi-open, \(g\)-open, \(rg\)-open, \(sg\)-open, \(\nu g\)-open] cover has a finite subcover.

(ii) \(T_0\) [resp: \(rT_0\), \(\nu T_0\) \([5]\), \(sT_0\), \(g_0\) \([35]\), \(rg_0\) \([11]\), \(sg_0\) \([34]\)] space if for each \(x \neq y \in X\), \(\exists U \in \tau(X)\) [resp: \(RO(X); \nu O(X); SO(X); GO(X); RGO(X); SGO(X)\)] containing either \(x\) or \(y\);

(iii) \(T_1\) [resp: \(rT_1\), \(\nu T_1\) \([5]\), \(sT_1\), \(g_1\) \([45]\), \(rg_1\) \([11]\), \(sg_1\) \([34]\)] space if for each \(x \neq y \in X\), \(\exists U, V \in \tau(X)\) [resp: \(RO(X); \nu O(X); SO(X); GO(X); RGO(X); SGO(X)\)] such that \(x \in U - V\) and \(y \in V - U\);

(iv) \(T_2\) [resp: \(rT_2\), \(\nu T_2\) \([5]\), \(sT_2\), \(g_2\) \([35]\), \(rg_2\) \([11]\), \(sg_2\) \([34]\)] space if for each \(x \neq y \in X\), \(\exists U, V \in \tau(X)\) [resp: \(RO(X); \nu O(X); SO(X); GO(X); RGO(X); SGO(X)\)] such that \(x \in U\), \(y \in V\) and \(U \cap V = \emptyset\);

(v) \(C_0\) [resp: \(rC_0, \nu C_0\) \([5]\), \(sC_0, gC_0\) \([7]\), \(rgC_0\) \([11]\), \(sgC_0\) \([10]\)] space if for each \(x \neq y \in X\), \(\exists U \in \tau(X)\) [resp: \(RO(X); \nu O(X); SO(X); GO(X); RGO(X); SGO(X)\)] whose closure contains either \(x\) or \(y\);

(vi) \(C_1\) [resp: \(rC_1\), \(\nu C_1\) \([5]\), \(sC_1\), \(gC_1\) \([7]\), \(rgC_1\) \([11]\), \(sgC_1\) \([10]\)] space if for each \(x \neq y \in X\),
$X, \exists U, V \in \tau(X)$ [resp: $RO(X); \nu O(X); SO(X); GO(X); RGO(X); SGO(X);]$ such that $x \in U$ and $y \in V$;

(vii) $C_2$ [resp: $rC_2$, $\nu C_2$ [5], $scC_2$, $gC_2$ [7], $rgC_2$ [11], $sgC_2$ [110] space if for each $x \neq y \in X, \exists U, V \in \tau(X)$ [resp: $RO(X); \nu O(X); SO(X); GO(X); RGO(X); SGO(X);]$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$;

(viii) $D_0$ [resp: $rD_0$, $\nu D_0$ [5], $sD_0$, $gD_0$ [7], $rgD_0$ [11], $sgD_0$ [110] space if for each $x \neq y \in X, \exists U \in D(X)$ [resp: $RDO(X); \nu DO(X); SDO(X); GDO(X); RGDO(X); SSGO(X);]$ containing either $x$ or $y$;

(ix) $D_1$ [resp: $rD_1$, $\nu D_1$ [5], $sD_1$, $gD_1$ [7], $rgD_1$ [11], $sgD_1$ [110] space if for each $x \neq y \in X, \exists U, V \in \tau(X)$ [resp: $RO(X); \nu O(X); SO(X); GO(X); RGO(X); SGO(X);$] $\exists x \in U - V; y \in V - U$;

(x) $D_2$ [resp: $rD_2$, $\nu D_2$ [5], $sD_2$, $gD_2$ [7], $rgD_2$ [11], $sgD_2$ [110] space if for each $x \neq y \in X, \exists U, V \in \tau(X)$ [resp: $RO(X); \nu O(X); SO(X); GO(X); RGO(X); SGO(X);$] $\exists x \in U - V; y \in V - U$ and $U \cap V = \emptyset$;

(xi) $R_0$ [resp: $rR_0$, $\nu R_0$ [5], $sR_0$, $gR_0$ [7], $rgR_0$ [11], $sgR_0$ [110] space if for each $x \neq y \in X, \exists U \in \tau(X)$ [resp: $RO(X); \nu O(X); SO(X); GO(X); RGO(X); SGO(X);$] $\exists x \in U \in \tau(X); x \in U \in \nu O(X); x \in U \in SO(X); x \in U \in GO(X); x \in U \in RGO(X); x \in U \in SGO(X)$;

(xii) $R_1$ [resp: $rR_1$, $\nu R_1$ [5], $sR_1$, $gR_1$ [7], $rgR_1$ [11], $sgR_1$ [110] space for $x, y \in X \ni \exists \{x\} \neq \{y\}$ [resp: $\exists x \neq y \ni \nu \{x\} \neq \nu \{y\}; \exists s \{x\} \neq s \{y\}; \exists g \{x\} \neq g \{y\}; \exists \nu g \{x\} \neq \nu g \{y\}; \exists s \{x\} \neq s \{y\}; \forall \nu \exists g \{x\} \neq g \{y\}; \exists g \{x\} \neq g \{y\}; \exists s \{x\} \neq s \{y\}; \exists g \{x\} \neq g \{y\}; \exists g \{x\} \neq g \{y\}; \exists s \{x\} \neq s \{y\}; \exists g \{x\} \neq g \{y\}; \exists g \{x\} \neq g \{y\}$] such that $U \cap V \ni \{x\} \ni \{y\}$ and $U \cap V \ni \{y\} \ni \{x\}$ are disjoint.

Theorem 2.1.

(i) If $x$ is a $\nu g$-limit point of any $A \subset X$, then every $\nu g$-neighbourhood of $x$ contains infinitely many distinct points.

(ii) Let $A \subseteq Y \subseteq X$ and $Y$ is regularly open subspace of $X$ then $A$ is $\nu g$-open in $X$ iff $A$ is $\nu g$-open in $\tau Y$.

Theorem 2.2. If $f$ is $\nu g$-continuous [resp: $\nu g$-irresolute,$\nu g$-homeomorphism] and $G$ is open [resp: $\nu g$-open $[\nu g$-closed]] set in $Y$, then $f^{-1}(G)$ is $\nu g$-open [resp: $\nu g$-open $[\nu g$-closed]] in $X$.

Theorem 2.3. Let $Y$ and $\{X_\alpha : \alpha \in I\}$ be topological spaces. Let $f : Y \to \Pi X_\alpha$ be a function. If $f$ is $\nu g$-continuous, then $\pi_\alpha \circ f : Y \to X_\alpha$ is $\nu g$-continuous.

Theorem 2.4. If $Y$ is $\nu T_4$ and $\{X_\alpha : \alpha \in I\}$ be topological spaces. Let $f : Y \to \Pi X_\alpha$ be a function, then $f$ is $\nu g$-continuous if $\nu g$-continuous when $\pi_\alpha \circ f : Y \to X_\alpha$ is $\nu g$-continuous.

Corollary 2.1. Let $f_\alpha : X_\alpha \to Y$ be a function and let $f : \Pi X_\alpha \to \Pi Y_\alpha$ be defined by $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$. If $f$ is $\nu g$-continuous then each $f_\alpha$ is $\nu g$-continuous.

Corollary 2.2. For each $\alpha$, let $X_\alpha$ be $\nu T_4$ and let $f_\alpha : X_\alpha \to Y$ be a function and let $f : \Pi X_\alpha \to \Pi Y_\alpha$ be defined by $f((x_\alpha)_{\alpha \in I}) = (f_\alpha(x_\alpha))_{\alpha \in I}$, then $f$ is $\nu g$-continuous if each $f_\alpha$ is $\nu g$-continuous.
§3. \(\nu g_i\) spaces, \(i = 0, 1, 2\)

**Definition 3.1.** \(X\) is said to be

(i) a \(\nu g_0\) space if for each pair of distinct points \(x, y\) of \(X\), there exists a \(\nu g\)-open set \(G\) containing either \(x\) or \(y\);

(ii) a \(\nu g_1\) space if for each pair of distinct points \(x, y\) of \(X\), there exists a \(\nu g\)-open set \(G\) containing \(x\) but not \(y\) and a \(\nu g\)-open set \(H\) containing \(y\) but not \(x\);

(iii) a \(\nu g_2\) space if for each pair of distinct points \(x, y\) of \(X\), there exists disjoint \(\nu g\)-open sets \(G\) and \(H\) such that \(G\) containing \(x\) but not \(y\) and \(H\) containing \(y\) but not \(x\).

\[
\nu g_i \Rightarrow T_i \Rightarrow g_i \Rightarrow \nu g_i,
\]

**Note 3.1.**

(i) \(\nu g_i \Rightarrow \nu g_i\) for \(i = 0, 1, 2\). But the converse is not true in general.

(ii) \(X\) is \(\nu g_2 \Rightarrow X\) is \(\nu g_1 \Rightarrow X\) is \(\nu g_0\).

**Example 3.1.** Let \(X = \{a, b, c\}\) and

(i) \(\tau = \{\emptyset, \{a, c\}, X\}\) then \(X\) is \(\nu g_1\) but not \(\tau - T_0\) and \(T_0\), \(i = 0, 1, 2\);

(ii) \(\tau = \{\emptyset, \{a\}, \{a, c\}, X\}\) then \(X\) is not \(\nu g_i\) for \(i = 0, 1, 2\).

**Remark 3.1.** If \(X\) is \(\nu - T_2\) then \(\nu - T_1\) and \(\nu g_i\) are one and the same for \(i = 0, 1, 2\).

**Proof.** Since \(X\) is \(\nu - T_2\), every \(\nu g\)-closed set is \(\nu\)-closed and so the proof is straightforward from the respective definitions.

**Theorem 3.1.**

(i) Every open subspace [resp: regular open] of \(\nu g_i\) space is \(\nu g_i\) for \(i = 0, 1, 2\);

(ii) The product of \(\nu g_i\) spaces is again \(\nu g_i\) for \(i = 0, 1, 2\).

**Theorem 3.2.**

(i) \(X\) is \(\nu g_0\) iff \(\forall x \in X, \exists U \in \nu \text{GO}(X)\) containing \(x\) \(\supset\) the subspace \(U\) is \(\nu g_0\).

(ii) \(X\) is \(\nu g_0\) iff distinct points of \(X\) have disjoint \(\nu g\)-closures.

**Theorem 3.3.** The followings are equivalent:

(i) \(X\) is \(\nu g_1\),

(ii) Each one point set is \(\nu g\)-closed,

(iii) Each subset of \(X\) is the intersection of all \(\nu g\)-open sets containing it,

(iv) For any \(x \in X\), the intersection of all \(\nu g\)-open sets containing the point is the set \(\{x\}\).

**Theorem 3.4.** If \(X\) is \(\nu g_1\) then distinct points of \(X\) have disjoint \(\nu g\)-closures.

**Theorem 3.5.** Suppose \(x\) is a \(\nu g\)-limit point of a subset of \(A\) of a \(\nu g_1\) space \(X\). Then every neighbourhood of \(x\) contains infinitely many distinct points of \(A\).

**Theorem 3.6.** \(X\) is \(\nu g_2\) iff the intersection of all \(\nu g\)-closed, \(\nu g\)-neighbourhoods of each point of the space is reduced to that point.

**Proof.** Let \(X\) be \(\nu g_2\) and \(x \in X\), then for each \(y \neq x\) in \(X\), \(\exists U, V \in \nu \text{GO}(X) \ni x \in U, y \in V\) and \(U \cap V = \emptyset\). Since \(x \in U - V\), hence \(X - V\) is a \(\nu g\)-closed, \(\nu g\)-neighbourhood of \(x\) to which \(y\) does not belong. Consequently, the intersection of all \(\nu g\)-closed, \(\nu g\)-neighbourhoods of \(x\) is reduced to \(\{x\}\).

Conversely let \(y \neq x\) in \(X\), then by hypothesis there exists a \(\nu g\)-closed, \(\nu g\)-neighbourhood \(U\) of \(x\) such that \(y \notin U\). Now \(\exists G \in \nu \text{GO}(X) \ni x \in G \subset U\). Thus \(G\) and \(X - U\) are disjoint \(\nu g\)-open sets containing \(x\) and \(y\) respectively. Hence \(X\) is \(\nu g_2\).
Theorem 3.7. If to each point \( x \in X \), there exist a \( \nu g \)-closed, \( \nu g \)-open subset of \( X \) containing \( x \) which is also a \( \nu g_2 \) subspace of \( X \), then \( X \) is \( \nu g_2 \).

**Proof.** Let \( x \in X, U \) a \( \nu g \)-closed, \( \nu g \)-open subset of \( X \) containing \( x \) and which is also a \( \nu g_2 \) subspace of \( X \), then the intersection of all \( \nu g \)-closed, \( \nu g \)-neighbourhoods of \( x \) in \( U \) is reduced to \( x \). \( U \) being \( \nu g \)-closed, \( \nu g \)-open, these are \( \nu g \)-closed, \( \nu g \)-neighbourhoods of \( x \) in \( X \). Thus the intersection of all \( \nu g \)-closed, \( \nu g \)-neighbourhoods of \( x \) is reduced to \( \{ x \} \). Hence by theorem 3.6, \( X \) is \( \nu g_2 \).

Theorem 3.8. If \( X \) is \( \nu g_2 \), then the diagonal \( \Delta \) in \( X \times X \) is \( \nu g \)-closed.

**Proof.** Suppose \((x, y) \in X \times X - \Delta \). As \((x, y) \notin \Delta \) and \( x \neq y \). Since \( X \) is \( \nu g_2 \), \( \exists U, V \in \nu GO(X) \ni x \in U, y \in V \) and \( U \cap V = \phi \). \( U \cap V = \phi \Rightarrow (U \times V) \cap \Delta = \phi \) and therefore \((U \times V) \subset X \times X - \Delta \). Further \((x, y) \in (U \times V) \) and \((U \times V) \) is \( \nu g \)-open in \( X \times X \) gives \( X \times X - \Delta \) is \( \nu g \)-open. Hence \( \Delta \) is \( \nu g \)-closed.

Corollary 3.1.

(i) In an \( T_1 \) [resp: \( rT_1; \nu T_1; sT_1; g_1; r g_1; s g_1 \)] space, each singleton set is \( \nu g \)-closed;

(ii) if \( X \) is \( T_1 \) [resp: \( rT_1; \nu T_1; sT_1; g_1; r g_1; s g_1 \)] then distinct points of \( X \) have disjoint \( \nu g \)-closures;

(iii) if \( X \) is \( T_2 \) [resp: \( r T_2; \nu T_2; s T_2; g_2; r g_2; s g_2 \)] then the diagonal \( \Delta \) in \( X \times X \) is \( \nu g \)-closed.

Theorem 3.9. In \( \nu g_2 \)-space, \( \nu g \)-limits of sequences, if exist, are unique.

Theorem 3.10. In a \( \nu g_2 \) space, a point and disjoint \( \nu g \)-compact subspace can be separated by disjoint \( \nu g \)-open sets.

**Proof.** Let \( X \) be a \( \nu g_2 \) space, \( x \in X \) and \( C \) a \( \nu g \)-compact subspace of \( X \) not containing \( x \). Let \( y \in C \) then for \( x \neq y \) in \( X \), there exist disjoint \( \nu g \)-open neighborhoods \( G_x \) and \( H_y \). Allowing this for each \( y \) in \( C \), we obtain a class \( \{ H_y \} \) whose union covers \( C \); and since \( C \) is \( \nu g \)-compact, some finite subclass, which we denote by \( \{ H_i, i = 1 \text{ to } n \} \) covers \( C \). If \( G_i \) is \( \nu g \)-neighborhood of \( x \) corresponding to \( H_i \), we put \( G = \bigcup_{i=1}^{n} G_i \) and \( H = \bigcap_{i=1}^{n} H_i \), satisfying the required properties.

Theorem 3.11. Every \( \nu g \)-compact subspace of a \( \nu g_2 \) space is \( \nu g \)-closed.

**Proof.** Let \( C \) be \( \nu g \)-compact subspace of a \( \nu g_2 \) space. If \( x \) be any point in \( C^c \), by above theorem \( x \) has a \( \nu g \)-neighborhood \( G \ni x \in G \subset C^c \). This shows that \( C^c \) is the union of \( \nu g \)-open sets and therefore \( C^c \) is \( \nu g \)-open. Thus \( C \) is \( \nu g \)-closed.

Corollary 3.2.

(i) Show that in a \( T_2 \) [resp: \( rT_2; \nu T_2; sT_2; g_2; r g_2; s g_2 \)] space, a point and disjoint compact [resp: nearly-compact; \( \nu \)-compact; semi-compact; \( g \)-compact; \( r g \)-compact; \( s g \)-compact] subspace can be separated by disjoint \( \nu g \)-open sets;

(ii) every compact [resp: nearly-compact; \( \nu \)-compact; semi-compact; \( g \)-compact; \( r g \)-compact; \( s g \)-compact] subspace of a \( T_2 \) [resp: \( rT_2; \nu T_2; sT_2; g_2; r g_2; s g_2 \)] space is \( \nu g \)-closed.

Theorem 3.12. If \( f \) is injective, \( \nu g \)-irresolute and \( Y \) is \( \nu g_i \), then \( X \) is \( \nu g_i, i = 0, 1, 2 \).

**Proof.** Obvious from the definitions and so omitted.

Corollary 3.3.

(i) If \( f \) is injective, \( \nu g \)-continuous and \( Y \) is \( T_i \) then \( X \) is \( \nu g_i, i = 0, 1, 2 \);

(ii) if \( f \) is injective, \( r \)-irresolute \([r \text{-continuous}] \) and \( Y \) is \( r T_i \) then \( X \) is \( \nu g_i, i = 0, 1, 2 \);

(iii) the property of being a space is \( \nu g \)-closed is a \( \nu g \)-topological property;

(iv) let \( f \) is a \( \nu g \)-homeomorphism, then \( X \) is \( \nu g_i \) if \( Y \) is \( \nu g_i, i = 0, 1, 2 \).
Theorem 3.13. Let $X$ be $T_1$ and $f : X \to Y$ be $\nu g$-closed subjection. Then $X$ is $\nu g_1$.

Theorem 3.14. Every $\nu g$-irresolute map from a $\nu g$-compact space into a $\nu g_2$ space is $\nu g$-closed.

Proof. Suppose $f : X \to Y$ is $\nu g$-irresolute where $X$ is $\nu g$-compact and $Y$ is $\nu g_2$. Let $C \subset X$ be closed, then $C \subset X$ is $\nu g$-closed and hence $C$ is $\nu g$-compact and so $f(C)$ is $\nu g$-compact. But then $f(C)$ is $\nu g$-closed in $Y$. Hence the image of any $\nu g$-closed set in $X$ is $\nu g$-closed set in $Y$. Thus $f$ is $\nu g$-closed.

Theorem 3.15. Any $\nu g$-irresolute bijection from a $\nu g$-compact space onto a $\nu g_2$ space is a $\nu g$-homeomorphism.

Proof. Let $f : X \to Y$ be a $\nu g$-irresolute bijection from a $\nu g$-compact space onto a $\nu g_2$ space. Let $G$ be an $\nu g$-open subset of $X$. Then $X - G$ is $\nu g$-closed and hence $f(X - G)$ is $\nu g$-closed. Since $f$ is bijective $f(X - G) = Y - f(G)$. Therefore $f(G)$ is $\nu g$-open in $Y$. This means that $f$ is $\nu g$-open. Hence $f$ is bijective $\nu g$-irresolute and $\nu g$-open. Thus $f$ is $\nu g$-homeomorphism.

Corollary 3.4. Any $\nu g$-continuous bijection from a $\nu g$-compact space onto a $\nu g_2$ space is a $\nu g$-homeomorphism.

Theorem 3.16. The followings are equivalent:

(i) $X$ is $\nu g_2$;

(ii) for each pair $x \neq y \in X$, $\exists$ an $\nu g$-open, $\nu g$-closed set $V \ni x \in V$ and $y \notin V$;

(iii) for each pair $x \neq y \in X$, $\exists f : X \to [0, 1]$ such that $f(x) = 0$, $f(C) = 1$ and $f$ is $\nu g$-continuous.

Theorem 3.17. If $f : X \to Y$ is $\nu g$-irresolute and $Y$ is $\nu g_2$ then

(i) the set $A = \{ (x_1, x_2) : f(x_1) = f(x_2) \}$ is $\nu g$-closed in $X \times X$;

(ii) $G(f)$, graph of $f$ is $\nu g$-closed in $X \times Y$.

Proof. (i) Let $A = \{ (x_1, x_2) : f(x_1) = f(x_2) \}$. If $(x_1, x_2) \in X \times X - A$, then $f(x_1) \neq f(x_2) \Rightarrow \exists$ disjoint $V_1, V_2 \in \nu gO(Y) \ni f(x_1) \in V_1$ and $f(x_2) \in V_2$, then by $\nu g$-irresoluteness of $f$, $f^{-1}(V_j) \in \nu gO(X, x_j)$ for each $j$. Thus $(x_1, x_2) \in f^{-1}(V_1) \times f^{-1}(V_2) \in \nu gO(X \times X)$. Therefore $f^{-1}(V_1) \times f^{-1}(V_2) \subset X \times X - A \Rightarrow X \times X - A$ is $\nu g$-open. Hence $A$ is $\nu g$-closed.

(ii) Let $(x, y) \notin G(f) \Rightarrow y \neq f(x) \Rightarrow \exists$ disjoint $\nu g$-open sets $V$ and $W \ni x \in V$ and $y \in W$. Since $f$ is $\nu g$-irresolute, $3U \in \nu gO(X) \ni x \in U$ and $f(U) \subset W$. Therefore we obtain $(x, y) \notin U \times V \subset X \times Y - G(f)$. Hence $X \times Y - G(f)$ is $\nu g$-open. Hence $G(f)$ is $\nu g$-closed in $X \times Y$.

Theorem 3.18. If $f : X \to Y$ is $\nu g$-open and the set $A = \{ (x_1, x_2) : f(x_1) = f(x_2) \}$ is closed in $X \times X$. Then $Y$ is $\nu g_2$.

Theorem 3.19. Let $Y$ and $\{ X_\alpha : \alpha \in I \}$ be topological spaces. If $f : Y \to \Pi X_\alpha$ be a $\nu g$-continuous function and $Y$ is $\nu T_1$, then $\Pi X_\alpha$ and each $X_\alpha$ are $\nu g_i$, $i = 0, 1, 2$.

Problem 3.1. If $Y$ be a $\nu g_2$ space and $A$ be regular-open subspace of $X$. If $f : (A, \tau_A) \to (Y, \sigma)$ is $\nu g$-irresolute. Does there exist any extension $F : (X, \tau) \to (Y, \sigma)$.

Theorem 3.20. Let $X$ be an arbitrary space, $R$ an equivalence relation in $X$ and $p : X \to X/R$ the identification map. If $R \subset X \times X$ is $\nu g$-closed in $X \times X$ and $p$ is $\nu g$-open map, then $X/R$ is $\nu g_2$.

Proof. Let $p(x), p(y)$ be distinct members of $X/R$. Since $x$ and $y$ are not related,
$R \in \nu GC(X \times X)$. There are $U, V \in \nu GO(X) \ni x \in U, y \in V$ and $U \times V \subseteq R^c$. Thus $p(U), p(V)$ are disjoint and also $\nu g$-open in $X/R$ since $p$ is $\nu g$-open.

**Theorem 3.21.** The following four properties are equivalent:

(i) $X$ is $\nu g_2$;

(ii) Let $x \in X$. For each $y \neq x$, $\exists U \in \nu GO(X) \ni x \in U$ and $y \notin \nu g(U)$;

(iii) for each $x \in X$, $\cap \{ \nu g(U) \mid U \in \nu GO(X) \land x \in U \} \ni \{ x \}$;

(iv) the diagonal $\Delta = \{(x, x) / x \in X\}$ is $\nu g$-closed in $X \times X$.

**Proof.** (i) $\Rightarrow$ (ii). Let $x \neq y \in X$, then there are disjoint $\nu g$-open sets $U, V \in \nu GO(X) \ni x \in U$ and $y \in V$. Clearly $V^c$ is $\nu g$-closed, $\nu g(U) \subseteq V^c, y \notin V^c$ and therefore $y \notin \nu g(U)$.

(ii) $\Rightarrow$ (iii). If $y \neq x$, then $\exists U \in \nu GO(X), x \in U$ and $y \notin \nu g(U)$. So $y \notin \cap \{ \nu g(U) \mid U \in \nu GO(X) \land x \in U \}$.

(iii) $\Rightarrow$ (iv). We prove $\Delta^c$ is $\nu g$-open. Let $(x, y) \notin \Delta$. Then $y \neq x$ and $\cap \{ \nu g(U) \mid U \in \nu GO(X) \land x \in U \} = \{ x \}$, there is some $U \in \nu GO(X)$ with $x \in U$ and $y \notin \nu g(U)$. Since $U \cap \{ \nu g(U) \}$ is $\nu g$-open set such that $(x, y) \in U \times \{ \nu g(U) \}$ is $\nu g$-open set such that $(x, y) \in U \times \{ \nu g(U) \}$.

(iv) $\Rightarrow$ (i). Let $(x, y) \notin \Delta$ and thus $\exists U, V \in \nu GO(X) \ni (x, y) \in U \times V$ and $(U \times V) \cap \Delta = \phi$. Clearly, for the $U, V \in \nu GO(X)$ we have $x \in U, y \in V$ and $U \cap V = \phi$.

### §4. $\nu g_3$ and $\nu g_4$ Spaces

**Definition 4.1.** $X$ is said to be

(i) a $\nu g_3$ space if for every $\nu g$-closed sets $F$ and a point $x \notin F$, $\exists$ disjoint $U, V \in \nu O(X) \ni F \subseteq U, x \in V$;

(ii) a $\nu g_4$ space if for every $\nu g$-closed sets $F$ and a point $x \notin F$, $\exists$ disjoint $U, V \in \nu GO(X) \ni F \subseteq U, x \in V$;

(iii) a $\nu g_4$ space if for each pair of disjoint $\nu g$-closed sets $F$ and $H$, $\exists$ disjoint $U, V \in \nu O(X) \ni F \subseteq U, H \subseteq V$;

(iv) a $\nu g_4$ space if for each pair of disjoint $\nu g$-closed sets $F$ and $H$, $\exists$ disjoint $U, V \in \nu gO(X) \ni F \subseteq U, H \subseteq V$.

**Note 4.1.**

(i) Every $\nu T_3$ space is $\nu g_3$ space but not conversely;

(ii) every $\nu T_4$ space is $\nu g_4$ space but not conversely;

(iii) every $\nu g_3$ space is $\nu g_3$ space but not conversely;

(iv) every $\nu g_4$ space is $\nu g_4$ space but not conversely.

**Example 4.1.** Let $Y$ and $Z$ be disjoint infinite sets and let $X = Y \cup Z$ and $\tau = \{ \phi, Y, Z, X \}$. Clearly $X$ is locally indiscrete and thus every $\nu$-closed set is clopen, hence $X$ is $\nu T_3$. If $\phi \neq A \subseteq X$ and $x \in Y - A$ then $A$ is $\nu g$-closed, but $A$ and $x$ cannot be separated by disjoint $\nu$-open sets, thus $X$ is not $\nu g_3$. Similarly $X$ is $\nu T_4$, but not $\nu g_4$.

**Example 4.2.** Let $X = \{ a, b, c \}$ and

(i) $\tau = \{ \phi, \{ a \}, \{ b \}, \{ a, b \}, X \}$ then $X$ is $\nu g_3; \nu g_4$ and $\nu T_i$ for $i = 3, 4$;

(ii) $\tau = \{ \phi, \{ a \}, X \}$ then $X$ is $\nu g_3$ but not $\nu g_4$ and $\nu T_i$ for $i = 3, 4$.

**Lemma 4.1.** $X$ is $\nu g$-regular iff $X$ is $\nu$-regular and $\nu - T \frac{1}{2}$.
**Proof.** $X$ is $\nu g$-regular, then obviously $X$ is $\nu$-regular. Let $A \subseteq X$ be $\nu g$-closed. For each $x \notin A$, $\exists \nu g \in \nu O(X, x) \ni V_x \cap A = \emptyset$. If $V = \bigcup \{V_x : x \notin A\}$, then $V$ is $\nu$-open and $V = X - A$. Hence $A$ is $\nu$-closed implies $X$ is $\nu - T_\sharp$.

**Theorem 4.1.** If $X$ is $\nu g_3$. Then for each $x \in X$ and each $U \in \nu O(X, x)$, $\exists V \in \nu O(X, x) \ni \overline{\nu(A)} \subseteq U$.

**Proof.** Let $x \in U \in \nu O(X, x)$. Let $B = X - U$, then $B$ is $\nu g$-closed and by $\nu g$-regularity of $X$, $\exists$ disjoint $V, W \in \nu O(X) \ni x \in V \& B \subseteq W$. Then $\nu(\overline{V}) \cap B = \emptyset \Rightarrow \nu(\overline{V}) \subseteq X - B$.

**Theorem 4.2.** The followings are equivalent:

(i) $X$ is $\nu g_3$;

(ii) $\forall x \in X$ and $\forall G \in \nu GO(X, x)$, $\exists U \in \nu O(X) \ni x \in U \subseteq \overline{\nu(U)} \subseteq G$;

(iii) $\exists F \in \nu GC(X)$, the intersection of all $\nu$-closed $\nu$-neighbourhoods of $F$ is exactly $F$;

(iv) for every set $A$ and $B \in \nu GO(X) \ni A \cap B \neq \emptyset$, $\exists G \in \nu O(X) \ni A \cap G \neq \emptyset$ and $\nu(\overline{G}) \subseteq B$;

(v) for $A \neq \emptyset$ and $B \in \nu GC(X)$ with $A \cap B = \emptyset$, $\exists$ disjoint $G, H \in \nu O(X) \ni A \subseteq G$ and $B \subseteq H$.

**Theorem 4.3.** If $X$ is $\nu g g_3$. Then for each $x \in X$ and each $U \in \nu GO(X, x)$, $\exists V \in \nu GO(X, x) \ni \nu g(\overline{A}) \subseteq U$.

**Proof.** Let $x \in U \in \nu O(X, x)$. Let $B = X - U$, then $B$ is $\nu g$-closed and $\nu g$-regularity of $X$, $\exists$ disjoint $V, W \in \nu GO(X) \ni x \in V \& B \subseteq W$. Then $\nu(\overline{V}) \cap B = \emptyset \Rightarrow \nu(\overline{V}) \subseteq X - B$.

**Corollary 4.1.** If $X$ is $T_\sharp$ [resp: $rT_\sharp$; $sT_\sharp$; $rg_3$; $sg_3$]. Then for each $x \in X$ and each $\nu g$-open neighborhood $U$ of $x$ there exists a $\nu g$-open neighborhood $V$ of $x$ such that $\nu g(\overline{V}) \subseteq U$.

**Theorem 4.4.** If $f$ is $\nu$-closed, $\nu g$-irresolute bijection. Then $X$ is $\nu g g_3$ iff $Y$ is $\nu g g_3$.

**Proof.** Let $F$ be closed set in $X$ and $x \notin F$, then $f(x) \notin f(F)$ and $f(F)$ is $\nu g$-closed in $Y$. By $\nu g_3$ of $Y$, $\exists V, W \in \nu GO(Y) \ni f(x) \in V$ and $f(F) \subseteq W$. Hence $x \in f^{-1}(V)$ and $F \subseteq f^{-1}(W)$, where $f^{-1}(V)$ and $f^{-1}(W)$ are disjoint $\nu g$-open sets in $X$ (by $\nu g$-irresoluteness of $f$).

Hence $X$ is $\nu g g_3$.

Conversely, $X$ be $\nu g g_3$ and $K$ any $\nu g$-closed in $Y$ with $y \notin K$, then $f^{-1}(K)$ is $\nu g$-closed in $X \ni f^{-1}(y) \notin f^{-1}(K)$. By $\nu g_3$ of $X$, $\exists$ disjoint $V, W \in \nu GO(X) \ni f^{-1}(y) \in V$ and $f^{-1}(K) \subseteq W$. Hence $y \in f(V)$ and $K \subseteq f(W) \ni f(V)$ and $f(W)$ are disjoint $\nu g$-open sets in $X$. Thus $Y$ is $\nu g g_3$.

**Theorem 4.5.** $X$ is $\nu g$-normal iff $\forall F \in \nu GC(X)$ and a $\nu g$-open set $G \in \nu GO(X, A)$, $\exists V \in \nu O(X) \ni F \subseteq V \subseteq \nu \overline{V} \subseteq G$.

**Theorem 4.6.** $X$ is $\nu g$-normal iff $\forall$ disjoint $A, B \in \nu GC(X), \exists$ disjoint $U, V \in \nu GO(X) \ni A \subseteq U$ and $B \subseteq V$.

**Proof.** Necessity: Follows from the fact that every $\nu$-open set is $\nu g$-open.

Sufficiency: Let $A, B$ be disjoint $\nu g$-closed sets and $U, V$ are disjoint $\nu g$-open sets such that $A \subseteq U$ and $B \subseteq V$. Since $U$ and $V$ are $\nu g$-open sets, $A \subseteq U$ and $B \subseteq V \Rightarrow A \subseteq \nu(U)\circ$ and $B \subseteq \nu(V)\circ$. Hence $\nu(U)\circ$ and $\nu(V)\circ$ are disjoint $\nu g$-open sets satisfying the axiom of $\nu g$-normality.

