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Neutrosophic SuperHyperAlgebra And New Types of Topologies

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Global Knowledge

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FORWARD

In general, a system S (that may be a company, association, institution, society, country, etc.) is formed by sub-systems S_i { or $P(S)$, the powerset of S }, and each sub-system S_i is formed by sub-sub-systems S_{ij} { or $P(P(S)) = P^2(S)$ } and so on. That's why the n-th PowerSet of a Set S { defined recursively and denoted by $P^n(S) = P(P^{n-1}(S)$ } was introduced, to better describes the organization of people, beings, objects etc. in our real world.

The n-th PowerSet was used in defining the SuperHyperOperation, SuperHyperAxiom, and their corresponding Neutrosophic SuperHyperOperation, Neutrosophic SuperHyperAxiom in order to build the SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra. In general, in any field of knowledge, one in fact encounters **SuperHyperStructures**, https://fs.unm.edu/SuperHyperAlgebra.pdf.

Also, six new types of topologies have been introduced in the last years (2019-2022), such as: Refined Neutrosophic Topology, Refined Neutrosophic Crisp Topology, NeutroTopology, AntiTopology, SuperHyperTopology, and Neutrosophic SuperHyperTopology, http://fs.unm.edu/TT/.

Florentin Smarandache, Memet Şahin, Derya Bakbak, Vakkas Uluçay & Abdullah Kargın

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Preface

Neutrosophic set has been derived from a new branch of philosophy, namely Neutrosophy. Neutrosophic set is capable of dealing with uncertainty, indeterminacy and inconsistent information. Neutrosophic set approaches are suitable to modeling problems with uncertainty, indeterminacy and inconsistent information in which human knowledge is necessary, and human evaluation is needed.

Neutrosophic set theory firstly proposed in 1998 by Florentin Smarandache, who also developed the concept of single valued neutrosophic set, oriented towards real world scientific and engineering applications. Since then, the single valued neutrosophic set theory has been extensively studied in books and monographs introducing neutrosophic sets and its applications, by many authors around the world. Also, an international journal - Neutrosophic Sets and Systems started its journey in 2013.

http://fs.unm.edu/neutrosophy.htm.

This first volume collects original research and applications from different perspectives covering different areas of neutrosophic studies, such as decision-making, neutroalgebra, neutro metric, and some theoretical papers.

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Chapter One

New Type Hyper Groups, New Type SuperHyper Groups and Neutro-New Type SuperHyper Groups

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ABSTRACT

In this chapter, a new type Hyper groups are defined, corresponding basic properties and examples for new type Hyper groups are given and proved. Moreover, new type Hyper groups groups and are compared to hyper groups and groups. New type Hyper groups are shown to have a more general structure according to Hyper groups and groups. Also, new type SuperHyper groups are defined, corresponding basic properties and examples for new type SuperHyper are given and proved. Furthermore, we defined neutro-new type SuperHyper groups.

Keywords: SuperHyper Structure, New type Hyper groups, New type SuperHyper groups, Neutro-new type SuperHyper groups

INTRODUCTION

Hyperstructures [1] are defined by Marty in 1934. Hyperstructures are a extended and a new form of classical structures. Corsini obtained hypergroups [2] in 1993. So, many researchers have made studies on this subject [3-7]. Recently, Hashemi studied Hyper JK-algebras [8]; Muhiuddin et al. obtained Hyperstructure Theory Applied to BF-Algebras [9].

Neutrosophic theory, consisting of neutrosophic logic and neutrosophic sets, was defined by Florentin Smarandache in 1998. In neutrosophic set theory, there are T, I and F graphs (membership function, performance function and membership function, respectively) for each element. These functions can be set independently. For this reason, neutrosophic logic and neutrosophic sets are used in decision-making problems in almost all branches of science. So, many researchers have made studies on this subject [11 -20, 38-45].

Florentin Smarandache introduced new research areas in neutrosophy, which he called neutro-structures and anti-structures, respectively, in 2019 [21, 22]. When evaluating <A> as an element (concept, attribute, idea, proposition, theory, etc.), during the neutrosification process, he worked on three regions; two opposites corresponding to <A> and <antiA> and also a neutral (indeterminate) <neutA> (also called <neutralA>). A neutro-algebra consists of at least one neutro-operation (indeterminate for other items and false for other items) or it is an algebra well-defined for some items (also called internally defined), indeterminate for others, and externally defined for others. Therefore, the subject attracted the attention of many researchers [23–32]. Recently, Al-Tahan et al. studied some neutroHyperstructures [33]; Ibrahim and Agboola obtained NeutroHyperGroups [34].

Florentin Smarandache introduced new research areas, which he called SuperHyperstructures [35] in 2022. Recently, Hamidi studied Superhyper BCK-Algebras [36]; Jahanpanah and Daneshpayeh obtained Superhyper BE-Algebras [37].

In the second section, basic definitions on Hypergrup [2], SuperHyperoperation [35] are given. In the third chapter, new type Hyper groups are defined, corresponding basic properties and examples for new type Hyper groups are given and proved. Moreover, new type Hyper groups are compared to hyper group and group. New type Hyper groups are shown to have a more general structure according to Hyper groups and group. In the fourth section, new type SuperHyper groups are defined, corresponding basic properties and examples for new type SuperHyper groups are given and proved. In the fifth section, we defined neutro-new type SuperHyper groups. In the last section, results and suggestions are given.

BACKGROUND

Definition 1. [21]

i) [Law of neutro-well defined]

There exists a double (b, n) ∈ (G, G) such that b # n ∈ G [degree of truth T] and there exist a double (u, v) ∈ (G, G) such that u # v = indeterminate [degree of indeterminacy I], or there exist a double (p, q) ∈ (G, G) such that p # q ∉ G [degree of outer-defined F], where (T, I, F) is different from (1,0,0) and (0,0,1). Because (1,0,0) represents the classical well-defined law (100% well-defined law; T =1, I = 0, F = 0), while (0,0,1) represents the outer-defined law (i.e. 100% outer-defined law, or T=0, I=0, F =1).

ii) [Axiom of neutro-associativity]

There exists a triplet (b, n, m) ∈ (G, G, G) such that b # (n # m) = (b # n) # m [degree of truth T], and there exist two triplets (p, q, r) ∈ (G, G, G) such that p # (q # r) or (p # q) # r = indeterminate [degree of indeterminacy I], or there exist (u, v, w) ∈ (G, G, G) or u # (v # w) ≠ (u # v) # w [degree of falsehood F], where (T, I, F) is different from (1,0,0) and (0,0,1). Because (1,0,0) represents the classical law (100% true law; T =1, I = 0, F = 0), while (0,0,1) represents the anti- law (i.e. 100% false law, or T=0, I=0, F =1).

iii) [Axiom of existence of the neutro-identity element]

For an element a ∈ G, there exists e ∈ G such that a # e = e # a = a [degree of truth T], and for two elements b, c ∈ G, there exists an e ∈ G such that [b # e or e # b = indeterminate (degree of indeterminacy I) or c # e ≠ c ≠ e # c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

iv) [Axiom of existence of the neutro-inverse element]

For an element a ∈ G, there exists u ∈ G such that a # u = u # a = a (degree of truth T), and for two elements b, c ∈ G, there exists u ∈ G such that [b # u or u # b = indeterminate (degree of indeterminacy I) or c # u ≠ c ≠ u # c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

v) [Axiom of neutro-commutativity]

There exists a double (b, n) ∈ (G, G) such that b # n = n # b (degree of truth T) and there exist two doubles (u, v), (p, q) ∈ (G, G) such that [u # v or v # u = indeterminate (degree of indeterminacy I) or p # q ≠ q # p (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

Definition 2. [21] A neutro-group is a neutro-algebraic structure which possesses at least one of the axioms {i – iv} of Definition 1 and it is an alternative to classical group.

Definition 3. [21] A neutro-commutative group is a neutro – algebraic structure which possesses at least one of the axioms {i – v} of Definition 1 and it is an alternative to classical commutative group.

Definition 4. [21] Let H be a non-empty set and ∘: H ×H → P^*(H) be a hyperoperation. The couple (H, ∘) is called a hypergroupoid. For any two non-empty subsets A and B of H and x ∈ H, we define

$$A \circ B = \bigcup_{a\in A, b\in B} a \circ b, \; A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

Where, P^*(H) is power set of H and ∅ ∉ P^*(H).

Definition 5. [2] A hypergroupoid (H, ∘) is called a semihypergroup if for all a, b, c ∈ H,

$$(a \circ b) \circ c = a \circ (b \circ c)$$

A hypergroupoid (H, ∘) is called a quasihypergroup if for all a ∈ H,

$$a \circ H = H \circ a = H.$$

This condition is also called the reproduction axiom.

Definition 6. [2] A hypergroupoid (H, ∘) which is both a semihypergroup and a quasi-hypergroup is called a hypergroup.

Definition 7. [35] Let X be a nonempty set. Then (X, $o^*_{(m,n)}$) is called an (m, n)-super hyperalgebra, where

$$o^*_{(m,n)}: X^m \to P^n_*(X)$$

is called an (m, n)-super hyperoperation, $P^n_*(X)$ is the n^{th}-powerset of the set X, $\emptyset \notin P^n_*(X)$, for any subset A of $P^n_*(X)$, we identify {A} with A, m, n ≥ 1 and

$$X^m = X \times X \times \ldots \times X \text{ (m times)},$$

$$P^n_*(X) = P(P(\ldots P(X)).$$

Let $o^*_{(m,n)}: X^m \to P^n_*(X)$ is an (m, n)-super hyperoperation on X and $A_1, \ldots, A_m$ subsets of X. We define $o^*_{(m,n)}(A_1, \ldots, A_m) = \bigcup_{x_i \in A_i} o^*_{(m,n)}(x_1, \ldots, x_m)$.

If $\emptyset \in P^n_*(X)$, $o^*_{(m,n)}: X^m \to P^n_*(X)$ is called a neutrosophic (m, n)-super hyperoperation.

Also, it is shown that $o^*_{(m,n)}: X^m \to P^n(X)$

Definition 8. [35] Let $o^*_{(m,n)}: H^m \to P^n_*(H)$ be an (m, n)-super hyperalgebra. Strong SuperHyperAssociativity, for all $x_1, \ldots, x_m, y_1, \ldots, y_{m-1} \in H$,

$$o^*_{(m,n)}(o^*_{(m,n)}(x_1, \ldots, x_m), y_1, \ldots, y_{m-1}) = o^*_{(m,n)}(x_1, o^*_{(m,n)}(x_2, \ldots, x_m), y_1, \ldots, y_{m-1})$$

$$= o^*_{(m,n)}(x_1, x_2\, o^*_{(m,n)}(x_3, \ldots, x_m), y_1, \ldots, y_{m-1})$$

$$= o^*_{(m,n)}(x_1, \ldots, x_{m-1}\, o^*_{(m,n)}(x_m, y_1, \ldots, y_{m-1})$$

NEW TYPE HYPER GROUPS

Definition 9. Let H be a non-empty set and #: H×H → P^*(H) be a hyperoperation. If the following conditions are satisfied, then (H, #) is called a new type hyper group.

i) For all h, k ∈ H, h#k∈ P^*(H).

ii) For all h, k, m ∈ H, h #(k#m) = (h#k)#m

iii) For all h ∈ H, there is an e element such that

h#e = e#h = h

iv) For all h ∈ H, there is an h^{-1} element such that

h#h^{-1} = h^{-1}#h = e

Corollary 10. In Definition 9, we take H instead of P^*(H), then (H, #) is a group.

Corallary 11. It is clear that H ∈ P^*(H). Thus, every groups are a new type hyper group. But, the opposite is not always true.

Corollary 12. Let (H, #) be a new type hyper group. If (H, #) satisfies the condition

i) For all h ∈ H, h#H = H#h = H

then, (H, #) is a hyper group.

Example 13. Let H = {a, b, c, {a, b, c}} be a set.

#	a	b	c	{a, b, c}
a	{a, b, c}	b	c	a
b	a	{a, b, c}	c	b
c	a	b	{a, b, c}	c
{a, b, c}	a	b	c	{a, b, c}

i) It is clear that for all h, k ∈ H, h#k ∈ P^*(H).

ii) It is clear that for all h, k, m ∈ H, h #(k#m) = (h#k)#m

iii) For all h ∈ H, there is an e = {a, b, c} element such that

h#e = e#h = h

iv) For all h ∈ H, there is an h^{-1} = h element such that

h#h^{-1} = h^{-1}#h = e

Thus, (H, #) is a new type hyper group.

NEW TYPE SUPERHYPER GROUPS

Definition 14. Let H be a non-empty set and $o^*_{(m,n)}$: $H^m \to P^n_*$(H) be a superhyperoperation. (H, $o^*_{(m,n)}$) is called a new type superhyper group if the following conditions are satisfied.

i) For all $x_1,\dots,x_m$∈ H$o^*_{(m,n)}(x_1,\dots,x_m) \in P^n_*$(H)

ii) Strong SuperHyperAssociativity, for all $x_1,\dots,x_m$, $y_1,\dots,y_{m-1}$∈ H,

$$o^*_{(m,n)}(o^*_{(m,n)}(x_1,\dots,x_m), y_1,\dots,y_{m-1}) = o^*_{(m,n)}(x_1, o^*_{(m,n)}(x_2,\dots,x_m), y_1,\dots,y_{m-1})$$

$$= o^*_{(m,n)}(x_1,x_2\, o^*_{(m,n)}(x_3,\dots,x_m), y_1,\dots,y_{m-1})$$

$$= o^*_{(m,n)}(x_1,\dots,x_{m-1}\, o^*_{(m,n)}(x_m, y_1,\dots,y_{m-1})$$

iii) For all x ∈ H, there is an e element of H such that

$$o^*_{(m,n)}(x,e,e,\dots,e) = o^*_{(m,n)}(e,x,e,\dots,e) = \dots = o^*_{(m,n)}(e,e,e,\dots,x,e) = o^*_{(m,n)}(e,e,e,\dots e,x) = x$$

iv) For all x ∈ H, there is a x^{-1} element of H such that

$o^*_{(m,n)}(x, x^{-1}, x^{-1}, \ldots, x^{-1}) = o^*_{(m,n)}(x^{-1}, x, x^{-1}, \ldots, x^{-1})$

$$= \ldots = o^*_{(m,n)}(x^{-1}, x^{-1}, x^{-1}, \ldots, x, x^{-1})$$

$$= o^*_{(m,n)}(x^{-1}, x^{-1}, x^{-1}, \ldots x^{-1}, x) = e$$

Corollary 15. In Definition 14, we take m = 2, n = 1, then (H, $o^*_{(m,n)}$) is a new type hyper group.

Corallary 16. Let (H, $o^*_{(m,n)}$) be a new type superhyper group. If the following condition is satisfied, then (H, $o^*_{(m,n)}$) is a superhyper group.

i) For all a ∈ H

$H = o^*_{(m,n)}(a, H, H, \ldots, H) = o^*_{(m,n)}(H, a, H, H, \ldots, H)$

$$= \ldots = o^*_{(m,n)}(H, H, \ldots, H, a, H)$$

$$= o^*_{(m,n)}(H, H, H, \ldots, H, a)$$

NEUTRO-NEW TYPE SUPERHYPER GROUPS

In this section, the symbol "$=_{NC}$" will be used for situations where equality is uncertain. For example, if it is not certain whether "a" and "b" are equal, then it is denoted by $a =_{NC} b$.

Definition 17. Let H be a non-empty set and $o^*_{(m,n)}: H^m \rightarrow P^n_*(H)$ be a neutro-function. If at least one of the following {i, ii, iii} conditions is satisfied, then (H, $o^*_{(m,n)}$) is called a neutro-new type superhyper group.

i) For some $x_i \in A_i$,

$$o^*_{(m,n)}(A_1, \ldots, A_m) = \cup_{x_i \in A_i} o^*_{(m,n)}(x_1, \ldots, x_m) \neq \emptyset \in P^n_*(H) \text{ (degree of truth T)}$$

and For some $z_i \in A_i$, $y_i \in A_i$,

$$(o^*_{(m,n)}(A_1, \ldots, A_m) = \bigcup_{x_i \in A_i} o^*_{(m,n)}(z_1, \ldots, z_m) = \emptyset \notin P^n_*(\text{H}) \text{ (degree of falsity F)}$$

or

$$o^*_{(m,n)}(A_1, \ldots, A_m) = \bigcup_{y_i \in A_i} o^*_{(m,n)}(y_1, \ldots, y_m) =_{\text{NC}} \emptyset \notin P^n_*(\text{H}) \text{ (degree of indeterminacy I)).}$$

Where (T, I, F) is different from (1,0,0) and (0,0,1).

ii) For some $x_1, \ldots, x_m$, $y_1, \ldots, y_{m-1} \in$ H,

$$o^*_{(m,n)}(o^*_{(m,n)}(x_1, \ldots, x_m), y_1, \ldots, y_{m-1}) = o^*_{(m,n)}(x_1, o^*_{(m,n)}(x_2, \ldots, x_m), y_1, \ldots, y_{m-1})$$

$$= o^*_{(m,n)}(x_1, x_2\, o^*_{(m,n)}(x_3, \ldots, x_m), y_1, \ldots, y_{m-1})$$

$$= o^*_{(m,n)}(x_1, \ldots, x_{m-1}\, o^*_{(m,n)}(x_m, y_1, \ldots, y_{m-1})$$

(degree of truth T)

and for some $k_1, \ldots, k_m$, $l_1, \ldots, l_{m-1} \in$ H, $z_1, \ldots, z_m$, $t_1, \ldots, t_{m-1} \in$ H,

$$(o^*_{(m,n)}(o^*_{(m,n)}(k_1, \ldots, k_m), l_1, \ldots, l_{m-1}) \neq o^*_{(m,n)}(k_1, o^*_{(m,n)}(k_2, \ldots, k_m), l_1, \ldots, l_{m-1})$$

$$\neq o^*_{(m,n)}(k_1, k_2\, o^*_{(m,n)}(k_3, \ldots, k_m), l_1, \ldots, l_{m-1})$$

$$\neq o^*_{(m,n)}(k_1, \ldots, k_{m-1}\, o^*_{(m,n)}(k_m, l_1, \ldots, l_{m-1})$$

(degree of falsity F)

or

$$(o^*_{(m,n)}(o^*_{(m,n)}(z_1, \ldots, z_m), y_1, \ldots, y_{m-1}) =_{\text{NC}} o^*_{(m,n)}(z_1, o^*_{(m,n)}(z_2, \ldots, z_m), t_1, \ldots, t_{m-1})$$

$$=_{\text{NC}} o^*_{(m,n)}(z_1, z_2\, o^*_{(m,n)}(z_3, \ldots, z_m), t_1, \ldots, t_{m-1})$$

$=_{NC}\ o^*_{(m,n)}(z_1, \ldots, z_{m-1}\ o^*_{(m,n)}(z_m, t_1, \ldots, t_{m-1})$

(degree of Indeterminacy F)).

Where (T, I, F) is different from (1,0,0) and (0,0,1).

iii) For some x ∈ H, there is an e element of H such that

$o^*_{(m,n)}(x, e, e, \ldots, e) = o^*_{(m,n)}(e, x, e, \ldots, e) = \ldots = o^*_{(m,n)}(e, e, e, \ldots, x, e) = o^*_{(m,n)}(e, e, e, \ldots e, x) = x$

(degree of truth T)

and for some y ∈ H, z ∈ H,

$(o^*_{(m,n)}(y, e, e, \ldots, e) \neq o^*_{(m,n)}(e, y, e, \ldots, e) \neq \ldots \neq o^*_{(m,n)}(e, e, e, \ldots, y, e) \neq o^*_{(m,n)}(e, e, e, \ldots e, y) \neq y$

(degree of falsity F)

or

$(\ o^*_{(m,n)}(\ z, e, e, \ldots, e)\ =_{NC}\ o^*_{(m,n)}(\ e, z, e, \ldots, e)\ =_{NC}\ \ldots\ =_{NC}\ o^*_{(m,n)}(\ e, e, e, \ldots, z, e) =_{NC}\ o^*_{(m,n)}($
$e, e, e, \ldots e, z) =_{NC}\ z$

(degree of indeterminacy F)).

Where (T, I, F) is different from (1,0,0) and (0,0,1).

iv) For some x ∈ H, there is a x^{-1} element of H such that

$o^*_{(m,n)}(x, x^{-1}, x^{-1}, \ldots, x^{-1}) = o^*_{(m,n)}(x^{-1}, x, x^{-1}, \ldots, x^{-1})$

$= \ldots = o^*_{(m,n)}(x^{-1}, x^{-1}, x^{-1}, \ldots, x, x^{-1})$

$= o^*_{(m,n)}(x^{-1}, x^{-1}, x^{-1}, \ldots x^{-1}, x) = e$

(degree of truth T)

and for some y ∈ H, z ∈ H,

$$(o^*_{(m,n)}(y, x^{-1}, x^{-1}, \ldots, x^{-1}) \neq o^*_{(m,n)}(x^{-1}, y, x^{-1}, \ldots, x^{-1})$$

$$\neq \ldots \neq o^*_{(m,n)}(x^{-1}, x^{-1}, x^{-1}, \ldots, y, x^{-1})$$

$$\neq o^*_{(m,n)}(x^{-1}, x^{-1}, x^{-1}, \ldots x^{-1}, y) \neq e$$

or

$$(o^*_{(m,n)}(z, x^{-1}, x^{-1}, \ldots, x^{-1}) =_{NC} o^*_{(m,n)}(x^{-1}, z, x^{-1}, \ldots, x^{-1})$$

$$=_{NC} \ldots =_{NC} o^*_{(m,n)}(x^{-1}, x^{-1}, x^{-1}, \ldots, z, x^{-1})$$

$$=_{NC} o^*_{(m,n)}(x^{-1}, x^{-1}, x^{-1}, \ldots x^{-1}, z) =_{NC} e$$

(degree of indeterminacy F)).

Note 18. From Definition 17, the neutro-new type superhypergroup differrent from the new type superhypergroup. Neutro-new type superhypergroup are given as an alternative to new type superhypergroup. But, for a neutro-new type superhypergroup, instead of the ones that are not met in Definition 17, new type superhypergroup conditions are valid.

Example 19. Let H ={h, k}be a set. $o^*_{(2,2)}$: $H^2 \longrightarrow P^2_{\cup}$ (H) is a superhyperoperation such that

$$o^*_{(2,2)}(X_1, X_2) = (X_1 \cap X_2) \cup (X_1 \cup X_2)^C$$

Where, $o^{\cup}_{(2,2)}$ is satisfied the condition i in Definition 17. Because, if $X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 =$ H, then

$$o^*_{(2,2)}(X_1, X_2) = \emptyset \notin (H, o^*_{(2,2)}).$$

Thus, $(H, o^*_{(2,2)})$ is a neutro-new type superhypergroup. But, $(H, o^*_{(2,2)})$ is not a new type superhypergroup.

Example 20. Let H ={h, k}be a set. $o^{\#}_{(2,2)}$: $H^2 \rightarrow P^2_{\cup}$(H) is a superhyperoperation such that

$$o^{\#}_{(2,2)}(X_1, X_2) = (X_1 \setminus X_2) \cup (X_2 \setminus X_1)$$

Where, $o^{\#}_{(2,2)}$ is satisfied the condition i in Definition 17. Because, if $X_1 \cap X_2 = \emptyset$, then

$o^{\#}_{(2,2)}(X_1, X_2) = \emptyset \notin (H, o^{\#}_{(2,2)})$.

Thus, $(H, o^{\#}_{(3,2)})$ is a neutro-new type superhypergroup. But, $(H, o^{\#}_{(2,2)})$ is not a new type superhypergroup.

Theorem 21. Neutro-new type superhyper groups can be obtained from every new type superhyper group.

Proof. Let (H, $o^{*}_{(m,n)}$) be a new type superhyper group such that

$o^{*}_{(m,n)}$: $H^m \rightarrow P^n_*$(H),

It is clear that $\emptyset \notin P^*$(H). We assume that for any h ∉H such that

h≠ ∅ and $o^{*}_{(m,n)}(a, \ldots, x_m) = \emptyset \notin P^*$(H).

Thus, (H∪{h}, $o^{*}_{(m,n)}$) satisfies condition i from Definition 17. Thus, (H∪{h}, $o^{*}_{(m,n)}$) is a neutro-new type superhyper group.

CONCLUSIONS

In this chapter, the new type superhyper group is defined and relevant basic properties are given. Similarities and differences between the hyper group and superhyper group are discussed. Also, the neutro-new type superhyper group is defined and relevant basic properties are given. Similarities and differences between the neutro-new type superhyper group and new type superhyper group are discussed. Researchers can make use of this chapter to define new type superhyper ring, new type superhyper field, new type superhyper modules, neutro- new type superhyper ring, neutro- new type superhyper field, neutro- new type superhyper modules.

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Chapter Two

SuperHyper Groups and Neutro–SuperHyper Groups

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ABSTRACT

In this chapter, SuperHyper groups are defined, corresponding basic properties and examples for SuperHyper are given and proved. Moreover, SuperHyper groups and are compared to each other. SuperHyper groups are shown to have a more general structure according to Hyper groups. In addition, it is shown that a Hyper group can be obtained from every SuperHyper groups Also, Neutro-SuperHyper groups are defined, corresponding basic properties and examples for Neutro-SuperHyper groups are given and proved. Neutro-SuperHyper groups are shown to have a more general structure according to SuperHyper groups. Thus, (T, I, F) components which constitute the neutrosophic theory are added to SuperHyper groups and a new structure is obtained.

Keywords: SuperHyper Structure, Hyper groups, SuperHyper groups, Neutro- SuperHyper groups

INTRODUCTION

Marty defined hyperstructures [1] in 1934. Hyperstructures are an extended and a new form of classical structures. Corsini obtained hypergroups [2] in 1993. So, many researchers have made studies on this subject [3-7]. Recently, Kanwal et al. studied On Cyclic LA-Hypergroups [8]; Fasino and Freni introduced Hypergroup Theory and Algebrization of Incidence Structures [9];

Smarandache defined neutrosophic logic and the concept of neutrosophic set in 1998 [10]. In the concept of neutrosophic logic and neutrosophic sets, there is a degree of membership T, a degree of uncertainty I and a degree of falsity F. These degrees are defined independently from each other. A neutrosophic value has the form (T, I, F). In other words, in neutrosophy, a situation is handled according to its accuracy, its falsehood, and its uncertainty. Therefore, neutrosophic logic and neutrosophic clusters help us explain many uncertainties in our lives. So, many researchers have made studies on this subject [11-20, 38-65].

Florentin Smarandache introduced new research areas in neutrosophy, which he called neutro-structures and anti-structures, respectively, in 2019 [21, 22]. When evaluating <A> as an element (concept, attribute, idea, proposition, theory, etc.), during the neutrosification process, he worked on three regions; two opposites corresponding to <A> and <antiA> and also a neutral (indeterminate) <neutA> (also called <neutralA>). A neutro-algebra consists of at least one neutro-operation (indeterminate for other items and false for other items) or it is an algebra well-defined for some items (also called internally defined), indeterminate for others, and externally defined for others. Therefore, the subject attracted the attention of many researchers [23–32]. Recently, Al-Tahan et al. studied some neutroHyperstructures [33]; Ibrahim and Agboola obtained NeutroHyperGroups [34].

Florentin Smarandache introduced new research areas, which he called SuperHyperstructures [35] in 2022. Recently, Hamidi studied Superhyper BCK-Algebras [36]; Jahanpanah and Daneshpayeh obtained Superhyper BE-Algebras [37].

In the second section, basic definitions on Hypergrup [2], SuperHyperoperation [35], definitions of neutro-group is given [29]. In the third chapter, SuperHyper groups are defined, corresponding basic properties and examples for SuperHyper are given and proved. Moreover, SuperHyper groups and are compared to each other. SuperHyper groups are shown to have a more general structure according to Hyper groups. In the fourth section, Neutro-SuperHyper groups are defined, corresponding basic properties and examples for Neutro-SuperHyper groups are given and proved. Neutro-SuperHyper groups are shown to have a more general structure according to SuperHyper groups. In the last section, results and suggestions are given.

BACKGROUND

Definition 1. [21]

i) [Law of neutro-well defined]

There exists a double (b, n) ∈ (G, G) such that b # n ∈ G [degree of truth T] and there exist a double (u, v) ∈ (G, G) such that u # v = indeterminate [degree of indeterminacy I], or there exist a double (p, q) ∈ (G, G) such that p # q ∉ G [degree of outer-defined F], where (T, I, F) is different from (1,0,0) and (0,0,1). Because (1,0,0) represents the classical well-defined law (100% well-defined law; T =1, I = 0, F = 0), while (0,0,1) represents the outer-defined law (i.e. 100% outer-defined law, or T=0, I=0, F =1).

ii) [Axiom of neutro-associativity]

There exists a triplet (b, n, m) ∈ (G, G, G) such that b # (n # m) = (b # n) # m [degree of truth T], and there exist two triplets (p, q, r) ∈ (G, G, G) such that p # (q # r) or (p # q) # r = indeterminate [degree of indeterminacy I], or there exist (u, v, w) ∈ (G, G, G) or u # (v # w) ≠ (u # v) # w [degree of falsehood F], where (T, I, F) is different from (1,0,0) and (0,0,1). Because (1,0,0) represents the classical law (100% true law; T =1, I = 0, F = 0), while (0,0,1) represents the anti- law (i.e. 100% false law, or T=0, I=0, F =1).

iii) [Axiom of existence of the neutro-identity element]

For an element a ∈ G, there exists e ∈ G such that a # e = e # a = a [degree of truth T], and for two elements b, c ∈ G, there exists an e ∈ G such that [b # e or e # b = indeterminate (degree of indeterminacy I) or c # e ≠ c ≠ e # c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

iv) [Axiom of existence of the neutro-inverse element]

For an element a ∈ G, there exists u ∈ G such that a # u = u # a = a (degree of truth T), and for two elements b, c ∈ G, there exists u ∈ G such that [b # u or u # b = indeterminate (degree of indeterminacy I) or c # u ≠ c ≠ u # c (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

v) [Axiom of neutro-commutativity]

There exists a double (b, n) ∈ (G, G) such that b # n = n # b (degree of truth T) and there exist two doubles (u, v), (p, q) ∈ (G, G) such that [u # v or v # u = indeterminate (degree of indeterminacy I) or p # q ≠ q # p (degree of falsehood F)], where (T, I, F) is different from (1,0,0) and (0,0,1).

Definition 2. [21] A neutro-group is a neutro-algebraic structure which possesses at least one of the axioms {i – iv} of Definition 1 and it is an alternative to classical group.

Definition 3. [21] A neutro-commutative group is a neutro – algebraic structure which possesses at least one of the axioms {i – v} of Definition 1 and it is an alternative to classical commutative group.

Definition 4. [21] Let H be a non-empty set and ∘: H ×H → P^*(H) be a hyperoperation. The couple (H, ∘) is called a hypergroupoid. For any two non-empty subsets A and B of H and x ∈ H, we define

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \quad A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

Where, $\emptyset \notin P^*(H)$.

Definition 5. [2] A hypergroupoid (H, ∘) is called a semihypergroup if for all a, b, c ∈ H,

$$(a \circ b) \circ c = a \circ (b \circ c)$$

A hypergroupoid (H, ∘) is called a quasihypergroup if for all a ∈ H,

$$a \circ H = H \circ a = H.$$

This condition is also called the reproduction axiom.

Definition 6. [2] A hypergroupoid (H, ∘) which is both a semihypergroup and a quasi-hypergroup is called a hypergroup.

Definition 7. [35] Let X be a nonempty set. Then (X, $o^*_{(m,n)}$) is called an (m, n)-superhyperalgebra, where

$$o^*_{(m,n)}: \mathrm{X}^{\mathrm{m}} \rightarrow P^n_*(\mathrm{X})$$

is called an (m, n)-superhyperoperation, P^n_*(X) is the n^{th}-powerset of the set X, $\emptyset \notin P^n_*$(X), for any subset A of P^n_*(X), we identify {A} with A, m, n ≥ 1 and

$$X^m = \mathrm{X} \times \mathrm{X} \times \ldots \times \mathrm{X} \text{ (m times)},$$

$$P^n_*(\mathrm{X}) = \mathrm{P}(\mathrm{P}(\ldots\mathrm{P}(\mathrm{X})).$$

Let $o^*_{(m,n)}$: $\mathrm{X}^{\mathrm{m}} \rightarrow P^n_*$(X) is an (m, n)-super hyperoperation on X and $A_1, \ldots, A_m$ subsets of X. We define $o^*_{(m,n)}(A_1, \ldots, A_m) = \bigcup_{x_i \in A_i} o^*_{(m,n)}\,(x_1, \ldots, x_m)$.

If $\emptyset \in P^n_*$(X), $o^*_{(m,n)}$: $\mathrm{X}^{\mathrm{m}} \rightarrow P^n_*$(X) is called a neutrosophic (m, n)-superhyperoperation.

Also, it is shown that $o^*_{(m,n)}$: $\mathrm{X}^{\mathrm{m}} \rightarrow P^n$(X)

Definition 8. [35] Let $o^*_{(m,n)}$: $\mathrm{H}^{\mathrm{m}} \rightarrow P^n_*$ (H) be an (m, n)-superhyperalgebra. Strong SuperHyperAssociativity, for all $x_1, \ldots, x_m, y_1, \ldots, y_{m-1} \in$ H,

$$o^*_{(m,n)}(o^*_{(m,n)}(x_1, \ldots, x_m), y_1, \ldots, y_{m-1}) = o^*_{(m,n)}(x_1, o^*_{(m,n)}(x_2, \ldots, x_m), y_1, \ldots, y_{m-1})$$

$$= o^*_{(m,n)}(x_1, x_2\ o^*_{(m,n)}(x_3, \ldots, x_m), y_1, \ldots, y_{m-1})$$

$$= o^*_{(m,n)}(x_1, \ldots, x_{m-1}\ o^*_{(m,n)}(x_m, y_1, \ldots, y_{m-1})$$

SUPERHYPER GROUPS

Definition 9. Let H be a non-empty set and $o^*_{(m,n)}$: $H^m \rightarrow P^n_*(H)$ be a superhyperoperation. The couple (H, $o^*_{(m,n)}$) is called a superhyper groupoid. For any two non-empty subsets A and B of H and x ∈ H, we define

$$o^*_{(m,n)}(A_1, \ldots, A_m) = \cup_{x_i \in A_i} o^*_{(m,n)} (x_1, \ldots, x_m).$$

Where, $\emptyset \notin P^*(H)$.

If $\emptyset \in P^*(H)$, then (H, $o^*_{(m,n)}$) is called a neutrosophic superhyper groupoid.

Note 10. From Definition 4 and Definition 7, we obtain defininiton of superhyper groupoid.

Example 11. Let H ={a, b}be a set. $o^{\cup}_{(3,2)}$: $H^3 \rightarrow P^2_{\cup}(H)$ is a superhyperoperation such that

$$o^{\cup}_{(3,2)}(x_1, x_2, x_3) = \cup^3_{i=1}\{x_i\}.$$

For example, $o^{\cup}_{(3,2)}$(a, a, b) = {a}∪{a}∪{b} = {a, b}

Where, $P^2_{\cup}(H)$ = P(P(H))

P(H) = {a, b, {a, b}}

P(P(H)) = {a, b, {a, b}, {a, {a, b}}, {b, {a, b}}, {a, b, {a, b}}}.

Thus, $o^{\cup}_{(3,2)}(x_1, x_2, x_3) \in P^2_{\cup}(H)$

Hence, (H, $o^{\cup}_{(3,2)}$) is a superhyper groupoid.

Theorem 12. Let H be a non-empty finity set, $o^{\cup}_{(m,n)}$: $H^m \rightarrow P^n_{\cup}(H)$ be a superhyperoperation such that

$$o^{\cup}_{(m,n)}(x_1, \ldots, x_m) = \cup^m_{i=1}\{x_i\}.$$

Then, (H, $o^{\cup}_{(m,n)}$) is a superhyper groupoid.

Proof: It is clear that for all $x_i \in P^n_{\cup}$(H),

$$o^{\cup}_{(m,n)}(x_1, \ldots, x_m) = \cup_{i=1}^{m}\{x_i\} \in P^n_{\cup}(\mathrm{H}).$$

Thus, (H, $o^{\cup}_{(m,n)}$) is a superhyper groupoid.

Theorem 13. Let H be a non-empty finity set, $o^{\cap}_{(m,n)}$: $\mathrm{H}^{\mathrm{m}} \rightarrow P^n_{\cap}$(H) be a superhyperoperation such that

$$o^{\cap}_{(m,n)}(x_1, \ldots, x_m) = \cap_{i=1}^{m}\{x_i\}$$

Then, (H, $o^{\cap}_{(m,n)}$) is not a superhyper groupoid. But, (H, $o^{\cap}_{(m,n)}$) is a neutrosophic superhyper groupoid. Where, s(H)>1. (s(H) is element number of H)

Proof: We assume that $\{x_1\}, \{x_2\}, \ldots, \{x_m\}$ sets are discrete. Thus,

$$o^{\cap}_{(m,n)}(x_1, \ldots, x_m) = \cap_{i=1}^{m}\{x_i\} = \emptyset \notin P^n_{\cap}(\mathrm{H}).$$

Hence, Then, (H, $o^{\cap}_{(m,n)}$) is not a superhyper groupoid and (H, $o^{\cap}_{(m,n)}$) is a neutrosophic superhyper groupoid.

Definition 14. Let (H, $o^{*}_{(m,n)}$) is called a superhypergroupoid. If (H, $o^{*}_{(m,n)}$) is satisfied the strong SuperHyperAssociativity, then (H, $o^{*}_{(m,n)}$) is called a supersemihyper group.

If $\emptyset \in P^{*}$(H), then (H, $o^{*}_{(m,n)}$) is called a neutrosophic supersemihyper group.

Note 15. From Definition 5 and Definition 8, we obtain defininiton of superhypersemihyper group.

Example 16. From Example 11, (H, $o^{\cup}_{(3,2)}$) is a superhypergroupoid. Also, it is clear that (H, $o^{\cup}_{(3,2)}$) is satisfies the strong superHyperAssociativity such that for all $x_1, \ldots, x_m$, $y_1, \ldots, y_{m-1} \in$ H,

$o^{\cup}_{(3,2)}(o^{\cup}_{(3,2)}(x_1, \ldots, x_m), y_1, \ldots, y_{m-1}) = o^{\cup}_{(3,2)}(o^{\cup}_{(3,2)}(x_2, \ldots, x_m), y_1, \ldots, y_{m-1})$

$= o^{\cup}_{(3,2)}(x_1, x_2\ o^{\cup}_{(3,2)}(x_3, \ldots, x_m), y_1, \ldots, y_{m-1})$

$= o^{\cup}_{(3,2)}(x_1, \ldots, x_{m-1}\ o^{\cup}_{(3,2)}(x_m, y_1, \ldots, y_{m-1})$.

Hence, (H, $o^{\cup}_{(3,2)}$) is a supersemihyper group.

Definition 17. Let (H, $o^{*}_{(m,n)}$) be a superhyper groupoid. For all a ∈ H, If

H = $o^{*}_{(m,n)}$(a, H, H, …, H) = $o^{*}_{(m,n)}$(H, a, H, H, …, H)

= … = $o^{*}_{(m,n)}$(H, H, …, H, a, H)

= $o^{*}_{(m,n)}$(H, H, H, …, H, a)

then, (H, $o^{*}_{(m,n)}$) is called a superquasihyper group.

If ∅ ∈ P^{*}(H), then (H, $o^{*}_{(m,n)}$) is called a neutrosophic superquasihyper group.

Where, from Defininiton 7, for all a, $x_2, \ldots, x_m$ a ∈ H,

$$o^{*}_{(m,n)}(\text{a, H, H, …, H}) = \bigcup_{x_i \in A_i} o^{*}_{(m,n)}\,(\text{a},\ x_2, \ldots, x_m).$$

Note 18. From Definition 5, we obtain defininiton of superhypersemihypergroup.

Example 19. From Example 14, (H, $o^{\cup}_{(3,2)}$) is a superhypergroupoid. Also, it is clear that for a∈ H, (H, $o^{\cup}_{(3,2)}$) is satisfies the

H = $o^{\cup}_{(3,2)}$(a, H, H, …, H) = $o^{\cup}_{(3,2)}$(H, a, H, H, …, H)

= … = $o^{\cup}_{(3,2)}$(H, H, …, H, a, H)

= $o^{\cup}_{(3,2)}$(H, H, H, …, H, a)

Hence, (H, $o^{\cup}_{(3,2)}$) is a superquasihyper group.

Definition 20. A hypergroupoid (H, $o^{*}_{(m,n)}$) which is both a supersemihypergroup and a superquasihypergroup is called a superhypergroup.

If $\emptyset \in P^{*}$(H), then (H, $o^{*}_{(m,n)}$) is called a neutrosophic superhyper group.

Note 21. From Definition 6, we obtain defininiton of superhypersemihypergroup.

Example 22. From Example 19, Example 16, and Example 11; (H, $o^{\cup}_{(3,2)}$) is a superhypergroup.

Theorem 23. Let H be a non-empty finity set, $o^{\cup}_{(m,n)}$: $\mathrm{H}^{\mathrm{m}} \rightarrow P^{n}_{\cup}$(H) be a superhyperoperation such that

$$o^{\cup}_{(m,n)}(x_1, . . . , x_m) = \cup_{i=1}^{m}\{x_i\}.$$

Then, (H, $o^{\cup}_{(m,n)}$) is a superhypergroup.

Proof: From Theorem 13, (H, $o^{\cup}_{(m,n)}$) is a superhyper groupoid. Also, for all $x_1, \ldots, x_m$, $y_1, \ldots, y_{m-1} \in$ H,

$$o^{\cup}_{(m,n)}(o^{\cup}_{(m,n)}(x_1, \ldots, x_m), y_1, \ldots, y_{m-1}) = o^{\cup}_{(m,n)}(x_1, o^{\cup}_{(m,n)}(x_2, \ldots, x_m), y_1, \ldots, y_{m-1})$$

$$= o^{\cup}_{(m,n)}(x_1, x_2\ o^{\cup}_{(m,n)}(x_3, \ldots, x_m), y_1, \ldots, y_{m-1})$$

$$= o^{\cup}_{(m,n)}(x_1, \ldots, x_{m-1}\ o^{\cup}_{(m,n)}(x_m, y_1, \ldots, y_{m-1}).$$

Thus, (H, $o^{\cup}_{(m,n)}$) is a supersemihyper groupoid. Furthermore, for all a ∈ H,

$$\mathrm{H} = o^{\cup}_{(m,n)}(\mathrm{a, H, H, \ldots, H}) = o^{\cup}_{(m,n)}(\mathrm{H, a, H, H, \ldots, H})$$

$$= \ldots = o^{\cup}_{(m,n)}(\mathrm{H, H, \ldots, H, a, H})$$

$$= o^{\cup}_{(m,n)}(\mathrm{H, H, H, \ldots, H, a}).$$

Thus, (H, $o^{\cup}_{(m,n)}$) is a superquasihyper groupoid.

Hence, from Definition 20, (H, $o^{\cup}_{(m,n)}$) is a superhyper group.

Corollary 24. Let H be a non-empty finity set, $o^{\cap}_{(m,n)}$: $\mathrm{H^m} \to P^n_{\cap}$(H) be a superhyperoperation such that

$$o^{\cap}_{(m,n)}(x_1, \ldots, x_m) = \cap_{i=1}^{m}\{x_i\}$$

Then, From Theorem 13, (H, $o^{\cap}_{(m,n)}$) is not a superhypergroup. Where, s(H)>1. (s(H) is element number of H)

NEUTRO-SUPERHYPER GROUPS

In this section, the symbol "$=_{NC}$" will be used for situations where equality is uncertain. For example, if it is not certain whether "a" and "b" are equal, then it is denoted by a $=_{NC}$ b.

