STRUCTURAL PROPERTIES OF NEUTROSOPHIC ABEL-GRASSMANN'S GROUPOIDS

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Abstract. In this paper, we have introduced the notion of neutrosophic $(2,2)$-regular, neutrosophic strongly regular neutrosophic $\mathcal{AG}$-groupoids and investigated these structures. We have shown that neutrosophic regular, neutrosophic intra-regular and neutrosophic strongly regular $\mathcal{AG}$-groupoid are the only generalized classes of an $\mathcal{AG}$-groupoid. Further we have shown that non-associative regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular and strongly regular neutrosophic $\mathcal{AG}^*$-groupoids do not exist.

Keywords. A neutrosophic $\mathcal{AG}$-groupoid, left invertive law, medial law and paramedial law.

Introduction

We know that in every branch of science there are lot of complications and problems arise which affluence with uncertainties and impaction. most of these problems and complications are relating with human life. These problems also play pivotal role for being subjective and classical. For Instance, commonly used methods are not sufficient to apply on these for the reason that these problems can not deal with various involved ambiguities in it.. To solve these complications, concept of fuzzy sets was published by Lotfi A.Zadeh in 1965, which has a wide range of applications in various fields such as engineering , artificial intelligence, control engineering, operation research, management science, robotics and many more. It give us model the uncertainty present in a phenomena that do not have sharp boundaries. many papers on fuzzy sets have been appeared which shows the importance and its applications to the set theory, algebra, real analysis,measure theory and topology etc. fuzzy set theory is applied in many real applications to handle uncertainty.

In literature, a lot of theories have been developed to contend with uncertainty, imprecision and vagueness. In which, theory of probability, rough set theory fuzzy set theory, intiutionistic fuzzy sets etc, have played imperative role to cope with diverse types of uncertainties and imprecision entrenched in a system. But all these above theories were not sufficient tool to deal with indeterminate and inconsistent

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information in believe system. For Instance, F.Samrandache noticed that the law of excluded middle are presently inactive in the modern logics and getting inspired with sport games (winning/tie/defeating), voting system (yes/ NA/no), decision making (making a decision/hesitating/not making) etc, In 1995 F. Samrandache developed a new concept called neutrosophic set (NS) which is basically generalization of fuzzy sets and intiutionistic fuzzy sets. NS can be described by membership degree, and indeterminate degree and non-membership degree. This theory with its hybrid structures have proven efficient tool in different fields such as control theory, databases, medical diagnosis problem, decision making problem, physics and topology etc.

Since indeterminacy almost exists everywhere in this world, the neutrosophics found place into contemporary research. The fundamental theory of neutrosophic set, proposed by Smarandache. Salama et al. provide a natural basis for trating mathematically the neutrosophic phenomena which presents pervasively in our real world and for developing new branches of neutrosophic mathematics.The neutrosophic logic is an extended idea of neutrosophy.. By giving representation to indeterminates, the introduction of neutrosophic theory has put forth a significant concept. Uncertainty or indeterminacy proved to be one of the most important factor in approximately all real-world problems. Fuzzy theory is used when uncertainty is modeled and when there is indeterminancy involved we use neutrosophic theory. Most of fuzzy models dealing with the analysis and study of unsupervised data, make use of the directed graphs or bipartite graphs. Thus the use of graphs in fuzzy models becomes inevitable. The neutrosophic models are basically fuzzy models that authorize the factor of indeterminancy.

The neutrosophic algebraic structures have defined very recently. Basically, Vasantha K andasmy and Florentin Smarandache present the concept of neutrosophic algebraic structures by using neutrosophic theory. A number of the neutrosophic algrebraic structures introduced and considered include neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, neutrosophic bigroupoids and neutrosophic AG-groupoids.

Abel Grassmann's groupoid abbreviated as an AG-groupoid is a groupoid whose elements satisfies the left invertive law i.e $(ab)c = (cb)a$ for all $a, b, c \in S$. An AG-groupoid is a non associative and non-commutative algebraic structure mid way between a groupoid and commutative semigroup. AG-groupoids generalizes the concept of commutative semigroup and have an important application within the theory of flocks.

An $\mathcal{AG}$-groupoid [10], is a groupoid $\mathcal{S}$ holding the left invertive law

$$(ab)c = (cb)a, \text{ for all } a, b, c \in \mathcal{S}. \tag{1}$$

Basic Laws of $\mathcal{AG}$-groupoid

This left invertive law has been obtained by introducing braces on the left of ternary commutative law $abc = cba$.

In an $\mathcal{AG}$-groupoid, the medial law holds [2]

$$(ab)(cd) = (ac)(bd), \text{ for all } a, b, c, d \in \mathcal{S}. \tag{2}$$

In an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, the paramedial law holds [7]

$$(ab)(cd) = (dc)(ba), \text{ for all } a, b, c, d \in \mathcal{S}. \tag{3}$$

Further if an $\mathcal{AG}$-groupoid contains a left identity, the following law holds [7]

$$a(bc) = b(ac), \text{ for all } a, b, c \in \mathcal{S}. \tag{4}$$

Since the world is full of indeterminacy, the neutrosophics found their place into contemporary research. In 1995, Florentin Smarandache introduced the idea of neutrosophy. Neutrosophic logic is an extension of fuzzy logic. In 2003 W.B Vasantha Kandasamy and Florentin Smarandache introduced algebraic structures (such as neutrosophic semigroup, neutrosophic ring, etc.). Madad Khan et al., for the first time introduced the idea of a neutrosophic LA-semigroup in [4]. Moreover $SUI = \{a + bI$: where a, $b \in S$ and I is literal indeterminacy such that $I^2 = I\}$ becomes neutrosophic LA-semigroup under the operation defined as:

$(a + bI) * (c + dI) = ac + bdI$ for all $(a + bI)$, $(c + dI) \in SUI$. That is $(SUI, *)$ becomes neutrosophic LA-semigroup. They represented it by $N(S)$.

$$[(a_1 + a_2I)(b_1 + b_2I)](c_1 + c_2I) = [(c_1 + c_2I)(b_1 + b_2I)](a_1 + a_2I), \tag{1}$$

holds for all $(a_1 + a_2I)$, $(b_1 + b_2I)$, $(c_1 + c_2I) \in N(S)$.

It is since than called the neutrosophic left invertive law. A neutrosophic groupoid satisfying the left invertive law is called a neutrosophic left almost semigroup and is abbreviated as neutrosophic LA-semigroup.

In a neutrosophic LA-semigroup $N(S)$ medial law holds i.e

$$[(a_1{+}a_2I)(b_1{+}b_2I)][(c_1{+}c_2I)(d_1{+}d_2I)] = [(a_1{+}a_2I)(c_1{+}c_2I)][(b_1{+}b_2I)(d_1{+}d_2I)], \tag{2}$$

holds for all $(a_1 + a_2I)$, $(b_1 + b_2I)$, $(c_1 + c_2I)$, $(d_1 + d_2I) \in N(S)$.

There can be a unique left identity in a neutrosophic LA-semigroup. In a neutrosophic LA-semigroup $N(S)$ with left identity $(e + eI)$ the following laws hold for all $(a_1 + a_2I)$, $(b_1 + b_2I)$, $(c_1 + c_2I)$, $(d_1 + d_2I) \in N(S)$.

$$[(a_1{+}a_2I)(b_1{+}b_2I)][(c_1{+}c_2I)(d_1{+}d_2I)] = [(d_1{+}d_2I)(b_1{+}b_2I)][(c_1{+}c_2I)(a_1{+}a_2I)], \tag{3}$$

$$[(a_1{+}a_2I)(b_1{+}b_2I)][(c_1{+}c_2I)(d_1{+}d_2I)] = [(d_1{+}d_2I)(c_1{+}c_2I)][(b_1{+}b_2I)(a_1{+}a_2I)], \tag{4}$$

and

$$(a_1 + a_2I)[(b_1 + b_2I)(c_1 + c_2I)] = (b_1 + b_2I)[(a_1 + a_2I)(c_1 + c_2I)]. \tag{5}$$

for all $(a_1 + a_2I)$, $(b_1 + b_2I)$, $(c_1 + c_2I) \in N(S)$.

(3) is called neutrosophic paramedial law and a neutrosophic LA semigroup satisfies (5) is called neutrosophic AG^{**}-groupoid.

Now, $(a + bI)^2 = a + bI$ implies $a + bI$ is idempotent and if holds for all $a + bI \in N(S)$ then $N(S)$ is called idempotent neutrosophic LA-semigroup.

This structure is closely related with a neutrosophic commutative semigroup, because if a Neutrosophic $\mathcal{AG}$-groupoid contains a right identity, then it becomes a commutative semigroup. Define the binary operation "$\bullet$" on a commutative inverse semigroup $N(\mathcal{S})$ as

$$(a_1 + a_2I) \bullet (b_1 + b_2I) = (b_1 + b_2I)(a_1 + a_2I)^{-1}$$

for all $a_1 + a_2I, b_1 + b_2I \in N(\mathcal{S})$

then $(N(\mathcal{S}), \bullet)$ becomes an $\mathcal{AG}$-groupoid.

A neutrosophic $\mathcal{AG}$-groupoid $(\mathcal{S}, \cdot)$ with neutrosophic left identity becomes a neutrosophic semigroup $\mathcal{S}$ under new binary operation "$\circ$" defined as

$$(x_1 + x_2I) \circ (y_1 + y_2I) = [(x_1 + x_2I)(a_1 + a_2I)](y_1 + y_2I)$$

for all $x_1 + x_2I, y_1 + y_2I \in N(\mathcal{S})$.