**Theorem 4.7.** The followings are equivalent:

(i) $X$ is $\nu$-normal;
(ii) for any pair of disjoint closed sets A and B, \( \exists \) disjoint U, V \( \in \nu GO(X) \)and \( \exists A \subseteq U \) and \( B \subseteq V \);

(iii) for every closed set A and an open B \( \supseteq A \), \( \exists U \in \nu GO(X) \) \( \supseteq A \subseteq U \subseteq \nu(U) \subseteq B \);

(iv) for every closed set A and a \( \nu g \)-open B \( \supseteq A \), \( \exists U \in \nu O(X) \) \( \supseteq A \subseteq U \subseteq \nu(U) \subseteq (B)^o \);

(v) for every \( \nu g \)-closed set A and every open B \( \supseteq A \), \( \exists U \in \nu O(X) \) \( \supseteq A \subseteq \nu(A) \subseteq U \subseteq \nu(U) \subseteq B \).

**Proof.** (i) \( \Rightarrow \) (ii). Let A and B be two disjoint closed subsets of X. Since X is \( \nu g \)-normal, \( \exists \) disjoint U, V \( \in \nu O(X) \) \( \supseteq A \subseteq U \) and \( B \subseteq V \). Since \( \nu g \)-open sets are \( \nu g \)-open sets, it follows that U, V \( \in \nu GO(X) \) \( \supseteq A \subseteq U \) and \( B \subseteq V \).

(ii) \( \Rightarrow \) (iii). Let A be a closed subset of X and \( B \in \tau(X) \supseteq A \subseteq B \). Then A and \( X - B \) are disjoint closed subsets of X. Therefore, \( \exists \) disjoint U, V \( \in \nu GO(X) \), \( A \subseteq U \) and \( X - B \subseteq V \). Thus \( A \subseteq U \subseteq X - V \subseteq B \). Since B is open and \( X - V \) is \( \nu g \)-closed, therefore \( \nu(X - V) \subseteq B \). Hence A \( \subseteq U \subseteq \nu(U) \subseteq B \).

(iii) \( \Rightarrow \) (iv). Let A be a closed subset of X and \( B \in \nu GO(X) \supseteq A \subseteq B \). Since B is \( \nu g \)-open and A is closed, therefore A \( \subseteq B^o \). In view of theorem 2 of Maheshwari and Prasad, there exists a \( \nu \)-open set U such that \( A \subseteq U \subseteq \nu(U) \subseteq B^o \).

(iv) \( \Rightarrow \) (v). Let A be any \( \nu g \)-closed subset of X and B be an open set such that A \( \subseteq B \). A \( \subseteq B \Rightarrow A \subseteq B \). By theorem 2 of Maheshwari and Prasad, \( \exists U \in \nu O(X) \supseteq A \subseteq U \subseteq \nu(U) \subseteq B^o \).

(v) \( \Rightarrow \) (i). Let A and B be disjoint closed subsets of X. Then A is \( \nu g \)-closed and A \( \subseteq X - B \). Therefore, \( \exists U \in \nu O(X) \supseteq A \subseteq \nu(A) \subseteq U \subseteq \nu(U) \subseteq B \). Thus A \( \subseteq U \), \( B \subseteq X - \nu(U) \), which is \( \nu \)-open and \( U \cap (X - \nu(U)) = \phi \). Hence X is \( \nu g \)-normal.

**Theorem 4.8.** The followings are equivalent:

(i) X is \( \nu g \)-normal;

(ii) for every A \( \in \nu GC(X) \) and every \( \nu g \)-open set containing A, there exists a \( \nu \)-clopen set V such that A \( \subseteq V \subseteq U \).

**Theorem 4.9.** Let X be an almost normal space and \( F \cap A = \phi \) where F is regularly closed and A is \( \nu g \)-closed, then there exists disjoint open sets U and V such that F \( \subseteq U \), \( B \subseteq V \).

**Theorem 4.10.** X is almost normal iff for every disjoint regular closed set F and a closed set A, there exists disjoint \( \nu g \)-open sets in X such that F \( \subseteq U \), \( B \subseteq V \).

**Proof.** Necessity: Follows from the fact that every open set is \( \nu g \)-open.

Sufficiency: Let F, A be disjoint subsets of X such that F is regular closed and A is closed, there exists disjoint \( \nu g \)-open sets in X such that F \( \subseteq U \), \( B \subseteq V \). Hence F \( \subseteq U^o \), \( B \subseteq V^o \), where \( U^o \) and \( V^o \) are disjoint open sets. Hence X is almost regular.

**Theorem 4.11.** The followings are equivalent:

(i) X is almost normal;

(ii) for every regular closed set A and for every \( \nu g \)-open set B containing A, \( \exists U \in \tau(X) \supseteq A \subseteq U \subseteq \nu(U) \subseteq B \);

(iii) for every \( \nu g \)-closed set A and for every regular-open set B containing A, \( \exists U \in \tau(X) \supseteq A \subseteq U \subseteq \nu(U) \subseteq B \);

(iv) for every pair of disjoint regularly closed set A and \( \nu g \)-closed set B, \( \exists U, V \in \tau(X) \supseteq U \cap V = \phi \).
§5. $\nu g$-$R_i$, spaces $i = 0, 1$

**Definition 5.1.** Let $x \in X$. Then
(i) $\nu g$-kernel of $x$ is defined and denoted by $\text{Ker}_{\nu g} \{x\} = \cap \{U : U \in \nu GO(X) \text{ and } x \in U\};$
(ii) $\text{Ker}_{\nu g} F = \cap \{U : U \in \nu GO(X) \text{ and } F \subseteq U\}.$

**Lemma 5.1.** Let $A \subseteq X$, then $\text{Ker}_{\nu g} \{A\} = \{x \in X : \nu g \{x\} \cap A \neq \phi\}.$

**Lemma 5.2.** Let $x \in X$. Then $y \in \text{Ker}_{\nu g} \{x\}$ iff $x \in \nu g \{y\}.$

**Proof.** Suppose that $y \notin \text{Ker}_{\nu g} \{x\}$. Then $\exists V \in \nu GO(X)$ containing $x \ni y \notin V$. Therefore we have $x \notin \nu g \{y\}$. The proof of converse part can be done similarly.

**Lemma 5.3.** For any points $x \neq y \in X$, the followings are equivalent:
(1) $\text{Ker}_{\nu g} \{x\} \neq \text{Ker}_{\nu g} \{y\};$
(2) $\nu g \{x\} \neq \nu g \{y\}.$

**Proof.** (1) $\Rightarrow$ (2). Let $\text{Ker}_{\nu g} \{x\} \neq \text{Ker}_{\nu g} \{y\}$, then $\exists z \in X \ni z \in \text{Ker}_{\nu g} \{x\}$ and $z \notin \text{Ker}_{\nu g} \{y\}.$ From $z \in \text{Ker}_{\nu g} \{x\}$ it follows that $x \cap \nu g \{z\} \neq \phi \Rightarrow x \in \nu g \{z\}.$ By $z \notin \text{Ker}_{\nu g} \{y\}$, we have $\{y\} \cap \nu g \{z\} = \phi$. Since $x \in \nu g \{z\}, \nu g \{x\} \subseteq \nu g \{z\}$ and $\{y\} \cap \nu g \{x\} = \phi$. Therefore $\nu g \{x\} \neq \nu g \{y\}$. Now $\text{Ker}_{\nu g} \{x\} \neq \text{Ker}_{\nu g} \{y\} \Rightarrow \nu g \{x\} \neq \nu g \{y\}$.

(2) $\Rightarrow$ (1). If $\nu g \{x\} \neq \nu g \{y\}$. Then $\exists z \in X \ni z \in \nu g \{x\}$ and $z \notin \nu g \{y\}$. Then $\exists$ a $\nu g$-open set containing $z$ and therefore containing $x$ but not $y$, namely, $y \notin \text{Ker}_{\nu g} \{x\}$. Hence $\text{Ker}_{\nu g} \{x\} \neq \text{Ker}_{\nu g} \{y\}.$

**Definition 5.2.** $X$ is said to be
(i) $\nu g$-$R_0$ iff $\nu g \{x\} \subseteq G$ whenever $x \in G \in \nu GO(X),$
(ii) weakly $\nu g$-$R_0$ iff $\cap \nu g \{x\} = \phi,$
(iii) $\nu g$-$R_1$ iff for $x$, $y \in X \ni \nu g \{x\} \neq \nu g \{y\}$ $\exists$ disjoint $U, V \in \nu GO(X) \ni \nu g \{x\} \subseteq U$ and $\nu g \{y\} \subseteq V.$

**Example 5.1.**
(i) Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$, then $X$ is weakly $\nu g$-$R_0$ and $\nu g$-$R_i$, $i = 0, 1;$
(ii) For $\tau = \{\phi, \{a\}, \{a, b, c\}, X\}$, then $X$ is not weakly $\nu g$-$R_0$ and $\nu g$-$R_i$, $i = 0, 1.$

**Remark 5.1.**
(i) $r R_i \Rightarrow g R_i \Rightarrow g R_i \Rightarrow \nu g R_i$, $i = 0, 1;$
(ii) Every weakly-$R_0$ space is weakly $\nu g$-$R_0$.

**Lemma 5.1.** Every $\nu g$-$R_0$ space is weakly $\nu g$-$R_0$.
Converse of the above lemma is not true in general by the following examples.

**Example 5.2.** Let $X = \{a, b, c\}$ and $\tau = \{\phi, \{a\}, \{a, b\}, X\}$. Clearly, $X$ is weakly $\nu g$-$R_0$, since $\cap \nu g \{x\} = \phi$. But it is not $\nu g$-$R_0$, for $\{a\} \subseteq X$ is $\nu g$-open and $\nu g \{a\} = \{a, c\} \subseteq \{a\}.$

**Theorem 5.1.** Every $\nu g$-regular space $X$ is $\nu-T_2$ and $\nu R_0$.

**Proof.** Let $X$ be $\nu g$-regular and let $x \neq y \in X$. By lemma 4.1, $\{x\}$ is either $\nu$-open or $\nu$-closed. If $\{x\}$ is $\nu$-open, $\{x\}$ is $\nu g$-open and hence $\nu$-clopen. Thus $\{x\}$ and $X - \{x\}$ are separating $\nu$-open sets. Similar argument, for $\{x\}$ is $\nu$-closed gives $\{x\}$ and $X - \{x\}$ are separating $\nu$-closed sets. Thus $X$ is $\nu$-$T_2$ and $\nu R_0$.

**Theorem 5.2.** $X$ is $\nu g$-$R_0$ iff given $x \neq y \in X; \nu g \{x\} \neq \nu g \{y\}.$

**Proof.** Let $X$ be $\nu g$-$R_0$ and let $x \neq y \in X$. Suppose $U$ is a $\nu g$-open set containing $x$ but not $y$, then $y \in \nu g \{y\} \subset X - U$ and so $x \notin \nu g \{y\}$. Hence $\nu g \{x\} \neq \nu g \{y\}.$
Conversely, let \( x \neq y \in X \ni \nu g(x) \neq \nu g(y) \Rightarrow \nu g(x) \subseteq X - \nu g(y) = U \text{(say)} \) a \( \nu g \)-open set in \( X \). This is true for every \( \nu g(x) \). Thus \( \nu g(x) \subseteq U \) where \( x \in \nu g(x) \subseteq U \in \nu gO(\tau) \), which in turn implies \( \nu g(x) \subseteq U \) where \( x \in U \in \nu gO(X) \). Hence \( X \) is \( \nu g-R_0 \).

**Theorem 5.3.** \( X \) is weakly \( \nu g-R_0 \) iff \( Ker_{\nu g} \{ x \} \neq X \) for any \( x \in X \).

**Proof.** Let \( x_0 \in X \ni \ker_{\nu g} \{ x_0 \} = X \). This means that \( x_0 \) is not contained in any proper \( \nu g \)-open subset of \( X \). Thus \( x_0 \) belongs to the \( \nu g \)-closure of every singleton set. Hence \( x_0 \in \cap \nu g(x) \), a contradiction.

Conversely, assume \( Ker_{\nu g} \{ x \} \neq X \) for any \( x \in X \). If there is an \( x_0 \in X \ni x_0 \in \cap \nu g(x) \), then every \( \nu g \)-open set containing \( x_0 \) must contain every point of \( X \). Therefore, the unique \( \nu g \)-open set containing \( x_0 \) is \( X \). Hence \( Ker_{\nu g} \{ x_0 \} = X \), which is a contradiction. Thus \( X \) is weakly \( \nu g-R_0 \).

**Theorem 5.4.** The following statements are equivalent:

(i) \( X \) is \( \nu g-R_0 \) space;

(ii) for each \( x \in X, \nu g(x) \subseteq Ker_{\nu g} \{ x \} \);

(iii) for any \( \nu g \)-closed set \( F \) and a point \( x \notin F \), \( \exists U \in \nu gO(X) \ni x \notin U \) and \( F \subseteq U \);

(iv) each \( \nu g \)-closed set \( F \) can be expressed as \( F = \cap \{ G : \nu g \text{-open and } F \subseteq G \} \);

(v) each \( \nu g \)-open set \( G \) can be expressed as \( G = \cup \{ A : A \text{ is } \nu g \text{-closed and } A \subseteq G \} \);

(vi) for each \( \nu g \)-closed set \( F, x \notin F \) implies \( \nu g(x) \cap F = \emptyset \).

**Proof.** (i) \( \Rightarrow \) (ii). For any \( x \in X \), we have \( Ker_{\nu g} \{ x \} = \cap \{ U : U \in \nu gO(X) \text{ and } x \in U \} \).

Since \( X \) is \( \nu g-R_0 \), each \( \nu g \)-open set containing \( x \) contains \( \nu g(x) \). Hence \( \nu g(x) \subseteq Ker_{\nu g} \{ x \} \).

(ii) \( \Rightarrow \) (iii). Let \( x \notin F \in \nu gC(X) \). Then for any \( y \in F, \nu g(y) \subseteq F \) and so \( x \notin \nu g(y) \Rightarrow y \notin \nu g(x) \) that is \( \exists U_y \in \nu gO(X) \ni y \in U_y \) and \( x \notin U_y \). Hence \( U = \cup \{ U_y : U_y \text{ is } \nu g \text{-open } y \in U_y \text{ and } x \notin U_y \} \). Then \( U \in \nu gO(X) \ni x \notin U \) and \( F \subseteq U \).

(iii) \( \Rightarrow \) (iv). Let \( F \in \nu gC(X) \) and \( N = \cap \{ G : G \in \nu gO(X) \text{ and } F \subseteq G \} \). Then \( F \subseteq N \Rightarrow (1) \). Let \( x \notin F \), then by (iii) \( \exists G \in \nu gO(X) \ni x \notin G \) and \( F \subseteq G \), hence \( x \notin N \) which implies \( x \in N \Rightarrow x \in F \). Hence \( N \subseteq F \Rightarrow (2) \).

Therefore from (1) and (2), each \( \nu g \)-closed set \( F = \cap \{ G : G \text{ is } \nu g \text{-open and } F \subseteq G \} \)

(iv) \( \Rightarrow \) (v). obvious.

(v) \( \Rightarrow \) (vi). Let \( x \notin F \in \nu gC(X) \). Then \( X - F = G \) is a \( \nu g \)-open set containing \( x \). Then by (v), \( G \) can be expressed as the union of \( \nu g \)-closed sets \( A \) contained in \( G \), and so there is an \( M \in \nu gC(X) \ni x \in M \subseteq G \); and hence \( \nu g(x) \subseteq G \) which implies \( \nu g(x) \cap F = \emptyset \).

(vi) \( \Rightarrow \) (i). Let \( x \in G \in \nu gO(X) \). Then \( x \notin (X - G) \), which is a \( \nu g \)-closed set. Therefore by (vi) \( \nu g(x) \cap (X - G) = \emptyset \), which implies that \( \nu g(x) \subseteq G \). Thus \( X \) is \( \nu g-R_0 \) space.

**Theorem 5.5.** If \( f \) is \( \nu g \)-closed one-one function and if \( X \) is weakly \( \nu g-R_0 \), so is \( Y \).

**Theorem 5.6.** If \( X \) is weakly \( \nu g-R_0 \), then for every space \( Y, X \times Y \) is weakly \( \nu g-R_0 \).

**Proof.** \( \cap \nu g((x,y)) \subseteq \cap \{ \nu g(x) \times \nu g(y) \} = \cap \{ \nu g(x) \times \nu g(y) \} \subseteq \phi \times Y = \emptyset \). Hence \( X \times Y \) is \( \nu g-R_0 \).

**Corollary 5.1.**

(i) If \( X \) and \( Y \) are weakly \( \nu g-R_0 \), then \( X \times Y \) is weakly \( \nu g-R_0 \);

(ii) if \( X \) and \( Y \) are \( \text{(weakly-)}R_0 \), then \( X \times Y \) is weakly \( \nu g-R_0 \);

(iii) if \( X \) and \( Y \) are \( \nu g-R_0 \), then \( X \times Y \) is weakly \( \nu g-R_0 \);

(iv) if \( X \) is \( \nu g-R_0 \) and \( Y \) are weakly \( R_0 \), then \( X \times Y \) is weakly \( \nu g-R_0 \).
Theorem 5.7. $X$ is $νg$-$R_0$ iff for any $x, y ∈ X, νg[x] ≠ νg[y] ⇒ νg[x] ∩ νg[y] = φ$.

**Proof.** Let $X$ is $νg$-$R_0$ and $x, y ∈ X ⊇ νg[x] ≠ νg[y]$. Then $∃ z ∈ νg[x], z ∉ νg[y]$. There exists $V ∈ νGO(X) ⊇ y / V$ and $z ∈ V$, hence $x ∈ V$. Therefore, $x ∉ νg[y]$. Thus $x ∈ [G y]^c ∈ νGO(X)$, which implies $νg[x] ⊂ [νg[y]]^c$ and $νg[x] ∩ νg[y] = φ$. The proof for otherwise is similar.


Theorem 5.8. $X$ is $νg$-$R_0$ iff for any points $x, y ∈ X, Ker_{νg}[x] ≠ Ker_{νg}[y] ⇒ Ker_{νg}[x] ∩ Ker_{νg}[y] = φ$.

**Proof.** Suppose $X$ is $νg$-$R_0$. Thus by lemma 5.3, for any $x, y ∈ X$ if $Ker_{νg}[x] ≠ Ker_{νg}[y]$ then $νg[x] ≠ νg[y]$. Assume that $z ∈ Ker_{νg}[x] ∩ Ker_{νg}[y]$. By $z ∈ Ker_{νg}[x]$ and lemma 5.2, it follows that $x ∈ νg[z]$. Since $x ∈ νg[z], νg[x] = νg[z]$. Similarly, we have $νg[y] = νg[z] = νg[x]$. This is a contradiction. Therefore, we have $Ker_{νg}[x] ∩ Ker_{νg}[y] = φ$.

Conversely, let $x, y ∈ X, νg[x] ≠ νg[y]$, then by lemma 5.3, $Ker_{νg}[x] ≠ Ker_{νg}[y]$. Hence by hypothesis $Ker_{νg}[x] ∩ Ker_{νg}[y] = φ$ which implies $νg[x] ∩ νg[y] = φ$. Because $z ∈ νg[x]$ implies that $x ∈ Ker_{νg}[z]$ and $Ker_{νg}[x] ∩ Ker_{νg}[z] ≠ φ$. Therefore by theorem 5.7, $X$ is a $νg$-$R_0$ space.

Theorem 5.9. The following properties are equivalent:

1. $X$ is a $νg$-$R_0$ space;
2. For any $A ≠ φ$ and $G ∈ νGO(X, τ) ⊃ A ∩ G ≠ φ$, $∃ F ∈ νGC(X, τ) ⊃ A ∩ F ≠ φ$ and $F ⊂ G$.

**Proof.**

(1) ⇒ (2). Let $A ≠ φ$ and $G ∈ νGO(X, τ) ⊃ A ∩ G ≠ φ$. There exists $x ∈ A ∩ G$. Since $x ∈ G ∈ νGO(X), νg[x] ⊂ G$. Set $F = νg[x]$, then $F ∈ νGC(X)$. $F ⊂ G$ and $A ∩ F ≠ φ$;

(2) ⇒ (1). Let $G ∈ νGO(X)$ and $x ∈ G$. By (2) $νg[x] ⊂ G$. Hence $X$ is $νg$-$R_0$.

Theorem 5.10. The following properties are equivalent:

1. $X$ is a $νg$-$R_0$ space;
2. $x ∈ νg[y]$, for any points $x$ and $y$ in $X$.

**Proof.** (1) ⇒ (2). Assume $X$ is $νg$-$R_0$. Let $x ∈ νg[y]$ and $D$ be any $νg$-open set such that $y ∈ D$. Now by hypothesis, $x ∈ D$. Therefore, every $νg$-open set which contain $x$ contains $y$. Hence $y ∈ νg[x]$.

(2) ⇒ (1). Let $x ∈ U ∈ νGO(X)$. If $y / U$, then $x / νg[y]$ and hence $y / νg[x]$. This implies that $νg[x] ⊂ U$. Hence $X$ is $νg$-$R_0$.

Theorem 5.11. The following properties are equivalent:

1. $X$ is a $νg$-$R_0$ space;
2. If $F$ is $νg$-closed, then $F = Ker_{νg}(F)$;
3. If $F$ is $νg$-closed and $x ∈ F$, then $Ker_{νg}[x] ⊂ F$;
4. If $x ∈ X$, then $Ker_{νg}[x] ⊂ νg[x]$.

**Proof.** (1) ⇒ (2). Let $x / F ∈ νGC(X) ⇒ (X − F) ∈ νGO(X, x)$. For $X$ is $νg$-$R_0$, $νg(x) ⊂ (X − F)$. Thus $νg(x) ∩ F = φ$ and $x / Ker_{νg}(F)$. Hence $Ker_{νg}(F) = F$. 

(2) $\Rightarrow$ (3). $A \subseteq B \Rightarrow \text{Ker} \nu_g(A) \subseteq \text{Ker} \nu_g(B)$. Therefore, by (2) $\text{Ker} \nu_g\{x\} \subseteq \text{Ker} \nu_g(F) = F$.

(3) $\Rightarrow$ (4). Since $x \in \nu_g\{x\}$ and $\nu_g\{x\}$ is $\nu_g$-closed, by (3) $\text{Ker} \nu_g\{x\} \subseteq \nu_g\{x\}$.

(4) $\Rightarrow$ (1). Let $x \in \nu_g\{y\}$. Then by lemma 5.2, $y \in \text{Ker} \nu_g\{x\}$. Since $x \in \nu_g\{x\}$ and $\nu_g\{x\}$ is $\nu_g$-closed, by (4) we obtain $y \in \text{Ker} \nu_g\{x\} \subseteq \nu_g\{x\}$. Therefore $x \in \nu_g\{y\}$ implies $y \in \nu_g\{x\}$.

The converse is obvious and $X$ is $\nu_g$-$R_0$.

**Corollary 5.2.** The following properties are equivalent:

(1) $X$ is $\nu_g$-$R_0$;

(2) $\nu_g\{x\} = \text{Ker} \nu_g\{x\} \forall x \in X$.

**Proof.** Straightforward from theorems 5.4 and 5.11.

Recall that a filter base $F$ is called $\nu_g$-convergent to a point $x$ in $X$, if for any $\nu_g$-open set $U$ of $X$ containing $x$, there exists $B \in F$ such that $B \subseteq U$.

**Lemma 5.4.** Let $x$ and $y$ be any two points in $X$ such that every net in $X$ $\nu_g$-converging to $y$ $\nu_g$-converges to $x$. Then $x \in \nu_g\{y\}$.

**Proof.** Suppose that $x_n = y$ for each $n \in N$. Then $\{x_n\}_{n \in N}$ is a net in $\nu_g\{y\}$. Since $\{x_n\}_{n \in N} \nu_g$-converges to $y$, then $\{x_n\}_{n \in N} \nu_g$-converges to $x$ and this implies that $x \in \nu_g\{y\}$.

**Theorem 5.12.** The following statements are equivalent:

(1) $X$ is a $\nu_g$-$R_0$ space;

(2) If $x, y \in X$, then $y \in \nu_g\{x\}$ iff every net in $X$ $\nu_g$-converging to $y$ $\nu_g$-converges to $x$.

**Proof.**

(1) $\Rightarrow$ (2). Let $x, y \in X \ni y \in \nu_g\{x\}$. Suppose that $\{x_n\}_{n \in A}$ is a net in $X \ni \{x_n\}_{n \in A} \nu_g$-converges to $y$. Since $y \in \nu_g\{x\}$, by theorem 5.7 we have $\nu_g\{x\} = \nu_g\{y\}$. Therefore $x \in \nu_g\{y\}$. This means that $\{x_n\}_{n \in A} \nu_g$-converges to $x$.

Conversely, let $x, y \in X$ such that every net in $X$ $\nu_g$-converging to $y$ $\nu_g$-converges to $x$. Then $x \in \nu_g\{y\}$ by theorem 5.4. By theorem 5.7, we have $\nu_g\{x\} = \nu_g\{y\}$. Therefore $y \in \nu_g\{x\}$.

(2) $\Rightarrow$ (1). Let $x, y \in X \ni \nu_g\{x\} \cap \nu_g\{y\} \neq \emptyset$. Let $z \in \nu_g\{x\} \cap \nu_g\{y\}$. So $\exists$ a net $\{x_n\}_{n \in A}$ in $\nu_g\{x\} \ni \{x_n\}_{n \in A} \nu_g$-converges to $z$. Since $z \in \nu_g\{y\}$, then $\{x_n\}_{n \in A} \nu_g$-converges to $y$. It follows that $y \in \nu_g\{x\}$. Similarly we obtain $x \in \nu_g\{y\}$. Therefore $\nu_g\{x\} = \nu_g\{y\}$. Hence, $X$ is $\nu_g$-$R_0$.

**Theorem 5.13.**

(i) Every subspace of $\nu_g$-$R_1$ space is again $\nu_g$-$R_1$;

(ii) Product of any two $\nu_g$-$R_1$ spaces is again $\nu_g$-$R_1$.

**Theorem 5.14.** $X$ is $\nu_g$-$R_1$ iff given $x \neq y \in X$, $\nu_g\{x\} \neq \nu_g\{y\}$.

**Theorem 5.15.** Every $\nu_g$ space is $\nu_g$-$R_1$.

The converse is not true. However, we have the following result.

**Theorem 5.16.** Every $\nu_g_1$, $\nu_g$-$R_1$ space is $\nu_g$.

**Proof.** Let $x \neq y \in X$. Since $X$ is $\nu_g_1$, $\{x\}$ and $\{y\}$ are $\nu_g$-closed sets $\ni \nu_g\{x\} \neq \nu_g\{y\}$.

Since $X$ is $\nu_g$-$R_1$, $\exists U, V \ni \nu G(X), \ni x \in U, y \in V$. Hence $X$ is $\nu_g$.

**Corollary 6.1.** $X$ is $\nu_g$ iff it is $\nu_g$-$R_1$ and $\nu_g_1$.

**Theorem 5.17.** The followings are equivalent

(i) $X$ is $\nu_g$-$R_1$;

(ii) $X$ is $\nu_g$-$R_0$;

(iii) $X$ is $\nu_g$-$R_1$ and $\nu_g$-$R_0$.

The converse is obvious and $X$ is $\nu_g$-$R_0$. The proof follows straightforward from theorems 5.4 and 5.11.
(ii) \( \cap \nu g\{x\} = \{x\} \);

(iii) For every \( x \in X \), intersection of all \( \nu g \)-neighborhoods of \( x \) is \( \{x\} \).

**Proof.** (i) \( \Rightarrow \) (ii). Let \( y \neq x \in X \ni y \in \nu g\{x\} \). Since \( X \) is \( \nu g-R_1 \), \( \exists U \in \nu gO(X) \ni y \in U \), \( x \notin U \) or \( x \in U \), \( y \notin U \). In either case \( y \notin \nu g\{x\} \). Hence \( \cap \nu g\{x\} = \{x\} \).

(ii) \( \Rightarrow \) (iii). If \( y \neq x \in X \), then \( x \notin \cap \nu g\{y\} \), so there is a \( \nu g \)-open set containing \( x \) but not \( y \). Therefore \( y \) does not belong to the intersection of all \( \nu g \)-neighborhoods of \( x \). Hence intersection of all \( \nu g \)-neighborhoods of \( x \) is \( \{x\} \).

(iii) \( \Rightarrow \) (i). Let \( x \neq y \in X \), by hypothesis, \( y \) does not belong to the intersection of all \( \nu g \)-neighborhoods of \( x \) and \( x \) does not belong to the intersection of all \( \nu g \)-neighborhoods of \( y \), which implies \( \nu g\{x\} \neq \nu g\{y\} \). Hence \( X \) is \( \nu g-R_1 \).

**Theorem 5.18.** The followings are equivalent:

(i) \( X \) is \( \nu g-R_1 \);

(ii) For each pair \( x, y \in X \ni \nu g\{x\} \neq \nu g\{y\} \), \( \exists \) a \( \nu g \)-clopen set \( V \ni x \in V \) and \( y \notin V \);

(iii) For each pair \( x, y \in X \ni \nu g\{x\} \neq \nu g\{y\} \), \( \exists f : X \to [0,1] \ni f(x) = 0 \) and \( f(C) = 1 \) and \( f \) is \( \nu g \)-continuous.

**Proof.**

(i) \( \Rightarrow \) (ii). Let \( x, y \in X \ni \nu g\{x\} \neq \nu g\{y\} \), \( \exists \) disjoint \( U, W \in \nu gO(X) \ni \nu g\{x\} \subset U \) and \( \nu g\{y\} \subset W \) and \( V = \nu g\{U\} \) is \( \nu g \)-open and \( \nu g \)-closed such that \( x \in V \) and \( y \notin V \).

(ii) \( \Rightarrow \) (iii). Let \( x, y \in X \) such that \( \nu g\{x\} \neq \nu g\{y\} \), and let \( V \) be \( \nu g \)-open and \( \nu g \)-closed such that \( x \in V \) and \( y \notin V \). Then \( f : X \to [0,1] \) defined by \( f(z) = 0 \) if \( z \in V \) and \( f(z) = 1 \) if \( z \notin V \) satisfied the desired properties.

(iii) \( \Rightarrow \) (i). Let \( x, y \in X \ni \nu g\{x\} \neq \nu g\{y\} \), let \( f : X \to [0,1] \) be \( \nu g \)-continuous, \( f(x) = 0 \) and \( f(y) = 1 \). Then \( U = f^{-1}([0,1]) \) and \( V = f^{-1}((1,\infty)) \) are disjoint \( \nu g \)-open and \( \nu g \)-closed sets in \( X \), such that \( \nu g\{x\} \subset U \) and \( \nu g\{y\} \subset V \).

**Theorem 5.19.** If \( X \) is \( \nu g-R_1 \), then \( X \) is \( \nu g-R_0 \).

**Proof.** Let \( x \in U \in \nu gO(X) \). If \( y \notin U \), then \( \nu g\{x\} \neq \nu g\{y\} \). Hence, \( \exists \) a \( \nu g \)-open \( V_y \ni \nu g\{y\} \subset V_y \) and \( x \notin V_y \), \( \Rightarrow y \notin \nu g\{x\} \). Thus \( \nu g\{x\} \subset U \). Therefore \( X \) is \( \nu g-R_0 \).

**Theorem 5.20.** \( X \) is \( \nu g-R_i \) if \( x, y \in X \), \( Ker_{\nu g}\{x\} \neq Ker_{\nu g}\{y\} \), \( \exists \) disjoint \( U, V \in \nu gO(X) \ni \nu g\{x\} \subset U \) and \( \nu g\{y\} \subset V \).

§6. \( \nu g-C_i \) and \( \nu g-D_i \) spaces, \( i = 0,1,2 \)

**Definition 6.1.** \( X \) is said to be a

(i) \( \nu g-C_0 \) space if for each pair of distinct points \( x, y \) of \( X \) there exists a \( \nu g \)-open set \( G \) whose closure contains either \( x \) or \( y \).

(ii) \( \nu g-C_1 \) space if for each pair of distinct points \( x, y \) of \( X \) there exists a \( \nu g \)-open set \( G \) whose closure containing \( x \) but not \( y \) and a \( \nu g \)-open set \( H \) whose closure containing \( y \) but not \( x \).

(iii) \( \nu g-C_2 \) space if for each pair of distinct points \( x, y \) of \( X \) there exists disjoint \( \nu g \)-open sets \( G \) and \( H \) such that \( G \) containing \( x \) but not \( y \) and \( H \) containing \( y \) but not \( x \).

**Note 6.1.** \( \nu g-C_2 \Rightarrow \nu g-C_1 \Rightarrow \nu g-C_0 \) but converse need not be true in general.
Example 6.1. Let \( X = \{a,b,c,d\} \) and \( \tau = \{\phi, \{a\}, \{a,b\}, \{c,d\}, \{a,c,d\}, X\} \), then \( X \) is \( \nu_g\)-\( C_i \), \( i = 0, 1, 2 \).

Theorem 6.1.

(i) Every subspace of \( \nu_g\)-\( C_i \) space is \( \nu_g\)-\( C_i \).

(ii) Every \( \nu_g \) spaces is \( \nu_g\)-\( C_i \).

(iii) Product of \( \nu_g\)-\( C_i \) spaces is \( \nu_g\)-\( C_i \).

Theorem 6.2. Let \((X, \tau)\) be any \( \nu_g\)-\( C_i \) space and \( A \) be an any non empty subset of \( X \) then \( A \) is \( \nu_g\)-\( C_i \) iff \((A, \tau_A)\) is \( \nu_g\)-\( C_i \).

Theorem 6.3.

(i) If \( X \) is \( \nu_g\)-\( C_i \) then each singleton set is \( \nu_g\)-closed.

(ii) In an \( \nu_g\)-\( C_i \) space disjoint points of \( X \) have disjoint \( \nu_g\)-closures.

Definition 6.2. \( A \subset X \) is called a \( \nu_g \) Difference(Shortly \( \nu_g\)-\( D \)-set) if there are two \( U, V \in \nu GO(X, \tau) \) such that \( U \neq X \) and \( A = U - V \).