Definition 25. Let H be a non-empty set and $o^{*}_{(m,n)}$: $\mathrm{H^m} \to P^n_{*}$(H) be a neutro-function. If at least one of the following {i, ii, iii} conditions is satisfied, then (H, $o^{*}_{(m,n)}$) is called a neutro-superhypergroup.

i) For some $x_i \in A_i$,

$$o^{*}_{(m,n)}(A_1, \ldots, A_m) = \cup_{x_i \in A_i} o^{*}_{(m,n)} (x_1, \ldots, x_m) \neq \emptyset \in P^n_{*}(\mathrm{H}) \text{ (degree of truth T)}$$

and For some $z_i \in A_i$, $y_i \in A_i$,

$$(o^{*}_{(m,n)}(A_1, \ldots, A_m) = \cup_{x_i \in A_i} o^{*}_{(m,n)} (z_1, \ldots, z_m) = \emptyset \notin P^n_{*}(\mathrm{H}) \text{ (degree of falsity F)}$$

or

$$o^{*}_{(m,n)}(A_1, \ldots, A_m) = \cup_{y_i \in A_i} o^{*}_{(m,n)} (y_1, \ldots, y_m) =_{NC} \emptyset \notin P^n_{*}(\mathrm{H}) \text{ (degree of indeterminacy I)).}$$

Where (T, I, F) is different from (1,0,0) and (0,0,1).

ii) For some $x_1, \ldots, x_m$, $y_1, \ldots, y_{m-1} \in$ H,

$$o^*_{(m,n)}(o^*_{(m,n)}(x_1, \ldots, x_m), y_1, \ldots, y_{m-1}) = o^*_{(m,n)}(x_1, o^*_{(m,n)}(x_2, \ldots, x_m), y_1, \ldots, y_{m-1})$$

$$= o^*_{(m,n)}(x_1, x_2\ o^*_{(m,n)}(x_3, \ldots, x_m), y_1, \ldots, y_{m-1})$$

$$= o^*_{(m,n)}(x_1, \ldots, x_{m-1}\ o^*_{(m,n)}(x_m, y_1, \ldots, y_{m-1})$$

(degree of truth T)

and for some $k_1, \ldots, k_m$, $l_1, \ldots, l_{m-1} \in$ H, $z_1, \ldots, z_m$, $t_1, \ldots, t_{m-1} \in$ H,

$$(o^*_{(m,n)}(o^*_{(m,n)}(k_1, \ldots, k_m), l_1, \ldots, l_{m-1}) \neq o^*_{(m,n)}(k_1, o^*_{(m,n)}(k_2, \ldots, k_m), l_1, \ldots, l_{m-1})$$

$$\neq o^*_{(m,n)}(k_1, k_2\ o^*_{(m,n)}(k_3, \ldots, k_m), l_1, \ldots, l_{m-1})$$

$$\neq o^*_{(m,n)}(k_1, \ldots, k_{m-1}\ o^*_{(m,n)}(k_m, l_1, \ldots, l_{m-1})$$

(degree of falsity F)

or

$$(o^*_{(m,n)}(o^*_{(m,n)}(z_1, \ldots, z_m), y_1, \ldots, y_{m-1}) =_{NC} o^*_{(m,n)}(z_1, o^*_{(m,n)}(z_2, \ldots, z_m), t_1, \ldots, t_{m-1})$$

$$=_{NC} o^*_{(m,n)}(z_1, z_2\ o^*_{(m,n)}(z_3, \ldots, z_m), t_1, \ldots, t_{m-1})$$

$$=_{NC} o^*_{(m,n)}(z_1, \ldots, z_{m-1}\ o^*_{(m,n)}(z_m, t_1, \ldots, t_{m-1})$$

(degree of Indeterminacy F)).

Where (T, I, F) is different from (1,0,0) and (0,0,1).

iii) For some a ∈ H

H = $o^*_{(m,n)}$(a, H, H, …, H) = $o^*_{(m,n)}$(H, a, H, H, …, H)

$= \ldots = o^{*}_{(m,n)}$(H, H, …, H, a, H)

$= o^{*}_{(m,n)}$(H, H, H, …, H, a)

(degree of truth T)

and for some b ∈ H, c ∈ H,

(H $\neq o^{*}_{(m,n)}$(b, H, H, …, H) $\neq o^{*}_{(m,n)}$(H, b, H, H, …, H)

$\neq \ldots \neq o^{*}_{(m,n)}$(H, H, …, H, b, H)

$\neq o^{*}_{(m,n)}$(H, H, H, …, H, b)

(degree of falsity F)

or

(H$=_{NC}$ $o^{*}_{(m,n)}$(c, H, H, …, H) $=_{NC}$ $o^{*}_{(m,n)}$(H, c, H, H, …, H)

$=_{NC} \ldots =_{NC}$ $o^{*}_{(m,n)}$(H, H, …, H, c, H)

$=_{NC}$ $o^{*}_{(m,n)}$(H, H, H, …, H, c)

(degree of falsity F)).

Where (T, I, F) is different from (1,0,0) and (0,0,1).

Note 26. From Definition 24, the neutro-superhypergroup differrent from the superhypergroup. neutro-superhypergroup are given as an alternative to superhypergroup. But, for a neutro-superhypergroup, instead of the ones that are not met in Definition 24, classical superhypergroup conditions are valid.

Example 27. Let H ={a, b}be a set. $o^{\cap}_{(3,2)}$: $H^3 \rightarrow P^2_{\cap}$(H) is a neutron-function such that

$$o^{\cap}_{(3,2)}(x_1, x_2, x_3) = \bigcap_{i=1}^{3}\{x_i\}.$$

Where, $P^2_{\cup}$(H) = P(P(H))

P(H) = {a, b, {a, b}}

P(P(H)) = {a, b, {a, b}, {a, {a, b}}, {b, {a, b}}, {a, b, {a, b}}}.

Also,

$o^{\cap}_{(3,2)}$(a, a, a) = {a}∩{a}∩{a} = {a}∈ $P^2_{\cap}$(H) .

$o^{\cap}_{(3,2)}$(a, a, b) = {a}∩{a}∩{b} =∅ ∉ $P^2_{\cap}$(H).

Thus, (H, $o^{\cap}_{(3,2)}$) satisfies condition i from Definition 24. Hence, (H, $o^{\cap}_{(3,2)}$) is a neutro-superhypergroup.

Corollary 28. From Theorem 16, H, $o^{\cap}_{(3,2)}$) is not a neutro-superhypergroup. But, from Example 26, (H, $o^{\cap}_{(3,2)}$) is a neutro-superhypergroup.

Theorem 29. Neutro-superhyper groups can be obtained from every superhyper group.

Proof. Let (H, $o^{*}_{(m,n)}$) be a superhyper group such that

$$o^{*}_{(m,n)}: \mathrm{H}^{\mathrm{m}} \to P^{n}_{*}(\mathrm{H}), \quad o^{*}_{(m,n)}(A_1, \ldots, A_m) = \bigcup_{x_i \in A_i} o^{*}_{(m,n)}(x_1, \ldots, x_m)$$

It is clear that ∅ ∉ P^{*}(H). We assume that for any a≠ ∅ element

$$o^{*}_{(m,n)}(a, \ldots, x_m) = \emptyset \notin P^{*}(\mathrm{H}).$$

Thus, (H∪{a}, $o^{*}_{(m,n)}$) satisfies condition i from Definition 24. Thus, (H∪{a}, $o^{*}_{(m,n)}$) is a neutro-superhyper group.

Corollary 30. Let (H, $o^{*}_{(m,n)}$) be a neutrosophic superhyper group. Then, (H, $o^{*}_{(m,n)}$) is a neutro-superhyper group.

CONCLUSIONS

In this chapter, the superhyper group is defined and relevant basic properties are given. Similarities and differences between the hyper group and superhyper group are discussed. Also, the neutro-superhyper group is defined and relevant basic properties are given. Similarities and differences between the neutro-superhyper group and superhyper group are discussed. Researchers can make use of this chapter to define superhyper ring, superhyper field, superhyper modules, neutro-superhyper ring, neutro-superhyper field, neutro-superhyper modules.

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Chapter Three

Fixed Point Theorem for Compatible Mappings of Type (I) and (II) in Neutrosophic Metric Spaces

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Abstract

In this manuscript, we take the concept of compatible mappings in neutrosophic metric spaces. We define the relation between two pair of mappings which are Compatible of type (II) if and only if pair of mappings are Compatible of type (I) and also prove that for four mappings common fixed point theorem under the compatible mappings condition of type (I) and (II) in the complete neutrosophic metric spaces. We also prove some non-trivial examples which support our result. In an application we use our established result to find the unique solution to an integral equation in dynamic market equilibrium economics.

Keywords: neutrosophic metric spaces, compatible mappings of types (α) and (β), compatible mappings of type (I) and (II).

MSC: 54H25, 47H10.

1. Introduction

The concept of quantum particle physics and fuzzy topology may have important applications, given by Elnaschie [1, 2]. Zadeh [3] introduced the notion of fuzzy sets (FS). Kramosil and Michalek [4] used the notion of FS and defined the notion of fuzzy metric spaces (FMSs).
Kaleva and Seikkala [5] introduced the concept of FMS and proved the distance between two points in a FMS is a non-negative, upper semi continuous, normal and convex fuzzy

number. Deng [6] established the fuzzy pseudo-metric spaces, between two fuzzy points defined the metric. Erceg [7] describes a uniformity for a metric space on a FS, used the definition of uniformity given by Hutton and defined the Conjugate pseudo-metrics. Lowen [8] presented a class of mathematical functions that can be used to calculate the distance between FS and explained the relation to the ordinary pseudo metrics Fang [9] gave some new fixed point (FP) theorems for contractive type mappings in FMS. George and veeramani [10] defined a Hausdorff topology and some known results of metric spaces including Baire's theorem on FMS. Grabiec [11] expand the well-known Banach FP theorems and Edelstein to FMS. Mihet [12] extended results on fuzzy contractive mappings to Edelstein fuzzy contractive mappings. Many research treating imprecision and uncertainty have been developed and studied [41-59].

Alaca [13] used the idea of intuitionistic fuzzy sets (IFs), and defined the notion of intuitionistic fuzzy metric space (IFMS). Turkoglu et.al [14] defined R-weakly commuting mappings in IFMSs and proved the intuitionistic fuzzy version of Pants theorem. Abbas and Jungck [15] established the existence of coincidence points and common FPs for mappings satisfying certain contractive conditions, without appealing to continuity, in a cone metric space. Park [16, 17] defined the notion of IFMS as a natural generalization of fuzzy metric spaces and proved some results of metric spaces including the Uniform limit theorem for IFMSs and Baire's theorem. Saleem [18] introduced the notion of intuitionistic extended fuzzy b-metric-like spaces, established some FP theorems.
Farheen et al. [19] introduced the concept of intuitionistic fuzzy double controlled metric spaces that generalized the concept of intuitionistic fuzzy b-metric spaces. Alaca et al [20] gave some new definitions of compatible mapping in IFMSs. Saadati and Park [21] defined precompact set in IFMSs and proved that any subset of an IFMS is compact if and only if it is complete and precompact. Also defined intuitionistic fuzzy metrizable spaces topologically complete and defined intuitionistic fuzzy normed spaces and fuzzy boundedness for linear operators and proved that every finite dimensional intuitionistic fuzzy normed space is complete. Pant [22] introduced the notion of R-weak commutativity of mappings and proved two common FP theorems for pair of mappings.
Turkoglu et al. [23] formulated the definition of compatible maps and compatible maps of types (α) and (β) in IFMSs and give some relations between the concepts of compatible maps and compatible maps of types (α) and (β). Turkoglu et.al [24] proved a common FP theorem for compatible maps of type (α) on FMS. Simsek and Kirisci [25] used the notion of neutrosophic metric space (NMS) and proved various FP theorems. Ishtiaq [26] introduced the concept of an orthogonal NMS. Uddin [27] derived the concept of controlled neutrosophic metric-like spaces as a generalization of neutrosophic metric spaces. Ahin et al. [30, 31] provided certain transformations based on centroid points between single valued neutrosophic numbers as well as according to the truth, indeterminacy, and falsity values of single valued neutrosophic numbers. By expanding the idea of Q-neutrosophic soft expert sets and defining the related ideas and fundamental operations of complement, subset, union, intersection, AND, and OR, Hassan et al. [32, 40] made it possible to convert soft expert sets from being one dimensional to being two dimensional. Ulucay et al. [33] introduced the term "time-neutrosophic soft expert set" and described its fundamental operations, including complement, union, intersection, AND, and OR, as well as looked into some of its characteristics. Ulucay et al. [34] created a ranking technique based on the outranking relations of bipolar neutrosophic numbers and provided a new outranking

methodology for multi-attribute decision-making issues in the bipolar neutrosophic environment.
For improved neutrosophic sets, Ulucay et al. [35] presented a new distance-based similarity measure. According to Broumi et al. [36], the features of complex neutrosophic sets were employed to measure the fluctuation and uncertainty of neutrosophic sets. In order to create an algorithm for a neutrosophic soft expert multisets (NSEM) decision-making approach that enables for a more effective decision-making process, Bakbak et al. [37] designed a NSEMs aggregation operator. In addition to examining the desirable aspects of the outranking relations and developing a ranking approach for multi-criteria decision-making situations, Ulucay et al. [38, 39] introduced the notion of the neutrosophic soft expert graph. In recent years, the academic community has witnessed growing research interests in uncertainty set theory [60-79].
The basic aim of this article,

i) We generalize the results of (Alaca et al [20]) in the content of NMS
ii) we obtain the concepts of B-compatible and A-compatible and some examples for different compatible mappings in NMS.
iii) We established the concept of compatible mappings of type (I), (II) in NMS.
iv) We also give an application that supports our main result.

2. Preliminaries

In this section, we discuss some basic definitions which are help to understand our main result.

Definition 2.1: [28] A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ is satisfying the following conditions:

a) $*$ is continuous;
b) $n * 1 = n$ for all $n \in [0,1]$;
c) $*$ is commutative associative;
d) $n * g \leq c * d$ whenever $n \leq c, g \leq d$ for all $n, g, c, d \in [0,1]$.

Definition 2.2: [28] A binary operation $\Diamond: [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm if $\diamond$ is satisfying the following conditions:

1) $\Diamond$ is continuous;
2) $n \Diamond 0 = 0 \Diamond n = n$ for all $n \in [0,1]$.
3) $n \Diamond g \leq c \Diamond d$ whenever $n \leq c, g \leq d$ and $n, g, c, d \in [0,1]$,
4) $\Diamond$ is commutative associative;

The following definition was introduced by Alaca et al. [1].

Definition 2.3: [17] A 5-tuple $(\mathcal{L}, \beta_i, \rho_i, *, \Diamond)$ is called IFMS if $\mathcal{L}$ is an arbitrary nonempty set, $*$ is a CTN $\Diamond$ is a CTCN and β, ρ are fs on $\mathcal{L}^2 \times (0, \infty)$ satisfying the following conditions:

IFM1) $\beta_i(\ell, \varpi, \tau) + \rho_i(\ell, \varpi, \tau) \leq 1$ for all $\ell, \varpi \in \mathcal{L}$ and $\tau > 0$;

IFM2) $\beta_i(\ell, \varpi, 0) = 0$ for all $\ell, \varpi \in \mathcal{L}$;
IFM3) $\beta_i(\ell, \varpi, \tau) = 1$ for all $\ell, \varpi \in \mathcal{L}$ and $\tau > 0$ if and only if $\ell = \varpi$;
IFM4) $\beta_i(\ell, \varpi, \tau) = \beta_i(\varpi, \ell, \tau)$ for all $\ell, \varpi \in \mathcal{L}$ and $\tau > 0$;
IFM5) $\beta_i(\ell, \varpi, \tau) * \beta_i(\varpi, z, s) \leq \beta_i(\ell, z, \tau + s)$ for all $\ell, \varpi, z \in \mathcal{L}$ and s, $\tau > 0$;
IFM6) for all $\ell, \varpi \in \mathcal{L}, \beta_i(\ell, \varpi, .): [0, \infty) \rightarrow [0,1]$is left continuous;
IFM7) $\lim_{\tau \to \infty} \beta_i(\ell, \varpi, \tau) = 1$ for all $\ell, \varpi \in \mathcal{L}$ and $\tau > 0$;
IFM8) $\rho_i(\ell, \varpi, 0) = 1$ for all $\ell, \varpi \in \mathcal{L}$;
IFM9) $\rho_i(\ell, \varpi, \tau) = 0$ for all $\ell, \varpi \in \mathcal{L}$ and $\tau > 0$ if and only if $\ell = \varpi$;
IFM10) $\rho_i(\ell, \varpi, \tau) = \rho_i(\varpi, \ell, \tau)$ for all $\ell, \varpi \in \mathcal{L}$ and $\tau > 0$;
IFM11) $\rho_i(\ell, \varpi, \tau) \lozenge \rho_i(\varpi, z, s) \geq \rho_i(\ell, z, \tau + s)$ for all $\ell, \varpi, z \in \mathcal{L}$ and $s, \tau > 0$;
IFM12) for all $\ell, \varpi \in \mathcal{L}, \rho_i(\ell, \varpi, .): [0, \infty) \rightarrow [0,1]$is right continuous;
IFM13) $\lim_{\tau \to \infty} \rho_i(\ell, \varpi, \tau) = 0$for all ℓ, ϖ in $\mathcal{L}$;

Then the pair (β_i, ρ_i) is said to be intuitionistic fuzzy metric on $\mathcal{L}$.

Definition 2.4: [29] A 6-tuple $(\mathcal{L}, \beta, \rho, \omega, *, \lozenge)$ is said to be a NMS if $\mathcal{L}$ is an arbitrary nonempty set, $*$ is a CTN $\lozenge$ is a CTCN and β, ρ and ω are fuzzy sets on $\mathcal{L}^2 \times (0, \infty)$ satisfying the following conditions:

NMS1) $\beta(\ell, \varpi, \tau) + \rho(\ell, \varpi, \tau) + \omega(\ell, \varpi, \tau) \leq 3$ for all $\ell, \varpi \in \mathcal{L}$ and $\tau > 0$;
NMS2) $\beta(\ell, \varpi, 0) = 0$ for all $\ell, \varpi \in \mathcal{L}$;
NMS3) $\beta(\ell, \varpi, \tau) = 1$ for all $\ell, \varpi \in \mathcal{L}$ and $\tau > 0$ if and only if $\ell = \varpi$;
NMS4) $\beta(\ell, \varpi, \tau) = \beta(\varpi, \ell, \tau)$ for all $\ell, \varpi \in \mathcal{L}$ and $\tau > 0$;
NMS5) $\beta(\ell, \varpi, \tau) * \beta(\varpi, z, s) \leq \beta(\ell, z, \tau + s)$ for all $\ell, \varpi, z \in \mathcal{L}$ and s, $\tau > 0$;
NMS6) for all $\ell, \varpi \in \mathcal{L}, \beta(\ell, \varpi, .): [0, \infty) \rightarrow [0,1]$is left continuous;
NMS7) $\lim_{\tau \to \infty} \beta(\ell, \varpi, \tau) = 1$ for all $\ell, \varpi \in \mathcal{L}$ and $\tau > 0$;
NMS8) $\rho(\ell, \varpi, 0) = 1$ for all $\ell, \varpi \in \mathcal{L}$;
NMS9) $\rho(\ell, \varpi, \tau) = 0$ for all $\ell, \varpi \in \mathcal{L}$ and $\tau > 0$ if and only if $\ell = \varpi$;
NMS10) $\rho(\ell, \varpi, \tau) = \rho(\varpi, \ell, \tau)$ for all $\ell, \varpi \in \mathcal{L}$ and $\tau > 0$;
NMS11) $\rho(\ell, \varpi, \tau) \lozenge \rho(\varpi, z, s) \geq \rho(\ell, z, \tau + s)$ for all $\ell, \varpi, z \in \mathcal{L}$ and $s, \tau > 0$;
NMS12) for all $\ell, \varpi \in \mathcal{L}, \rho(\ell, \varpi, .): [0, \infty) \rightarrow [0,1]$is right continuous;
NMS13) $\lim_{\tau \to \infty} \rho(\ell, \varpi, \tau) = 0$for all ℓ, ϖ in $\mathcal{L}$;
NMS14) $\omega(\ell, \varpi, 0) = 1$ for all $\ell, \varpi \in \mathcal{L}$;
NMS15) $\omega(\ell, \varpi, \tau) = 0$ for all $\ell, \varpi \in \mathcal{L}$ and $\tau > 0$ if and only if $\ell = \varpi$;
NMS16) $\omega(\ell, \varpi, \tau) = \omega(\varpi, \ell, \tau)$ for all $\ell, \varpi \in \mathcal{L}$ and $\tau > 0$;
NMS17) $\omega(\ell, \varpi, \tau) \lozenge \omega(\varpi, z, s) \geq \omega(\ell, z, \tau + s)$ for all $\ell, \varpi, z \in \mathcal{L}$ and $s, \tau > 0$;
NMS18) for all $\ell, \varpi \in \mathcal{L}, \omega(\ell, \varpi, .): [0, \infty) \rightarrow [0,1]$is right continuous;
NMS19) $\lim_{\tau \to \infty} \omega(\ell, \varpi, \tau) = 0$for all ℓ, ϖ in $\mathcal{L}$.

Then (β, ρ, ω) is said to be neutrosophic metric on $\mathcal{L}$.

Definition 2.5: [29] Let $(\mathcal{L}, \beta, \rho, \omega, *, \lozenge)$ be a NMS. Then,

i) a sequence $\{\ell_\kappa\}$ in $\mathcal{L}$ is said to be Cauchy sequence if, for all $\tau > 0$ and $\hbar > 0$

$$\lim_{\kappa \to \infty} \beta(\ell_{\kappa+\hbar}, \ell_\kappa, \tau) = 1,$$
$$\lim_{\kappa \to \infty} \rho(\ell_{\kappa+\hbar}, \ell_\kappa, \tau) = 0,$$
$$\lim_{\kappa \to \infty} \omega(\ell_{\kappa+\hbar}, \ell_\kappa, \tau) = 0.$$

ii) a sequence $\{\ell_\kappa\}$ in $\mathcal{L}$ is called convergent to a point $\ell \in \mathcal{L}$ if, for all $\tau > 0$,

$$\lim_{\kappa \to \infty} \beta(\ell_\kappa, \ell, \tau) = 1,$$

$$\lim_{\kappa \to \infty} \rho(\ell_\kappa, \ell, \tau) = 0,$$

$$\lim_{\kappa \to \infty} \omega(\ell_\kappa, \ell, \tau) = 0.$$

iii) is a complete if and only if every Cauchy sequence in $\mathcal{L}$ is convergent.

Definition 2.6: [23] The maps $\mathcal{A}$ and $\mathcal{E}$ are called compatible, if $\mathcal{A}$ and $\mathcal{E}$are self-mappings in IFMS $(\mathcal{L}, \beta_i, \rho_i, *, \lozenge)$ for all $\tau > 0$,

$$\lim_{\kappa \to \infty} \beta_i(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = 1,$$

$$\lim_{\kappa \to \infty} \rho_i(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = 0,$$

so that $\lim_{\kappa \to \infty} \mathcal{A}\ell_\kappa = \lim_{\kappa \to \infty} \mathcal{E}\ell_\kappa = z$ for some $z \in \mathcal{L}$, wherever $\{\ell_\kappa\}$ is a sequence in $\mathcal{L}$.

Definition 2.7: [23] Suppose that $\mathcal{A}$ and $\mathcal{E}$ are maps from an IFMS $(\mathcal{L}, \beta_i, \rho_i, *, \lozenge)$ into itself. for all $\tau > 0$, the maps $\mathcal{A}$ and $\mathcal{E}$ are said to be compatible of type (α).

$$\lim_{\kappa \to \infty} \beta_i(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 1 \text{ and } \lim_{\kappa \to \infty} \rho_i(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0,$$

$$\lim_{\kappa \to \infty} \beta_i(\mathcal{E}\mathcal{A}\ell_\kappa, \mathcal{A}\mathcal{A}\ell_\kappa, \tau) = 1 \text{ and } \lim_{\kappa \to \infty} \rho_i(\mathcal{E}\mathcal{A}\ell_\kappa, \mathcal{A}\mathcal{A}\ell_\kappa, \tau) = 0$$

Wherever $\{\ell_\kappa\}$ is a sequence in $\mathcal{L}$ such that $\lim_{\kappa \to \infty} \mathcal{A}\ell_\kappa = \lim_{\kappa \to \infty} \mathcal{E}\ell_\kappa = z$ for some $z \in \mathcal{L}$.

Definition 2.8: [23] Suppose that $\mathcal{A}$ and $\mathcal{E}$ be maps from an IFMS $(\mathcal{L}, \beta_i, \rho_i, *, \lozenge)$ into itself, for all $\tau > 0$ and the maps $\mathcal{A}$ and $\mathcal{E}$ are called compatible type (β) if,

$$\lim_{\kappa \to \infty} \beta_i(\mathcal{A}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 1 \text{ and } \lim_{\kappa \to \infty} \rho_i(\mathcal{A}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0$$

Wherever $\{\ell_\kappa\}$ is a sequence in $\mathcal{L}$ such that $\lim_{\kappa \to \infty} \mathcal{A}\ell_\kappa = \lim_{\kappa \to \infty} \mathcal{E}\ell_\kappa = z$ for some $z \in \mathcal{L}$.

Definition 2.9: [20] Suppose that $\mathcal{A}$ and $\mathcal{E}$ be mappings from an IFMS $(\mathcal{L}, \beta_i, \rho_i, *, \lozenge)$ into itself. The pair $(\mathcal{A}, \mathcal{E})$ is called $\mathcal{A}$-Compatible if, for all $\tau > 0$,

$$\lim_{\kappa\to\infty} \beta_i(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 1 \text{ and } \lim_{\kappa\to\infty} \rho_i(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0.$$

Wherever $\{\ell_\kappa\}$ is a sequence in $\mathcal{L}$ such that $\lim_{\kappa\to\infty} \mathcal{A}\ell_\kappa = \lim_{\kappa\to\infty} \mathcal{E}\ell_\kappa = z$ for some $z \in \mathcal{L}$.

Definition 2.10: [20] Suppose that $\mathcal{A}$ and $\mathcal{E}$ be mappings from an IFMS $(\mathcal{L}, \beta_i, \rho_i, *, \lozenge)$into itself. Then the pair $(\mathcal{A}, \mathcal{E})$ is called $\mathcal{E}$-Compatible if and only if $(\mathcal{E}, \mathcal{A})$ is $\mathcal{E}$-compatible.

Definition 2.11: [20] Assume that $\mathcal{A}$ and $\mathcal{E}$ be mappings from an IFMS $(\mathcal{L}, \beta_i, \rho_i, *, \lozenge)$into itself. for all $\tau > 0$ and the pair $(\mathcal{A}, \mathcal{E})$ is called Compatible of type (I).

$$\lim_{\kappa\to\infty} \beta_i(\mathcal{A}\mathcal{E}\ell_\kappa, z, \lambda\tau) \leq \beta_i(\mathcal{E}z, z, \tau) \text{and } \lim_{\kappa\to\infty} \rho_i(\mathcal{A}\mathcal{E}\ell_\kappa, z, \lambda\tau) \geq \rho_i(\mathcal{E}z, z, \tau)$$

such that $\lim_{\kappa\to\infty} \mathcal{A}\ell_\kappa = \lim_{\kappa\to\infty} \mathcal{E}\ell_\kappa = z$ for some $z \in \mathcal{L}$ wherever $\lambda \epsilon (0,1]$ and $\{\ell_\kappa\}$ is a sequence in $\mathcal{L}$.

Definition 2.12: [20] The pair $(\mathcal{A}, \mathcal{E})$ is called Compatible of type (II) iff $(\mathcal{E}, \mathcal{A})$ is compatible of type (I) and we suppose that $\mathcal{A}$ and $\mathcal{E}$ be mappings from an IFMS $(\mathcal{L}, \beta_i, \rho_i, *, \lozenge)$ into itself.

3. Main Result

In our main section, we discuss some definition and important results in NMS and also discuss some non trivial examples which are satisfying our results.

Lemma 3.1: Let $(\mathcal{L}, \beta, \rho, \omega, *, \lozenge)$ be a NMS and $\{\varpi_\kappa\}$ be a sequence in $\mathcal{L}$. If there exists a number $k \in (0,1)$ such that

$$\left.\begin{array}{r} \beta(\varpi_{\kappa+2}, \varpi_{\kappa+1}, k\tau) \geq \beta(\varpi_{\kappa+1}, \varpi_\kappa, \tau), \\ \rho(\varpi_{\kappa+2}, \varpi_{\kappa+1}, k\tau) \leq \rho(\varpi_{\kappa+1}, \varpi_\kappa, \tau), \\ \omega(\varpi_{\kappa+2}, \varpi_{\kappa+1}, k\tau) \leq \omega(\varpi_{\kappa+1}, \varpi_\kappa, \tau), \end{array}\right\} \quad (1)$$

for all $\tau > 0$ and $\kappa = 1,2,3, ...$, then $\{\varpi_\kappa\}$ is a Cauchy sequence in $\mathcal{L}$.

Proof: we use induction and inquelity (1) with the help of Alaca et al. [1], we have, for all $\tau > 0$ and $\kappa = 1,2, ...$,

$$\beta(\varpi_{\kappa+1}, \varpi_{\kappa+2}, \tau) \geq \beta\left(\varpi_1, \varpi_2, \frac{\tau}{k^\kappa}\right), \quad (2)$$

$$\rho(\varpi_{\kappa+1}, \varpi_{\kappa+2}, \tau) \leq \rho\left(\varpi_1, \varpi_2, \frac{\tau}{k^\kappa}\right), \quad (3)$$

$$\omega(\varpi_{\kappa+1},\varpi_{\kappa+2},\tau) \leq \omega\left(\varpi_1,\varpi_2,\frac{\tau}{k^{\kappa}}\right). \quad (4)$$

By using the above inequalities and Definition 2.4 for any positive integer $\hbar$ and real number $\tau > 0$, we have

$$\beta(\varpi_\kappa,\varpi_{\kappa+\hbar},\tau) \geq \beta\left(\varpi_\kappa,\varpi_{\kappa+1},\frac{\tau}{\hbar}\right)^{\hbar-\text{times}} * \ldots * \beta\left(\varpi_{\kappa+\hbar-1},\varpi_{\kappa+\hbar},\frac{\tau}{\hbar}\right)$$

$$\geq \beta\left(\varpi_1,\varpi_2,\frac{\tau}{\hbar k^{\kappa-1}}\right)^{\hbar-\text{times}} * \ldots * \beta\left(\varpi_1,\varpi_2,\frac{\tau}{\hbar k^{\kappa+\hbar-2}}\right),$$

$$\rho(\varpi_\kappa,\varpi_{\kappa+\hbar},\tau) \leq \rho\left(\varpi_\kappa,\varpi_{\kappa+1},\frac{\tau}{\hbar}\right)^{\hbar-\text{times}} \Diamond \ldots \Diamond\, \rho\left(\varpi_{\kappa+\hbar-1},\varpi_{\kappa+\hbar},\frac{\tau}{\hbar}\right)$$

$$\leq \rho\left(\varpi_1,\varpi_2,\frac{\tau}{\hbar k^{\kappa-1}}\right)^{\hbar-\text{times}} \Diamond \ldots \Diamond\, \rho\left(\varpi_1,\varpi_2,\frac{\tau}{\hbar k^{\kappa+\hbar-2}}\right),$$

and

$$\omega(\varpi_\kappa,\varpi_{\kappa+\hbar},\tau) \leq \omega\left(\varpi_\kappa,\varpi_{\kappa+1},\frac{\tau}{\hbar}\right)^{\hbar-\text{times}} \Diamond \ldots \Diamond\, \omega\left(\varpi_{\kappa+\hbar-1},\varpi_{\kappa+\hbar},\frac{\tau}{\hbar}\right)$$

$$\leq \omega\left(\varpi_1,\varpi_2,\frac{\tau}{\hbar k^{\kappa-1}}\right)^{\hbar-\text{times}} \Diamond \ldots \Diamond\, \omega\left(\varpi_1,\varpi_2,\frac{\tau}{\hbar k^{\kappa+\hbar-2}}\right).$$

Therefore, by Definition 2.4 we obtain

$$\lim_{\kappa\to\infty} \beta(\varpi_\kappa,\varpi_{\kappa+\hbar},\tau) \geq 1^{\hbar-\text{times}} * \ldots * 1 \geq 1,$$

$$\lim_{\kappa\to\infty} \rho(\varpi_\kappa,\varpi_{\kappa+\hbar},\tau) \leq 0^{\hbar-\text{times}} \Diamond \ldots \Diamond\, 0 \leq 0,$$

$$\lim_{\kappa\to\infty} \omega(\varpi_\kappa,\varpi_{\kappa+\hbar},\tau) \leq 0^{\hbar-\text{times}} \Diamond \ldots \Diamond\, 0 \leq 0.$$

Which implies that $\{\varpi_\kappa\}$ is a Cauchy sequence in $\mathcal{L}$. This complete the proof.

Lemma 3.2: Suppose that $(\mathcal{L},\beta,\rho,\omega,*,\Diamond)$ be a NMS and for all $\ell,\varpi \in \mathcal{L}, \tau > 0$ and if for a number $k \in (0,1)$,

$$\beta(\ell,\varpi,k\tau) \geq \beta(\ell,\varpi,\tau),$$
$$\rho(\ell,\varpi,k\tau) \leq \rho(\ell,\varpi,\tau),$$
$$\omega(\ell,\varpi,k\tau) \leq \omega(\ell,\varpi,\tau),$$

Then $\ell = \varpi$.

Proof: Same as [31]

Definition 3.1: Let $\mathcal{A}$ and $\mathcal{E}$ be maps from a NMS $(\mathcal{L},\beta,\rho,\omega,*,\Diamond)$ into itself. The maps $\mathcal{A}$ and $\mathcal{E}$ are said to be compatible if, for all $\tau > 0$,

$$\lim_{\kappa\to\infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa,\mathcal{E}\mathcal{A}\ell_\kappa,\tau) = 1,$$

$$\lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = 0,$$

$$\lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = 0.$$

such that $\lim_{\kappa\longrightarrow\infty} \mathcal{A}\ell_\kappa = \lim_{\kappa\longrightarrow\infty} \mathcal{E}\ell_\kappa = z$ for some $z \in \mathcal{L}$ wherever $\{\ell_\kappa\}$ is a sequence in $\mathcal{L}$.

Definition 3.2: The maps $\mathcal{A}$ and $\mathcal{E}$ are called compatible of type (α) if we assume that $\mathcal{A}$ and $\mathcal{E}$ be maps from a NMS $(\mathcal{L}, \beta, \rho, \omega, *, \Diamond)$ into itself, for all $\tau > 0$,

$$\lim_{\kappa\longrightarrow\infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 1\,,$$

$$\lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0\,,$$

$$\lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0,$$

and

$$\lim_{\kappa\longrightarrow\infty} \beta(\mathcal{E}\mathcal{A}\ell_\kappa, \mathcal{A}\mathcal{A}\ell_\kappa, \tau) = 1,$$

$$\lim_{\kappa\to\infty} \rho(\mathcal{E}\mathcal{A}\ell_\kappa, \mathcal{A}\mathcal{A}\ell_\kappa, \tau) = 0\,,$$

$$\lim_{\kappa\to\infty} \omega(\mathcal{E}\mathcal{A}\ell_\kappa, \mathcal{A}\mathcal{A}\ell_\kappa, \tau) = 0.$$

such that $\lim_{\kappa\longrightarrow\infty} \mathcal{A}\ell_\kappa = \lim_{\kappa\longrightarrow\infty} \mathcal{E}\ell_\kappa = z$ for some $z \in \mathcal{L}$ wherever $\{\ell_\kappa\}$ is a sequence in $\mathcal{L}$.

Definition 3.3: Suppose that $\mathcal{A}$ and $\mathcal{E}$ be maps from a NMS $(\mathcal{L}, \beta, \rho, \omega, *, \Diamond)$ into itself. The maps $\mathcal{A}$ and $\mathcal{E}$ are called to be compatible type (β) if, for all $\tau > 0$,

$$\lim_{\kappa\longrightarrow\infty} \beta(\mathcal{A}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 1,$$
$$\lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0\,,$$
$$\lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0.$$

for some $z \in \mathcal{L}$,wherever $\{\ell_\kappa\}$ is a sequence in $\mathcal{L}$ and $\lim_{\kappa\longrightarrow\infty} \mathcal{A}\ell_\kappa = \lim_{\kappa\longrightarrow\infty} \mathcal{E}\ell_\kappa = z$.

Proposition 3.1: Suppose that $(\mathcal{L}, \beta, \rho, \omega, *, \lozenge)$ be a NMS with $\tau * \tau \geq \tau$ and $(1-\tau) \lozenge (1-\tau)$ for all $\tau \in [0,1]$, Then $\mathcal{A}$ and $\mathcal{E}$ are compatible iff they are compatible mappings of type (α) and $\mathcal{A}$ and $\mathcal{E}$ be continuous mappings from $\mathcal{L}$ into itself.

Proof: Let $\mathcal{A}$ and $\mathcal{E}$ are compatible and suppose that $\{\ell_\kappa\}$ is a sequence in $\mathcal{L}$ such that

$$\lim_{\kappa\to\infty} \mathcal{A}\ell_\kappa = \lim_{\kappa\to\infty} \mathcal{E}\ell_\kappa = z$$

for some $z \in \mathcal{L}$. Since $\mathcal{A}$ and $\mathcal{E}$ be continuous. We get

$$\lim_{\kappa\to\infty} \mathcal{A}\mathcal{A}\ell_\kappa = \lim_{\kappa\to\infty} \mathcal{A}\mathcal{E}\ell_\kappa = \mathcal{A}z,$$

$$\lim_{\kappa\to\infty} \mathcal{E}\mathcal{A}\ell_\kappa = \lim_{\kappa\to\infty} \mathcal{E}\mathcal{E}\ell_\kappa = \mathcal{E}z.$$

since $\mathcal{A}$ and $\mathcal{E}$ are compatible,

$$\lim_{\kappa\to\infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = 1,$$
$$\lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = 0,$$
$$\lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = 0,$$

for all $\tau > 0$. we get

$$\beta(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) \geq \beta\left(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \frac{\tau}{2}\right) * \beta\left(\mathcal{E}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \frac{\tau}{2}\right),$$

$$\rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) \leq \rho\left(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \frac{\tau}{2}\right) \lozenge \rho\left(\mathcal{E}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \frac{\tau}{2}\right),$$

$$\omega(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) \leq \omega\left(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \frac{\tau}{2}\right) \lozenge \omega\left(\mathcal{E}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \frac{\tau}{2}\right),$$

for all $\tau > 0$. This implies that

$$\lim_{\kappa\to\infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) \geq 1 * 1 \geq 1,$$

$$\lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) \leq 0 \lozenge 0 \leq 0,$$

$$\lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) \leq 0 \lozenge 0 \leq 0.$$

It fellows that

$$\lim_{\kappa\to\infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 1,$$
$$\lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0,$$
$$\lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0.$$

for all $\tau > 0$,

$$\lim_{\kappa\to\infty} \beta(\mathcal{E}\mathcal{A}\ell_\kappa, \mathcal{A}\mathcal{A}\ell_\kappa, \tau) = 1,$$

$$\lim_{\kappa\to\infty} \rho(\mathcal{E}\mathcal{A}\ell_\kappa, \mathcal{A}\mathcal{A}\ell_\kappa, \tau) = 0,$$

$$\lim_{\kappa\to\infty} \omega(\mathcal{E}\mathcal{A}\ell_\kappa, \mathcal{A}\mathcal{A}\ell_\kappa, \tau) = 0.$$

where $\mathcal{A}$ and $\mathcal{E}$ are compatible of type (α). Conversely, we consider $\mathcal{A}$ and $\mathcal{E}$ are compatible of type (α) and let $\{\ell_\kappa\}$ be a sequence in $\mathcal{L}$ and $\lim_{\kappa\to\infty} \mathcal{A}\ell_\kappa = \lim_{\kappa\to\infty} \mathcal{E}\ell_\kappa = z$ for some $z \in \mathcal{L}$ and $\mathcal{A}$ and $\mathcal{E}$ are continuous.

we get,

$$\lim_{\kappa\to\infty} \mathcal{A}\mathcal{A}\ell_\kappa = \lim_{\kappa\to\infty} \mathcal{A}\mathcal{E}\ell_\kappa = \mathcal{A}z,$$

$$\lim_{\kappa\to\infty} \mathcal{E}\mathcal{A}\ell_\kappa = \lim_{\kappa\to\infty} \mathcal{E}\mathcal{E}\ell_\kappa = \mathcal{E}z.$$

Where $\mathcal{A}$ and $\mathcal{E}$ are compatible of type (α), for all $\tau > 0$, we obtain,

$$\lim_{\kappa\to\infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 1,$$

$$\lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0,$$

$$\lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0,$$

and

$$\lim_{\kappa\to\infty} \beta(\mathcal{E}\mathcal{A}\ell_\kappa, \mathcal{A}\mathcal{A}\ell_\kappa, \tau) = 1,$$

$$\lim_{\kappa\to\infty} \rho(\mathcal{E}\mathcal{A}\ell_\kappa, \mathcal{A}\mathcal{A}\ell_\kappa, \tau) = 0,$$

$$\lim_{\kappa\to\infty} \omega(\mathcal{E}\mathcal{A}\ell_\kappa, \mathcal{A}\mathcal{A}\ell_\kappa, \tau) = 0,$$

Thus, from the inequality

$$\beta(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) \geq \beta\left(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \frac{\tau}{2}\right) * \beta\left(\mathcal{E}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \frac{\tau}{2}\right),$$

$$\rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) \leq \rho\left(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \frac{\tau}{2}\right) \lozenge \rho\left(\mathcal{E}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \frac{\tau}{2}\right),$$

$$\omega(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) \leq \omega\left(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \frac{\tau}{2}\right) \lozenge \omega\left(\mathcal{E}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \frac{\tau}{2}\right),$$

for all $\tau > 0$, it follow that

$$\lim_{\kappa\to\infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) \geq 1 * 1 \geq 1,$$

$$\lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) \leq 0 \lozenge 0 \leq 0,$$

$$\lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) \leq 0 \lozenge 0 \leq 0,$$

for all $\tau > 0$, which implies that

$$\lim_{\kappa\to\infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = 1,$$

$$\lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = 0,$$

$$\lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = 0.$$

so, A and B are compatible and hence proved.

Proposition 3.2: Suppose that $(\mathcal{L}, \beta, \rho, \omega, *, \lozenge)$ be a NMS with $\tau * \tau \geq \tau$ and $(1-\tau) \lozenge (1-\mathrm{t}) \leq (1-\mathrm{t})$ for all $\tau \in [0,1]$ then $\mathcal{A}$, $\mathcal{E}$ are compatible maps type (β) and assume that $\mathcal{A}$ and $\mathcal{E}$ be maps of continuous from $\mathcal{L}$ into itself.

Proof: Same as [31].

Proposition 3.3: Suppose $(\mathcal{L}, \beta, \rho, \omega, *, \lozenge)$ be a NMS with $\tau * \tau \geq \tau$ and $(1-\tau) \lozenge (1-\tau) \leq (1-\tau)$ for all $\tau \in [0,1]$ and Then $\mathcal{A}$ and $\mathcal{E}$ are compatible maps type (β) of iff they are compatible map of type (α) and let $\mathcal{A}$ and $\mathcal{E}$ be continuous maps from $\mathcal{L}$ into itself.
Proof: Same lines as [31].

Definition 3.4: Suppose that $\mathcal{A}$ and $\mathcal{E}$ be mappings from a NMS $(\mathcal{L}, \beta, \rho, \omega, *, \lozenge)$ into itself. The pair $(\mathcal{A}, \mathcal{E})$ is called $\mathcal{A}$-Compatible if, for all $\tau > 0$,

$$\lim_{\kappa\to\infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 1,$$

$$\lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0,$$

$$\lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0.$$

Wherever $\{\ell_\kappa\}$ is a sequence in $\mathcal{L}$ such that $\lim_{\kappa\to\infty} \mathcal{A}\ell_\kappa = \lim_{\kappa\to\infty} \mathcal{E}\ell_\kappa = z$ for some $z \in \mathcal{L}$.

Definition 3.5: Suppose that $\mathcal{A}$ and $\mathcal{E}$ be mappings from a NMS $(\mathcal{L}, \beta, \rho, \omega, *, \lozenge)$ into itself. Then the pair $(\mathcal{A}, \mathcal{E})$ is said to $\mathcal{E}$-Compatible iff $(\mathcal{E}, \mathcal{A})$ is $\mathcal{E}$-compatible.

Definition 3.6: Suppose that $\mathcal{A}$ and $\mathcal{E}$ be mappings from a NMS $(\mathcal{L}, \beta, \rho, \omega, *, \Diamond)$ into itself. We use the pair $(\mathcal{A}, \mathcal{E})$ is called Compatible of type (I) if, for all $\tau > 0$,

$$\lim_{\kappa\to\infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa, z, \lambda\tau) \le \beta(\mathcal{E}z, z, \tau),$$

$$\lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, z, \lambda\tau) \ge \rho(\mathcal{E}z, z, \tau),$$

$$\lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{E}\ell_\kappa, z, \lambda\tau) \ge \omega(\mathcal{E}z, z, \tau).$$

Wherever $\lambda \epsilon (0,1]$ and $\{\ell_\kappa\}$ is a sequence in $\mathcal{L}$ and $\lim_{\kappa\to\infty} \mathcal{A}\ell_\kappa = \lim_{\kappa\to\infty} \mathcal{E}\ell_\kappa = z$ for some $z \in \mathcal{L}$.