It is easy to show that "$\circ$" is associative

$$\begin{aligned}
& [(x_1 + x_2I) \circ (y_1 + y_2I)] \circ (z_1 + z_2I) \\
= \; & [[[(x_1 + x_2I)(a_1 + a_2I)](y_1 + y_2I)](a_1 + a_2I)](z_1 + z_2I) \\
= \; & [[(z_1 + z_2I)(a_1 + a_2I)][[(x_1 + x_2I)(a_1 + a_2I)](y_1 + y_2I)]] \\
= \; & [(x_1 + x_2I)(a_1 + a_2II)[[(z_1 + z_2I)(a_1 + a_2I)](y_1 + y_2I)] \\
= \; & [(x_1 + x_2I)(a_1 + a_2I)[(y_1 + y_2I)(a_1 + a_2I)](z_1 + z_2I)] \\
= \; & (x_1 + x_2I) \circ [(y_1 + y_2I) \circ (z_1 + z_2I)].
\end{aligned}$$

Hence $(\mathcal{S}, \circ)$ is a neutrosophic semigroup. The Connections discussed above make this non-associative structure interesting and useful.

1. Regularities in Neutrosophic $\mathcal{AG}$-groupoids

An element $a + bI$ of a Neutrosophic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called a **regular** element of $N(\mathcal{S})$ if there exists $x_1 + x_2I \in N(\mathcal{S})$ such that $a + bI = [(a + bI) * (x_1 + x_2I)](a + bI)$ and $\mathcal{S}$ is called regular if all elements of $\mathcal{S}$ are regular.

An element $a + bI$ of an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called a **weakly regular** element of $\mathcal{S}$ if there exist $x_1 + x_2I, y_1 + y_2I \in N(\mathcal{S})$ such that $a + bI = [(a + bI)(x_1 + x_2I)][(a + bI)(y_1 + y_2I)]$ and $N(\mathcal{S})$ is called weakly regular if all elements of $\mathcal{S}$ are weakly regular.

An element $a + bI$ of a Neutrosophic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called an **intra-regular** element of $N(\mathcal{S})$ if there exist $x_1 + x_2I, y_1 + y_2I \in N(\mathcal{S})$ such that $a + bI = [(x_1 + x_2I)(a + bI)^2](y_1 + y_2I)$ and $N(\mathcal{S})$ is called intra-regular if all elements of $N(\mathcal{S})$ are intra-regular.

An element $a+bI$ of a Neutrosophic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called a **right regular** element of $N(\mathcal{S})$ if there exists $x_1 + x_2I \in N(\mathcal{S})$ such that $a + bI = (a + bI)^2(x_1 + x_2I) = [(a + bI)(a + bI)](x_1 + x_2I)$ and $N(\mathcal{S})$ is called right regular if all elements of $N(\mathcal{S})$ are right regular.

An element $a + bI$ of a Neutrosophic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called a **left regular** element of $\mathcal{S}$ if there exists $x_1 + x_2I \in N(\mathcal{S})$ such that $a + bI = (x_1 + x_2I)(a + bI)^2 = (x_1 + x_2I)[(a + bI)(a + bI)]$ and $N(\mathcal{S})$ is called left regular if all elements of $N(\mathcal{S})$ are left regular.

An element $a + bI$ of a Neutrosophic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called a **left quasi regular** element of $N(\mathcal{S})$ if there exist $x_1 + x_2I, y_1 + y_2I \in N(\mathcal{S})$ such that $a + bI = [(x_1 + x_2I)(a + bI)][(y_1 + y_2I)(a + bI)]$ and $N(\mathcal{S})$ is called left quasi regular if all elements of $\mathcal{S}$ are left quasi regular.

An element $a + bI$ of a Neutrosophic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called a **completely regular** element of $N(\mathcal{S})$ if $a + bI$ is regular, left regular and right regular. $N(\mathcal{S})$ is called completely regular if it is regular, left and right regular.

An element $a + bI$ of a Neutrosopic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called a **(2,2)-regular** element of $N(\mathcal{S})$ if there exists $x_1 + x_2I \in N(\mathcal{S})$ such that $a + bI = [(a + bI)^2(x_1 +$

$x_2I)](a+bI)^2$ and $N(\mathcal{S})$ is called $(2,2)$-regular $\mathcal{AG}$-groupoid if all elements of $N(\mathcal{S})$ are $(2,2)$-regular.

An element $a+bI$ of a Neutrosophic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called a **strongly regular** element of $N(\mathcal{S})$ if there exists $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x_1+x_2I)](a+bI)$ and $(a+bI)(x_1+x_2I) = (x_1+x_2I)(a+bI)$. $N(\mathcal{S})$ is called strongly regular Neutrosophic $\mathcal{AG}$-groupoid if all elements of $N(\mathcal{S})$ are strongly regular.

A Neutrosophic $\mathcal{AG}$-groupoid $N(\mathcal{S})$ is called Neutrosophic $\mathcal{AG}^*$-groupoid if the following holds [6]

$$[(a_1+a_2I)(b_1+b_2I)](c_1+c_2I) = (b_1+b_2I)[(a_1+a_2I)(c_1+c_2I)] \tag{5}$$

for all $a_1+a_2I, b_1+b_2I, c_1+c_2I \in N(\mathcal{S})$.

In Neutrosophic $\mathcal{AG}^*$-groupoid $\mathcal{S}$, the following law holds [10]

$$(x_1x_2)(x_3x_4) = (x_{p(1)}x_{p(2)})(x_{p(3)}x_{p(4)}), \tag{6}$$

where $\{p(1),p(2),p(3),p(4)\}$ means any permutation on the set $\{1,2,3,4\}$. It is an easy consequence that if $\mathcal{S}=\mathcal{S}^2$, then $\mathcal{S}$ becomes a commutative semigroup.

A Neutrosophic $\mathcal{AG}$-groupoid may or may not contains a left identity. The left identity of a Neutrosophic $\mathcal{AG}$-groupoid allow us to introduce the inverses of elements in a Neutrosophic $\mathcal{AG}$-groupoid. If an $\mathcal{AG}$-groupoid contains a left identity, then it is unique [7].

Example 1. *Let us consider a Neutrosophic $\mathcal{AG}$-groupoid*
$\mathcal{S} = \{1+1I,\ 1+2I,\ 1+3I,\ 2+1I,\ 2+2I,\ 2+3I,\ 3+1I,\ 3+2I,\ 3+3I\}$ *in the following multiplication table.*

$*$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+1I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+2I$	$1+3I$	$1+1I$	$1+2I$	$2+2I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$
$1+3I$	$1+2I$	$1+3I$	$1+1I$	$2+3I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$
$2+1I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$
$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$
$2+3I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$
$3+1I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$
$3+2I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$
$3+3I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$

Clearly $\mathcal{S}$ is non-commutative and non-associative, because $bc \neq cb$ and $(cc)a \neq c(ca)$. Note that $\mathcal{S}$ has no left identity.

Lemma 1. *If $N(\mathcal{S})$ is a regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular or strongly regular neutrosophic$\mathcal{AG}$-groupoid, then $N(S)$= $N(\mathcal{S})^2$.*

Proof. Let $N(\mathcal{S})$ be a neutrosophic regular $\mathcal{AG}$-groupoid, then $N(\mathcal{S})^2 \subseteq N(\mathcal{S})$ is obvious. Let $a+bI \in N(\mathcal{S})$, then since $N(\mathcal{S})$ is regular so there exists $x+yI \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x+yI)](a+bI)$. Now

$$a+bI = [(a+bI)(x+yI)](a+bI) \in N(\mathcal{S})\mathcal{N}(\mathcal{S}) = N(\mathcal{S})^2.$$

Similarly if $N(\mathcal{S})$ is weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular or strongly regular, then we can show that $N(\mathcal{S}) = N(\mathcal{S})^2$. $\square$

The converse is not true in general, because in Example **??**, $N(\mathcal{S}) = N(\mathcal{S})^2$ holds but $N(\mathcal{S})$ is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular and strongly regular, because $d_1 + d_2I \in N(\mathcal{S})$ is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular and strongly regular.

Theorem 1. *If $N(\mathcal{S})$ is a Neutrosophic $\mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then $N(\mathcal{S})$ is intra-regular if and only if for all $a+bI \in N(\mathcal{S})$, $a+bI = [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)]$ holds for some $x_1+x_2I, z_1+z_2I \in N(\mathcal{S})$.*

Proof. Let $N(\mathcal{S})$ be an intra-regular $Neutrosophic\ \mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I)$. Now by using Lemma 1, $y_1+y_2I = (u_1+u_2I)(v_1+v_2I)$ for some $u_1+u_2I, v_1+v_2I \in N(\mathcal{S})$.

$$
\begin{aligned}
& a+bI \\
= \;& [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= \;& [(x_1+x_2I)[(a+bI)(a+bI)]](y_1+y_2I) \\
= \;& [(a+bI)[(x_1+x_2I)(a+bI)]](y_1+y_2I) \\
= \;& [(y_1+y_2I)[(x_1+x_2I)(a+bI)]](a+bI) \\
= \;& [(y_1+y_2I)[(x_1+x_2I)(a+bI)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= \;& [[(u_1+u_2I)(v_1+v_2)][(x+yI)(a+bI)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I), \\
= \;& [[(a+bI)(x_1+x_2I)][(v_1+v_2)(u_1+u_2I)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= \;& [[(a+bI)(x_1+x_2I)](t_1+t_2I)][(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= \;& [[(x_1+x_2I)(a+bI)^2](y_1+y_2I)](t_1+t_2I)][(a+bI)(x_1+x_2I)] \\
= \;& [[(t_1+t_2I)(y_1+y_2I)][(x_1+x_2I)(a+bI)^2]][(a+bI)(x_1+x_2I)] \\
= \;& [[(a+bI)^2(x_1+x_2I)][(y_1+y_2I)(t_1+t_2I)]][(a+bI)(x_1+x_2I)] \\
= \;& [[(a+bI)^2(x_1+x_2I)][(s_1+s_2I)]][(a+bI)(x_1+x_2I)], \\
= \;& [[(s_1+s_2I)(x_1+x_2I)](a+bI)^2][(a+bI)(x_1+x_2I)] \\
= \;& [[(s_1+s_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(a+bI)(x_1+x_2I)] \\
= \;& [[(a+bI)(a+bI)][(x_1+x_2I)(s_1+s_2I)]][(a+bI)(x_1+x_2I)] \\
= \;& [[(a+bI)(a+bI)](w_1+w_2I)[(a+bI)(x_1+x_2I)], \\
= \;& [[(w_1+w_2I)(a+bI)](a+bI)][(a+bI)(x_1+x_2I)] \\
= \;& [(z_1+z_2I)(a+bI)][(a+bI)(x_1+x_2I)], \\
= \;& [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)]
\end{aligned}
$$

where $(w_1+w_2I)(a+bI) = (z_1+z_2I) \in N(S)$ where $(x_1+x_2I)(s_1+s_2I) = (w_1+w_2I) \in N(S)$ where $(y_1+y_2I)(t_1+t_2I) = (s_1+s_2I) \in N(S)$ where $(v_1+v_2)(u_1+u_2I) = (t_1+t_2I) \in N(S)$ where $[(u_1+u_2I)(v_1+v_2) = (y_1+y_2I) \in N(S)$ $\square$