Clearly every \( \nu_g\)-open set \( U \) different from \( X \) is a \( \nu_g\)-\( D \)-set if \( A = U \) and \( V = \phi \).

Definition 6.3. \( X \) is said to be a

(i) \( \nu_g\)-\( D \)-\( 0 \) if for any pair of distinct points \( x \) and \( y \) of \( X \) there exist a \( \nu_g\)-\( D \)-set in \( X \) containing \( x \) but not \( y \) or a \( \nu_g\)-\( D \)-set in \( X \) containing \( y \) but not \( x \);

(ii) \( \nu_g\)-\( D \)-\( 1 \) if for any pair of distinct points \( x \) and \( y \) in \( X \) there exist a \( \nu_g\)-\( D \)-set of \( X \) containing \( x \) but not \( y \) and a \( \nu_g\)-\( D \)-set in \( X \) containing \( y \) but not \( x \);

(iii) \( \nu_g\)-\( D \)-\( 2 \) if for any pair of distinct points \( x \) and \( y \) of \( X \) there exists disjoint \( \nu_g\)-\( D \)-sets \( G \) and \( H \) in \( X \) containing \( x \) and \( y \), respectively.

Example 6.2. Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, \{a\}, \{b, c\}, X\} \) then \( X \) is \( \nu_g\)-\( D_i \), \( i = 0, 1, 2 \).

Remark 6.1. (i) If \( X \) is \( T_1 \), then it is \( \nu_g \), \( i = 0, 1, 2 \) and converse is false;

(ii) If \( X \) is \( \nu_g \), then it is \( \nu g_{i-1} \), \( i = 1, 2 \);

(iii) If \( X \) is \( \nu g \), then it is \( \nu g_{D_i} \), \( i = 0, 1, 2 \);

(iv) If \( X \) is \( \nu g_{D_i} \), then it is \( \nu g_{D_{i-1}} \), \( i = 1, 2 \).

Theorem 6.4. The following statements are true:

(i) \( X \) is \( \nu g\)-\( D \)-\( 0 \) iff it is \( \nu g \);

(ii) \( X \) is \( \nu g\)-\( D \)-\( 1 \) iff it is \( \nu g\)-\( D \)-\( 2 \).

Proof. (i) The sufficiency is stated in remark 6.1(iii). To prove necessity, let \( X \) be \( \nu g\)-\( D \)-\( 0 \).

Then for each \( x \neq y \in X \), at least one of them, say \( x \), belong to a \( \nu g\)-\( D \)-set \( G \) but \( y \notin G \). Let \( G = U_1 - U_2 \) where \( U_1 \neq X \) and \( U_1, U_2 \in \nu GO(X) \). Then \( x \in U_1 \) and for \( y \notin G \) we have two cases:

(a) \( y \notin U_1 \);

(b) \( y \in U_1 \) and \( y \notin U_2 \).

In case (a), \( x \in U_1 \) but \( y \notin U_1 \); In case (b), \( y \in U_2 \) but \( x \notin U_2 \). Hence \( X \) is \( \nu g \).

(ii) Sufficiency. Remark 6.1(iv). Necessity. Suppose \( X \) is \( \nu g\)-\( D \)-\( 1 \). Then for each \( x \neq y \in X \), we have \( \nu g\)-\( D \)-sets \( G_1, G_2 \ni x \in G_1; y \notin G_1 \); \( y \in G_2, x \notin G_2 \). Let \( G_1 = U_1 - U_2, G_2 = U_3 - U_4 \).

From \( x \notin G_2 \), it follows that either \( x \notin U_3 \) or \( x \in U_3 \) and \( x \in U_4 \). We discuss the two cases separately.

(1) \( x \notin U_3 \). By \( y \notin G_1 \) we have two subcases:
(a) \( y \notin U_1 \). From \( x \in U_1 - U_2 \), it follows that \( x \in U_1 - (U_2 \cup U_3) \) and by \( y \in U_3 - U_4 \) we have \( y \in U_3 - (U_1 \cup U_4) \). Therefore \((U_1 - (U_2 \cup U_3)) \cap (U_3 - (U_1 \cup U_4)) = \emptyset\).

(b) \( y \in U_1 \) and \( y \in U_2 \). We have \( x \in U_1 - U_2, \ y \in U_2, \ (U_1 - U_2) \cap U_2 = \emptyset \).

(2) \( x \in U_3 \) and \( x \in U_4 \). We have \( y \in U_3 - U_4, \ x \in U_4, \ (U_3 - U_4) \cap U_4 = \emptyset \).

Therefore \( X \) is \( \nu g\)-\( D_2 \).

**Corollary 6.1.** If \( X \) is \( \nu g\)-\( D_1 \), then it is \( \nu g_0 \).

**Proof.** Remark 6.1(iv) and theorem 6.2(i).

**Definition 6.4.** A point \( x \in X \) which has \( X \) as the unique \( \nu g \)-neighborhood is called \( \nu g c.c \) point.

**Theorem 6.5.** For a \( \nu g_0 \) space \( X \) the followings are equivalent:

1. \( X \) is \( \nu g\)-\( D_1 \);
2. \( X \) has no \( \nu g c.c \) point.

**Proof.** (1) \( \Rightarrow \) (2). Since \( X \) is \( \nu g\)-\( D_1 \), each point \( x \) of \( X \) is contained in a \( \nu g\)-\( D \)-set \( O = U - V \) and thus in \( U \). By definition \( U \neq X \). This implies that \( x \) is not a \( \nu g c.c \) point.

(2) \( \Rightarrow \) (1). If \( X \) is \( \nu g_0 \), then for each \( x \neq y \in X \), at least one of them, \( x \) (say) has a \( \nu g \)-neighborhood \( U \) containing \( x \) and not \( y \). Thus \( U \) which is different from \( X \) is a \( \nu g\)-\( D \)-set. If \( X \) has no \( \nu g c.c \) point, then \( y \) is not a \( \nu g c.c \) point. This means that there exists a \( \nu g \)-neighborhood \( V \) of \( y \) such that \( V \neq X \). Thus \( y \in (V - U) \) but not \( x \) and \( V - U \) is a \( \nu g\)-\( D \)-set. Hence \( X \) is \( \nu g\)-\( D_1 \).

**Corollary 6.2.** A \( \nu g_0 \) space \( X \) is \( \nu g\)-\( D_1 \) iff there is a unique \( \nu g c.c \) point in \( X \).

**Proof.** Only uniqueness is sufficient to prove. If \( x_0 \) and \( y_0 \) are two \( \nu g c.c \) points in \( X \) then since \( X \) is \( \nu g_0 \), at least one of \( x_0 \) and \( y_0 \) say \( x_0 \), has a \( \nu g \)-neighbourhood \( U \) such that \( x_0 \in U \) and \( y_0 \notin U \), hence \( U \neq X \), \( x_0 \) is not a \( \nu g c.c \) point, a contradiction.

**Remark 6.2.** It is clear that an \( \nu g_0 \) space \( X \) is not \( \nu g\)-\( D_1 \) iff there is a unique \( \nu g c.c \) point in \( X \). It is unique because if \( x \) and \( y \) are both \( \nu g c.c \) point in \( X \), then at least one of them say \( x \) has a \( \nu g \)-neighborhood \( U \) containing \( x \) but not \( y \). But this is a contradiction since \( U \neq X \).

**Definition 6.5.** \( X \) is \( \nu g \)-symmetric if for \( x \) and \( y \) in \( X \), \( x \in \nu g \{ y \} \) implies \( y \in \nu g \{ x \} \).

**Theorem 6.6.** \( X \) is \( \nu g \)-symmetric if \( \{ x \} \) is \( \nu g \)-closed for each \( x \in X \).

**Proof.** Assume that \( x \in \nu g \{ y \} \) but \( y \notin \nu g \{ x \} \). This means that \( \nu g \{ x \}^c \) contains \( y \). This implies that \( \nu g \{ y \} \subset \nu g \{ x \}^c \). Now \( \nu g \{ x \}^c \) contains \( x \) which is a contradiction.

Conversely, suppose that \( \{ x \} \subset E \in \nu G O (X) \) but \( \nu g \{ x \} \) is not a subset of \( E \). This means that \( \nu g \{ x \} \) and \( E^c \) are not disjoint. Let \( y \) belong to their intersection. Now we have \( x \in \nu g \{ y \} \) which is a subset of \( E^c \) and \( x \notin E \). But this is a contradiction.

**Corollary 6.3.** If \( X \) is a \( \nu g_1 \), then it is \( \nu g \)-symmetric.

**Proof.** Follows from theorem 3.3 and theorem 6.6.

**Corollary 6.4.** The following are equivalent:

1. \( X \) is \( \nu g \)-symmetric and \( \nu g_0 \);
2. \( X \) is \( \nu g_1 \).

**Proof.** By corollary 6.3 and remark 6.1 it suffices to prove only (1) \( \Rightarrow \) (2). Let \( x \neq y \) and by \( \nu g_0 \), we may assume that \( x \in G_1 \subset \{ y \}^c \) for some \( G_1 \in \nu G O (X) \). Then \( x \notin \nu g \{ y \} \) and hence \( y \notin \nu g \{ x \} \). There exists a \( G_2 \in \nu G O (X) \) such that \( y \in G_2 \subset \{ x \}^c \) and \( X \) is a \( \nu g_1 \) space.

**Theorem 6.7.** For an \( \nu g \)-symmetric space \( X \) the following are equivalent:
(1) $X$ is $νg_0$;
(2) $X$ is $νg-D_1$;
(3) $X$ is $νg_1$.

Proof. (1) $⇒$ (3) corollary 6.4 and (3) $⇒$ (2) $⇒$ (1) remark 6.1.

Theorem 6.8. If $f$ is a $νg$-irresolutive surjective function and $E$ is a $νg-D$-set in $Y$, then the inverse image of $E$ is a $νg-D$-set in $X$.

Proof. Let $E$ be a $νg-D$-set in $Y$. Then there are $νg$-open sets $U_1, U_2 ∈ νGO(Y) ⊇ E = U_1 − U_2$ and $U_1 ≠ Y$. By the $νg$-irresoluteness of $f, f^{−1}(U_1)$ and $f^{−1}(U_2)$ are $νg$-open in $X$. Since $U_1 ≠ Y$, we have $f^{−1}(U_1) ≠ X$. Hence $f^{−1}(E) = f^{−1}(U_1) − f^{−1}(U_2)$ is a $νg-D$-set.

Theorem 6.9. If $Y$ is $νg-D_1$ and $f$ is $νg$-irresolutive and bijective, then $X$ is $νg-D_1$.

Proof. Suppose that $Y$ is a $νg-D_1$ space. Let $x ≠ y ∈ X$. Since $f$ is injective and $Y$ is $νg-D_1$, there exist $νg$-sets $G_x$ and $G_y$ of $Y$ containing $f(x)$ and $f(y)$ respectively, such that $f(y) ∉ G_x$ and $f(x) ∉ G_y$. By theorem 6.8, $f^{−1}(G_x)$ and $f^{−1}(G_y)$ are $νg$-sets in $X$ containing $x$ and $y$ respectively. Hence $X$ is a $νg-D_1$ space.

Theorem 6.10. $X$ is $νg-D_1$ iff for each $x ≠ y ∈ X$, $∃ νg$-irresolutive surjective function $f$, where $Y$ is a $νg-D_1$ space $∃ f(x) ≠ f(y)$.

Proof. Necessity. For every $x ≠ y ∈ X$, it suffices to take the identity function on $X$.

Sufficiency. Let $x ≠ y ∈ X$. By hypothesis, $∃ νg$-irresolutive, surjection $f$ from $X$ onto a $νg-D_1$ space $Y$ $∃ f(x) ≠ f(y)$. Therefore, $∃$ disjoint $νg-D$-sets $G_x$ and $G_y$ in $Y$ $∃ f(x) ∈ G_x$ and $f(y) ∈ G_y$. Since $f$ is $νg$-irresolutive and surjective, by theorem 6.8, $f^{−1}(G_x)$ and $f^{−1}(G_y)$ are disjoint $νg$-sets in $X$ containing $x$ and $y$ respectively. Therefore $X$ is $νg-D_1$ space.

Corollary 6.5. Let $\{X_α/α ∈ I\}$ be any family of topological spaces. If $X_α$ is $νg-D_1$ for each $α ∈ I$, then the product $ΠX_α$ is $νg-D_1$.

Proof. Let $(x_α)$ and $(y_β)$ be any pair of distinct points in $ΠX_α$. Then there exists an index $β ∈ I$ $x_β ≠ y_β$. The natural projection $P_β : ΠX_α → X_β$ is almost continuous and almost open and $P_β((x_α)) = P_β((y_α))$. Since $X_β$ is $νg-D_1$, $ΠX_α$ is $νg-D_1$.

Conclusion. In this paper we defined new separation axioms using $νg$-open sets and studied their interrelations with other separation axioms.

Acknowledgement. Author is thankful to referees for critical comments for the development of the paper.

References


More on $\nu g$-separation axioms

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Abstract In this paper by using $\nu g$-open sets, I define almost $\nu g$-normality and mild $\nu g$-normality also we continue the study of further properties of $\nu g$-normality. We show that these three axioms are regular open hereditary. I also define the class of almost $\nu g$-irresolute mappings and show that $\nu g$-normality is invariant under almost $\nu g$-irresolute $M$-$\nu g$-open continuous surjection.

Keywords $\nu g$-open, semiopen, semipreopen, almost normal, mildly normal, $M$-$\nu g$-closed, $M$-$\nu g$-open, rc-continuous.

2000 Mathematics Subject Classification: 54D15, 54D10

§1. Introduction

In 1967, A. Wilansky has introduced the concept of US spaces. In 1968, C. E. Aull studied some separation axioms between $T_1$ and $T_2$ spaces, namely, $S_1$ and $S_2$. In 1982, S. P. Arya et al have introduced and studied the concept of semi-US spaces and also they made study of $s$-convergence, sequentially semi-closed sets, sequentially $s$-compact notions. G. B. Navlagi studied $P$-Normal Almost-$P$-Normal, Mildly-$P$-Normal and Pre-US spaces. Recently S. Balasubramanian and P. Aruna Swathi Vyjayanthi studied $\nu$-Normal Almost-$\nu$-Normal, Mildly-$\nu$-Normal and $\nu$-US spaces. $\nu g$-open sets and $\nu g$-continuous mappings were introduced in 2009 and 2011 by S. Balasubramanian. The purpose of this paper is to examine the normality axioms, $\nu g$-US, $\nu g$-$S_1$ and $\nu g$-$S_2$ spaces. $\nu g$-convergence, sequentially $\nu g$-compact, sequentially $\nu g$-continuous maps, and sequentially sub $\nu g$-continuous maps are also introduced and studied in this paper. All notions and symbols which are not defined in this paper may be found in the appropriate references. Throughout the paper $X$ and $Y$ denote topological spaces on which no separation axioms are assumed explicitly stated.

§2. Preliminaries

Definition 2.1. A $\subset X$ is called
(i) closed [resp: Semi closed; $\nu$-closed] if its complement if is open [resp: semi open; $\nu$-open];
(ii) $ro\alpha$-open [$\nu$-open] if $U \in \alpha O(X)$ [$RO(X)$] such that $U \subset A \subset \alpha (U)$ [$U \subset A \subset \alpha(U)$];
(iii) semi-$\theta$-open if it is the union of semi-regular sets and its complement is semi-$\theta$-closed;
whenever there exists a subsequence \( ν_{cl} \not\in x \) such that

\( \{0 < x < f(nk) : f(nk) \not\in A\} \subseteq A \);  

(v) \( g\)-closed \([rg\)-closed; \( g'\)-closed; \( \bar{g}\)-closed\)] if \( \bar{A} \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open \([gs\)-open; semi-open\] in \( X \);  

(vi) \( sg\)-closed \([gs\)-closed\)] if \( s(\bar{A}) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is semi-open \([open; regular-open\] in \( X \);  

(vii) \( pg\)-closed \([gp\)-closed; \( gpr\)-closed\)] if \( p(\bar{A}) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is pre-open \([open; regular-open\] in \( X \);  

(viii) \( og\)-closed \([go\)-closed; \( ro\)-closed; \( rao\)-closed; \( \alpha gs\)-closed; \( ga\)-closed\)] if \( \alpha(\bar{A}) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open \([\alpha\)-open; \( ra\)-open; \( r\)-open; semi-open; \( gs\)-open\] in \( X \);  

(ix) \( vg\)-closed if \( \nu(\bar{A}) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \nu\)-open in \( X \);  

(x) The family of all \( vg\)-open sets of \( X \) containing point \( x \) is denoted by \( \nu GO(X, x) \).

**Definition 2.2.** Let \( A \subseteq X \). Then a point \( x \) is said to be a

(i) limit point of \( A \) if each open set containing \( x \) contains some point \( y \) of \( A \) such that \( x \neq y \);  

(ii) \( T_0 \)-limit point \([20] \) of \( A \) if each open set containing \( x \) contains some point \( y \) of \( A \) such that \( cl\{x\} \neq cl\{y\} \), or equivalently, such that they are topologically distinct;  

(iii) \( \nu T_0 \)-limit point \([11] \) of \( A \) if each open set containing \( x \) contains some point \( y \) of \( A \) such that \( \nu cl\{x\} \neq \nu cl\{y\} \), or equivalently, such that they are topologically distinct.

**Definition 2.3.** \([10] \) A function \( f \) is said to be

(i) almost-\( \nu \)-irresolute if for each \( x \in X \) and each \( \nu \)-neighborhood \( V \) of \( f(x) \), \( \nu(f^{-1}(V)) \) is a \( \nu \)-neighborhood of \( x \);  

(ii) sequentially \( \nu \)-continuous at \( x \in X \) if \( f(x_n) \rightarrow^{\nu} f(x) \) whenever \( x_n \rightarrow^{\nu} X \). If \( f \) is sequentially \( \nu \)-continuous at all \( x \in X \), then \( f \) is said to be sequentially \( \nu \)-continuous;  

(iii) sequentially nearly \( \nu \)-continuous if for each point \( x \in X \) and each sequence \( x_n \rightarrow^{\nu} x \) in \( X \), there exists a subsequence \( x_{nk} \) of \( x_n \) such that \( f(x_{nk}) \rightarrow^{\nu} f(x) \).

(iv) sequentially sub-\( \nu \)-continuous if for each \( x \in X \) and each sequence \( x_n \rightarrow^{\nu} x \) in \( X \), there exists a subsequence \( x_{nk} \) of \( x_n \) and a point \( y \in Y \) such that \( f(x_{nk}) \rightarrow^{\nu} y \);

(v) sequentially \( \nu \)-compact preserving if the image \( f(K) \) of every sequentially \( \nu \)-compact set \( K \) of \( X \) is sequentially \( \nu \)-compact in \( Y \).

**§3. \( \nu g-T_0 \) limit point**

**Definition 3.1.** In \( X \), a point \( x \) is said to be a \( \nu g-T_0 \)-limit point of \( A \) if each \( \nu g \)-open set containing \( x \) contains some point \( y \) of \( A \) such that \( \nu g\{x\} \neq \nu g\{y\} \), or equivalently, such that they are topologically distinct with respect to \( \nu g \)-open sets.

**Example 3.1.** Since regular open \( \Rightarrow \nu \)-open \( \Rightarrow \nu g \)-open, then \( r-T_0 \)-limit point \( \Rightarrow \nu-T_0 \)-limit point \( \Rightarrow \nu g-T_0 \)-limit point.

**Definition 3.2.** A set \( A \) together with all its \( \nu g-T_0 \)-limit points is denoted by \( T_0-\nu g(\bar{A}) \).

**Lemma 3.1.** If \( x \) is a \( \nu g-T_0 \)-limit point of a set \( A \) then \( x \) is \( \nu g \)-limit point of \( A \).

**Lemma 3.2** If \( X \) is \( \nu g-T_0 \)-space then every \( \nu g-T_0 \)-limit point and every \( \nu g \)-limit point are equivalent.

**Corollary 3.1.**
(i) If $X$ is $r$-$T_0$-space then every $\nu g$-$T_0$-limit point and every $\nu g$-limit point are equivalent; 
(ii) If $X$ is $\nu T_0$-space then every $\nu g$-$T_0$-limit point and every $\nu g$-limit point are equivalent.

**Theorem 3.1.** For $x \neq y \in X$, 
(i) $x$ is a $\nu g$-$T_0$-limit point of $\{y\}$ iff $x \notin \nu g[y]$ and $y \in \nu g[x]$; 
(ii) $x$ is not a $\nu g$-$T_0$-limit point of $\{y\}$ iff either $x \in \nu g[y]$ or $\nu g[x] = \nu g[y]$; 
(iii) $x$ is not a $\nu g$-$T_0$-limit point of $\{y\}$ iff either $x \in \nu g[y]$ or $y \in \nu g[x]$.

**Corollary 3.2.** 
(i) If $x$ is a $\nu g$-$T_0$-limit point of $\{y\}$, then $y$ cannot be a $\nu g$-limit point of $\{x\}$; 
(ii) If $\nu g[x] = \nu g[y]$, then neither $x$ is a $\nu g$-$T_0$-limit point of $\{y\}$ nor $y$ is a $\nu g$-$T_0$-limit point of $\{x\}$; 
(iii) If a singleton set $A$ has no $\nu g$-$T_0$-limit point in $X$, then $\nu g[A] = \nu g[x]$ for all $x \in \nu g[A]$.

**Lemma 3.3.** In $X$, if $x$ is a $\nu g$-limit point of a set $A$, then in each of the following cases, $X$ becomes $\nu g$-$T_0$-limit point of $A(\{x\} \neq A)$.

- (i) $\nu g[x] \neq \nu g[y]$ for $y \in A$, $x \neq y$; 
- (ii) $\nu g[x] = \{x\}$; 
- (iii) $X$ is a $\nu g$-$T_0$-space; 
- (iv) $A - \{x\}$ is $\nu g$-open.

**Corollary 3.3.** In $X$, if $x$ is a limit point [resp: $r$-limit point; $\nu$-limit point] of a set $A$, then in each of the following cases $x$ becomes $\nu g$-$T_0$-limit point of $A (\{x\} \neq A)$.

- (i) $\nu g[x] \neq \nu g[y]$ for $y \in A$, $x \neq y$; 
- (ii) $\nu g[x] = \{x\}$; 
- (iii) $X$ is a $\nu g$-$T_0$-space; 
- (iv) $A - \{x\}$ is $\nu g$-open.

§4. $\nu g$-$T_0$ and $\nu g$-$R_i$ axioms, $i = 0, 1$

In view of lemma 3.3(iii), $\nu g$-$T_0$-axiom implies the equivalence of the concept of limit point of a set with that of $\nu g$-$T_0$-limit point of the set. But for the converse, if $x \in \nu g[y]$ then $\nu g[x] \neq \nu g[y]$ in general, but if $x$ is a $\nu g$-$T_0$-limit point of $\{y\}$, then $\nu g[x] = \nu g[y]$.

**Lemma 4.1.** In a space $X$, a limit point $x$ of $\{y\}$ is a $\nu g$-$T_0$-limit point of $\{y\}$ iff $\nu g[x] \neq \nu g[y]$.

This lemma leads to characterize the equivalence of $\nu g$-$T_0$-limit point and $\nu g$-limit point of a set as the $\nu g$-$T_0$-axiom.

**Theorem 4.1.** The following conditions are equivalent:

- (i) $X$ is a $\nu g$-$T_0$ space; 
- (ii) Every $\nu g$-limit point of a set $A$ is a $\nu g$-$T_0$-limit point of $A$; 
- (iii) Every $r$-limit point of a singleton set $\{x\}$ is a $\nu g$-$T_0$-limit point of $\{x\}$; 
- (iv) For any $x, y \in X$, $x \neq y$ if $x \in \nu g[y]$, then $x$ is a $\nu g$-$T_0$-limit point of $\{y\}$.

**Note 4.1.** In a $\nu g$-$T_0$-space $X$ if every point of $X$ is a $r$-limit point of $X$, then every point of $X$ is $\nu g$-$T_0$-limit point of $X$. But a space $X$ in which each point is a $\nu g$-$T_0$-limit point of $X$ is not necessarily a $\nu g$-$T_0$-space.
Theorem 4.2. The following conditions are equivalent:
(i) $X$ is a $νg−R_0$ space;
(ii) For any $x, y ∈ X$, if $x ∈ νg\{y\}$, then $x$ is not a $νg-T_0$-limit point of $\{y\}$;
(iii) A point $νg$-closure set has no $νg-T_0$-limit point in $X$;
(iv) A singleton set has no $νg-T_0$-limit point in $X$.

Since every $r-R_0$-space is $νg-R_0$-space, we have the following Corollary.

Corollary 4.1. The following conditions are equivalent:
(i) $X$ is a $r−R_0$ space;
(ii) For any $x, y ∈ X$, if $x ∈ νg\{y\}$, then $X$ is not a $νg-T_0$-limit point of $\{y\}$;
(iii) A point $νg$-closure set has no $νg-T_0$-limit point in $X$;
(iv) A singleton set has no $νg-T_0$-limit point in $X$.

Since every $ν-R_0$-space is $νg-R_0$-space, we have the following Corollary.

Corollary 4.2. The following conditions are equivalent:
(i) $X$ is a $ν−R_0$ space;
(ii) For any $x, y ∈ X$, if $x ∈ νg\{y\}$, then $X$ is not a $νg-T_0$-limit point of $\{y\}$;
(iii) A point $νg$-closure set has no $νg-T_0$-limit point in $X$;
(iv) A singleton set has no $νg-T_0$-limit point in $X$.

Theorem 4.3. In a $νg-R_0$ space $X$, a point $x$ is $νg-T_0$-limit point of $A$ iff every $νg$-open set containing $x$ contains in finitely many points of $A$ with each of which $x$ is topologically distinct.

If $νg-R_0$ space is replaced by $r-R_0$ space in the above theorem, we have the following corollaries.

Corollary 4.3. (a) In an $r-R_0$-space $X$,
(i) If a point $x$ is $r-T_0$-limit point of a set then every $νg$-open set containing $X$ contains infinitely many points of $A$ with each of which $X$ is topologically distinct;
(ii) If a point $x$ is $νg-T_0$-limit point of a set then every $νg$-open set containing $X$ contains infinitely many points of $A$ with each of which $X$ is topologically distinct.

(b) In an $ν-R_0$-space $X$,
(i) If a point $x$ is $r-T_0$-limit point of a set then every $νg$-open set containing $x$ contains infinitely many points of $A$ with each of which $x$ is topologically distinct;
(ii) If a point $x$ is $νg-T_0$-limit point of a set then every $νg$-open set containing $x$ contains infinitely many points of $A$ with each of which $x$ is topologically distinct.

Theorem 4.4. $X$ is $νg-R_0$ space iff a set $A$ of the form $A = \cup νg\{x_i \; i=1 \; \text{to} \; n\}$, a finite union of point closure sets has no $νg-T_0$-limit point.

Corollary 4.4. (a) If $X$ is $r-R_0$ space and
(i) If $A = \cup νg\{x_i \; i=1 \; \text{to} \; n\}$, a finite union of point closure sets has no $νg-T_0$-limit point;
(ii) If $X = \cup νg\{x_i \; i=1 \; \text{to} \; n\}$, then $X$ has no $νg-T_0$-limit point.
(b) If $X$ is $ν-R_0$ space and
(i) If $A = \cup νg\{x_i \; i=1 \; \text{to} \; n\}$, a finite union of point closure sets has no $νg-T_0$-limit point;
(ii) If $X = \cup νg\{x_i \; i=1 \; \text{to} \; n\}$, then $X$ has no $νg-T_0$-limit point.

Theorem 4.5. The following conditions are equivalent:
(i) $X$ is $νg-R_0$-space;
(ii) For any \( X \) and a set \( A \) in \( X \), \( x \) is a \( \nu_{\text{g}} \)-\( T_0 \)-limit point of \( A \) iff every \( \nu_{\text{g}} \)-open set containing \( x \) contains infinitely many points of \( A \) with each of which \( X \) is topologically distinct.

Various characteristic properties of \( \nu_{\text{g}} \)-\( T_0 \)-limit points studied so far is enlisted in the following Theorem for a ready reference.

**Theorem 4.6.** In a \( \nu_{\text{g}} \)-\( R_0 \)-space, we have the following:

(i) A singleton set has no \( \nu_{\text{g}} \)-\( T_0 \)-limit point in \( X \);

(ii) A finite set has no \( \nu_{\text{g}} \)-\( T_0 \)-limit point in \( X \);

(iii) A point \( \nu_{\text{g}} \)-closure has no set \( \nu_{\text{g}} \)-\( T_0 \)-limit point in \( X \);

(iv) A finite union point \( \nu_{\text{g}} \)-closure sets have no set \( \nu_{\text{g}} \)-\( T_0 \)-limit point in \( X \);

(v) for \( x, y \in X \), \( x \in T_0 - \nu_{\text{g}}(y) \) iff \( x = y \);

(vi) for any \( x, y \in X \), \( x \neq y \) iff neither \( X \) is \( \nu_{\text{g}} \)-\( T_0 \)-limit point of \( \{y\} \) nor \( y \) is \( \nu_{\text{g}} \)-\( T_0 \)-limit point of \( \{x\} \);

(vii) for any \( x, y \in X \), \( x \neq y \) iff \( T_0 - \nu_{\text{g}}(x) \cap T_0 - \nu_{\text{g}}(y) = \emptyset \);

(viii) any point \( x \in X \) is a \( \nu_{\text{g}} \)-\( T_0 \)-limit point of a set \( A \) in \( X \) iff every \( \nu_{\text{g}} \)-open set containing \( X \) contains infinitely many points of \( A \) with each which \( X \) is topologically distinct.

**Theorem 4.7.** \( X \) is \( \nu_{\text{g}} \)-\( R_1 \) iff for any \( \nu_{\text{g}} \)-open set \( U \) in \( X \) and points \( x, y \) such that \( x \in X - U, y \in U \), there exists a \( \nu_{\text{g}} \)-open set \( V \) in \( X \) such that \( y \in V \subset U, x \notin V \).

**Lemma 4.2.** In \( \nu_{\text{g}} \)-\( R_1 \) space \( X \), if \( x \) is a \( \nu_{\text{g}} \)-\( T_0 \)-limit point of \( X \), then for any non empty \( \nu_{\text{g}} \)-open set \( U \), there exists a non empty \( \nu_{\text{g}} \)-open set \( V \) such that \( V \subset U, x \notin \nu_{\text{g}}(V) \).

**Lemma 4.3.** In a \( \nu_{\text{g}} \)-regular space \( X \), if \( X \) is a \( \nu_{\text{g}} \)-\( T_0 \)-limit point of \( X \), then for any non empty \( \nu_{\text{g}} \)-open set \( U \), there exists a non empty \( \nu_{\text{g}} \)-open set \( V \) such that \( \nu_{\text{g}}(V) \subset U, x \notin \nu_{\text{g}}(V) \).

**Corollary 4.5.** In a regular space \( X \),

(i) If \( X \) is a \( \nu_{\text{g}} \)-\( T_0 \)-limit point of \( X \), then for any non empty \( \nu_{\text{g}} \)-open set \( U \), there exists a non empty \( \nu_{\text{g}} \)-open set \( V \) such that \( \nu_{\text{g}}(V) \subset U, x \notin \nu_{\text{g}}(V) \);

(ii) If \( x \) is a \( T_0 \)-limit point of \( X \), then for any non empty \( \nu_{\text{g}} \)-open set \( U \), there exists a non empty \( \nu_{\text{g}} \)-open set \( V \) such that \( \nu_{\text{g}}(V) \subset U, x \notin \nu_{\text{g}}(V) \).

**Theorem 4.8.** If \( X \) is a \( \nu_{\text{g}} \)-compact \( \nu_{\text{g}} \)-\( R_1 \)-space, then \( X \) is a Baire space.

**Proof.** Let \( \{A_n\} \) be a countable collection of \( \nu_{\text{g}} \)-closed sets of \( X \), each \( A_n \) having empty interior in \( X \). Take \( A_1 \), since \( A_1 \) has empty interior, \( A_1 \) does not contain any \( \nu_{\text{g}} \)-open set say \( U_0 \). Therefore we can choose a point \( y \in U_0 \) such that \( y \notin A_1 \). For \( X \) is \( \nu_{\text{g}} \)-regular, and \( y \in (X - A_1) \cap U_0 \), a \( \nu_{\text{g}} \)-open set, we can find a \( \nu_{\text{g}} \)-open set \( U_1 \) in \( X \) such that \( y \in U_1, \nu_{\text{g}}(U_1) \subset (X - A_1) \cap U_0 \). Hence \( U_1 \) is a non empty \( \nu_{\text{g}} \)-open set in \( X \) such that \( \nu_{\text{g}}(U_1) \subset U_0 \) and \( \nu_{\text{g}}(U_1) \cap A_1 = \emptyset \). Continuing this process, in general, for given non empty \( \nu_{\text{g}} \)-open set \( U_{n-1} \), we can choose a point of \( U \) which is not in the \( \nu_{\text{g}} \)-closed set \( A_n \) and a \( \nu_{\text{g}} \)-open set \( U_n \) containing this point such that \( \nu_{\text{g}}(U_n) \subset U_{n-1} \) and \( \nu_{\text{g}}(U_n) \cap A_n = \emptyset \). Thus we get a sequence of nested non empty \( \nu_{\text{g}} \)-closed sets which satisfies the finite intersection property. Therefore \( \cap \nu_{\text{g}}(U_n) \neq \emptyset \). Then some \( x \in \cap \nu_{\text{g}}(U_n) \) which in turn implies that \( x \in U_n - 1 \) as \( \nu_{\text{g}}(U_n) \subset U_{n-1} \) and \( x \notin A_n \) for each \( n \).