Definition 3.7: The pair $(\mathcal{A}, \mathcal{E})$ is called Compatible of type (II) iff $(\mathcal{E}, \mathcal{A})$ is compatible of type (I). If $\mathcal{A}$ and $\mathcal{E}$ be mappings from a NMS $(\mathcal{L}, \beta, \rho, *, \Diamond)$ into itself.

Proposition 3.4: The pair $(\mathcal{A}, \mathcal{E})$ is $\mathcal{A}$ Compatible (resp., $\mathcal{E}$ -compatible), they are compatible of type (I) (resp., of type (II)). Suppose that $(\mathcal{L}, \beta, \rho, \omega, *, \Diamond)$ be a NMS and $\mathcal{A}, \mathcal{E}$ be mappings from $\mathcal{L}$ to itself and $\mathcal{E}$ (resp,. $\mathcal{A}$) is continuous.

Proof: Let the pair $(\mathcal{A}, \mathcal{E})$ is $\mathcal{A}$-compatible and let $\{\ell_\kappa\}$ be a sequence in $\mathcal{L}$ and $\lim_{\kappa\to\infty} \mathcal{A}\ell_\kappa = \lim_{\kappa\to\infty} \mathcal{E}\ell_\kappa = z$ for some $z \epsilon \mathcal{L}$. Since $\mathcal{E}$ is continous, we get

$$\lim_{\kappa\to\infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 1 = \lim_{\kappa\to\infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}z, \tau),$$
$$\lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0 = \lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}z, \tau),$$
$$\lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0 = \lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}z, \tau).$$

Further,

$$\beta(\mathcal{E}z, z, \tau) \ge \beta\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right) * \beta\left(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}z, \frac{\tau}{2}\right),$$
$$\rho(\mathcal{E}z, z, \tau) \le \rho\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right) \Diamond \rho\left(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}z, \frac{\tau}{2}\right),$$
$$\omega(\mathcal{E}z, z, \tau) \le \omega\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right) \Diamond \omega\left(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}z, \frac{\tau}{2}\right).$$

Then, we get

$$\beta(\mathcal{E}z, z, \tau) \ge \lim_{\kappa\to\infty}\left(\beta\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right) * \beta\left(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}z, \frac{\tau}{2}\right)\right)$$
$$= \lim_{\kappa\to\infty} \beta\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right),$$
$$\rho(\mathcal{E}z, z, \tau) \le \lim_{\kappa\to\infty}\left(\rho\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right) \Diamond \rho\left(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}z, \frac{\tau}{2}\right)\right)$$
$$= \lim_{\kappa\to\infty} \rho\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right),$$
$$\omega(\mathcal{E}z, z, \tau) \le \lim_{\kappa\to\infty}\left(\omega\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right) \Diamond \omega\left(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}z, \frac{\tau}{2}\right)\right)$$
$$= \lim_{\kappa\to\infty} \omega\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right).$$

We get,

$$\lim_{\kappa\to\infty} \beta\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right) \le \beta(\mathcal{E}z, z, \tau),$$
$$\lim_{\kappa\to\infty} \rho\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right) \ge \rho(\mathcal{E}z, z, \tau),$$
$$\lim_{\kappa\to\infty} \omega\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right) \ge \omega(\mathcal{E}z, z, \tau).$$

The sequence $\{\ell_\kappa\}$ in $\mathcal{L}$ holds for every choice inequality with the corresponding $z \in \mathcal{L}$ The pair $(\mathcal{A}, \mathcal{E})$ is compatible of type (I) hence proved.

Proposition 3.5: Suppose that $(\mathcal{L}, \beta, \rho, \omega, *, \lozenge)$ be a NMS and $\mathcal{A}, \mathcal{E}$ be mappings from $\mathcal{L}$ into itself with $\mathcal{E}$ (resp,. $\mathcal{A}$) is countinous. If the pair $(\mathcal{A}, \mathcal{E})$ is Compatible of type (I) and type (II) and $\lim_{\kappa\to\infty} \mathcal{A}\mathcal{E}\ell_\kappa = z$ $(res., \lim_{\kappa\to\infty} \mathcal{E}\mathcal{A}\ell_\kappa = z)$, then it is $\mathcal{A}$-compatible (resp., $\mathcal{E}$-compatible) for every sequence $\{\ell_\kappa\}$ in $\mathcal{L}$ and $\lim_{\kappa\to\infty} \mathcal{A}\ell_\kappa = \lim_{\kappa\to\infty} \mathcal{E}\ell_\kappa = z$ for some $z \in \mathcal{L}$.

Proof: Let the pair $(\mathcal{A}, \mathcal{E})$ is compatible of type (I) and let $\{\ell_\kappa\}$ be a sequence in $\mathcal{L}$ and $\lim_{\kappa\to\infty} \mathcal{A}\ell_\kappa = \lim_{\kappa\to\infty} \mathcal{E}\ell_\kappa = z$ for some $z \in \mathcal{L}$ and $\mathcal{E}$ is continous, we obtain,

$$\beta(\mathcal{E}z, z, \tau) \ge \lim_{\kappa\to\infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa, z, \lambda\tau),$$
$$\rho(\mathcal{E}z, z, \tau) \le \lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, z, \lambda\tau),$$
$$\omega(\mathcal{E}z, z, \tau) \le \lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{E}\ell_\kappa, z, \lambda\tau).$$

then

$$\lim_{\kappa\to\infty} \beta(\mathcal{E}\mathcal{E}\ell_\kappa, z, \tau) \ge \lim_{\kappa\to\infty} \left(\beta\left(\mathcal{E}z, z, \frac{\tau}{2}\right) * \beta\left(\mathcal{E}\mathcal{E}\ell_\kappa, \mathcal{E}z, \frac{\tau}{2}\right)\right)$$
$$= \beta\left(\mathcal{E}z, z, \frac{\tau}{2}\right),$$
$$\lim_{\kappa\to\infty} \rho(\mathcal{E}\mathcal{E}\ell_\kappa, z, \tau) \le \lim_{\kappa\to\infty} \left(\rho\left(\mathcal{E}z, z, \frac{\tau}{2}\right) \lozenge \rho\left(\mathcal{E}\mathcal{E}\ell_\kappa, \mathcal{E}z, \frac{\tau}{2}\right)\right)$$
$$= \rho\left(\mathcal{E}z, z, \frac{\tau}{2}\right),$$
$$\lim_{\kappa\to\infty} \omega(\mathcal{E}\mathcal{E}\ell_\kappa, z, \tau) \le \lim_{\kappa\to\infty} \left(\omega\left(\mathcal{E}z, z, \frac{\tau}{2}\right) \lozenge \omega\left(\mathcal{E}\mathcal{E}\ell_\kappa, \mathcal{E}z, \frac{\tau}{2}\right)\right)$$
$$= \omega\left(\mathcal{E}z, z, \frac{\tau}{2}\right).$$

Furthermore,

$$\lim_{\kappa\to\infty} \beta(\mathcal{E}\mathcal{E}\ell_\kappa, z, \tau) \ge \lim_{\kappa\to\infty} \beta\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\lambda\tau}{2}\right),$$
$$\lim_{\kappa\to\infty} \rho(\mathcal{E}\mathcal{E}\ell_\kappa, z, \tau) \le \lim_{\kappa\to\infty} \rho\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\lambda\tau}{2}\right),$$
$$\lim_{\kappa\to\infty} \omega(\mathcal{E}\mathcal{E}\ell_\kappa, z, \tau) \le \lim_{\kappa\to\infty} \omega\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\lambda\tau}{2}\right),$$

Then, we get

$$\beta(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) \ge \beta\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right) * \beta\left(\mathcal{E}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right),$$
$$\rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) \le \rho\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right) \lozenge \rho\left(\mathcal{E}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right),$$
$$\omega(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) \le \omega\left(\mathcal{A}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right) \lozenge \omega\left(\mathcal{E}\mathcal{E}\ell_\kappa, z, \frac{\tau}{2}\right).$$

Thus, as $\kappa \to \infty$,

$$\lim_{\kappa\to\infty}\beta(\mathcal{A}\mathcal{E}\ell_\kappa,\mathcal{E}\mathcal{E}\ell_\kappa,\tau) \geq \lim_{\kappa\to\infty}\left(\beta\left(\mathcal{A}\mathcal{E}\ell_\kappa,z,\frac{\tau}{2}\right)*\beta\left(\mathcal{E}\mathcal{E}\ell_\kappa,z,\frac{\tau}{2}\right)\right)$$
$$\leq \lim_{\kappa\to\infty}\left(\beta\left(\mathcal{A}\mathcal{E}\ell_\kappa,z,\frac{\tau}{2}\right)*\beta\left(\mathcal{A}\mathcal{E}\ell_\kappa,z,\frac{\lambda\tau}{4}\right)\right)$$
$$= 1,$$
$$\lim_{\kappa\to\infty}\rho(\mathcal{A}\mathcal{E}\ell_\kappa,\mathcal{E}\mathcal{E}\ell_\kappa,\tau) \leq \lim_{\kappa\to\infty}\left(\rho\left(\mathcal{A}\mathcal{E}\ell_\kappa,z,\frac{\tau}{2}\right)\lozenge\rho\left(\mathcal{E}\mathcal{E}\ell_\kappa,z,\frac{\tau}{2}\right)\right)$$
$$\leq \lim_{\kappa\to\infty}\left(\rho\left(\mathcal{A}\mathcal{E}\ell_\kappa,z,\frac{\tau}{2}\right)\lozenge\rho\left(\mathcal{A}\mathcal{E}\ell_\kappa,z,\frac{\lambda\tau}{4}\right)\right)$$
$$= 0.$$
$$\lim_{\kappa\to\infty}\omega(\mathcal{A}\mathcal{E}\ell_\kappa,\mathcal{E}\mathcal{E}\ell_\kappa,\tau) \leq \lim_{\kappa\to\infty}\left(\omega\left(\mathcal{A}\mathcal{E}\ell_\kappa,z,\frac{\tau}{2}\right)\lozenge\omega\left(\mathcal{E}\mathcal{E}\ell_\kappa,z,\frac{\tau}{2}\right)\right)$$
$$\leq \lim_{\kappa\to\infty}\left(\omega\left(\mathcal{A}\mathcal{E}\ell_\kappa,z,\frac{\tau}{2}\right)\lozenge\omega\left(\mathcal{A}\mathcal{E}\ell_\kappa,z,\frac{\lambda\tau}{4}\right)\right)$$
$$= 0.$$

Therefore,

$$\lim_{\kappa\to\infty}\beta(\mathcal{A}\mathcal{E}\ell_\kappa,\mathcal{E}\mathcal{E}\ell_\kappa,\tau) = 1,$$
$$\lim_{\kappa\to\infty}\rho(\mathcal{A}\mathcal{E}\ell_\kappa,\mathcal{E}\mathcal{E}\ell_\kappa,\tau) = 0,$$
$$\lim_{\kappa\to\infty}\omega(\mathcal{A}\mathcal{E}\ell_\kappa,\mathcal{E}\mathcal{E}\ell_\kappa,\tau) = 0.$$

these are 0 and 1 for any sequence ℓ_κ in $\mathcal{L}$ that is $z \epsilon \mathcal{L}$ and this limit always exists. Hence the pair $(\mathcal{A},\mathcal{E})$ is $\mathcal{A}$- Compatible. Proved.

Example 3.1: Suppose that $\mathcal{L} = [0,1]$ and $*$ be the CTN and $\lozenge$ be the CTCN describe by $n * g = \min\{n, g\}$ and $n \lozenge g = \max\{n, g\}$ respectively for all $n, g \epsilon [0,1]$. For each $\tau \epsilon (0,\infty)$ and $\ell, \varpi \epsilon \mathcal{L}$, define (β,ρ) by

$$\beta(\ell,\varpi,\tau) = \begin{cases}\dfrac{\tau}{\tau+|\ell-\varpi|}, \tau > 0\\ 0,\ \ \tau = 0\end{cases},$$

$$\rho(\ell,\varpi,\tau) = \begin{cases}\dfrac{|\ell-\varpi|}{\tau+|\ell-\varpi|}, \tau > 0\\ 1,\ \ \tau = 0\end{cases},$$

$$\omega(\ell,\varpi,\tau) = \begin{cases}\dfrac{|\ell-\varpi|}{\tau}, \tau > 0\\ 1,\ \ \tau = 0\end{cases},$$

so, $(\mathcal{L}, \beta, \rho, \omega, *, \lozenge)$ is a NMS, where $\lozenge$ and $*$ are define by and $n \lozenge g = \max\{n, g\}$ and $n * g = \min\{n, g\}$ respectively. Suppose that $\mathcal{A}$ and $\mathcal{E}$ be describe the $\mathcal{A}\ell = 0$ for $\frac{1}{3} < \ell < \frac{1}{2}$, $\mathcal{A}\ell = 1$ for $0 \leq \ell \leq \frac{1}{3}$ and $\frac{1}{2} \leq \ell \leq 1$ and $\mathcal{E}\ell = \ell$ for all $\ell \epsilon \mathcal{L}$ Suppose that $\{\ell_\kappa\}$ be a sequence in $\mathcal{L}$ and $\lim_{\kappa\to\infty}\mathcal{A}\ell_\kappa = \lim_{\kappa\to\infty}\mathcal{E}\ell_\kappa = z$.

and $z \epsilon \{1\}$ and $\lim_{\kappa \to \infty} \ell_\kappa = 1$, we obtain

$$\lim_{\kappa \to \infty} \beta(\mathcal{A}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = 1,$$

$$\lim_{\kappa \to \infty} \rho(\mathcal{A}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = 0,$$

$$\lim_{\kappa \to \infty} \omega(\mathcal{A}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = 0,$$

and

$$\lim_{\kappa \to \infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = 1,$$

$$\lim_{\kappa \to \infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = 0,$$

$$\lim_{\kappa \to \infty} \omega(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = 0,$$

and also we have

$$\lim_{\kappa \to \infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 1,$$

$$\lim_{\kappa \to \infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0,$$

$$\lim_{\kappa \to \infty} \omega(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0.$$

Similarly

$$\lim_{\kappa \to \infty} \beta(\mathcal{A}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 1,$$

$$\lim_{\kappa \to \infty} \rho(\mathcal{A}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0,$$

$$\lim_{\kappa \to \infty} \omega(\mathcal{A}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = 0.$$

Compatible of type (α), compatible type (β) thus $(\mathcal{A}, \mathcal{E})$ is compatible, $\mathcal{A}$-compatible, $\mathcal{E}$-compatible. Moreover, $z = 0$ is a FP of $\mathcal{A}$. Finally the pair $(\mathcal{A}, \mathcal{E})$ is compatible of type (II) and type (I).

The result of Proposition 3.4 need not to be true $\mathcal{E}$ is not continuous this statement shows the example that given below.

Example 3.2: Suppose that $\mathcal{L} = [0,2]$ with the usual metric. For each $\tau > 0$ and $\ell, \varpi \epsilon \mathcal{L}$, describe (β, ρ, ω) by

$$\beta(\ell, \varpi, \tau) = \begin{cases} \dfrac{\tau}{\tau + |\ell - \varpi|}, \tau > 0 \\ 0, \quad \tau = 0 \end{cases},$$

$$\rho(\ell,\varpi,\tau) = \begin{cases} \dfrac{|\ell-\varpi|}{\tau+|\ell-\varpi|}, \tau > 0 \\ 1, \ \tau = 0 \end{cases},$$

$$\omega(\ell,\varpi,\tau) = \begin{cases} \dfrac{|\ell-\varpi|}{\tau}, \tau > 0 \\ 1, \ \tau = 0 \end{cases}.$$

Clearly, $(\mathcal{L}, \beta, \rho, \omega, *, \Diamond)$ is a NMS, where $*$ and $\Diamond$are define by $n * g = \{n, g\}$ and $n \Diamond g = \min\{1, n + g\}$ respectively. Suppose that $\mathcal{A}$ and $\mathcal{E}$ be defined as $\mathcal{E}\ell = 1$ for $\ell \neq 1$, $\mathcal{E}\ell = 2$ for $\ell = 1$, $\mathcal{A}\ell = 1$ for all $\ell \in \mathcal{L}$ and Then $\mathcal{E}$ is not continuous at $z = 1$. We take that the pair $(\mathcal{A}, \mathcal{E})$ is compatible of type (II), but not of type (I), of type (α), $\mathcal{E}$-compatible, $\mathcal{A}$-compatible or compatible.

To see this, we suppose that $\{\ell_\kappa\}$ is a sequence in $\mathcal{L}$ such that $\lim_{\kappa\to\infty} \mathcal{A}\ell_\kappa = \lim_{\kappa\to\infty} \mathcal{E}\ell_\kappa = z$. We defining of $\mathcal{A}$ and $\mathcal{E}, z \in \{1\}$. Where $\mathcal{A}$ and $\mathcal{E}$ agree on $\mathcal{L}/\{1\}$, we use $\ell_\kappa \to 1$. Now, $\mathcal{A}\mathcal{E}\ell_\kappa = 1, \mathcal{E}\mathcal{A}\ell_\kappa = 2, \mathcal{A}\mathcal{A}\ell_\kappa = 1, \mathcal{E}\mathcal{E}\ell_\kappa = 2$, $\mathcal{E}1 = 2$ and $\mathcal{A}1 = 1$ Thus, for $\tau > 0$,

$$\lim_{\kappa\to\infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = \frac{\tau}{\tau+1} < 1,$$

$$\lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = \frac{1}{\tau+1} > 0,$$

$$\lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = \frac{1}{\tau} > 0.$$

and

$$\lim_{\kappa\to\infty} \beta(\mathcal{A}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = \frac{\tau}{\tau+1} < 1,$$

$$\lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = \frac{1}{\tau+1} > 0,$$

$$\lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{A}\ell_\kappa, \tau) = \frac{1}{\tau} > 0.$$

Similarly,

$$\lim_{\kappa\to\infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = \frac{\tau}{\tau+1} < 1,$$

$$\lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = \frac{1}{\tau+1} > 0,$$

$$\lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{E}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = \frac{1}{\tau} > 0.$$

and

$$\lim_{\kappa\to\infty} \beta(\mathcal{A}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = \frac{\tau}{\tau+1} < 1,$$

$$\lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = \frac{1}{\tau+1} > 0,$$

$$\lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{A}\ell_\kappa, \mathcal{E}\mathcal{E}\ell_\kappa, \tau) = \frac{1}{\tau} > 0.$$

Type (β) $\mathcal{A}$-compatible, $\mathcal{E}$-compatible or compatible. so the pair $(\mathcal{A}, \mathcal{E})$ is none of compatible of type (α), also for $\tau > 0$,

$$\beta(\mathcal{E}1,1,\tau) = \frac{\tau}{\tau+1} < 1 = \lim_{\kappa\to\infty} \beta(\mathcal{A}\mathcal{E}\ell_\kappa, 1, \tau),$$

$$\rho(\mathcal{E}1,1,\tau) = \frac{1}{\tau+1} > 0 = \lim_{\kappa\to\infty} \rho(\mathcal{A}\mathcal{E}\ell_\kappa, 1, \tau),$$

$$\omega(\mathcal{E}1,1,\tau) = \frac{1}{\tau} > 0 = \lim_{\kappa\to\infty} \omega(\mathcal{A}\mathcal{E}\ell_\kappa, 1, \tau).$$

and

$$\beta(\mathcal{A}1,1,\tau) = 1 > \frac{\tau}{\tau+1} = \lim_{\kappa\to\infty} \beta(\mathcal{E}\mathcal{A}\ell_\kappa, 1, \tau),$$

$$\rho(\mathcal{A}1,1,\tau) = 0 < \frac{1}{\tau+1} = \lim_{\kappa\to\infty} \rho(\mathcal{E}\mathcal{A}\ell_\kappa, 1, \tau),$$

$$\omega(\mathcal{A}1,1,\tau) = 0 < \frac{1}{\tau} = \lim_{\kappa\to\infty} \omega(\mathcal{E}\mathcal{A}\ell_\kappa, 1, \tau).$$

Thus, pair of mappings $(\mathcal{A}, \mathcal{E})$ is compatible of not type (I), but are type (I).

Proposition 3.6: Suppose that $(\mathcal{L}, \beta, \rho, \omega, *, \Diamond)$ be a NMS and $\mathcal{A}, \mathcal{E}$ be self-mappings on $\mathcal{L}$. Let the pair $(\mathcal{A}, \mathcal{E})$ is compatible of type (II) and type (I) and $\mathcal{A}z = \mathcal{E}z$ for some $z \in \mathcal{L}$ and for $\tau > 0$ and $\lambda \in (0,1]$,

$$\beta(\mathcal{A}z, \mathcal{E}\mathcal{E}z, \tau) \geq \beta(\mathcal{A}z, \mathcal{A}\mathcal{E}z, \lambda\tau),$$
$$\rho(\mathcal{A}z, \mathcal{E}\mathcal{E}z, \tau) \leq \rho(\mathcal{A}z, \mathcal{A}\mathcal{E}z, \lambda\tau),$$
$$\omega(\mathcal{A}z, \mathcal{E}\mathcal{E}z, \tau) \leq \omega(\mathcal{A}z, \mathcal{A}\mathcal{E}z, \lambda\tau).$$

(resp.,

$$\beta(\mathcal{E}z, \mathcal{A}\mathcal{A}z, \tau) \geq \beta(\mathcal{E}z, \mathcal{E}\mathcal{A}z, \lambda\tau),$$
$$\rho(\mathcal{E}z, \mathcal{A}\mathcal{A}z, \tau) \leq \rho(\mathcal{E}z, \mathcal{E}\mathcal{A}z, \lambda\tau),$$
$$\omega(\mathcal{E}z, \mathcal{A}\mathcal{A}z, \tau) \leq \omega(\mathcal{E}z, \mathcal{E}\mathcal{A}z, \lambda\tau).$$

Proof: Suppose that $\{\ell_\kappa\}$ be a sequence in $\mathcal{L}$ describe the sequence $\ell_\kappa = z$ for $\kappa = 1,2, \ldots$ and $\mathcal{A}z = \mathcal{E}z$ for some $z \in \mathcal{L}$.Then we take $\lim_{\kappa\to\infty} \mathcal{A}\ell_\kappa = \lim_{\kappa\to\infty} \mathcal{E}\ell_\kappa = z$. Assume that the pair $(\mathcal{A}, \mathcal{E})$ is type (I) compatible, for $\tau > 0$, $\lambda \in (0,1]$,

$$\beta(\mathcal{A}z,\mathcal{E}\mathcal{E}z,\tau) \geq \lim_{\kappa\to\infty} \beta(\mathcal{A}z,\mathcal{A}\mathcal{E}\ell_\kappa,\lambda\tau) = \beta(\mathcal{A}z,\mathcal{A}\mathcal{E}z,\lambda\tau),$$
$$\rho(\mathcal{A}z,\mathcal{E}\mathcal{E}z,\tau) \leq \lim_{\kappa\to\infty} \rho(\mathcal{A}z,\mathcal{A}\mathcal{E}\ell_\kappa,\lambda\tau) = \rho(\mathcal{A}z,\mathcal{A}\mathcal{E}z,\lambda\tau),$$
$$\omega(\mathcal{A}z,\mathcal{E}\mathcal{E}z,\tau) \leq \lim_{\kappa\to\infty} \omega(\mathcal{A}z,\mathcal{A}\mathcal{E}\ell_\kappa,\lambda\tau) = \omega(\mathcal{A}z,\mathcal{A}\mathcal{E}z,\lambda\tau)$$

4. Fixed point theorem

In this part, we use the condition of compatible mapping of type (I) and (II) for satisfy a FP theorem for four mappings in a NMS.

Theorem 4.1: Suppose that $(\mathcal{L},\beta,\rho,\omega,*,\Diamond)$ be a complete NMS with $\tau * \tau \geq \tau$ and $(1-\tau)\Diamond(1-\tau) \leq (1-\tau)$ for all $\tau \in [0,1]$. Let $\mathcal{A},\mathcal{E},\complement$ and T are self-mappings on $\mathcal{L}$, so that

$$\mathcal{A}(\mathcal{L}) \subseteq \complement(\mathcal{L}) \text{ and } \mathcal{E}(\mathcal{L}) \subseteq T(\mathcal{L}), \qquad (5)$$

There exists a constant $k \in (0,1)$ such that

$$\beta(\mathcal{A}\ell,\mathcal{E}\varpi,k\tau) \geq \begin{pmatrix}\beta(\complement\ell,T\varpi,\tau) * \beta(\mathcal{A}\ell,\complement\ell,\tau) * \beta(\mathcal{E}\varpi,T\varpi,\tau)\\ * \beta(\mathcal{A}\ell,T\varpi,\alpha\tau) * \beta(\mathcal{E}\varpi,\complement\ell,(2-\alpha)\tau)\end{pmatrix}, \qquad (6)$$
$$\rho(\mathcal{A}\ell,\mathcal{E}\varpi,k\tau) \leq \begin{pmatrix}\rho(\complement\ell,T\varpi,\tau) \Diamond \rho(\mathcal{A}\ell,\complement\ell,\tau) \Diamond \rho(\mathcal{E}\varpi,T\varpi,\tau)\\ \Diamond \rho(\mathcal{A}\ell,T\varpi,\alpha\tau) \Diamond \rho(\mathcal{E}\varpi,\complement\ell,(2-\alpha)\tau)\end{pmatrix}, \qquad (7)$$
$$\omega(\mathcal{A}\ell,\mathcal{E}\varpi,k\tau) \leq \begin{pmatrix}\omega(\complement\ell,T\varpi,\tau) \Diamond \omega(\mathcal{A}\ell,\complement\ell,\tau) \Diamond \omega(\mathcal{E}\varpi,T\varpi,\tau)\\ \Diamond \omega(\mathcal{A}\ell,T\varpi,\alpha\tau) \Diamond \omega(\mathcal{E}\varpi,\complement\ell,(2-\alpha)\tau)\end{pmatrix}. \qquad (8)$$

For all $\ell,\varpi \in \mathcal{L}$, $\alpha \in (0,2)$ and $\tau > 0$. Assume that $\mathcal{A},\mathcal{E},\complement$ and T are fulfilling the equations given below:

- C1) $\mathcal{E}$ is continuous and the pairs $(\mathcal{E},T)$ and $(\mathcal{A},\complement)$ are compatible of type (II)
- C2) The pairs $(\mathcal{A},\complement)$ and $(\mathcal{E},T)$ are compatible of type (I) and T is continuous
- C3) $\mathcal{A}$ is continuous and the pairs $(\mathcal{A},\complement)$ and $(\mathcal{E},T)$ are compatible of type (II).
- C4) The pairs $(\mathcal{A},\complement)$ and $(\mathcal{E},T)$ are compatible of type (I) and $\complement$ is continuous

Then $\mathcal{A},\mathcal{E},\complement$ and T have a unique common FP in $\mathcal{L}$.

Proof: Suppose that ℓ_0 be an arbitrary point of $\mathcal{L}$ by (5), we take a sequence $\{\varpi_\kappa\}$ in $\mathcal{L}$ and

$$\varpi_{2\kappa} = T\ell_{2\kappa+1} = \mathcal{A}\ell_{2\kappa},\ \ \varpi_{2\kappa+1} = \complement\ell_{2\kappa+2} = \mathcal{E}\ell_{2\kappa+1},$$

for $\kappa = 0,1,2,\ldots$. Then, by (6), (7) and (8) for $\alpha = 1-\delta$, $\delta \in (0,1)$, we have

$$\beta(\mathcal{A}\ell_{2\kappa},\mathcal{E}\ell_{2\kappa+1},k\tau) \geq \begin{pmatrix}\beta(\complement\ell_{2\kappa},T\ell_{2\kappa+1},\tau) * \beta(\mathcal{A}\ell_{2\kappa},\complement\ell_{2\kappa},\tau)\\ * \beta(\mathcal{E}\ell_{2\kappa+1},T\ell_{2\kappa+1},\tau)\\ * \beta(\mathcal{A}\ell_{2\kappa},T\ell_{2\kappa+1},(1-\delta)\tau)\\ * \beta(\mathcal{E}\ell_{2\kappa+1},\complement\ell_{2\kappa},(1+\delta)\tau)\end{pmatrix},$$

$$\rho(\mathcal{A}\ell_{2\kappa},\mathcal{E}\ell_{2\kappa+1},k\tau) \leq \begin{pmatrix}\rho(\complement\ell_{2\kappa},T\ell_{2\kappa+1},\tau) \Diamond \rho(\mathcal{A}\ell_{2\kappa},\complement\ell_{2\kappa},\tau)\\ * \rho(\mathcal{E}\ell_{2\kappa+1},T\ell_{2\kappa+1},\tau)\\ \Diamond \rho(\mathcal{A}\ell_{2\kappa},T\ell_{2\kappa+1},(1-\delta)\tau)\\ \Diamond \rho(\mathcal{E}\ell_{2\kappa+1},\complement\ell_{2\kappa},(1+\delta)\tau)\end{pmatrix},$$

$$\omega(\mathcal{A}\ell_{2\kappa},\mathcal{E}\ell_{2\kappa+1},k\tau) \leq \begin{pmatrix} \omega(\complement\ell_{2\kappa},T\ell_{2\kappa+1},\tau) \lozenge \omega(\mathcal{A}\ell_{2\kappa},\complement\ell_{2\kappa},\tau) \\ * \omega(\mathcal{E}\ell_{2\kappa+1},T\ell_{2\kappa+1},\tau) \\ \lozenge \omega(\mathcal{A}\ell_{2\kappa},T\ell_{2\kappa+1},(1-\delta)\tau) \\ \lozenge \omega(\mathcal{E}\ell_{2\kappa+1},\complement\ell_{2\kappa},(1+\delta)\tau) \end{pmatrix}.$$

We get,

$$\beta(\varpi_{2\kappa},\varpi_{2\kappa+1},k\tau) \geq \begin{pmatrix} \beta(\varpi_{2\kappa-1},\varpi_{2\kappa},\tau) * \beta(\varpi_{2\kappa},\varpi_{2\kappa-1},\tau) \\ * \beta(\varpi_{2\kappa+1},\varpi_{2\kappa},\tau) * \beta(\varpi_{2\kappa},\varpi_{2\kappa},(1-\delta)\tau) \\ * \beta(\varpi_{2\kappa+1},\varpi_{2\kappa-1},(1+\delta)\tau) \end{pmatrix}$$

$$\geq \begin{pmatrix} \beta(\varpi_{2\kappa-1},\varpi_{2\kappa},\tau) * \beta(\varpi_{2\kappa},\varpi_{2\kappa+1},\tau) \\ * \beta(\varpi_{2\kappa+1},\varpi_{2\kappa},\delta\tau) \end{pmatrix},$$

$$\rho(\varpi_{2\kappa},\varpi_{2\kappa+1},k\tau) \leq \begin{pmatrix} \rho(\varpi_{2\kappa-1},\varpi_{2\kappa},\tau) \lozenge \rho(\varpi_{2\kappa},\varpi_{2\kappa-1},\tau) \\ \lozenge \rho(\varpi_{2\kappa+1},\varpi_{2\kappa},\tau) \lozenge \rho(\varpi_{2\kappa},\varpi_{2\kappa},(1-\delta)\tau) \\ \lozenge \rho(\varpi_{2\kappa+1},\varpi_{2\kappa-1},(1+\delta)\tau) \end{pmatrix}$$

$$\leq \begin{pmatrix} \rho(\varpi_{2\kappa-1},\varpi_{2\kappa},\tau) \lozenge \rho(\varpi_{2\kappa},\varpi_{2\kappa+1},\tau) \\ \lozenge \rho(\varpi_{2\kappa+1},\varpi_{2\kappa},\delta\tau) \end{pmatrix},$$

$$\omega(\varpi_{2\kappa},\varpi_{2\kappa+1},k\tau) \leq \begin{pmatrix} \omega(\varpi_{2\kappa-1},\varpi_{2\kappa},\tau) \lozenge \omega(\varpi_{2\kappa},\varpi_{2\kappa-1},\tau) \\ \lozenge \omega(\varpi_{2\kappa+1},\varpi_{2\kappa},\tau) \lozenge \omega(\varpi_{2\kappa},\varpi_{2\kappa},(1-\delta)\tau) \\ \lozenge \omega(\varpi_{2\kappa+1},\varpi_{2\kappa-1},(1+\delta)\tau) \end{pmatrix}$$

$$\leq \begin{pmatrix} \omega(\varpi_{2\kappa-1},\varpi_{2\kappa},\tau) \lozenge \omega(\varpi_{2\kappa},\varpi_{2\kappa+1},\tau) \\ \lozenge \omega(\varpi_{2\kappa+1},\varpi_{2\kappa},\delta\tau) \end{pmatrix}.$$

Furthermore, we get

$$\beta(\varpi_{2\kappa},\varpi_{2\kappa+1},k\tau) \geq \beta(\varpi_{2\kappa-1},\varpi_{2\kappa},\tau) * \beta(\varpi_{2\kappa+1},\varpi_{2\kappa},\tau) * \beta(\varpi_{2\kappa+1},\varpi_{2\kappa},\delta\tau),$$

$$\rho(\varpi_{2\kappa},\varpi_{2\kappa+1},k\tau) \leq \rho(\varpi_{2\kappa-1},\varpi_{2\kappa},\tau) \lozenge \rho(\varpi_{2\kappa+1},\varpi_{2\kappa},\tau) \lozenge \rho(\varpi_{2\kappa+1},\varpi_{2\kappa},\delta\tau),$$

$$\omega(\varpi_{2\kappa},\varpi_{2\kappa+1},k\tau) \leq \omega(\varpi_{2\kappa-1},\varpi_{2\kappa},\tau) \lozenge \omega(\varpi_{2\kappa+1},\varpi_{2\kappa},\tau) \lozenge \omega(\varpi_{2\kappa+1},\varpi_{2\kappa},\delta\tau).$$

Since CTN $*$ and CTCN $\lozenge$ are continuous $\beta(\ell,\varpi,.)$, $\rho(\ell,\varpi,.)$ and $\omega(\ell,\varpi,.)$ are continuous, suppose $\delta \to 1$, we get

$$\beta(\varpi_{2\kappa},\varpi_{2\kappa+1},k\tau) \geq \beta(\varpi_{2\kappa-1},\varpi_{2\kappa},\tau) * \beta(\varpi_{2\kappa+1},\varpi_{2\kappa},\tau),$$

$$\rho(\varpi_{2\kappa},\varpi_{2\kappa+1},k\tau) \leq \rho(\varpi_{2\kappa-1},\varpi_{2\kappa},\tau) \lozenge \rho(\varpi_{2\kappa+1},\varpi_{2\kappa},\tau),$$

$$\omega(\varpi_{2\kappa},\varpi_{2\kappa+1},k\tau) \leq \omega(\varpi_{2\kappa-1},\varpi_{2\kappa},\tau) \lozenge \omega(\varpi_{2\kappa+1},\varpi_{2\kappa},\tau).$$

Then, we get

$$\beta(\varpi_{2\kappa+1},\varpi_{2\kappa+2},k\tau) \geq \beta(\varpi_{2\kappa},\varpi_{2\kappa+1},\tau) * \beta(\varpi_{2\kappa+2},\varpi_{2\kappa+1},\tau),$$

$$\rho(\varpi_{2\kappa+1},\varpi_{2\kappa+2},k\tau) \leq \rho(\varpi_{2\kappa},\varpi_{2\kappa+1},\tau) \lozenge \rho(\varpi_{2\kappa+2},\varpi_{2\kappa+1},\tau),$$

$$\omega(\varpi_{2\kappa+1},\varpi_{2\kappa+2},k\tau) \leq \omega(\varpi_{2\kappa},\varpi_{2\kappa+1},\tau) \lozenge \omega(\varpi_{2\kappa+2},\varpi_{2\kappa+1},\tau).$$

we get, for $\varepsilon = 1,2, \ldots,$

$$\beta(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},k\tau) \geq \beta(\varpi_{\varepsilon},\varpi_{\varepsilon+1},\tau) * \beta(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},\tau),$$

$$\rho(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},k\tau) \leq \rho(\varpi_{\varepsilon},\varpi_{\varepsilon+1},\tau) \lozenge \rho(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},\tau),$$

$$\omega(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},k\tau) \leq \omega(\varpi_{\varepsilon},\varpi_{\varepsilon+1},\tau) \lozenge \omega(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},\tau).$$

Therefore, for $\varepsilon, \hbar = 1,2,\ldots,$

$$\beta(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},k\tau) \geq \beta(\varpi_{\varepsilon},\varpi_{\varepsilon+1},\tau) * \beta\left(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},\frac{\tau}{k^{\hbar}}\right),$$

$$\rho(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},k\tau) \leq \rho(\varpi_{\varepsilon},\varpi_{\varepsilon+1},\tau) \lozenge \rho\left(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},\frac{\tau}{k^{\hbar}}\right),$$

$$\omega(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},k\tau) \leq \omega(\varpi_{\varepsilon},\varpi_{\varepsilon+1},\tau) \lozenge \omega\left(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},\frac{\tau}{k^{\hbar}}\right).$$

as $\hbar \to \infty$,

$$\beta\left(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},\frac{\tau}{k^{\hbar}}\right) \to 1,$$

$$\rho\left(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},\frac{\tau}{k^{\hbar}}\right) \to 0,$$

$$\omega\left(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},\frac{\tau}{k^{\hbar}}\right) \to 0.$$

Then, we get, for $\varepsilon = 1,2, \cdots,$

$$\beta(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},k\tau) \geq \beta(\varpi_{\varepsilon},\varpi_{\varepsilon+1},\tau),$$

$$\rho(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},k\tau) \leq \rho(\varpi_{\varepsilon},\varpi_{\varepsilon+1},\tau),$$

$$\omega(\varpi_{\varepsilon+1},\varpi_{\varepsilon+2},k\tau) \leq \omega(\varpi_{\varepsilon},\varpi_{\varepsilon+1},\tau).$$

Hence by Lemma 3.1, $\{\varpi_\kappa\}$ is a Cauchy sequence in $\mathcal{L}$. Since $(\mathcal{L}, \beta, \rho, \omega, *, \lozenge)$ is complete, it converges to a point z in $\mathcal{L}$. Since $\{\mathcal{A}\ell_{2\kappa}\}, \{\mathcal{E}\ell_{2\kappa+1}\}, \{\mathrm{C}\ell_{2\kappa+2}\}$ and $\{T\ell_{2\kappa+1}\}$ are sub sequence of $\{\varpi_\kappa\}$. Thus, $\mathcal{A}\ell_{2\kappa}, \mathcal{E}\ell_{2\kappa+1}, \mathrm{C}\ell_{2\kappa+2}, T\ell_{2\kappa+1} \to z$ as $\kappa \to \infty$.

so, we take the equation (C4) that holds. The pair $(\mathcal{E}, T)$ is compatible of Type (I) and T is continuous, we get

$$\beta(Tz, z, \tau) \geq \lim_{\kappa\to\infty} \beta(\mathcal{E}T\ell_{2\kappa+1}, z, \lambda\tau),$$

$$\rho(Tz, z, \tau) \leq \lim_{\kappa\to\infty} \rho(\mathcal{E}T\ell_{2\kappa+1}, z, \lambda\tau),$$

$$\omega(Tz, z, \tau) \leq \lim_{\kappa\to\infty} \omega(\mathcal{E}T\ell_{2\kappa+1}, z, \lambda\tau).$$

$$TT\ell_{2\kappa+1} \to Tz.$$

Now, for $\alpha = 1$, setting $\ell = \ell_{2\kappa}$ and $\varpi = T\ell_{2\kappa+1}$ in (6), (7) and (8) we obtain

$$\beta(\mathcal{A}\ell_{2\kappa}, \mathcal{E}T\ell_{2\kappa+1}, k\tau) \geq \begin{pmatrix} \beta(\complement\ell_{2\kappa}, TT\ell_{2\kappa+1}, \tau) * \beta(\mathcal{A}\ell_{2\kappa}, \complement\ell_{2\kappa}, \tau) \\ * \beta(\mathcal{E}T\ell_{2\kappa+1}, TT\ell_{2\kappa+1}, \tau) \\ * \beta(\mathcal{A}\ell_{2\kappa}, TT\ell_{2\kappa+1}, \tau) \\ * \beta(\mathcal{E}T\ell_{2\kappa+1}, \complement\ell_{2\kappa}, \tau) \end{pmatrix}, \qquad (9)$$

$$\rho(\mathcal{A}\ell_{2\kappa}, \mathcal{E}T\ell_{2\kappa+1}, k\tau) \leq \begin{pmatrix} \rho(\complement\ell_{2\kappa}, TT\ell_{2\kappa+1}, \tau) \lozenge \rho(\mathcal{A}\ell_{2\kappa}, \complement\ell_{2\kappa}, \tau) \\ \lozenge \rho(\mathcal{E}T\ell_{2\kappa+1}, TT\ell_{2\kappa+1}, \tau) \\ \lozenge \rho(\mathcal{A}\ell_{2\kappa}, TT\ell_{2\kappa+1}, \tau) \\ \lozenge \rho(\mathcal{E}T\ell_{2\kappa+1}, \complement\ell_{2\kappa}, \tau) \end{pmatrix}, \qquad (10)$$

$$\omega(\mathcal{A}\ell_{2\kappa}, \mathcal{E}T\ell_{2\kappa+1}, k\tau) \leq \begin{pmatrix} \omega(\complement\ell_{2\kappa}, TT\ell_{2\kappa+1}, \tau) \lozenge \omega(\mathcal{A}\ell_{2\kappa}, \complement\ell_{2\kappa}, \tau) \\ \lozenge \omega(\mathcal{E}T\ell_{2\kappa+1}, TT\ell_{2\kappa+1}, \tau) \\ \lozenge \omega(\mathcal{A}\ell_{2\kappa}, TT\ell_{2\kappa+1}, \tau) \\ \lozenge \omega(\mathcal{E}T\ell_{2\kappa+1}, \complement\ell_{2\kappa}, \tau) \end{pmatrix}, \qquad (11)$$

Thus, we have to take limit as $\kappa \to \infty$ in above inequality, we get

$$\lim_{\kappa\to\infty} \beta\big(z, \mathcal{E}T\ell_{2\kappa+1,} k\tau\big) \geq \begin{pmatrix} \beta(z, Tz, \tau) * \beta(z, z, \tau) * \lim_{\kappa\to\infty} \beta\big(Tz, \mathcal{E}T\ell_{2\kappa+1,} \tau\big) \\ * \beta(z, Tz, T) * \lim_{\kappa\to\infty} \beta\big(z, \mathcal{E}T\ell_{2\kappa+1,} \tau\big) \end{pmatrix},$$

$$\lim_{\kappa\to\infty} \rho\big(z, \mathcal{E}T\ell_{2\kappa+1,} k\tau\big) \leq \begin{pmatrix} \rho(z, Tz, \tau) \lozenge \rho(z, z, \tau) \lozenge \lim_{\kappa\to\infty} \rho\big(Tz, \mathcal{E}T\ell_{2\kappa+1,} \tau\big) \\ \lozenge \rho(z, Tz, T) \lozenge \lim_{\kappa\to\infty} \rho\big(z, \mathcal{E}T\ell_{2\kappa+1,} \tau\big) \end{pmatrix},$$

$$\lim_{\kappa\to\infty} \omega\big(z, \mathcal{E}T\ell_{2\kappa+1,} k\tau\big) \leq \begin{pmatrix} \omega(z, Tz, \tau) \lozenge \omega(z, z, \tau) \lozenge \lim_{\kappa\to\infty} \omega\big(Tz, \mathcal{E}T\ell_{2\kappa+1,} \tau\big) \\ \lozenge \omega(z, Tz, T) \lozenge \lim_{\kappa\to\infty} \omega\big(z, \mathcal{E}T\ell_{2\kappa+1,} \tau\big) \end{pmatrix},$$