Proof. Conversely, let for all $a+bI \in N(\mathcal{S})$, $a+bI = [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)]$ holds for some $x_1+x_2I, z_1+z_2I \in N(\mathcal{S})$. Now by using (4), (1), (2) and (3), we have

$$\begin{aligned}
& a+bI \\
= & [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)] \\
= & [(a+bI)[(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= & [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= & [(a+bI)[(x_1+x_2I)(a+bI)](z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= & [[(x_1+x_2I)(a+bI)](z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)]](a+bI) \\
= & [[(x_1+x_2I)(a+bI)](z_1+z_2I)]^2(a+bI) \\
= & [[(x_1+x_2I)(a+bI)]^2(z_1+z_2I)^2](a+bI) \\
= & [[(x_1+x_2I)^2(a+bI)^2][(z_1+z_2I)(z_1+z_2I)]](a+bI) \\
= & [[(x_1+x_2I)^2(z_1+z_2I)][(a+bI)^2(z_1+z_2I)]](a+bI) \\
= & [(a+bI)^2[[(x_1+x_2I)^2(z_1+z_2I)](z_1+z_2I)](a+bI) \\
= & [[(a+bI)(a+bI)][(x_1+x_2I)^2(z_1+z_2I)](z_1+z_2I)](a+bI) \\
= & [[(a+bI)(a+bI)][[(z_1+z_2I)(z_1+z_2I)](x_1+x_2I)^2]](a+bI) \\
= & [[(a+bI)(a+bI)][(z_1+z_2I)^2(x_1+x_2I)^2]](a+bI) \\
= & [[(x_1+x_2I)^2(z_1+z_2I)^2][(a+bI)(a+bI)]](a+bI) \\
= & [(t_1+t_2I)[(a+bI)(a+bI)]](a+bI), \\
= & [(t_1+t_2I)(a+bI)^2](u_1+u_2I)
\end{aligned}$$

where $[(x_1+x_2I)^2(z_1+z_2I)^2] = (t_1+t_2I) \in N(S)$ and $(a+bI) = (u_1+u_2I) \in N(S)$ where $(a+bI) = (u_1+u_2I) \in N(S)$ □

Proof. Thus $N(\mathcal{S})$ is intra-regular. □

Theorem 2. *If $N(\mathcal{S})$ is a Neutrosophic $\mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then the following are equivalent.*

(*i*) $N(\mathcal{S})$ is weakly regular.
(*ii*) $N(\mathcal{S})$ is intra-regular.

Proof. (*i*) $\Longrightarrow$ (*ii*) Let $N(\mathcal{S})$ be a weakly regular *Neutrosophic* $\mathcal{AG}$-groupoid with left identity (*Neutrosophic*$\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$ and by Lemma 1, $x_1+x_2I = (u_1+u_2I)(v_1+v_2I)$ for some $u_1+u_2I, v_1+v_2I \in N(\mathcal{S})$. Let $(v_1+v_2I)(u_1+u_2I) = t_1+t_2I \in N(\mathcal{S})$. Now by using (3), (1), (4) and (2), we have

$$\begin{aligned}
a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= (x_1+x_2I)[(a+bI)^2(y_1+y_2I)] \\
&= [(u_1+u_2I)(v_1+v_2I)][(a+bI)^2(y_1+y_2I)] \\
&= [(y_1+y_2I)(a+bI)^2][(v_1+v_2I)(u_1+u_2I)] \\
&= [(y_1+y_2I)(a+bI)^2](t_1+t_2I)
\end{aligned}$$

Thus $N(\mathcal{S})$ is intra-regular.

(*ii*) $\Longrightarrow$ (*i*) Let $N(S)$ be a intra regular Neutrosophic AG-groupoid with left identity (Neutrosophic $A\mathcal{G}^{**}$-groupoid), then for any $a+bI \in N(S)$

$$\begin{aligned}
a+bI &= [(y_1+y_2I)(a+bI)^2](t_1+t_2I) \\
&= [(y_1+y_2I)(a+bI)^2][(v_1+v_2I)(u_1+u_2I)] \\
&= [(u_1+u_2I)(v_1+v_2I)][(a+bI)^2(y_1+y_2I)] \\
&= (x_1+x_2I)[(a+bI)^2(y_1+y_2I)] \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]
\end{aligned}$$

$\Longrightarrow$Thus $N(S)$ is weakly regular. □

Theorem 3. *If $N(\mathcal{S})$ is a Neutrosophic $\mathcal{AG}$-groupoid ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then the following are equivalent.*

(i) $N(\mathcal{S})$ is weakly regular.
(ii) $N(\mathcal{S})$ is right regular.

Proof. $(i) \Longrightarrow (ii)$ Let $N(\mathcal{S})$ be a weakly regular Neutrosophic $\mathcal{AG}$-groupoid ($\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = (a+bI)(x_1+x_2I)(a+bI)(y_1+y_2I)$ and let $(x_1+x_2I)(y_1+y_2I) = t_1+t_2I$ for some $t+tI \in N(\mathcal{S})$. Now by using (2), we have

$$\begin{aligned}
a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= (a+bI)^2(t_1+t_2I)
\end{aligned}$$

Thus $N(\mathcal{S})$ is right regular.
$(ii) \Longrightarrow (i)$ It follows from Lemma 1 and (2).

$$\begin{aligned}
a+bI &= (a+bI)^2(t_1+t_2I) \\
&= [(a+bI)(a+bI)](t_1+t_2I) \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]
\end{aligned}$$

where $(t_1+t_2I) = (x_1+x_2I)(y_1+y_2I) \in N(\mathcal{S})$. Thus $N(S)$ is weakly regular. □

Theorem 4. *If $N(\mathcal{S})$ is a Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then the following are equivalent.*
(i) $N(\mathcal{S})$ is weakly regular.
(ii) $N(\mathcal{S})$ is left regular.

Proof. $(i) \Longrightarrow (ii)$ Let $N(\mathcal{S})$ be a weakly regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$. Now by using (2) and (3), we have

$$\begin{aligned}
a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= [(y_1+y_2I)(x_1+x_2I)][(a+bI)(a+bI)] \\
&= [(y_1+y_2I)(x_1+x_2I)](a+bI)^2 \\
&= (t_1+t_2I)(a+bI)^2,
\end{aligned}$$

where $[(y_1+y_2I)(x_1+x_2I)] = (t_1+t_2I)$ for some $(t_1+t_2I) \in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is left regular.

$(ii) \Longrightarrow (i)$ It follows from Lemma 1, (3) and (2).

$$\begin{aligned} a+bI &= (t_1+t_2I)(a+bI)^2 \\ &= [(y_1+y_2I)(x_1+x_2I)](a+bI)^2 \\ &= [(y_1+y_2I)(x_1+x_2I)][(a+bI)(a+bI)] \\ &= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\ &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)], \end{aligned}$$

where($y_1+y_2I)(x_1+x_2I) = t_1+t_2I$ for some $t_1+t_2I \in N(\mathcal{S})$. Thus $N(S)$ is weakly regular. □

Theorem 5. *If $N(\mathcal{S})$ is a Neutrosophic $\mathcal{AG}$-groupoid with left identity $(Neutrosophic\mathcal{AG}^{**}$-groupoid$)$, then the following are equivalent.*

(i) $N(\mathcal{S})$ is weakly regular.

(ii) $N(\mathcal{S})$ is left quasi regular

Proof. $(i) \Longrightarrow (ii)$Let $N(\mathcal{S})$ be a weakly regular $Neutrosophic\mathcal{AG}$-groupoid with left identity, then for $a+bI \in N(\mathcal{S})$ there exists $x_1+x_2I, y+yI \in N(\mathcal{S})$ such that $a+bI = [(a+b)(x_1+x_2I)][(a+b)(y_1+y_2I)]$

$$\begin{aligned} a+bI &= [(a+b)(x_1+x_2I)][(a+b)(y_1+y_2I)] \\ &= [(y_1+y_2I)(a+b)][(x_1+x_2I)(a+b)]. \end{aligned}$$

Thus $N(S)$ is left quasi regular.

$(ii) \Longrightarrow (i)$ Let $N(\mathcal{S})$ be a left quasi regular NeuIf $N(\mathcal{S})$ is a Neutrosophic $\mathcal{AG}$-groupoid with left identity then the following are equivalent.

$$\begin{aligned} a+bI &= [(y_1+y_2I)(a+b)][(x_1+x_2I)(a+b)] \\ &= [(a+b)(x_1+x_2I)][(a+b)(y_1+y_2I)]. \end{aligned}$$

Thus $N(S)$ is weakly regular. □

Theorem 6. *If $N(\mathcal{S})$ is a Neutrosophic $\mathcal{AG}$-groupoid with left identity, then the following are equivalent.*

(i) $N(\mathcal{S})$ is $(2,2)$-regular.

(ii) $N(\mathcal{S})$ is completely regular.