**Corollary 4.6.** If \( X \) is a compact \( \nu_{\text{g}} \)-\( R_1 \)-space, then \( X \) is a Baire space.

**Corollary 4.7.** Let \( X \) be a \( \nu_{\text{g}} \)-compact \( \nu_{\text{g}} \)-\( R_1 \)-space. If \( \{A_n\} \) is a countable collection of \( \nu_{\text{g}} \)-closed sets in \( X \), each \( A_n \) having non-empty \( \nu_{\text{g}} \)-interior in \( X \), then there is a point of \( X \) which is not in any of the \( A_n \).
Corollary 4.8. Let $X$ be a $\nu g$-compact $R_1$-space. If $\{A_n\}$ is a countable collection of $\nu g$-closed sets in $X$, each $A_n$ having non-empty $\nu g$-interior in $X$, then there is a point of $X$ which is not in any of the $A_n$.

Theorem 4.9. Let $X$ be a non empty compact $\nu g$-$R_1$-space. If every point of $X$ is a $\nu g$-$T_0$-limit point of $X$ then $X$ is uncountable.

Proof. Since $X$ is non empty and every point is a $\nu g$-$T_0$-limit point of $X$, $X$ must be infinite. If $X$ is countable, we construct a sequence of $\nu g$-open sets $\{V_n\}$ in $X$ as follows:

Let $X = V_1$, then for set $x_1$ is a $\nu g$-$T_0$-limit point of $X$, we can choose a non empty $\nu g$-open $V_2$ in $X$ such that $V_2 \subset V_1$ and $x_1 \notin \nu gV_2$. Next for $x_2$ and non empty $\nu g$-open set $V_2$, we can choose a non empty $\nu g$-open set $V_3$ in $X$ such that $V_3 \subset V_2$ and $x_2 \notin \nu gV_3$. Continuing this process for each $x_n$ and a non empty $\nu g$-open set $V_n$, we can choose a non empty $\nu g$-open set $V_{n+1}$ in $X$ such that $V_{n+1} \subset V_n$ and $x_n \notin \nu gV_{n+1}$.

Now consider the nested sequence of $\nu g$-closed sets $\nu gV_1 \supset \nu gV_2 \supset \nu gV_3 \supset \cdots \supset \nu gV_n \supset \cdots$, since $X$ is $\nu g$-compact and $\{\nu gV_n\}$ the sequence of $\nu g$-closed sets satisfy finite intersection property. By Cantors intersection theorem, there exists an $x$ in $X$ such that $x \in \nu gV_n$. Further $x \in X$ and $x \in V_1$, which is not equal to any of the points of $x$. Hence $X$ is uncountable.

Corollary 4.9. Let $X$ be a non empty $\nu g$-compact $\nu g$-$R_1$-space. If every point of $X$ is a $\nu g$-$T_0$-limit point of $x$, then $X$ is uncountable.

§5. $\nu g$-$T_0$-identification spaces and $\nu g$-separation axioms

Definition 5.1. Let $X$ be a topological space and let $\mathcal{R}$ be the equivalence relation on $X$ defined by $x \mathcal{R} y$ iff $\nu g\{x\} = \nu g\{y\}$.

Problem 5.1. Show that $x \mathcal{R} y$ iff $\nu g\{x\} = \nu g\{y\}$ is an equivalence relation.

Definition 5.1. The space $(X_0, Q(X_0))$ is called the $\nu g$-$T_0$-identification space of $(X, \tau)$, where $X_0$ is the set of equivalence classes of $\tau$ and $Q(X_0)$ is the decomposition topology on $X_0$.

Let $P_X : (X, \tau) \rightarrow (X_0, Q(X_0))$ denote the natural map.

Lemma 5.1. If $x \in X$ and $A \subset X$, then $x \in \nu gA$ iff every $\nu g$-open set containing $X$ intersects $A$.

Theorem 5.1. The natural map $P_X : (X, \tau) \rightarrow (X_0, Q(X_0))$ is closed, open and $P_X^{-1}(P_X(O)) = O$ for all $O \in PO(X, \tau)$ and $(X_0, Q(X_0))$ is $\nu g$-$T_0$.

Proof. Let $O \in PO(X, \tau)$ and let $C \subset P_X(O)$. Then there exists $x \in O$ such that $P_X(x) = C$. If $y \in C$, then $\nu g\{y\} = \nu g\{x\}$, which implies $y \in O$. Since $\tau \subset PO(X, \tau)$, then $P_X^{-1}(P_X(U)) = U$ for all $U \in \tau$, which implies $P_X$ is closed and open.

Let $G, H \subset X_0$ such that $G \neq H$ and let $x \in G$ and $y \in H$. Then $\nu g\{x\} \neq \nu g\{y\}$, which implies that $x \notin \nu g\{y\}$ or $y \notin \nu g\{x\}$, say $x \notin \nu g\{y\}$. Since $P_X$ is continuous and open, then $G \in A = P_X\{X - \nu g\{y\}\} \notin PO(X_0, Q(X_0))$ and $H \notin A$.

Theorem 5.2. The followings are equivalent:
(i) $X$ is $\nu gR_0$;
(ii) $X_0 = \{\nu g\{x\} : x \in X\}$;
(iii) $(X_0, Q(X_0))$ is $\nu gT_1$. 
Proof. (i) ⇒ (ii). Let \( C \subseteq X_0 \), and let \( x \in C \). If \( y \notin C \), then \( y = \nu g[y] = \nu g[x] \), which implies \( C = \nu g[x] \). If \( y \notin \nu g[x] \), then \( x = \nu g[y] \), since, otherwise, \( x \notin X - \nu g[y] \). This is a contradiction. Thus, if \( y \in \nu g[x] \), then \( x \in \nu g[y] \), which implies \( \nu g[x] \subset X - \nu g[y] \), which is a contradiction. Thus, if \( y \in \nu g[x] \), then \( x \in \nu g[y] \), which implies \( \nu g[y] = \nu g[x] \) and \( y \in C \). Hence \( X_0 = \{ \nu g[x] : x \in X \} \).

(ii) ⇒ (iii). Let \( A \not\subseteq B \in X_0 \). Then there exists \( x, y \in X \) such that \( A = \nu g[x] ; B = \nu g[y] \), and \( \nu g[x] \cap \nu g[y] = \phi \). Then \( A = \nu g[X - \nu g[y]] \in PO(X_0, Q(X_0)) \) and \( B \notin C \). Thus \( (X_0, Q(X_0)) \) is \( \nu g-T_1 \).

(iii) ⇒ (i). Let \( x \in U \in \nu gO(X) \). Let \( y \notin U \) and \( C_x, C_y \in X_0 \) containing \( x \) and \( y \) respectively. Then \( x \notin \nu g[y] \), which implies \( C_x \neq C_y \) and there exists \( \nu g \)-open set \( A \) such that \( C_x \notin A \) and \( C_y \notin A \). Since \( P_X \) is continuous and open, then \( y \in B = P_X^{-1}(A) \in \nu gO(X) \) and \( x \notin B \), which implies \( y \notin \nu g[x] \). Thus \( \nu g[x] \subseteq U \). This is true for all \( \nu g[x] \) implies \( \nu g[x] \subseteq A \). Hence \( X \in \nu g-R_0 \).

Theorem 5.3. \( X \) is \( \nu g - R_1 \) iff \( (X_0, Q(X_0)) \) is \( \nu g-T_2 \).

The proof is straightforward from using theorems 5.1 and 5.2 and is omitted.

§6. \( \nu g \)-open functions and \( \nu g-T_i \) spaces, \( i = 0, 1, 2 \)

Theorem 6.1. \( X \) is \( \nu g-T_i \), \( i = 0, 1, 2 \) iff there exists a \( \nu g \)-continuous, almost-open, 1-1 function \( f : X \to \nu g-T_i \) space, \( i = 0, 1, 2 \), respectively.

Theorem 6.2. If \( f : X \to Y \) is \( \nu g \)-continuous, \( \nu g \)-open, and \( x, y \in X \) such that \( \nu g[x] = \nu g[y] \), then \( \nu g[f(x)] = \nu g[f(y)] \).

Proof. Suppose \( \nu g[f(x)] \neq \nu g[f(y)] \). Then \( f(x) \notin \nu g[f(y)] \) or \( f(y) \notin \nu g[f(x)] \), say \( f(x) \notin \nu g[f(y)] \). Then \( f(x) \in A = Y - \nu g[f(y)] \in \nu gO(Y) \). If \( B = Y - A \nu g \), then \( f(x) \notin B \), and \( B \cap \nu g[f(y)] \neq \phi \), which implies \( f(y) \in B \), \( y \notin f^{-1}(B) \in \nu gO(X) \), and \( x \notin f^{-1}(B) \) which is a contradiction. Thus \( \nu gA = Y \). Since \( f(y) \notin A \), then \( y \notin (f^{-1}(A))^o \). If \( x \in (f^{-1}(A))^o \), then \( \{x\} \cup (f^{-1}(A))^o \) is \( \nu g \)-open containing \( x \) and not \( y \), which is a contradiction. Hence \( x \notin U = X - (f^{-1}(A))^o \) and \( \phi \neq f(U) \in \nu gO(Y) \). Then \( C = (f(U))^o \cap A^o = \phi \), for suppose not. Then \( f^{-1}(C) \in \nu gO(X) \), which implies \( f^{-1}(C) \subseteq (f^{-1}(C))^o \subseteq (f^{-1}(A))^o \), which is a contradiction. Hence \( C = \phi \), which contradicts \( \nu gA = Y \).

Theorem 6.3. The followings are equivalent:

(i) \( X \) is \( \nu g-T_0 \);

(ii) Elements of \( X_0 \) are singleton sets;

(iii) There exists a \( \nu g \)-continuous, \( \nu g \)-open, 1-1 function \( f : X \to Y \), where \( Y \) is \( \nu g-T_0 \).

Proof. (i) ⇒ (ii) and (ii) ⇒ (iii) are straightforward and is omitted.

(iii) ⇒ (i). Let \( x, y \in X \) such that \( f(x) \neq f(y) \), which implies \( \nu g[f(x)] \neq \nu g[f(y)] \). Then by theorem 6.2, \( \nu g[x] \neq \nu g[y] \). Hence \( X \) is \( \nu g-T_0 \).

Corollary 6.1. A space \( X \) is \( \nu g-T_i \), \( i = 1, 2 \) iff \( X \) is \( \nu g-T_i-1 \), \( i = 1, 2 \), respectively, and there exists a \( \nu g \)-continuous, \( \nu g \)-open, 1-1 function \( f : X \to \) a \( \nu g-T_0 \) space.

Definition 6.1. \( f : X \to Y \) is point-\( \nu g \)-closure 1-1 iff for \( x, y \in X \) such that \( \nu g(x) \neq \nu g(y) \), \( \nu g[f(x)] \neq \nu g[f(y)] \).

Theorem 6.4. (i) If \( f : X \to Y \) is point \( \nu g \)-closure 1-1 and \( X \) is \( \nu g-T_0 \), then \( f \) is 1-1;

(ii) If \( f : X \to Y \), where \( X \) and \( Y \) are \( \nu g-T_0 \), then \( f \) is point \( \nu g \)-closure 1-1 if \( f \) is 1-1.
**Proof.** Omitted.

The following result can be obtained by combining results for \( \nu g-T_0 \)-identification spaces, \( \nu g \)-induced functions and \( \nu g-T_i \) spaces, \( i = 1, 2 \).

**Theorem 6.5.** \( X \) is \( \nu g-R_i \), \( i = 0, 1 \) iff there exists a \( \nu g \)-continuous, almost-open point \( \nu g \)-closure 1-1 function \( f : X \to \nu g-R_i \) space, \( i = 0, 1 \), respectively.

### §7. \( \nu g \)-normal, almost \( \nu g \)-normal and mildly \( \nu g \)-normal spaces

**Definition 7.1.** A space \( X \) is said to be \( \nu g \)-normal if for any pair of disjoint closed sets \( F_1 \) and \( F_2 \), there exist disjoint \( \nu g \)-open sets \( U \) and \( V \) such that \( F_1 \subset U \) and \( F_2 \subset V \).

**Note 7.1.** From the above Definition we have the following implication diagram.

\[
\begin{array}{c}
gr-T_4 \quad g-T_4 \\
\downarrow \quad \downarrow \quad \searrow \\
gr\alpha-T_4 \rightarrow r\alpha-T_4 \rightarrow rg-T_4 \rightarrow \nu g-T_4 \leftrightarrow sg-T_4 \leftrightarrow \beta g-T_4 \\
\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\
\alpha-T_4 \rightarrow \nu-T_4 \searrow \\
r-T_4 \rightarrow \pi-T_4 \rightarrow T_4 \rightarrow \alpha-T_4 \rightarrow s-T_4 \rightarrow \beta-T_4 \\
\searrow \quad \downarrow \searrow \\
\pi g-T_4 \quad p-T_4 \rightarrow \omega-T_4 \neq ga-T_4 \rightarrow \alpha g-T_4 \\
\searrow \quad \searrow \quad \searrow \\
gp-T_4 \leftrightarrow pg-T_4 \quad r\omega-T_4
\end{array}
\]

**Example 7.1** Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, \{a\}, \{b, c\}, X\} \). Then \( X \) is \( \nu g \)-normal.

**Example 7.2** Let \( X = \{a, b, c, d\} \) and \( \tau = \{\emptyset, \{b, d\}, \{a, b, d\}, \{b, c, d\}, X\} \). Then \( X \) is \( \nu g \)-normal and is not normal.

We have the following characterization of \( \nu g \)-normality.

**Theorem 7.1.** For a space \( X \) the followings are equivalent:

(a) \( X \) is \( \nu g \)-normal;

(b) For every pair of open sets \( U \) and \( V \) whose union is \( X \), there exist \( \nu g \)-closed sets \( A \) and \( B \) such that \( A \subset U, B \subset V \) and \( A \cup B = X \);

(c) For every closed set \( F \) and every open set \( G \) containing \( F \), there exists a \( \nu g \)-open set \( U \) such that \( F \subset U \subset \nu g(U) \subset G \).

**Proof.** (a) \( \Rightarrow \) (b). Let \( U \) and \( V \) be a pair of open sets in a \( \nu g \)-normal space \( X \) such that \( X = U \cup V \). Then \( X \neq U \cup V \) and \( X \subset V \) are disjoint closed sets. Since \( X \) is \( \nu g \)-normal there exist disjoint \( \nu g \)-open sets \( U_1 \) and \( V_1 \) such that \( X - U \subset U_1 \) and \( X - V \subset V_1 \). Let \( A = X - U_1, B = X - V_1 \). Then \( A \) and \( B \) are \( \nu g \)-closed sets such that \( A \subset U, B \subset V \) and \( A \cup B = X \).

(b) \( \Rightarrow \) (c). Let \( F \) be a closed set and \( G \) be an open set containing \( F \). Then \( X - F \) and \( G \) are open sets whose union is \( X \). Then by (b), there exist \( \nu g \)-closed sets \( W_1 \) and \( W_2 \) such that \( W_1 \subset X - F \) and \( W_2 \subset G \) and \( W_1 \cup W_2 = X \). Then \( F \subset X - W_1, X - G \subset X - W_2 \) and \( (X - W_1) \cap (X - W_2) = \emptyset \). Let \( U = X - W_1 \) and \( V = X - W_2 \). Then \( U \) and \( V \) are disjoint
\(u_{\nu g}\)-open sets such that \(F \subset U \subset X - V \subset G\). As \(X - V\) is \(u_{\nu g}\)-closed set, we have \(u_{\nu g}(U) \subset X - V\) and \(F \subset U \subset u_{\nu g}(U) \subset G\).

(e) \(\Rightarrow\) (a). Let \(F_1\) and \(F_2\) be any two disjoint closed sets of \(X\). Put \(G = X - F_2\), then \(F_1 \cap G = \emptyset\). \(F_1 \subset G\) where \(G\) is an open set. Then by (e), there exists a \(u_{\nu g}\)-open set \(U\) of \(X\) such that \(F_1 \subset U \subset u_{\nu g}(U) \subset G\). It follows that \(F_2 \subset X - u_{\nu g}(U) = V\), say, then \(V\) is \(u_{\nu g}\)-open and \(U \cap V = \emptyset\). Hence \(F_1\) and \(F_2\) are separated by \(u_{\nu g}\)-open sets \(U\) and \(V\). Therefore \(X\) is \(u_{\nu g}\)-normal.

**Theorem 7.2.** A regular open subspace of a \(u_{\nu g}\)-normal space is \(u_{\nu g}\)-normal.

**Proof.** Let \(Y\) be a regular open subspace of a \(u_{\nu g}\)-normal space \(X\). Let \(A\) and \(B\) be disjoint closed subsets of \(Y\). By \(u_{\nu g}\)-normality of \(X\), there exist disjoint \(u_{\nu g}\)-open sets \(U\) and \(V\) in \(X\) such that \(A \subset U\) and \(B \subset V\), \(U \cap Y\) and \(V \cap Y\) are \(u_{\nu g}\)-open in \(Y\) such that \(A \subset U \cap Y\) and \(B \subset V \cap Y\). Hence \(Y\) is \(u_{\nu g}\)-normal.

**Example 7.3.** Let \(X = \{a, b, c\}\) with \(\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}\) is \(u_{\nu g}\)-normal and \(u_{\nu g}\)-regular.

Now, we define the following.

**Definition 7.2.** A function \(f : X \rightarrow Y\) is said to be almost-\(u_{\nu g}\)-irresolute if for each \(x \in X\) and each \(u_{\nu g}\)-neighborhood \(V\) of \(f(x)\), \(u_{\nu g}(f^{-1}(V))\) is a \(u_{\nu g}\)-neighborhood of \(X\).

Clearly every \(u_{\nu g}\)-irresolute map is almost \(u_{\nu g}\)-irresolute.

The Proof of the following lemma is straightforward and hence omitted.

**Lemma 7.1.** \(f\) is almost \(u_{\nu g}\)-irresolute iff \(f^{-1}(V) \subset u_{\nu g} - \text{int}(u_{\nu g}(f^{-1}(V)))\) for every \(V \in u_{\nu g}GO(Y)\).

Now we prove the following.

**Lemma 7.2.** \(f\) is almost \(u_{\nu g}\)-irresolute iff \(f(u_{\nu g}(U)) \subset u_{\nu g}(f(U))\) for every \(U \in u_{\nu g}GO(X)\).

**Proof.** Let \(U \in u_{\nu g}GO(X)\). Suppose \(y \notin u_{\nu g}(f(U))\). Then there exists \(V \in u_{\nu g}GO(y)\) such that \(V \cap f(U) = \emptyset\). Hence \(f^{-1}(V) \cap U = \emptyset\). Since \(U \in u_{\nu g}GO(X)\), we have \(u_{\nu g}(u_{\nu g}(f^{-1}(V))) \cap u_{\nu g}(U) = \emptyset\). Then by lemma 7.1, \(f^{-1}(V) \cap u_{\nu g}(U) = \emptyset\) and hence \(V \cap f(u_{\nu g}(U)) = \emptyset\). This implies that \(y \notin f(u_{\nu g}(U))\).

Conversely, if \(V \in u_{\nu g}(Y)\), then \(W = X - u_{\nu g}(f^{-1}(V)) \in u_{\nu g}(X)\). By hypothesis, \(f(u_{\nu g}(W)) \subset u_{\nu g}(f(W))\) and hence \(X - u_{\nu g}(u_{\nu g}(f^{-1}(V))) = u_{\nu g}(W) \subset f^{-1}(u_{\nu g}(f(W))) \subset f^{-1}(u_{\nu g}(f(X - f^{-1}(V)))) \subset f^{-1}(u_{\nu g}(X - Y)) = f^{-1}(X - Y) = X - f^{-1}(V)\). Therefore, \(f^{-1}(V) \subset u_{\nu g}(u_{\nu g}(f^{-1}(V)))\). By lemma 7.1, \(f\) is almost \(u_{\nu g}\)-irresolute.

Now we prove the following result on the invariance of \(u_{\nu g}\)-normality.

**Theorem 7.3.** If \(f\) is an \(M\)-\(u_{\nu g}\)-open continuous almost \(u_{\nu g}\)-irresolute function from a \(u_{\nu g}\)-normal space \(X\) onto a space \(Y\), then \(Y\) is \(u_{\nu g}\)-normal.

**Proof.** Let \(A\) be a closed subset of \(Y\) and \(B\) be an open set containing \(A\). Then by continuity of \(f\), \(f^{-1}(A)\) is closed and \(f^{-1}(B)\) is an open set of \(X\) such that \(f^{-1}(A) \subset f^{-1}(B)\).

As \(X\) is \(u_{\nu g}\)-normal, there exists a \(u_{\nu g}\)-open set \(U\) in \(X\) such that \(f^{-1}(A) \subset U \subset u_{\nu g}(U) \subset f^{-1}(B)\).

Then \(f(f^{-1}(A)) \subset f(U) \subset f(u_{\nu g}(U)) \subset f(f^{-1}(B))\). Since \(f\) is \(M\)-\(u_{\nu g}\)-open almost \(u_{\nu g}\)-irresolute surjection, we obtain \(A \subset f(U) \subset u_{\nu g}(f(U)) \subset B\). Therefore by theorem 7.1 \(Y\) is \(u_{\nu g}\)-normal.

**Lemma 7.3.** A mapping \(f\) is \(M\)-\(u_{\nu g}\)-closed if and only if for each subset \(B\) in \(Y\) and for each \(u_{\nu g}\)-open set \(U\) in \(X\) containing \(f^{-1}(B)\), there exists a \(u_{\nu g}\)-open set \(V\) containing \(B\) such that \(f^{-1}(V) \subset U\).
Now we prove the following.

**Theorem 7.4.** If \( f \) is an \( M \)-\( \nu g \)-closed continuous function from a \( \nu g \)-normal space onto a space \( Y \), then \( Y \) is \( \nu g \)-normal.

Proof of the theorem is routine and hence omitted.

Now in view of lemma 2.2 \([19]\) and lemma 7.3, we prove the following result.

**Theorem 7.5.** If \( f \) is an \( M \)-\( \nu g \)-closed map from a weakly Hausdorff \( \nu g \)-normal space \( X \) onto a space \( Y \) such that \( f^{-1}(y) \) is \( S \)-closed relative to \( X \) for each \( y \in Y \), then \( Y \) is \( \nu g \)-\( T_2 \).

Proof. Let \( y_1 \) and \( y_2 \) be any two distinct points of \( Y \). Since \( X \) is weakly Hausdorff, \( f^{-1}(y_1) \) and \( f^{-1}(y_2) \) are disjoint closed subsets of \( X \) by lemma 2.2 \([19]\). As \( X \) is \( \nu g \)-normal, there exist disjoint \( \nu g \)-open sets \( V_1 \) and \( V_2 \) such that \( f^{-1}(y_i) \subset V_i \), for \( i = 1, 2 \). Since \( f \) is \( M \)-\( \nu g \)-closed, there exist \( \nu g \)-open sets \( U_i \) containing \( y_i \) such that \( f^{-1}(U_i) \subset V_i \) for \( i = 1, 2 \). Then it follows that \( U_1 \cap U_2 = \emptyset \). Hence \( Y \) is \( \nu g \)-\( T_2 \).

**Theorem 7.6.** For a space \( X \) we have the following:

(a) If \( X \) is normal then for any disjoint closed sets \( A \) and \( B \), there exist disjoint \( \nu g \)-open sets \( U, V \) such that \( A \subset U \) and \( B \subset V \);

(b) if \( X \) is normal then for any closed set \( A \) and any open set \( V \) containing \( A \), there exists an \( \nu g \)-open set \( U \) of \( X \) such that \( A \subset U \subset \nu g(V) \subset V \).

**Definition 7.2.** \( X \) is said to be almost \( \nu g \)-normal if for each closed set \( A \) and each regular closed set \( B \) such that \( A \cap B = \emptyset \), there exist disjoint \( \nu g \)-open sets \( U \) and \( V \) such that \( A \subset U \) and \( B \subset V \).

**Note 7.2.** From the above definition we have the following implication diagram.

\[
\begin{align*}
\text{Al-}g-T_4 & \quad \text{Al-gs-T}_4 \\
\downarrow & \quad \downarrow & \quad \searrow \\
\text{Al-rga-T}_4 & \quad \text{Al-rga-T}_4 & \quad \text{Al-rg-T}_4 & \quad \text{Al-\nu g-T}_4 & \quad \text{Al-sq-T}_4 & \quad \text{Al-\beta g-T}_4 \\
& \quad \downarrow & \quad \uparrow & \quad \uparrow & \quad \uparrow & \quad \uparrow \\
\text{Al-}r\alpha-T_4 & \quad \text{Al-r\alpha-T}_4 & \quad \searrow & \quad \uparrow & \quad \searrow & \quad \uparrow \\
\text{Al-r-T}_4 & \quad \text{Al-\pi-T}_4 & \quad \text{Al-T}_4 & \quad \text{Al-\alpha-T}_4 & \quad \text{Al-s-T}_4 & \quad \text{Al-\beta-T}_4 \\
& \quad \searrow & \quad \downarrow & \quad \searrow & \quad \searrow & \quad \searrow \\
\text{Al-\pi g-T}_4 & \quad \text{Al-p-T}_4 & \quad \not\leftrightarrow & \quad \text{Al-\omega-T}_4 & \quad \not\leftrightarrow & \quad \text{Al-\alpha g-T}_4 & \quad \text{Al-\alpha g-T}_4 \\
& \quad \downarrow & \quad \searrow & \quad \not\leftrightarrow & \quad \searrow & \quad \searrow & \quad \searrow \\
\text{Al-gp-T}_4 & \quad \text{Al-gpg-T}_4 & \quad \text{Al-r\omega-T}_4 & \quad \text{Al-r\omega-T}_4
\end{align*}
\]

**Example 7.4.** Let \( X = \{a, b, c\} \) and \( \tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\} \). Then \( X \) is almost \( \nu g \)-normal and \( \nu g \)-normal.

Now, we have characterization of almost \( \nu g \)-normality in the following.

**Theorem 7.7.** For a space \( X \) the following statements are equivalent:

(a) \( X \) is almost \( \nu g \)-normal;

(b) for every pair of sets \( U \) and \( V \), one of which is open and the other is regular open whose union is \( X \), there exist \( \nu g \)-closed sets \( G \) and \( H \) such that \( G \subset U, H \subset V \) and \( G \cup H = X \);

(c) for every closed set \( A \) and every regular open set \( B \) containing \( A \), there is a \( \nu g \)-open set \( V \) such that \( A \subset V \subset \nu g(V) \subset B \).
Proof. (a) ⇒ (b). Let \( U \) be an open set and \( V \) be a regular open set in an almost \( \nu g \)-normal space \( X \) such that \( U \cup V = X \). Then \( (X - U) \) is closed set and \( (X - V) \) is regular closed set with \( (X - U) \cap (X - V) = \emptyset \). By almost \( \nu g \)-normality of \( X \), there exist disjoint \( \nu g \)-open sets \( U_1 \) and \( V_1 \) such that \( X - U \subset U_1 \) and \( X - V \subset V_1 \). Let \( G = X - U_1 \) and \( H = X - V_1 \). Then \( G \) and \( H \) are \( \nu g \)-closed sets such that \( G \subset U, H \subset V \) and \( G \cup H = X \).

(b) ⇒ (c) and (c) ⇒ (a) are obvious.

One can prove that almost \( \nu g \)-normality is also regular open hereditary. Almost \( \nu g \)-normality does not imply almost \( \nu g \)-regularity in general. However, we observe that every almost \( \nu g \)-normal \( \nu g \)-R\(_0\) space is almost \( \nu g \)-regular.

Next, we prove the following.

Theorem 7.8. Every almost regular, \( \nu \)-compact space \( X \) is almost \( \nu g \)-normal.

Recall that \( f : X \to Y \) is called \( rc \)-continuous if inverse image of regular closed set is regular closed.

Now, we state the invariance of almost \( \nu g \)-normality in the following.

Theorem 7.9. If \( f \) is continuous \( M \)-\( \nu g \)-open \( rc \)-continuous and almost \( \nu g \)-irresolute surjection from an almost \( \nu g \)-normal space \( X \) onto a space \( Y \), then \( Y \) is almost \( \nu g \)-normal.

Definition 7.3. A space \( X \) is said to be mildly \( \nu g \)-normal if for every pair of disjoint regular closed sets \( F_1 \) and \( F_2 \) of \( X \), there exist disjoint \( \nu g \)-open sets \( U \) and \( V \) such that \( F_1 \subset U \) and \( F_2 \subset V \).

Note 7.3. From the above Definition we have the following implication diagram.

\[
\begin{array}{cccc}
\text{Mild-g-T}_4 & \text{Mild-gs-T}_4 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\text{Mild-rog-T}_4 & \text{Mild-rgα-T}_4 & \text{Mild-rg-T}_4 & \text{Mild-\nu g-T}_4 & \text{Mild-sg-T}_4 & \text{Mild-βg-T}_4 \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\text{Mild-rα-T}_4 & \text{Mild-\nu T}_4 & \text{Mild-\nu g-T}_4 & \text{Mild-\alpha T}_4 & \text{Mild-\pi T}_4 & \text{Mild-β T}_4 \\
\text{Mild-\pi g-T}_4 & \text{Mild-p-T}_4 & \text{Mild-\omega T}_4 & \text{Mild-\nu g-T}_4 & \text{Mild-\alpha T}_4 & \text{Mild-\omega T}_4 \\
\end{array}
\]

We have the following characterization of mild \( \nu g \)-normality.

Theorem 7.10. For a space \( X \) the following are equivalent.

(a) \( X \) is mildly \( \nu g \)-normal;
(b) for every pair of regular open sets \( U \) and \( V \) whose union is \( X \), there exist \( \nu g \)-closed sets \( G \) and \( H \) such that \( G \subset U, H \subset V \) and \( G \cup H = X \);
(c) for any regular closed set \( A \) and every regular open set \( B \) containing \( A \), there exists a \( \nu g \)-open set \( U \) such that \( A \subset U \subset \overline{\nu g(U)} \subset B \);
(d) for every pair of disjoint regular closed sets, there exist \( \nu g \)-open sets \( U \) and \( V \) such that \( A \subset U, B \subset V \) and \( \nu g(U) \cap \nu g(V) = \emptyset \).

This theorem may be proved by using the arguments similar to those of theorem 7.7.
Also, we observe that mild \( \nu g \)-normality is regular open hereditary.
We define the following:

**Definition 7.4.** A space $X$ is weakly $νg$-regular if for each point $x$ and a regular open set $U$ containing $\{x\}$, there is a $νg$-open set $V$ such that $x ∈ V ⊂ \overline{V} ⊂ U$.

**Theorem 7.11.** If $f : X → Y$ is an $M-νg$-open $rc$-continuous and almost $νg$-irresolute function from a mildly $νg$-normal space $X$ onto a space $Y$, then $Y$ is mildly $νg$-normal.

**Proof.** Let $A$ be a regular closed set and $B$ be a regular open set containing $A$. Then by $rc$-continuity of $f$, $f^{-1}(A)$ is a regular closed set contained in the regular open set $f^{-1}(B)$. Since $X$ is mildly $νg$-normal, there exists a $νg$-open set $V$ such that $f^{-1}(A) ⊂ V ⊂ νg(\overline{V}) ⊂ f^{-1}(B)$ by theorem 7.10. As $f$ is $M-νg$-open and almost $νg$-irresolute surjection, it follows that $f(V) ∈ νgO(Y)$ and $A ⊂ f(V) ⊂ νg(\overline{f(V)}) ⊂ B$. Hence $Y$ is mildly $νg$-normal.

**Theorem 7.12.** If $f : X → Y$ is $rc$-continuous, $M-νg$-closed map from a mildly $νg$-normal space $X$ onto a space $Y$, then $Y$ is mildly $νg$-normal.

§8. $νg$-US spaces

**Definition 8.1.** A sequence $< x_n >$ is said to $νg$-converges to $x ∈ X$, written as $< x_n > →^{νg} x$ if $< x_n >$ is eventually in every $νg$-open set containing $x$.

Clearly, if a sequence $< x_n > →^{ν} Y$ of $X$, then $< x_n > →^{νg}$ to $X$.

**Definition 8.2.** A space $X$ is said to be $νg$-US if every sequence $< x_n >$ in $X$ $νg$-converges to a unique point.

**Theorem 8.1.** Every $νg$-US space is $νg$-$T_1$.  

**Proof.** Let $X$ be $νg$-US space. Let $x ≠ y ∈ X$. Consider the sequence $< x_n >$ where $x_n = x$ for every $n$. Clearly, $< x_n > →^{νg} X$. Also, since $x ≠ y$ and $X$ is $νg$-US, $< x_n > ↑^{νg} y$, i.e. there exists a $νg$-open set $V$ containing $y$ but not $X$. Similarly, if we consider the sequence $< y_n >$ where $y_n = y$ for all $n$, and proceeding as above we get a $νg$-open set $U$ containing $X$ but not $y$. Thus, the space $X$ is $νg$-$T_1$.

**Theorem 8.2.** Every $νg$-$T_2$ space is $νg$-US.

**Proof.** Let $X$ be $νg$-$T_2$ space and $< x_n >$ be a sequence in $X$. If $< x_n >$ $νg$-converge to two distinct points $X$ and $y$. That is, $< x_n >$ is eventually in every $νg$-open set containing $X$ and also in every $νg$-open set containing $y$. This is contradiction since $X$ is $νg$-$T_2$ space. Hence the space $X$ is $νg$-US.

**Definition 8.3.** A set $F$ is sequentially $νg$-closed if every sequence in $F$ $νg$-converges to a point in $F$.

**Theorem 8.3.** $X$ is $νg$-US iff the diagonal set is a sequentially $νg$-closed subset of $X × X$.

**Proof.** Let $X$ be $νg$-US. Let $< x_n, x_n >$ be a sequence in $Δ$. Then $< x_n >$ is a sequence in $X$. As $X$ is $νg$-US, $< x_n > →^{νg} x$ for a unique $x ∈ X$, i.e. if $< x_n > →^{νg} x$ and $y$. Thus, $x = y$. Hence $Δ$ is sequentially $νg$-closed set.