Thus, we get

$$\lim_{\kappa\to\infty} \beta\big(z, \mathcal{E}T\ell_{2\kappa+1,} k\tau\big) \geq \begin{pmatrix} \beta(z, Tz, \tau) * \lim_{\kappa\to\infty} \beta\big(Tz, \mathcal{E}T\ell_{2\kappa+1,} \tau\big) \\ * \lim_{\kappa\to\infty} \beta\big(z, \mathcal{E}T\ell_{2\kappa+1,} \tau\big) \end{pmatrix}$$

$$\geq \begin{pmatrix} \beta(z, Tz, \tau) * \beta\left(z, Tz, \frac{\tau}{2}\right) * \lim_{\kappa\to\infty} \beta\left(z, \mathcal{E}T\ell_{2\kappa+1,} \frac{\tau}{2}\right) \\ * \lim_{\kappa\to\infty} \beta\left(z, \mathcal{E}T\ell_{2\kappa+1,} \tau\right) \end{pmatrix}$$

$$\geq \begin{pmatrix} \lim_{\kappa\to\infty} \beta(z, \mathcal{E}T\ell_{2\kappa+1}, \lambda\tau) * \lim_{\kappa\to\infty} \beta\left(Tz, \mathcal{E}T\ell_{2\kappa+1}, \frac{\lambda\tau}{2}\right) \\ * \lim_{\kappa\to\infty} \beta\left(z, \mathcal{E}T\ell_{2\kappa+1,} \frac{\tau}{2}\right) * \lim_{\kappa\to\infty} \beta\left(z, \mathcal{E}T\ell_{2\kappa+1,} \tau\right) \end{pmatrix},$$

$$\lim_{\kappa\to\infty} \rho\left(z, \mathcal{E}T\ell_{2\kappa+1,} k\tau\right) \leq \begin{pmatrix} \rho(z, Tz, \tau) \lozenge \lim_{\kappa\to\infty} \rho\left(Tz, \mathcal{E}T\ell_{2\kappa+1,} \tau\right) \\ \lozenge \lim_{\kappa\to\infty} \rho(z, \mathcal{E}T\ell_{2\kappa+1,} \tau) \end{pmatrix}$$

$$\leq \begin{pmatrix} \rho(z, Tz, \tau) \lozenge \rho\left(z, Tz, \frac{\tau}{2}\right) \lozenge \lim_{\kappa\to\infty} \rho\left(z, \mathcal{E}T\ell_{2\kappa+1,} \frac{\tau}{2}\right) \\ \lozenge \lim_{\kappa\to\infty} \rho(z, \mathcal{E}T\ell_{2\kappa+1,} \tau) \end{pmatrix}$$

$$\leq \begin{pmatrix} \lim_{\kappa\to\infty} \rho(z, \mathcal{E}T\ell_{2\kappa+1}, \lambda\tau) \lozenge \lim_{\kappa\to\infty} \rho\left(Tz, \mathcal{E}T\ell_{2\kappa+1}, \frac{\lambda\tau}{2}\right) \\ \lozenge \lim_{\kappa\to\infty} \rho\left(z, \mathcal{E}T\ell_{2\kappa+1,} \frac{\tau}{2}\right) \lozenge \lim_{\kappa\to\infty} \rho\left(z, \mathcal{E}T\ell_{2\kappa+1,} \tau\right) \end{pmatrix},$$

Similarly,

$$\lim_{\kappa\to\infty} \omega\left(z, \mathcal{E}T\ell_{2\kappa+1,} k\tau\right) \leq \begin{pmatrix} \omega(z, Tz, \tau) \lozenge \lim_{\kappa\to\infty} \omega\left(Tz, \mathcal{E}T\ell_{2\kappa+1,} \tau\right) \\ \lozenge \lim_{\kappa\to\infty} \omega(z, \mathcal{E}T\ell_{2\kappa+1,} \tau) \end{pmatrix}$$

$$\leq \begin{pmatrix} \omega(z, Tz, \tau) \lozenge \omega\left(z, Tz, \frac{\tau}{2}\right) \lozenge \lim_{\kappa\to\infty} \omega\left(z, \mathcal{E}T\ell_{2\kappa+1,} \frac{\tau}{2}\right) \\ \lozenge \lim_{\kappa\to\infty} \omega(z, \mathcal{E}T\ell_{2\kappa+1,} \tau) \end{pmatrix}$$

$$\leq \begin{pmatrix} \lim_{\kappa\to\infty} \omega(z, \mathcal{E}T\ell_{2\kappa+1}, \lambda\tau) \lozenge \lim_{\kappa\to\infty} \omega\left(Tz, \mathcal{E}T\ell_{2\kappa+1}, \frac{\lambda\tau}{2}\right) \\ \lozenge \lim_{\kappa\to\infty} \omega\left(z, \mathcal{E}T\ell_{2\kappa+1,} \frac{\tau}{2}\right) \lozenge \lim_{\kappa\to\infty} \omega\left(z, \mathcal{E}T\ell_{2\kappa+1,} \tau\right) \end{pmatrix}.$$

As for $\lambda = 1$, in above inequality, then we get

$$\lim_{\kappa\to\infty} \beta\left(z, \mathcal{E}T\ell_{2\kappa+1,} k\tau\right) \geq \lim_{\kappa\to\infty} \beta\left(z, \mathcal{E}T\ell_{2\kappa+1,} \frac{\tau}{2}\right),$$

$$\lim_{\kappa\to\infty} \rho\left(z, \mathcal{E}T\ell_{2\kappa+1,} k\tau\right) \leq \lim_{\kappa\to\infty} \rho\left(z, \mathcal{E}T\ell_{2\kappa+1,} \frac{\tau}{2}\right),$$

$$\lim_{\kappa\to\infty} \omega\big(z, \mathcal{E}T\ell_{2\kappa+1,} k\tau\big) \leq \lim_{\kappa\to\infty} \omega\left(z, \mathcal{E}T\ell_{2\kappa+1,} \frac{\tau}{2}\right).$$

Therefore, $\lim_{\kappa\to\infty} \mathcal{E}T\ell_{2\kappa+1} = z$. Now using the compatibility of type (I), we have

$$\beta(Tz, z, \tau) \geq \lim_{\kappa\to\infty} \beta(z, \mathcal{E}T\ell_{2\kappa+1}, \lambda\tau) = 1,$$

$$\rho(Tz, z, \tau) \leq \lim_{\kappa\to\infty} \rho(z, \mathcal{E}T\ell_{2\kappa+1}, \lambda\tau) = 0,$$

$$\omega(Tz, z, \tau) \leq \lim_{\kappa\to\infty} \omega(z, \mathcal{E}T\ell_{2\kappa+1}, \lambda\tau) = 0,$$

and so $Tz = z$. Again we replacing ℓ by $\ell_{2\kappa}$ and ϖ by z in (6), (7) and (8) for $\alpha = 1$, we have

$$\beta(\mathcal{A}\ell_{2\kappa}, \mathcal{E}z, k\tau) \geq \begin{pmatrix} \beta(\mathrm{C}\ell_{2\kappa}, z, \tau) * \beta(\mathcal{A}\ell_{2\kappa}, \mathrm{C}\ell_{2\kappa}, \tau) \\ * \beta(\mathcal{E}z, z, \tau) * \beta(\mathcal{A}\ell_{2\kappa}, z, \tau) \\ * \beta(\mathcal{E}z, \mathrm{C}\ell_{2\kappa}, \tau) \end{pmatrix},$$

$$\rho(\mathcal{A}\ell_{2\kappa}, \mathcal{E}z, k\tau) \leq \begin{pmatrix} \rho(\mathrm{C}\ell_{2\kappa}, z, \tau) \lozenge \rho(\mathcal{A}\ell_{2\kappa}, \mathrm{C}\ell_{2\kappa}, \tau) \\ \lozenge \rho(\mathcal{E}z, z, \tau) \lozenge \rho(\mathcal{A}\ell_{2\kappa}, z, \tau) \\ \lozenge \rho(\mathcal{E}z, \mathrm{C}\ell_{2\kappa}, \tau) \end{pmatrix},$$

$$\omega(\mathcal{A}\ell_{2\kappa}, \mathcal{E}z, k\tau) \leq \begin{pmatrix} \omega(\mathrm{C}\ell_{2\kappa}, z, \tau) \lozenge \omega(\mathcal{A}\ell_{2\kappa}, \mathrm{C}\ell_{2\kappa}, \tau) \\ \lozenge \omega(\mathcal{E}z, z, \tau) \lozenge \omega(\mathcal{A}\ell_{2\kappa}, z, \tau) \\ \lozenge \omega(\mathcal{E}z, \mathrm{C}\ell_{2\kappa}, \tau) \end{pmatrix}.$$

as $\kappa \to \infty$, we acquire

$$\beta(\mathcal{E}z, z, k\tau) \geq \beta(\mathcal{E}z, z, \tau),$$

$$\rho(\mathcal{E}z, z, k\tau) \leq \rho(\mathcal{E}z, z, \tau),$$

$$\omega(\mathcal{E}z, z, k\tau) \leq \omega(\mathcal{E}z, z, \tau).$$

by Lemma 3.2, $\mathcal{E}z = z$. since $\mathcal{E}(\mathcal{L}) \sqsubseteq \mathrm{C}(\mathcal{L})$, there exists a point $u \epsilon \mathcal{L}$ and $\mathcal{E}z = \mathrm{C}u = z$. by (6), (7) and (8) for $\alpha = 1$, then

$$\beta(\mathcal{A}u, z, k\tau) \geq \begin{pmatrix} \beta(\mathrm{C}u, z, \tau) * \beta(\mathcal{A}u, \mathrm{C}u, \tau) \\ * \beta(z, z, \tau) * \beta(\mathcal{A}u, z, \tau) \\ * \beta(z, \mathrm{C}u, \tau) \end{pmatrix},$$

$$\rho(\mathcal{A}u, z, k\tau) \leq \begin{pmatrix} \rho(\mathrm{C}u, z, \tau) \lozenge \rho(\mathcal{A}u, \mathrm{C}u, \tau) \\ \lozenge \rho(z, z, \tau) \lozenge \rho(\mathcal{A}u, z, \tau) \\ \lozenge \rho(z, \mathrm{C}u, \tau) \end{pmatrix},$$

$$\omega(\mathcal{A}u, z, k\tau) \leq \begin{pmatrix} \omega(\complement u, z, \tau) \lozenge \omega(\mathcal{A}u, \complement u, \tau) \\ \lozenge \omega(z, z, \tau) \lozenge \omega(\mathcal{A}u, z, \tau) \\ \lozenge \omega(z, \complement u, \tau) \end{pmatrix}.$$

and

$$\beta(\mathcal{A}u, z, k\tau) \geq \beta(\mathcal{A}u, z, \tau),$$

$$\rho(\mathcal{A}u, z, k\tau) \leq \rho(\mathcal{A}u, z, \tau),$$

$$\omega(\mathcal{A}u, z, k\tau) \leq \omega(\mathcal{A}u, z, \tau).$$

and by Lemma 3.2, $u = z$,$\mathcal{A}u = \complement u = z$, by Proposition 3.6since the pair $(\mathcal{A}, \complement)$ is compatible of type (I) Therefore,

$$\beta(\mathcal{A}u, \complement\complement z, \tau) \geq \beta(\mathcal{A}u, \mathcal{A}\complement z, \tau),$$

$$\rho(\mathcal{A}u, \complement\complement z, \tau) \leq \rho(\mathcal{A}u, \mathcal{A}\complement z, \tau),$$

$$\omega(\mathcal{A}u, \complement\complement z, \tau) \leq \omega(\mathcal{A}u, \mathcal{A}\complement z, \tau).$$

and

$$\beta(z, \complement z, k\tau) \geq \beta(z, \mathcal{A}z, \tau),$$

$$\rho(z, \complement z, k\tau) \leq \rho(z, \mathcal{A}z, \tau),$$

$$\omega(z, \complement z, k\tau) \leq \omega(z, \mathcal{A}z, \tau).$$

Taking $\alpha = 1$, in inequality (6), (7) and (8) we have

$$\beta(\mathcal{A}z, z, k\tau) \geq \begin{pmatrix} \beta(\complement z, z, \tau) * \beta(\mathcal{A}z, \complement z, \tau) \\ * \beta(z, z, \tau) * \beta(\mathcal{A}z, z, \tau) \\ * \beta(z, \complement z, \tau) \end{pmatrix},$$

$$\rho(\mathcal{A}z, z, k\tau) \leq \begin{pmatrix} \rho(\complement z, z, \tau) \lozenge \rho(\mathcal{A}z, \complement z, \tau) \\ \lozenge \rho(z, z, \tau) \lozenge \rho(\mathcal{A}z, z, \tau) \\ \lozenge \rho(z, \complement z, \tau) \end{pmatrix},$$

$$\omega(\mathcal{A}z, z, k\tau) \leq \begin{pmatrix} \omega(\complement z, z, \tau) \lozenge \omega(\mathcal{A}z, \complement z, \tau) \\ \lozenge \omega(z, z, \tau) \lozenge \omega(\mathcal{A}z, z, \tau) \\ \lozenge \omega(z, \complement z, \tau) \end{pmatrix}.$$

Therefore,

$$\beta(\mathcal{A}z, z, k\tau) \geq \begin{pmatrix} \beta(\complement z, z, \tau) * \beta(\mathcal{A}z, \complement z, \tau) \\ * \beta(\mathcal{A}z, z, \tau) \end{pmatrix} \geq \beta\left(\mathcal{A}z, z, \frac{\tau}{2}\right),$$

$$\rho(\mathcal{A}z, z, k\tau) \leq \begin{pmatrix} \rho(\complement z, z, \tau) \lozenge \rho(\mathcal{A}z, \complement z, \tau) \\ \lozenge \rho(\mathcal{A}z, z, \tau) \end{pmatrix} \leq \rho\left(\mathcal{A}z, z, \frac{\tau}{2}\right),$$

$$\omega(\mathcal{A}z, z, k\tau) \leq \begin{pmatrix} \omega(\complement z, z, \tau) \lozenge \omega(\mathcal{A}z, \complement z, \tau) \\ \lozenge \omega(\mathcal{A}z, z, \tau) \end{pmatrix} \leq \omega\left(\mathcal{A}z, z, \frac{\tau}{2}\right).$$

and by Lemma 3.2, $\mathcal{A}z = z$. So , $\mathcal{A}z = \mathcal{E}z = \complement z = Tz = z$ and z is a common FP of $\mathcal{A}, \mathcal{E}, \complement$ and T. Easily verified by using the inequalities of (6), (7) and (8) for uniqueness of a common FP.

Example 4.1: Suppose that $\mathcal{L} = \left\{\frac{1}{\kappa} : \kappa = 1,2, \ldots\right\} \cup \{0\}$ with the usual metric and define (β, ρ, ω) , for all $\tau > 0$ and $\ell, \varpi \in \mathcal{L}$,

$$\beta(\ell, \varpi, \tau) = \begin{cases} \dfrac{\tau}{\tau + |\ell - \varpi|}, & \tau > 0 \\ 0, & \tau = 0 \end{cases},$$

$$\rho(\ell, \varpi, \tau) = \begin{cases} \dfrac{|\ell - \varpi|}{\tau + |\ell - \varpi|}, & \tau > 0 \\ 1, & \tau = 0 \end{cases},$$

$$\omega(\ell, \varpi, \tau) = \begin{cases} \dfrac{|\ell - \varpi|}{\tau}, & \tau > 0 \\ 1, & \tau = 0 \end{cases}$$

since, $(\mathcal{L}, \beta, \rho, \omega, *, \lozenge)$ is a complete NMS, $n \lozenge g = \max\{n, g\}$ and $n * g = \min\{n, g\}$. Let$\mathcal{A}, \mathcal{E}, \complement$ and T be described as $\mathcal{A}\ell = \frac{\ell}{4}, \complement\ell = \frac{\ell}{3}, \mathcal{E}\ell = \frac{\ell}{6}, T\ell = \frac{\ell}{3}$ for all $\ell \in \mathcal{L}$. Then we have

$$\mathcal{A}(\mathcal{L}) = \left\{\frac{1}{4\kappa} : \kappa = 1,2, \ldots\right\} \cup \{0\} \subseteq \left\{\frac{1}{2\kappa} : \kappa = 1,2, \ldots\right\} \cup \{0\} = \complement(\mathcal{L}),$$

$$\mathcal{E}(\mathcal{L}) = \left\{\frac{1}{6\kappa} : \kappa = 1,2, \ldots\right\} \cup \{0\} \subseteq \left\{\frac{1}{3\kappa} : \kappa = 1,2, \ldots\right\} \cup \{0\} = T(\mathcal{L}),$$

Also, the condition (6), (7) and (8) of Theorem 4.1 is fulfilled and $\mathcal{A}, \mathcal{E}, \complement$ and T are continuous. The pairs $(\mathcal{E}, T)$and$(\mathcal{A}, \complement)$ are compatible of type (I) and of type (II) such that $\lim_{\kappa\to\infty} \mathcal{A}\,\ell_\kappa = \lim_{\kappa\to\infty} \complement\ell_\kappa = \lim_{\kappa\to\infty} \mathcal{E}\ell_\kappa = \lim_{\kappa\to\infty} T\ell_\kappa = 0$ for some $0 \in \mathcal{L}$ If $\lim_{\kappa\to\infty} \ell_\kappa = 0$, where $\{\ell_\kappa\}$ is a sequence in $\mathcal{L}$. Thus all the conditions of Theorem 4.1 are satisfied and also 0 is the unique common FP of $\mathcal{A}, \mathcal{E}, \complement$ and T.

5. Application

Now we show how our established result can be used to find the unique solution to an integral equation in dynamic market equilibrium economics. Supply Q_β and demand Q_d, in many markets, current prices and pricing trends (whether prices are rising or dropping and whether they are rising or falling at an increasing or decreasing rate) have an impact. The economist, therefore, wants to know what the current price is $P(\vartheta)$, the first derivative $\frac{dP(\vartheta)}{d\vartheta}$, and the second derivative $\frac{d^2P(\vartheta)}{d\vartheta^2}$. Assume

$$Q_\beta = g_1 + \gamma_1 P(\vartheta) + e_1 \frac{dP(\vartheta)}{d\vartheta} + z_1 \frac{d^2P(\vartheta)}{d\vartheta^2},$$

$$Q_d = g_2 + \gamma_2 P(\vartheta) + e_2 \frac{dP(\vartheta)}{d\vartheta} + z_2 \frac{d^2P(\vartheta)}{d\vartheta^2}.$$

$g_1, g_2, \gamma_{1,} \gamma_2, e_1$ and e_2 are constants. If pricing clears the market at each point in time, comment on the dynamic stability of the market. In equilibrium, $Q_\beta = Q_d$. So,

$$g_1 + \gamma_1 P(\vartheta) + e_1 \frac{dP(\vartheta)}{d\vartheta} + z_1 \frac{d^2P(\vartheta)}{d\vartheta^2} = g_2 + \gamma_2 P(\vartheta) + e_2 \frac{dP(\vartheta)}{d\vartheta} + z_2 \frac{d^2P(\vartheta)}{d\vartheta^2}.$$

since

$$(z_1 - z_2)\frac{d^2P(\vartheta)}{d\vartheta^2} + (e_1 - e_2)d\frac{dP(\vartheta)}{d\vartheta} + (\gamma_1 - \gamma_2)P(\vartheta) = -(g_1 - g_2$$

Letting $z = z_1 - z_2, e = e_1 - e_2, \gamma = \gamma_1 - \gamma_2$ and $g = g_1 - g_2$ in above, we have

$$z\frac{d^2P(\vartheta)}{d\vartheta^2} + e\frac{dP(\vartheta)}{d\vartheta} + \gamma P(\vartheta) = -g,$$

Dividing through by $z, P(\vartheta)$ is governed by the following initial value problem

$$\begin{cases} P'' + \frac{e}{z}P'' + \frac{\gamma}{z}P(\vartheta) = -\frac{g}{z} \\ P(0) = 0 \\ P'(0) = 0, \end{cases} \qquad (12)$$

Where $\frac{e^2}{z} = \frac{4\gamma}{z}$ and $\frac{\gamma}{e} = \mu$ is a continuous function. It is easy to show that the problem (12) is equivalent to the integral equation:

$$P(\vartheta) = \int_0^T \xi(\vartheta, r) F(\vartheta, r, P(r))\, dr.$$

Where $\xi(\vartheta, r)$ is Green‘s function given by

$$\xi(\vartheta, r) = \begin{cases} r e^{\frac{\mu}{2}(\vartheta - r)} & \text{if } 0 \le r \le \vartheta \le T \\ \vartheta e^{\frac{\mu}{2}(r - \vartheta)} & \text{if } 0 \le \vartheta \le r \le \vartheta \le T. \end{cases}$$

We will show the existence of a solution to the integral equation:

$$P(\mathcal{v}) = \int_0^T G(\mathcal{v}, r, P(r))dr. \qquad (13)$$

Let $X = C\ ([0, T])$ set of real continuous functions defined on $[0, T]$ for $\mathcal{v} > 0$, we define

$$\beta(\ell, \varpi, \mathcal{v}) = \begin{cases} \dfrac{\mathcal{v}}{\mathcal{v} + |\ell - \varpi|}, & \tau > 0 \\ 0, & \tau = 0 \end{cases},$$

$$\rho(\ell, \varpi, \mathcal{v}) = \begin{cases} \dfrac{|\ell - \varpi|}{\mathcal{v} + |\ell - \varpi|}, & \tau > 0 \\ 1, & \tau = 0 \end{cases},$$

$$\omega(\ell, \varpi, \mathcal{v}) = \begin{cases} \dfrac{|\ell - \varpi|}{\mathcal{v}}, & \tau > 0 \\ 1, & \tau = 0 \end{cases}$$

For all $\ell, \varpi \epsilon \mathcal{L}$ with the CTN $' * '$ $n * g = \min\{n, g\}$ and CTCN $' \lozenge '$ $n \lozenge g = \max\{n, g\}$ and. It is easy to prove that $(\mathcal{L}, \beta, \rho, \omega, *, \lozenge)$ is complete NMS and so that $F : \mathcal{L} \rightarrow \mathcal{L}$ defined by

$$FP(\mathcal{v}) = \int_0^T G\ (\mathcal{v}, r, P(r)dr.$$

Theorem 5.1 Consider equation (13) and suppose that

(i) $G, H: [0, T] \times [0, T] \rightarrow \mathbb{R}^+$ are continuous functions,

(ii) There exist a continuous function $\xi: [0, T] \times [0, T] \rightarrow \mathbb{R}^+$ such that $\sup_{\mathcal{v} \in [0,T]} \int_0^T \xi(\mathcal{v}, r)dr \ \geq 1;$

(iii) $\left|G\big(\mathcal{v}, r, \ell(r)\big) - H\big(\mathcal{v}, r, \varpi(r)\big)\right| \leq k\xi(\mathcal{v}, r)|\ell(r) - \varpi(r)|$, for all $k \in (0,1)$

Then, the integral equation (13) has a unique solution. Where

$$D(\ell, \varpi) = \begin{pmatrix} \beta(\mathsf{C}\ell, T\varpi, \tau) * \beta(\mathcal{A}\ell, \mathsf{C}\ell, \tau) * \beta(\mathcal{E}\varpi, T\varpi, \tau) \\ * \beta(\mathcal{A}\ell, T\varpi, \alpha\tau) * \beta(\mathcal{E}\varpi, \mathsf{C}\ell, (2 - \alpha)\tau) \end{pmatrix},$$

$$E(\ell, \varpi) = \begin{pmatrix} \rho(\mathsf{C}\ell, T\varpi, \tau) \lozenge \rho(\mathcal{A}\ell, \mathsf{C}\ell, \tau) \lozenge \rho(\mathcal{E}\varpi, T\varpi, \tau) \\ \lozenge \rho(\mathcal{A}\ell, T\varpi, \alpha\tau) \lozenge \rho(\mathcal{E}\varpi, \mathsf{C}\ell, (2 - \alpha)\tau) \end{pmatrix},$$

and

$$B(\ell, \varpi) = \begin{pmatrix} \omega(\mathsf{C}\ell, T\varpi, \tau) \lozenge \omega(\mathcal{A}\ell, \mathsf{C}\ell, \tau) \lozenge \omega(\mathcal{E}\varpi, T\varpi, \tau) \\ \lozenge \omega(\mathcal{A}\ell, T\varpi, \alpha\tau) \lozenge \omega(\mathcal{E}\varpi, \mathsf{C}\ell, (2 - \alpha)\tau) \end{pmatrix}.$$

The pairs $(\mathcal{E}, T)$and$(\mathcal{A}, \mathsf{C})$ are compatible of type (I) and of type (II).

Proof: for $\ell, \varpi \in \mathcal{L}$, by using of assumptions, we have

$$\beta(\mathcal{A}\ell,\mathcal{E}\varpi,k\vartheta)=\frac{k\vartheta}{k\vartheta+\left|\int_0^T G\,(\vartheta,r,\ell(r)dr-\int_0^T H\,(\vartheta,r,\varpi(r)dr\right|}$$

$$\geq\frac{\vartheta}{\vartheta+|\ell(r)-\varpi(r)|}=D(\ell,\varpi).$$

$$\rho(\mathcal{A}\ell,\mathcal{E}\varpi,k\vartheta)=\frac{\left|\int_0^T G\,(\vartheta,r,\ell(r)dr-\int_0^T H\,(\vartheta,r,\varpi(r)dr\right|}{k\vartheta+\left|\int_0^T G\,(\vartheta,r,\ell(r)dr-\int_0^T H\,(\vartheta,r,\varpi(r)dr\right|}$$

$$\leq\frac{|\ell(r)-\varpi(r)|}{\vartheta+|\ell(r)-\varpi(r)|}=E(\ell,\varpi).$$

$$\omega(\mathcal{A}\ell,\mathcal{E}\varpi,k\vartheta)=\frac{\left|\int_0^T G\,(\vartheta,r,\ell(r)dr-\int_0^T H\,(\vartheta,r,\varpi(r)dr\right|}{k\vartheta}$$

$$\leq\frac{|\ell(r)-\varpi(r)|}{\vartheta}=B(\ell,\varpi).$$

all conditions of Theorem 4.1 are satisfied. Therefore, equation (13) has a unique fixed point.

6. **Conclusion**

we take the concept of compatible mappings in NMS and define the relation between two pair of mappings which are Compatible of type (II) if and only if pair of mappings are Compatible of type (I) and also prove that for four mappings common fixed point theorem under the compatible mappings condition of type (I) and (II) in the complete neutrosophic metric spaces also we give an application which are support our main result. In the future, we wish to use the control function and generalize these results in neutrosophic controlled metric spaces and neutrosophic double controlled metric spaces and trying to find the unique solution of different integral equations and differential equations.

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Chapter Four

A Study on Anti-Topological Neighbourhood and Anti-Topological Base

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ABSTRACT

In this paper, we introduce the concept of anti-topological neighbourhood and Anti-Topological-Base. Some examples of Anti-Neighbourhood and Anti-Base are given and we compare the theorems of the classical topological neighbourhood and Neutro-Topological-Neighbourhood with respect to Anti-Topological-Neighbourhood as well as classical topological base and Neutro-Topological-Neighbourhood with respect to Anti-Topological-Base.

KEYWORDS: Neutro-Topology, Neutro-Topological Neighbourhood, Neutro-Topological Base, Neutro-Topological sub-base, Anti-Topology, Anti-Topological Neighbourhood, Anti-Topological Base, Anti-Topological sub-Base.

INTRODUCTION

Topology is a significant subject of mathematics, hence it is surprising that topology's appreciation was delayed in the history of mathematics. Topology is the study of space characteristics that are unaffected by continuous deformation.

A key idea in mathematics, set theory, dates back to the work of Russian mathematician George Cantor (1877). We were able to investigate a variety of mathematical ideas thanks to set theory. However, there are a lot of unknowns in our life. The traditional logic of mathematics is frequently insufficient to resolve these difficulties. Then the idea of fuzzy sets was introduced by Zadeh (1965). It is a development of the traditional idea of a set. In his paper, he presented a hypothesis according to which fuzzy sets are sets with imprecise boundaries. In both directions, gradual changes from membership to nonmembership can be expressed using fuzzy sets. It offers meaningful representations of vague notions in everyday language in addition to a powerful and meaningful way to quantify uncertainties. a value in the discourse universe that indicates the fuzzy set's degree of membership. Real values in the closed range of 0 to 1 are used to represent these membership classifications. Chang (1968) discovered and popularised the theory of fuzzy topological spaces. The concepts for creating fuzzy topological spaces were provided by Lowen (1981). He provided the idea of fuzzy compression and two new functions, which allowed for the evident observation of further relationships between fuzzy topological spaces and topological spaces. A unique fuzzy topological space called the product spaces was discussed by Cheng-Ming (1985). He established a type of fuzzy points neighbourhood formation, such as the Q-neighbourhood, which is a crucial idea in fuzzy topological spaces. He also demonstrated how each fuzzy topological space is isomorphic topologically by a specific space of topology.

Atanassov (1996) introduced the concept of intuitionistic fuzzy sets as an extension of sets with better applicability. Coker (1997) developed the idea of intuitionistic smooth fuzzy topological spaces using the concept of intuitionistic fuzzy sets. The definitions of the intuitionistic smooth fuzzy topological spaces were first presented by Samanta and Mondal (1997).

Smarandache (1998) introduced the concept of a neutrosophic set for the first time. These concepts have three different degrees: T for membership, I for uncertainty, and F for non-membership. In other words, a situation is treated in neutrosophy in accordance with its trueness, falsity, and uncertainty. As a result, neutrosophic sets and logic enable us to make sense of a variety of uncertainties in our daily lives. On this topic, numerous studies have been conducted. Sahin et al. recently discovered some operations for neutrosophic sets with interval values; Neutrosophic multigroups and applications were researched by Ulucay et al (2019a); Q-neutrosophic soft expert set and its application were introduced by Hassan et al (2018). The acquisition of neutrosophic soft expert sets was introduced by Sahin et al (2015); Interval-valued refined neutrosophic sets and their applications were researched by Ulucay et al (2020b). Neutosophic set importance on deep transfer learning techniques was obtained by Khalifa et al. (2021); Generalised Hamming similarity measure based on neutrosophic quadraple numbers and its applications were researched by Kargin et al. (2021); In order to assess the quality of online education, Sahin et al. (2021a) obtain Hausdorff Measures on generalised set valued neutrosophic quadraple numbers and decision-making applications. The foundation for a wide family of novel mathematical ideas, including both their crisp and fuzzy counterparts, was laid by neutrosophy. Many research treating imprecision and uncertainty have been developed and studied[55-79]. The concepts of neutrosophic crisp set and neutrosophic crisp topological space were first developed by Salama et al. and Alblowi (2014). Neutron structures and antistructures are defined by Smarandache (2019). An algebraic structure can be divided into three regions, similar to neutrosophic logic: A, the set of elements that satisfy the conditions of the algebraic structure, the truth region; Neutro A, the set of elements that do not meet the conditions of the algebraic structure, the uncertainty region; and anti-A, the set of elements that do not satisfy the conditions of the algebraic structure, the inaccuracy region. By eliminating neutrosophic sets and neutrosophic numbers, the structure of neutrosophic logic has been translated to the structure of classical algebras. The academic world has seen a rise in interest in neutrosophic set theory research in recent years. As a result, it is possible to generate neutro-algebraic structures, which are more broadly structured than classical algebras. Additionally, the region of elements that do not conform to any of the classical algebras is also considered to have anti-algebraic structures. Recent research includes studies on neutro-algebra by Smarandache et al. (2020a), the neutrosophic triplet of BI-algebras by Razaei et al. (2020b), neutro-bck-algebra by Smarandache et al. (2020d), and neutro-hypergroups by Ibrahim et al. (2020b). In recent years, the academic community has witnessed growing research interests in uncertainty set theory [80-106].

In this chapter, Anti-Topology, neighbourhood and base are studied. The definition of Anti-Neighbourhood and comparison table of neighbourhood, Neutro-Neighbourhood and Anti-Neighbourhood with respect to the result of the classical topological neighbourhood is studied. Also, the definition of Anti-Base, Anti-sub-Base and the comparison table of base, Neutro-Base and Anti-Base as well as sub-base, Neutro-sub-Base, and Anti-sub-Base are given.

2. PRELIMINARIES

Definition 2.1. (Smarandache, 2020c) The NeutroSophication of the Law

1. Let X be a non-empty set and $*$ be a binary operation. For some elements $(a,b) \in (X,X)$, $(a*b) \in X$ (degree of well defined (T)) and for other elements $(x,y),(p,q) \in (X,X)$; [$x*y$ is indeterminate (degree of indeterminacy (I)), or $p*q \notin X$ (degree of outer-defined (F))], where (T, I, F) is different from (1,0,0) that represents the Classical Law, and from (0,0,1) that represents the Anti Law.
2. In Neutro Algebra, the classical well-defined for binary operation $*$ is divided into three regions: degree of well-defined (T), degree of indeterminacy (I) and degree of outer-defined (F) similar to neutrosophic set and neutrosophic logic.

Definition 2.2. (Şahin *et al.*, 2021b) Let X be the non-empty set and τ be a collection of subsets of X. Then τ is said to be a Neutro Topology on X and the pair (X,τ) is said to be a Neutro Topological space, if at least one of the following conditions hold good:

1. $[(\emptyset_N \in \tau, X_N \notin \tau)$ or $(X_N \in \tau, \emptyset_N \notin \tau)$] or $[\emptyset_N, X_N \in_{\sim} \tau]$.
2. For some n elements $a_1, a_2, \dots, a_n \in \tau$, $\bigcap_{i=1}^{n} a_i \in \tau$ [degree of truth T] and for other n elements $b_1, b_2, \dots, b_n \in \tau, p_1, p_2, \dots, p_n \in \tau$; $[(\bigcap_{i=1}^{n} b_i \notin \tau)$ [degree of

falsehood F] or ($\bigcap_{i=1}^{n} p_i$ is indeterminate (degree of indeterminacy I)], where n is finite; [where (T, I, F) is different from (1,0,0) that represents the Classical Axiom, and from (0,0,1) that represents the Anti Axiom].

3. For some n elements $a_1, a_2, \dots, a_n \in \tau, \bigcup_{i=1} a_i \in \tau$ [degree of truth T] and for other n elements $b_1, b_2, \dots, b_n \in \tau, p_1, p_2, \dots, p_n \in \tau$; [($\bigcup_{i=I} b_i \notin \tau$) [degree of falsehood F] or ($\bigcup_{i=I} p_i$ is indeterminate (degree of indeterminacy I)], where n is finite; [where (T, I, F) is different from (1,0,0) that represents the Classical Axiom, and from (0,0,1) that represents the Anti Axiom].

Definition 2.*3.* (Şahin *et al.*, 2021b) Let X be the non-empty set and τ be a collection of subsets of X. Then τ is said to be an Anti Topology on X and the pair (X, τ) is said to be an Anti Topological space, if at least one of the following conditions hold good:

1. $\emptyset_N, X_N \notin \tau$
2. For n elements $a_1, a_2, \dots, a_n \in \tau, \bigcap_{i=1}^{n} a_i \notin \tau$ [degree of falsehood F] where n is finite.
3. For some n elements $a_1, a_2, \dots, a_n \in \tau, \bigcup_{i=1} a_i \notin \tau$ [degree of falsehood F] where n is finite.

***Remark 1.* (Şahin *et al.*, 2021b)** The symbol "$\in_\sim$" will be used for situations where it is an unclear appurtenance (not sure if an element belongs or not to a set). For example, if it is not certain whether "a" is a member of the set P, then it is denoted by a $\in_\sim P$.

4. ANTI-TOPOLOGICAL- NEIGHBOURHOOD

Definition 4.1. Let (X, τ)be an Anti-Topological space and let $x \in X$. A subset N of X is said to be a τ-Anti-Neighbourhood of x if and only if there exists a τ-Anti-Open set G such that $x \in G \subset N$.

Example 1. Let $X = \{1, 2, 3, 4\}$ be a set and $\tau = \{\{1, 2\}, \{2,3\}, \{3,4\}\}$ be a collection of subsets of X.Then

i. It is clear that $\phi, X \notin \tau$

ii. Let $q_1 = \{1,2\},\ q_2 = \{2,3\}, q_3 = \{3,4\}$

Then $q_1 \cap q_2 = \{1,2\} \cap \{2,3\} = \{2\} \notin \tau$

$q_2 \cap q_3 = \{2,3\} \cap \{3,4\} = \{3\} \notin \tau$

$q_1 \cap q_3 = \{1,2\} \cap \{3,4\} = \phi \notin \tau$

iii. Let $q_1 = \{1,2\},\ q_2 = \{2,3\}, q_3 = \{3,4\}$

Then $q_1 \cup q_2 = \{1,2\} \cup \{2,3\} = \{1, 2, 3\} \notin \tau$

$q_2 \cup q_3 = \{2,3\} \cup \{3,4\} = \{2, 3, 4\} \notin \tau$

$q_1 \cup q_3 = \{1,2\} \cup \{3,4\} = \{1,2,3,4\} \notin \tau$

Therefore (X, τ) satisfies the conditions of Anti-Topological space.

(X, τ) is an Anti-Topological space.

τ-Anti-Neighbourhoods of 1 are $\{1,2\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}$

τ-Anti-Neighbourhoods of 2 are $\{1,2\}, \{2,3\}, \{1,2,3\}, \{1,2,4\}, \{2,3,4\}, \{1,2,3,4\}$

τ-Anti-Neighbourhoods of 3 are $\{2,3\},\{3,4\},\{1,2,3\},\{2,3,4\},\{1,3,4\},\{1,2,3,4\}$

τ-Anti-Neighbourhoods of 4 are $\{3,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}$

Now we compare the General topology, Neutro-Topology and Anti-Topology in terms of neighbourhood.

Here is the comparison table:

Table 1:Neighbourhood, Neutro-Neighbourhood and Anti-Neighbourhood

General Topology	Neutro-Topology	Anti-Topology
Theorem 1: A subset of a topological space is open if and only if it is neighbourhood of each of its points. Example: Let $X = \{1,2,3,4,5\}$ And $\tau = \{\phi,\{1\},\{1,2\},\{1,2,5\},$ $\{1,3,4\},\{1,2,3,4\},X\}$ Then, τ-neighbourhoods of 1 are: $\{1\},\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1$ $\{1,2,4,5\},\{1,3,4,5\}$, and X. τ-neighbourhoods of 2 are : $\{1,2\},\{1,2,3\},\{1,2,4\},\{1,2,5\},$ $\{1,2,3,4\},\{1,2,3,5\},\{1,2,4,5\},$ and X τ-neighbourhoods of 3 are: $\{1,3,4\},\{1,2,3,4\},\{1,3,4,5\},$ and X τ-neighbourhoods of 4 are: $\{1,3,4\},\{1,2,3,4\},\{1,3,4,5\},$ and X. τ-neighbourhoods of 5 are : $\{1,2,5\},\{1,2,3,5\},\{1,2,4,5\}$ and X Here X is neighbourhood of each of its points then we get that X is open since $X^0 = X$. Conversely, let X is open then it is seen from the above example that X is neighbourhood of each of its points since for all $x \in X$ there exists a $A \in \tau$ such that $x \in A \subseteq X$.	Result: A subset of a Neutro-Topological space is Neutro-Open if and only if it is Neutro-Neighbourhood of each of its points. Let $X = \{a,b,c,d\}$ and $\tau = \{\phi,\{a\}, \{a,b\},\{b,c\},$ $\{c,d\}\}$ τ - Neutro-Neighbourhoods of a are, $\{a\},\{a,b\},\{a,c\},\{a,d\},\{a,b,c\}$ $,\{a,c,d\},\{a,b,d\},X$. τ - Neutro-Neighbourhoods of b are: $\{a,b\},\{b,c\}\{a,b,c\},\{a,b,d\},$ $\{a,b,c\},X$. τ - Neutro-Neighbourhoods of c are: $\{c,d\},\{b,c\},\{a,c,d\},\{b,c,d\},$ $\{a,b,c\},X$. τ - Neutro-Neighbourhoods of d are: $\{c,d\},\{a,c,d\},\{b,c,d\},X$. We can see from the above discussion that $A = \{a,c,b\}$ is a τ -Neutro-Neighbourhood of each of its points. Consequently A is a Neutro-Open set since $A^0 = A$. a is interior point of A, since $a \in A$ and there exists $\{a\} \in \tau$ such that $a \in \{a\} \subseteq A$. b is interior point of A, since $b \in A$ and there exists $\{a,b\} \in \tau$ such that $b \in \{a,b\} \subseteq A$.	Result: A subset of a Anti-Topological space is Anti-Open if and only if it is Anti-Neighbourhood of each of its points. Let $X = \{1,2,3,4\}$ and $\tau = \{\{1,2\},\{2,3\},\{3,4\}\}$ τ- Anti-Neighbourhoods of 1 are, $\{1,2\},\{1,2,3\},\{1,2,4\},\{1,2,3,4\}$ τ-Anti-Neighbourhoods of 2 are: $\{1,2\},\{2,3\},\{1,2,3\},\{1,2,4\},$ $\{2,3,4\},\{1,2,3,4\}$ τ -Anti-Neighbourhoods of 3 are: $\{2,3\},\{3,4\},\{1,2,3\},\{1,2,3,4\},$ $\{1,3,4\}$ τ-Anti-Neighbourhoods of 4 are: $\{3,4\},\{1,3,4\},\{2,3,4\},$ $\{1,2,3,4\}$ We can see from the above discussion that $A = \{1,2,3\}$ is τ -Anti-Neighbourhood of each of its points. Consequently A is an Anti-Open set since $A^0 = A$. 1is interior point of A, since $1 \in A$ and there exists $\{1,2\} \in \tau$ such that $1 \in \{1,2\} \subseteq A$. 2 is interior point of A, since $2 \in A$ and there exists $\{1,2\} \in \tau$ such that $2 \in \{1,2\} \subseteq A$.

	c is interior point of A, since $c \in A$ and there exists $\{b,c\} \in \tau$ such that $c \in \{b,c\} \subseteq A$. d is not an interior point of A, since $d \notin A$ and there does not exists $G \in \tau$ such that $d \in G \subseteq A$. Therefore, $A^0 = A$. Conversely, $A = \{a,b,c\}$ is a Neutro-Open as $A^0 = A$. Clearly, it is τ- Neutro-Neighbourhood of each of its points.	3 is interior point of A, since $3 \in A$ and there exists $\{2,3\} \in \tau$ such that $3 \in \{2,3\} \subseteq A$. 4 is not an interior point of A, since $4 \notin A$ and there does not exists $G \in \tau$ such that $4 \in G \subseteq A$. Therefore, $A^0 = A$. Conversely, $A = \{1,2,3\}$ is a Anti-Open as $A^0 = A$. Clearly, it is τ- Anti-Neighbourhood of each of its points.
Theorem 2: Let X be a topological space, and for each $x \in X$, let $N(x)$ be the collection of all neighbourhoods of x. Then $[N0]: \forall x \in X, N(x) \neq \phi$ i.e. every point x has atleast on neighbourhood. $[N(1)]: N \in N(x) \Rightarrow x \in N$i.e. every neighbourhood of x contains x. $[N(2)]: N \in N(x), M \supset N \Rightarrow M \in N(x)$. i.e. every set containing a neighbourhood of x is a neighbourhood of x. $N[3]: N \in N(x), M \in N(x) \nRightarrow N \cap M \in N(x)$ $[N(4)]: N \in N(x) \Rightarrow \exists M \in N(x)$ such that, $M \subset N$ and $M \in N(y)\, \forall y \in M$	Result: Let X be a Neutro-Topological space, and for each $x \in X$, let $N(x)$ be the collection of all τ- Neutro-neighbourhoods of x. Then $[N0]: \forall x \in X, N(x) \neq \phi$ i.e. every point x has atleast a τ-Neutro-Neighbourhood. $[N(1)]: N \in N(x) \Rightarrow x \in N$ i.e. every τ-Neutro-Neighbourhood of x contains x. $[N(2)]: N \in N(x), M \supset N \Rightarrow M \in N(x)$. i.e. every set containing a τ-Neutro-Neighbourhood of x is τ-Neutro-Neighbourhood of x. $N[3]: N \in N(x), M \in N(x) \nRightarrow N \cap M \in N(x)$ To show this, an example is cited below: Example: Let $X = \{0,1,2,3\}$ and $\tau = \{\phi, X, \{1\}, \{2\}, \{2,3\}, \{1,3\}\}$. τ- Neutro-Neighbourhoods of 0 is X. τ- Neutro-Neighbourhoods of 1 are: $\{1\}, \{0,1\}, \{1,2\}, \{1,3\}, \{0,1,2\},$	Result: Let X be an Anti-Topological space, and for each $x \in X$, let $N(x)$ be the collection of all τ- Anti-neighbourhoods of x. Then $[N0]: \forall x \in X, N(x) \neq \phi$ i.e. every point x has atleast one τ -Anti - Neighbourhood. $[N(1)]: N \in N(x) \Rightarrow x \in N$ i.e. every τ -Anti-Neighbourhood of x contains x. $[N(2)]: N \in N(x), M \supset N \Rightarrow M \in N(x)$. i.e. every set containing a τ-Anti- Neighbourhood of x is τ -Anti -Neighbourhood of x. $N[3]: N \in N(x), M \in N(x) \nRightarrow N \cap M \in N(x)$ To show this, an example is cited below: Example: Let $X = \{1,2,3,4\}$ and $\tau = \{\{1,2\}, \{2,3\}, \{3,4\}\}$ τ- Anti-Neighbourhoods of 1 are: $\{1,2\}, \{1,2,3\}, \{1,2,4\}, \{1,2,3,4\}$ τ-Anti-Neighbourhoods of 2 are: $\{1,2\}, \{2,3\}, \{1,2,3\}, \{1,2,4\},$