Proof. $(i) \Longrightarrow (ii)$ Let $N(\mathcal{S})$ be a $(2,2)$-regular $Neutrosophic\mathcal{AG}$-groupoid with left identity, then for $a+bI \in N(\mathcal{S})$ there exists $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = [(a+bI)^2(x+yI)](a+bI)^2$. Now

$$\begin{aligned} a+bI &= [(a+bI)^2(x_1+x_2I)](a+bI)^2 \\ &= [(y_1+y_2I)(a+bI)^2], \end{aligned}$$

where $(a+bI)^2(x_1+x_2I) = y_1+y_2I \in N(\mathcal{S})$, and by using (3), we have

$$\begin{aligned} a+bI &= [(a+bI)^2(x_1+x_2I)][(a+bI)(a+bI)] \\ &= [(a+bI)(a+bI)][(x_1+x_2I)[(a+bI)^2] \\ &= (a+bI)^2(z_1+z_2I), \end{aligned}$$

where $(x_1+x_2I)(a+bI)^2 = z_1+z_2I \in N(\mathcal{S})$. And by using (3), (1) and (4), we have

$$\begin{aligned}
& a+bI \\
= & [(a+bI)^2[(x_1+x_2I)[(a+bI)(a+bI)]] \\
= & [(a+bI)(a+bI)][(x_1+x_2I)(a+bI)^2] \\
= & [(a+bI)(a+bI)][[(e_1+e_2I)(x_1+x_2I)][a+bI)(a+bI)]] \\
= & [(a+bI)(a+bI)][[(a+bI)(a+bI)][(x_1+x_2I)(e_1+e_2I)]] \\
= & [(a+bI)(a+bI)][(a+bI)^2(t_1+t_2I)] \\
= & [[(a+bI)^2(t_1+t_2I)](a+bI)](a+bI) \\
= & [[[(a+bI)(a+bI)](t_1+t_2I)](a+bI)](a+bI) \\
= & [[(t_1+t_2I)(a+bI)][(a+bI)](a+bI)]](a+bI) \\
= & [[(a+bI)(a+bI)][(t_1+t_2I)(a+bI)]](a+bI) \\
= & [[(a+bI)(t_1+t_2I)][(a+bI)(a+bI)]](a+bI) \\
= & [(a+bI)[[(a+bI)(t_1+t_2I)](a+bI)]](a+bI) \\
= & [(a+bI)(u_1+u_2I)](a+bI)
\end{aligned}$$

where $t_1+t_2I = (x_1+x_2I)(e_1+e_2I) \in N(\mathcal{S})$ & where $u_1+u_2I = (a+bI)^2(t_1+t_2I). \in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is neutrosophic left regular, right regular and regular, so $N(\mathcal{S})$ is completely regular.

$(ii) \Longrightarrow (i)$ Assume that $N(\mathcal{S})$ is a completely regular neutrosophic $\mathcal{AG}$-groupoid with left identity, then for any $a+bI \in N(\mathcal{S})$ there exist $x+xI, y+yI, z+zI \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x+xI)](a+bI)$, $a+bI = (a+bI)^2(y+yI)$ and $a+bI = (z+zI)(a+bI)^2$. Now by using (1), (4) and (3), we have

$$\begin{aligned}
& a+bI \\
= & [(a+bI)(x+xI)](a+bI) \\
= & [[(a+bI)^2(y+yI)](x+xI)][(z_1+z_2I)(a+bI)^2] \\
= & [[(x_1+x_2I)(y_1+y_2I)](a+bI)^2][(z_1+z_2I)(a+bI)^2] \\
= & [[(x_1+x_2I)(y_1+y_2I)][(a+bI)(a+bI)]][(z_1+z_2I)(a+bI)^2] \\
= & [[(a+bI)(a+bI)][(y_1+y_2I)(x_1+x_2I)]][(z_1+z_2I)(a+bI)^2] \\
= & [(a+bI)^2[(y_1+y_2I)(x_1+x_2I)]][(z_1+z_2I)(a+bI)^2] \\
= & [[(z_1+z_2I)(a+bI)^2][(y_1+y_2I)(x_1+x_2I)]](a+bI)^2 \\
= & [[(z_1+z_2I)(y_1+y_2I)][(a+bI)^2(x_1+x_2I)]](a+bI)^2 \\
= & [(a+bI)^2[[(z_1+z_2I)(y_1+y_2I)](x_1+x_2I)]](a+bI)^2 \\
= & [(a+bI)^2(v_1+v_2I)](a+bI)^2,
\end{aligned}$$

where $[(z_1+z_2I)(y_1+y_2I)][(x_1+x_2I)] = (v_1+v_2I) \in N(\mathcal{S})$. This shows that $N(\mathcal{S})$ is $(2,2)$-regular. □

Lemma 2. *Every weakly regular neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosopic\mathcal{AG}^{**}$-groupoid) is regular.*

Proof. Assume that $N(\mathcal{S})$ is a weakly regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$. Let

$(x_1+x_2I)(y_1+y_2I) = t_1+t_2I \in N(\mathcal{S})$ and $[(t_1+t_2I)[(y_1+y_2I)(x_1+x_2I)]](a+bI) = u_1+u_2I \in N(\mathcal{S})$. Now by using (1), (2), (3) and (4), we have

$$
\begin{aligned}
& a+bI \\
= \; & [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\
= \; & [[(a+bI)(y_1+y_2I)](x_1+x_2I)](a+bI) \\
= \; & [[(x_1+x_2I)(y_1+y_2I)](a+bI)](a+bI) \\
= \; & [(t_1+t_2I)(a+bI)](a+bI) \\
= \; & [(t_1+t_2I)[[(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]]](a+bI) \\
= \; & [(t_1+t_2I)[[(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)]]](a+bI) \\
= \; & [(t_1+t_2I)[[(y_1+y_2I)(x_1+x_2I)][(a+bI)(a+bI)]]](a+bI) \\
= \; & [(t_1+t_2I)[(a+bI)[[(y_1+y_2I)(x_1+x_2I)](a+bI)]]](a+bI) \\
= \; & [(a+bI)[(t_1+t_2I)[[(y_1+y_2I)(x_1+x_2I)](a+bI)]]](a+bI) \\
= \; & [(a+bI)(u_1+u_2I)](a+bI),
\end{aligned}
$$

where $[(t_1+t_2I)[[(y_1+y_2I)(x_1+x_2I)](a+bI)]] = u_1+u_2I \in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is regular. □

The converse of Lemma 2 is not true in general, as can be seen from the following example.

Example 2. [10] *Let us consider an $\mathcal{AG}$-groupoid $\mathcal{S} = \{1,2,3,4\}$ with left identity* 3 *in the following Cayley's table.*

$\bullet$	$1+1I$	$1+2I$	$1+3I$	$1+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$3+1I$	$3+2I$	$3+3I$	$3+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$
$1+1I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$	$4+2I$	$4+2I$	$4+4I$	$4+4I$	$4+2I$	$4+2I$	$4+4I$	$4+4I$
$1+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$4+2I$	$4+2I$	$4+2I$	$4+2I$	$4+2I$	$4+2I$	$4+2I$	$4+2I$
$1+3I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$
$1+4I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$4+1I$	$4+2I$	$4+1I$	$4+2I$	$4+1I$	$4+2I$	$4+1I$	$4+2I$
$2+1I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$
$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$
$2+3I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$
$2+4I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$
$3+1I$	$1+2I$	$1+2I$	$1+4I$	$1+4I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$	$3+2I$	$3+2I$	$3+4I$	$3+4I$	$4+2I$	$4+2I$	$4+4I$	$4+4I$
$3+2I$	$1+2I$	$1+2I$	$1+2I$	$1+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$3+2I$	$3+2I$	$3+2I$	$3+2I$	$4+2I$	$4+2I$	$4+2I$	$4+2I$
$3+3I$	$1+1I$	$1+2I$	$1+3I$	$1+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$3+1I$	$3+2I$	$3+3I$	$3+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$
$3+4I$	$1+1I$	$1+2I$	$1+1I$	$1+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$3+1I$	$3+2I$	$3+1I$	$3+2I$	$4+1I$	$4+2I$	$4+1I$	$4+2I$
$4+1I$	$1+2I$	$1+2I$	$1+4I$	$1+4I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$	$1+1I$	$1+2I$	$1+4I$	$1+4I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$
$4+2I$	$1+2I$	$1+2I$	$1+2I$	$1+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$1+1I$	$1+2I$	$1+2I$	$1+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$
$4+3I$	$1+1I$	$1+2I$	$1+3I$	$1+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$1+2I$	$1+2I$	$1+3I$	$1+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$
$4+4I$	$1+1I$	$1+2I$	$1+1I$	$1+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$1+2I$	$1+2I$	$1+1I$	$1+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$

Clearly $\mathcal{S}$ is regular, because $1 = (1.3).1$, $2 = (2.1).2$, $3 = (3.3).3$ and $4 = (4.1).4$, but $\mathcal{S}$ is not weakly regular, because $1 \in \mathcal{S}$ is not a weakly regular element of $\mathcal{S}$.

Theorem 7. *If $N(\mathcal{S})$ is a Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then the following are equivalent.*
(i) $N(\mathcal{S})$ is weakly regular.
(ii) $N(\mathcal{S})$ is completely regular.