Conversely, let $Δ$ be sequentially $νg$-closed. Let a sequence $< x_n > →^{νg} x$ and $y$. Hence $< x_n, x_n > →^{νg} (x, y)$. Since $Δ$ is sequentially $νg$-closed, $(x, y) ∈ Δ$ which means that $x = y$ implies space $X$ is $νg$-US.

**Definition 8.4.** A subset $G$ of a space $X$ is said to be sequentially $νg$-compact if every sequence in $G$ has a subsequence which $νg$-converges to a point in $G$.  

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Theorem 8.4. In a νg-US space every sequentially νg-compact set is sequentially νg-closed.

Proof. Let Y be a sequentially νg-compact subset of νg-US space X. Let \( < x_n > \) be a sequence in Y. Suppose that \( < x_n > \underset{\nu g}{\to} x \) converges to a point in \( X - Y \). Let \( < x_{n_p} > \) be subsequence of \( < x_n > \) that νg-converges to a point \( y \in Y \), since Y is sequentially νg-compact. Also, let a subsequence \( < x_{n_p} > \) of \( < x_n > \underset{\nu g}{\to} x \in X - Y \). Since \( < x_{n_p} > \) is a sequence in the νg-US space X, \( x = y \). Thus, Y is sequentially νg-closed set.

Next, we give a hereditary property of νg-US spaces.

Theorem 8.5. Every regular open subset of a νg-US space is νg-US.

Proof. Let X be a νg-US space and \( Y \subset X \) be an regular open set. Let \( < x_n > \) be a sequence in Y. Suppose that \( < x_n > \underset{\nu g}{\to} x, y \in Y \). We shall prove that \( < x_n > \underset{\nu g}{\to} x \) and \( y \in X \). Let U be any νg-open subset of X containing X and V be any νg-open set of X containing y. Then, \( U \cap Y \) and \( V \cap Y \) are νg-open sets in Y. Therefore, \( < x_n > \) is eventually in \( U \cap Y \) and \( V \cap Y \) and so in \( U \) and \( V \). Since X is νg-US, this implies that \( x = y \). Hence the subspace Y is νg-US.

Theorem 8.6. A space X is νg-T\(_2\) iff it is both νg-R\(_1\) and νg-US.

Proof. Let X be νg-T\(_2\) space. Then X is νg-R\(_1\) and νg-US by theorem 8.2. Conversely, let X be both νg-R\(_1\) and νg-US space. By theorem 8.1, X is both νg-T\(_1\) and νg-R\(_1\) and, it follows that space X is νg-T\(_2\).

Definition 8.5. A point y is a νg-cluster point of sequence \( < x_n > \) iff \( < x_n > \) is frequently in every νg-open set containing X.

The set of all νg-cluster points of \( < x_n > \) will be denoted by \( \nu g(\overline{x_n}) \).

Definition 8.6. A point y is νg-side point of a sequence \( < x_n > \) if y is a νg-cluster point of \( < x_n > \) but no subsequence of \( < x_n > \) νg-converges to y.

Now, we define the following.

Definition 8.7. A space X is said to be νg-S\(_1\) if it is νg-US and every sequence \( < x_n > \) νg-converges with subsequence of \( < x_n > \) νg-side points.

Definition 8.8. A space X is said to be νg-S\(_2\) if it is νg-US and every sequence \( < x_n > \) in X νg-converges which has no νg-side point.

Lemma 8.1. Every \( \nu g - S_1 \) space is νg-S\(_1\) and Every νg-S\(_1\) space is νg-US.

Now using the notion of sequentially continuous functions, we define the notion of sequentially νg-continuous functions.

Definition 8.9. A function f is said to be sequentially νg-continuous at \( x \in X \) if \( f(x_n) \underset{\nu g}{\to} f(x) \) whenever \( < x_n > \underset{\nu g}{\to} X \). If f is sequentially νg-continuous at all \( x \in X \), then f is said to be sequentially νg-continuous.

Theorem 8.7. Let f and g be two sequentially νg-continuous functions. If Y is νg-US, then the set \( A = \{ x | f(x) = g(x) \} \) is sequentially νg-closed.

Proof. Let Y be νg-US and suppose that there is a sequence \( < x_n > \) in A νg-converging to \( x \in X \). Since f and g are sequentially νg-continuous functions, \( f(x_n) \underset{\nu g}{\to} f(x) \) and \( g(x_n) \underset{\nu g}{\to} g(x) \). Hence \( f(x) = g(x) \) and \( x \in A \). Therefore, A is sequentially νg-closed.

Next, we prove the product theorem for νg-US spaces.

Theorem 8.8. Product of arbitrary family of νg-US spaces is νg-US.
Proof. Let $X = \prod_{\lambda \in \Lambda} X_\lambda$ where $X_\lambda$ is $\nu g$-US. Let a sequence $< x_n >$ in $X$ $\nu g$-converges to $x (= x_1)$ and $y (= y_1)$. Then the sequence $< x_n >$ $\nu g$-converges to $x_\lambda$ and $y_\lambda$ for all $\lambda \in \Lambda$. For suppose there exists a $\mu \in \Lambda$ such that $< x_{n\mu} > \rightarrow^{\nu g} x_\mu$. Then there exists a $\tau_\mu$-$\nu g$-open set $U$ containing $x_\mu$ such that $< x_{n\mu} >$ is not eventually in $U$. Consider the set $U = \prod_{\lambda \in \Lambda} X_\lambda \times U_\mu$. Then $U \in \nu GO(X, x)$. Also, $< x_n >$ is not eventually in $U$, which contradicts the fact that $< x_n >$ $\nu g$-converges to $x$. Thus we get $< x_{n\lambda} >$ $\nu g$-converges to $x_\lambda$ and $y_\lambda$ for all $\lambda \in \Lambda$. Since $X$ is $\nu g$-US for each $\lambda \in \Lambda$. Thus $x = y$. Hence $X$ is $\nu g$-US.

§9. Sequentially sub-$\nu g$-continuity

In this section we introduce and study the concepts of sequentially sub-$\nu g$-continuity, sequentially nearly $\nu g$-continuity and sequentially $\nu g$-compact preserving functions and study their relations and the property of $\nu g$-US spaces.

Definition 9.1. A function $f$ is said to be sequentially nearly $\nu g$-continuous if for each $x \in X$ and each sequence $< x_n > \rightarrow^{\nu g} x \in X$, there exists a subsequence $< x_{nk} >$ of $< x_n >$ such that $< f(x_{nk}) > \rightarrow^{\nu g} f(x)$.

Definition 9.2. A function $f$ is said to be sequentially sub-$\nu g$-continuous if for each $x \in X$ and each sequence $< x_n > \rightarrow^{\nu g} x \in X$, there exists a subsequence $< x_{nk} >$ of $< x_n >$ and a point $y \in Y$ such that $< f(x_{nk}) > \rightarrow^{\nu g} y$.

Definition 9.3. A function $f$ is said to be sequentially $\nu g$-compact preserving if $f(K)$ is sequentially $\nu g$-compact in $Y$ for every sequentially $\nu g$-compact set $K$ of $X$.

Lemma 9.1. Every function $f$ is sequentially sub-$\nu g$-continuous if $Y$ is a sequentially $\nu g$-compact.

Proof. Let $< x_n >$ be a sequence in $X$ $\nu g$-converging to a point $x$ of $X$. Then $\{ f(x_n) \}$ is a sequence in $Y$ and as $Y$ is sequentially $\nu g$-compact, there exists a subsequence $\{ f(x_{nk}) \}$ of $\{ f(x_n) \}$ $\nu g$-converging to a point $y \in Y$. Hence $f$ is sequentially sub-$\nu g$-continuous.

Theorem 9.1. Every sequentially nearly $\nu g$-continuous function is sequentially $\nu g$-compact preserving.

Proof. Let $f$ be sequentially nearly $\nu g$-continuous function and let $K$ be any sequentially $\nu g$-compact subset of $X$. Let $< y_n >$ be any sequence in $f(K)$. Then for each positive integer $n$, there exists a point $x_n \in K$ such that $f(x_n) = y_n$. Since $< x_n >$ is a sequence in the sequentially $\nu g$-compact set $K$, there exists a subsequence $< x_{nk} >$ of $< x_n >$ $\nu g$-converges to $x \in K$. By hypothesis, $f$ is sequentially nearly $\nu g$-continuous and hence there exists a subsequence $< x_j >$ of $< x_{nk} >$ such that $f(x_j) \rightarrow^{\nu g} f(x)$. Thus, there exists a subsequence $< y_j >$ of $< y_n >$ $\nu g$-converging to $f(x) \in f(K)$. This shows that $f(K)$ is sequentially $\nu g$-compact set in $Y$.

Theorem 9.2. Every sequentially $\nu g$-compact preserving function is sequentially sub-$\nu g$-continuous.

Proof. Suppose $f$ is a sequentially $\nu g$-compact preserving function. Let $x$ be any point of $X$ and $< x_n >$ any sequence in $X$ $\nu g$-converging to $x$. We shall denote the set $\{ x_n | n = 1, 2, 3, \cdots \}$ by $A$ and $K = A \cup \{ x \}$. Then $K$ is sequentially $\nu g$-compact since $x_n \rightarrow^{\nu g} x$. By hypothesis, $f$ is sequentially $\nu g$-compact preserving and hence $f(K)$ is a sequentially $\nu g$-compact set of $Y$. Since $\{ f(x_n) \}$ is a sequence in $f(K)$, there exists a subsequence $\{ f(x_{nk}) \}$
of \( \{f(x_n)\} \) \( \nu g \)-converging to a point \( y \in f(K) \). This implies that \( f \) is sequentially sub-\( \nu g \)-continuous.

**Theorem 9.3.** A function \( f : X \to Y \) is sequentially \( \nu g \)-compact preserving iff \( f/K : K \to f(K) \) is sequentially sub-\( \nu g \)-continuous for each sequentially \( \nu g \)-compact subset \( K \) of \( X \).

**Proof.** Suppose \( f \) is a sequentially \( \nu g \)-compact preserving function. Then \( f(K) \) is sequentially \( \nu g \)-compact set in \( Y \) for each sequentially \( \nu g \)-compact set \( K \) of \( X \). Therefore, by lemma 9.1 above, \( f/K : K \to f(K) \) is sequentially \( \nu g \)-continuous function.

Conversely, let \( K \) be any sequentially \( \nu g \)-compact set of \( X \). Let \( < y_n > \) be any sequence in \( f(K) \). Then for each positive integer \( n \), there exists a point \( x_n \in K \) such that \( f(x_n) = y_n \). Since \( < x_n > \) is a sequence in the sequentially \( \nu g \)-compact set \( K \), there exists a subsequence \( < x_{nk} > \) of \( < x_n > \) \( \nu g \)-converging to a point \( x \in K \). By hypothesis, \( f/K : K \to f(K) \) is sequentially sub-\( \nu g \)-continuous and hence there exists a subsequence \( < y_{nk} > \) of \( < y_n > \) \( \nu g \)-converging to a point \( y \in f(K) \). This implies that \( f(K) \) is sequentially \( \nu g \)-compact set in \( Y \). Thus, \( f \) is sequentially \( \nu g \)-compact preserving function.

The following corollary gives a sufficient condition for a sequentially sub-\( \nu g \)-continuous function to be sequentially \( \nu g \)-compact preserving.

**Corollary 9.1.** If \( f \) is sequentially sub-\( \nu g \)-continuous and \( f(K) \) is sequentially \( \nu g \)-closed set in \( Y \) for each sequentially \( \nu g \)-compact set \( K \) of \( X \), then \( f \) is sequentially \( \nu g \)-compact preserving function.

**Proof.** Omitted.

**Acknowledgment.** The author is thankful to the referees for their comments and suggestions for the development of the paper.

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On intuitionistic fuzzy pre-$\beta$-irresolute functions

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Abstract In this paper the concept of intuitionistic fuzzy pre-$\beta$-irresolute functions are introduced and studied. Besides giving characterizations and properties of this function, preservation of some intuitionistic fuzzy topological structure under this function are also given. We also study relationship between this function with other existing functions.

Keywords Intuitionistic Fuzzy $\beta$-open set, intuitionistic fuzzy pre-$\alpha$-irresolute, intuitionistic Fuzzy pre-$\beta$-irresolute.

2000 Mathematics Subject Classification: 54A40, 03E72

§1. Introduction

Ever since the introduction of Fuzzy sets by L. A. Zadeh [14], the Fuzzy concept has invaded almost all branches of mathematics. The concept of Fuzzy topological spaces was introduced and developed by C. L. Chang [2]. Atanassov [1] introduced the notion of intuitionistic fuzzy sets, Coker [4] introduced the intuitionistic fuzzy topological spaces. In this paper we have introduced the concept of intuitionistic Fuzzy pre-$\beta$-irresolute functions and studied their properties. Also we have given characterizations of intuitionistic fuzzy pre-$\beta$-irresolute functions. We also study relationship between this function with other existing functions.

§2. Preliminaries

Definition 2.1. [1] Let $X$ be a nonempty fixed set and $I$ the closed interval [0,1]. An intuitionistic fuzzy set (IFS) $A$ is an object of the following form

$$ A = \{< x, \mu_A(x), \nu_A(x) >; x \in X \}, $$

where the mappings $\mu_A(x) : X \rightarrow I$ and $\nu_A(x) : X \rightarrow I$ denote the degree of membership(namely) $\mu_A(x)$ and the degree of nonmembership (namely) $\nu_A(x)$ for each element $x \in X$ to the set $A$ respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x \in X$. 
Definition 2.2. Let $A$ and $B$ are IFS of the form $A = < x, \mu_A(x), \nu_A(x) >; x \in X$ and $B = < x, \mu_B(x), \nu_B(x) >; x \in X$. Then

(i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$;

(ii) $\overline{A}$ (or $A^c$) = $\{ < x, \nu_A(x), \mu_A(x) >; x \in X \}$;

(iii) $A \cap B = \{ < x, \mu_A(x) \wedge \mu_B(x), \nu_A(x) \vee \nu_B(x) >; x \in X \}$;

(iv) $A \cup B = \{ < x, \mu_A(x) \vee \mu_B(x), \nu_A(x) \wedge \nu_B(x) >; x \in X \}$.

We will use the notation $A = \{ < x, \mu_A(x), \nu_A(x) >; x \in X \}$ instead of $A = \{ < x, \mu_A(x), \nu_A(x) >; x \in X \}$.

Definition 2.3. Let $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$. An intuitionistic fuzzy point (IFP) $p_{(\alpha, \beta)}$ is an intuitionistic fuzzy set defined by $p_{(\alpha, \beta)}(x) = \begin{cases} (\alpha, \beta), & \text{if } x = p ; \\ (0,1), & \text{otherwise.} \end{cases}$

Definition 2.4. Let $p_{(\alpha, \beta)}$ be an IFP in IFTS $X$. An IFS $A$ in $X$ is called an intuitionistic fuzzy neighborhood (IFN) of $p_{(\alpha, \beta)}$ if there exists an IFOS $B$ in $X$ such that $p_{(\alpha, \beta)} \subseteq B \subseteq A$.

Definition 2.5. An intuitionistic fuzzy topology (IFT) in Coker’s sense on a nonempty set $X$ is a family $\tau$ of intuitionistic fuzzy sets in $X$ satisfying the following axioms:

(i) $0_\tau \in \tau$;

(ii) $G_1 \cap G_2 \in \tau$, for any $G_1, G_2 \in \tau$;

(iii) $\cup G_i \in \tau$ for any arbitrary family $\{ G_i; i \in J \} \subseteq \tau$.

In this paper by $(X, \tau)$ or simply by $X$ we will denote the intuitionistic fuzzy topological space (IFTS). Each IFS which belongs to $\tau$ is called an intuitionistic fuzzy open set (IFOS) in $X$. The complement $\overline{A}$ of an IFOS $A$ in $X$ is called an intuitionistic fuzzy closed set (IFCS) in $X$.

Definition 2.6. Let $(X, \tau)$ be an IFTS and $A = \{ < x, \mu_A(x), \nu_A(x) >; x \in X \}$ be an IFS in $X$. Then the intuitionistic fuzzy interior and intuitionistic fuzzy closure of $A$ are defined by

(i) $\text{int}(A) = \bigcup \{ C : C \text{ is an IFCS in } X \text{ and } C \supseteq A \}$;

(ii) $\text{cl}(A) = \bigcap \{ D : D \text{ is an IFOS in } X \text{ and } D \subseteq A \}$.

It can be also shown that $\text{cl}(A)$ is an IFCS, $\text{int}(A)$ is an IFOS in $X$ and $A$ is an IFS in $X$ if and only if $\text{cl}(A) = A$; $A$ is an IFOS in $X$ if and only $\text{int}(A) = A$.

Proposition 2.1. Let $(X, \tau)$ be an IFTS and $A, B$ be IFSSs in $X$. Then the following properties hold:

(i) $\text{cl} \overline{A} = \overline{\text{int}(A)}$, $\text{int} \overline{A} = \overline{\text{cl}(A)}$;

(ii) $\text{int}(A) \subseteq A \subseteq \text{cl}(A)$.

Definition 2.7. An IFS $A$ in an IFTS $X$ is called an intuitionistic fuzzy pre open set (IFPOS) if $A \subseteq \text{int}(\text{cl}(A))$. The complement of an IFPOS $A$ in IFTS $X$ is called an intuitionistic fuzzy pre closed set (IFPCS) in $X$. 


Definition 2.8.\textsuperscript{[6]} An IFS $A$ in an IFTS $X$ is called an intuitionistic fuzzy $\alpha$-open set (IFoOS) if and only if $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$. The complement of an IFoOS $A$ in $X$ is called intuitionistic fuzzy $\alpha$-closed (IFoCS) in $X$.

Definition 2.9.\textsuperscript{[6]} An IFS $A$ in an IFTS $X$ is called an intuitionistic fuzzy semi open set (IFSOS) if and only if $A \subseteq \text{cl}(\text{int}(A))$. The complement of an IFSOS $A$ in $X$ is called intuitionistic fuzzy semi closed (IFSCS) in $X$.

Definition 2.10.\textsuperscript{[13]} An IFS $A$ in an IFTS $X$ is called an intuitionistic fuzzy $\beta$-open set (IF$\beta$OS) (otherwise called as intuitionistic Fuzzy semi pre-open set) if and only if $A \subseteq \text{cl}(\text{int}(cA))$. The complement of an IF$\beta$OS $A$ in $X$ is called intuitionistic fuzzy $\beta$-closed (IF$\beta$CS) in $X$.

Definition 2.11.\textsuperscript{[6,13]} Let $f$ be a mapping from an IFTS $X$ into an IFTS $Y$. The mapping $f$ is called:

(i) intuitionistic fuzzy continuous if and only if $f^{-1}(B)$ is an IFOS in $X$, for each IFOS $B$ in $Y$;

(ii) intuitionistic fuzzy $\alpha$-continuous if and only if $f^{-1}(B)$ is an IFoOS in $X$, for each IFOS $B$ in $Y$;

(iii) intuitionistic fuzzy pre continuous if and only if $f^{-1}(B)$ is an IFPOS in $X$, for each IFOS $B$ in $Y$;

(iv) intuitionistic fuzzy semi continuous if and only if $f^{-1}(B)$ is an IFSOS in $X$, for each IFOS $B$ in $Y$;

(v) intuitionistic fuzzy $\beta$-continuous if and only if $f^{-1}(B)$ is an IF$\beta$OS in $X$, for each IFOS $B$ in $Y$.

Definition 2.12.\textsuperscript{[12]} Let $(X, \tau)$ be an IFTS and $A = \{< x, \mu_A(x), \nu_A(x) >; x \in X\}$ be an IFS in $X$. Then the intuitionistic fuzzy $\beta$-closure and intuitionistic fuzzy $\beta$-interior of $A$ are defined by

(i) $\beta\text{cl}(A) = \bigcap \{C : C$ is an IF$\beta$CS in $X$ and $C \supseteq A\}$;

(ii) $\beta\text{int}(A) = \bigcup \{D : D$ is an IF$\beta$OS in $X$ and $D \subseteq A\}$.

Definition 2.13. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ from a intuitionistic fuzzy topological space $(X, \tau)$ to another intuitionistic fuzzy topological space $(Y, \sigma)$ is said to be intuitionistic fuzzy $\beta$-irresolute if $f^{-1}(B)$ is an IF$\beta$OS in $(X, \tau)$ for each IF$\beta$OS $B$ in $(Y, \sigma)$.

Definition 2.14.\textsuperscript{[11]} A function $f : (X, \tau) \rightarrow (Y, \sigma)$ from a intuitionistic fuzzy topological space $(X, \tau)$ to another intuitionistic fuzzy topological space $(Y, \sigma)$ is said to be intuitionistic fuzzy pre irresolute if $f^{-1}(B)$ is an IFPOS in $(X, \tau)$ for each IFPOS $B$ in $(Y, \sigma)$.

Definition 2.15.\textsuperscript{[11]} A function $f : (X, \tau) \rightarrow (Y, \sigma)$ from a intuitionistic fuzzy topological space $(X, \tau)$ to another intuitionistic fuzzy topological space $(Y, \sigma)$ is said to be intuitionistic fuzzy pre-$\alpha$-irresolute if $f^{-1}(B)$ is an IFPOS in $(X, \tau)$ for each IFoOS $B$ in $(Y, \sigma)$.

Definition 2.16.\textsuperscript{[4,9]} Let $X$ be an IFTS. A family of $\{< x, \mu_G(x), \nu_G(x) >; i \in J\}$ intuitionistic fuzzy open sets (intuitionistic fuzzy pre-open sets) in $X$ satisfies the condition $I_\alpha = \cup \{< x, \mu_G(x), \nu_G(x) >; i \in J\}$ is called a intuitionistic fuzzy open (intuitionistic fuzzy pre-open) cover of $X$. A finite subfamily of a intuitionistic fuzzy open (intuitionistic fuzzy pre-open) cover $\{< x, \mu_G(x), \nu_G(x) >; i \in J\}$ of $X$ which is also a intuitionistic fuzzy open (intuitionistic fuzzy pre-open) cover of $X$ is called a finite subcover of $\{< x, \mu_G(x), \nu_G(x) >; i \in J\}$.
\[ i \in J \].

**Definition 2.17.** An IFTS \( X \) is called intuitionistic fuzzy precompact (pre Lindelof) if each intuitionistic fuzzy pre-open cover of \( X \) has a finite (countable) subcover for \( X \).

### §3. Intuitionistic Fuzzy pre-\( \beta \)-irresolute functions

**Definition 3.1.** A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) from an intuitionistic fuzzy topological space \((X, \tau)\) to another intuitionistic fuzzy topological space \((Y, \sigma)\) is said to be intuitionistic fuzzy pre-\( \beta \)-irresolute if \( f^{-1}(\beta) \) is an IFPOS in \((X, \tau)\) for each IF\( \beta OS \) \( B \) in \((Y, \sigma)\).

\[
\begin{array}{ccc}
\text{IF pre-\( \alpha \)-irresolute} & \text{IF pre continuous} \\
\downarrow & \downarrow \\
\text{IF pre-\( \beta \)-irresolute} & \downarrow & \downarrow \\
\text{IF pre irresolute} & \text{IF \( \beta \)-irresolute} & \text{IF \( \beta \)-continuous}
\end{array}
\]

**Proposition 3.1.** Every intuitionistic fuzzy pre-\( \beta \)-irresolute function is an intuitionistic fuzzy pre-\( \alpha \)-irresolute function.

**Proof.** Follows from the definitions.

However, the converse of the above proposition 3.1 needs not to be true, as shown by the following example.

**Example 3.1.** Let \( X = \{a, b\}, Y = \{c, d\}, \tau = \{0_-, 1_-, A\}, \sigma = \{0_-, 1_-, B\} \) where

\[
A = \{< x, (\frac{4}{5}, 0.7), (\frac{3}{4}, 0.7) >; x \in X\},
B = \{< y, (\frac{3}{5}, 0.7), (\frac{3}{4}, 0.5) >; y \in Y\},
C = \{< y, (\frac{4}{5}, 0.3), (\frac{1}{2}, 0.3) >; y \in Y\}.
\]

Define an intuitionistic fuzzy mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = c, f(b) = d \). \( A \) is an IF\( \alpha OS \) in \((Y, \sigma)\), since \( B \subseteq \text{int}(cl(int(B))) = B \). \( f^{-1}(B) = \{< x, (\frac{4}{5}, 0.7), (\frac{3}{4}, 0.7) >; x \in X\}, \) and \( \text{int}(cl(f^{-1}(B))) = 1_- \). Thus \( f^{-1}(B) \subseteq \text{int}(cl(f^{-1}(B))) \). Hence \( f^{-1}(B) \) is IFPOS in \( X \), which implies \( f \) is IF pre-\( \alpha \)-irresolute. \( C \) is an IF\( \beta OS \) in \((Y, \sigma)\) since \( C \subseteq cl(int(cl(C))) = 0_- \). \( f^{-1}(C) = \{< x, (\frac{4}{5}, 0.7), (\frac{3}{4}, 0.7) >; x \in X\} \) and \( \text{int}(cl(f^{-1}(C))) = 0_- \). So, \( f^{-1}(C) \) is an IFPOS in \( X \). Hence \( f \) is not IF pre-\( \beta \)-irresolute.

**Proposition 3.2.** Every intuitionistic fuzzy pre-\( \beta \)-irresolute is an intuitionistic fuzzy pre continuous.

**Proof.** Follows from the definitions.

However the converse of the above proposition 3.2 needs not to be true, in general as shown by the following example.

**Example 3.2.** Let \( X = \{a, b\}, Y = \{c, d\}, \tau = \{0_-, 1_-, A\}, \sigma = \{0_-, 1_-, B\} \) where

\[
A = \{< x, (\frac{4}{5}, 0.7), (\frac{3}{4}, 0.7) >; x \in X\},
B = \{< y, (\frac{3}{5}, 0.7), (\frac{3}{4}, 0.5) >; y \in Y\},
C = \{< y, (\frac{4}{5}, 0.3), (\frac{1}{2}, 0.3) >; y \in Y\}.
\]
Define an intuitionistic fuzzy mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = c, f(b) = d \). \( B \) is an \( IFOS \) in \((Y, \sigma)\). \( f^{-1}(B) = \{< x, (\frac{a}{0.6}, \frac{b}{0.3}) > | x \in X \} \) is an \( IFOS \) in \((X, \tau)\), since \( int(cl^{-1}(B)) = 1_\tau \) and \( f^{-1}(B) \subseteq int(clf^{-1}(B)) \). Hence \( f \) is an \( IF \) pre-continuous. By previous example, \( f \) is not \( IF \) pre-\( \beta \)-irresolute.

**Proposition 3.3.** Every intuitionistic fuzzy pre-\( \beta \)-irresolute is an intuitionistic fuzzy pre-irresolute function.

**Proof.** Follows from the definitions.

However the converse of the above proposition 3.3 needs not to be true, as shown by the following example.

**Example 3.3.** Let \( X = \{a, b\}, Y = \{c, d\}, \tau = \{0_\tau, 1_\tau, A\}, \sigma = \{0_\sigma, 1_\sigma, B\} \) where

\[
A = \{< x, (\frac{a}{0.7}, \frac{b}{0.4}) > | x \in X \},
\]

\[
B = \{< y, (\frac{c}{0.7}, \frac{d}{0.4}) > | y \in Y \},
\]

\[
C = \{< y, (\frac{d}{0.7}, \frac{c}{0.4}) > | y \in Y \}.
\]

Define an intuitionistic fuzzy mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = c, f(b) = d \). \( B \) is an \( IFOS \) in \((Y, \sigma)\), and \( B \) is an \( IFPOS \) in \( Y \), since \( B \subseteq int(cl(B)) \). \( f^{-1}(B) = \{< x, (\frac{a}{0.7}, \frac{b}{0.4}) > | x \in X \} \), and \( int(clf^{-1}(B)) = 1_\tau \). Thus \( f^{-1}(B) \subseteq int(clf^{-1}(B)) \). Hence \( f^{-1}(B) \) is \( IFPOS \) in \( X \), which implies \( f \) is \( IF \) pre-irresolute. \( C \) is an \( IFS \) in \( Y \). \( cl(int(cl(c))) = B^c \). Hence \( C \subseteq cl(int(cl(C))) \). Thus \( C \) is \( IF\beta OS \) in \( Y \). \( f^{-1}(C) = \{< x, (\frac{a}{0.7}, \frac{b}{0.4}) > | x \in X \} \). And \( int(clf^{-1}(C)) = 0_\sigma \). Since \( f^{-1}(C) \nsubseteq int(clf^{-1}(C)) \), \( f^{-1}(C) \) is not an \( IFPOS \) in \( X \). Hence \( f \) is not \( IF \) pre-\( \beta \)-irresolute.

**Proposition 3.4.** Every intuitionistic fuzzy pre-\( \beta \)-irresolute is an intuitionistic fuzzy \( \beta \)-irresolute function.

**Proof.** Follows from the definitions.

However the converse of the above proposition 3.4 needs not to be true, in general as shown by the following example.

**Example 3.4.** Let \( X = \{a, b\}, Y = \{c, d\}, \tau = \{0_\tau, 1_\tau, A\}, \sigma = \{0_\sigma, 1_\sigma, B\} \) where

\[
A = \{< x, (\frac{a}{0.7}, \frac{b}{0.4}) > | x \in X \},
\]

\[
B = \{< y, (\frac{c}{0.7}, \frac{d}{0.4}) > | y \in Y \}.
\]

Define an intuitionistic fuzzy mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = d, f(b) = c \). \( B \) is an \( IF\beta OS \) in \((Y, \sigma)\) since \( B \subseteq cl(int(cl(B))) = B^c \), and \( f^{-1}(B) = \{< x, (\frac{a}{0.7}, \frac{b}{0.4}) > | x \in X \} \), since \( f^{-1}(B) \subseteq cl(int(clf^{-1}(B))) \). Thus \( f^{-1}(B) \) is an \( IF\beta OS \) in \( X \). Thus \( f \) is an \( IF\beta \)-irresolute. \( B \) is an \( IF\beta OS \) in \((Y, \sigma)\), and \( int(clf^{-1}(B)) = A^c \). So \( f^{-1}(B) \) is an \( IF\beta OS \) in \( X \). Hence \( f^{-1}(B) \) is not \( IFPOS \) in \( X \). So \( f \) is not \( IF \) pre-\( \beta \)-irresolute function.

**Proposition 3.5.** Every intuitionistic fuzzy pre-\( \beta \)-irresolute function is an intuitionistic fuzzy \( \beta \)-continuous function.

**Proof.** Follows from the definitions.

However the converse of the above proposition 3.5 needs not to be true, as shown by the following example.

**Example 3.5.** Let \( X = \{a, b\}, Y = \{c, d\}, \tau = \{0_\tau, 1_\tau, A\}, \sigma = \{0_\sigma, 1_\sigma, B\} \) where

\[
A = \{< x, (\frac{a}{0.7}, \frac{b}{0.4}) > | x \in X \}.
\]
Define an intuitionistic fuzzy mapping \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = c, \ f(b) = d. \) \( B \) is an \( IFOS \) in \( (Y, \sigma). \) \( f^{-1}(B) = \{ < x, (\frac{a}{b}, \frac{b}{c}) > \mid x \in X \} \) and \( cl(int(clf^{-1}(B))) = 1. \) Thus \( f^{-1}(B) \subseteq cl(int(clf^{-1}(B))). \) Hence \( f^{-1}(B) \) is \( IFBOS \) in \( X, \) which implies \( f \) is \( IF \) \( \beta \)-continuous function. \( C \) is an \( IFBOS \) in \( (Y, \sigma) \) since \( C \subseteq cl(int(cl(C))) = 0. \) \( f^{-1}(C) = \{ < x, (\frac{a}{b}, \frac{b}{c}) > \mid x \in X \} \) and \( int(clf^{-1}(C))) = 0. \) So, \( f^{-1}(C) \not\subseteq int(clf^{-1}(C))). \) Thus \( f^{-1}(C) \) is not \( IFPOS \) in \( X. \) Hence \( f \) is not \( IF \) \( pre-\beta \)-irresolute.

**Theorem 3.1.** \( IF \) \( f \) is a function from an \( IFTS \ X, \tau \) to another \( IFTS \ Y, \sigma \) then the followings are equivalent:

(a) \( f \) is \( IF \) \( pre-\beta \)-irresolute,
(b) \( f^{-1}(B) \subseteq int(clf^{-1}(B)) \) for every \( IFBOS \) \( B \) in \( Y,
(c) \( f^{-1}(C) \) is \( IF \) \( pre \) closed in \( X \) for every \( IF \beta \)closed set \( C \) in \( Y,
(d) \( cl(intf^{-1}(D)) \subseteq f^{-1}(\beta cl(D)) \) for every \( IFS \) \( D \) of \( Y,
(e) \( f(cl(int(E))) \subseteq \beta cl(f(E)) \) for every \( IFS \) \( E \) of \( X.

**Proof.** (a) \( \Rightarrow \) (b). Let \( B \) be \( IFBOS \) in \( Y. \) By (a), \( f^{-1}(B) \) is \( IF \) pre open in \( X. \) \( \Rightarrow f^{-1}(B) \subseteq int(clf^{-1}(B)). \) Hence (a) \( \Rightarrow \) (b) is proved.

(b) \( \Rightarrow \) (c). Let \( C \) be any \( IF \beta CS \) in \( Y, \) then \( C \) be \( IFBOS \) in \( Y. \) By (b), \( f^{-1}(C) \subseteq int(clf^{-1}(C)). \) But \( f^{-1}(C) \subseteq int(cl(f^{-1}(C))) = \beta cl(clf^{-1}(C)) \) which implies \( f^{-1}(C) \subseteq \beta cl(clf^{-1}(C)). \) Hence \( f^{-1}(C) \) is \( IF \) pre closed in \( X. \) Hence (b) \( \Rightarrow \) (c) is proved.