	$\{0,1,3\}, \{1,2,3\}$, and X τ - Neutro-Neighbourhoods of 2 are: $\{2\}, \{0,2\}, \{1,2\}, \{2,3\}, \{0,1,2\}$, $\{1,2,3\}$, and X. τ - Neutro-Neighbourhoods of 3 are: $\{1,3\}, \{2,3\}, \{0,2,3\}, \{0,1,3\}$, $\{1,2,3\}$, and X. e.g. $\{0,1,3\}, \{0,2,3\} \in N(3)$ but $\{0,1,3\} \cap \{0,2,3\} = \{0,3\} \notin N(3)$. $[N(4)]$: $N \in N(x) \nRightarrow M \in N(x)$ such that $M \subset N$ and $M \in N(y)\ \forall y \in M$. e.g. $\{0,1,2\} \in N(1)$ then $\{0,1\} \in N(1)$ such that $\{0,1\} \subset \{0,1,2\}$ but $\{0,1\} \notin N(y)\ \forall y \notin M$.	$\{2,3,4\}, \{1,2,3,4\}$ τ-Anti-Neighbourhoods of 3 are: $\{2,3\}, \{3,4\}, \{1,2,3\}$, $\{1,2,3,4\}, \{1,3,4\}$ τ-Anti-Neighbourhoods of 4 are: $\{3,4\}, \{1,3,4\}, \{2,3,4\}$, $\{1,2,3,4\}$ e.g. $\{1,2\} \in N(2), \{2,3\} \in N(2)$ But $\{1,2\} \cap \{2,3\} = \{2\} \notin N(2)$. $[N(4)]$: $N \in N(x) \nRightarrow M \in N(x)$ such that $M \subset N$ and $M \in N(y)\ \forall y \in M$. e.g. $\{1,2,3,4\} \in N(4)$ then $\{1,3,4\} \in N(1)$ such that $\{1,3,4\} \subset \{1,2,3,4\}$ but $\{1,3,4\} \notin N(y)\ \forall y \epsilon M$.
Theorem 3: Let X be a non-empty set and with each $x \in X$, let there be associated a family $N(x)$ of subsets of X, called neighbourhoods, satisfying the following conditions: $[N0]$: $N(x) \neq \phi \forall x \in X$. $[N1]$: $N \in N(x) \Rightarrow x \in N$. $[N2]$: $N \in N(x), M \supset N \Rightarrow M \in N(x)$. $[N3]$: $N \in N(x), M \in N(x) \Rightarrow N \cap M \in N(x)$ $[N4]$: $N \in N(x) \Rightarrow \exists M \in N(x)$ such that $M \subset N$ and $M \in N(y)\ \forall y \in M$. Then there exists a unique topology τ on X in such a way that if $N^*(x)$ is the collection of neighbourhoods of x, defined by the topology τ, then $N^*(x) = N(x)$.	Result: Let X be a non-empty set and with each $x \in X$, let there be associated a family $N(x)$ of subsets of X, called τ -Neutro-Neighbourhoods. $[N0]$: $N(x) \neq \phi\ \forall x \in X$. $[N1]$: $N \in N(x) \Rightarrow x \in N$. $[N2]$: $N \in N(x), M \supset N \Rightarrow M \in N(x)$. $[N3]$: $N \in N(x), M \in N(x) \nRightarrow N \cap M \in N(x)$. To show this, an example is cited below: Let $X = \{0,1,2,3\}$ and $\tau = \{\phi, X, \{1\}, \{2\}, \{2,3\}, \{1,3\}\}$. τ - Neutro-Neighbourhoods of 0 is X. τ - Neutro-Neighbourhoods of 1 are: $\{1\}, \{0,1\}, \{1,2\}, \{1,3\}, \{0,1,2\}$, $\{0,1,3\}, \{1,2,3\}$ and X. τ- Neutro-Neighbourhoods of 2 are: $\{2\}, \{0,2\}, \{1,2\}, \{2,3\}, \{0,1,2\}$, $\{1,2,3\}$, and X.	Result: Let X be a non-empty set and with each $x \in X$, let there be associated a family $N(x)$ of subsets of X, called τ -Anti-Neighbourhoods. $[N0]$: $N(x) \neq \phi\ \forall x \in X$. $[N1]$: $N \in N(x) \Rightarrow x \in N$. $[N2]$: $N \in N(x), M \supset N \Rightarrow M \in N(x)$. $[N3]$: $N \in N(x), M \in N(x) \nRightarrow N \cap M \in N(x)$. To show this, an example is cited below: Let $X = \{a,b,c,d\}$ $\tau = \{\{a,b\}, \{a,c\}, \{c,d\}\}$ Anti-Neighbourhoods of a are, $\{a,b\}, \{a,c\}, \{a,b,c\}$, $\{a,b,d\}, \{a,c,d\}$ *and* X Anti-Neighbourhoods of b are, $\{a,b\}, \{a,b,c\}, \{a,b,d\}$ and X Anti-Neighbourhoods of c are, $\{a,c\}, \{c,d\}, \{a,b,c\}$, $\{a,c,d\}, \{b,c,d\}$ and X Anti-Neighbourhoods of d are,

	τ -Neutro-Neighbourhoods of 3 are : $\{1,3\},\{2,3\},\{0,2,3\},\{0,1,3\},\{1,2$ and X. $N[4]: N \in N(x) \nRightarrow M \in N(x)$ such that $M \subset N$ and $M \in N(y)\ \forall y \in M$ e.g. $\{0,1,2\} \in N(1), \{0,1\} \in N(1)$ Such that $\{0,1\} \subset \{0,1,2\}$ but $\{0,1\} \notin N(y)\ \forall y \in M$. Then there does not exists a unique Neutro-Topology τ on X in such a way that if $N^*(x)$ is the collection of τ - Neutro-Neighbourhoods of x, defined by the Neutro-Topology τ since all the properties are not satisfied by a τ -Neutro-Neighbourhood.	$\{c,d\},\{a,c,d\},\{b,c,d\}$ and X e.g. $\{a,b\},\{a,c\} \in N(a)$ but $\{a,b\} \cap \{a,c\} = \{a\} \notin N(a) N[4]: N \in N(x) \nRightarrow$ $M \in N(x)$ such that $M \subset N$ and $M \in N(y)\ \forall y \in M$ e.g. $X \in N(d)$ then there exists $\{b,c,d\} \in N(d)$ such that $\{b,c,d\} \subset X$ but $\{b,c,d\} \notin N(y) \forall\ y \in M$ since $\{b,c,d\} \notin N(b)$ Then there does not exists a unique Anti-Topology τ on X in such a way that if $N^*(x)$ is the collection of τ- Anti-Neighbourhoods of x , defined by the Anti-Topology τ since all the properties are not satisfied by τ- Anti-Neighbourhood.
Theorem 4: Let X be a non-empty set, and for each $x \in X$, let $N(x)$ be a nonempty collection of subsets of X satisfying the following conditions: $[M1]: N \in N(x) \Rightarrow c \in N$ $[M2]: N \in N(x), M \in N(x) \Rightarrow N \cap M \in N(x)$ Let τ consists of the empty and all those non-empty subsets G of X having the property that $x \in G$ implies that there exists a $N \in N(x)$ such that $x \in N \subset G$. Then τ is a topology for X.	Result: Let X be a non-empty set, and for each $x \in X$, let $N(x)$ be a non-empty collection of subsets of X. $[M1]: N \in N(x) \Rightarrow x \in N$. $[M2]: N \in N(X), M \in N(x) \nRightarrow N \cap M \in N(x)$ To show this, an example is cited below: Let $X = \{a,b,c,d\}$ $\tau = \{\phi, \{a\}, \{a,b\}, \{b,c\}, \{c,d\}\}$ τ -Neutro-Neighbourhoods of a are: $\{a\},\{a,b\},\{a,c\},\{a,d\}, \{a,b,c\},\{a,c,d\},\{a,b,d\}, X$. τ - Neutro-Neighbourhoods of b are: $\{a,b\},\{b,c\},\{a,b,c\},\{a,b,d\}, \{a,b,c\}, X$. τ - Neutro-Neighbourhoods of c are: $\{c,d\},\{b,c\},\{a,c,d\},\{b,c,d\}, \{a,b,c\}, X$. τ - Neutro-Neighbourhoods of d are: $\{c,d\},\{a,c,d\},\{b,c,d\}, X$. Now	Result: Let X be a non-empty set, and for each $x \in X$, let $N(x)$ be a non-empty collection of subsets of X. $[M1]: N \in N(x) \Rightarrow x \in N$. $[M2]: N \in N(X), M \in N(x) \nRightarrow N \cap M \in N(x)$ To show this, an example is cited below: Let $X = \{a,b,c,d\}$ $\tau = \{\{a,b\},\{a,c\},\{c,d\}\}$ Anti-Neighbourhoods of a are, $\{a,b\},\{a,c\},\{a,b,c\}, \{a,b,d\},\{a,c,d\}$ and X Anti-Neighbourhoods of b are, $\{a,b\},\{a,b,c\},\{a,b,d\}$ and X Anti-Neighbourhoods of c are, $\{a,c\},\{c,d\},\{a,b,c\}$, $\{a,c,d\},\{b,c,d\}$ and X Anti-Neighbourhoods of d are, $\{c,d\},\{a,c,d\},\{b,c,d\}$ *and* X e.g. $\{a,b\},\{a,c\} \in N(a)$ But $\{a,b\} \cap \{a,c\} = \{a\} \notin N(a)$

	$\{a,b\} \in N(b)$ $\{b,c\} \in N(b)$ but $\{a,b\} \cap \{b,c\} = \{b\} \notin N(b)$ Therefore the second condition is not satisfied by Neutro-Neighbourhood.	Therefore the second condition is not satisfied by Anti-Neighbourhood.

ANTI-TOPOLOGICAL- BASE

Definition 4.2. Let (X,τ) be an Anti-Topological space. Then a non-empty sub-collection B of subsets of X is said to be an Anti-Base for some Anti-Topology on X if the following conditions satisfied:

1. For all $x \in X$ there exists $A \in B$ such that $x \in A$
2. For some $A_1, A_2 \in B$ and for $x \in A_1 \cap A_2$ there may not exists $A_3 \in B$ such that $x \in A_3 \in A_1 \cap A_2$

Example 2. Let $X = \{a,b,c,d\}$ and $\tau = \{\{a,b\},\{a,c\},\{b,c\},\{c,d\},\{b,d\},\{a,d\}\}$ be a collection of subsets of X.

Let $B = \{\{a,b\},\{b,c\},\{c,d\}\}$.

Then B is an Anti-Topological-Base since

1. For all $x \in X$ there exists $A \in B$ such that $x \in A$
2. Let $\{a,b\},\{b,c\} \in B$ but $\{a,b\} \cap \{b,c\} = \{b\}$, we can not get any $A \in B$ such that $x \in A \subseteq \{b\}$

Now we compare the General-topology, Neutro-Topology and Anti-topology in terms of base.

Here is the comparison table:

Table 2: Base, Neutro-Base and Anti-Base

General Topology	Neutro-Topology	Anti-Topology
Theorem 1: Let (X,τ) be a topological space. A sub-collection B of τ is a base for τ if and only if every τ-open set can be expressed as the union of members of B.	Result: Let (X,τ) be a Neutro-Topological space. A sub-collection B of τ is a Neutro-Base for τ if every Neutro-Open set can be expressed as the union of members of B but the converse is not true. To show that the converse part is not true an example is cited below. Example : Let $X = \{a,b,c,d\}$ $\tau = \{\phi,\{a\},\{a,b\},\{b,c\},\{c,d\}\}$ $B = \{\{a\},\{a,b\},\{c,d\}\}$ Here not every τ -Neutro-	Result: In Anti-topology the theorem is not satisfied since (1)We can not express any Anti-Open set as the union of members of anti-Base. (2)The converse part is not true in Anti-Topology because of the third condition of Anti-Topology.

	Open set can be expressed as the union of members of B. It is seen that $\{b,c\}$ can not be expressed as the union of members of B.	
Theorem 2: Let (X,τ) be a topological space and B be a base for τ. Then B has the following properties: (1) For every $x \in X$, there exists a $A \in B$ such that $x \in B$ i.e $X = \cup\{A : A \in B\}$ (2) For every $A_1, A_2 \in B$ and every point $x \in A_1 \cap A_2$ there exists a $A \in B$ such that $x \in A \subset A_1 \cap A_2$ that is the intersection of any two members of B is a union of members of B.	Result: Let (X,τ) be a Neutro-Topological space and B be a Neutro-Base for τ. Then B has the following properties : (1) For every $x \in X$ there exists a $A \in B$ such that $x \in B$ that is $X = \cup\{A : A \in B\}$ Therefore we get $X = \cup\{A : A \in B\}$. (2) For some $A_1, A_2 \in B$ and every point $x \in A_1 \cap A_2$ there exists $A \in B$ such that $x \in A \subset A_1 \cap A_2$ and for some $A_1, A_2 \in B$ for any $x \in A_1 \cap A_2$ there does not exist a A such that $x \in A \subset A_1 \cap A_2$. Example: Let $X = \{a,b,c,d\}$ $\tau = \{\{a\},\{a,b\},\{c,d\}\}$ (1) For all $x \in X$ there exists $A \in B$ such that $X = \cup\{A : A \in B\}$ (2) $\{a\},\{a,b\} \in B$, then there exists $\{a\} \in B$ such that $a \in \{a\} \subset \{a\} \cap \{a, b\}$. Again $\{a,b\},\{c,d\} \in B$ then $\{a,b\} \cap \{c,d\} = \phi$ but there does not exist any $A \in B$ such that $x \in A \subset \{a,b\} \cap \{c,d\}$. Therefore the second condition of the theorem is not satisfied by all $A \in B$.	Result: Let (X,τ) be an Anti-Topological space and B be an Anti-Base for τ. Then B has the following properties : (1) For every $x \in X$ there exists a $A \in B$ such that $x \in B$ that is $X = \cup\{A : A \in B\}$ Therefore, we get $X = \cup\{A : A \in B\}$. (2) For every $A_1, A_2 \in B$ and every point $x \in A_1 \cap A_2$ there may not exists $A \in B$ such that $x \in A \subset A_1 \cap A_2$. Example: Let $X = \{a,b,c,d\}$ $\tau = \{\{a,b\},\{a,c\},\{b,c\},\{c,d\},\{b,d\},\{a,d\}\}$ $B = \{\{a,b\},\{b,c\},\{c,d\}\}$ Let $A_1 = \{a,b\}, A_2 = \{b,c\}$ Then $A_1 \cap A_2$={b} But there does not exists any $A \in B$ such that $x \in A \subset A_1 \cap A_2$ Therefore the second condition is different in Anti-Topological space.
Theorem 3: Let X be a non-empty set and let B be a collection of subsets of X satisfying the following conditions : (1) For every $x \in X$,there exists $A \in B$ such that $x \in B$ $i.e$ $X = \cup\{A : A \in B\}$ (2)	Result: Let X be a non-empty set and let B be a collection of subsets of X. Then (1) For every $x \in X$, there exists a $A \in B$ such that $x \in B$ i.e. $X = \cup\{A : A \in B\}$ (2) For some $A_1, A_2 \in B$ and	Result: Let X be a non-empty set and let B be a collection of subsets of X. Then (1) For every $x \in X$, there exists a $A \in B$ such that $x \in B$ i.e. $X = \cup\{A : A \in B\}$

For every $A_1 \in B, A_2 \in B$ and every point $x \in A_1 \cap A_2$ there exists a $A \in B$ such that $x \in A \subset A_1 \cap A_2$ that is the intersection of any two members of B is a union of members of B. Then there exists a unique topology τ for X such that B is a base for τ.	every point $x \in A_1 \cap A_2$ there exists $A \in B$ such that $x \in A \subset A_1 \cap A_2$ and for some $A_1, A_2 \in B$ for any $x \in A_1 \cap A_2$ there may not exist A such that $x \in A \subset A_1 \cap A_2$. Let $X = \{a,b,c,d,e\}$ $\tau = \{\phi, \{a\}, \{a,b\}, \{b,c\}, \{b,d\}, \{c,d\}, \{c,d,e\}, \{a,b,c,d\}\}$ $B = \{\phi, \{a\}, \{a,b\}, \{b,d\}, \{c,d\}, \{c,d,e\}, \{a,b,c,d\}\}$ Now $\{a\}, \{a,b\} \in B$ then $\{a\} \cap \{a,b\} = \{a\}$. For $a \in \{a\}, \exists \{a\} \in B$ such that $x \in \{a\} \subset \{a\} \cap \{a,b\}$. But $\{a,b\}, \{b,d\} \in B, \{a,b\} \cap \{b,d\} = \{b\}$But there does not exists any $A \in B \subset \{a,b\} \cap \{b,d\}$. Then there exists a unique Neutro -Topology τ for X such that B is a Neutro-Base for τ if the above conditions satisfied by B. So it is seen that the second condition is not satisfied.	(2) For every $A_1, A_2 \in B$ and for $x \in A_1 \cap A_2$ there may not exists $A \in B$ such that $x \in A \subset A_1 \cap A_2$. Example: Let $X = \{a,b,c,d\}$ $\tau = \{\{a,b\}, \{b,c\}, \{a,d\}, \{c,d\}, \{a,c\}, \{b,d\}\}$ $B = \{\{a,b\}, \{a,d\}, \{b,c\}\}$ Let $A_1 = \{a,b\}, A_2 = \{a,d\}$ Then $A_1 \cap A_2 = \{a,b\} \cap \{a,d\} = \{a\}$ There does not exists any $A \in B$ such that $x \in A \subset A_1 \cap A_2$ Thus the second condition is different in Anti-Topological space.

ANTI-TOPOLOGICAL- SUB-BASE

Definition 4.3. Let (X, τ) be an anti-topological space. A collection B' of subsets of X is called an Anti-sub-Base for the Anti-Topology τ if and only if $B' \subset \tau$ and finite intersection of members of B' form an Anti-Base for τ

Example 3. Let $X = \{a,b,c,d,e,f\}$

$\tau = \{\{a,b\}, \{b,c\}, \{c,d\}, \{e,f\}, \{d,e\}, \{c,f\}, \{a,d,f\}, \{a,c,e\}\}$

$B' = \{\{a,b\}, \{c,d\}, \{e,f\}, \{d,e\}, \{b,c\}, \{c,f\}, \{a,c,e\}\}$

The finite intersection of members of B' are,

$B_0 = \{\{a\}, \{d\}, \{e\}, \{c\}, \{b\}, \{f\}\}$

Then B_0 is a base since

1. For all $x \in X$ there exists $A \in B$ such that $x \in A$
2. $\{a\}, \{d\} \in B_0$, then$\{a\} \cap \{d\} = \phi$

Clearly it is seen that there does not exists any x such that $x \in A \subset A_1 \cap A_2$

Now we compare the General topology, Neutro-Topology and Anti-Topology in terms of sub-base.

Here is the comparison table:

Table 3: Sub-base, Neutro-sub-Base and Anti-sub-Base

General Topology	Neutro-Topology	Anti-Topology
Theorem 1: Let B^* be a non-empty collection of subsets of a non-empty set X. Then B_0 is a sub-base for a unique topology τ for X, that is finite intersections of members of B^* form a base for τ.	Result: Let B^* be a non-empty collection of subsets of a non-empty set X. Then B_0 is a Neutro-sub-Base for a unique Neutro Topology τ for X, that is finite intersections of members of B^* form a base for τ if the following conditions are satisfied by B_0: (1) For every $x \in X, \exists A \in B_0$ such that $x \in B_0$ such that $X = \cup\{A : A \in B_0\}$ (2) For every $A_1 \in B_0, A_2 \in B_0$, every point $x \in A_1 \cap A_2 \exists A \in B_0$ such that $x \in A \subset A_1 \cap A_2$ and for some $A_1, A_2 \in B_0$ there may not exist any $A_3 \in B$ such that $x \in A_3 \subseteq A_1 \cap A_2$ Let $X = \{a, b, c, d, e, f\}$ T= $\{\phi, \{a, b\}, \{b, c\}, \{d, e\}, \{c, d, e\}, \{a, b, c, d\}, \{e, f\}\}$ Let $B^* = \{\{a, b\}, \{b, c\}, \{d, e\}, \{e, f\}, \{c, d, e\}, \{a, b, c, d\}\}$ The finite intersection of members of B^* are $B_0 = \{\{b\}, \{d\}, \{e\}, \{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{e, f\}, \{a, b, c, d\}\}$ Then $B_0$is a Neutro-Base since 1.For every $x \in X, \exists A \in B_0$ such that $x \in B_0$. 2. $\{a, b\} \in B_0, \{a, b, c, d\} \in Bo\{a, b\} \cap \{a, b, c, d\} = \{a, b\}$, then for every $x \in \{a, b\} \exists$ a $A \in B_0$ such that $x \in B_0 \subset \{a, b\} \cap \{a, b, c, d\}$. Again, $\{b, c\}, \{c, d\} \in B_0$ $\{b, c\} \cap \{c, d\} = \{c\}$ but there does not exist any $A_3 \in B_0$ such that $x \in A_3 \subseteq A_1 \cap A_2$ Therefore, the second condition of Neutro-sub-Base is not satisfied here.	Result: Let B^* be a non-empty collection of subsets of a non-empty set X. Then B_0 is an Anti-sub-Base for a unique Anti-Topology τ for X, that is finite intersections of members of B^* form a base for τ if the following conditions are satisfied by B_0: (1) For every $x \in X, \exists A \in B_0$ such that $x \in B_0$ such that $X = \cup\{A : A \in B_0\}$ (2) For every $A_1 \in B_0, A_2 \in B_0$, every point x $\in A_1 \cap A_2$there may not exist A $\in B_0$ such that $x \in A \subset A_1 \cap A_2$ Example: Let $X = \{a, b, c, d, e, f\}$ $\tau = \{\{a, b\}, \{b. c\}, \{c, d\}, \{d, e\}, \{e, f\}, \{c, f\}, \{a, d, f\}, \{b, d, f\}\}$ $B^* = \{\{a, b\}, \{, d\}, \{b, c\}, \{e, f\}, \{d, e\}, \{c, f\}, \{a, d, f\}\}$ Then the finite intersection of members of B^* are B_0= $\{\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}\}$ Then (1) For every $x \in X, \exists A \in B_0$ such that $x \in B_0$ such that $X = \cup\{A : A \in B_0\}$ (2) $\{a\}, \{b\} \in B_0$ But $\{a\} \cap \{b\} = \phi$ Therefore, there does not exists any $A \in B_0$ such that $x \in A \subset A_1 \cap A_2$

5. CONCLUSION

In this study, we introduced the notion of Anti-Topological Neighbourhood and Anti-Topological Base via Anti-Topology. We have discussed some theorems of neighbourhood and base. Similarities and differences between neighbourhood, Neutro-Neighbourhood and Anti-Neighbourhood as well as base, Neutro-Base and Anti-Base, sub-base, Neutro-sub-Base and Anti-sub-Base are discussed. We get that a discrete topology and an indiscrete topology can not be an Anti Topology since clearly in both cases X and ϕ belongs to them.

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Chapter Five

Neutrosophic n- normed linear space

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ABSTRACT

The principal objective of this study is to extend the application of cubic n-norms, specifically within the domain of neutrosophic n-normed linear spaces (NCS). This entails adapting the notion of intuitionistic n-norms to harmonize with the neutrosophic framework. The research also involves a comprehensive exploration of Cauchy and convergent sequences within neutrosophic n-normed spaces. To enhance comprehension, visual aids in the form of growth diagrams are incorporated to illustrate normed linear structures in a lucid and accessible manner. Moreover, this investigation introduces level sets for the innovative construct referred to as n-normed linear space (NRC), which aligns with the notion of NCS. The formulation of these level sets is supported by rigorous and concrete mathematical demonstrations. Additionally, the concept of automata NCS is presented, providing an application of NCS.

.KEYWORDS: Automata fuzzy n - norm, cubic n - norm, neutrosophic n - norm, automata neutrosophic n - norm.

1.INTRODUCTION

Dimension extension is an intriguing concept in the realm of functional analysis. It all began with the groundbreaking idea of extending normed linear spaces to two dimensions and n dimensions, inspired by Gahler's pioneering work [1,2]. His research sparked interest in many researchers who furthered the development of Banach space theory in n dimensions.

In their research, Narayanan and Vijayabalaji [4] embarked on the task of extending the notion of n - NRC into a domain known as fuzzy n - NRC, which integrates fuzzy theory with n - NRC principles. Subsequently, Vijayabalaji and Thillaigovindan [8] took the initiative to redefine and expand upon the concept of f- n - NRC by incorporating t-norms and t-co-norms. A valuable resource for scholars in the field of fuzzy n - NRC can be originate in the book authored by Thillaigovindan et al. [7]. This book extensively explores various intriguing generalizations of f-n- NRC, including intuitionistic fuzzy n - NRC [9] and interval-valued fuzzy n- NRC [10]. In recent years, the academic community has witnessed growing research interests in uncertainty set theory [13-44].

In a pioneering work, Jun [3] lead the novel concept of cubic sets, which represents a fusion of fuzzy sets and interval-valued fuzzy sets. This innovative idea served as the inspiration for Vijayabalaji [12] to further advance the field by introducing cubic n - NRC (C- n - NRC). These C- n - NRC provide a unified framework encompassing all the previously mentioned normed structures.

Naturally, researchers wondered if these structures could be further generalized. The present work addresses this question by leveraging the remarkable structure of neutrosophic sets (NS) [5]. NS serves as a comprehensive generalization of all existing uncertainty theories, offering lucid solutions to various problems. Many research treating imprecision and uncertainty have been developed and studied [45-64].

It is noteworthy to observe that there has been a notable absence of concrete examples illustrating the application of fuzzy n - normed space thus far. Recently, Vijayabalaji and Punniyamoorthy [11] addressed this gap by demonstrating the application of fuzzy n - NRC through integration with automata theory. This development served as a motivating factor for us to extend the application of NCS. Consequently, we have lead the concept of automata NCS, and further provided an illustrative application of this concept.

This research presents the introduction of NCS in Section 2, which serves as a generalization of all the aforementioned structures, accompanied by an illustrative example. Section 3 also presents the growth diagram of normed linear structures and provides insights into Cauchy sequences and convergent sequences in NCS. Additionally, we introduce level sets for NCS with essential results.

Section 4 introduces a novel concept of NCS utilizing automata theory, effectively merging the principles of NCS with this theoretical framework. Section 5 expounds upon the matrix representation of input strings. Section 6 outlines the operations applicable to the matrix representation of the string. Moving forward to Section 7, we present an algorithm for identifying the optimal finite automata NCS, offering a clear and detailed illustration through an example. Section 8 delineates potential directions for future research, while Section 9 provides concluding remarks on the entirety of this work.

In the context of a given linear space, we typically denote the elements of X^n as (x_1, x_2,...,x_n). For the sake of simplicity and ease of reference, we will use the term 'Φ' to denote these elements throughout this chapter. Furthermore, we will denote different combinations of these elements as follows.

$(\tau_1, \ldots, \tau_{n-1}, \tau_n) = (\Phi - 1, \tau_n)$, $(\tau_1, \ldots, \tau_{n-1}, c\,\tau_n) = (\Phi - 1, c\,\tau_n)$,
$(\tau_1, \ldots, \tau_{n-1}, \tau'_n) = (\Phi - 1, \tau'_n)$ and $(\tau_1, \ldots, \tau_{n-1}, \tau_n + \tau'_n) = (\Phi - 1, \tau_n + \tau'_n)$.
.
These conventions will be applied consistently throughout the chapter, as appropriate.

2. NEUTROSOPHIC n- NORMED LINEAR SPACE(NCS)

This section is devoted to the introduction of the concept of NCS. This structure is being formulated as a seamless extension that encompasses all pre-existing fuzzy, intuitionistic, and cubic structures within the domain of n- NRC.

Definition 3.1. A neutrosophic n-normed linear space, abbreviated as NCS, is represented as S = $\{(\mathbf{X}, \mathbf{M}(\Phi, \kappa), \mathbf{P}(\Phi, \kappa), \mathbf{H}(\Phi, \kappa)) \mid (\Phi, \kappa) = (\tau_1, \ldots, \tau_n, t) \in X^{\mathbf{n}} \times [0, \infty)\}$, where **X** is a linear space over a field F, * is a continuous t-norm, $\oplus$ is a continuous t-co-norm, and **M**, **P**, and **H** are neutrosophic sets on $\mathbf{X} \times [0, \infty)$. In this context, **M** represents the truth-membership function, **P** represents the falsity-membership function, and **H** represents the indeterminacy-membership function. These functions satisfy the following conditions:

1. Complementarity: $0 \leq \mathbf{M}(\Phi, \kappa) + \mathbf{P}(\Phi, \kappa) + \mathbf{H}(\Phi, \kappa) \leq 3$.

2. Linear Dependency: $\mathbf{M}(\Phi, \kappa) = 1 \Leftrightarrow$ if the elements $\tau_1, \ldots, \tau_n$ in X are linearly dependent.

3. Permutation Invariance holds for **M**.

4. Scaling Property for N: For $c \neq 0$ in the field F, $\mathbf{M}(\Phi - 1, c\,x_n, \kappa) = \mathbf{M}(\Phi - 1, \frac{t}{|c|})$.

5. Fuzzy Triangle Inequality for **M**: $\mathbf{M}(\Phi - 1, \tau_n, \nu) * \mathbf{M}(\Phi - 1, \tau'_n, \kappa) \leq \mathbf{M}(\Phi - 1, \tau_n + \tau'_n, \nu + \kappa)$.

6. Continuity for **M**: $\mathbf{M}(\Phi, \kappa) = 1$ is continuous in t.

7. Complementarity for **P**: $\mathbf{P}(\Phi, \kappa) = 0 \Leftrightarrow$ the elements $\tau_1, \ldots, \tau_n$ in X are linearly dependent.

8. Permutation Invariance holds for **P**.

9. Scaling Property for **P**: For $c \neq 0$ in the field F, **P** $(\Phi - 1, c\,\tau_n, \kappa) =$ **P** $(\Phi - 1, \tau_n, \frac{\kappa}{|c|})$.

10. Fuzzy Triangle Inequality for **P**: **P** $(E\Phi - 1, \tau_n, \nu) \oplus$ **P** $(\Phi - 1, \tau'_n, \kappa) \leq$ **P** $(\Phi - 1, \tau_n + \tau'_n, \nu + \kappa)$.

11. Continuity for **P**: **P** $(\Phi, \kappa) = 0$ is continuous in t.

12. Complementarity for **H**: **H** $(\Phi, \kappa) = 0 \Leftrightarrow$ the elements $\tau_1, \ldots, \tau_n$ in X are linearly dependent.

13. Permutation Invariance holds for **H**.

14. Scaling Property for **H**: For $c \neq 0$ in the field F, **H** $(\Phi - 1, c\,\tau_n, \kappa) =$ **H** $(\Phi - 1, \tau_n, \frac{\kappa}{|c|})$.

15. Fuzzy Triangle Inequality for **H**: **H** $(\Phi - 1, \tau_n, \nu) \oplus$ **H** $(\Phi - 1, \tau'_n, \kappa) \leq$ **H** $(\Phi - 1, \tau_n + \tau'_n, \nu + \kappa)$.

16. Continuity for **H**: **H** $(\Phi, \kappa) = 0$ is continuous in t.

In essence, a NCS incorporates truth, falsity, and indeterminacy membership functions to capture uncertainty in linear spaces.

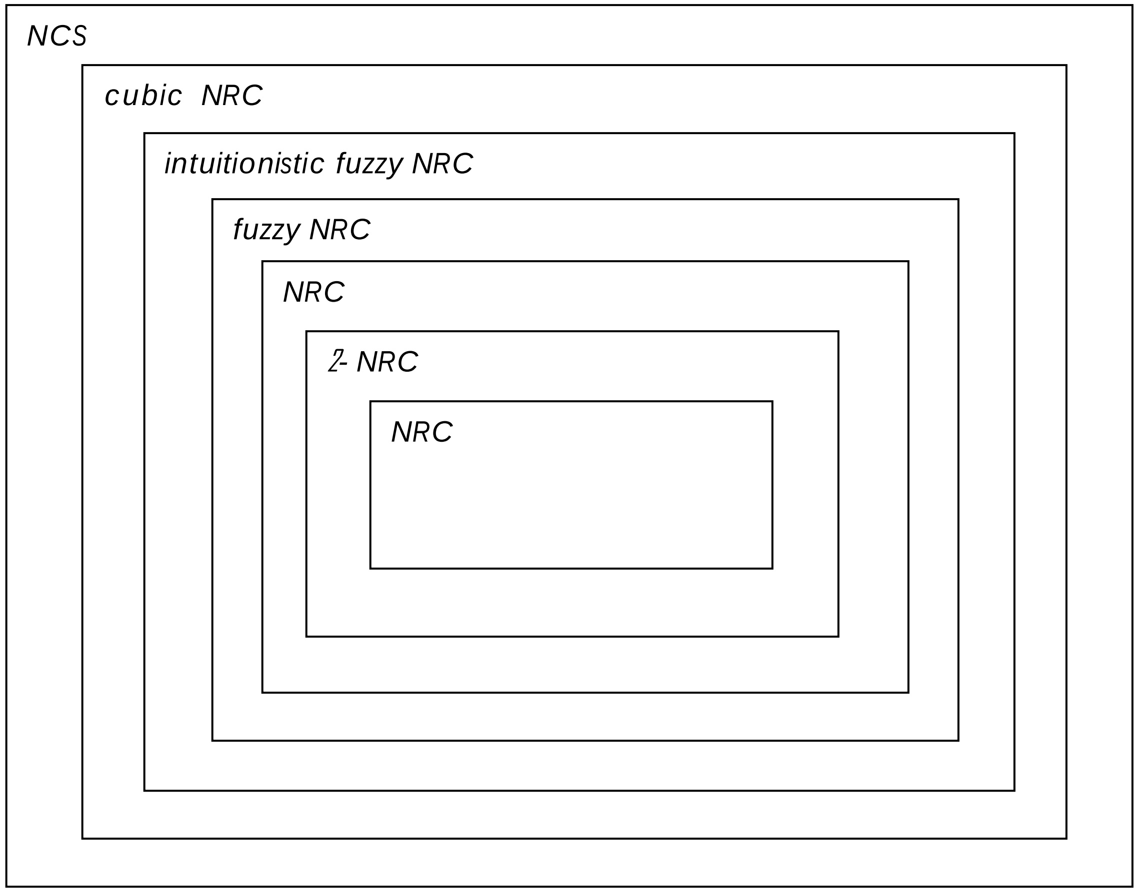
To substantiate the definition provided above, we give the following illustrative example.

Example 3.2. Consider an NRC denoted as $(X, \|\bullet, \bullet, \ldots, \bullet\|)$. In this space, we define the binary operations as follows: $a * b = min\{a, b\}$ and $a \oplus b = max\{a, b\}$, for all $a, b \in [0, 1]$.

Additionally, we set the membership functions as follows: **M** $(\Phi, \kappa) = \frac{\kappa}{\kappa + \|\Phi\|}$, **P** $(\Phi, \kappa) = \frac{\|\Phi\|}{\kappa + \|\Phi\|}$ and **H** $(\Phi, \kappa) = \frac{\|\Phi\|}{\kappa}$.

With these definitions in place, we can construct a NCS S.

The development of this normed linear space is visually depicted as follows.



Definition 3.3. In a NCS, S a sequence $\{\tau_n\}$ is considered to converge to τ if, for any given positive real numbers $\omega > 0$ and $\kappa > 0$, where $0 < \omega < 1$, there exists an integer $n_0 \in N$ (the set of natural numbers) such that the following conditions hold for all $n \geq n_0$.

1. The truth-membership function $\mathbf{M}(\Phi - 1, \tau_n - \tau, \kappa) > 1 - \omega$.
2. The falsity-membership function $\mathbf{P}(\Phi - 1, \tau_n - \tau, \kappa) < \omega$.
3. The indeterminacy-membership function $\mathbf{H}(\Phi - 1, \tau_n - \tau, \kappa) < \omega$.

Theorem 3.4. In a NCS, S a sequence $\{\tau_n\}$ converges to τ if and only if the truth-membership function $\mathbf{M}(\Phi - 1, \tau_n - \tau, \kappa) \to 1$, the falsity-membership function $\mathbf{P}(\Phi - 1, \tau_n - \tau, \kappa) \to 0$ and the indeterminacy-membership function $\mathbf{H}(\Phi - 1, \tau_n - \tau, \kappa) \to 0$ as $n \to \infty$.

Proof. Let's consider the sequence $\{\tau_n\}$ that converges to x in S. Fix $t > 0$. Let the sequence $\{\tau_n\}$ converges to τ in S.

According to Definition 3.3, for any given positive real numbers $\omega > 0$ and $\kappa > 0$, where $0 < \omega < 1$, $\exists$ an integer $n_0 \in N$ (the set of natural numbers) such that the following conditions hold for all $n \geq n_0$:

1. $\mathbf{M}(\Phi - 1, \tau_n - \tau, \kappa) > 1 - \omega$.
2. $\mathbf{P}(\Phi - 1, \tau_n - \tau, \kappa) < \omega$.
3. $\mathbf{H}(\Phi - 1, \tau_n - \tau, \kappa) < \omega$.

$\Rightarrow$ As $n \to \infty$, we have $\mathbf{M}(\Phi - 1, \tau_n - \tau, \kappa) \to 1$, $\mathbf{P}(\Phi - 1, \tau_n - \tau, \kappa) \to 0$ and $\mathbf{H}(\Phi - 1, \tau_n - \tau, \kappa) \to 0$.

Conversely, assume that $\mathbf{M}(\Phi - 1, \tau_n - \tau, \kappa) \to 1$, $\mathbf{P}(\Phi - 1, \tau_n - \tau, \kappa) \to 0$ and $\mathbf{H}(\Phi - 1, \tau_n - \tau, \kappa) \to 0$ as $n \to \infty$.

Then for every ω, $0 < \omega < 1$, $\exists$ an integer n_0 such that $1 - \mathbf{M}(\Phi - 1, \tau_n - \tau, \kappa) < \omega$, $\mathbf{P}(\Phi - 1, \tau_n - \tau, \kappa) < \omega$ and $\mathbf{H}(\Phi - 1, \tau_n - \tau, \kappa) < \omega$.

Thus $\mathbf{M}(\Phi - 1, \tau_n - \tau, \kappa) > 1 - \omega$, $\mathbf{P}(\Phi - 1, \tau_n - \tau, \kappa) < \omega$ and $\mathbf{H}(\Phi - 1, \tau_n - \tau, \kappa) < \omega$ for all $n \geq n_0$.

Hence $\{\tau_n\}$ converges to τ in S.

Definition 3.4. In a NCS, S, a sequence $\{\tau_n\}$ is considered to be a Cauchy sequence if, for any given positive real numbers $\omega > 0$ and $\kappa > 0$, where $0 < \omega < 1$, there exists an integer $n_0 \in N$ (the set of natural numbers) such that for all $n, k \geq n_0$, the following conditions hold:

1. $\mathbf{M}(\Phi - 1, \tau_n - \tau_k, \kappa) > 1 - \omega$.
2. $\mathbf{P}(\Phi - 1, \tau_n - \tau_k, \kappa) < \omega$.
3. $\mathbf{H}(\Phi - 1, \tau_n - \tau_k, \kappa) < \omega$.

Theorem 3.4. In a NCS S, every convergent sequence is a Cauchy sequence.

Proof.

Given that $\{\tau_n\}$ converges to τ in S.

Let $t > 0$ and $p \in (0,1)$. Choose $\kappa \in (0,1)$ so that $(1-\omega) * (1-\omega) > 1-\in$ and $\omega \oplus \omega < \in$.

As $\{\tau_n\}$ converges to τ

Then there exists an integer $n_0 \in N$ such that for all $n \geq n_0$,

1. $\mathbf{M}(\Phi - 1, \tau_n - \tau, \kappa) > 1 - \omega$
2. $\mathbf{P}(\Phi - 1, \tau_n - \tau, \kappa) < \omega$ and
3. $\mathbf{H}(\Phi - 1, \tau_n - \tau, \kappa) < \omega$.

Note that

$\mathbf{M}(\Phi - 1, \tau_n - \tau_k, \kappa)$

$= \mathbf{M}(\Phi - 1, \tau_n - \tau + \tau - \tau_k, \frac{\kappa}{2} + \frac{\kappa}{2})$

$\geq \mathbf{M}(\Phi - 1, \tau_n - \tau, \frac{t}{2}) * \mathbf{M}(\Phi - 1, \tau_n - \tau_k, \frac{\kappa}{2})$

$\geq (1-\omega) * (1-\omega)$

$> 1 - \in$, for all $n, k \geq n_0$.

Also

$\mathbf{P}(\Phi - 1, \tau_n - \tau_k, \kappa)$

$= \mathbf{P}(\Phi - 1, \tau_n - \tau + \tau - \tau_k, \frac{\kappa}{2} + \frac{\kappa}{2})$

$\leq \mathbf{P}(\Phi - 1, \tau_n - \tau, \frac{\kappa}{2}) \oplus \mathbf{P}(\Phi - 1, \tau_n - \tau_k, \frac{\kappa}{2})$

$< \omega \oplus \omega$

$< \in$, for all $n, k \geq n_0$

and

$\mathbf{H}(\Phi - 1, \tau_n - \tau_k, \kappa)$

$= \mathbf{H}(\Phi - 1, \tau_n - \tau + \tau - \tau_k, \frac{\kappa}{2} + \frac{\kappa}{2})$

$\leq \mathbf{H}(\Phi - 1, \tau_n - \tau, \frac{\kappa}{2}) \oplus \mathbf{M}(\Phi - 1, \tau_n - \tau_k, \frac{\kappa}{2})$

$< \omega \oplus \omega$

$< \in$, for all n, k $\geq n_0$.

So $\{\tau_n\}$ is a Cauchy sequence in S.

Definition 3.5. In a NCS S, a sequence is regarded as complete if every Cauchy sequence contained within it converges.

Remark 3.6. It is important to acknowledge that there may exist Cauchy sequences in S that do not converge. For example, let's consider the sequence in example 3.2. Now, if $\{\tau_n\}$ is a sequence in S, then $\{\tau_n\}$ is a Cauchy sequence in an η- NRC if and only if $\{\tau_n\}$ is a Cauchy sequence in S. Similarly, $\{\tau_n\}$ is a convergent sequence in an η- NRC if and only if $\{\tau_n\}$ is a convergent sequence in S.

Remark 3.7. In S, if every Cauchy sequence has a convergent subsequence, then it is referred to as being complete.

In fuzzy algebraic structures and even in fuzzy topological structures, the concept of level sets plays a pivotal role in extending these structures to higher dimensions. In the case of the NCS structure as well, the formation of level sets follows a similar pattern.

Definition 3.8. Given a NCS, S, its level set is defined as follows.

$\|\Phi\|_\eta = \inf\{\kappa : \mathbf{H}(\Phi,\kappa) \geq \eta,\ \mathbf{M}(\Phi,\kappa) < 1-\eta$ and $\mathbf{P}(\Phi,\kappa) < 1-\eta, \eta \in (0,1)\}$.

We define this set as η- n- NRC of S.

It's significant to emphasize that the provided definition is applicable under the condition (17). This conditional-based approach for level sets is a unique feature not typically found in fuzzy algebraic structures. Condition (17) states that $\mathbf{H}(\Phi,\kappa) > 0$, $\mathbf{M}(\Phi,\kappa) > 0$, and $\mathbf{P}(\Phi,\kappa) > 0$ when the elements $\tau_1, \ldots, \tau_n$ in the range $\eta \in (0,1)$ are linearly dependent.

Theorem 3.9. The level set defined in Definition 3.8, along with the condition (17), constitutes a η- n- NRC.

Proof. To substantiate this statement, let's verify the four conditions for a - η - NRC as follows:

(1) $\| \Phi \|_\eta = 0$

$\Rightarrow$ inf { κ : **M** (Φ, κ)≥ η , **P** (Φ, κ) < 1 – η and **H** (Φ, κ) < 1 – $\eta, \eta \in (0,1)$ }

$\Rightarrow$ **M** (Φ, κ)≥ η , **P** (Φ, κ) < 1 – η and **H** (Φ, κ) < 1 – $\eta, \eta \in (0,1)$

$\Rightarrow$ **M** (Φ, κ) > 0 , **P** (Φ, κ) > 0 and **H** (Φ, κ) > $0, \eta \in (0,1)$

$\Rightarrow$ $\tau_1, \ldots, \tau_n$ are linearly dependent, from condition (17).