Proof. $(i) \Longrightarrow (ii)$
Let $N(\mathcal{S})$ be a weakly regular Neutrosophic $\mathcal{AG}$-groupoid ($\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = (a+bI)(x_1+x_2I)(a+bI)(y_1+y_2I)$ and let $(x_1+x_2I)(y_1+y_2I) = t_1+t_2I$ for some $t+tI \in N(\mathcal{S})$. Now by using (2), we have

$$\begin{aligned} a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\ &= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\ &= (a+bI)^2(t_1+t_2I), \end{aligned}$$

where $(x_1+x_2I)(y_1+y_2I) = t_1+t_2I \in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is right regular.
Let $N(\mathcal{S})$ be a weakly regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$. Now by using (2) and (3), we have

$$\begin{aligned} a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\ &= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\ &= [(y_1+y_2I)(x_1+x_2I)][(a+bI)(a+bI)] \\ &= [(y_1+y_2I)(x_1+x_2I)](a+bI)^2 \\ &= (t_1+t_2I)(a+bI)^2, \end{aligned}$$

where($y_1+y_2I)(x_1+x_2I) = t_1+t_2I$ for some $t_1+t_2I \in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is left regular.
Assume that $N(\mathcal{S})$ is a weakly regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$. Let $(x_1+x_2I)(y_1+y_2I) = t_1+t_2I \in N(\mathcal{S})$ and $[(t_1+t_2I)[(y_1+y_2I)(x_1+x_2I)]](a+bI) = u_1+u_2I \in N(\mathcal{S})$. Now by using (1), (2), (3) and (4), we have

$$\begin{aligned} a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\ &= [[(a+bI)(y_1+y_2I)](x_1+x_2I)](a+bI) \\ &= [[(x_1+x_2I)(y_1+y_2I)](a+bI)](a+bI) \\ &= [(t_1+t_2I)(a+bI)](a+bI) \\ &= [(t_1+t_2I)[[(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]]](a+bI) \\ &= [(t_1+t_2I)[[(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)]]](a+bI) \\ &= [(t_1+t_2I)[[(y_1+y_2I)(x_1+x_2I)][(a+bI)(a+bI)]]](a+bI) \\ &= [(t_1+t_2I)[(a+bI)[[(y_1+y_2I)(x_1+x_2I)](a+bI)]]](a+bI) \\ &= [(a+bI)[(t_1+t_2I)[[(y_1+y_2I)(x_1+x_2I)](a+bI)]]](a+bI) \\ &= [(a+bI)(u_1+u_2I)](a+bI), \end{aligned}$$

where $[(t_1+t_2I)[[(y_1+y_2I)(x_1+x_2I)](a+bI)]] = (u_1+u_2I) \in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is regular. Thus $N(S)$ is completely regular.
$(ii) \Longrightarrow (i)$

Assume that $N(S)$ is completely regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $t_1+t_2I \in N(\mathcal{S})$such that $a+bI = (a+bI)^2(x_1+x_2I)$, $a+bI = (y_1+y_2I)(a+bI)^2$, $a+bI = [(a+bI)(z_1+z_2I)](a+bI)$

$$\begin{aligned} a+bI &= (a+bI)^2(x_1+x_2I) \\ &= [(a+bI)(a+bI)][(v_1+v_2I)(u_1+u_2I)] \\ &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)], \end{aligned}$$

where $(x_1+x_2I) = (v_1+v_2I)(u_1+u_2I) \in N(\mathcal{S})$. Thus $N(S)$ is neutrosophic weakly regular

$$\begin{aligned} a+bI &= (x_1+x_2I)(a+bI)^2 \\ &= [(u_1+u_2I)(v_1+v_2I)](a+bI)^2 \\ &= [(u_1+u_2I)(v_1+v_2I)[(a+bI)(a+bI)] \\ &= [(a+bI)(a+bI)][(u_1+u_2I)(v_1+v_2I)] \\ &= [(a+bI)(u_1+u_2I)][(a+bI)(v_1+v_2I)], \end{aligned}$$

where$(x_1+x_2I) = (u_1+u_2I)(v_1+v_2I)$ for some $(x_1+x_2I) \in N(\dot{\mathcal{S}})$. Thus $N(S)$ is weakly regular.

$$\begin{aligned} & a+bI \\ = {} & [(a+bI)(z_1+z_2I)](a+bI) \\ = {} & [[(a+bI)^2(x_1+x_2I)](z_1+z_2I)][(y_1+y_2I)(a+bI)^2] \\ = {} & [[(z_1+z_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(y_1+y_2I)[(a+bI)(a+bI)]] \\ = {} & [(a+bI)[[(z_1+z_2I)(x_1+x_2I)](a+bI)]][(a+bI)[(y_1+y_2I)(a+bI)]] \\ = {} & [(a+bI)(t_1+t_2I)][(a+bI)(w_1+w_2I)], \end{aligned}$$

where $t_1+t_2I = [[(z_1+z_2I)(x_1+x_2I)](a+bI)]] \in N(\mathcal{S})$ & $(w_1+w_2I) = [(y_1+y_2I)(a+bI)] \in N(S)$. Thus $N(\mathcal{S})$ is weakly regular. □

Lemma 3. *Every strongly regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid) is completely regular.*

Proof. Assume that $N(\mathcal{S})$ is a strongly regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exists $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x_1+x_2I)](a+bI)$ and $(a+bI)(x_1+x_2I) = (x_1+x_2I)(a+bI)$. Now by using (1), we have

$$\begin{aligned} a+bI &= [(a+bI)(x_1+x_2I)](a+bI) \\ &= [(x_1+x_2I)(a+bI)](a+bI) \\ &= [(a+bI)(a+bI)](x_1+x_2I) \\ &= (a+bI)^2(x_1+x_2I). \end{aligned}$$

This shows that $N(\mathcal{S})$ is right regular and by Theorems 3 and 7, it is clear to see that $N(\mathcal{S})$ is completely regular. □

Note that a completely regular Neutrosophic$\mathcal{AG}$-groupoid need not to be a strongly regular Neutrosophic $\mathcal{AG}$-groupoid, as can be seen from the following example.

Theorem 8. *In a Neutrosophic $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), the following are equivalent.*
(*i*) *$N(\mathcal{S})$ is weakly regular.*
(*ii*) *$N(\mathcal{S})$ is intra-regular.*
(*iii*) *$N(\mathcal{S})$ is right regular.*
(*iv*) *$N(\mathcal{S})$ is left regular.*
(*v*) *$N(\mathcal{S})$ is left quasi regular.*
(*vi*) *$N(\mathcal{S})$ is completely regular.*
(*vii*) *For all $a+bI \in N(\mathcal{S})$, there exist $x+xI, y+yI \in N(\mathcal{S})$ such that $a+bI = [(x_{11}+x_2I)(a+bI)][(a+bI)(y_1+y_2I)]$.*

Proof. $(i) \Longrightarrow (ii)$ Let $N(\mathcal{S})$ be weakly regular $Neutrosophic$ $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$ and by Lemma 1, $x_1+x_2I = (u_1+u_2I)(v_1+v_2I)$ for some $u_1+u_2I, v_1+v_2I \in N(\mathcal{S})$. Let $(v_1+v_2I)(u_1+u_2I) = t_1+t_2I \in N(\mathcal{S})$. Now by using (3), (1), (4) and (2), we have

$$\begin{aligned}
a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= (x_1+x_2I)[(a+bI)^2(y_1+y_2I)] \\
&= [(u_1+u_2I)(v_1+v_2I)][(a+bI)^2(y_1+y_2I)] \\
&= [(y_1+y_2I)(a+bI)^2][(v_1+v_2I)(u_1+u_2I)] \\
&= [(y_1+y_2I)(a+bI)^2](t_1+t_2I)
\end{aligned}$$

where $(v_1+v_2I)(u_1+u_2I) = (t_1+t_2I) \in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is intra-regular
.$(ii) \Longrightarrow (iii)$ Let $N(\mathcal{S})$ be a weakly regular $Neutrosophic$ $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I)$

$$\begin{aligned}
a+bI &= [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
&= [[(u_1+u_2I)(v_1+v_2I)][(a+bI)(a+bI)]](y_1+y_2I) \\
&= [(a+bI)^2((v_1+v_2I)(u_1+u_2I))](y_1+y_2I) \\
&= [(y_1+y_2I)((v_1+v_2I)(u_1+u_2I))](a+bI)^2 \\
&= [(y_1+y_2I)((v_1+v_2I)(u_1+u_2I))][(a+bI)(a+bI)] \\
&= [(a+bI)(a+bI)][((v_1+v_2I)(u_1+u_2I))(y_1+y_2I)] \\
&= (a+bI)^2(s_1+s_2I),
\end{aligned}$$

where $x_1+x_2I = (u_!+u_2I)(v_1+v_2I) \in N(\mathcal{S})$ & $s_1+s_2I = [(y_1+y_2I)[v_1+v_2I)(u_1+u_2I)]] \in N(\mathcal{S})$. Thus $N(S)$ is right regular
$(iii) \Longrightarrow (iv)$ Let $N(\mathcal{S})$ be a right regular $Neutrosophic$ $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = (a+b)^2(x_1+x_2I)$

$$\begin{aligned} a+bI &= (a+b)^2(x_1+x_2I) \\ &= [(a+bI)(a+bI)](x_1+x_2I) \\ &= [(a+bI)(a+bI)][(u_1+u_2I)(v_1+v_2I)] \\ &= [(v_1+v_2I)(u_1+u_2I)][a+bI)(a+bI)] \\ &= (y_1+y_2I)(a+bI)^2 \end{aligned}$$

where $y_1+y_2I = [(v_1+v_2I)(u_1+u_2I)] \in N(\mathcal{S})$. Thus $N(S)$ is left regular

$(iv) \implies (v)$ Let $N(\mathcal{S})$ be a left regular $Neutrosophic$ $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = (x_1+x_2I)(a+bI)^2$

$$\begin{aligned} a+bI &= (x_1+x_2I)(a+bI)^2 \\ &= [(u_1+u_2I)(v_1+v_2I)][(a+bI)(a+bI)] \\ &= [(u_1+u_2I)(a+bI)][(v_1+v_2I)(a+bI)] \end{aligned}$$

Thus $N(S)$ is left quasi regular

$(v) \implies (vi)$ Let $N(\mathcal{S})$ be a left quasi regular $Neutrosophic$ $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]$

$$\begin{aligned} a+bI &= [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)] \\ &= [(a+bI)(a+bI)][(y_1+y_2I)(x_1+x_2I)] \\ &= (a+bI)^2(v_1+v_2I) \end{aligned}$$

where $v_1+v_2I = [(y_1+y_2I)(x_1+x_2I)] \in N(\mathcal{S})$

Thus $N(S)$ is right regular $\implies (1)$

Let $N(\mathcal{S})$ be a left quasi regular $Neutrosophic$ $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]$

$$\begin{aligned} a+bI &= [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)] \\ &= [(x_1+x_2I)(y_1+y_2I)][(a+bI)(a+bI)] \\ &= (u_1+u_2I)(a+bI)^2 \end{aligned}$$

where $(u_1+u_2I) = [(x_1+x_2I)(y_1+y_2I)] \in N(\mathcal{S})$

Thus $N(S)$ is left regular$\implies (2)$

Let $N(\mathcal{S})$ be a left quasi regular $Neutrosophic$ $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]$

$$\begin{aligned} a+bI &= [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)] \\ &= [(a+bI)(a+bI)][(y_1+y_2I)(x_1+x_2I)] \\ &= [((y_1+y_2I)(x_1+x_2I))(a+bI)](a+bI) \\ &= [(v_1+v_2I)(a+bI)](a+bI), \\ &= [(v_1+v_2I)[(x_1+x_2I)(a+bI))((y_1+y_2I)(a+bI)]](a+bI) \\ &= [(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)][(a+bI)(a+bI)]](a+bI) \\ &= [(v_1+v_2I)[(a+bI)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]]](a+bI) \\ &= [(a+bI)[(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]]](a+bI) \\ &= [(a+bI)(t_1+t_2I)](a+bI) \end{aligned}$$

where $(v_1+v_2I)=(y_1+y_2I)(x_1+x_2I)\in N(\mathcal{S})$ & where $t_1+t_2I=[(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]]\in N(\mathcal{S})$

Thus $N(S)$ is regular $\Longrightarrow$ (3)

By (1).(2) & (3) $N(S)$ is completely regular.