(c) \( \Rightarrow \) (d). Let \( D \) be \( IFS \) in \( X, \) then \( \beta cl(D) \) is a \( IF \) \( \beta \)closed in \( Y. \) \( \Rightarrow f^{-1}(\beta cl(D)) \) is \( IF \) pre closed in \( X. \) Then \( cl(intf^{-1}(\beta cl(D))) \subseteq f^{-1}(\beta cl(D)). \) Thus we have \( cl(intf^{-1}(D)) \subseteq f^{-1}(\beta cl(D)). \) Hence (c) \( \Rightarrow (d) \) is proved.

(d) \( \Rightarrow \) (e). Let \( E \) be an \( IFCS \) in \( X. \) \( cl(int(E)) \subseteq cl(int(f^{-1}(f(E)))) \subseteq cl(int(f^{-1}(\beta cl(f(E))))) \subseteq f^{-1}(\beta cl(f(E))))) \), then \( cl(int(E)) \subseteq f^{-1}(\beta cl(f(E))). \) We get \( f(cl(int(E))) \subseteq \beta cl(f(E)). \) Hence (d) \( \Rightarrow \) (e) is proved.

(e) \( \Rightarrow \) (a). Let \( B \) be \( IF \beta \)open set in \( Y. \) Then \( f^{-1}(1B) = \beta f^{-1}(B) \) is an \( IFCS \) in \( X. \) By (e), \( f(cl(int(f^{-1}(B)))) \subseteq \beta cl(f^{-1}(B)) \subseteq \beta cl(B) = \beta intB. \)

Thus

\[
f(cl(int(f^{-1}(B)))) \subseteq \beta intB.
\]

Consider,

\[
\int(cl(f^{-1}(B))) = cl(cl(f^{-1}(B))) = cl(cl(intf^{-1}(B)))) = cl(int(clf^{-1}(B)))
\]

\[
\subseteq f^{-1}(f(cl(int(clf^{-1}(B))))).
\]

By (1), (2), \( \int(cl(f^{-1}(B))) \subseteq f^{-1}(f(cl(intf^{-1}(B)))) \subseteq f^{-1}(B) = \beta intB. \) Thus, \( f^{-1}(B) \subseteq int(clf^{-1}(B)). \) Hence \( f^{-1}(B) \) is an \( IF \) pre open in \( X. \) Therefore \( f \) is \( IF \) \( pre-\beta \)-irresolute. Hence (e) \( \Rightarrow \) (a) is proved.
§4. Properties of intuitionistic fuzzy pre-\(\beta\)-irresolute functions

The following four lemmas are given here for convenience of the reader.

**Lemma 4.1.**[4] Let \( f : X \to Y \) be a mapping, and \( A_\alpha \) be a family of IF sets of \( Y \). Then
(a) \( f^{-1}(\bigcup A_\alpha) = \bigcup f^{-1}(A_\alpha) \),
(b) \( f^{-1}(\bigcap A_\alpha) = \bigcap f^{-1}(A_\alpha) \).

**Lemma 4.2.**[7] Let \( f : X_i \to Y_i \) be a mapping and \( A, B \) are IFSs of \( Y_1 \) and \( Y_2 \) respectively then \( (f_1 \times f_2)^{-1}(A \times B) = f_1^{-1}(A) \times f_2^{-1}(B) \).

**Lemma 4.3.**[7] Let \( g : X \to X \times Y \) be a graph of a mapping \( f : (X, \tau) \to (Y, \sigma) \). If \( A \) and \( B \) are IFSs of \( X \) and \( Y \) respectively, then \( g^{-1}(1_\times \times B) = (1_\times \cap f^{-1}(B)) \).

**Lemma 4.4.**[7] Let \( X \) and \( Y \) be intuitionistic fuzzy topological spaces, then \( (X, \tau) \) is product related to \( (Y, \sigma) \) if for any IFS \( C \) in \( X \), \( D \) in \( Y \) whenever \( \overline{\nearrow} A \supseteq \overline{\nearrow} C \), \( \overline{\nearrow} B \supseteq \overline{\nearrow} D \) implies \( \overline{\nearrow} A \times 1_\times \cup 1_\times \times \overline{\nearrow} B \supseteq \overline{\nearrow} C \times D \) there exists \( A_1 \in \tau \), \( B_1 \in \sigma \) such that \( \overline{\nearrow} A_1 \supseteq \overline{\nearrow} C \) and \( \overline{\nearrow} B_1 \supseteq \overline{\nearrow} D \) and \( \overline{\nearrow} A_1 \times 1_\times \times \overline{\nearrow} B_1 = \overline{\nearrow} A \times 1_\times \cup 1_\times \times \overline{\nearrow} B \).

**Lemma 4.5.** Let \( X \) and \( Y \) be intuitionistic fuzzy topological spaces such that \( X \) is product related to \( Y \). Then the product \( A \times B \) of \( IF\beta OS \) \( A \) in \( X \) and a \( IF\beta OS \) \( B \) in \( Y \) is a \( IF\beta OS \) in Fuzzy product spaces \( X \times Y \).

**Theorem 4.1.** Let \( f : X \to Y \) be a function and assume that \( X \) is product related to \( Y \). If the graph \( g : X \to X \times Y \) of \( f \) is \( IF \) pre-\(\beta\)-irresolute, so is \( f \).

**Proof.** Let \( B \) be \( IF\beta OS \) in \( Y \). Then by lemma 4.3 \( f^{-1}(B) = 1_\times \cap f^{-1}(B) = g^{-1}(1_\times \times B) \).

Now \( 1_\times \times B \) is a \( IF\beta OS \) in \( X \times Y \). Since \( g \) is \( IF \) pre-\(\beta\)-irresolute then \( g^{-1}(1_\times \times B) \) is \( IF \) pre open in \( X \). Hence \( f^{-1}(B) \) is \( IF \) pre open in \( X \). Thus \( f \) is \( IF \) pre-\(\beta\)-irresolute.

**Theorem 4.2.** If a function \( f : X \to \Pi Y_i \) is a \( IF \) pre-\(\beta\)-irresolute, then \( P_i \circ f : X \to Y_i \) is \( IF \) pre-\(\beta\)-irresolute, where \( P_i \) is the projection of \( \Pi Y_i \) onto \( Y \).

**Proof.** Let \( B_i \) be any \( IF\beta OS \) of \( Y_i \). Since \( P_i \) is \( IF \) continuous and \( IFOS \), it is \( IF\beta OS \).

Now \( P_i : \Pi Y_i \to Y_i \); \( P_i^{-1}(B_i) \) is \( IF\beta OS \) in \( \Pi Y_i \). Therefore, \( P_i \) is \( IF \) \( \beta \)-irresolute function. Now \( (P_i \circ f)^{-1}(B_i) = f^{-1}(P_i^{-1}(B_i)) \), since \( f \) is \( IF \) pre-\(\beta\)-irresolute and \( P_i^{-1}(B_i) \) is \( IF\beta OS \), \( f^{-1}(P_i^{-1}(B_i)) \) is \( IFPOS \) in \( X \). Hence \( (P_i \circ f) \) is \( IF \) pre-\(\beta\)-irresolute.

**Theorem 4.3.** If \( f_1 : X_1 \to Y_i, (i = 1, 2) \) are \( IF \) pre-\(\beta\)-irresolute and \( X_1 \) is product related to \( X_2 \) then \( f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2 \) is \( IF \) pre-\(\beta\)-irresolute.

**Proof.** Let \( C = \bigcup(A_i \times B_i) \) where \( A_i \) and \( B_i \) are \( IF \beta \) open sets of \( Y_1 \) and \( Y_2 \) respectively. Since \( Y_1 \) is product related to \( Y_2 \), by previous lemma 4.5, that \( C = \bigcup(A_i \times B_i) \) is \( IF\beta \)-open of \( Y_1 \times Y_2 \). Then by lemma 4.1.4 and 4.2 we have \( (f_1 \times f_2)^{-1}(C) = (f_1 \times f_2)^{-1}(A_i \times B_i) = \bigcup (f_1^{-1}(A_i) \times f_2^{-1}(B_i)) \). Since \( f_1 \) and \( f_2 \) are \( IF \) pre-\(\beta\)-irresolute, \( (f_1 \times f_2)^{-1}(C) \) is an \( IFPOS \) in \( X_1 \times X_2 \) and hence \( f_1 \times f_2 \) is \( IF \) pre-\(\beta\)-irresolute function.

**Definition 4.1.**[5] Let \( (X, \tau) \) be any \( IFTS \) and let \( A \) be any \( IFS \) in \( X \). Then \( A \) is called \( IF \) dense set if \( cl(A) = 1_\times \) and \( A \) is called nowhere \( IF \) dense set if \( int(cl(A)) = 0_\times \).

**Theorem 4.4.** If a function \( f : (X, \tau) \to (Y, \sigma) \) is \( IF \) pre-\(\beta\)-irresolute, then \( f^{-1}(A) \) is \( IF \) pre closed in \( X \) for any nowhere \( IF \) dense set \( A \) of \( Y \).

**Proof.** Let \( A \) be any nowhere \( IF \) dense set in \( Y \). Then \( int(cl(A)) = 0_\times \). Now, \( \overline{int(cl(A))} = 1_\times \Rightarrow cl(\overline{int(A)}) = 1_\times \) which implies \( cl(int(\overline{A})) = 1_\times \).
Since \( \text{int}1_\sim = 1_\sim, \text{cl}(\text{int}(\bar{A})) \subseteq \text{cl}(\text{int}(\bar{A})) \). So \( \bar{A} \subseteq \text{cl}(\text{int}(\bar{A})) \). Hence \( \bar{A} \subseteq \text{cl}(\text{int}(\bar{A})) \). Then \( \bar{A} \) is \( IFS\beta OS \) in \( Y \). Since \( f \) is \( IF \) \( \beta \)-irresolute, \( f^{-1}(\bar{A}) \) is \( IF \) \( \beta \)-irresolute in \( X \).

**Theorem 4.5.** A mapping \( f : X \to Y \) from an \( IFTS \) \( X \) into an \( IFTS \) \( Y \) is \( IF \) \( \beta \)-irresolute if and only if for each \( IFP \) \( p_{\alpha,\beta} \) in \( X \) and \( IFOS \) \( B \) in \( Y \) such that \( f(p_{\alpha,\beta}) \subseteq B \), there exists an \( IFTPOS \) \( A \) in \( X \) such that \( p_{\alpha,\beta} \in A \) and \( f(A) \subseteq B \).

**Proof.** Let \( f \) be any \( IF \) \( \beta \)-irresolute mapping, \( p_{\alpha,\beta} \) be an \( IFP \) in \( X \) and \( B \) be any \( IFOS \) in \( Y \) such that \( f(p_{\alpha,\beta}) \subseteq B \). Then \( p_{\alpha,\beta} \in f^{-1}(B) \). Let \( A = f^{-1}(B) \). Then \( A \) is an \( IFTPOS \) in \( X \) which containing \( IFP \) \( p_{\alpha,\beta} \) and \( f(A) = f(f^{-1}(B)) \subseteq B \).

Conversely, let \( B \) be an \( IFOS \) in \( Y \) and \( p_{\alpha,\beta} \) be \( IFP \) in \( X \) such that \( p_{\alpha,\beta} \in f^{-1}(B) \). According to assumption there exists \( IFPOS \) \( A \) in \( X \) such that \( p_{\alpha,\beta} \in A \) and \( f(A) \subseteq B \). Hence \( p_{\alpha,\beta} \in A \subseteq f^{-1}(B) \). We have \( p_{\alpha,\beta} \in A \subseteq \text{int}(clA) \subseteq \text{int}(clf^{-1}(B)) \). Therefore, \( f^{-1}(B) \subseteq \text{int}(clf^{-1}(B)) \). So \( f \) is \( IF \) \( \beta \)-irresolute mapping.

**Theorem 4.6.** A mapping \( f : X \to Y \) from an \( IFTS \) \( X \) into an \( IFTS \) \( Y \) is \( IF \) \( \beta \)-irresolute if and only if for each \( IFP \) \( p_{\alpha,\beta} \) in \( X \) and \( IFOS \) \( B \) in \( Y \) such that \( f(p_{\alpha,\beta}) \subseteq B \), \( cl(f^{-1}(B)) \) is \( IFN \) of \( IFP \) \( p_{\alpha,\beta} \) in \( X \).

**Proof.** Let \( f \) be any \( IF \) \( \beta \)-irresolute mapping, \( p_{\alpha,\beta} \) be an \( IFP \) in \( X \) and \( B \) be any \( IFOS \) in \( Y \) such that \( f(p_{\alpha,\beta}) \subseteq B \). Then \( p_{\alpha,\beta} \in f^{-1}(B) \subseteq \text{int}(cl(f^{-1}(B))) \subseteq cl(f^{-1}(B)). \)

Hence \( cl(f^{-1}(B)) \) is \( IFN \) of \( p_{\alpha,\beta} \) in \( X \).

Conversely, let \( B \) be any \( IFOS \) in \( Y \) and \( p_{\alpha,\beta} \) be \( IFP \) in \( X \) such that \( f(p_{\alpha,\beta}) \subseteq B \). Then \( p_{\alpha,\beta} \subseteq f^{-1}(B) \). According to assumption \( cl(f^{-1}(B)) \) is \( IFN \) of \( IFP \) \( p_{\alpha,\beta} \) in \( X \). So \( cl(f^{-1}(B)) \subseteq \text{int}(cl(f^{-1}(B))) \). Hence \( f^{-1}(B) \) is \( IFPOS \) in \( X \). Therefore \( f \) is \( IF \) \( \beta \)-irresolute.

**Theorem 4.7.** The followings hold for functions \( f : X \to Y \) and \( g : Y \to Z \).

(i) If \( f \) is \( IF \) \( \beta \)-irresolute and \( \beta \)-irresolute then \( g \circ f \) is \( IF \) \( \beta \)-irresolute.

(ii) If \( f \) is \( IF \) \( \beta \)-irresolute and \( \beta \)-irresolute then \( g \circ f \) is \( IF \) \( \beta \)-continuous.

(iii) If \( f \) is \( IF \) \( \beta \)-irresolute and \( \beta \)-irresolute then \( g \circ f \) is \( IF \) \( \beta \)-irresolute.

**Proof.** (i) Let \( B \) be an \( IFOS \) in \( Z \). Since \( g \) is \( IF \) \( \beta \)-irresolute, \( g^{-1}(B) \) is \( IFOS \) in \( Y \). Now \( (g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B)) \). Since \( f \) is \( IF \) \( \beta \)-irresolute, \( f^{-1}(g^{-1}(B)) \) is \( IFPOS \) in \( X \). Hence \( g \circ f \) is \( IF \) \( \beta \)-irresolute.

(ii) Let \( B \) be \( IFOS \) in \( Z \). Since \( g \) is \( IF \) \( \beta \)-continuous, \( g^{-1}(B) \) is \( IFOS \) in \( Y \). Now \( (g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B)) \). Since \( f \) is \( IF \) \( \beta \)-irresolute, \( f^{-1}(g^{-1}(B)) \) is \( IFPOS \) in \( X \) which implies \( g \circ f \) is \( IF \) \( \beta \)-continuous.

(iii) Let \( B \) be an \( IFOS \) in \( Z \). Since \( g \) is \( IF \) \( \beta \)-irresolute, \( g^{-1}(B) \) is \( IFPOS \) in \( Y \). Now \( (g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B)) \). Since \( f \) is \( IF \) \( \beta \)-irresolute, \( f^{-1}(g^{-1}(B)) \) is \( IFPOS \) in \( X \). Hence \( g \circ f \) is \( IF \) \( \beta \)-irresolute.

§5. Preservation of some intuitionistic fuzzy topological structure

**Definition 5.1.** An IFTS \( (X, \tau) \) is \( IFS\beta \)-disconnected if there exists intuitionistic fuzzy \( \beta \)-open sets \( A, B \) in \( X \), \( A \not\subseteq 1_\sim, B \not\subseteq 1_\sim \) such that \( A \cup B = 1_\sim \) and \( A \cap B = 0_\sim \). If \( X \) is not
**Definition 5.2.** Let $X$ be an IFTS. A family of $\{< x, \mu_{G_i}(x), \nu_{G_i}(x) >; i \in J \}$ intuitionistic fuzzy $\beta$-open sets in $X$ satisfies the condition $1_\sim = \bigcup \{< x, \mu_{G_i}(x), \nu_{G_i}(x) >; i \in J \}$ is called a $IF\beta$-cover of $X$. A finite subfamily of a $IF\beta$-open cover $\{< x, \mu_{G_i}(x), \nu_{G_i}(x) >; i \in J \}$ of $X$ which is also a $IF\beta$-open cover of $X$ is called a finite subcover of $\{< x, \mu_{G_i}(x), \nu_{G_i}(x) >; i \in J \}$.

**Definition 5.3.** A space $X$ is called Intuitionistic Fuzzy $\beta$-compact (Lindelof) if every intuitionistic fuzzy $\beta$-open cover of $X$ has a finite (countable) subcover.

**Theorem 5.1.** Every surjective Intuitionistic Fuzzy pre-$\beta$-irresolute image of a intuitionistic fuzzy precompact space is intuitionistic fuzzy $\beta$-compact.

**Proof.** Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be intuitionistic fuzzy pre-$\beta$-irresolute mapping of a intuitionistic fuzzy precompact space $(X, \tau)$ onto a space $(Y, \sigma)$. Let $\{G_i(i \in I)\}$ be any intuitionistic fuzzy pre-$\beta$-open cover of $(Y, \sigma)$. Then $\{f^{-1}(G_i)(i \in I)\}$ is a intuitionistic fuzzy pre open cover of $X$. Since $X$ is intuitionistic fuzzy precompact, there exists a finite subfamily $\{f^{-1}(G_{i_j})(j = 1, 2, \cdots, n)\}$ of $\{f^{-1}(G_i)(i \in I)\}$ which covers $X$. It follows that $\{G_{i_j}(j = 1, 2, \cdots, n)\}$ is a finite subfamily of $\{G_i(i \in I)\}$ which covers $Y$. Hence $Y$ is intuitionistic fuzzy $\beta$-compact.

**References**


On some trigonometric and hyperbolic functions evaluated on circulant matrices

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Abstract In this paper, we investigate the outcome when the trigonometric functions \( \sin x \), \( \cos x \), \( e^x \) and the hyperbolic functions \( \sinh x \) and \( \cosh x \) are evaluated on circulant matrices.

Keywords circulant matrix, trigonometric functions, hyperbolic functions.

2000 Mathematics Subject Classification: 54D15, 54D10

§1. Introduction

Given any sequence of numbers, \( c_0, c_1, \cdots, c_{n-1} \), we can form circulant matrices. From [2], circulant matrices have four types: the right circulant, the left circulant, the skew-right circulant and the skew-left circulant and take the following forms, respectively:

\[
RCIRC_n(\vec{c}) = \begin{bmatrix}
    c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
    c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2} \\
    c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\
    c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0
\end{bmatrix},
\]

(1)

\[
LCIRC_n(\vec{c}) = \begin{bmatrix}
    c_0 & c_1 & c_2 & \cdots & c_{n-2} & c_{n-1} \\
    c_1 & c_2 & c_3 & \cdots & c_{n-1} & c_0 \\
    c_2 & c_3 & c_4 & \cdots & c_0 & c_1 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-4} & c_{n-3} \\
    c_{n-1} & c_0 & c_1 & \cdots & c_{n-3} & c_{n-2}
\end{bmatrix},
\]

(2)
In each matrix, \( \bar{c} = (c_0, c_1, \cdots, c_{n-1}) \) is called the circulant vector.

The said matrices have significant applications in different fields. These fields include physics, image processing, probability and statistics, number theory, geometry, numerical solutions of ordinary and partial differential equations [2], frequency analysis, signal processing, digital encoding, graph theory [4], and time-series analysis [3].

\section{2. Preliminaries}

In this section, we shall use \( \text{diag}(c_0, c_1, \cdots, c_{n-1}) \) to denote the diagonal matrix

\[
\begin{pmatrix}
    c_0 & 0 & 0 & \cdots & 0 & 0 \\
    0 & c_1 & 0 & \cdots & 0 & 0 \\
    0 & 0 & c_2 & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & 0 & 0 & \cdots & c_{n-2} & 0 \\
    0 & 0 & 0 & \cdots & 0 & c_{n-1}
\end{pmatrix},
\]

and \( \text{adiag}(c_0, c_1, \cdots, c_{n-1}) \) to denote the anti diagonal matrix

\[
\begin{pmatrix}
    0 & 0 & 0 & \cdots & 0 & c_0 \\
    0 & 0 & 0 & \cdots & c_1 & 0 \\
    0 & 0 & 0 & \cdots & 0 & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & c_{n-2} & 0 & \cdots & 0 & 0 \\
    c_{n-1} & 0 & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]
Note that these matrices are not necessarily circulant.

Now, we define the Fourier matrix to establish the relationship of the circulant matrices.

**Definition 2.1.** The unitary matrix $F_n$ given by

$$F_n = \frac{1}{\sqrt{n}} \begin{pmatrix}
1 & 1 & 1 & \cdots & 1 & 1 \\
1 & \omega & \omega^2 & \cdots & \omega^{n-2} & \omega^{n-1} \\
1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-2)} & \omega^{2(n-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \omega^{n-2} & \omega^{2(n-2)} & \cdots & \omega^{(n-2)(n-2)} & \omega^{(n-1)(n-2)} \\
1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-2)(n-1)} & \omega^{(n-1)(n-1)}
\end{pmatrix},$$

(5)

where $\omega = e^{2i\pi/n}$ is called the Fourier matrix.

Here are the equations that show the relationship between the circulant matrices:

$$RCIRC_n(\tilde{c}) = F_nDF_n^{-1},$$

(6)

where $\tilde{c} = (c_0, c_1, \cdots, c_{n-1})$ and $D = \text{diag}(d_0, d_1, \cdots, d_{n-1})$.

$$LCIRC_n(\tilde{c}) = HIRCIRC_n(\tilde{c}),$$

(7)

where $\tilde{c} = (c_0, c_1, \cdots, c_{n-1})$, $D = \text{diag}(d_0, d_1, \cdots, d_{n-1})$, $\Pi = \begin{pmatrix}
1 & \mathbb{O}_1 \\
\mathbb{O}_1^T & \tilde{I}_{n-1}
\end{pmatrix}$ (an $n \times n$ matrix), $\tilde{I}_{n-1} = \text{adiag}(1, 1, \cdots, 1, 1)$ (an $(n-1) \times (n-1)$ matrix), $\mathbb{O}_1 = (0, 0, \cdots, 0, 0)$ (an $(n-1) \times 1$ matrix).

$$SRCIRC_n(\tilde{c}) = \Delta F_nDF_n^{-1} \Delta^{-1},$$

(8)

where $\tilde{c} = (c_0, c_1, \cdots, c_{n-1})$, $D = \text{diag}(d_0, d_1, \cdots, d_{n-1})$, $\Delta = \text{diag}(1, \theta, \cdots, \theta^{n-1})$, and $\theta = e^{i\pi/n}$.

$$SLCIRC_n(\tilde{c}) = \Sigma SRCIRC_n(\tilde{c}),$$

(9)

where $\tilde{c} = (c_0, c_1, \cdots, c_{n-1})$, $D = \text{diag}(d_0, d_1, \cdots, d_{n-1})$, $\Sigma = \begin{pmatrix}
1 & \mathbb{O}_1 \\
\mathbb{O}_1^T & -\tilde{I}_{n-1}
\end{pmatrix}$ (an $n \times n$ matrix), $\tilde{I}_{n-1} = \text{adiag}(1, 1, \cdots, 1, 1)$ (an $(n-1) \times (n-1)$ matrix), $\mathbb{O}_1 = (0, 0, \cdots, 0, 0)$.

The matrices $\Pi$ and $\Sigma$ are orthogonal meaning $\Pi = \Pi^T = \Pi^{-1}$ and $\Sigma = \Sigma^T = \Sigma^{-1}$. In case of right multiplication with these matrices, we have the following results from [1]:

1. $RCIRC_n(\tilde{c})\Pi = LCIRC_n(\tilde{\gamma})$ where $\tilde{c} = (c_0, c_1, \cdots, c_{n-1})$ and $\tilde{\gamma} = (c_0, c_{n-1}, c_{n-2}, \cdots, c_2, c_1)$.

2. $SRCIRC_n(\tilde{c})\Sigma = SLCIRC_n(\tilde{\rho})$ where $\tilde{c} = (c_0, c_1, \cdots, c_{n-1})$ and $\tilde{\rho} = (c_0, -c_{n-1}, -c_{n-2}, \cdots, -c_2, -c_1)$. 
§2. Preliminary results

For the rest of the paper we shall use the following notations:

1. The set of complex right circulant matrices
\[ \text{RCIRC}_n(\mathbb{C}) = \{ \text{RCIRC}_n(\vec{c}) | \vec{c} \in \mathbb{C}^n \} , \]

2. The set of complex left circulant matrices
\[ \text{LCIRC}_n(\mathbb{C}) = \{ \text{LCIRC}_n(\vec{c}) | \vec{c} \in \mathbb{C}^n \} , \]

3. The set of complex skew-right circulant matrices
\[ \text{SRCIRC}_n(\mathbb{C}) = \{ \text{SRCIRC}_n(\vec{c}) | \vec{c} \in \mathbb{C}^n \} , \]

4. The set of complex skew-left circulant matrices
\[ \text{SLCIRC}_n(\mathbb{C}) = \{ \text{SLCIRC}_n(\vec{c}) | \vec{c} \in \mathbb{C}^n \} , \]

5. \( e^x = E[x] \),

6. \( \sin x = S[x] \),

7. \( \cos x = C[x] \),

8. \( \sinh x = Sh[x] \),

9. \( \cosh x = Ch[x] \).

From [2], it has been shown that:

1. the sum of circulant matrices of the same type is a circulant matrix of the same type,

2. the product of right circulant matrices is a right circulant matrix,

3. the product of skew-right circulant matrices is a skew-right circulant matrix.

For the other products, we have the following lemmas which will be used to prove our results:

Lemma 2.1. The product of two left circulant matrices is a right circulant matrix.

Proof.
\[
\text{LCIRC}_n(\vec{a})\text{LCIRC}_n(\vec{b}) = \Pi \text{RCIRC}_n(\vec{a})\Pi \text{RCIRC}_n(\vec{b}) = \Pi \text{LCIRC}_n(\vec{a})\text{RCIRC}_n(\vec{b}) = \text{RCIRC}_n(\vec{a})\text{RCIRC}_n(\vec{b}) \in \text{RCIRC}_n(\mathbb{C}).
\]

Lemma 2.2. The product of a left circulant and a right circulant is a left circulant matrix.
Proof.
Case 1
\[
\text{RCIRC}_n(\vec{a})\text{LCIRC}_n(\vec{b}) = \text{RCIRC}_n(\vec{a})\Pi\text{RCIRC}_n(\vec{b}) = \text{LCIRC}_n(\vec{a})\text{RCIRC}_n(\vec{b}) = \Pi\text{RCIRC}_n(\vec{a})\text{RCIRC}_n(\vec{b}) \in \text{LCIRC}_n(\mathbb{C}).
\]
Case 2
\[
\text{LCIRC}_n(\vec{b})\text{RCIRC}_n(\vec{a}) = \Pi\text{RCIRC}_n(\vec{a})\text{RCIRC}_n(\vec{b}) \in \text{LCIRC}_n(\mathbb{C}).
\]

Lemma 2.3. The product of two skew-left circulant matrices is a skew-right circulant matrix.
Proof.
\[
\text{SLCIRC}_n(\vec{a})\text{SLCIRC}_n(\vec{b}) = \Sigma\text{SRCIRC}_n(\vec{a})\Sigma\text{SRCIRC}_n(\vec{b}) = \Sigma\text{SLCIRC}_n(\vec{a})\text{SRCIRC}_n(\vec{b}) = \text{SRCIRC}_n(\vec{a})\text{SRCIRC}_n(\vec{b}) \in \text{SRCIRC}_n(\mathbb{C}).
\]

Lemma 2.4. The product of a skew-left circulant and a skew-right circulant is a skew-left circulant matrix.
Proof.
Case 1
\[
\text{SRCIRC}_n(\vec{a})\text{SLCIRC}_n(\vec{b}) = \Sigma\text{SRCIRC}_n(\vec{a})\Sigma\text{SRCIRC}_n(\vec{b}) = \Sigma\text{SRCIRC}_n(\vec{a})\text{SRCIRC}_n(\vec{b}) \in \text{SLCIRC}_n(\mathbb{C}).
\]
Case 2
\[
\text{SLCIRC}_n(\vec{b})\text{SRCIRC}_n(\vec{a}) = \Sigma\text{SRCIRC}_n(\vec{a})\Sigma\text{SRCIRC}_n(\vec{b}) \in \text{SLCIRC}_n(\mathbb{C}).
\]

Lemma 2.5.
If \(k\) is odd, \(\text{LCIRC}_n^k(\vec{a}) \in \text{LCIRC}_n^k(\mathbb{C})\); 
If \(k\) is even, \(\text{LCIRC}_n^k(\vec{a}) \in \text{RCIRC}_n^k(\mathbb{C})\).

Lemma 2.6.
If \(k\) is odd, \(\text{SLCIRC}_n^k(\vec{a}) \in \text{SLCIRC}_n^k(\mathbb{C})\); 
If \(k\) is even, \(\text{SLCIRC}_n^k(\vec{a}) \in \text{SRCIRC}_n^k(\mathbb{C})\).
§3. Main results

**Theorem 3.1.** $E[RCIRC_n(\vec{c})]$ is a right circulant matrix.

**Proof.**

\[
E[RCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} \frac{RCIRC^k_n(\vec{c})}{k!} = \sum_{k=0}^{+\infty} \frac{F_n DF_n^{-1}}{k!} = F_n \left( \sum_{k=0}^{+\infty} \frac{D^k}{k!} \right) F_n^{-1}.
\]

It takes the form of Eq. 6, so it is a right circulant matrix.

**Theorem 3.2.** $E[LCIRC_n(\vec{c})]$ is a sum of right circulant matrix and a left circulant matrix.

**Proof.**

\[
E[LCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} \frac{LCIRC^k_n(\vec{c})}{k!} = \sum_{k=0}^{+\infty} \frac{LCIRC_{2k}^n(\vec{c})}{(2k)!} + \sum_{k=0}^{+\infty} \frac{LCIRC_{2k+1}^n(\vec{c})}{(2k+1)!}.
\]

The first summand is a right circulant matrix because the powers of $LCIRC_n(\vec{c})$ are even while the second summand is a left circulant matrix because the powers of $LCIRC_n(\vec{c})$ are odd.

**Theorem 3.3.** $E[SRCIRC_n(\vec{c})]$ is a skew-right circulant matrix.

**Proof.**

\[
E[SRCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} \frac{SRCIRC^k_n(\vec{c})}{k!} = \sum_{k=0}^{+\infty} \frac{\Delta F_n DF_n^{-1} \Delta^{-1}}{k!} = \Delta F_n \left( \sum_{k=0}^{+\infty} \frac{D^k}{k!} \right) F_n^{-1} \Delta^{-1}.
\]

It takes the form of Eq. 8, so it is a skew-right circulant matrix.

**Theorem 3.4.** $E[SLCIRC_n(\vec{c})]$ is a sum of skew-right circulant matrix and a skew-left circulant matrix.

**Proof.**

\[
E[SLCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} \frac{SLCIRC^k_n(\vec{c})}{k!} = \sum_{k=0}^{+\infty} \frac{SLCIRC_{2k}^n(\vec{c})}{(2k)!} + \sum_{k=0}^{+\infty} \frac{SLCIRC_{2k+1}^n(\vec{c})}{(2k+1)!}.
\]
The first summand is a skew-right circulant matrix because the powers of $SLCIRC_n(\vec{c})$ are even while the second summand is a skew-left circulant matrix because the powers of $SLCIRC_n(\vec{c})$ are odd.

**Theorem 3.5.** $S[RCIRC_n(\vec{c})]$ is a right circulant matrix.

**Proof.**

$$S[RCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} (-1)^k \frac{RCIRC_n^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{+\infty} (-1)^k \frac{F_n DF_n^{-1}^{2k+1}}{(2k+1)!}$$

$$= F_n \sum_{k=0}^{+\infty} (-1)^k \frac{D^{2k+1}}{(2k+1)!} F_n^{-1}.$$

It takes the form of Eq. 6, so it is a right circulant matrix.

**Theorem 3.6.** $S[LCIRC_n(\vec{c})]$ is a left circulant matrix.

**Proof.**

$$S[LCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} (-1)^k \frac{LCIRC_n^{2k+1}}{(2k+1)!}.$$

Since the powers of $LCIRC_n(vecc)$ are odd, it is a left circulant matrix.

**Theorem 3.7.** $S[SRCIRC_n(\vec{c})]$ is a skew-right circulant matrix.

**Proof.**

$$S[SRCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} (-1)^k \frac{SRCIRC_n^{2k+1}}{(2k+1)!}$$

$$= \sum_{k=0}^{+\infty} (-1)^k \frac{\Delta F_n DF_n^{-1} \Delta^{-1}^{2k+1}}{(2k+1)!}$$

$$= \Delta F_n \sum_{k=0}^{+\infty} (-1)^k \frac{D^{2k+1}}{(2k+1)!} F_n^{-1} \Delta^{-1}.$$

It takes the form of Eq. 8, so it is a skew-right circulant matrix.

**Theorem 3.8.** $S[SLCIRC_n(\vec{c})]$ is a skew-left circulant matrix.

**Proof.**

$$S[SLCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} (-1)^k \frac{SLCIRC_n^{2k+1}}{(2k+1)!}.$$

Since the powers of $SLCIRC_n(vecc)$ are odd, it is a left circulant matrix.