Conversely, we assume that $\tau_1, \ldots, \tau_n$ are linearly dependent.

$\Rightarrow$ **M** (Φ, κ) = 1, **P** (Φ, κ) = 0 and **H** (Φ, κ) = 0 , from Definition 2.1.

$\Rightarrow$ inf { κ : **M** (Φ, κ)≥ η , **P** (Φ, κ) < 1 – η and **H** (Φ, κ) < 1 – $\eta, \eta \in (0,1)$ }=

0

$\Rightarrow$ $\| \Phi \|_\eta = 0$.

(2) As **M** (Φ, κ), **P** (Φ, κ) and **H** (Φ, κ) is an unvarying in any permutation of $\tau_1, \ldots, \tau_n$, it is evident to note that $\| E \|_\eta$ is an unvarying in any permutation of $\tau_1, \ldots, \tau_n$.

(3) $\| \Phi - 1, c\tau_n \|_\eta$

= inf {s: **M** (Φ – 1, c τ_n, ν)≥ η , **P** (Φ – 1,c τ_n, ν) < 1 – η

and **H** (Φ – 1, c τ_n, ν) < 1 – $\eta, \eta \in (0,1)$}

= inf {ν: **M** (Φ – 1, τ_n, $\frac{\nu}{|c|}$)≥ η, **P** (Φ – 1, τ_n, $\frac{\nu}{|c|}$) < 1 – η

and **H** (Φ – 1, τ_n, $\frac{\nu}{|c|}$) < 1 – η, $\eta \in (0,1)$}

Let $\kappa = \frac{\nu}{|c|}$. Then

$\| E - 1, c\tau_n \|_\eta$

= inf { κ |c|: **M** (Φ – 1, τ_n, κ)≥ η, **P** (Φ – 1, τ_n, κ) < 1 – η

and **H** (Φ – 1, τ_n, κ) < 1 – $\eta, \eta \in (0,1)$}

$=|c| \inf \{ \kappa: \mathbf{M}(\Phi - 1, \tau_n, \kappa) \geq \eta, \mathbf{P}(\Phi - 1, \tau_n, \kappa) < 1 - \eta$

and $\mathbf{H}(\Phi - 1, \tau_n, \kappa) < 1 - \eta, \eta \in (0,1)\}$

$= |c| \, \| \Phi - 1, \tau_n \|_\eta$.

(4) $\| \Phi - 1, \tau_n \|_\eta + \| \Phi - 1, \tau_n' \|_\eta$

$= \inf \{ \kappa : \mathbf{M}(\Phi - 1, \tau_n, \kappa) \geq \eta, \mathbf{P}(\Phi - 1, \tau_n, \kappa) < 1 - \eta$

and $\mathbf{H}(\Phi - 1, \tau_n, \kappa) < 1 - \eta, \eta \in (0,1) \}$

$+ \inf \{ s : \mathbf{M}(\Phi - 1, \tau_n', \nu) \geq \eta, \mathbf{P}(\Phi - 1, \tau_n', \nu) < 1 - \eta$

and $\mathbf{H}(\Phi - 1, \tau_n', \nu) < 1 - \eta, \eta \in (0,1) \}$

$= \inf \{ \kappa + \nu: \mathbf{M}(\Phi - 1, \tau_n, \kappa) \geq \eta, \mathbf{P}(\Phi - 1, \tau_n', \nu) \geq \eta, \mathbf{H}(\Phi - 1, \tau_n, \kappa) < 1 - \eta,$

$\mathbf{M}(\Phi - 1, \tau_n', \nu) < 1 - \eta, \mathbf{P}(\Phi - 1, \tau_n, \kappa) < 1 - \eta, \mathbf{H}(\Phi - 1, \tau_n', \nu) < 1 - \eta, \eta \in (0,1) \}$

$\geq \inf \{ \kappa + \nu : \mathbf{M}(\Phi - 1, \tau_n + \tau_n', \kappa + \nu) \geq \eta, \mathbf{P}(\Phi - 1, \tau_n + \tau_n', \kappa + \nu) \geq \eta$ and

$\mathbf{H}(\Phi - 1, \tau_n + \tau_n', \kappa + \nu) < 1 - \eta, \eta \in (0,1) \}$

$= \inf \{\omega : \mathbf{M}(\Phi - 1, \tau_n + \tau_n', \omega) \geq \eta, \mathbf{P}(\Phi - 1, \tau_n + \tau_n', \omega) \geq \eta$ and

$\mathbf{H}(\Phi - 1, \tau_n + \tau_n', \omega)\ 1 - \eta, \eta \in (0,1)\}, \omega = \kappa + \nu$

$= \| \Phi - 1, \tau_n + \tau_n' \|_\eta$.

4. AUTOMATA NCS

This section endeavors to integrate automata theory with Neutrosophic n- normed linear space (NCS) in the subsequent manner.

Definition 4.1. A deterministic finite automata *NCS* is the quintuple $M = (\mathrm{N}_i(E,t), \Sigma, \theta, \mathrm{N}_1(E,t), \mathrm{F})$, where $\mathrm{N}_i(E,t) = \{\mathrm{N}_i(E,t)/i = 1,2,\dots,k\}$. That is, $\mathrm{N}_i(E,t)$ is the non-empty set of states of *NCS*.

Σ: collections of input symbols.

θ: $\mathrm{N}_i(E,t) \times \Sigma \times [0,1] \to \mathrm{N}_i(E,t)$ is the fuzzy changeover function.

$\mathrm{N}_1(E,t)$: starting state.

F: collections of final states and F is a subset of $\mathrm{N}_i(E,t)$.

Definition 4.2. A non-deterministic finite automata *NCS* is the 5-tuple $M = (\mathrm{N}_i(E,t), \Sigma, \theta, \mathrm{N}_1(E,t), \mathrm{F})$. Where

$\mathrm{N}_i(E,t) = \{\mathrm{N}_i(E,t)/i = 1,2,\dots,k\}$. That is, $\mathrm{N}_i(E,t)$is the non-empty set of states of *NCS*.

Σ: collections of input symbols.

θ: $\mathrm{N}_i(E,t) \times \Sigma \times [0,1] \to 2^{\mathrm{N}_i(E,t)}$ is the fuzzy changeover function.

$\mathrm{N}_1(E,t)$: starting state.

F: collections of final states and F is a subset of $\mathrm{N}_i(E,t)$.

5. MATRIX FORM OF THE INPUT STRING

In this section we define the matrix form of the input string of finite automata *NCS*.

Definition 5.1. If the matrix form of the special changeover function of the deterministic finite automata *NCS* is defined as

$$T_M(NCS) = \theta\left(\mathrm{N}_i(E,t), a, \mathrm{N}_j(E,t)\right) = \begin{cases} (0,1], if\ \theta(\mathrm{N}_i(E,t), a) = \mathrm{N}_j(E,t) \\ 0, if\ \theta(\mathrm{N}_i(E,t), a) \neq \mathrm{N}_j(E,t) \end{cases}$$

Then $\theta = \begin{matrix} & \mathrm{N}_1 & \mathrm{N}_2 & \mathrm{N}_n \\ \mathrm{N}_1 & & & \\ \mathrm{N}_2 & & & \\ \mathrm{N}_m & & & \end{matrix} \begin{bmatrix} \theta_{11} & \cdots & \theta_{1n} \\ \vdots & \ddots & \vdots \\ \theta_{m1} & \cdots & \theta_{mn} \end{bmatrix}.$

Similarly we can define the matrix form of the input string of non-deterministic cases.

Example 5.2.

Consider the finite automata *NCS* $M = (\mathrm{N}_i(E,t), \Sigma, \theta, \mathrm{N}_1(E,t), \mathrm{F})$ where $\mathrm{N}_i(E,t) = \{\mathrm{N}_1, \mathrm{N}_2, \mathrm{N}_3, \mathrm{N}_4\}$, $\Sigma = \{a, b\}$, Starting state = N_1, F = $\{\mathrm{N}_3\}$ and the special changeover matrix θ of two different input strings are given by

$$\theta_{ij}(a) = \begin{array}{c} \\ \mathrm{N}_1 \\ \mathrm{N}_2 \\ \mathrm{N}_3 \\ \mathrm{N}_4 \end{array} \begin{array}{cccc} \mathrm{N}_1 & \mathrm{N}_2 & \mathrm{N}_3 & \mathrm{N}_4 \\ \end{array}$$

$$\theta_{ij}(a) = \begin{array}{c} \mathrm{N}_1 \\ \mathrm{N}_2 \\ \mathrm{N}_3 \\ \mathrm{N}_4 \end{array} \begin{bmatrix} [0.4,0.2,0.6] & [0,0.4,0.9] & [0.9,0.1,0.6] & [0,0,0] \\ [0.7,0.4,1.0] & [0,0,0] & [0,0,0] & [0.5,0.8,0.1] \\ [0.8,0.3,0.5] & [0,0,0] & [0,0,0] & [0,0,0] \\ [0,0,0] & [0,0,0] & [0.1,0.2,0.3] & [0,0,0] \end{bmatrix}$$

$$\theta_{ij}(b) = \begin{array}{c} \mathrm{N}_1 \\ \mathrm{N}_2 \\ \mathrm{N}_3 \\ \mathrm{N}_4 \end{array} \begin{bmatrix} [0,0,0] & [0,0.4,0.9] & [0.4,0,0.3] & [0.3,0.4,0.9] \\ [0.5,0.5,1.0] & [0,0,0] & [0,0,0] & [0.6,0.1,0.3] \\ [0.7,0.1,0.1] & [0,0,0] & [0,0,0] & [0,0,0] \\ [0,0,0] & [0.2,0.4,0.7] & [0.3,0.5,0.6] & [1.0,0,0.2] \end{bmatrix}$$

(columns $\mathrm{N}_1, \mathrm{N}_2, \mathrm{N}_3, \mathrm{N}_4$)

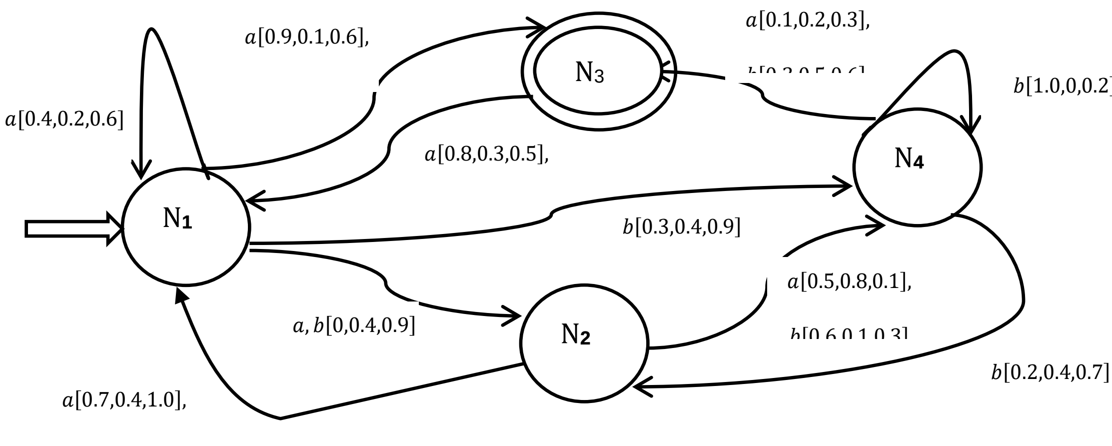


Figure 1: Changeover diagram for finite automata NCS

6. OPERATIONS ON MATRIX FORM OF THE STRING

In this section, we demonstrate various operations performed on the matrix representation of the string..

Definition 6.1. Matrix of concatenation of the string

Let $\theta_{ij}(a)$ and $\theta_{ij}(b)$ be the special changeover matrix of the finite automata *NCS*. The special changeover matrix of concatenation of the string is defined as

$$\theta_{ij}(ab) = \theta_{ij}(a \cup b) = \begin{cases} [\mathsf{M}\,\mathbf{min}(\bullet), \mathsf{P}\,\mathbf{max}(\bullet), \mathsf{H}\,\mathbf{max}(\bullet)], & \theta_{ij}(a), \theta_{ij}(b) \neq [0,0,0] \\ 0, & \theta_{ij}(a) \text{ or } \theta_{ij}(b) = [0,0,0] \end{cases}$$ for all i and j.

Example 6.2. From example 5.2, the matrix of the concatenation of the strings a and b is

$$\theta_{ij}(ab) = \begin{matrix} \\ N_1 \\ N_2 \\ N_3 \\ N_4 \end{matrix} \begin{matrix} N_1 & N_2 & N_3 & N_4 \\ \left[\begin{matrix} [0,0.2,0.6] & [0,0.4,0.9] & [0.4,0.1,0.6] & [0,0,0] \\ [0.5,0.5,1.0] & [0,0,0] & [0,0,0] & [0.5,0.8,0.3] \\ [0.7,0.3,0.5] & [0,0,0] & [0,0,0] & [0,0,0] \\ [0,0,0] & [0,0,0] & [0.1,0.5,0.6] & [0,0,0] \end{matrix}\right] \end{matrix}$$

Definition 6.3. Matrix of addition of the string

Let $\theta_{ij}(a)$ and $\theta_{ij}(b)$ be the special changeover matrix of the finite automata *NCS*. The special changeover matrix of addition of the string is defined as

$$\theta_{ij}(a+b) = \theta_{ij}(a \cup b) = \begin{cases} [\mathsf{M}\,\mathbf{max}(\bullet), \mathsf{P}\,\mathbf{min}(\bullet), \mathsf{H}\,\mathbf{min}(\bullet)], & \theta_{ij}(a), \theta_{ij}(b) \neq [0,0,0] \\ 0, & \theta_{ij}(a) \text{ or } \theta_{ij}(b) = [0,0,0] \end{cases}$$ for all i and j.

Example 6.4. From example 5.2 and 6.2, the matrix of the addition of the strings a and ab is

$$\hat{\hat{\delta}}_{ij}(a+ab) = \begin{matrix} \\ N_1 \\ N_2 \\ N_3 \\ N_4 \end{matrix} \begin{matrix} N_1 & N_2 & N_3 & N_4 \\ \left[\begin{matrix} [0.4,0.2,0.6] & [0,0.4,0.9] & [0.9,0.1,0.6] & [0,0,0] \\ [0.5,0.5,1.0] & [0,0,0] & [0,0,0] & [0.5,0.8,0.1] \\ [0.8,0.3,0.5] & [0,0,0] & [0,0,0] & [0,0,0] \\ [0,0,0] & [0,0,0] & [0.1,0.2,0.3] & [0,0,0] \end{matrix}\right] \end{matrix}$$

7. APPLICATION

This section commences with the presentation of an algorithm for identifying the optimal finite automata NCS, elucidated through a relevant example.

Algorithm 7.1.

Step 1: Define the finite automata NCS.

Step 2: Construct the matrix of the input strings.

Step 3: Determine the set of accepted strings.

Step 4: Create the matrix of the accepted strings, employing the matrix operation of concatenation of the strings.

Step 5: Eliminate any redundant paths of the accepted strings.

Step 6: Identify the optimal machine from the assumed machine.

Example 7.2.

Problem statement

In the digital realm, all tasks are interconnected within a single network. This poses a challenge for search engines, which may encounter complexities. To streamline the process and mitigate these complications, we implement a strategy to eliminate redundant searches and identify an optimal search engine.

Step 1:

We designate the strings concluding with a or b, utilizing the finite automata Neutrosophic n- normed linear space (NCS) as illustrated in Example 5.2.

Step 2: The matrix form of the input strings are

$$\theta_{ij}(a) = \begin{array}{c} \\ N_1 \\ N_2 \\ N_3 \\ N_4 \end{array} \begin{array}{c} \begin{array}{cccc} N_1 & N_2 & N_3 & N_4 \end{array} \\ \begin{bmatrix} [0.4,0.2,0.6] & [0,0.4,0.9] & [0.9,0.1,0.6] & [0,0,0] \\ [0.7,0.4,1.0] & [0,0,0] & [0,0,0] & [0.5,0.8,0.1] \\ [0.8,0.3,0.5] & [0,0,0] & [0,0,0] & [0,0,0] \\ [0,0,0] & [0,0,0] & [0.1,0.2,0.3] & [0,0,0] \end{bmatrix} \end{array}$$

$$\theta_{ij}(b) = \begin{array}{c} \\ N_1 \\ N_2 \\ N_3 \\ N_4 \end{array} \begin{array}{c} \begin{array}{cccc} N_1 & N_2 & N_3 & N_4 \end{array} \\ \begin{bmatrix} [0,0,0] & [0,0.4,0.9] & [0.4,0,0.3] & [0.3,0.4,0.9] \\ [0.5,0.5,1.0] & [0,0,0] & [0,0,0] & [0.6,0.1,0.3] \\ [0.7,0.1,0.1] & [0,0,0] & [0,0,0] & [0,0,0] \\ [0,0,0] & [0.2,0.4,0.7] & [0.3,0.5,0.6] & [1.0,0,0.2] \end{bmatrix} \end{array}$$

Step 3: The set of accepted strings are strings that end with a or b.

Step 4: The matrix form of the accepted string ab is

$$\theta_{ij}(ab) = \begin{array}{c} \\ N_1 \\ N_2 \\ N_3 \\ N_4 \end{array} \begin{array}{c} \begin{array}{cccc} N_1 & N_2 & N_3 & N_4 \end{array} \\ \begin{bmatrix} [0,0.2,0.6] & [0,0.4,0.9] & [0.4,0.1,0.6] & [0,0,0] \\ [0.5,0.5,1.0] & [0,0,0] & [0,0,0] & [0.5,0.8,0.3] \\ [0.7,0.3,0.5] & [0,0,0] & [0,0,0] & [0,0,0] \\ [0,0,0] & [0,0,0] & [0.1,0.5,0.6] & [0,0,0] \end{bmatrix} \end{array}$$

Step 5: We remove the path of the string ab from the state N_1 to N_4, from the state N_4 to N_2 and from the state N_4 to N_4.

Step 6: The required optimum machine is

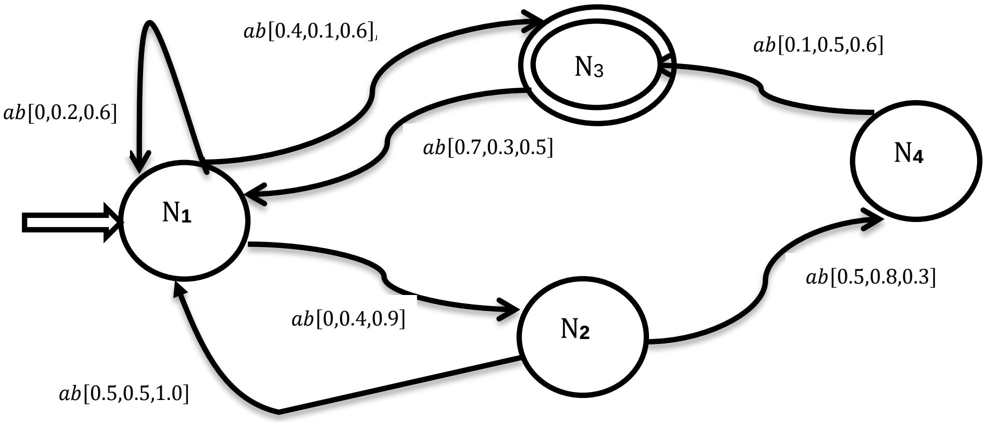


Figure 2: Changeover diagram for optimum finite automata NCS M

8. FUTURE RESEARCH DIRECTIONS

This structure namely neutrosophic n - NRC can be further generalized to neutrosophic n - Banach space. Further several operators can be constructed by using neutrosophic n - Banach space. There are lot of scope to form neutrosphic n - inner product space and it can be correlated with this new structure namely NCS.

9. CONCLUSION

This endeavor focuses on establishing the concept of a NCS as a natural extension of the cubic NRC. To facilitate better comprehension, a growth diagram of normed structures is provided. Application of NCS using automata theory is also provided. In addition to the previous section, our attention now turns towards deducing the open mapping theorem and closed graph theorem within the framework of NCS. Furthermore, we have intentions to derive the Hahn-Banach theorem for our novel structure in the near future.

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Chapter Six

NeutroSets, NetroRelations and It's Applications

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ABSTRACT

In this section, the concept of NeutroSet will be introduced for the first time using the concepts of NeutroAlgebra and Anti-Algebra defined by Smarandache. Then, the properties of this set structure and operations on sets have been studied by giving examples. Additionally, the definition of NeutroRelation, its properties, and results will be given with examples.

Keywords: Neutrosophic sets, Algebra, NeutroAlgebra, Anti-Algebra, Partial Algebra, Relation.

INTRODUCTION

The concept of fuzzy sets was first introduced by Zadeh in 1965[1], and since then this concept has been used in modelling many problems encountered in real life. In traditional fuzzy set logic, where X is a space and A is a subset of X, $\mu_A(x) \in [0,1]$ represents a single value. (where $\mu: X \longrightarrow [0,1]$ is the membership function of the fuzzy set). In some cases, the grade of membership itself is uncertain and difficult to define with a value. Thus, to eliminate the uncertainty of the grade of membership in fuzzy set logic, interval-valued fuzzy set logic was proposed [2]. Later, in 1986, Atasanov defined intuitionistic fuzzy sets, which are a generalization of these two concepts [3]. According to this definition, where $t_A(x)$ is the truth value of the membership grade and $f_A(x)$ is the false value of the membership grade, we have $0 \leq t_A(x) + f_A(x) \leq 1$ for $t_A(x), f_A(x) \in [0,1]$.

The concept of Neutrosophy was first defined by Smarandache in 1995[4]. In his paper a new branch of philosophy is defined, called neutrosophy, which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. According to this logic, any idea T is true, I is uncertain, and F is false; where T, I, F are standard or non-standard subsets included in the non-standard unit interval $]0^{-}, 1^{+}[$. Fuzzy set is used to tackle the uncertainty using the membership grade, whereas neutrosophic set is used to tackle uncertainty using the truth, indeterminacy and falsity membership grades which are considered as independent. Neutrosophic set constitutes a further generalisation of clasic sets, fuzzy sets, interval-valued fuzzy sets, intuitionistic fuzzy sets, picture fuzzy sets, Pythagorean fuzzy sets, and spherical fuzzy sets, amongst others. Since then, this logic has been applied in various domains of science and engineering. Later, as a result of this work, F Smarandache and her colleagues studied single-valued neutrosophic sets[5]. The logic of neutrosophy, which is used to solve many uncertain problems we encounter in daily life, attracts the attention of scientists in every field, and day by day, its effectiveness is being used in medicine, law, robot programming technique, artificial intelligence, engineering applications, sociology, psychology, etc. Its use in areas is becoming widespread. Many research treating imprecision and uncertainty have been developed and studied [15-33].

Recently, Florentin Smarandache generalized the classical Algebraic Structures to NeutroAlgebraic Structures and AntiAlgebraic Structures in 2019[6]. In another study[7], he proved that the NeutroAlgebra is a generalization of Partial Algebra. He considered < A > as an item (concept, attribute, idea, proposition, theory, etc.). Through the process of neutrosphication, he split the nonempty space and worked onto three regions two opposite ones corresponding to <A> and <AntiA>, and one corresponding to neutral (indeterminate) <NeutA> between the opposites, regions that may or may not be disjoint depending on the application, but their union equals the whole space.

A NeutroAlgebra is an algebra which has at least one NeutroOperation that is well-defined for some elements, indeterminate for others, and outer-defined for the others or one NeutroAxiom (axiom that is true for some elements, indeterminate for other elements, and false for the other elements). A Partial Algebra is an algebra that has at least one partial operation (well-defined for some elements, and indeterminate for other elements), and all its axioms are classical (i.e., the axioms are true for all elements). Through a theorem he proved that NeutroAlgebra is a generalization of Partial Algebra, and examples of NeutroAlgebras that are not partial algebras were given. Also, the NeutroFunction and NeutroOperation were introduced.

In recent studies on NeutroAlgebraic structures, Agboola, A. examined NeutroGroup and some of its properties[8]. Again, Agboola, A. expanded this group definition and defined the concept of NeutroRing in another study[9]. Later, Ibrahim, Muritala and colleagues defined the concept of NeutroVectorSpaces[10]. As a result, Şahin, M., and his colleagues defined the concept of Neutro-R Module[11] and later the concepts of Neutro-G Module and Anti-G Module[12]. However, Olgun, N and their colleagues also studied homomorphisms by defining the concept of Neutro Ordered R-module[13]. In recent years, the academic community has witnessed growing research interests in uncertainty set theory [34-64].

In the light of the above studies, it has been observed that NeutroSet has not been defined in the studies carried out so far. In the studies, the classical set definition was used and the algebraic structures on the operations defined on this set were examined. In this study, a definition of NeutroSet, which has not been made before, will be made and this concept will be introduced with examples. Then, the concept of NeutroRelation will be defined and its properties will be given with examples.

BACKGROUND

This section presents some basic definitions, results of the relations and the neutrosophy..

Definition 1. [14] : A and B are sets, the cartesian product of A and B is defined to be the set

$$A \times B = \{ (a,b) \mid a \in A \land b \in B \}$$

More generally if $A_1, A_2, A_3, \dots, A_n$ are sets we define their cartesian product by

$$A_1 \times A_2 \times A_3 \times \dots \times A_n = \{ (a_1, a_2, \dots, a_n) \mid a_1 \in A_{1,\dots,} a_n \in A_n \}.$$

Definition 2. [14] A unary relation on a set *A* is defined to be a subset of A.

Definition 3. [14] A n-ary relation on A , for $n > 1$, is a subset of the *n*-fold cartesian product of $A \times A \times \dots \times A$.

Notice that an n-ary relation on A is a unary relation on the n-fold product of $A \times A \times \dots \times A$.

This formal definition provide a concrete realization within set theory of the intuitive concept of a relation.

However, as is often the case in set theory, having seen how a concept may be defined set theoretically, we revert at once to the more familiar notation. For example, if R is some property that applies to pairs of elements of a set A we often speak of ''the binary relation R on A'', though strictly speaking the relation concerned is the set

$$\{ (a,b) \mid a \in A \land b \in A \land R(a,b) \}.$$

Also common is the tacit identification of such a property R with the relation it defines, so that $R(a,b)$ and $(a,b) \in R$ mean the same. Indeed, in the specific case of binary relation, It sometimes go even further, writing aRb instead of $R(a,b)$. In the case of ordering relations we rarely write $< (a,b)$ or $(a,b) \in <$ though from a set theoretic point of view, both could be said to be more accurate than the more common notation $< b$.

Binary relation play a particularly important role in set theory and, indeed in mathematics as a whole.

There are several properties that apply to binary relations.

Definition 4. [14] Let R be any binary relation on a set A. We say

R is reflexive if $(\forall a \in A)\quad (aRa)$;

R is symmetric if $(\forall a, b \in A)\quad (aRb \rightarrow bRa)$;

R is antisymmetric if $(\forall a, b \in A)\quad (aRb \wedge a \neq b) \rightarrow \neg(bRa)$;

R is connected if $(\forall a, b \in A)\quad (a \neq b) \rightarrow (aRb \vee bRa)$

R is transitive if $(\forall a, b, c \in A)\quad (aRb \wedge bRc) \rightarrow (aRc)$;

Definition 5. [14] A binary relation on a set is said to be an equivalence relation just in case it is reflexive, symmetric and transitive.

If R is an equivalence relation on a set A, the equivalence class of an element a of A under the equivalence relation R is defined to be the set

$$[a] = [a]_R = \{b \in A \ \mid \ aRb\}$$

Result: [14] Let R be an equivalence relation on a set A. Then R partitions A into a collection of disjoint equivalence classes.

Definition 6. [14] A partial ordering of a set A is a binary relation on a which is reflexive, antisymmetric and transitive. Usually (but not always) partial orderings are denoted by the symbol $\leq$.

A partially ordered set, or poset consist of a set A together with partial ordering $\leq$ of A. More formally, we define the poset to be ordered pair $(A, \leq)$.

Definition 7. [7] We recall that in neutrosophy we have for an item <A> , its opposite <antiA>, and in between them their neutral <neutA>.

We denoted by <nonA>=<neutA>U<antiA> , where U means union, and <nonA> means what is not<A> Or <nonA> is refined/split into two parts: <neutA> and <antiA>.

The neutrosophic triplet of<A> is: (<A>,<neutA>,<antiA>), with <nonA>=<neutA>U<antiA>.

Definition 8. [7] Let U be a universe of discourse, endowed with some well-defined laws, a non-empty set S ⊆U and an Axiom α, defined on S, using these laws. Then:

1) If all elements of S verify the axiom α, we have a Classical Axiom, or simply we say Axiom.

2) If some elements of S verify the axiom α and others do not, we have a NeutroAxiom (which is also called NeutAxiom).

3) If no elements of S verify the axiom α, then we have an AntiAxiom.

The Neutrosophic Triplet Axioms are: (Axiom, NeutroAxiom, AntiAxiom) with

NeutroAxiom ⋃ AntiAxiom = NonAxiom, and NeutroAxiom ⋂ AntiAxiom = φ (empty set), where ⋂ means intersection.

Theorem 9. [7] The Axiom is 100% true, the NeutroAxiom is partially true (its truth degree > 0) and partially false (its falsehood degree > 0), and the AntiAxiom is 100% false.

Theorem 10. [7]: Let d: {Axiom, NeutroAxiom, AntiAxiom} → [0 ,1] represent the degree of negation function. The NeutroAxiom represents a degree of partial negation {d ∊ (0, 1)} of the Axiom, while the AntiAxiom represents a degree of total negation {d = 1} of the Axiom.

We denote by <A>= Axiom; <neutA>= NeutroAxiom (or NeutAxiom); <antiA>= AntiAxiom; and <nonA>= NonAxiom in the Neutrosophic Representation.

Similarly, as in Neutrosophy, NonAxiom is refined/split into two parts: NeutroAxiom and AntiAxiom.

Definition 11. [7] Let U be a universe of discourse, and a non-empty set $\subseteq U$, endowed with a well-defined binary law $*$ on U. For any $x, y \in S$, one has $x * y \in S$. This is called Classical Binary Operation. If there exist at least two elements $a, b \in S$ such that $a * b \in S$ and there exist at least other two elements $c, d \in S$ such that $c * \mathrm{d} \notin S$ then it is called Neutro Defined Binary Operation.

Main result

NeutroSets

In this section we define a NeutroSet first time by using the definition of a single valued neutrosophic set and NeutroAlgebra.

Definition 12. Let X be a space of points (some objects) with a generic element in X denoted by x. We define a NeutroSet S in X with

< <S>, <Neut S>, <Anti S> >

where <S> is classic elements of S, <Neut S> is the partial elements (a truth- membership function T_S, an indeterminancy-membership function I_S and a falsity-membership function F_S) of S, and <Anti S> is the non-elements of S. <Neut S>= $\{(T_S(x), I_S(x),\ F_S(x)) : x \in S\ \}$

We know $T_S(x)$, $I_S(x)$ and $F_S(x)$ are real standart or non standart subsets of $]0^-, 1^+[$. that is

$T_S: X \longrightarrow]0^-, 1^+[$ $\quad I_S: X \longrightarrow]0^-, 1^+[$ and $F_S: X \longrightarrow]0^-, 1^+[$

There is no restriction on the sum of $T_S(x)$, $I_S(x)$ and $F_S(x)$ so,

$$0^- \le supT_S(x) + supI_S(x) + supF_S(x) \le 3^+$$

Example 13. Let our space X be pieces of iron placed on A4 paper. When a material with a magnetic effect is placed in the middle of the paper, we will see that some iron pieces are affected by this magnetism and move. Let's define the iron pieces that stick to the material with this magnetic effect as a classical set. Therefore, the definition of the classical set is insufficient to distinguish between iron pieces affected by this material and iron pieces that are not affected at all. In this case, if we consider the neutrosophic cluster logic for the less affected iron pieces, we can exactly model the space we are working in with the modeling we call neutro cluster. The difference between the set newly defined here and the Neutrosophic set is that the known operations of the set apply in the same way to the operations on elements that definitely belong to the set and those that do not definitely belong to the set. Here, neutrosophic logic is used for uncertain situations. The problem is modelled in Figure 1 below. In this Figure 1 , the entire A4 paper is modelled as the X space, the inner circle as the special material, the outer circle as the domain of the material, and the x_i 's as the iron pieces on the paper.

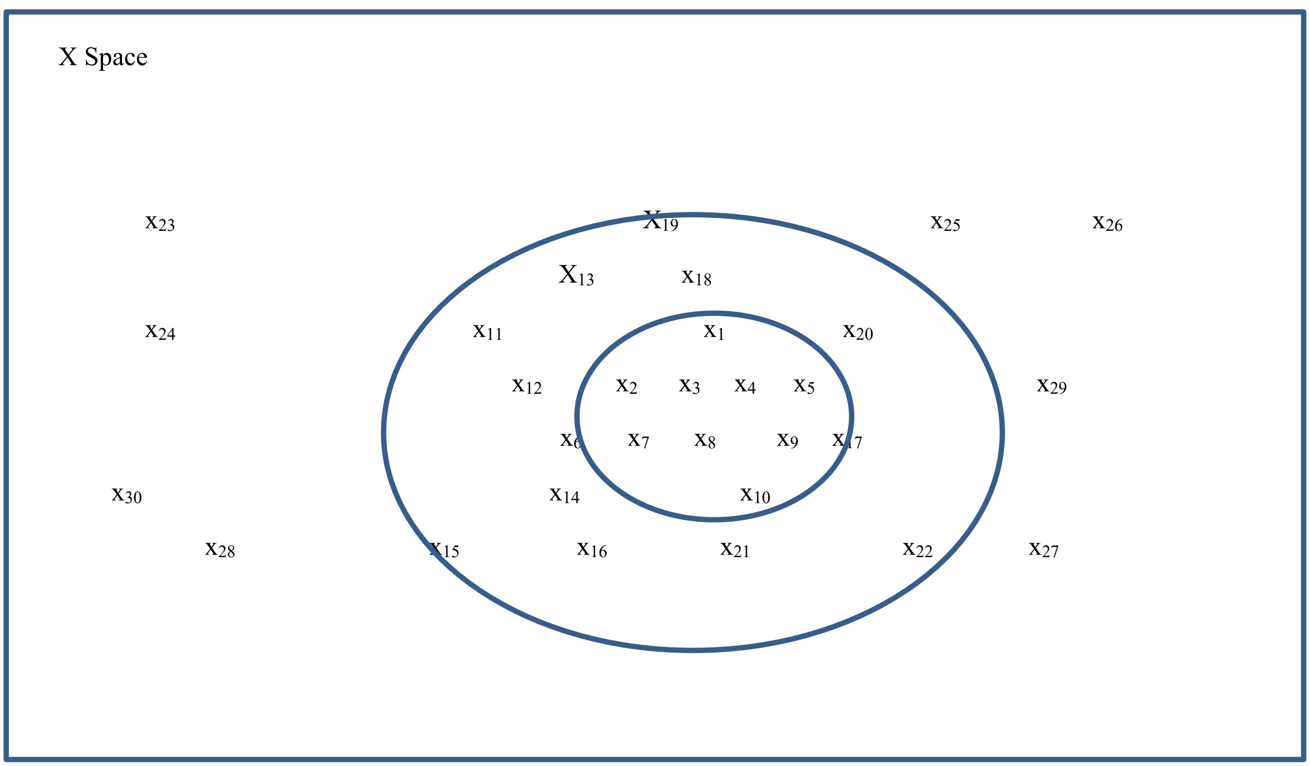

Figure 1

Let a set S in X space be defined as the set of elements that are affected, unaffected, or containing uncertainty by the material that has a magnetic effect. Then S NeutroSet is defined by

S= <<S Classical set>, <Neutrosophic Set>, <Anti S>>

$=<<x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}>, <(x_{11},0.4,0.7,0.3),(x_{12},0.6,0.5,0.2),(x_{13},0.5,0.5,0.3), (x_{14},0.7,0.4,0.1),(x_{15},0.2,0.2,0.7),(x_{16},0.6,0.5,0.1),(x_{17},0.5,0.5,0.2),(x_{18},0.6,0.4,0.3),(x_{19},0.2, 0.2,0.8),(x_{20},0.7,0.1,0,4), (x_{21},0.7,0.3,0.1),(x_{22},0.3,0.3,0.7 >,< x_{23}, x_{24}, x_{25}, x_{26}, x_{27}, x_{28}, x_{29}, x_{30} >>$

Definition 14. The complement of a NeutroSet S is denoted by S^C and defined by

$$< <AntiS>, <Neut^c\ S>, <S> >$$

where $<Neut^c\ S>=\{(1-T_S(x), 1-I_S(x),\ 1-F_S(x)) \ :\quad x \in S\ \}$.

Example 15. If NeutroSet S in Example 13 is taken, then the complement of a NeutroSet S is found as the set

$S^C=<< x_{23}, x_{24}, x_{25}, x_{26}, x_{27}, x_{28}, x_{29}, x_{30} >, <(x_{11},0.6,0.3,0.7),(x_{12},0.4,0.5,0.8),(x_{13},0.5,0.5,0.7), (x_{14},0.3,0.6,0.9),(x_{15},0.8,0.8,0.3),(x_{16},0.4,0.5,0.9),(x_{17},0.5,0.5,0.8),(x_{18},0.4,0.6,0.7),(x_{19},0.8, 0.8,0.2),(x_{20},0.3,0.9,0,6), (x_{21},0.3,0.7,0.9),(x_{22},0.7,0.7,0.3 >,<x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}>>$.

Proposition 16. The complement of the complement of a NeutroSet S is itself. That is $(S^C)^C=S$.

Proof: It is obtain the easily from the Definition 14 .

Definition 17. Let W and S be two NeutroSets. If

i) $< W > \subseteq < S >$ and $< Anti\ S > \subseteq < Anti\ W >$
ii) $infT_W(x) \leq infT_S(x), supT_W(x) \leq supT_S(x)$
iii) $infF_W(x) \leq infF_S(x), supF_W(x) \leq supF_S(x)$
then W is a NeutroSubset of the NeutroSet S.

Example 18. If NeutroSet S in Example 13 and $W=<< x_1, x_2, x_3, x_4, x_5>, <(x_{11},0.4,0.7,0.3),(x_{13},0.5,0.5,0.3), (x_{18},0.6,0.4,0.3),(x_{19},0.2,0.2,0.8),(x_{20},0.7,0.1,0,4)>,< x_6, x_7, x_8, x_9, x_{10}, x_{23}, x_{24}, x_{25}, x_{26}, x_{27}, x_{28}, x_{29}, x_{30}>>$ is taken, then it is obtain that W is a NeutroSubset of the NeutroSet S.

Definition 19. Let W and S be two NeutroSets. The union of two NeutroSets is defined by

$$< < W \cup S >,\ < Neut(W \cup S) >,\ < Anti(W \cup S) >$$

where
$$T_{(W \cup S)}(x) = \max\{ T_w(x), T_S(x)\}$$
$$I_{(W \cup S)}(x) = \max\{ I_w(x), I_S(x)\}$$

$$F_{(W \cup S)}(x) = \max\{F_w(x), F_S(x)\}$$

for all x in X.

Example 20. Let X={a,b,d,e,f,g,m,x,y,c,t,z,k,h,j} be any space and let W, S be NeutroSets of X. If

W=<<a,b,e,x,y>, <(c,0.5,0.4,0.3),(t,0.7,0.2,0.4),(z,0.3,0.4,0.6),(k,0.2,0.2,0.8)>,< d,f,g,m,>>

S==<<a,d,f,m,x,>, <(c,0.6,0.5,0.2),(h,0.4,0.6,0.3),(j,0.4,0.6,0.5),(k,0.6,0.3,0.3)>,< b,e,y,g>> are taken

then it is obtain that

Neutro($W \cup S$)=<<a,b,e,x,y,d,f,m>,<(c,0.6,0.5,0.4),(t,0.7,0.2,0.4),(z,0.3,0.4,0.6),(k,0.6,0.3,0.8),(h,0.4,0.6,0.3),(j,0.4,0.6,0.5)>,< g>>.

Corollary 21: $Anti(W \cup S)$ is not equal to $Anti(W) \cup Anti(S)$.

Proof: Looking at Example 20, this result can be easily obtained.

Definition 22. Let W and S be two NeutroSets. The intersection of two NeutroSets is defined by

$$<< W \cap S >, < Neut(W \cap S) >, < Anti(W \cap S) >$$

where
$$T_{(W \cap S)}(x) = \min\{T_w(x), T_S(x)\}$$

$$I_{(W \cap S)}(x) = \min\{I_w(x), I_S(x)\}$$

$$F_{(W \cap S)}(x) = \min\{F_w(x), F_S(x)\}$$

for all x in X.

Example 23. Let X={a,b,d,e,f,g,m,x,y,c,t,z,k,h,j} be any space and let W, S be NeutroSets of X. If

W=<<a,b,e,x,y>, <(c,0.5,0.4,0.3),(t,0.7,0.2,0.4),(z,0.3,0.4,0.6),(k,0.2,0.2,0.8)>,< d,f,g,m,>>

S==<<a,d,f,m,x,>, <(c,0.6,0.5,0.2),(h,0.4,0.6,0.3),(j,0.4,0.6,0.5),(k,0.6,0.3,0.3)>,< b,e,y,g>> are taken

then it is obtain that

Neutro($W \cap S$)=<<a,x>,<(c,0.5,0.4,0.2),(k,0.2,0.2,0.3)>,< b,d,e,f,g,m,y>>.

Corollary 24: $Anti(W \cap S)$ is not equal to $Anti(W) \cap Anti(S)$.

Proof: Looking at Example 23, this result can be easily obtained.

Properties Of NeutroSet Operations

Let A, B, C be NeutroSets in the universal NeutroSet U. Then we have

i) (Commutativity) $A \cup B = B \cup A$, $A \cap B = B \cap A$ and $A \times B = B \times A$.
ii) (Associativity) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$ and $A \times (B \times C) = (A \times B) \times C$.
iii) (Distributivity) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cap B = B \cap A$ and $A \times B = B \times A$.
iv) $A \cup A = A$, $A \cap A = A$
v) $A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$, $\cup U = U$, and $A \cap U = A$.
vi) (De Morgan Laws) $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$
vii) $(A^c)^c = A$.

The above properties are easily obtained if the definitions of neutro sets are used.

NeutroRelation

In this section, we define a NeutroRelation by using the NeutroSet.

Definition 25. : A and B are NeutroSets, the cartesian product of A and B is defined to be the NeutroSet

$$A \times B = \{ (a, b) \mid a \in A \land b \in B \}$$

More generally if $A_1, A_2, A_3, \ldots, A_n$ are sets we define their cartesian product by

$$A_1 \times A_2 \times A_3 \times \ldots \times A_n = \{ (a_1, a_2, \ldots, a_n) \mid a_1 \in A_{1, \ldots,} a_n \in A_n \}.$$

Definition 26. An a binary relation on a set *A* is defined to be a subset of $\times A$.

Definition 27. A n-ary relation on A , for $n > 1$, is a subset of the *n*-fold cartesian product of $A \times A \times \ldots \times A$.

Notice that an n-ary relation on A is a unary relation on the n-fold product of $A \times A \times \ldots \times A$.

Example 28. Let X={a,b,d,e,f,g,m,x,y,c,t,z,k,h,j} be any space and let A=<<a,b,e,x,y>, <(c,0.5,0.4,0.3),(t,0.7,0.2,0.4),(z,0.3,0.4,0.6),(k,0.2,0.2,0.8)>,< d,f,g,m,>> be NeutroSet of X. Then a NeutroRelation is defined any subset of the NeutroSubsets of $A \times A$.

Conclusions

In the studies on NeutroAlgebra carried out so far, the classical set definition has been used and the algebraic structures related to the operations defined on this set have been examined. In this study, NeutroSet was defined for the first time. Additionally, this concept and its results are introduced with examples. Then, the concept of NeutroRelationship is defined and its features are given with examples. By using these new concepts, some algebraic structures can be established and new studies can be carried out in the future.

Future Research Directions

The authors hope that the proposed the concept of NeotroSet can be applied to the definition of newly defined algebraic structures.

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Chapter Seven

A Review Hybrid Structure of Neutrosophy and Machine Learning Algorithms for Different Types of Problems

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Abstract

In recent years, machine learning has been widely used for regression and classification problems in engineering, finance, agriculture, image recognition, Natural Language Processing (NLP), health, etc. With this interest, many studies have been conducted to increase efficiency and examine the impact of different perspectives on the learning process. However, few papers exist on using Neutrosophy with machine learning algorithms. The data preprocessing step is important for artificial neural networks to process data. Neutrosophy adds a different perspective to the data preprocessing step and offers a highly effective solution for handling noisy, corrupted, incomplete, ambiguous data. Neutrosophic Logic (NL) labeled data as true (T), indeterminacy (I), and false (F) in the data preprocessing step to create a neutrosophic space. Then, the hybrid structure is achieved by applying the desired machine learning methods. Promising results have been obtained using the concept of neutrosophy with machine learning algorithms in the computer field. In this paper, a literature review was conducted to analyze how neurosophic is used with machine learning algorithms and for which type of problems.