$(vi)\Longrightarrow(i)$ Let $N(\mathcal{S})$ be a complete regular *Neutrosophic* $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI\in N(\mathcal{S})$ there exist $x_1+x_2I\in N(\mathcal{S})$ such that $a+bI=(a+bI)^2(x_1+x_2I)$, $a+bI=(y_1+y_2I)(a+bI)^2$, $a+bI=[(a+bI)(z_1+z_2I)]\ (a+bI)$

$$\begin{aligned} a+bI &= (a+bI)^2(x_1+x_2I) \\ &= [(a+bI)(a+bI)][(u_1+u_2I)(v_1+v_2I)] \\ &= [(a+bI)(u_1+u_2I)][(a+bI)(v_1+v_2I)] \end{aligned}$$

where $(x_1+x_2I)=[(u_1+u_2I)(v_1+v_2I)]\in N(\mathcal{S})$. Thus $N(S)$ is weakly regular.

$$\begin{aligned} a+bI &= (y_1+y_2I)(a+bI)^2 \\ &= [(v_1+v_2I)(u_1+u_2I)][(a+bI)(a+bI)] \\ &= [(a+bI)(u_1+u_2I)][(a+bI)(v_1+v_2I)] \end{aligned}$$

where $(y_1+y_2I)=[(v_1+v_2I)(u_1+u_2I)]\in N(\mathcal{S})$. Thus $N(S)$ is neutrosophic weakly regular.

$$\begin{aligned} & a+bI \\ =\ & [(a+bI)(z_1+z_2I)](a+bI) \\ =\ & [[(a+bI)^2(x_1+x_2I)](z_1+z_2I)][(y_1+y_2I)(a+bI)^2] \\ =\ & [[(z_1+z_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(y_1+y_2I)[(a+bI)(a+bI)]] \\ =\ & [(a+bI)[[(z_1+z_2I)(x_1+x_2I)](a+bI)]][(a+bI)[(y_1+y_2I)(a+bI)]] \\ =\ & [(a+bI)(t_1+t_2I)][(a+bI)(w_1+w_2I)] \end{aligned}$$

where $t_1+t_2I=[[(z_1+z_2I)(x_1+x_2I)](a+bI)]]\in N(\mathcal{S})$ & $(w_1+w_2I)=[(y_1+y_2I)(a+bI)]\in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is neutrosophic weakly regular.

$(ii)\Longrightarrow(vii)$ Let $N(\mathcal{S})$ be an intra-regular *Neutrosophic* $\mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then for any $a+bI\in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I\in N(\mathcal{S})$ such that $a+bI=[(x_1+x_2I)(a+bI)^2](y_1+y_2I)$. Now by using Lemma 1,

$y_1+y_2I=(u_1+u_2I)(v_1+v_2I)$ for some $u_1+u_2I, v_1+v_2I \in N(\mathcal{S})$. Thus by using (4), (1) and (3), we have

$$\begin{aligned}
& a+bI \\
= & [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= & [(x_1+x_2I)[(a+bI)(a+bI)]](y_1+y_2I) \\
= & [(a+bI)[(x_1+x_2I)(a+bI)]](y_1+y_2I) \\
= & [(y_1+y_2I)[(x_1+x_2I)(a+bI)]](a+bI) \\
= & [(y_1+y_2I)[(x_1+x_2I)(a+bI)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= & [[(u_1+u_2I)(v_1+v_2)][(x+yI)(a+bI)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= & [[(a+bI)(x_1+x_2I)][(v_1+v_2)(u_1+u_2I)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= & [[(a+bI)(x_1+x_2I)](t_1+t_2I)][(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= & [[(x_1+x_2I)(a+bI)^2](y_1+y_2I)](t_1+t_2I)][(a+bI)(x_1+x_2I)] \\
= & [[(t_1+t_2I)(y_1+y_2I)][(x_1+x_2I)(a+bI)^2]][(a+bI)(x_1+x_2I)] \\
= & [[(a+bI)^2(x_1+x_2I)][(y_1+y_2I)(t_1+t_2I)]][(a+bI)(x_1+x_2I)] \\
= & [[(a+bI)^2(x_1+x_2I)][(s_1+s_2I)][(a+bI)(x_1+x_2I)] \\
= & [[(s_1+s_2I)(x_1+x_2I)](a+bI)^2][(a+bI)(x_1+x_2I)] \\
= & [[(s_1+s_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(a+bI)(x_1+x_2I)] \\
= & [[(a+bI)(a+bI)][(x_1+x_2I)(s_1+s_2I)]][(a+bI)(x_1+x_2I)] \\
= & [[(a+bI)(a+bI)](w_1+w_2I)[(a+bI)(x_1+x_2I)] \\
= & [[(w_1+w_2I)(a+bI)](a+bI)][(a+bI)(x_1+x_2I)] \\
= & [(z_1+z_2I)(a+bI)][(a+bI)(x_1+x_2I)] \\
= & [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)]
\end{aligned}$$

where $[(u_1+u_2I)(v_1+v_2)=(y_1+y_2I)\in N(S)$ & where $(v_1+v_2)(u_1+u_2I)=(t_1+t_2I)\in N(S)$ & where $(y_1+y_2I)(t_1+t_2I)=(s_1+s_2I)\in N(S)$ & where $(x_1+x_2I)(s_1+s_2I)=(w_1+w_2I)\in N(S)$ & where $(w_1+w_2I)(a+bI)=(z_1+z_2I)\in N(S)$

$(vii)\Longrightarrow(ii)$ let for all $a+bI\in N(\mathcal{S})$, $a+bI=[(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)]$ holds for some $x+xI, z+zI\in N(\mathcal{S})$. Now by using (4), (1), (2) and (3), we

have

$$
\begin{aligned}
& a+bI \\
= & [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)] \\
= & [(a+bI)[(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= & [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= & [(a+bI)[(x_1+x_2I)(a+bI)](z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= & [[(x_1+x_2I)(a+bI)](z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)]](a+bI) \\
= & [[(x_1+x_2I)(a+bI)](z_1+z_2I)]^2(a+bI) \\
= & [[(x_1+x_2I)(a+bI)]^2(z_1+z_2I)^2](a+bI) \\
= & [[(x_1+x_2I)^2(a+bI)^2][(z_1+z_2I)(z_1+z_2I)]](a+bI) \\
= & [[(x_1+x_2I)^2(z_1+z_2I)][(a+bI)^2(z_1+z_2I)]](a+bI) \\
= & [(a+bI)^2[[(x_1+x_2I)^2(z_1+z_2I)](z_1+z_2I)](a+bI) \\
= & [[(a+bI)(a+bI)][(x_1+x_2I)^2(z_1+z_2I)](z_1+z_2I)](a+bI) \\
= & [[(a+bI)(a+bI)][[(z_1+z_2I)(z_1+z_2I)](x_1+x_2I)^2]](a+bI) \\
= & [[(a+bI)(a+bI)][(z_1+z_2I)^2(x_1+x_2I)^2]](a+bI) \\
= & [[(x_1+x_2I)^2(z_1+z_2I)^2][(a+bI)(a+bI)]](a+bI) \\
= & [(t_1+t_2I)[(a+bI)(a+bI)]](a+bI), \\
= & [(t_1+t_2I)(a+bI)^2](u_1+u_2I)
\end{aligned}
$$

where $[(x_1+x_2I)^2(z_1+z_2I)^2] = (t_1+t_2I) \in N(S)$ and $(a+bI) = (u_1+u_2I) \in N(S)$ where $(a+bI) = (u_1+u_2I) \in N(S)$. Thus $N(S)$ is neutrosophic intra regular. □

Remark 1. *Every intra-regular, right regular, left regular, left quasi regular and completely regular $\mathcal{AG}$-groupoids with left identity ($\mathcal{AG}^{**}$-groupoids) are regular.*

The converse of above is not true in general. Indeed, from Example 2, regular $\mathcal{AG}$-groupoid with left identity is not necessarily intra-regular.

Theorem 9. *In a Neutrosophic $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, the following are equivalent.*
(i) $N(\mathcal{S})$ is weakly regular.
(ii) $N(\mathcal{S})$ is intra-regular.
(iii) $N(\mathcal{S}$ is right regular.
(iv) $N(\mathcal{S})$ is left regular.
(v) $N(\mathcal{S})$ is left quasi regular.
(vi) $N(\mathcal{S})$ is completely regular.
(vii) For all $a+bI \in N(\mathcal{S})$, there exist $x+xI, y+yI \in N(\mathcal{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)][(a+bI)(y_1+y_2I)]$.
(viii) $\mathcal{S}$ is $(2,2)$-regular.