**Theorem 3.9.** $C[RCIRC_n(\vec{c})]$ is a right circulant matrix.
Proof.

\[ C[RCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} (-1)^k RCIRC_{2k}^n(\vec{c}) \] 
\[ = \sum_{k=0}^{+\infty} (-1)^k \frac{F_n D F_n^{-1}}{(2k)!}^{2k} \] 
\[ = F_n \left[ \sum_{k=0}^{+\infty} (-1)^k \frac{D^{2k}}{(2k)!} \right] F_n^{-1}. \]

It takes the form Eq. 6, so it is a right circulant matrix.

Theorem 3.10. \( C[LCIRC_n(\vec{c})] \) is a right circulant matrix.

Proof.

\[ C[LCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} (-1)^k \frac{LCIRC_{2k}^n(\vec{c})}{(2k)!}. \]

Since the powers of \( LCIRC_n(\vec{c}) \) are even, it is a right circulant matrix.

Theorem 3.11. \( C[SRCIRC_n(\vec{c})] \) is a skew-right circulant matrix.

Proof.

\[ C[SRCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} (-1)^k \frac{SRCIRC_{2k}^n(\vec{c})}{(2k)!}. \]

It takes the form Eq. 8, so it is a right circulant matrix.

Theorem 3.12. \( C[SLCIRC_n(\vec{c})] \) is a skew-right circulant matrix.

Proof.

\[ C[SLCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} (-1)^k \frac{SLCIRC_{2k}^n(\vec{c})}{(2k)!}. \]

Since the powers of \( SLCIRC_n(\vec{c}) \) are even, it is a skew-right circulant matrix.

Theorem 3.13. \( Sh[RCIRC_n(\vec{c})] \) is a right circulant matrix.

Proof.

\[ Sh[RCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} \frac{RCIRC_{2k+1}^n(\vec{c})}{(2k+1)!} \] 
\[ = \sum_{k=0}^{+\infty} \frac{F_n D F_n^{-1}}{(2k+1)!}^{2k+1} \] 
\[ = F_n \left[ \sum_{k=0}^{+\infty} \frac{D^{2k+1}}{(2k+1)!} \right] F_n^{-1}. \]
It takes the form of Eq. 6, so it is a right circulant matrix.

**Theorem 3.14.** \( Sh[LCIRC_n(\vec{c})] \) is a left circulant matrix.

**Proof.**

\[
Sh[LCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} \frac{[LCIRC_n^{2k+1}(\vec{c})]}{(2k+1)!}.
\]

Because the powers of \( LCIRC_n(\vec{c}) \) are odd, it is a left circulant matrix.

**Theorem 3.15.** \( Sh[SRCIRC_n(\vec{c})] \) is a skew-right circulant matrix.

**Proof.**

\[
Sh[SRCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} \frac{\Delta F_n [DF_n^{-\frac{1}{2}} \Delta^{-1}]^{2k+1}}{(2k+1)!} = \Delta F_n \left[ \sum_{k=0}^{+\infty} \frac{D^{2k+1}}{(2k+1)!} \right] F_n^{-\frac{1}{2}} \Delta^{-1}.
\]

It takes the form of Eq. 8, so it is a skew-right circulant matrix.

**Theorem 3.16.** \( Sh[SLCIRC_n(\vec{c})] \) is a skew-left circulant matrix.

**Proof.**

\[
Sh[SLCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} \frac{[SLCIRC_n^{2k+1}(\vec{c})]}{(2k+1)!}.
\]

Because the powers of \( SLCIRC_n(\vec{c}) \) are odd, it is a skew-left circulant matrix.

**Theorem 3.17.** \( Ch[RCIRC_n(\vec{c})] \) is a right circulant matrix.

**Proof.**

\[
Ch[RCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} \frac{[RCIRC_n^{2k}]}{(2k)!} = \sum_{k=0}^{+\infty} \frac{[F_n DF_n^{-\frac{1}{2}}]^{2k}}{(2k)!} = F_n \left[ \sum_{k=0}^{+\infty} \frac{D^{2k}}{(2k)!} \right] F_n^{-\frac{1}{2}}.
\]

It takes the form of Eq. 6, so it is a right circulant matrix.

**Theorem 3.18.** \( Ch[LCIRC_n(\vec{c})] \) is a right circulant matrix.

**Proof.**

\[
Ch[LCIRC_n(\vec{c})] = \sum_{k=0}^{+\infty} \frac{[LCIRC_n^{2k}]}{(2k)!}.
\]

Because the powers of \( LCIRC_n(\vec{c}) \) are even, it is a right circulant matrix.

**Theorem 3.19.** \( Ch[SRCIRC_n(\vec{c})] \) is a skew-right circulant matrix.
Proof.

\[
Ch[SRCRC_n(\vec{c})] = \sum_{k=0}^{+\infty} \frac{[SRCRC_n^{2k}]}{(2k)!}
\]

\[
= \sum_{k=0}^{+\infty} \frac{[\Delta F_n D F_n^{-1} \Delta^{-1}]^{2k}}{(2k)!}
\]

\[
= \Delta F_n \left[ \sum_{k=0}^{+\infty} \frac{D^{2k}}{(2k)!} \right] F_n^{-1} \Delta^{-1}.
\]

It takes the form of Eq. 8, so it is a skew-right circulant matrix.

**Theorem 3.20.** \(Ch[SLCRC_n(\vec{c})]\) is a skew-right circulant matrix.

**Proof.**

\[
Ch[SLCRC_n(\vec{c})] = \sum_{k=0}^{+\infty} \frac{[SLCRC_n^{2k}]}{(2k)!}.
\]

Because the powers of \(SLCRC_n(\vec{c})\) are even, it is a skew-right circulant matrix.

§4. Conclusion

Right and skew-right circulant matrices remains invariant on their type when evaluated on trigonometric and hyperbolic functions that are used in this paper. On the other hand, the left and skew-left circulant matrices change their type depending on the parity of the trigonometric and hyperbolic function.

References


Generalization of an intuitionistic fuzzy $G_{str}$ open sets in an intuitionistic fuzzy grill structure spaces

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Abstract The purpose of this paper is to introduce the concepts of an intuitionistic fuzzy grill, intuitionistic fuzzy $G$ structure space, intuitionistic fuzzy $\delta G_{str}$ set and intuitionistic fuzzy $\alpha G_{str}$ open set. The concepts of an intuitionistic fuzzy $E_{\alpha G_{str}}$ continuous function, intuitionistic fuzzy $\alpha G_{str}$-$T_i$ space, $i = 0, 1, 2$ and intuitionistic fuzzy $\alpha G_{str}$-co-closed graphs are defined. Some interesting properties are established.

Keywords Intuitionistic fuzzy grill, intuitionistic fuzzy $G$ structure space, intuitionistic fuzzy $\delta G_{str}$ set and intuitionistic fuzzy $\alpha G_{str}$, ($\alpha G_{str}$, semi $G_{str}$, pre $G_{str}$, regular $G_{str}$ and $\beta G_{str}$) open set, intuitionistic fuzzy $\alpha G_{str}$ exterior, intuitionistic fuzzy $E_{\alpha G_{str}}$ continuous function, intuitionistic fuzzy $\alpha G_{str}$-$T_i$ space, $i = 0, 1, 2$ and intuitionistic fuzzy $\alpha G_{str}$-co-closed graphs.

2010 Mathematics Subject Classification: 54A40, 03E72

§1. Introduction and preliminaries

The concept of fuzzy sets was introduced by Zadeh [7] and later Atanassov [1] generalized the idea to intuitionistic fuzzy sets. On the other hand, Coker [2] introduced the notions of an intuitionistic fuzzy topological spaces, intuitionistic fuzzy continuity and some other related concepts. The concept of an intuitionistic fuzzy $\alpha$-closed set was introduced by H. Gurcay and D. Coker [5]. The concept of fuzzy grill was introduced by Sumita Das, M. N. Mukherjee [6]. Erdal Ekici [4] studied slightly precontinuous functions, separation axioms and pre-co-closed graphs in fuzzy topological space. In this paper, the concepts of an intuitionistic fuzzy grill, intuitionistic fuzzy $G$ structure space, intuitionistic fuzzy $\delta G_{str}$ set and intuitionistic fuzzy $\alpha G_{str}$ open set are introduced. Some interesting properties of separation axioms in intuitionistic fuzzy grill structure space with intuitionistic fuzzy $E_{\alpha G_{str}}$ continuous function are established.

§2. Preliminaries

Definition 2.1, [1] Let $X$ be a nonempty fixed set and $I$ be the closed interval $[0, 1]$. An intuitionistic fuzzy set (IFS) $A$ is an object of the following form $A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \}$. The membership function $\mu_A(x)$ and non-membership function $\gamma_A(x)$ of $A$ at point $x$ must satisfy $0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$. If $\mu_A(x) > \gamma_A(x)$ for all $x \in X$, then $A$ is a fuzzy open set in $X$. If $\mu_A(x) + \gamma_A(x) = 1$ for all $x \in X$, then $A$ is a fuzzy closed set in $X$. If $\mu_A(x) = 1$ and $\mu_A(x) + \gamma_A(x) < 1$ for all $x \in X$, then $A$ is a fuzzy semi-open set in $X$. Similarly, $A$ is a fuzzy semi-closed set in $X$ if $\mu_A(x) = 0$ and $\mu_A(x) + \gamma_A(x) < 1$ for all $x \in X$. If $\mu_A(x) = 1$ and $\mu_A(x) + \gamma_A(x) < 1$ for all $x \in X$, then $A$ is a fuzzy regular-open set in $X$. If $\mu_A(x) = 1$ and $\mu_A(x) + \gamma_A(x) < 1$ for all $x \in X$, then $A$ is a fuzzy regular-closed set in $X$.
\(X\), where the mappings \(\mu_A : X \rightarrow I\) and \(\gamma_A : X \rightarrow I\) denote the degree of membership (namely \(\mu_A(x)\)) and the degree of nonmembership (namely \(\gamma_A(x)\)) for each element \(x \in X\) to the set \(A\), respectively, and \(0 \leq \mu_A(x) + \gamma_A(x) \leq 1\) for each \(x \in X\). Obviously, every fuzzy set \(A\) on a nonempty set \(X\) is an \(IFS\) of the following form, \(A = \{\langle x, \mu_A(x), 1 - \mu_A(x) \rangle : x \in X\}\).

For the sake of simplicity, we shall use the symbol \(A = \{\langle x, \mu_A, \gamma_A \rangle\}\) for the intuitionistic fuzzy set \(A = \{\langle x, \mu_A(x), \gamma_A(x) \rangle : x \in X\}\).

**Definition 2.2.** Let \(X\) be a nonempty set and the \(IFSs\) \(A\) and \(B\) in the form \(A = \{\langle x, \mu_A(x), \gamma_A(x) \rangle : x \in X\}\) and \(B = \{\langle x, \mu_B(x), \gamma_B(x) \rangle : x \in X\}\). Then

(i) \(A \subseteq B\) iff \(\mu_A(x) \leq \mu_B(x)\) and \(\gamma_A(x) \geq \gamma_B(x)\) for all \(x \in X\),

(ii) \(A \cap B = \{\langle x, \mu_A(x) \land \mu_B(x), \gamma_A(x) \lor \gamma_B(x) \rangle : x \in X\}\),

(iii) \(A \cup B = \{\langle x, \mu_A(x) \lor \mu_B(x), \gamma_A(x) \land \gamma_B(x) \rangle : x \in X\}\).

**Definition 2.3.** The \(IFSs\) \(0_\sim\) and \(1_\sim\) are defined by \(0_\sim = \{\langle x, 0, 1 \rangle : x \in X\}\) and \(1_\sim = \{\langle x, 1, 0 \rangle : x \in X\}\).

**Definition 2.4.** An intuitionistic fuzzy topology (\(IFT\)) in Coker’s sense on a nonempty set \(X\) is a family \(\tau\) of \(IFSs\) in \(X\) satisfying the following axioms:

(i) \(0_\sim, 1_\sim \in \tau\),

(ii) \(G_1 \cap G_2 \in \tau\) for any \(G_1, G_2 \in \tau\),

(iii) \(\cup G_i \in \tau\) for arbitrary family \(\{G_i \mid i \in I\} \subseteq \tau\).

In this case the ordered pair \((X, \tau)\) is called an intuitionistic fuzzy topological space (\(IFTS\)) on \(X\) and each \(IFS\) in \(\tau\) is called an intuitionistic fuzzy open set (\(IFOS\)). The complement \(\overline{A}\) of an \(IFS\) \(A\) in \(X\) is called an intuitionistic fuzzy closed set (\(IFCS\)) in \(X\).

**Definition 2.5.** Let \(A\) be an \(IFS\) in \(IFTS\) \(X\). Then \(\text{int}(A) = \bigcup\{G \mid G\) is an \(IFOS\) in \(X\) and \(G \subseteq A\}\) is called an intuitionistic fuzzy interior of \(A\); \(d(A) = \bigcap\{G \mid G\) is an \(IFCS\) in \(X\) and \(G \supseteq A\}\) is called an intuitionistic fuzzy closure of \(A\).

**Proposition 2.6.** For any \(IFS\) \(A\) in \((X, \tau)\) we have

(i) \(d(\overline{A}) = \overline{\text{int}(A)}\),

(ii) \(\text{int}(\overline{A}) = d(A)\).

**Corollary 2.1.** Let \(A, A_i (i \in J)\) be \(IFSs\) in \(X, B, B_j (j \in K)\) \(IFSs\) in \(Y\) and \(f : X \rightarrow Y\) a function. Then

(i) \(A \subseteq f^{-1}(f(A))\) (If \(f\) is injective, then \(A = f^{-1}(f(A))\)),

(ii) \(f(f^{-1}(B)) \subseteq B\) (If \(f\) is surjective, then \(f(f^{-1}(B)) = B\)),

(iii) \(f^{-1}(\bigcup B_j) = \bigcup f^{-1}(B_j)\),

(iv) \(f^{-1}(\bigcap B_j) = \bigcap f^{-1}(B_j)\),

(v) \(f^{-1}(1_\sim) = 1_\sim\),

(vi) \(f^{-1}(0_\sim) = 0_\sim\),

(vii) \(f^{-1}(B) = \overline{f^{-1}(B)}\).

**Definition 2.7.** Let \(X\) be a nonempty set and \(x \in X\) a fixed element in \(X\). If \(r \in \mathbb{I}_0, s \in \mathbb{I}_1\) are fixed real numbers such that \(r + s \leq 1\), then the \(IFS\) \(x_{r,s} = \{x, x_r, 1 - x_{1-s}\}\) is called an intuitionistic fuzzy point (\(IFP\)) in \(X\), where \(r\) denotes the degree of membership of \(x_{r,s}\), \(s\) denotes the degree of nonmembership of \(x_{r,s}\) and \(x \in X\) the support of \(x_{r,s}\). The \(IFP\) \(x_{r,s}\) is contained in the \(IFS\) \(A(x_{r,s} \in A)\) if and only if \(r < \mu_A(x), s > \gamma_A(x)\).
Definition 2.8. An IFSs $A$ and $B$ are said to be quasi coincident with the the IFS $A\#B$ if and only if there exists an element $x \in X$ such that $\mu_A(x) + \gamma_A(x) = \mu_B(x) + \gamma_B(x)$.

Definition 2.9. Let $A$ be an IFS of an IFTS $X$. Then $A$ is called an intuitionistic fuzzy $\alpha$-open set (IFOS) if $A \subseteq \text{int} \text{(cl}(\text{int}(A))$. The complement of an intuitionistic fuzzy $\alpha$-open set is called an intuitionistic fuzzy $\alpha$-closed set (IFS).

§3. Intuitionistic fuzzy operators with respect to an intuitionistic fuzzy grills

Definition 3.1. Let $\zeta^X$ be the collection of all intuitionistic fuzzy sets in $X$. A collection $\mathcal{F} \subseteq \zeta^X$ is said to be an intuitionistic fuzzy stack on $X$ if $A \subseteq B$ and $A \in \mathcal{F}$ then $B \in \mathcal{F}$.

Definition 3.2. Let $\zeta^X$ be the collection of all intuitionistic fuzzy sets in $X$. An intuitionistic fuzzy grill $\mathcal{G}$ on $X$ is an intuitionistic fuzzy stack on $X$ if $\mathcal{G}$ satisfies the following conditions:

(i) $0_\zeta \notin \mathcal{G}$,

(ii) If $A, B \in \zeta^X$ and $A \cup B \in \mathcal{G}$, then $A \in \mathcal{G}$ or $B \in \mathcal{G}$.

Definition 3.3. Let $(X, T)$ be an intuitionistic fuzzy topological space and let $\zeta^X$ be the collection of all intuitionistic fuzzy sets in $X$. Let $\mathcal{G}$ be an intuitionistic fuzzy grill on $X$. A function $\Phi_{\mathcal{G}} : \zeta^X \to \zeta^X$ is defined by

$$\Phi_{\mathcal{G}}(A) = \bigcup \{ IFint(A) \cap U \mid A \cap U \in \mathcal{G}, U \in T \},$$

for each $A \in \zeta^X$. The function $\Phi_{\mathcal{G}}$ is an intuitionistic fuzzy operator associated with an intuitionistic fuzzy grill $\mathcal{G}$ and an intuitionistic fuzzy topology $T$.

Remark 3.1. Let $(X, T)$ be an intuitionistic fuzzy topological space. Let $\Phi_{\mathcal{G}}$ be an intuitionistic fuzzy grill $\mathcal{G}$ on $X$. Let $\Phi_{\mathcal{G}}$ be an intuitionistic fuzzy operator associated with an intuitionistic fuzzy grill $\mathcal{G}$ and an intuitionistic fuzzy topology $T$. Then

(i) $\Phi_{\mathcal{G}}(0_\zeta) = 0_\zeta = \Phi_{\mathcal{G}}(1_\zeta)$,

(ii) If $A, B \in \zeta^X$ and $A \subseteq B$, then $\Phi_{\mathcal{G}}(A) \subseteq \Phi_{\mathcal{G}}(B)$.

Proof. The proof of (i) and (ii) are follows from the definition of $\Phi_{\mathcal{G}}$.

Definition 3.4. Let $(X, T)$ be an intuitionistic fuzzy topological space and let $\mathcal{G}$ be an intuitionistic fuzzy grill on $X$. Let $\Psi_{\mathcal{G}}$ be an intuitionistic fuzzy operator associated with an intuitionistic fuzzy grill $\mathcal{G}$ and an intuitionistic fuzzy topology $T$. A function $\Psi_{\mathcal{G}} : T \to \zeta^X$ is defined by $\Psi_{\mathcal{G}}(A) = A \cup IFint(\overline{\mathcal{G}})$ for each $A \in T$. The function $\Psi_{\mathcal{G}}$ is an intuitionistic fuzzy operator associated with $\mathcal{G}$.

Definition 3.5. Let $(X, T)$ be an intuitionistic fuzzy topological space and let $\mathcal{G}$ be an intuitionistic fuzzy grill on $X$. Let $\Psi_{\mathcal{G}}$ be an intuitionistic fuzzy operator associated with $\mathcal{G}$. A collection $\mathcal{G}_str = \{ A \mid \Psi_{\mathcal{G}}(A) = A \} \cup \{ 1_\zeta \}$ is said to be an intuitionistic fuzzy $\mathcal{G}$ structure on $X$. Then $(X, \mathcal{G}_str)$ is said to be an intuitionistic fuzzy $\mathcal{G}$ structure space. Every member of $\mathcal{G}_str$ is an intuitionistic fuzzy $\mathcal{G}_str$ open set (in short, $IF\mathcal{G}_str OS$) and the complement of an intuitionistic fuzzy $\mathcal{G}_str$ open set is an intuitionistic fuzzy $\mathcal{G}_str$ closed set (in short, $IF\mathcal{G}_str CS$).
**Definition 3.6.** Let $(X, \mathcal{G}_{str})$ be an intuitionistic fuzzy $\mathcal{G}$ structure space and let $A \in \zeta^X$. Then

(i) the intuitionistic fuzzy $\mathcal{G}_{str}$ closure of $A$ is denoted and defined by

$$IF_{\mathcal{G}_{str}} cl(A) = \cap \{B \in \zeta^X \mid B \supseteq A \text{ and } \overline{B} \in \mathcal{G}_{str}\},$$

(ii) the intuitionistic fuzzy $\mathcal{G}_{str}$ interior of $A$ is denoted and defined by

$$IF_{\mathcal{G}_{str}} int(A) = \cup \{B \in \zeta^X \mid B \subseteq A \text{ and } B \in \mathcal{G}_{str}\}.$$

**Remark 3.1.** Let $(X, \mathcal{G}_{str})$ be an intuitionistic fuzzy $\mathcal{G}$ structure space. For any $A, B \in \zeta^X$,

(i) $IF_{\mathcal{G}_{str}} cl(A) = A$ if and only if $A$ is an intuitionistic fuzzy $\mathcal{G}_{str}$ closed set,

(ii) $IF_{\mathcal{G}_{str}} int(A) = A$ if and only if $A$ is an intuitionistic fuzzy $\mathcal{G}_{str}$ open set,

(iii) $IF_{\mathcal{G}_{str}} int(A) \subseteq A \subseteq IF_{\mathcal{G}_{str}} cl(A),$

(iv) $IF_{\mathcal{G}_{str}} int(1_\sim) = 1_\sim = IF_{\mathcal{G}_{str}} cl(1_\sim)$ and $IF_{\mathcal{G}_{str}} int(0_\sim) = 0_\sim = IF_{\mathcal{G}_{str}} cl(0_\sim),$

(v) $IF_{\mathcal{G}_{str}} cl(\overline{A}) = IF_{\mathcal{G}_{str}} cl(\overline{A})$ and $IF_{\mathcal{G}_{str}} cl(\overline{A}) = IF_{\mathcal{G}_{str}} int(A).$

**§4. Properties of an intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ set and $\alpha_{\mathcal{G}_{str}}$ open set in an intuitionistic fuzzy $\mathcal{G}$ structure spaces**

**Definition 4.1.** Let $(X, \mathcal{G}_{str})$ be an intuitionistic fuzzy $\mathcal{G}$ structure space and let $A \in \zeta^X$. Then $A$ is said be an intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ set (in short, $IF_{\delta_{\mathcal{G}_{str}}} S$) if

$$IF_{\mathcal{G}_{str}} int(IF_{\mathcal{G}_{str}} cl(A)) \subseteq IF_{\mathcal{G}_{str}} cl(IF_{\mathcal{G}_{str}} int(A)).$$

**Proposition 4.1.** Let $(X, \mathcal{G}_{str})$ be an intuitionistic fuzzy $\mathcal{G}$ structure space. Then

(i) The complement of an intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ set is an intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ set,

(ii) Finite union of intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ sets is an intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ set,

(iii) Finite intersection of intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ sets is an intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ set,

(iv) Every intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ sets is an intuitionistic fuzzy $\mathcal{G}_{str}$ open set.

**Proof.** (i) Let $A$ be an intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ set. Then

$$IF_{\mathcal{G}_{str}} int(IF_{\mathcal{G}_{str}} cl(A)) \subseteq IF_{\mathcal{G}_{str}} cl(IF_{\mathcal{G}_{str}} int(A)).$$

Taking complement on both sides, we have

$$IF_{\mathcal{G}_{str}} int(IF_{\mathcal{G}_{str}} cl(A)) \supseteq IF_{\mathcal{G}_{str}} cl(IF_{\mathcal{G}_{str}} int(A)),

IF_{\mathcal{G}_{str}} cl(IF_{\mathcal{G}_{str}} cl(A)) \supseteq IF_{\mathcal{G}_{str}} cl(IF_{\mathcal{G}_{str}} int(A)),

IF_{\mathcal{G}_{str}} cl(IF_{\mathcal{G}_{str}} cl(A)) \supseteq IF_{\mathcal{G}_{str}} cl(IF_{\mathcal{G}_{str}} int(A)).$$

Hence $\overline{A}$ is an intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ set.

(ii) Let $A$ and $B$ be any two intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ sets. Then

$$IF_{\mathcal{G}_{str}} int(IF_{\mathcal{G}_{str}} cl(A)) \subseteq IF_{\mathcal{G}_{str}} cl(IF_{\mathcal{G}_{str}} int(A)), \quad (1)$$
The complement of an intuitionistic fuzzy pre

\[ IF_{gs_{str}}, \text{int}(IF_{gs_{str}}, \text{cl}(B)) \subseteq IF_{gs_{str}}, \text{cl}(IF_{gs_{str}}, \text{int}(B)). \tag{2} \]

Now, taking union of 3.1 and 3.2, we have

\[ IF_{gs_{str}}, \text{int}(IF_{gs_{str}}, \text{cl}(A)) \cup IF_{gs_{str}}, \text{int}(IF_{gs_{str}}, \text{cl}(B)) \]

\[ \subseteq IF_{gs_{str}}, \text{cl}(IF_{gs_{str}}, \text{int}(A)) \cup IF_{gs_{str}}, \text{cl}(IF_{gs_{str}}, \text{int}(B)), \]

\[ IF_{gs_{str}}, \text{int}(IF_{gs_{str}}, \text{cl}(A) \cup IF_{gs_{str}}, \text{cl}(B)) \subseteq IF_{gs_{str}}, \text{cl}(IF_{gs_{str}}, \text{int}(A) \cup IF_{gs_{str}}, \text{int}(B)), \]

\[ IF_{gs_{str}}, \text{int}(IF_{gs_{str}}, \text{cl}(A \cup B)) \subseteq IF_{gs_{str}}, \text{cl}(IF_{gs_{str}}, \text{int}(A \cup B)). \]

Hence \( A \cup B \) is an intuitionistic fuzzy \( \delta_{gs_{str}} \) set.

(iii) The proof is obvious by taking complement of (ii).

(iv) It is obvious.

**Note 4.1.** In general, arbitrary union of an intuitionistic fuzzy \( \delta_{gs_{str}} \) set needs not to be intuitionistic fuzzy \( \delta_{gs_{str}} \) set.

**Definition 4.2.** Let \( (X, \mathcal{G}_{gs}) \) be an intuitionistic fuzzy \( \mathcal{G} \) structure space and let \( A \in \zeta^X \).

Then \( A \) is said to be an

(i) intuitionistic fuzzy semi\( \mathcal{G}_{gs} \) open set (in short, \( IF_{\mathcal{G}_{gs}}, OS \)) if \( A \subseteq IF_{\mathcal{G}_{gs}}, \text{cl}(IF_{\mathcal{G}_{gs}}, \text{int}(A)) \).

The complement of an intuitionistic fuzzy semi\( \mathcal{G}_{gs} \) open set is said to be an intuitionistic fuzzy semi\( \mathcal{G}_{gs} \) closed set (in short, \( IF_{\mathcal{G}_{gs}}, CS \)).

(ii) intuitionistic fuzzy pre\( \mathcal{G}_{gs} \) open set (in short, \( IF_{\mathcal{G}_{gs}}, OS \)) if \( A \subseteq IF_{\mathcal{G}_{gs}}, \text{int}(IF_{\mathcal{G}_{gs}}, \text{cl}(A)) \).

The complement of an intuitionistic fuzzy pre\( \mathcal{G}_{gs} \) open set is said to be an intuitionistic fuzzy pre\( \mathcal{G}_{gs} \) closed set (in short, \( IF_{\mathcal{G}_{gs}}, CS \)).

(iii) intuitionistic fuzzy \( \alpha_{gs_{str}} \) open set (in short, \( IF_{\alpha_{gs_{str}}}, OS \)) if

\[ A \subseteq IF_{\mathcal{G}_{gs}}, \text{int}(IF_{\mathcal{G}_{gs}}, \text{cl}(IF_{\mathcal{G}_{gs}}, \text{int}(A))). \]

The complement of an intuitionistic fuzzy \( \alpha_{gs_{str}} \) open set is said to be an intuitionistic fuzzy \( \alpha_{gs_{str}} \) closed set (in short, \( IF_{\alpha_{gs_{str}}}, CS \)).

(iv) intuitionistic fuzzy \( \beta_{gs_{str}} \) open set (in short, \( IF_{\beta_{gs_{str}}}, OS \)) if

\[ A \subseteq IF_{\mathcal{G}_{gs}}, \text{cl}(IF_{\mathcal{G}_{gs}}, \text{int}(IF_{\mathcal{G}_{gs}}, \text{cl}(A))). \]

The complement of an intuitionistic fuzzy \( \beta_{gs_{str}} \) open set is said to be an intuitionistic fuzzy \( \beta_{gs_{str}} \) closed set (in short, \( IF_{\beta_{gs_{str}}}, CS \)).

(v) intuitionistic fuzzy regular\( \mathcal{G}_{gs} \) open set (in short, \( IF_{R_{\mathcal{G}_{gs}}}, OS \)) if

\[ A = IF_{\mathcal{G}_{gs}}, \text{int}(IF_{\mathcal{G}_{gs}}, \text{cl}(A)). \]

The complement of an intuitionistic fuzzy regular\( \mathcal{G}_{gs} \) open set is said to be an intuitionistic fuzzy regular\( \mathcal{G}_{gs} \) closed set (in short, \( IF_{R_{\mathcal{G}_{gs}}}, CS \)).

**Note 4.2.** The family of all intuitionistic fuzzy semi\( \mathcal{G}_{gs} \), (resp. pre\( \mathcal{G}_{gs} \), \( \alpha_{gs_{str}} \), \( \beta_{gs_{str}} \) and regular\( \mathcal{G}_{gs} \)) open sets are denoted by \( IF_{\mathcal{G}_{gs}}, O(X) \) (resp. \( IF_{\alpha_{gs_{str}}}, O(X) \), \( IF_{\beta_{gs_{str}}}, O(X) \) and \( IF_{R_{\mathcal{G}_{gs}}}, O(X) \)).

**Definition 4.3.** Let \( (X, \mathcal{G}_{gs}) \) be an intuitionistic fuzzy \( \mathcal{G} \) structure space and let \( A \in \zeta^X \).

Then
(i) the intuitionistic fuzzy $\alpha_{str}$ closure of $A$ is denoted and defined by
$$IF_{\alpha_{str}}(A) = \cap\{B \in \xi^X \mid B \supseteq A \text{ and } \overline{B} \in IF_{\alpha_{str}}, O(X)\};$$
(ii) the intuitionistic fuzzy $\alpha_{str}$ interior of $A$ is denoted and defined by
$$IF_{\alpha_{str}}(A) = \cup\{B \in \xi^X \mid B \subseteq A \text{ and } B \in IF_{\alpha_{str}}, O(X)\}.$$

**Remark 4.1.** Let $(X, \mathcal{G}_{str})$ be an intuitionistic fuzzy $\mathcal{G}$ structure space. For any $A, B \in \xi^X$,

(i) $IF_{\alpha_{str}}(A) = A$ if and only if $A$ is an intuitionistic fuzzy $\alpha_{str}$ closed set,
(ii) $IF_{\alpha_{str}}(A) = A$ if and only if $A$ is an intuitionistic fuzzy $\alpha_{str}$ open set,
(iii) $IF_{\alpha_{str}}(A) \subseteq A \subseteq IF_{\alpha_{str}}(A)$,
(iv) $IF_{\alpha_{str}}(A) = 1_{\alpha_{str}} = IF_{\alpha_{str}}(0_{\alpha_{str}})$ and $IF_{\alpha_{str}}(A) = 0_{\alpha_{str}} = IF_{\alpha_{str}}(0_{\alpha_{str}})$,
(v) $IF_{\alpha_{str}}(A) = IF_{\alpha_{str}}(A) \cup IF_{\alpha_{str}}(A)$ and $IF_{\alpha_{str}}(A) = IF_{\alpha_{str}}(A)$.

**Proposition 4.2.** Let $(X, \mathcal{G}_{str})$ be an intuitionistic fuzzy $\mathcal{G}$ structure space. Then

(i) Every intuitionistic fuzzy regular $\mathcal{G}_{str}$ open set is an intuitionistic fuzzy $\mathcal{G}_{str}$ open set;
(ii) Every intuitionistic fuzzy intuitionistic fuzzy $\mathcal{G}_{str}$ open set is an intuitionistic fuzzy semi $\mathcal{G}_{str}$ (resp. pre $\mathcal{G}_{str}$, $\alpha_{str}$, and $\beta_{str}$) open set.

**Proof.** It is obvious.

**Remark 4.2.** The converse of the proposition 4.2 needs not to be true as shown in example 4.1.

**Example 4.1.** Let $X = \{a, b\}$ be a nonempty set. Let $A = \langle x, (a, b), (a, b) \rangle$, $B = \langle x, (a, b), (a, b) \rangle$, $C = \langle x, (a, b), (a, b) \rangle$ and $E = \langle x, (a, b), (a, b) \rangle$ be intuitionistic fuzzy sets on $X$. The family $T = \{0_{\alpha_{str}}, 1_{\alpha_{str}}, A, B, C, D\}$ is an intuitionistic fuzzy topology on $X$ and the family $\mathcal{G} = \{G \in \xi^X \mid 0.2 \leq \mu_G(x) \leq 1 \text{ and } 0 \leq \gamma_G(x) \leq 0.8\}$ is an intuitionistic fuzzy grill on $X$. Then the family $\mathcal{G}_{str} = \{0_{\alpha_{str}}, 1_{\alpha_{str}}, A\}$ is an intuitionistic fuzzy $\mathcal{G}$ structure on $X$. Therefore, $(X, \mathcal{G}_{str})$ is an intuitionistic fuzzy $\mathcal{G}$ structure space. Now,

(i) $F = \langle x, (a, b), (a, b) \rangle$ is an intuitionistic fuzzy $\mathcal{G}_{str}$ open set but needs not to be an intuitionistic fuzzy regular $\mathcal{G}_{str}$ open set in $X$.
(ii) $H = \langle x, (a, b), (a, b) \rangle$ is an intuitionistic fuzzy $\alpha_{str}$ (resp. semi $\mathcal{G}_{str}$) open set but needs not to be an intuitionistic fuzzy $\mathcal{G}_{str}$ open set in $X$.
(iii) $K = \langle x, (a, b), (a, b) \rangle$ is an intuitionistic fuzzy pre $\mathcal{G}_{str}$ (resp. $\beta_{str}$) open set but needs not to be an intuitionistic fuzzy $\mathcal{G}_{str}$ open set in $X$.