Keywords: Neutrosophy, Machine learning, Artificial neural network.

Introduction

Several theories for uncertainty express situations where lack of information, truth, or falsity is uncertain. These theories are fuzzy, intuitionistic fuzzy, rough set, plithogenic sets, neutrosophy, etc. [1]. Neutrosophy was introduced by Florentin Smarandache in 1995.

Some statements are only partially true, not completely true for events and situations in our daily lives. This situation shows the opposite of the classical theorem, which states that it shows a 100% true situation in a space in any field of science [2]. Smarandache proposes the Neutro and Anti Theorem as an alternative to the classical theorem for any branch of science [3]. A neutrosophic set is a branch of neutrosophy that studies neutrality and the interaction of the ratios of neutrality. It calls the structure containing the relations and properties defined to determine the neutrosophic set of a frame, the resulting determination of which is called the Single-Valued Neutrosophic Set (SVNS). Neutrosophic sets are a generalization of intuitionistic fuzzy sets and fuzzy sets. While a Neutrosophic set (NS) corresponds to the general concept of sets in Neutrosophy, NL includes the concepts of true, false, and indeterminate, which are neutrosophic components. SVNS is a set obtained by systematically evaluating classical, fuzzy, and interval-valued fuzzy sets. With SVNS, properties of transactions and relationships are obtained [4].

When dealing with unsupervised data, ambiguities can be encountered, which can be dealt with in practice through the concept of neutrosophy as long as there is a relationship between the data. However, Fuzzy Cognitive Maps (FCMs) are needed for Neutrosophic Cognitive Maps (NCMs). NCMs are created using neutrosophic fields, graphs, vector spaces, and matrices [5].

The normalization, discretization, feature engineering, feature selection, noise reduction, outlier detection, normalization, missing value filling, and converting categorical values into numerical values that the algorithms can understand (many machine learning algorithms work with numerical values) [6]. From an engineering point of view, applying the neutrosophic set to data sets via set theory operators is realized in the data preprocessing step. Neutrosophic set theory can treat ambiguous, inconsistent, incomplete, ambiguous, and inaccurate datasets in computer science [7]. In deep learning, the results are usually labeled as true and false, while in neutrosophy, uncertainty is added to these situations, so the output layer must be set for three positions.

Data, referred to as big data, is collected through many environments and devices with the development of the internet and technology. Thanks to the development of storage structures and software, enormous amounts of data with more attributes can be stored. However, reducing the attributes has become necessary due to the processing costs and time requirements of many attributes, and it can be done via NS. In addition, considering it as a hybrid of the rough set theory proposed by Broumi [8] and the Rough Neutrosophic Set (RNS), it provides a better solution than NS in uncertain and insufficient data sets [9]. NL introduces uncertainty by interpreting the outcomes of a situation as the human brain interprets them. In recent years, the academic community has witnessed growing research interests in uncertainty set theory [46-90]. Neutrosophic logic differs from intuitionistic fuzzy logic by distinguishing between absolute and relative truth. A new parallelized filter feature technique based on rough neutrosophic set theory (Sp-RSNT) is proposed by [9], integrating NS and RNS to handle uncertain and missing data while using the feature reduction method. All NS, SVNS, NL, RNS, and Sp-RSNT help achieve promising results for some problems, efficiency, reducing loss, and accuracy.

Very few studies review Neutrosophy and machine learning papers [10]. The main contribution of this paper is to provide information on how to use Neutrosophic approaches for which problem types and in which machine learning steps, summarized in four different groups. This paper will contribute to the studies that research the solution to uncertainties, give an idea to researchers, and prevent wasting time; these groups answer the following questions.

1. Which Neutrosophic methods are used in data preprocessing methods in the literature?
2. Which type of machine learning is used in the literature with neutrosophy?
3. Which types of methods like Neutrosophy, Neutrosophy with statistical methods, and Neutrosophy with machine learning are used in the literature?
4. Which type of problems are used neutrosophy or machine learning with neutrosophy?

The rest of this paper is organized as follows: In the 'Literature' section, reviewed studies that use the Neutrosophy approach to machine learning algorithms and the contribution of their results. Section 'Conclusion' includes recommendations and general conceptions.

LITERATURE REVIEW

In the literature, numerous studies show that Neutrosophy plays an important role in complex disease diagnosis with small data samples where machine learning methods are not used. A few of them, for example, use the neutrosophy refined (multiple) method with several rows in a table labeled T, F, and I [11]. Another study was conducted on a case study with a three-way decision model by creating Single-Valued Neutrosophic Probabilistic Rough Multisets (SVNPRMs) on two universes using SVNM and probabilistic rough sets (PRS) together [12], and a few additional studies are [13], [14].

Neutrosophy with machine learning examples is mainly used in medical image detection for disease diagnosis [15]. Still, since there are cases where the disease is unclear, applying Neutrosophy has provided an advantage to machine learning methods in this problem. They used a multi-attribute group decision-making approach for single-valued triangular neutrosophic numbers (TNNs) [16] in combination with the Convolutional Neural Networks (CNN) method from Deep CNN (DCNNs) for classification and image segmentation of image data of a skin disease called melanoma [17].

The study used a neutrosophic set and theory to convert medical images for diagnosing COVID 19 and different types of viruses from the grayscale image to the neutrosophic domain. They labeled the images as true, false, and ambiguous. They then experimented with these labeled data in Deep Transfer Learning (DTL) models on Restnet18, Googlenet, and Alexnet, achieving 87.1% accuracy, suggesting using neutrosophic sets [18]. Additionally, [18] using neutrosophy achieved 1% better results in classification error than [19] not using neutrosophy. In [20], the authors use spatial and neutrosophic descriptors with pre-trained network parameters (VGGNet, GoogleNet, AlexNet, ResNet, and DenseNet) as feature extractors with the CNN method. Then, using Long Short-Term Memory (LSTM) and Bi-directional LSTM (Bi-LSTM) network layers, they train the classification problem and obtain the results. The paper results from the proposed system are 96.3% accurate and 95.75% precise.

Extract features using grayscale images because the image is too big. Adding neutrosophic closeness values [-1.1] to regularly distributed inputs ensured anomaly and linear separability. Then, Neutrosophic Support Vector Machine (N-SVM) was used to account for undefined or outliers input and compare the results of a conventional Support

Vector Machine (SVM). So, their N-SVM classification accuracy results outperform SVM [21].

They reformulated SVM for neutrosophy and proposed a solution to the sensitivity of SVM to noise and outliers. Their results show that their proposed SVM outperforms the traditional SVM in classification accuracy and Matthews correlation coefficient (MCC) [22].

They [23] proposed a deep neural network base consisting of residual blocks with a softmax block after each block, segmented the image using the single-valued pentagonal neutrosophic number (SVPNN) method, and labeled malignant or non-malicious using machine learning. When they compared their proposed segmentation and classification method with K-Nearest Neighbor (KNN), YOLO, Decision Tree (DT), SVM, Multilayer Perceptron (MLP), Random Forest (RF), Bayesian Network (BN), and Naive Bayes (NB) algorithms, they obtained better results accuracy. Results found that the accuracy increased from 91% to 99.50% for the PH2 dataset, from 91.50% to 99.33% for ISIC 2017, from 90.53% to 98.56% for ISIC 2018, and from 90.35% to 99.04% for ISIC 2019 with the data preprocessing step in 4 datasets.

They used the Neutrosophic Graph Cut-based Segmentation (NGCS) method in the data preprocessing phase of cervical cancer data. They used the Neutrosophic C-Means Clustering Technique (NCMCT) to determine uncertainty membership with NGCS. As a result, they achieved better accuracy than the traditional Graph-cut technique by using Principal Component Analysis (PCA) on the dataset and SVM for classification [24].

With NS, the noisy data for each pixel is labeled as uncertainty, thus creating a two-path network, which is also used in N-CNN. Weights were updated by merging these two parallel paths in CNN. They concluded that this new N-CNN resulted in better results than the traditional CNN [25].

Detecting anomalies in time series data collected with Industrial Internet of Things (IIoT) tools can offer a solution to both security problems and problems caused by the size of the data. A heuristic-based neutrosophic model is proposed in [26] for the anomaly detection problem. In the proposed model, neutrosophy data preprocessing is used to widen the difference between abnormalities and normals, and the data set is processed so that the data set is represented by T, F, and I in the neutrosophic matrix. In the data preprocessing step, the normal distribution of the data was obtained with the neutrosophy method in the multi-feature data space. Since the data set is a time series, they conducted experiments using variable-length time windows. They proved that better results are obtained in anomaly detection with the unsupervised structure they call Time2Event.

The hybrid structure of machine learning models with neutrosophy is used in sentiment analysis problems, especially [27], [28], [29]. Neutrosophic sentiment analysis models NLP, speech, and text sentiment, and it will also contribute to the studies when researching the solution to contain uncertainties [30]. Sentiment analysis tools divide posts on social media into two groups, labeled as positive and negative. Neutrosophic helps to understand social media better by adding a third status, which is neutral [28]. They proposed the concept of multi refined neutrosophic set (MRNS) and added the concepts of strong, weak, and uncertain to the existing concepts of T, F, and I by adding three elements for T and F. Thus, the set has seven features with different proportions of values [29].

In the preprocessing step, they [31] used Pre-trained Language Models (PLMs) (Bidirectional Encoder Representations from Transformers (BERT), A Lite BERT (ALBERT), A Robustly Optimized BERT Approach (RoBERTa), and MPNet) and Bi-LSTM. After that, using SVNS values, they defined a membership function for each emotion as neutral, aggressive, and non-aggressive. They used clustering techniques, the Gaussian Mixture Model (GMM), and k-means. With their proposed model, they identified positive and negative extremes. They marked the data at a certain distance as neutral in K-means, saving time and resources and finding a result equivalent to the most recently developed models.

The analytic network process (ANP), a generalization of the Analytic Hierarchy Process (AHP), is called Neutrosophic ANP (N-ANP) using SVNSs. In data preprocessing, they created a network structure that captures the complex interdependencies and interrelationships between attributes from the questionnaires stored in matrices. To obtain a suitable benchmark for the entities and NL and Multi-Criteria Decision-Making (MCDM), using self-attribute reduction associated with multi-attribute utility theory (MAUT), this data set used decision trees (DT), K-Nearest Neighbors (KNN) and NB algorithms [32]. They used a drone selection problem, MCDM, with decision-making applications for Neutrosophic, Evaluation of Mixed Data (EVAMIX), and CRITIC [33].

They [34] proposed the Neutrosophic Gamma Distribution (NGD) model for dealing with uncertain statistical datasets since gamma distribution is insufficient in some applications when dealing with uncertain data. They constructed an estimation framework to handle the uncertain parameters of the NGD, analyzed it with cooling system downtime data, and evaluated the Monte Carlo simulation. They concluded that the NGD is more flexible than the gamma distribution.

Linear regression has traditionally been widely used in many fields with qualitative data. Linear regression, one of the regression analyses used in machine learning, is used when the data consists of one or more predictors and independent variables. The dependent variable or variables are predicted using independent variables [35].

They introduced the concept of correlation and correlation coefficients for the uncertainty and imprecision of data using neutrosophic clusters [36]. Using a linear regression model, they then applied these coefficients and relationships to neutrosophic data [37].

They prepared the dataset unsupervised with NCM and FCM, worked on many different example problems, and did not use machine learning methods [5].

The most widely used machine learning methods with Neutrosophy papers are CNN [20], [25]; LSTM [20]; Support Vector Machine (SVM) [10], [21], [24], [22], [38]; N-SVM [21], [22]; NB [32]; Decision Trees (DT) [32]; MLP [39]; K-NN classifier [10], [32], [38]; Bi-LSTM [28], [31], [40]; K-Means [27], [31]; Gaussian Mixture Model (GMM) [31]; Gated Recurrent Units (GRU) [28], CNN Bi-LSTM [41], Bidirectional Encoder Representations from Transformers (BERT) [40], a multivalued neutrosophic convolutional LSTM (MVN-ConvLSTM) [42], A Lite BERT (ALBERT) [40], A Robustly Optimised BERT Approach (RoBERTa) [40], Masked and Permuted Pre-training for Language Understanding (MPNet) [40].

Table 1: Preprocessing methods of Neutrosophy

Data preprocessing methods	Type of Neutrosophy
Classification	NS and NL [7], TNN [17]
Reducing the size of attributes through attribute selection	Neutrosophic Cognitive Maps (NCM) [5], Rough Neutrosophic Set Theory (Sp-RSNT) [9], NL- MAUT [32], MRSS [29]
Handle noisy, corrupted, incomplete, and ambiguous data	RNS [8], SVNS [4], [31], hybrid RNS-SVNS [43],N-ANP [32]
Image segmentation	NGCS and NCMCT [24], Triangular Neutrosophic Number (TNN) [17], SVPNN and PNN [23]
Regression	Neutrosophic regression [37]

In Table 1, studies that include examples of the use of neutrosophy in the data preprocessing step are collected.

Table 2: Type of machine learning

Type of machine learning	
unsupervised	[5], [17], [20], [25], [41]
supervised	[18], [21], [23], [24], [31], [32], [39], [40]
Regression	

In Table 2, different learning algorithms are used depending on whether the data used, if labled is called supervised learning or unlabled is unsupervised learning Regression is used when the data are floating numbers.

Table 3: Type of hybrid methods

Type of methods	
Neutrosophy	[11], [12], [13], [15], [17], [19], [26], [41]
Neutrosophy with statistical methods	[14], [34]
Neutrosophy with machine learning	[17], [18], [21], [23], [25], [28], [31], [32], [40], [41]

Table 3 lists the papers where neutrosophy is used alone and hybrid with machine learning or statistical methods.

Table 4: Type of problems

Type of problem	Neutrosophy	Neutrosophy with machine learning
Medical applications	[11], [12], [13], [14], [15]	[17], [18], [20], [21], [23], [24], [39]
Image processing	[17], [19], [24]	[21], [23], [24], [25], [39]
Time series dataset	[26]	
Sentiment analysis (NLP, speech, and text)	[27], [28], [29]	[31], [40], [41]
Decision support systems	[33]	[32]
Anomaly detection	[26]	
IIoT		[26], [38]
Instruction detection		[42]
Discrimination		[22]
Regression	[44], [45]	

In Table 4 applying Neutrosophy to problems such as medical applications, image processing, time series dataset, instruction detection, sentiment analysis (NLP, speech and text), decision support systems, anomaly detection, IIoT and regression are grouped according to whether machine learning is used or not.

CONCLUSIONS

Neutrosophy offers a remarkable solution as it provides a new perspective in data preprocessing and problem solving for many problems that can be solved with machine learning. It is an effective method as it surpasses traditional methods in many applications. This paper examines the use of Neutrosophic in machine learning algorithms at which stage and for which problems. These problems are medical applications image processing, time series dataset, sentiment analysis (NLP, speech, and text), decision support systems, anomaly detection, IIoT, instruction detection, discrimination, and regression. It has been applied in the data preprocessing step and by using it in to a few machine learning algorithms. In particular, different types of machine learning algorithms such as SVM, CNN, K-NN, MLP and LSTM are used.

Abbreviations

AHP: Analytic Hierarchy Process

ANP: Analytic Network Process

ALBERT: A Lite Bidirectional Encoder Representations from Transformers

BERT: Bidirectional Encoder Representations from Transformers

Bi-LSTM: Bi-directional Long Short-Term Memory

BN: Bayesian Network

CNN: Convolutional Neural Network

DCNNs: Deep Convolutional Neural Networks

DT: Decision Tree

DTL: Deep Transfer Learning

EVAMIX: Evaluation of Mixed Data

F: False

FCMs: Fuzzy Cognitive Maps

GMM: Gaussian Mixture Model

GRU: Gated Recurrent Units

I: Indeterminacy

IIoT: Industrial Internet of Things

KNN: K-Nearest Neighbors

LSTM: Long Short-Term Memory

MCC: Matthews correlation coefficient

MCDM: Multi-Criteria Decision-Making

MLP: Multilayer Perceptron

MPNet: Masked and Permuted Pre-training for Language Understanding

MRNS: Multi Refined Neutrosophic Set

NB: Naive Bayes

NCMs: Neutrosophic Cognitive Maps

NCMCT: Neutrosophic C-Means Clustering Technique

NGCS: Neutrosophic Graph Cut-based Segmentation

NGD: Neutrosophic Gamma Distribution

NL: Neutrosophic logic

NLP: Natural Language Processing

NS: Neutrosophic set

N-SVM: Neutrosophic Support Vector Machine

PLMs: Pre-trained Language Models

RF: Random Forest

RoBERTa: Robustly Optimized BERT Approach

RNS: Rough Neutrosophic Set

Sp-RSNT: Parallelized Filter Feature Technique Based on Rough Neutrosophic Set Theory

SVM: Support Vector Machine

SVNPRMs: Single-Valued Neutrosophic Probabilistic Rough Multisets

SVNS: Single-Valued Neutrosophic Set

PCA: Principal Component Analysis

PRS: Probabilistic Rough Sets

T: True

TNNs: Triangular Neutrosophic Numbers

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Chapter Eight

A Decision-making Method under Trapezoidal Fuzzy Multi-Numbers Based on Centroid Point and Circumcenter of Centroids

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Abstract

In this study, we propose a novel decision-making method based on trapezoidal fuzzy multi-numbers and their different centroid points. The proposed method aims to handle decision problems involving multiple criteria or attributes, each characterized by fuzzy information in the form of trapezoidal fuzzy multi-numbers. To do this, we first give some basic notions and operations of trapezoidal fuzzy multi-numbers (TFM-numbers). Secondly, we give different centroid points of TFM-numbers including desired properties. Then, we give an algorithm to solve multi-criteria decision-making problems by using proposed centroid points under TFM-numbers. Finally, we give an application to show the usage of the method on a real-life problem with TFM-numbers.

Keywords: decision-making, fuzzy logic, trapezoidal fuzzy multi-numbers, centroid points, uncertainty.

1. Introduction

Fuzzy set theory, introduced by Zadeh [48] in 1965, extends the classical set theory to handle uncertain information. Then, it has been applied in various fields. For example, transportation planning [29], agro-industrial engineering, technology applications [21], education [32, 33], and law [6]. In time, some special types of fuzzy sets have been introduced such as fuzzy numbers with operations proposed and their relationships studied in [14]. An overview of works on fuzzy numbers provided and extended known operations of fuzzy sets given in [15]. The median method introduced to find the best solution for a transportation problem in [41]. Yun et al. [47] generalized triangular fuzzy numbers based on Zadeh's extension principle. Another examples on trapezoidal and triangular fuzzy numbers can be found in [1, 3, 7, 8, 11, 12, 13, 16, 22, 27, 28].

Due to the membership values of fuzzy sets being in [0,1], they may not provide complete information in some problems where each element can have different membership values. Therefore, a different generalization of fuzzy sets called multi-fuzzy sets (fuzzy bags) was introduced by Yager [44]. Then, Miyamoto [23, 24], Sebastian, and Ramakrishnan [34-36] further expanded Yager's concept to handle uncertainty. Also, there have been numerous studies on multi-fuzzy sets, such as [25, 26, 37-40, 42, 43]. In 2018, by using the real number set $\mathbb{R}$, as a universe set in fuzzy multi-sets, Ulucay et al. [44] developed trapezoidal fuzzy multi-numbers. Then, many authors have studied TFM-numbers. For example, on similarity measures [18, 30, 31, 45], on distance measures [13,18] and on aggregation operators [13,19,21]. As we know, no studies have been introduced on TFM-numbers related to centroid points. To fill this gap, we studied on TFM-numbers based on centroid points

2. Preliminaries

In this section, we give some basic notions related to fuzzy set, fuzzy number, fuzzy multi-set, and trapezoidal fuzzy multi-set which are needed for the rest of the paper.

Definition 2.1 [48] Let X be a non-empty set. A fuzzy set F on X is defined as follows:

$$F = \{(x, \mu_F(x)) : x \in X\}$$

where $\mu_F : X \longrightarrow [0,1]$ for $x \in X$.

Definition 2.2 [17] Let $\eta_A \in [0,1]$ and $a, b, c, d \in \mathbb{R}$ such that $a \leq b \leq c \leq d$. Then, a generalized trapezoidal fuzzy number (GTF-number) $A = \langle (a, b, c, d); \eta_A \rangle$ is a special fuzzy set on the real number set $\mathbb{R}$, whose membership functions are defined as follows:

$$\mu_A(x) = \begin{cases} (x-a)\eta_A/(b-a) & a \leq x < b \\ \eta_A & b \leq x \leq c \\ (d-x)\eta_A/(d-c) & c < x \leq d \\ 0 & otherwise \end{cases}$$

If $\eta_A = 1$, then A is called a trapezoidal fuzzy number and denoted by $A = \langle (a, b, c, d) \rangle$.

Definition 2.3 [9] Let $A = \langle (a, b, c, d); \eta_A \rangle$ be a GTF-number with its membership function $\eta_A(x)$. Centroid point of A is denoted by $\mathrm{C(x(A), y(A))}$ and given as follows:

$$x(A) = \frac{\int_a^b x \frac{(x-a)\eta_A}{(b-a)} dx + \int_b^c \eta_A x dx + \int_c^d x \frac{(d-x)\eta_A}{(d-c)} dx}{\int_a^b \frac{(x-a)\eta_A}{(b-a)} dx + \int_b^c \eta_A dx + \int_c^d \frac{(d-x)\eta_A}{(d-c)} dx},$$

$$y(A)=\frac{\int_0^{\eta_A} y\frac{y(b-a)+a\eta_A}{\eta_A}dy-\int_0^{\eta_A} y\frac{d\eta_A-(d-c)\eta_A}{\eta_A}dy}{\int_0^{\eta_A} \frac{y(b-a)+a\eta_A}{\eta_A}dy-\int_0^{\eta_A} \frac{d\eta_A-(d-c)\eta_A}{\eta_A}dy}$$

Theorem 2.4 [9] Let $A=\langle(a,b,c,d);\eta_A\rangle$ be a GTF-number. Centroid point of *A,* $C(x(A),y(A))$, computed as follows:

$$x(A)=\frac{(c^2+d^2-a^2-b^2+cd-ab)}{3(c+d-a-b)},\qquad y(A)=\frac{\eta_A(2b+a-d-2c)}{3(b+a-d-c)}$$

Definition 2.5 [10] Let $A=\langle(a,b,c,d);\eta_A\rangle$ be a GTF-number. Score of A, denoted by s(A), is defined as follows:

$$s(A)=x(A).y(A)$$

where

$$x(A)=\frac{\int_a^b x\frac{(x-a)\eta_A}{(b-a)}dx+\int_b^c \eta_A x dx+\int_c^d x\frac{(d-x)\eta_A}{(d-c)}dx}{\int_a^b \frac{(x-a)\eta_A}{(b-a)}dx+\int_b^c \eta_A dx+\int_c^d \frac{(d-x)\eta_A}{(d-c)}dx},$$

$$y(A)=\frac{\int_0^{\eta_A} y\frac{y(b-a)+a\eta_A}{\eta_A}dy-\int_0^{\eta_A} y\frac{d\eta_A-(d-c)\eta_A}{\eta_A}dy}{\int_0^{\eta_A} \frac{y(b-a)+a\eta_A}{\eta_A}dy-\int_0^{\eta_A} \frac{d\eta_A-(d-c)\eta_A}{\eta_A}dy}$$

Definition 2.6 [34] Let X be a non-empty set. A fuzzy-multi set G on X is defined as follows:

$$G=\{(x,\mu_G^1(x),\ \mu_G^2(x),\dots,\mu_G^i(x),\dots):x\in X\}$$

where $\mu_G^i: X\rightarrow[0,1]$ for all i ∈ {1, 2, ..., p} and $x\in X$.

Definition 2.7 [44] Let $\eta_A^i\in[0,1]$ ($i\in\{1,2,\dots,T\}$) and $a,b,c,d\in\mathbb{R}$ such that $a\le b\le c\le d$. Then, a trapezoidal fuzzy multi-number (TFM-number) $A=\langle(a,b,c,d);\eta_A^1,\eta_A^2,\dots,\eta_A^T\rangle$ is a special fuzzy multi-set on the real number set ℝ, whose membership functions are defined as follows:

$$\mu_A^i(x)=\begin{cases}\frac{(x-a)}{(b-a)}\eta_A^i, & a\le x<b\\ \eta_A^i, & b\le x\le c\\ \frac{(d-x)}{(d-c)}\eta_A^i, & c<x\le d\\ 0, & otherwise,\end{cases}$$

Definition 2.8 [44] Let $A_1 = \langle (a_1, b_1, c_1, d_1); \eta_{A_1}^1, \eta_{A_1}^2, \dots, \eta_{A_1}^T \rangle$ and

$A_2 = \langle (a_2, b_2, c_2, d_2); \eta_{A_2}^1, \eta_{A_2}^2, \dots, \eta_{A_2}^T \rangle$ be two TFM-numbers and $\gamma \geq 0$ be any real number. Then,

i. $A_1 + A_2 = \langle (a_1 + a_2, b_1 + b_2, c_1 + c_2, d_1 + d_2);$

$$\eta_{A_1}^1 + \eta_{A_2}^1 - \eta_{A_1}^1 \eta_{A_2}^1, \eta_{A_1}^2 + \eta_{A_2}^2 - \eta_{A_1}^2 \eta_{A_2}^2, \dots, \eta_{A_1}^T + \eta_{A_2}^T - \eta_{A_1}^T \eta_{A_2}^T \rangle$$

ii. $A_1 . A_2 =$

$$\begin{cases} \langle (a_1 a_2, b_1 b_2, c_1 c_2, d_1 d_2); \eta_{A_1}^1 \eta_{A_2}^1, \eta_{A_1}^2 \eta_{A_2}^2, \dots, \eta_{A_1}^T \eta_{A_2}^T \rangle \ (d_1 > 0, d_2 > 0) \\ \langle (a_1 d_2, b_1 c_2, c_1 b_2, d_1 a_2); \eta_{A_1}^1 \eta_{A_2}^1, \eta_{A_1}^2 \eta_{A_2}^2, \dots, \eta_{A_1}^T \eta_{A_2}^T \rangle \ (d_1 < 0, d_2 > 0) \\ \langle (d_1 d_2, c_1 c_2, b_1 b_2, a_1 a_2); \eta_{A_1}^1 \eta_{A_2}^1, \eta_{A_1}^2 \eta_{A_2}^2, \dots, \eta_{A_1}^T \eta_{A_2}^T \rangle \ (d_1 < 0, d_2 < 0) \end{cases}$$

iii. $\gamma A_1 = \langle (\gamma a_1, \gamma b_1, \gamma c_1, \gamma d_1); 1 - (1 - \eta_{A_1}^1)^\gamma, 1 - (1 - \eta_{A_1}^2)^\gamma, \dots, 1 - (1 - \eta_{A_1}^T)^\gamma.$

iv. ${A_1}^\gamma = \langle (a_1^\gamma, b_1^\gamma, c_1^\gamma, d_1^\gamma); (\eta_{A_1}^1)^\gamma, (\eta_{A_1}^2)^\gamma, \dots, (\eta_{A_1}^T)^\gamma \rangle$

Definition 2.9 [44] Let $A = \langle (a, b, c, d); \eta_A^1, \eta_A^2, \dots, \eta_A^T \rangle$ be a TFM-number. Then,

i. If $a > 0$, A is called a positive TFM-number,
ii. If $d > 0$, A is called a negative TFM-number,
iii. If $a < 0$ and $d > 0$, A is called neither a positive nor negative TFM-number.

Throughout the paper, we will work on positive TFM-numbers.

Definition 2.10 [44] Let $A_i = \langle (a_i, b_i, c_i, d_i); \eta_{A_i}^1, \eta_{A_i}^2, \dots, \eta_{A_i}^T \rangle$ be a collection of TFM-numbers and $w = (w_1, w_2, \dots, w_n)^T$ their weight vector. Then, the trapezoidal fuzzy multi-geometric operator ($TFMWG$) is defined as follows:

$$TFMWG(A_1, A_2, \dots, A_n) = {A_1}^{w_1} \text{ x } {A_2}^{w_2} \text{ x} \dots \text{x } {A_n}^{w_n}$$

Theorem 2.11 [44] Let $A_i = \langle (a_i, b_i, c_i, d_i); \eta_{A_i}^1, \eta_{A_i}^2, \dots, \eta_{A_i}^T \rangle$ be a collection of TFM-numbers and $w = (w_1, w_2, \dots, w_n)^T$ their weight vector. Aggregated value by using the TFM weighted geometric ($TFMWG$) operator is also a TFM-number and computed as follows:

$TFMWG(A_1, A_2, \dots, A_n)$

$$= \langle \left(\prod_{i=1}^{n} {a_i}^{w_i}, \prod_{i=1}^{n} {b_i}^{w_i}, \prod_{i=1}^{n} {c_i}^{w_i}, \prod_{i=1}^{n} {d_i}^{w_i} \right); \prod_{i=1}^{n} (\eta_{A_i}^1)^{w_i}, \prod_{i=1}^{n} (\eta_{A_i}^2)^{w_i}, \dots, \prod_{i=1}^{n} (\eta_{A_i}^T)^{w_i} \rangle$$

3. Some Centroid Point of TFM-numbers

In this section, we propose some novel centroid point and score definitions of TFM-numbers for decision-making problems. Also, we give some properties of the definitions.

The following definition firstly proposed by Chu and Tsao [10] for trapezoidal fuzzy numbers and triangular fuzzy numbers. We extended the definition to TFM-numbers as follows:

Definition 3.1 Let $A = \langle (a, b, c, d); \eta_A^1, \eta_A^2, \ldots, \eta_A^T \rangle$ be a TFM-number. Centroid points of A is denoted $C(A)$, is given as

$$C(A) = (C_1(x_1(A), y_1(A)), C_2(x_2(A), y_2(A)), C_3(x_3(A), y_3(A)), C_4(x_4(A), y_4(A)))$$

where

$$x_i(A) = \frac{\int_a^b x\frac{(x-a)\eta_A^i}{(b-a)}dx + \int_b^c \eta_A^i x dx + \int_c^d x\frac{(d-x)\eta_A^i}{(d-c)}dx}{\int_a^b \frac{(x-a)\eta_A^i}{(b-a)}dx + \int_b^c \eta_A^i dx + \int_c^d \frac{(d-x)\eta_A^i}{(d-c)}dx}, (i = 1,2, \ldots, T),$$

$$y_i(A) = \frac{\int_0^{\eta_A^i} y\frac{y(b-a)+a\eta_A^i}{\eta_A^i}dy - \int_0^{\eta_A^i} y\frac{d\eta_A^i-(d-c)\eta_A^i}{\eta_A^i}dy}{\int_0^{\eta_A^i} \frac{y(b-a)+a\eta_A^i}{\eta_A^i}dy - \int_0^{\eta_A^i} \frac{d\eta_A^i-(d-c)\eta_A^i}{\eta_A^i}dy}, \ (i = 1,2, \ldots, T.$$

Theorem 3.2. Let $A = \langle (a, b, c, d); \eta_A^1, \eta_A^2, \ldots, \eta_A^T \rangle$ be a TFM-number. Centroid points of A, denoted by $C(A)$, is computed with

$$C(A) = (C_1(x_1(A), y_1(A)), C_2(x_2(A), y_2(A)), C_3(x_3(A), y_3(A)), C_4(x_4(A), y_4(A)))$$

where

$$x_i(A) = \frac{(c^2+d^2-a^2-b^2+cd-ab)}{3(c+d-a-b)}, \ y_i(A) = \frac{\eta_A^i(2b+a-d-2c)}{3(a+b-d-c)}, \ (i = 1,2, \ldots, T).$$

Definition 3.3 Let $A = \langle (a, b, c, d); \eta_A^1, \eta_A^2, \ldots, \eta_A^T \rangle$ be a TFM-number and $(C_1(x_1(A), y_1(A)), C_2(x_2(A), y_2(A)), C_3(x_3(A), y_3(A)), C_4(x_4(A), y_4(A)))$ be the centroid points of A. Then,

i. 1. Score function of A denoted by $S_1(A)$ is given as;

$$S_1(A) = \frac{\sum_{i=1}^T x_i(A) y_i(A)}{T}, (i = 1,2, \ldots, T)$$

where

$$x_i(A) = \frac{(c^2+d^2-a^2-b^2+cd-ab)}{3(c+d-a-b)},\ y_i(A) = \frac{\eta_A^i(2b+a-d-2c)}{3(a+b-d-c)},\ (i = 1,2,\dots,T).$$

The following definition firstly proposed by Cheng [9] for triangular fuzzy numbers and we extended the definition to TFM-numbers as;

ii. 2. Score function of A denoted by $S_2(A)$ is given as;

$$S_2(A) = \frac{\sum_{i=1}^{T}\sqrt{x^2{}_i(A)+y^2{}_i(A)}}{T}$$

where

$$x_i(A) = \frac{(c^2+d^2-a^2-b^2+cd-ab)}{3(c+d-a-b)},\ y_i(A) = \frac{\eta_A^i(2b+a-d-2c)}{3(a+b-d-c)},\ (i = 1,2,\dots,T).$$

The following method proposed by Bakar and Gegov [4] for trapezoidal fuzzy numbers and triangular fuzzy numbers. We extended the method to TFM-numbers as:

iii. 3. Score function of A denoted by $S_3(A)$ is given as;

$$S_3(A) = dist(A_x) \times dist(A_y)$$

where

$dist(A_x) = |d - x_i(A)| + |x_i(A) - a|$ and shows spreading of A horizontally and

$dist(A_y) = \frac{\sum_{i=1}^{T} y_i(A)}{T}$ and it shows the spreading of A vertically.

Here, since $a \leq x_i(A) \leq d$, we get

$$dist(A_x) = |d - x_i(A)| + |x_i(A) - a| = d - x_i(A) + x_i(A) - a = d - a$$

Property 3.4 Let $A = \langle(a,b,c,d);\eta_A^1,\eta_A^2,\dots,\eta_A^T\rangle$ and $B = \langle(a,b,c,d);\eta_B^1,\eta_B^2,\dots,\eta_B^T\rangle$ be two TFM-numbers. If $\eta_A^i > \eta_B^i$ $(i = 1,2,\dots,T)$, then $S_3(A) > S_3(B)$.

Proof: Since $dist(A_x) = |d - x_i(A)| + |x_i(A) - a| = d - a$ and

$dist(B_x) = |d - x_i(B)| + |x_i(B) - a| = d - a$, we have

$$dist(A_x) = dist(B_x)$$

On the other hand,

since $\eta_A^i > \eta_B^i$ $(i = 1,2,\dots,T)$, we get

$$y_i(A) = \frac{\sum_{i=1}^{T} y_i(A)}{T} > \frac{\sum_{i=1}^{T} y_i(B)}{T} = y_i(B)\ (i = 1,2,\dots,T).$$

This means $dist(A_y) > dist(B_y)$. As a result;

$$S_3(A) = dist(A_x) \times dist(A_y) > dist(B_x) \times dist(B_y) = S_3(B)$$

Property 3.5 Let A, and B be two TFM-numbers The score function S_k $(k = 1,2,3)$ obviously satisfies the following conditions:

i. $S_k(A) > 0$ (non-negativity)

ii. $S_k(A + B) = S_k(B + A)$ (commutativity)

Definition 3.6 Let $A = \langle (a, b, c, d); \eta_A^1, \eta_A^2, \ldots, \eta_A^T \rangle$ be a TFM-number, $P_i\left(\overline{x}_i(A), \overline{y}_i(A)\right)$ $(i = 1,2, \ldots, T)$ be circumcenter of centroids in Figure 1 and

$$C(A) = (C_1(x_1(A), y_1(A)), C_2(x_2(A), y_2(A)), C_3(x_3(A), y_3(A)), C_4(x_4(A), y_4(A)))$$

be centroid of A. Then, based on the circumcenter of the centroids of A denoted by $\left(P_1(\overline{x}_1(A), \overline{y}_1(A)), P_2(\overline{x}_2(A), \overline{y}_2(A)), \ldots, P_T(\overline{x}_T(A), \overline{y}_T(A))\right)$, 4. score function of A denoted by $S_4(A)$ is given as;

$$S_4(A) = \frac{\sum_{i=1}^{T} \sqrt{\left((x_i(A) - \overline{x}_i(A)\right)^2 + \left((y_i(A) - \overline{y}_i(A)\right)^2}}{T}$$

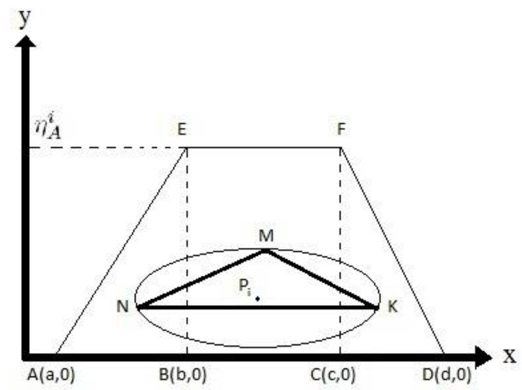

Figure 1: Centroids' circumcenter

There are several methods to find the center of the circumcenter of the triangle given in Figure 1. Here, an algorithm is presented for finding the center of the circumcircle of the triangle by utilizing the midpoint and slope of the two sides of the formed triangle along with their corresponding equations.

By using Azman and Abdullah [2], we give following algorithm to found $P_i(\overline{x}_i(A), \overline{y}_i(A))$: as;

Algorithm 1

Step 1 Find the centroid points of each part of the trapezoid as follows for all i $(i = 1,2,\ldots,T)$:

As is known, the abscissa and ordinate of the centroid of a triangle are respectively the averages of the abscissas and ordinates of the triangle's vertices' coordinates. Similarly, the abscissa and ordinate of the centroid of a rectangle are respectively the averages of the abscissas and ordinates of the rectangle's vertices' coordinates.

Centroid of the $\triangle ABE$: $N(\frac{a+2b}{3}, \frac{\eta_A^i}{3})$

Centroid of the $ACFE$: $M(\frac{b+c}{2}, \frac{\eta_A^i}{2})$

Centroid of the $\triangle CDF$: $K(\frac{2c+d}{3}, \frac{\eta_A^i}{3})$

Step 2 Find the midpoints of $[NM]$ and $[MK]$, denoted by R and S, respectively, given in Figure 2:

Midpoint of $[NM]$: $R(\frac{2a+7b+3c}{12}, \frac{5\eta_A^i}{12})$

Midpoint of $[MK]$: $S(\frac{3b+7c+2d}{12}, \frac{5\eta_A^i}{12})$

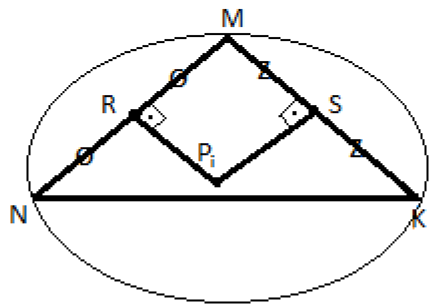

Figure 2: Centroids' circumcenter

Step 3 Find the slopes of $[NM]$ and $[MK]$, denoted by m_{NM} and m_{MK}, respectively as follows:

Slope of $[NM]$: $m_{NM} = \frac{-\eta_A^i}{2a+b-3c}$

Slope of $[MK]$: $m_{MK} = \frac{\eta_A^i}{3b-c-2d}$

Step 4 Find the slopes of $[P_iR]$ and $[P_iS]$ as follows:

Since $[P_iR] \perp [NM]$ and $[P_iS] \perp [MK]$, we get that:

$m_{P_iR} = \frac{2a+b-3c}{\eta_A^i}$ and $m_{P_iS} = \frac{-3b+c+2d}{\eta_A^i}$

Step 5 Find the equation of the $[P_iR]$ and $[P_iS]$ as follows:

Equation of the $[P_iR]$: $\ell_{P_iR}: y = \frac{2a+b+3c}{\eta_A^i}(x - \frac{2a+7b-3c}{12}) + \frac{5\eta_A^i}{12}$

Equation of the $[P_iS]$: $\ell_{P_iS}: y = \frac{-3b+c+2d}{\eta_A^i}(x - \frac{3b+7c+2d}{12}) + \frac{5\eta_A^i}{12}$

Step 6 Find the intersection of ℓ_{P_iR} and ℓ_{P_iS}. Intersection point is circumcenter of centroids:

$$\ell_{P_iR} \cap \ell_{P_iS} = P_i(\frac{\alpha\beta - \gamma\sigma}{12(\alpha-\gamma)}, \frac{\alpha\beta\gamma - \alpha\gamma\sigma + 5\alpha(\eta_A^i)^2 - 5\gamma(\eta_A^i)^2}{12\eta_A^i(\alpha-\gamma)})$$

where,

$$\alpha = 2a + b - 3c,$$

$$\beta = 2a + 7b + 3c,$$

$$\gamma = -3b + c + 2d,$$

$$\sigma = 3b + 7c + 2d$$

Definition 3.7 Let $A = \langle (a, b, c, d); \eta_A^1, \eta_A^2, \dots, \eta_A^T \rangle$ be a TFM-number. Centroid of the centroid points of A denoted by $R\,(A)$, is given as follows:

$$R\,(A) = \left(R_1(\dot{x}_1(A), \dot{y}_1(A)), R_2(\dot{x}_2(A), \dot{y}_2(A)), \dots, R_T(\dot{x}_T(A), \dot{y}_T(A))\right)$$

where,

$\dot{x}_i(A) = \frac{2a+7b+7c+2d}{18}$, $\dot{y}_i(A) = \frac{7\eta_A^i}{18}$ $(i = 1,2,\dots,T)$ which are found by following process:

Inspired by Babu et al. [3], we give a process to find the $R_i(\dot{x}_i(A), \dot{y}_i(A))$ as;

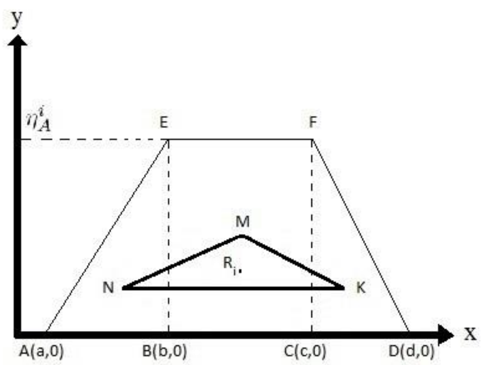

Figure 3: Centroid of the centroids

Centroid of the $\triangle ABE$: $N(\frac{a+2b}{3}, \frac{\eta_A^i}{3})$

Centroid of the $\square ACFE$: $M(\frac{b+c}{2}, \frac{\eta_A^i}{2})$

Centroid of the $\triangle CDF$: $K(\frac{2c+d}{3}, \frac{\eta_A^i}{3})$

As seen in Figure 3, R_i is the centroid of the ABE triangle. Since we know the coordinates of three vertices of the triangle, we can easily find the coordinates of the R_i as follows:

$$R_i\left(\frac{2a+7b+7c+2d}{18}, \frac{7\eta_A^i}{18}\right)$$

Definition 3.8 Let $A = \langle(a,b,c,d);\eta_A^1,\eta_A^2,\ldots,\eta_A^T\rangle$ be a TFM-number. Based on the centroid of the centroid points of A denoted by $\left(R_1(\dot{x}_1(A),\dot{y}_1(A)),R_2(\dot{x}_2(A),\dot{y}_2(A)),\ldots,R_T(\dot{x}_T(A),\dot{y}_T(A))\right)$, 5.Score function of A denoted by $S_5(A)$ is computed as follows:

$$S_5(A) = \frac{\sum_{i=1}^{T}\dot{x}_i(A).\dot{y}_i(A)}{T}, \ (i = 1,2,\ldots,T)$$

where

$\dot{x}_i(A) = \frac{2a+7b+7c+2d}{18}$ and $\dot{y}_i(A) = \frac{7\eta_A^i}{18}$

Property 3.9 Let $A = \langle(a,b,c,d);\eta_A^1,\eta_A^2,\ldots,\eta_A^T\rangle$ be a TFM-number. If, η_A^i=1 $(i = 1,2,\ldots,T)$, then, the score function of A is linear.