Proof. $(i) \Longrightarrow (ii)$ Let $N(\mathcal{S})$ be a weakly regular *Neutrosophic* $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$ and by Lemma 1, $x_1+x_2I = (u_1+u_2I)(v_1+v_2I)$ for some $u_1+u_2I, v_1+v_2I \in N(\mathcal{S})$. Let $(v_1+v_2I)(u_1+u_2I) = t_1+t_2I \in N(\mathcal{S})$. Now by using (3), (1), (4) and (2), we

have

$$\begin{aligned} a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\ &= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\ &= (x_1+x_2I)[(a+bI)^2(y_1+y_2I)] \\ &= [(u_1+u_2I)(v_1+v_2I)][(a+bI)^2(y_1+y_2I)] \\ &= [(y_1+y_2I)(a+bI)^2][(v_1+v_2I)(u_1+u_2I)] \\ &= [(y_1+y_2I)(a+bI)^2](t_1+t_2I) \end{aligned}$$

where $(v_1+v_2I)(u_1+u_2I) = (t_1+t_2I) \in N(\mathcal{S})$. Thus $N(\mathcal{S})$ is intra-regular.

$(ii) \Longrightarrow (iii)$ Let $N(\mathcal{S})$ be a intra regular $Neutrosophic$ $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I)$

$$\begin{aligned} a+bI &= [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\ &= [[(u_1+u_2I)(v_1+v_2I)][(a+bI)(a+bI)]](y_1+y_2I), \\ &= [(a+bI)^2((v_1+v_2I)(u_1+u_2I))](y_1+y_2I) \\ &= [(y_1+y_2I)((v_1+v_2I)(u_1+u_2I))](a+bI)^2 \\ &= [(y_1+y_2I)((v_1+v_2I)(u_1+u_2I))][(a+bI)(a+bI)] \\ &= [(a+bI)(a+bI)][((v_1+v_2I)(u_1+u_2I))(y_1+y_2I)] \\ &= (a+bI)^2(s_1+s_2I) \end{aligned}$$

where $x_1+x_2I = (u_!+u_2I)(v_1+v_2I) \in N(\mathcal{S})$ & where $s_1+s_2I = [(y_1+y_2I)[v_1+v_2I)(u_1+u_2I)]] \in N(\mathcal{S})$. Thus $N(S)$ is right regular

$(iii) \Longrightarrow (iv)$ Let $N(\mathcal{S})$ be a right regular $Neutrosophic$ $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = (a+b)^2(x_1+x_2I)$

$$\begin{aligned} a+bI &= (a+b)^2(x_1+x_2I) \\ &= [(a+bI)(a+bI)](x_1+x_2I) \\ &= [(a+bI)(a+bI)][(u_1+u_2I)(v_1+v_2I)] \\ &= [(v_1+v_2I)(u_1+u_2I)][a+bI)(a+bI)] \\ &= (y_1+y_2I)(a+bI)^2 \end{aligned}$$

where $y_1+y_2I = [(v_1+v_2I)(u_1+u_2I)] \in N(\mathcal{S})$. Thus $N(S)$ is left regular

$(iv) \Longrightarrow (v)$ Let $N(\mathcal{S})$ be a left regular $Neutrosophic$ $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = (x_1+x_2I)(a+bI)^2$

$$\begin{aligned} a+bI &= (x_1+x_2I)(a+bI)^2 \\ &= [(u_1+u_2I)(v_1+v_2I)][(a+bI)(a+bI)] \\ &= [(u_1+u_2I)(a+bI)][(v_1+v_2I)(a+bI)] \end{aligned}$$

Thus $N(S)$ is left quasi regular

$(v) \Longrightarrow (vi)$ Let $N(\mathcal{S})$ be a left quasi regular *Neutrosophic* $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]$

$$\begin{aligned} a+bI &= [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)] \\ &= [(a+bI)(a+bI)][(y_1+y_2I)(x_1+x_2I)] \\ &= (a+bI)^2(v_1+v_2I) \end{aligned}$$

where $v_1+v_2I = [(y_1+y_2I)(x_1+x_2I)] \in N(\mathcal{S})$. Thus $N(S)$ is neutrosophic right regular. Let $N(\mathcal{S})$ be a left quasi regular *Neutrosophic* $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]$

$$\begin{aligned} a+bI &= [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)] \\ &= [(x_1+x_2I)(y_1+y_2I)][(a+bI)(a+bI)] \\ &= (u_1+u_2I)(a+bI)^2 \end{aligned}$$

where $(u_1+u_2I) = [(x_1+x_2I)(y_1+y_2I)] \in N(\mathcal{S})$. Thus $N(S)$ is neutrosophic left regular.

Let $N(\mathcal{S})$ be a neutrosophic left quasi regular *Neutrosophic* $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I \in N(\mathcal{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]$

$$\begin{aligned} & a+bI \\ = \ & [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)] \\ = \ & [(a+bI)(a+bI)][(y_1+y_2I)(x_1+x_2I)] \\ = \ & [((y_1+y_2I)(x_1+x_2I))(a+bI)](a+bI) \\ = \ & [(v_1+v_2I)(a+bI)](a+bI) \\ = \ & [(v_1+v_2I)[(x_1+x_2I)(a+bI))((y_1+y_2I)(a+bI)]](a+bI) \\ = \ & [(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)][(a+bI)(a+bI)]](a+bI) \\ = \ & [(v_1+v_2I)[(a+bI)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]]](a+bI) \\ = \ & [(a+bI)[(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]]](a+bI) \\ = \ & [(a+bI)(t_1+t_2I)](a+bI) \end{aligned}$$

where $(v_1+v_2I) = (y_1+y_2I)(x_1+x_2I) \in N(\mathcal{S})$ & where $t_1+t_2I = [(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]] \in N(\mathcal{S})$. Thus $N(S)$ is regular $\Longrightarrow (3)$

By (1).(2) & (3) $N(S)$ is neutrosophic completely regular.

$(vi) \Longrightarrow (i)$ Assume that $N(S)$ is neutrosophic completely regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $t_1+t_2I \in N(\mathcal{S})$such that $a+bI = (a+bI)^2(x_1+x_2I)$, $a+bI = (y_1+y_2I)(a+bI)^2$, $a+bI = [(a+bI)(z_1+z_2I)](a+bI)$

$$
\begin{aligned}
& a+bI \\
= \; & [(a+bI)(z_1+z_2I)](a+bI) \\
= \; & [[(a+bI)^2(x_1+x_2I)](z_1+z_2I)][(y_1+y_2I)(a+bI)^2] \\
= \; & [[(z_1+z_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(y_1+y_2I)[(a+bI)(a+bI)]] \\
= \; & [(a+bI)[[(z_1+z_2I)(x_1+x_2I)](a+bI)]][(a+bI)[(y_1+y_2I)(a+bI)]] \\
= \; & [(a+bI)(t_1+t_2I)][(a+bI)(w_1+w_2I)]
\end{aligned}
$$

Thus $N(S)$ is neutrosophic weakly regular.

$(ii) \Longrightarrow (vii)$Let $N(\mathcal{S})$ be a neutrosophic intra-regular $Neutrosophic\, \mathcal{AG}$-groupoid with left identity ($\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathcal{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I)$. Now by using Lemma 1, $y+yI = (u_1+u_2I)(v_1+v_2I)$ for some $u+uI, v+vI \in N(\mathcal{S})$.

$$
\begin{aligned}
& a+bI \\
= \; & [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= \; & [(x_1+x_2I)[(a+bI)(a+bI)]](y_1+y_2I) \\
= \; & [(a+bI)[(x_1+x_2I)(a+bI)]](y_1+y_2I) \\
= \; & [(y_1+y_2I)[(x_1+x_2I)(a+bI)]](a+bI) \\
= \; & [(y_1+y_2I)[(x_1+x_2I)(a+bI)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= \; & [[(u_1+u_2I)(v_1+v_2)][(x+yI)(a+bI)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I), \\
= \; & [[(a+bI)(x_1+x_2I)][(v_1+v_2)(u_1+u_2I)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= \; & [[(a+bI)(x_1+x_2I)](t_1+t_2I)][(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
= \; & [[(x_1+x_2I)(a+bI)^2](y_1+y_2I)](t_1+t_2I)][(a+bI)(x_1+x_2I)] \\
= \; & [[(t_1+t_2I)(y_1+y_2I)][(x_1+x_2I)(a+bI)^2]][(a+bI)(x_1+x_2I)] \\
= \; & [[(a+bI)^2(x_1+x_2I)][(y_1+y_2I)(t_1+t_2I)]][(a+bI)(x_1+x_2I)] \\
= \; & [[(a+bI)^2(x_1+x_2I)][(s_1+s_2I)]][(a+bI)(x_1+x_2I)], \\
= \; & [[(s_1+s_2I)(x_1+x_2I)](a+bI)^2][(a+bI)(x_1+x_2I)] \\
= \; & [[(s_1+s_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(a+bI)(x_1+x_2I)] \\
= \; & [[(a+bI)(a+bI)][(x_1+x_2I)(s_1+s_2I)]][(a+bI)(x_1+x_2I)] \\
= \; & [[(a+bI)(a+bI)](w_1+w_2I)[(a+bI)(x_1+x_2I)], \\
= \; & [[(w_1+w_2I)(a+bI)](a+bI)][(a+bI)(x_1+x_2I)] \\
= \; & [(z_1+z_2I)(a+bI)][(a+bI)(x_1+x_2I)], \\
= \; & [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)]
\end{aligned}
$$

where $(w_1+w_2I)(a+bI) = (z_1+z_2I) \in N(S)$ where $(x_1+x_2I)(s_1+s_2I) = (w_1+w_2I) \in N(S)$ where $(y_1+y_2I)(t_1+t_2I) = (s_1+s_2I) \in N(S)$ where $(v_1+v_2)(u_1+u_2I) = (t_1+t_2I) \in N(S)$ where $[(u_1+u_2I)(v_1+v_2) = (y_1+y_2I) \in N(S)$