**Proposition 4.3.** Every intuitionistic fuzzy regular $\mathcal{G}_{str}$ open set is an intuitionistic fuzzy $\delta_{str}$ open set.

**Proof.** Let $A$ be an intuitionistic fuzzy regular $\mathcal{G}_{str}$ open set. Then $IF_{\mathcal{G}_{str}}(A) \subseteq IF_{\mathcal{G}_{str}}(A)$. Since every intuitionistic fuzzy regular $\mathcal{G}_{str}$ open set is an intuitionistic fuzzy $\mathcal{G}_{str}$ open set, $IF_{\mathcal{G}_{str}}(A) \subseteq IF_{\mathcal{G}_{str}}(A)$. Hence $A$ is an intuitionistic fuzzy $\delta_{str}$ open set.

**Remark 4.3.** The converse of the proposition 4.3 needs not to be true as shown in example 4.2.

**Example 4.2.** Let $X = \{a, b\}$ be a nonempty set. Let $A = \langle x, (a, b), (a, b) \rangle$, $B = \langle x, (a, b), (a, b) \rangle$, $C = \langle x, (a, b), (a, b) \rangle$ and $E = \langle x, (a, b), (a, b) \rangle$ be intuitionistic fuzzy sets on $X$. The family $T = \{0_{\alpha_{str}}, 1_{\alpha_{str}}, A, B, C, D\}$ is an intuitionistic fuzzy
topology on $X$ and the family $\mathcal{G} = \{ G \in \zeta^X \mid 0.2 \leq \mu_G(x) \leq 1 \text{ and } 0 \leq \gamma_G(x) \leq 0.8 \}$ is an intuitionistic fuzzy grill on $X$. Then the family $\mathcal{G}_{str} = \{ \emptyset, 1 \cdots, A \}$ is an intuitionistic fuzzy $\mathcal{G}$ structure on $X$. Therefore, $(X, \mathcal{G}_{str})$ is an intuitionistic fuzzy $\mathcal{G}$ structure space. Now, $F = \{ ( x, (\frac{a_i}{0.7}, \frac{b_i}{0.1}), (\frac{a_i}{0.2}, \frac{b_i}{0.7}) ) \}$ is an intuitionistic fuzzy $\delta_{\mathcal{G}_{str}}$ set but needs not to be an intuitionistic fuzzy regular $\mathcal{G}_{str}$ open set in $X$.

**Proposition 4.4.** Every intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$ open set is an intuitionistic fuzzy semi $\mathcal{G}_{str}$ open set.

**Proof.** Let $A$ be an intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$ open set. Then

$$A \subseteq IF_{\mathcal{G}_{str}} int(IF_{\mathcal{G}_{str}} cl(IF_{\mathcal{G}_{str}} int(A))) \subseteq IF_{\mathcal{G}_{str}} cl(IF_{\mathcal{G}_{str}} int(A)).$$

Hence $A$ is an intuitionistic fuzzy semi $\mathcal{G}_{str}$ open set.

**Remark 4.4.** The converse of the proposition 4.4 needs not to be true as shown in example 4.3.

**Proposition 4.5.** Every intuitionistic fuzzy pre$\mathcal{G}_{str}$ open set is an intuitionistic fuzzy $\beta_{\mathcal{G}_{str}}$ open set.

**Proof.** Let $A$ be an intuitionistic fuzzy pre$\mathcal{G}_{str}$ set. Then

$$A \subseteq IF_{\mathcal{G}_{str}} int(IF_{\mathcal{G}_{str}} cl(A)) \subseteq IF_{\mathcal{G}_{str}} cl(IF_{\mathcal{G}_{str}} int(A)) \subseteq IF_{\mathcal{G}_{str}} cl(IF_{\mathcal{G}_{str}} int(A)).$$

Hence $A$ is an intuitionistic fuzzy $\beta_{\mathcal{G}_{str}}$ open set.

**Remark 4.5.** The converse of the proposition 4.5 needs not to be true as shown in example 4.3.

**Example 4.3.** Let $X = \{ a, b \}$ be a nonempty set. Let $A_1 = \{ ( x, (\frac{a_i}{0.2}, \frac{b_i}{0.3}), (\frac{a_i}{0.7}, \frac{b_i}{0.7}) ) \}$, $A_2 = \{ ( x, (\frac{a_i}{0.3}, \frac{b_i}{0.6}), (\frac{a_i}{0.2}, \frac{b_i}{0.7}) ) \}$, $A_3 = \{ ( x, (\frac{a_i}{0.7}, \frac{b_i}{0.6}), (\frac{a_i}{0.3}, \frac{b_i}{0.7}) ) \}$,

$$A_4 = \{ ( x, (\frac{a_i}{0.7}, \frac{b_i}{0.7}), (\frac{a_i}{0.7}, \frac{b_i}{0.2}) ) \}, A_5 = \{ ( x, (\frac{a_i}{0.7}, \frac{b_i}{0.3}), (\frac{a_i}{0.7}, \frac{b_i}{0.7}) ) \}, A_6 = \{ ( x, (\frac{a_i}{0.7}, \frac{b_i}{0.3}), (\frac{a_i}{0.7}, \frac{b_i}{0.7}) ) \},$$

$$A_7 = \{ ( x, (\frac{a_i}{0.7}, \frac{b_i}{0.7}), (\frac{a_i}{0.7}, \frac{b_i}{0.7}) ) \}, A_8 = \{ ( x, (\frac{a_i}{0.7}, \frac{b_i}{0.3}), (\frac{a_i}{0.7}, \frac{b_i}{0.7}) ) \}, A_9 = \{ ( x, (\frac{a_i}{0.7}, \frac{b_i}{0.3}), (\frac{a_i}{0.7}, \frac{b_i}{0.7}) ) \},$$

$$A_{10} = \{ ( x, (\frac{a_i}{0.7}, \frac{b_i}{0.7}), (\frac{a_i}{0.7}, \frac{b_i}{0.7}) ) \}, A_{11} = \{ ( x, (\frac{a_i}{0.7}, \frac{b_i}{0.7}), (\frac{a_i}{0.7}, \frac{b_i}{0.7}) ) \}, A_{12} = \{ ( x, (\frac{a_i}{0.7}, \frac{b_i}{0.7}), (\frac{a_i}{0.7}, \frac{b_i}{0.7}) ) \},$$

and $A_{13} = \{ ( x, (\frac{a_i}{0.7}, \frac{b_i}{0.7}), (\frac{a_i}{0.7}, \frac{b_i}{0.7}) ) \}$ be intuitionistic fuzzy sets on $X$.

The family $T = \{ 0, \ldots, 1, 1_i, i = 1, 2, \cdots, 13 \}$ is an intuitionistic fuzzy topology on $X$ and the family $\mathcal{G} = \{ G \in \zeta^X \mid 0.1 \leq \mu_G(x) \leq 1 \text{ and } 0 \leq \gamma_G(x) \leq 0.9 \}$ is an intuitionistic fuzzy grill on $X$. Then the family $\mathcal{G}_{str} = \{ 0, \ldots, 1, 1_i, A_1, A_2, A_3, A_7 \}$ is an intuitionistic fuzzy $\mathcal{G}$ structure on $X$. Therefore, $(X, \mathcal{G}_{str})$ is an intuitionistic fuzzy $\mathcal{G}$ structure space. Now,

(i) $B = \{ ( x, (\frac{a_i}{0.7}, \frac{b_i}{0.7}), (\frac{a_i}{0.7}, \frac{b_i}{0.7}) ) \}$ is an intuitionistic fuzzy semi $\mathcal{G}_{str}$ open set but needs not to be an intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$ open set in $X$.

(ii) $C = \{ ( x, (\frac{a_i}{0.7}, \frac{b_i}{0.7}), (\frac{a_i}{0.7}, \frac{b_i}{0.7}) ) \}$ is an intuitionistic fuzzy $\beta_{\mathcal{G}_{str}}$ open set but needs not to be an intuitionistic fuzzy pre$\mathcal{G}_{str}$ open set in $X$.

**Remark 4.6.** From the diagram, the following implications hold:
§5. Separation axioms in an intuitionistic fuzzy \( \mathcal{G} \) structure space

**Definition 5.1.** Let \((X, \mathcal{G}_{str})\) be an intuitionistic fuzzy \( \mathcal{G} \) structure space. Then an intuitionistic fuzzy set \( A \) is said to be an intuitionistic fuzzy \( \mathcal{G}_{str} \) (resp. \( \alpha_{\mathcal{G}_{str}} \)) clopen set if and only if it is both intuitionistic fuzzy \( \mathcal{G}_{str} \) open (resp. \( \alpha_{\mathcal{G}_{str}} \)) and intuitionistic fuzzy \( \mathcal{G}_{str} \) closed (resp. \( \alpha_{\mathcal{G}_{str}} \)).

**Definition 5.2.** Let \((X, \mathcal{G}_{str})\) be an intuitionistic fuzzy \( \mathcal{G} \) structure space. Then an intuitionistic fuzzy set \( A \) is said to be an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{str}} \) exterior of \( A \) if \( IFExt_{\alpha_{\mathcal{G}_{str}}} (A) = IF_{\alpha_{\mathcal{G}_{str}}} \text{int}(\overline{A}) \).

**Remark 5.1.** Let \((X, \mathcal{G}_{str})\) be an intuitionistic fuzzy \( \mathcal{G} \) structure space. Then

(i) If \( A \) is an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{str}} \) clopen set, then \( IFExt_{\alpha_{\mathcal{G}_{str}}} (A) = A = \overline{A} \);

(ii) If \( A \subseteq B \) then \( IFExt_{\alpha_{\mathcal{G}_{str}}} (A) \supseteq IFExt_{\alpha_{\mathcal{G}_{str}}} (B) \);

(iii) \( IFExt_{\alpha_{\mathcal{G}_{str}}} (1_{\sim}) = 0_{\sim} \) and \( IFExt_{\alpha_{\mathcal{G}_{str}}} (0_{\sim}) = 1_{\sim} \).

**Definition 5.3.** Let \((X, \mathcal{G}_{1str})\) and \((Y, \mathcal{G}_{2str})\) be any two intuitionistic fuzzy \( \mathcal{G} \) structure spaces. Let \( f : (X, \mathcal{G}_{1str}) \to (Y, \mathcal{G}_{2str}) \) be an intuitionistic fuzzy function. Then

(i) intuitionistic fuzzy \( \alpha_{\mathcal{G}_{str}} \) continuous function if for each intuitionistic fuzzy point \( x_{r,s} \) in \( X \) and each intuitionistic fuzzy \( \mathcal{G}_{2str} \) open set \( B \) in \( Y \) containing \( f(x_{r,s}) \), there exists an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{1str}} \) open set \( A \) in \( X \) containing \( x_{r,s} \) such that \( f(A) \subseteq B \);

(ii) intuitionistic fuzzy \( E_{\alpha_{\mathcal{G}_{str}}} \) continuous function if for each intuitionistic fuzzy point \( x_{r,s} \) in \( X \) and each intuitionistic fuzzy \( \mathcal{G}_{2str} \) clopen set \( B \) in \( Y \) containing \( f(x_{r,s}) \), there exists an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{1str}} \) clopen set \( A \) in \( X \) containing \( x_{r,s} \) such that \( f(IFExt_{\alpha_{\mathcal{G}_{str}}} (A)) \subseteq B \);

(iii) intuitionistic fuzzy \( \alpha_{\mathcal{G}_{str}} \) open function if for each intuitionistic fuzzy \( \mathcal{G}_{1str} \) open set \( A \) in \( X \), \( f^{-1}(A) \) is an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{str}} \) open set \( A \) in \( Y \).

**Proposition 5.1.** Let \((X, \mathcal{G}_{1str})\) and \((Y, \mathcal{G}_{2str})\) be any two intuitionistic fuzzy \( \mathcal{G} \) structure spaces. Let \( f : (X, \mathcal{G}_{1str}) \to (Y, \mathcal{G}_{2str}) \) be an intuitionistic fuzzy function. Then the followings are equivalent:

(i) \( f \) is intuitionistic fuzzy \( E_{\alpha_{\mathcal{G}_{str}}} \) continuous function;
(ii) \( f^{-1}(A) \) is an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{1,str}} \) open set in \( X \), for each intuitionistic fuzzy \( \mathcal{G}_{2,str} \) clopen set \( A \) in \( Y \);

(iii) \( f^{-1}(A) \) is an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{1,str}} \) closed set in \( X \), for each intuitionistic fuzzy \( \mathcal{G}_{2,str} \) clopen set \( A \) in \( Y \);

(iv) \( f^{-1}(A) \) is an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{1,str}} \) clopen set in \( X \), for each intuitionistic fuzzy \( \mathcal{G}_{2,str} \) clopen set \( A \) in \( Y \).

**Proof.** (i) \( \Rightarrow \) (ii). Let \( B \) be an intuitionistic fuzzy \( \mathcal{G}_{2,str} \) clopen set in \( Y \) and let \( x_{r,s} \in f^{-1}(B) \). Since \( f(x_{r,s}) \in B \), by (i), there exists an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{1,str}} \) clopen set \( A \) in \( X \) containing \( x_{r,s} \) such that

\[
f(\text{IFExt}_{\alpha_{\mathcal{G}_{1,str}}}(A)) = f(\text{IFext}_{\alpha_{\mathcal{G}_{1,str}}} \text{int}(A)) \subseteq B,
\]

Thus

\[
f^{-1}(B) = \bigcup_{x_{r,s} \in f(\text{IFext}_{\alpha_{\mathcal{G}_{1,str}}} \text{int}(A))} \text{IFext}_{\alpha_{\mathcal{G}_{1,str}}} \text{int}(A).
\]

This implies that

\[
f^{-1}(B) = \bigcup_{x_{r,s} \in \text{IFext}_{\alpha_{\mathcal{G}_{1,str}}} \text{int}(A)} \text{IFext}_{\alpha_{\mathcal{G}_{1,str}}} \text{int}(A).
\]

Thus \( f^{-1}(B) \) is an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{1,str}} \) open set in \( X \).

(ii) \( \Rightarrow \) (iii). Let \( B \) be an intuitionistic fuzzy \( \mathcal{G}_{2,str} \) clopen set in \( Y \). Then \( \overline{B} \) is an intuitionistic fuzzy \( \mathcal{G}_{2,str} \) clopen set in \( Y \). Thus, \( f^{-1}(\overline{B}) = f^{-1}(\overline{B}) \). Since \( f^{-1}(B) \) is an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{1,str}} \) open set in \( X \), \( f^{-1}(\overline{B}) \) is an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{1,str}} \) closed set in \( X \).

(iii) \( \Rightarrow \) (iv). The proof is easy.

(iv) \( \Rightarrow \) (v). Let \( B \) be an intuitionistic fuzzy \( \mathcal{G}_{2,str} \) clopen set in \( Y \) containing \( f(x_{r,s}) \). By (iv), \( f^{-1}(B) \) is an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{1,str}} \) open set in \( X \). If we take \( A = f^{-1}(B) \), then \( f(A) \subseteq B \). By remark 5.1, \( f(\text{IFext}_{\alpha_{\mathcal{G}_{1,str}}}(A)) \subseteq B \).

**Remark 5.2.** Let \( (X, \mathcal{G}_{1,str}) \) and \( (Y, \mathcal{G}_{2,str}) \) be any two intuitionistic fuzzy \( \mathcal{G} \) structure spaces. If \( f : (X, \mathcal{G}_{1,str}) \to (Y, \mathcal{G}_{2,str}) \) is an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{1,str}} \) exterior set connected function, then \( f \) is an intuitionistic fuzzy \( \alpha_{\mathcal{G}_{2,str}} \) continuous function.

**Definition 5.4.** An intuitionistic fuzzy \( \mathcal{G} \) structure space \( (X, \mathcal{G}_{str}) \) is said to be an

(i) intuitionistic fuzzy \( \mathcal{G}_{str,clo-T_0} \) if for each pair of distinct intuitionistic fuzzy points \( x_{r,s} \) and \( y_{m,n} \) in \( X \), there exists an intuitionistic fuzzy \( \mathcal{G}_{str} \) clopen set \( A \) of \( X \) containing one intuitionistic fuzzy point \( x_{r,s} \) but not \( y_{m,n} \);

(ii) intuitionistic fuzzy \( \mathcal{G}_{str,clo-T_1} \) if for each pair of distinct intuitionistic fuzzy points \( x_{r,s} \) and \( y_{m,n} \) in \( X \), there exist intuitionistic fuzzy \( \mathcal{G}_{str} \) clopen sets \( A \) and \( B \) containing \( x_{r,s} \) and \( y_{m,n} \) respectively such that \( y_{m,n} \notin A \) and \( x_{r,s} \notin B \);

(iii) intuitionistic fuzzy \( \mathcal{G}_{str,clo-T_2} \) if for each pair of distinct intuitionistic fuzzy points \( x_{r,s} \) and \( y_{m,n} \) in \( X \), there exist intuitionistic fuzzy \( \mathcal{G}_{str} \) clopen sets \( A \) and \( B \) containing \( x_{r,s} \) and \( y_{m,n} \) respectively such that \( A \cap B = \emptyset \);

(iv) intuitionistic fuzzy \( \alpha_{\mathcal{G}_{str},clo-regular} \) if for each intuitionistic fuzzy \( \mathcal{G}_{str} \) clopen set \( A \) and an intuitionistic fuzzy point \( x_{r,s} \notin A \), there exist disjoint intuitionistic fuzzy \( \alpha_{\mathcal{G}_{str}} \) open sets \( B \) and \( C \) such that \( A \subseteq B \) and \( x_{r,s} \in C \);

(v) intuitionistic fuzzy \( \alpha_{\mathcal{G}_{str},clo-normal} \) if for each pair of disjoint intuitionistic fuzzy \( \mathcal{G}_{str} \) clopen sets \( A \) and \( B \) in \( X \), there exist disjoint intuitionistic fuzzy \( \alpha_{\mathcal{G}_{str}} \) open sets \( C \) and \( D \) such that \( A \subseteq C \) and \( B \subseteq D \).
Definition 5.5. An intuitionistic fuzzy $\mathcal{G}$ structure space $(X, \mathcal{G}_{str})$ is said to be an an
(i) intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$,-$T_0$ if for each pair of distinct intuitionistic fuzzy points $x_{r,s}$ and
$y_{m,n}$ in $X$, there exists an intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$, open set $A$ of $X$ containing one intuitionistic fuzzy
point $x_{r,s}$ but not $y_{m,n}$;
(ii) intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$,-$T_1$ if for each pair of distinct intuitionistic fuzzy points $x_{r,s}$ and
$y_{m,n}$ in $X$, there exist intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$, open sets $A$ and $B$ containing $x_{r,s}$ and $y_{m,n}$
respectively such that $y_{m,n} \notin A$ and $x_{r,s} \notin B$;
(iii) intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$,-$T_2$ if for each pair of distinct intuitionistic fuzzy points $x_{r,s}$ and
$y_{m,n}$ in $X$, there exist an intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$, open sets $A$ and $B$ containing $x_{r,s}$ and
$y_{m,n}$ respectively such that $A \cap B = 0_{\sim}$;
(iv) intuitionistic fuzzy strongly $\alpha_{\mathcal{G}_{str}}$-regular if for each intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$, closed
set $A$ and an intuitionistic fuzzy point $x_{r,s} \notin A$, there exist disjoint intuitionistic fuzzy $\mathcal{G}_{str}$ open
sets $B$ and $C$ such that $A \subseteq B$ and $x_{r,s} \in C$;
(v) intuitionistic fuzzy strongly $\alpha_{\mathcal{G}_{str}}$-normal if for each pair of disjoint intuitionistic fuzzy
$\alpha_{\mathcal{G}_{str}}$, closed sets $A$ and $B$ in $X$, there exist disjoint intuitionistic fuzzy $\mathcal{G}_{str}$ open sets $C$ and $D$
such that $A \subseteq C$ and $B \subseteq D$.

Proposition 5.2. Let $(X, \mathcal{G}_{1str})$ and $(Y, \mathcal{G}_{2str})$ be any two intuitionistic fuzzy $\mathcal{G}$ structure
spaces. Let $f : (X, \mathcal{G}_{1str}) \to (Y, \mathcal{G}_{2str})$ be an injective, intuitionistic fuzzy $E_{\alpha_{\mathcal{G}_{str}}}$ continuous
function.

(i) If $(Y, \mathcal{G}_{2str})$ is intuitionistic fuzzy $\mathcal{G}_{2str}$,clo-$T_0$, then $(X, \mathcal{G}_{1str})$ is intuitionistic fuzzy
$\alpha_{\mathcal{G}_{str}}$,-$T_0$;
(ii) if $(Y, \mathcal{G}_{2str})$ is intuitionistic fuzzy $\mathcal{G}_{2str}$,clo-$T_1$, then $(X, \mathcal{G}_{1str})$ is intuitionistic fuzzy
$\alpha_{\mathcal{G}_{str}}$,-$T_1$;
(iii) if $(Y, \mathcal{G}_{2str})$ is intuitionistic fuzzy $\mathcal{G}_{2str}$,clo-$T_2$, then $(X, \mathcal{G}_{1str})$ is intuitionistic fuzzy
$\alpha_{\mathcal{G}_{str}}$,-$T_2$.

Proof. (i) Suppose that $(Y, \mathcal{G}_{2str})$ is an intuitionistic fuzzy $\mathcal{G}_{2str}$,clo-$T_0$ space. For any
distinct intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in $X$, there exists an intuitionistic fuzzy $\mathcal{G}_{2str}$
clopen set $A$ in $Y$ such that $f(x_{r,s}) \in A$ and $f(y_{r,s}) \notin A$. Since $f$ is an intuitionistic fuzzy
$E_{\alpha_{\mathcal{G}_{str}}}$ continuous function, $f^{-1}(A)$ is an intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$, open set in $X$ such that
$x_{r,s} \in f^{-1}(A)$. This implies that $(X, \mathcal{G}_{1str})$ is an intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$,-$T_0$ space.

(ii) Suppose that $(Y, \mathcal{G}_{2str})$ is an intuitionistic fuzzy $\mathcal{G}_{2str}$,clo-$T_1$ space. For any distinct
intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in $X$, there exist an intuitionistic fuzzy $\mathcal{G}_{2str}$ clopen
sets $A$ and $B$ in $Y$ such that $f(x_{r,s}) \in A$, $f(x_{r,s}) \notin B$, $f(y_{r,s}) \notin A$ and $f(y_{r,s}) \in B$. Since $f$
is an intuitionistic fuzzy $E_{\alpha_{\mathcal{G}_{str}}}$, continuous function, $f^{-1}(A)$ and $f^{-1}(B)$ are intuitionistic fuzzy
$\alpha_{\mathcal{G}_{str}}$, open sets in $X$ respectively such that $x_{r,s} \in f^{-1}(A)$, $x_{r,s} \notin f^{-1}(B)$, $y_{r,s} \notin f^{-1}(A)$ and
$y_{r,s} \in f^{-1}(B)$. This implies that $(X, \mathcal{G}_{1str})$ is an intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$,-$T_1$ space.

(iii) Suppose that $(Y, \mathcal{G}_{2str})$ is an intuitionistic fuzzy $\mathcal{G}_{2str}$,clo-$T_2$ space. For any distinct
intuitionistic fuzzy points $x_{r,s}$ and $y_{m,n}$ in $X$, there exist intuitionistic fuzzy $\mathcal{G}_{2str}$ clopen sets $A$
and $B$ in $Y$ such that $f(x_{r,s}) \in A$, $f(x_{r,s}) \notin B$, $f(y_{r,s}) \notin A$, $f(y_{r,s}) \in B$ and $A \cap B = 0_{\sim}$. Since
$f$ is an intuitionistic fuzzy $E_{\alpha_{\mathcal{G}_{str}}}$, continuous function, $f^{-1}(A)$ and $f^{-1}(B)$ are intuitionistic fuzzy
$\alpha_{\mathcal{G}_{str}}$, open sets in $X$, containing $x_{r,s}$ and $y_{r,s}$ respectively such that

$$f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(0_{\sim}) = 0_{\sim}.$$
This implies that \((X, \mathcal{G}_{\text{str}})\) is an intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}}, T_2\) space.

**Proposition 5.3.** Let \((X, \mathcal{G}_{\text{str}})\) and \((Y, \mathcal{G}_{\text{str}})\) be any two intuitionistic fuzzy \(\mathcal{G}\) structure spaces. Let \(f : (X, \mathcal{G}_{\text{str}}) \rightarrow (Y, \mathcal{G}_{\text{str}})\) be an intuitionistic fuzzy function.

(i) If \(f\) is an injective, intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) open function and intuitionistic fuzzy \(E_{\alpha_{\mathcal{G}_{\text{str}}}},\) continuous function from intuitionistic fuzzy strongly \(\alpha_{\mathcal{G}_{\text{str}}},\) -regular space \((X, \mathcal{G}_{\text{str}})\) onto an intuitionistic fuzzy \(\mathcal{G}\) structure space \((Y, \mathcal{G}_{\text{str}})\), then \((Y, \mathcal{G}_{\text{str}})\) is an intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) clo-regular space;

(ii) if \(f\) is an injective, intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) open function and intuitionistic fuzzy \(E_{\alpha_{\mathcal{G}_{\text{str}}}},\) continuous function from intuitionistic fuzzy strongly \(\alpha_{\mathcal{G}_{\text{str}}},\) -normal space \((X, \mathcal{G}_{\text{str}})\) onto an intuitionistic fuzzy \(\mathcal{G}\) structure space \((Y, \mathcal{G}_{\text{str}})\), then \((Y, \mathcal{G}_{\text{str}})\) is an intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) clo-normal space.

**Proof.** (i) Let \(A\) be an intuitionistic fuzzy \(\mathcal{G}_{\text{str}}\) clopen set in \(Y\) such that \(y_{m,n} \notin A\). Take \(y_{m,n} = f(x_{r,s})\). Since \(f\) is an intuitionistic fuzzy \(E_{\alpha_{\mathcal{G}_{\text{str}}}},\) continuous function, \(B = f^{-1}(A)\) is an intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) closed sets in \(X\) such that \(x_{r,s} \notin B\). Since \((X, \mathcal{G}_{\text{str}})\) is an intuitionistic fuzzy strongly \(\alpha_{\mathcal{G}_{\text{str}}},\) -regular space, there exist disjoint intuitionistic fuzzy \(\mathcal{G}_{\text{str}}\) open sets \(C\) and \(D\) such that \(B \subseteq C\) and \(x_{r,s} \in D\). Since \(f\) is an intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) open function, we have \(A = f(B) \subseteq f(C)\) and \(y_{m,n} = f(x_{r,s}) \in f(D)\) such that \(f(C)\) and \(f(D)\) are disjoint intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) open sets in \(Y\). This implies that \((Y, \mathcal{G}_{\text{str}})\) is an intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) clo-regular space.

(ii) Let \(A_1\) and \(A_2\) be disjoint intuitionistic fuzzy \(\mathcal{G}_{\text{str}}\) clopen sets in \(Y\). Since \(f\) is an intuitionistic fuzzy \(E_{\alpha_{\mathcal{G}_{\text{str}}}},\) continuous function, \(f^{-1}(A_1)\) and \(f^{-1}(A_2)\) are intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) closed sets in \(X\). Take \(B = f^{-1}(A_1)\) and \(C = f^{-1}(A_2)\). This implies that \(B \cap C = \emptyset\).

Since \((X, \mathcal{G}_{\text{str}})\) is an intuitionistic fuzzy strongly \(\alpha_{\mathcal{G}_{\text{str}}},\) -normal space, there exist disjoint intuitionistic fuzzy \(\mathcal{G}_{\text{str}}\) open sets \(D_1\) and \(D_2\) such that \(B \subseteq D_1\) and \(C \subseteq D_2\). Since \(f\) is an intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) open function, we have \(A_1 = f(B) \subseteq f(D_1)\) and \(A_2 = f(C) \subseteq f(D_2)\) such that \(f(D_1)\) and \(f(D_2)\) are disjoint intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) open sets in \(Y\). This implies that \((Y, \mathcal{G}_{\text{str}})\) is an intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) clo-normal space.

**Definition 5.6.** Let \((X, \mathcal{G}_{\text{str}})\) and \((Y, \mathcal{G}_{\text{str}})\) be any two intuitionistic fuzzy \(\mathcal{G}\) structure spaces. Let \(f : X \rightarrow Y\) be an intuitionistic fuzzy function. An intuitionistic fuzzy graph \(G(f) = \{(x_{r,s}, f(x_{r,s}) : x_{r,s} \in \xi_X, \zeta_Y\}\) of an intuitionistic fuzzy function \(f\) is an intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) co-closed graph if and only if for each \((x_{r,s}, y_{m,n}) \in \xi_X \times \zeta_Y \setminus G(f)\), there exist an intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) open set \(A\) in \(X\) containing \(x_{r,s}\) and an intuitionistic fuzzy \(\mathcal{G}_{\text{str}}\) clopen set \(B\) in \(Y\) containing \(y_{m,n}\) such that \(f(A) \cap B = \emptyset\).

**Proposition 5.4.** Let \((X, \mathcal{G}_{\text{str}}), (Y, \mathcal{G}_{\text{str}})\) and \((X \times Y, \mathcal{G}_{\text{str}})\) be any three intuitionistic fuzzy \(\mathcal{G}\) structure spaces. Let \(f : (X, \mathcal{G}_{\text{str}}) \rightarrow (Y, \mathcal{G}_{\text{str}})\) be an intuitionistic fuzzy function.

(i) If \(f\) is an intuitionistic fuzzy \(E_{\alpha_{\mathcal{G}_{\text{str}}}},\) continuous function and \((Y, \mathcal{G}_{\text{str}})\) is an intuitionistic fuzzy intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) -T\(_2\) space, then \(G(f)\) is an intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) co-closed graph in \(\xi_X \times \zeta_Y\);

(ii) if \(f\) is an intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) continuous function and \((Y, \mathcal{G}_{\text{str}})\) is an intuitionistic fuzzy intuitionistic fuzzy \(\mathcal{G}_{\text{str}},\) clo-\(-T_1\) space, then \(G(f)\) is an intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) co-closed graph in \(\xi_X \times \zeta_Y\);

(iii) If \(f\) is an intuitionistic fuzzy injective function has an intuitionistic fuzzy \(\alpha_{\mathcal{G}_{\text{str}}},\) co-
closed graph in $\zeta^X \times \zeta^Y$. Then $(X, \mathcal{F}_{1str})$ is an intuitionistic fuzzy $\alpha_{\mathcal{G}_{1str}}$-$T_1$ space.

**Proof.** (i) Let $(x_{r,s}, y_{m,n}) \in \zeta^X \times \zeta^Y \setminus G(f)$ and $f(x_{r,s}) \notin y_{m,n}$. Since $(Y, \mathcal{G}_{2str})$ is an intuitionistic fuzzy $\alpha_{\mathcal{G}_{2str}}$-$T_2$ space, there exist intuitionistic fuzzy $\mathcal{G}_{2str}$ clopen sets $A$ and $B$ in $Y$ containing $f(x_{r,s})$ and $y_{m,n}$ respectively such that $A \cap B = \emptyset$. Since $f$ is an intuitionistic fuzzy $E_{\alpha_{\mathcal{G}_{str}}}$ continuous function, there exists an intuitionistic fuzzy $\alpha_{\mathcal{G}_{1str}}$ open set $C$ in $X$ containing $x_{r,s}$ such that $f(E_{\alpha_{\mathcal{G}_{str}}} (C)) = f(C) \subseteq A$ and $f(C) \cap A = \emptyset$. This implies that $G(f)$ is an intuitionistic fuzzy $\alpha_{\mathcal{G}_{1str}}$-$co$-closed graph in $\zeta^X \times \zeta^Y$.

(ii) Let $(x_{r,s}, y_{m,n}) \in \zeta^X \times \zeta^Y \setminus G(f)$ and $f(x_{r,s}) \notin y_{m,n}$. Since $(Y, \mathcal{G}_{2str})$ is an intuitionistic fuzzy $\alpha_{\mathcal{G}_{2str}}$-$T_2$ space, there exists an intuitionistic fuzzy $\mathcal{G}_{2str}$ clopen set $A$ in $Y$ such that $f(x_{r,s}) \in A$ and $f(x_{r,s}) \notin A$. Since $f$ is an intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$ continuous function, there exists an intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$ open set $B$ in $X$ containing $x_{r,s}$ such that $f(B) \subseteq A$. Therefore, $f(B) \cap \overline{A} = \emptyset$, and $\overline{A}$ is an intuitionistic fuzzy $\mathcal{G}_{2str}$ clopen in $Y$ containing $y_{m,n}$. This implies that $G(f)$ is an intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$-$co$-closed set in $\zeta^X \times \zeta^Y$.

(iii) Let $x_{r,s}$ and $y_{m,n}$ be any two intuitionistic fuzzy points in $X$. Then $(x_{r,s}, f(x_{r,s})) \in \zeta^X \times \zeta^Y \setminus G(f)$. Since $G(f)$ is an intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$-$co$-closed graph in $\zeta^X \times \zeta^Y$, there exists an intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$ open set $A$ in $X$ and an intuitionistic fuzzy $\mathcal{G}_{2str}$ clopen set $B$ in $Y$ such that $x_{r,s} \in A, f(y_{m,n}) \in B$ and $f(A) \cap B = \emptyset$. Since $f$ is an intuitionistic fuzzy injective function, $A \cap f^{-1}(B) = \emptyset$, and $f^{-1}(B)$ is an intuitionistic fuzzy $\mathcal{G}_{2str}$ clopen set in $Y$ containing $y_{r,s}$ and $y_{m,n} \notin A$. Thus $(X, \mathcal{F}_{1str})$ is an intuitionistic fuzzy $\alpha_{\mathcal{G}_{str}}$-$T_1$ space.

**References**


SCIENTIA MAGNA
An international journal

ISBN 9781599731810