$(vii) \Longrightarrow (ii)$let for all $a+bI \in N(\mathcal{S})$, $a+bI = [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)]$ holds for some $x_1+x_2I, z_1+z_2I \in N(\mathcal{S})$. Now by using (4), (1), (2) and (3), we

have

$$\begin{aligned}
& a+bI \\
= & [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)] \\
= & [(a+bI)[(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= & [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= & [(a+bI)[(x_1+x_2I)(a+bI)](z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
= & [[(x_1+x_2I)(a+bI)](z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)]](a+bI) \\
= & [[(x_1+x_2I)(a+bI)](z_1+z_2I)]^2(a+bI) \\
= & [[(x_1+x_2I)(a+bI)]^2(z_1+z_2I)^2](a+bI) \\
= & [[(x_1+x_2I)^2(a+bI)^2][(z_1+z_2I)(z_1+z_2I)]](a+bI) \\
= & [[(x_1+x_2I)^2(z_1+z_2I)][(a+bI)^2(z_1+z_2I)]](a+bI) \\
= & [(a+bI)^2[[(x_1+x_2I)^2(z_1+z_2I)](z_1+z_2I)](a+bI) \\
= & [[(a+bI)(a+bI)][(x_1+x_2I)^2(z_1+z_2I)](z_1+z_2I)](a+bI) \\
= & [[(a+bI)(a+bI)][[(z_1+z_2I)(z_1+z_2I)](x_1+x_2I)^2]](a+bI) \\
= & [[(a+bI)(a+bI)][(z_1+z_2I)^2(x_1+x_2I)^2]](a+bI) \\
= & [[(x_1+x_2I)^2(z_1+z_2I)^2][(a+bI)(a+bI)]](a+bI) \\
= & [(t_1+t_2I)[(a+bI)(a+bI)]](a+bI), \\
= & [(t_1+t_2I)(a+bI)^2](u_1+u_2I)
\end{aligned}$$

where $[(x_1+x_2I)^2(z_1+z_2I)^2] = (t_1+t_2I) \in N(S)$ and $(a+bI) = (u_1+u_2I) \in N(S)$ where $(a+bI) = (u_1+u_2I) \in N(S)$. Thus $N(S)$ is intra regular.

$(vi) \Longrightarrow (viii)$ Assume that $N(S)$ is neutrosophic completely regular Neutrosophic $\mathcal{AG}$-groupoid with left identity ($Neutrosophic\mathcal{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathcal{S})$ there exist $t_1+t_2I \in N(\mathcal{S})$ such that $a+bI = (a+bI)^2(x_1+x_2I)$, $a+bI = (y_1+y_2I)(a+bI)^2$, $a+bI = [(a+bI)(z_1+z_2I)](a+bI)$

$$\begin{aligned}
a+bI &= (a+bI)^2(x_1+x_2I) \\
&= [(a+bI)(a+bI)][(v_1+v_2I)(u_1+u_2I)] \\
&= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]
\end{aligned}$$

where $(x_1+x_2I) = (v_1+v_2I)(u_1+u_2I) \in N(\mathcal{S})$. Thus $N(S)$ is weakly regular

$$\begin{aligned}
a+bI &= (x_1+x_2I)(a+bI)^2 \\
&= [(u_1+u_2I)(v_1+v_2I)](a+bI)^2 \\
&= [(u_1+u_2I)(v_1+v_2I)][(a+bI)(a+bI)] \\
&= [(a+bI)(a+bI)][(u_1+u_2I)(v_1+v_2I))] \\
&= [(a+bI)(u_1+u_2I)][(a+bI)(v_1+v_2I)]
\end{aligned}$$

where$(x_1+x_2I)=(u_1+u_2I)(v_1+v_2I)$ for some $(x_1+x_2I)\in N(S)$. Thus $N(S)$ is weakly regular.

$$\begin{aligned}
& a+bI \\
= \;& [(a+bI)(z_1+z_2I)](a+bI) \\
= \;& [[(a+bI)^2(x_1+x_2I)](z_1+z_2I)][(y_1+y_2I)(a+bI)^2] \\
= \;& [[(z_1+z_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(y_1+y_2I)[(a+bI)(a+bI)]] \\
= \;& [(a+bI)[[(z_1+z_2I)(x_1+x_2I)](a+bI)]][(a+bI)[(y_1+y_2I)(a+bI)]] \\
= \;& [(a+bI)(t_1+t_2I)][(a+bI)(w_1+w_2I)]
\end{aligned}$$

where $t_1+t_2I=[[(z_1+z_2I)(x_1+x_2I)](a+bI)]]\in N(\mathcal{S})$ & $(w_1+w_2I)=[(y_1+y_2I)(a+bI)]\in N(\mathcal{S})$. Thus $N(S)$ is weakly regular.

$(viii)\implies(vi)$ Let $N(\mathcal{S})$ be a neutrosophic $(2,2)$-regular $Neutrosophic\mathcal{AG}$-groupoid with left identity, then for $a+bI\in N(\mathcal{S})$ there exists $x_1+x_2I\in N(\mathcal{S})$ such that $a+bI=[(a+bI)^2(x+yI)](a+bI)^2$. Now

$$\begin{aligned}
a+bI &= [(a+bI)^2(x_1+x_2I)](a+bI)^2 \\
&= [(y_1+y_2I)(a+bI)^2]
\end{aligned}$$

where $(a+bI)^2(x_1+x_2I)=(y_1+y_2I)\in N(\mathcal{S})$ and by using (3), we have

$$\begin{aligned}
a+bI &= (a+bI)^2[(x_1+x_2I)[(a+bI)(a+bI)]] \\
&= [(a+bI)(a+bI)][(x_1+x_2I)[(a+bI)^2]] \\
&= (a+bI)^2(z_1+z_2I)
\end{aligned}$$

where $(x_1+x_2I)(a+bI)^2=(z_1+z_2I)\in N(\mathcal{S})$. And by using (3), (1) and (4), we have

$$\begin{aligned}
& a+bI \\
= \;& [(a+bI)^2[(x_1+x_2I)[(a+bI)(a+bI)]] \\
= \;& [(a+bI)(a+bI)][(x_1+x_2I)(a+bI)^2] \\
= \;& [(a+bI)(a+bI)][[(e_1+e_2I)(x_1+x_2I)][a+bI)(a+bI)]] \\
= \;& [(a+bI)(a+bI)][[(a+bI)(a+bI)][(x_1+x_2I)(e_1+e_2I)]] \\
= \;& [(a+bI)(a+bI)][(a+bI)^2(t_1+t_2I)] \\
= \;& [[(a+bI)^2(t_1+t_2I)](a+bI)](a+bI) \\
= \;& [[[(a+bI)(a+bI)](t_1+t_2I)](a+bI)](a+bI) \\
= \;& [[(t_1+t_2I)(a+bI)][(a+bI)](a+bI)]](a+bI) \\
= \;& [[(a+bI)(a+bI)][(t_1+t_2I)(a+bI)]](a+bI) \\
= \;& [[(a+bI)(t_1+t_2I)][(a+bI)(a+bI)]](a+bI) \\
= \;& [(a+bI)[[(a+bI)(t_1+t_2I)](a+bI)]](a+bI) \\
= \;& [(a+bI)(u_1+u_2I)](a+bI)
\end{aligned}$$

where $t_1+t_2I=(x_1+x_2I)(e_1+e_2I)\in N(\mathcal{S})$
& $u_1+u_2I=(a+bI)^2(t_1+t_2I).\in N(\mathcal{S})$

Thus $N(\mathcal{S})$ is left regular, right regular and regular, so $N(\mathcal{S})$ is completely regular.

$(vi)\implies(viii)$ Assume that $N(\mathcal{S})$ is a completely regular Neutrosophic $\mathcal{AG}$-groupoid with left identity, then for any $a+bI\in N(\mathcal{S})$ there exist $x+xI,y+yI,z+$

$zI \in N(\mathcal{S})$ such that $a+bI = [(a+bI)(x_1+x_2I)](a+bI)$, $a+bI = (a+bI)^2(y_1+y_2I)$ and $a+bI = (z_1+z_2I)(a+bI)^2$. Now by using (1), (4) and (3), we have

$$\begin{aligned}
& a+bI \\
= & [(a+bI)(x_1+x_2I)](a+bI) \\
= & [[(a+bI)^2(y_1+y_2I)](x_1+x_2I)][(z_1+z_2I)(a+bI)^2] \\
= & [[(x_1+x_2I)(y_1+y_2I)](a+bI)^2][(z_1+z_2I)(a+bI)^2] \\
= & [[(x_1+x_2I)(y_1+y_2I)][(a+bI)(a+bI)]][(z_1+z_2I)(a+bI)^2] \\
= & [[(a+bI)(a+bI)][(y_1+y_2I)(x_1+x_2I)]][(z_1+z_2I)(a+bI)^2] \\
= & [(a+bI)^2[(y_1+y_2I)(x_1+x_2I)]][(z_1+z_2I)(a+bI)^2] \\
= & [[(z_1+z_2I)(a+bI)^2][(y_1+y_2I)(x_1+x_2I)]](a+bI)^2 \\
= & [[(z_1+z_2I)(y_1+y_2I)][(a+bI)^2(x_1+x_2I)]](a+bI)^2 \\
= & [(a+bI)^2[[(z_1+z_2I)(y_1+y_2I)](x_1+x_2I)]](a+bI)^2 \\
= & [(a+bI)^2(v_1+v_2I)](a+bI)^2
\end{aligned}$$

where $[(z_1+z_2I)(y_1+y_2I)][(x_1+x_2I)] = (v_1+v_2I) \in N(\mathcal{S})$. Thus $N(S)$ is $(2,2)$-regular. □

Remark 2. $(2,2)$*-regular and strongly regular* $\mathcal{AG}$*-groupoids with left identity are regular.*

The converse of above is not true in general, as can be seen from Example **??**.

Theorem 10. *Regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular,* $(2,2)$*-regular and strongly regular* $\mathcal{AG}^*$*-groupoids becomes semigroups.*

Proof. It follows from (6) and Lemma 1. □

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