Proof Let $A = \langle(a_1,b_1,c_1,d_1);\eta_A^1,\eta_A^2,\ldots,\eta_A^T\rangle$ and $B = \langle(a_2,b_2,c_2,d_2);\eta_B^1,\eta_B^2,\ldots,\eta_B^T\rangle$ be two TFM-numbers, η_A^i=η_B^i =1 $(i = 1,2,\ldots,T)$ and $\gamma_1,\gamma_1 \in \mathbb{R}$. We need to show the following equality which means the linearity of S_5:

$$S_5(\gamma_1 A + \gamma_2 B) = \gamma_1 S_5(A) + \gamma_2 S_5(B)$$

Since η_A^i=η_B^i =1 $(i = 1,2,\ldots,T)$, A and B are trapezoidal fuzzy numbers and denoted by

$A = (a_1,b_1,c_1,d_1)$ and $B = (a_2,b_2,c_2,d_2)$. Thus,

$$S_5(\gamma_1 A + \gamma_2 B) = \frac{2(\gamma_1 a_1 + \gamma_2 a_2) + 7(\gamma_1 b_1 + \gamma_2 b_2) + 7(\gamma_1 c_1 + \gamma_2 c_2) + 2(\gamma_1 d_1 + \gamma_2 d_2)}{18}\frac{7}{18}$$

$$=\frac{2\gamma_1 a_1+7\gamma_1 b_1+7\gamma_1 c_1+2\gamma_1 d_1+2\gamma_2 a_2+7\gamma_2 b_2+7\gamma_2 c_2+2\gamma_2 d_2}{18}\frac{7}{18}$$

$$=\frac{\gamma_1(2a_1+7b_1+7c_1+2d_1)+\gamma_2(2a_2+7b_2+7c_2+2d_2)}{18}\frac{7}{18}$$

$=\gamma_1 S_5(A)+\gamma_2 S_5(B)$

Property 3.10 Let $A,$ and B be two TFM-numbers. The score function S_k $(k = 1,2,\ldots,5)$ generally doesn't satisfy the following conditions:

1. $S_k(A+B)= S_k(A)+S_k(B)$
2. $S_k(A.B)= S_k(A).S_k(B)$

Proof The proof will be presented for S_1. It will be enough to give a counter-example for proof.

Let $A = \langle(2,3,4,5); 0.4,0.1,0.2,0.7\rangle$ and $B = \langle(1,2,4,7); 0.2,0.7,0.6,0.4\rangle$ be two TFM-numbers. Then, we have

$A+B = \langle(3,5,8,12); 0.52,0.73,0.68,0.82\rangle$ and $A.B = \langle(2,6,16,35); 0.08,0.07,0.12,0.28\rangle$. Additionally, if we find scores of $A, B, A+B$, and $A.B$ as follows, respectively:

$S_1(A) = 0.57$, $S_1(B) = 0.81$, $S_1(A+B) = 2.31$ and $S_1(A.B) = 0.99$.

Therefore, we have

1. $S_1(A+B)=2.31 \neq 0.57+0.81 = S_1(A)+S_1(B)$ and
2. $S_1(A.B)=0.99 \neq 0.57\text{x}0.81 = S_1(A).S_1(B)$

For S_2, S_3, S_4 and S_5 the proof can be similarly done.

Example 3.11 Let $A = \langle(2,3,4,8); 0.1,0.4,0.8,0.5\rangle$ be a TFM-number. Then,

we find centroid points of A by using **Definition 3.1** as;

$(C_1(x_1(A), y_1(A)), C_2(x_2(A), y_2(A)), C_3(x_3(A), y_3(A)), C_4(x_4(A), y_4(A)))$

$= (C_1(4.42,0.03), C_2(4.42,0.19), C_3(4.42,0.38), C_4(4.42,0.24))$

we find circumcenters of the centroids of A by using **Algorithm 1** as;

$(P_1(\overline{x}_1(A), \overline{y}_1(A)), P_2(\overline{x}_2(A), \overline{y}_2(A)), P_3(\overline{x}_3(A), \overline{y}_3(A)), P_4(\overline{x}_4(A), \overline{y}_4(A)))$

$= (P_1(4,-45.67), P_2(4,-44.85), P_3(4,-42.95), P_4(4,-44.46))$

we find the centroid of the centroid points of A by using **Definition 3.7** as;

$R\,(A) = (R_1(\dot{x}_1(A), \dot{y}_1(A)), R_2(\dot{x}_2(A), \dot{y}_2(A)), R_3(\dot{x}_3(A), \dot{y}_3(A)), R_4(\dot{x}_4(A), \dot{y}_4(A)))$

$=(R_1(3.83,0.03), R_2(3.83,0.15), R_3(3.83,0.31), R_4(3.83,0.19))$

Therefore,

i. 1. Score value of A is computed as;

$$S_1(A) = \frac{\sum_{i=1}^{4} x_i(A) y_i(A)}{4}$$

$$= \frac{x_1(A).\, y_1(A) + x_2(A).\, y_2(A) + x_3(A).\, y_3(A) + x_4(A).\, y_4(A)}{4}$$

$$= \frac{4.42\text{x}0.03 + 4.42\text{x}0.19 + 4.42\text{x}0.38 + 4.42\text{x}0.24}{4}$$

$=0.92$

ii. 2. Score value of A is computed as;

$$S_2(A) = \frac{\sum_{i=1}^{4} \sqrt{x^2{}_i(A) + y^2{}_i(A)}}{4}$$

$$= \frac{\sqrt{x^2{}_1(A)+y^2{}_1(A)} + \sqrt{x^2{}_2(A)+y^2{}_2(A)} + \sqrt{x^2{}_3(A)+y^2{}_3(A)} + \sqrt{x^2{}_4(A)+y^2{}_4(A)}}{4}$$

$$= \frac{\sqrt{4.42^2+0.03^2} + \sqrt{4.42^2+0.19^2} + \sqrt{4.42^2+0.38^2} + \sqrt{4.42^2+0.24^2}}{4}$$

$= 4.42$

iii. 3. Score value of A is computed as;

$dist(A_x) = d - a = 8 - 2 = 6,$

$$dist(A_y) = \frac{\sum_{i=1}^{T} y_i(A)}{T}$$

$$= \frac{\sum_{i=1}^{4} y_i(A)}{4}$$

$$= \frac{y_1(A)+y_2(A)+y_3(A)+y_4(A)}{4}$$

$$= \frac{0.03+0.19+0.38+0.24}{4}$$

$= 0.21$

$S_3(A) = dist(A_x) \times dist(A_y)$

$= 6 \times 0.21$

$= 1.26$

iv. 4. Score value of A is computed as;

$$S_4(A) = \frac{\sum_{i=1}^{4} \sqrt{\left((x_i(A) - \bar{x}_i(A)\right)^2 + \left((y_i(A) - \bar{y}_i(A)\right)^2}}{4}$$

$$=\frac{\left(\begin{array}{c}\sqrt{\left((x_1(A)-\bar{x}_1(A)\right)^2+\left((y_1(A)-\bar{y}_1(A)\right)^2}+\sqrt{\left((x_2(A)-\bar{x}_2(A)\right)^2+\left((y_2(A)-\bar{y}_2(A)\right)^2}\\+\sqrt{\left((x_3(A)-\bar{x}_3(A)\right)^2+\left((y_3(A)-\bar{y}_3(A)\right)^2}+\sqrt{\left((x_4(A)-\bar{x}_4(A)\right)^2+\left((y_4(A)-\bar{y}_4(A)\right)^2}\end{array}\right)}{4}$$

$$=\frac{\left(\begin{array}{c}\sqrt{(4.42-4)^2+\left(0.03-(-45.67)\right)^2}+\sqrt{(4.42-4)^2+\left(0.19-(-44.85)\right)^2}\\+\sqrt{(4.42-4)^2+(0.38-(-42.95))^2}+\sqrt{(4.42-4)^2+(0.24-(-44.46))^2}\end{array}\right)}{4}$$

=44.69

v. 5. Score value of A is computed as;

$$S_5(A)=\frac{\sum_{i=1}^{4}\dot{x}_i(A).\dot{y}_i(A)}{4}$$

$$=\frac{\dot{x}_1(A).\dot{y}_1(A)+\dot{x}_2(A).\dot{y}_2(A)+\dot{x}_3(A).\dot{y}_3(A)+\dot{x}_4(A).\dot{y}_4(A)}{4}$$

$$=\frac{3.83\text{x}0.03+3.83\text{x}0.15+3.83\text{x}0.31+3.83\text{x}0.19}{4}$$

=0.65

Definition 3.12 Let $A=\langle(a_1,b_1,c_1,d_1);\eta_A^1,\eta_A^2,\dots,\eta_A^T\rangle$, $B=\langle(a_2,b_2,c_2,d_2);\eta_B^1,\eta_B^2,\dots,\eta_B^T\rangle$ be two TFM-numbers and S_k $(k=1,2,\dots,5)$ be score functions given in the previous sections. Then,

1. If $S_k(A)>S_k(B)$ then B is smaller than A, denoted by $A\succ B$
2. If $S_k(B)>S_k(A)$ then A is smaller than B, denoted by $A\prec B$
3. If $S_k(A)=S_k(B)$ then A is similar to B, denoted by $A\simeq B$

3. An Approach to Decision-Making Problems

In this section, we propose a method to solve multi-criteria decision-making problems and give a numerical example.

Definition 4.1 [19] Let $X=\{x_1,x_2,\dots,x_m\}$ be set of alternatives, $C=\{c_1,c_2,\dots,c_n\}$ be set of criteria and $v=(v_1,v_2,\dots,v_n)$ be weights vector such that $v_j>0$ and $\sum_{j=1}^{n}v_j=1$. Then, the characteristic of the alternative x_i on criteria c_j is represented by the TFM-number $A_{ij}=\langle(a_{ij},b_{ij},c_{ij},d_{ij});\eta_{A_{ij}}^1,\eta_{A_{ij}}^2,\dots,\eta_{A_{ij}}^T\rangle$. All the possible values that the alternative x_i $(i=1,2,\dots,m)$ satisfies the criteria c_j $(j=1,2,\dots,n)$ represented in the following TFM decision matrix $(A_{ij})_{mxn}$;

$$(A_{ij})_{mxn}=\begin{pmatrix}A_{11}&A_{12}&\cdots&A_{1n}\\A_{21}&A_{22}&\cdots&A_{2n}\\\vdots&\vdots&\ddots&\vdots\\A_{m1}&A_{m2}&\cdots&A_{mn}\end{pmatrix}_{mxn}$$

Linguistic terms	TFM-numbers
Definitely-low(DL)	⟨(0.01,0.05,0.10,0.15);0.9,0.85,0.70,0.75⟩
Too-Low(TL)	⟨(0.05,0.10,0.15,0.20);0.80,0.75,0.70,0.70⟩
Very-Low(VL)	⟨(0.10,0.15,0.18,0.25);0.78,0.81,0.69,0.71⟩
Low(L)	⟨(0.12,0.20,0.20,0.30);0.76,0.65,0.67,0.65⟩
Fairly-low(FL)	⟨(0.15,0.23,0.25,0.35);0.65,0.70,0.60,0.60⟩
Medium(M)	⟨(0.25,0.30,0.35,0.40);0.45,0.40,0.55,0.50⟩
Fairly-high(FH)	⟨(0.30,0.35,0.40,0.45);0.60,0.45,0.60,0.55⟩
High(H)	⟨(0.40,0.45,0.50,0.55);0.70,0.50,0.65,0.70⟩
Very-High(VH)	⟨(0.45,0.55,0.65,0.75);0.80,0.60,0.70,0.75⟩
Too-High(TH)	⟨(0.50,0.60,0.70,0.80);0.90,0.70,0.80,0.95⟩
Definitely-high(DH)	⟨(0.70,0.80,0.90,1.00);0.95,0.80,0.90,1.00⟩

Table 1: TFM-numbers of linguistic terms

Algorithm 2

Step 1 Present TFM decision matrix showing results of the evaluation of the expert based upon the characteristic of the alternative x_i $(i = 1,2,\ldots,m)$ satisfies the attribute c_j $(i = 1,2,\ldots,n)$ based on linguistic terms Table 1 as follows:

$$(A_{ij})_{mxn} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix}_{mxn}$$

Step 2 For all i $(i = 1,2,\ldots,m)$find the aggregation values according to the $TFMWG$ operator, to obtain the ultimate performance value corresponding to the alternative x_i $(i = 1,2,\ldots,m)$ according to attribute c_j $(i = 1,2,\ldots,n)$ as follows:

$$A_i = TFMWG(A_{i1}, A_{i2}, \ldots, A_{in}), \ (i = 1,2,\ldots,m)$$

Step 3 Find centroid points of A_i $(i = 1,2,\ldots,m)$ by using **Definition 3.1**, circumcenter of the centroids by using **Algorithm 1** and centroid of the centroids by using **Definition 3.7**, respectively as;

$(C_1(x_1(A_i), y_1(A_i)), C_2(x_2(A_i), y_2(A_i)), \ldots, C_T(x_T(A_i), y_T(A_i)))$

$(P_1(\bar{x}_1(A_i), \bar{y}_1(A_i)), P_2(\bar{x}_2(A_i), \bar{y}_2(A_i)), \ldots, P_T(\bar{x}_T(A_i), \bar{y}_T(A_i)))$

$(R_1(\dot{x}_1(A_i), \dot{y}_1(A_i)), R_2(\dot{x}_2(A_i), \dot{y}_2(A_i)), \ldots, R_T(\dot{x}_T(A_i), \dot{y}_T(A_i)))$ $(i = 1,2,\ldots,m)$.

Step 4 Find the scores S_k $(k = 1,2,\ldots,5)$ of each A_i $(i = 1,2,\ldots,m)$denoted by $\boldsymbol{S_k(A_i)}$ by using the given methods.

Step 5 Rank all the alternatives $A_i\ \ (i = 1,2,\dots,m)$ and select the best one, in accordance with the score of each A_i. The bigger the score, the better the alternatives A_i as seen in **Definition 3.12.**

4.1 Numerical Example

To show the usefulness of the proposed method, we give the following application.

Example 4.2 With the modernization of agriculture and the diversification of needs, an agricultural company has produced five agricultural pesticides ($X = \{x_1, x_2, x_3, x_4, x_5\}$) suitable for modern farming. However, due to high production costs, the company has a budget sufficient to produce only one of these alternatives. To decide which agricultural pesticide to produce, the protection performance of these five alternatives during all four seasons of the year (autumn, winter, spring, and summer) will be considered by taking the following four criteria into account while making this evaluation.

i. Yield on crops (c_1)
ii. Sales amount (c_2)
iii. Side effect to soil (c_3)
iv. Adaptation to seasonal conditions (c_4)

The weight vector of the attributes is w=(0.3,0.2,0.4,0.1). The company considers the alternatives in the context of the linguistic terms given in Table 1. Since the evaluation will be conducted across four seasons, Each TFM-numbers in the decision matrix has four membership degrees. The process of finding the best alternative is given as follows:

Step 1 Alternatives and attributes evaluated by the company and results of the evaluation are presented in the TFM decision matrix $(A_{ij})_{5\times4}$ as follows:

$$(A_{ij})_{5\times4} =$$

$$\begin{pmatrix}
\langle(0.10,0.15,0.18,0.25);0.78,0.81,0.69,0.71\rangle & \langle(0.15,0.23,0.25,0.35);0.65,0.70,0.60,0.60\rangle & \langle(0.30,0.35,0.40,0.45);0.60,0.45,0.60,0.55\rangle & \langle(0.45,0.55,0.65,0.75);0.80,0.60,0.70,0.75\rangle \\
\langle(0.05,0.10,0.15,0.20);0.80,0.75,0.70,0.70\rangle & \langle(0.12,0.20,0.20,0.30);0.76,0.65,0.67,0.65\rangle & \langle(0.25,0.30,0.35,0.40);0.45,0.40,0.55,0.50\rangle & \langle(0.40,0.45,0.50,0.55);0.70,0.50,0.65,0.70\rangle \\
\langle(0.12,0.20,0.20,0.30);0.76,0.65,0.67,0.65\rangle & \langle(0.50,0.60,0.70,0.80);0.90,0.70,0.80,0.95\rangle & \langle(0.12,0.20,0.20,0.30);0.76,0.65,0.67,0.65\rangle & \langle(0.25,0.30,0.35,0.40);0.45,0.40,0.55,0.50\rangle \\
\langle(0.25,0.30,0.35,0.40);0.45,0.40,0.55,0.50\rangle & \langle(0.05,0.10,0.15,0.20);0.80,0.75,0.70,0.70\rangle & \langle(0.50,0.60,0.70,0.80);0.90,0.70,0.80,0.95\rangle & \langle(0.25,0.30,0.35,0.40);0.45,0.40,0.55,0.50\rangle \\
\langle(0.50,0.60,0.70,0.80);0.90,0.70,0.80,0.95\rangle & \langle(0.70,0.80,0.90,1.00);0.95,0.80,0.90,1.00\rangle & \langle(0.40,0.45,0.50,0.55);0.70,0.50,0.65,0.70\rangle & \langle(0.12,0.20,0.20,0.30);0.76,0.65,0.67,0.65\rangle
\end{pmatrix}$$

Step 2 For all i (i=1,2,...,5), the aggregation values according to the TFMWG operator are computed, to obtain the ultimate performance value corresponding to the alternative x_i (i=1,2,...,5) as follows:

$A_1 = TFMWG(A_{11}, A_{12}, A_{13}, A_{14})$

=⟨(0.196,0.261,0.301,0.378); 0.404,0.382,0.600,0.483⟩

$A_2 = TFMWG(A_{21}, A_{22}, A_{23}, A_{24}$

=⟨(0.140,0.207,0.252,0.317); 0.345,0.365,0.478,0.396⟩

$A_3 = TFMWG(A_{31}, A_{32}, A_{33}, A_{34})$

= ⟨(0.172,0.259,0.272,0.356); 0.355,0.362,0.523,0.463⟩

$A_4 = TFMWG(A_{41}, A_{42}, A_{43}, A_{44})$

= ⟨(0.239,0.318,0.390,0.459); 0.497,0.456,0.591,0.579⟩

$A_5 = TFMWG(A_{51}, A_{52}, A_{53}, A_{54})$

= ⟨(0.424,0.508,0.568,0.653); 0.821,0.761,0.812,0.719⟩

Step 3 Centroid points of A_i $(i = 1,2, \dots ,5)$ computed with the help of the formula of $x_i(A)$ and $y_i(A)$ given in **Definition 3.1** as follows:

For A_1

$(C_1(0.284, 0.143), C_2(0.284, 0.194), C_3(0.284, 0.278), C_4(0.284, 0.243))$

For A_2

$(C_1(0.228, 0.124), C_2(0.228, 0.193), C_3(0.228, 0.304), C_4(0.228, 0.204))$

For A_3

$(C_1(0.270, 0.103), C_2(0.270, 0.206), C_3(0.270, 0.287), C_4(0.270, 0.256))$

For A_4

$(C_1(0.351, 0.198), C_2(0.351, 0.278), C_3(0.351, 0.310), C_4(0.351, 0.292))$

For A_5

$(C_1(0.538, 0.297), C_2(0.538, 0.299), C_3(0.538, 0.398), C_4(0.538, 0.402))$

The circumcenter of centroids computed by **Algorithm 1** as follows:

For A_1

$(P_1(0.282, 0.159), P_2(0.282, 0.149), P_3(0.282, 0.241), P_4(0.282, 0.198))$

For A_2

$(P_1(0.228, 0.135), P_2(0.228, 0.142), P_3(0.228, 0.191), P_4(0.228, 0.187))$

For A_3

$(P_1(0.268, 0.136), P_2(0.268, 0.151), P_3(0.268, 0.221), P_4(0.268, 0.191))$

For A_4

$(P_1(0.352, 0.193), P_2(0.352, 0.169), P_3(0.352, 0.235), P_4(0.352, 0.221))$

For A_5

$(P_1(0.537, 0.298), P_2(0.537, 0.251), P_3(0.537, 0.297), P_4(0.537, 0.332))$

And

Centroid of the centroids computed by **Definition 3.7** as follows:

For A_1

$(R_1(0.282,0.022), R_2(0.282,0.021), R_3(0.282,0.033), R_4(0.282,0.026))$

For A_2

$(R_1(0.229,0.019), R_2(0.229,0.020), R_3(0.229,0.026), R_4(0.229,0.022))$

For A_3

$(R_1(0.265,0.019), R_2(0.265,0.020), R_3(0.265,0.029), R_4(0.265,0.025))$

For A_4

$(R_1(0.352,0.027), R_2(0.352,0.025), R_3(0.352,0.032), R_4(0.352,0.032))$

For A_5

$(R_1(0.538,0.045), R_2(0.538,0.042), R_3(0.538,0.045), R_4(0.538,0.039))$

Step 4

Scores S_k $(k = 1,2, \ldots ,5)$ of each A_i denoted by $\boldsymbol{S_k(A_i)}$ $(i = 1,2, \ldots, m)$ given as follows:

$$S_1(A_1) = \frac{\sum_{i=1}^{4} x_i(A)y_i(A)}{4}$$

$$=\frac{x_1(A)y_1(A)+x_2(A)y_2(A)+x_3(A)y_3(A)+x_4(A)y_4(A)}{4}$$

=0.062

Similarly,

$S_1(A_2)$=0.044

$S_1(A_3)$=0.058

$S_1(A_4)$=0.089

$S_1(A_5)$=0.188

$$S_2(A_1) = \frac{\sum_{i=1}^{4} \sqrt{x^2{}_i(A) + y^2{}_i(A)}}{4}$$

$$= \frac{\sqrt{x^2{}_1(A) + y^2{}_1(A)} + \sqrt{x^2{}_2(A) + y^2{}_2(A)} + \sqrt{x^2{}_3(A) + y^2{}_3(A)} + \sqrt{x^2{}_4(A) + y^2{}_4(A)}}{4}$$

$= 0.378$

Similarly,

$S_2(A_2)$=0.295

$S_2(A_3)$=0.330

$S_2(A_4)$=0.441

$S_2(A_5)$=0.656

$$S_3(A_1) = dist(A_{1\,x}) \times dist\left(A_{1\,y}\right)$$

$$= (d-a)\frac{\sum_{i=1}^{4} y_i(A)}{4}$$

=0.045

Similarly,

$S_3(A_2)$=0.034

$S_3(A_3)$=0.046

$S_3(A_4)$=0.063

$S_3(A_5)$=0.089

$$S_4(A_1) = \frac{\sum_{i=1}^{4} \sqrt{\left((x_i(A) - \bar{x}_i(A)\right)^2 + \left((y_i(A) - \bar{y}_i(A)\right)^2}}{4}$$

$$= \frac{\left(\begin{array}{c}\sqrt{\left((x_1(A)-\bar{x}_1(A)\right)^2+\left((y_1(A)-\bar{y}_1(A)\right)^2}+\sqrt{\left((x_2(A)-\bar{x}_2(A)\right)^2+\left((y_2(A)-\bar{y}_2(A)\right)^2} \\ +\sqrt{\left((x_3(A)-\bar{x}_3(A)\right)^2+\left((y_3(A)-\bar{y}_3(A)\right)^2}+\sqrt{\left((x_4(A)-\bar{x}_4(A)\right)^2+\left((y_4(A)-\bar{y}_4(A)\right)^2}\end{array}\right)}{4}$$

=0.057

Similarly,

$S_4(A_2)$=0.049

$S_4(A_3)$=0.058

$S_4(A_4)$=0.066

$S_4(A_5)$=0.075

$$S_5(A_1) = \frac{\sum_{i=1}^{4} \dot{x}_i(A).\dot{y}_i(A)}{4}$$

$$= \frac{\dot{x}_1(A).\dot{y}_1(A) + \dot{x}_2(A).\dot{y}_2(A) + \dot{x}_3(A).\dot{y}_3(A) + \dot{x}_4(A).\dot{y}_4(A)}{4}$$

=0.0071

Similarly,

$S_5(A_2)$=0.0049

$S_5(A_3)$=0.0061

$S_5(A_4)$=0.0102

$S_5(A_5)$=0.0229

Step 5

We rank all the alternatives A_i $(i = 1,2,\ldots,5)$ according to their scores as follows:

$S_1(A_5) > S_1(A_4) > S_1(A_3) > S_1(A_1) > S_1(A_2) \Rightarrow A_5 > A_4 > A_3 > A_1 > A_2$

$S_2(A_5) > S_2(A_4) > S_2(A_1) > S_2(A_3) > S_2(A_2) \Rightarrow A_5 > A_4 > A_1 > A_3 > A_2$

$S_3(A_5) > S_3(A_4) > S_3(A_3) > S_3(A_1) > S_3(A_2) \Rightarrow A_5 > A_4 > A_3 > A_1 > A_2$

$S_4(A_5) > S_4(A_4) > S_4(A_3) > S_4(A_1) > S_4(A_2) \Rightarrow A_5 > A_4 > A_3 > A_1 > A_2$

$S_5(A_5) > S_5(A_4) > S_5(A_1) > S_5(A_3) > S_5(A_2) \Rightarrow A_5 > A_4 > A_1 > A_3 > A_2$

Therefore the best alternative is A_5. If the company doesn't want to select A_5, then it should select the A_4 as the second best alternative.

5. Conclusion

In this study, we proposed a decision-making method based on the circumcenter of centroids and centroid points. Through the utilization of trapezoidal fuzzy multi-numbers, the proposed method captured the vagueness associated with decision criteria, allowing for a more accurate representation of uncertainty. By employing the circumcenter of centroids and centroid points as a measure of central tendency, the method facilitated the evaluation and comparison of alternatives. The method provided an overall assessment of each alternative, enabling decision-makers to identify the most preferable options. The method successfully integrated the various fuzzy criteria and provided a clear preference order, assisting the decision-makers in choosing the most suitable choices for decision-makers. The proposed decision-making methods offer several advantages. They allow decision-makers to explicitly model uncertainty, consider multiple criteria, and capture the trade-offs among them.

Future research could focus on extending the proposed method to handle decision problems with a larger number of criteria or incorporating additional types of fuzzy numbers. Furthermore, the applicability of the method in different domains and real-world scenarios could be explored to validate its robustness and effectiveness.

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Chapter Nine

Multi-criteria decision-making method based on intuitionistic trapezoidal fuzzy multi-numbers and some harmonic aggragation operators: Application of Architucture

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ABSTRACT

It becomes even more complex with complex architectural problems, and decision-making methods are needed, and it is understood how important decision-making methods are. While the use of decision-making methods in the field of engineering is dominant, their use in the field of architecture is becoming more and more widespread. It can be listed as reaching an optimum solution with the targeted and designed alternatives with these methods, evolving the design process, allowing recycling, controlling these processes and creating data for architecture in the future. In this chapter, intuitionistic trapezoidal fuzzy multi-numbers weighted harmonic mean (ITFMNWHM) is developed to aggregate the decision information. The desirable properties of this operator are presented in detail. Further, we develop an approach to multi-citeria decision-making (MCDM) problem on the basis of the proposed developed aggregation operator. And then, we developed a score function for intuitionistic trapezoidal fuzzy multi-numbers. Finally, the effectiveness and applicability of our proposed MCDM model, as well as comparison analysis with other approaches are illustrated with a practical example.

Keywords: intuitionistic fuzzy sets, intuitionistic fuzzy multi-sets, intuitionistic trapezoidal fuzzy multi-numbers, weighted harmonic aggragation operators, multi-criteria decision-making.

1. Introduction

The nature of the data is typically uncertain and such that, technology, economics, artificial intelligence and healthcare etc. imprecise in many real world problems. In the field of decision analysis, it is extremely challenging to arrive at precise conclusions based on evidence that is hazy or ambiguous. Generally, most of the existing methods provide deterministic solutions to the optimization problems under a uncertanity environment. But in practice, the decision-making may not have specific, accurate and comprehensive idea on these solutions. Since decision making problems which contain uncertain are difficult to model and solve, and it is a need for us to develop some mathematical theories.

The concept of fuzzy sets was first initiated by Zadeh [1] to manage uncertainty in real life. It has emerged that one component is insufficient to represent some special types of information. In this situation, a component namely non-membership value is invited to illustrate the information properly and in addition to this new component Atanassov [2] first defined the intuitionistic fuzzy set. Because of its ability to measure the fuzziness in a quite precise and comprehensive manner, intuitionistic fuzzy set theory has achieved a great deal. In some ambiguous circumstances, however, the sum of the grades of positive membership and negative membership can exceed 1, which is not suitable for intuitionistic fuzzy set. Yager [3] conducted the first study on the fuzzy multisets. They defined the concept of fuzzy multisets and basic operations including desired properties. Then, Shinoj and John [4] introduced intuitionistic fuzzy multisets based on fuzzy multisets and intuitionistic fuzzy sets. As a result, the multisets have been gradually drawn attention by the scholars [5-7]. Although the fuzzy multi-number and intuitionistic fuzzy multi-number are important tools to model problems involving uncertainty, these theories are inadequate to model some uncertainties. Therefore, many extended forms of the theories have been studied on fuzzy numbers [8-14], intuitionistic fuzzy numbers [15-18], fuzzy multi-numbers [19-24], and other fuzzy sets [25-28], but very few methods consider value of the uncertainty in the occurrences are more than one. The theories have studied in various areas such as [32-46].

Over the course of the past few decades, there has been a growing interest in the strategies for constructing novel aggeration operators to merge information. Harmonic mean operator is the one of the basic operators. Because of their effectiveness and numerous benefits, aggeration operators have developed into an essential component of the decision-making process. The harmonic mean is also used to reduce the influence on the average of elements in a data array that has very high values than others. It is very usable when there are anomalous alternative preferences made by decision makers. In most cases, these aggeration operators are predicated on a variety of operational rules that are designed to combine a limited number of neutrosophic numbers into a single neutrosophic number. In the literature, there few fuzzy harmonic operators developed by some researchers Aydın et al. [47], Shit et al. [48], Zhao et al. [49] and Xu [50].

In order to use the concept of fuzzy multi sets to define an uncertain quantity or a quantity difficult to quantify, in Ulucay et al. [19] the authors put forward the concept of trapezoidal fuzzy multi-numbers (TFM-numbers). They developed some harmonic aggregation operators of TFM-numbers.

2. Preliminary

This section firstly introduces several the known definitions and propositions that would be helpful for better study of this paper.

Definition 2.1 (Zimmermann 1993) A t-norm is a function $t: [0,1] \times [0,1] \to [0,1]$ which satisfies the following properties:

v. $t(0,0) = 0$ and $t(\mu_{X_1}(x),1) = t(1, \mu_{X_1}(x)) = \mu_{X_1}(x),\ x \in E$

vi. If $\mu_{X_1}(x) \leq \mu_{X_3}(x)$ and $\mu_{X_2}(x) \leq \mu_{X_4}(x)$, then
$t(\mu_{X_1}(x), \mu_{X_2}(x)) \leq t(\mu_{X_3}(x), \mu_{X_4}(x))$

vii. $t(\mu_{X_1}(x), \mu_{X_2}(x)) = t(\mu_{X_2}(x), \mu_{X_1}(x))$

viii. $t(\mu_{X_1}(x), t(\mu_{X_2}(x), \mu_{X_3}(x))) = t(t(\mu_{X_1}(x), \mu_{X_2}(x), \mu_{X_3}(x))$

Definition 2.2 (Zimmermann 1993) An s-norm is a function $s: [0,1] \times [0,1] \to [0,1]$ which satisfies the following properties:

v. $s(1,1) = 1$ and $s(\mu_{X_1}(x),0) = s(0, \mu_{X_1}(x)) = \mu_{X_1}(x), x \in E$

vi. if $\mu_{X_1}(x) \leq \mu_{X_3}(x)$ and $\mu_{X_2}(x) \leq \mu_{X_4}(x)$, then
$s(\mu_{X_1}(x), \mu_{X_2}(x)) \leq s(\mu_{X_3}(x), \mu_{X_4}(x))$

vii. $s(\mu_{X_1}(x), \mu_{X_2}(x)) = s(\mu_{X_2}(x), \mu_{X_1}(x))$
$s(\mu_{X_1}(x), s(\mu_{X_2}(x), \mu_{X_3}(x))) = s(s(\mu_{X_1}(x), \mu_{X_2})(x), \mu_{X_3}(x))$

Definition 2.3 [1] The fuzzy sets defined on a non-empty Y as objects having the form $F = \{\langle y, \varphi_F(y)\rangle : y \in Y\}$ where the functions $\varphi_F : Y \to [0,1]$ for $y \in Y$.

Definition 2.4 [6] Let Y be a non-empty set. A multi-fuzzy set G on Y is defined as $G = \left\{\left\langle y, \varphi_G^1(y), \varphi_G^2(y), ..., \varphi_G^i(y), ...\right\rangle : y \in Y\right\}$ where $\varphi_G^i : Y \to [0,1]$ for all $i \in \{1,2,...,p\}$ and $y \in Y$.

Definition 2.5 [19] An ITFM number $\tilde{a} = \left\langle [a,b,c,d]; \left(\varphi_A^1, \varphi_A^2, ..., \varphi_A^p\right), \left(\sigma_A^1, \sigma_A^2, ..., \sigma_A^p\right)\right\rangle$ on □ (The set of all ITFM-number on □ will be denoted by Ω.) is characterized by membership functions and non-membership functions are defined as, respectively:

$$\mu_A^i(y)=\begin{cases}(y-a)\varphi_A^i/(b-a), & a\le y\le b\\ \varphi_A^i, & b\le y\le c\\ (d-y)\varphi_A^i/(d-c), & c\le y\le d\\ 0, & \text{otherwise}\end{cases}$$

$$v_A^i(y)=\begin{cases}\dfrac{(b-y)+\sigma_A^i(y-a)}{(b-a)}, & a\le y\le b\\ \sigma_A^i, & b\le y\le c\\ \dfrac{(y-c)+\sigma_A^i(d-y)}{(d-c)}, & c\le y\le d\\ 1, & \text{otherwise.}\end{cases}$$

Definition 2.6 [19] Let $A=\left\langle [a_1,b_1,c_1,d_1];\left(\varphi_A^1,\varphi_A^2,...,\varphi_A^p\right),\left(\sigma_A^1,\sigma_A^2,...,\sigma_A^p\right)\right\rangle$,
$B=\left\langle [a_2,b_2,c_2,d_2];\left(\varphi_B^1,\varphi_B^2,...,\varphi_B^p\right),\left(\sigma_B^1,\sigma_B^2,...,\sigma_B^p\right)\right\rangle\in\Omega$ and $\gamma\neq 0$ be any real number. Then,

1. $A+B=\left\langle [a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2];\right.$
$\left.\left(s(\varphi_A^1,\varphi_B^1),s(\varphi_A^2,\varphi_B^2),...,s(\varphi_A^P,\varphi_B^P)\right),\left(t(\sigma_A^1,\sigma_B^1),t(\sigma_A^2,\sigma_B^2),...,t(\sigma_A^P,\sigma_B^P)\right)\right\rangle.$

2. $A-B=\left\langle [a_1-a_2,\ b_1-b_2,\ c_1-c_2,\ d_1-d_2];\right.$
$\left.\left(s(\varphi_A^1,\varphi_B^1),s(\varphi_A^2,\varphi_B^2),...,s(\varphi_A^P,\varphi_B^P)\right),\left(t(\sigma_A^1,\sigma_B^1),t(\sigma_A^2,\sigma_B^2),...,t(\sigma_A^P,\sigma_B^P)\right)\right\rangle.$

3. $$A.B=\begin{cases}\left\langle [a_1a_2,b_1b_2,c_1c_2,d_1d_2];\left(t(\varphi_A^1,\varphi_B^1),t(\varphi_A^2,\varphi_B^2),...,t(\varphi_A^p,\varphi_B^p)\right),\left(s(\sigma_A^1,\sigma_B^1),s(\sigma_A^2,\sigma_B^2),...,s(\sigma_A^P,\sigma_B^P)\right)\right\rangle,(\mathrm{d}_1>0,\mathrm{d}_2>0)\\ \left\langle [a_1d_2,b_1c_2,c_1b_2,d_1a_2];\left(t(\varphi_A^1,\varphi_B^1),t(\varphi_A^2,\varphi_B^2),...,t(\varphi_A^p,\varphi_B^p)\right),\left(s(\sigma_A^1,\sigma_B^1),s(\sigma_A^2,\sigma_B^2),...,s(\sigma_A^P,\sigma_B^P)\right)\right\rangle,(\mathrm{d}_1<0,\mathrm{d}_2>0)\\ \left\langle [d_1d_2,c_1c_2,b_1b_2,a_1a_2];\left(t(\varphi_A^1,\varphi_B^1),t(\varphi_A^2,\varphi_B^2),...,t(\varphi_A^p,\varphi_B^p)\right),\left(s(\sigma_A^1,\sigma_B^1),s(\sigma_A^2,\sigma_B^2),...,s(\sigma_A^P,\sigma_B^P)\right)\right\rangle,(\mathrm{d}_1<0,\mathrm{d}_2<0)\end{cases}$$

4.

$$A/B=\begin{cases}\left\langle\left[\frac{a_1}{d_2},\frac{b_1}{c_2},\frac{c_1}{b_2},\frac{d_1}{a_2}\right];\left(t(\varphi_A^1,\varphi_B^1),t(\varphi_A^2,\varphi_B^2),...,t(\varphi_A^p,\varphi_B^p)\right),\left(s(\sigma_A^1,\sigma_B^1),s(\sigma_A^2,\sigma_B^2),...,s(\sigma_A^P,\sigma_B^P)\right)\right\rangle,(d_1>0,d_2>0)\\ \left\langle\left[\frac{d_1}{d_2},\frac{c_1}{c_2},\frac{b_1}{b_2},\frac{a_1}{a_2}\right];\left(t(\varphi_A^1,\varphi_B^1),t(\varphi_A^2,\varphi_B^2),...,t(\varphi_A^p,\varphi_B^p)\right),\left(s(\sigma_A^1,\sigma_B^1),s(\sigma_A^2,\sigma_B^2),...,s(\sigma_A^P,\sigma_B^P)\right)\right\rangle,(d_1<0,d_2>0)\\ \left\langle\left[\frac{d_1}{a_2},\frac{c_1}{b_2},\frac{b_1}{c_2},\frac{a_1}{d_2}\right];\left(t(\varphi_A^1,\varphi_B^1),t(\varphi_A^2,\varphi_B^2),...,t(\varphi_A^p,\varphi_B^p)\right),\left(s(\sigma_A^1,\sigma_B^1),s(\sigma_A^2,\sigma_B^2),...,s(\sigma_A^P,\sigma_B^P)\right)\right\rangle,(d_1<0,d_2<0)\end{cases}$$

5. $\gamma A=\left\langle\left[\gamma a_1,\gamma b_1,\gamma c_1,\gamma d_1\right];\left(1-(1-\varphi_A^1)^\gamma,1-(1-\varphi_A^2)^\gamma,...,1-(1-\varphi_A^p)^\gamma\right),\left((\sigma_A^1)^\gamma,(\sigma_A^2)^\gamma,...,(\sigma_A^P)^\gamma\right)\right\rangle(\gamma\geq 0).$

6.

$$A^\gamma=\left\langle\left[a_1^\gamma,b_1^\gamma,c_1^\gamma,d_1^\gamma\right];\left((\varphi_A^1)^\gamma,(\varphi_A^2)^\gamma,...,(\varphi_A^P)^\gamma\right),\left(\left(1-(1-\sigma_A^1)^\gamma,1-(1-\sigma_A^2)^\gamma,...,1-(1-\sigma_A^p)^\gamma\right)\right)\right\rangle(\gamma\geq 0).$$

Definition 2.7 [19] Let $A=\left\langle\left[a_1,b_1,c_1,d_1\right];\left(\varphi_A^1,\varphi_A^2,...,\varphi_A^p\right),\left(\sigma_A^1,\sigma_A^2,...,\sigma_A^p\right)\right\rangle\in\Omega.$. Then, the normalized ITFM-number of A is given by

$$\overline{A}=\left\langle\left[\frac{a_1}{a_1+b_1+c_1+d_1},\frac{b_1}{a_1+b_1+c_1+d_1},\frac{c_1}{a_1+b_1+c_1+d_1},\frac{d_1}{a_1+b_1+c_1+d_1}\right];\left(\varphi_A^1,\varphi_A^2,...,\varphi_A^p\right),\left(\sigma_A^1,\sigma_A^2,...,\sigma_A^p\right)\right\rangle.$$

Definition 2.8 (Xu 2009) Let $x_1, x_2, x_3, \dots, x_n$ be n real numbers. Then, harmonic mean operator

$$M_{harmonic}(x_1,x_2,x_3,\dots,x_n)=\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\dots+\frac{1}{x_n}} \tag{8}$$

$$=\frac{n}{\sum_{j=1}^{n}\frac{1}{x_j}}$$

Definition 2.9 (Xu 2009) Let $x_1, x_2, x_3, \dots, x_n$ be n real numbers. Then, weighted harmonic mean operator

$$M_{weighted\ harmonic}(x_1,x_2,x_3,\dots,x_n)=\frac{n}{\frac{w_1}{x_1}+\frac{w_2}{x_2}+\frac{w_3}{x_3}+\dots+\frac{w_n}{x_n}} \tag{9}$$

$$= \frac{n}{\sum_{j=1}^{n} \frac{w_j}{x_j}}$$

where $w = (w_1, w_2, w_3, \dots, w_n)^T$ is a weight vector of x_j $(j = 1,2,3, \dots, n)$, $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

3. Some weight harmonic mean operators for ITFM-numbers

Definition 3.1 Let $\mathcal{L}_r = \langle [\mathrm{a_r}, \mathrm{b_r}, \mathrm{c_r}, \mathrm{d_r}]; (\mu^1_{\mathcal{L}_r}, \mu^2_{\mathcal{L}_r}, \dots, \mu^{\mathrm{p}}_{\mathcal{L}_r}), (\vartheta^1_{\mathcal{L}_r}, \vartheta^2_{\mathcal{L}_r}, \dots, \vartheta^{\mathrm{p}}_{\mathcal{L}_r}) \rangle$ be a collection of ITFM-numbers for $(r = 1,2,3, \dots, n)$. A mapping $f^{w}_{\mathrm{ITFMNWHM}}: {\mathcal{L}_r}^n \to \mathcal{L}$ is called intuitionistic trapezoidal fuzzy multi-numbers weighted harmonic mean (ITFMNWHM) operator if it satisfies:

$$\mathrm{ITFMNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n) = \frac{1}{\sum_{r=1}^{n} \frac{w_r}{\mathcal{L}_r}} \tag{10}$$

where $w = (w_1, w_2, w_3, \dots, w_n)^T$ is the associated weight vector of $\mathcal{L}_r$ for $r = 1,2,3, \dots, n$ and

$$\sum_{r=1}^{n} w_r = 1.$$

Theorem 3.2 Let $\mathcal{L}_r = \langle [\mathrm{a_r}, \mathrm{b_r}, \mathrm{c_r}, \mathrm{d_r}]; (\mu^1_{\mathcal{L}_r}, \mu^2_{\mathcal{L}_r}, \dots, \mu^{\mathrm{p}}_{\mathcal{L}_r}), (\vartheta^1_{\mathcal{L}_r}, \vartheta^2_{\mathcal{L}_r}, \dots, \vartheta^{\mathrm{p}}_{\mathcal{L}_r}) \rangle$ be a collection of ITFM-numbers for $r = 1,2,3, \dots, n$, $k = 1,2.3, \dots p$ and the associated weight vector of $\mathcal{L}_r$ is $w = (w_1, w_2, w_3, \dots, w_n)^T$ for $\sum_{r=1}^{n} w_r = 1$ then

$$\mathrm{ITFMNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n) = \frac{1}{\frac{w_1}{\mathcal{L}_1} + \frac{w_2}{\mathcal{L}_2} + \dots + \frac{w_n}{\mathcal{L}_n}}$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{n} \frac{w_r}{\mathrm{a}_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{\mathrm{b}_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{\mathrm{c}_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{\mathrm{d}_r}} \right]; \left(\frac{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu^1_{\mathcal{L}_r})^{\mathrm{w}_r} - \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu^1_{\mathcal{L}_r})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu^1_{\mathcal{L}_r})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu^1_{\mathcal{L}_r})^{\mathrm{w_r}}} \right.\right.$$

$$\left. \frac{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu^2_{\mathcal{L}_r})^{\mathrm{w}_r} - \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu^2_{\mathcal{L}_r})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu^2_{\mathcal{L}_r})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu^2_{\mathcal{L}_r})^{\mathrm{w_r}}}, \dots, \frac{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu^{\mathrm{p}}_{\mathcal{L}_r})^{\mathrm{w}_r} - \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu^{\mathrm{p}}_{\mathcal{L}_r})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu^{\mathrm{p}}_{\mathcal{L}_r})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu^{\mathrm{p}}_{\mathcal{L}_r})^{\mathrm{w_r}}} \right)$$

$$\left(\frac{2 \prod_{\mathrm{r=1}}^{\mathrm{n}} (\vartheta^1_{\mathcal{L}_r})^{\mathrm{w}_r}}{\prod_{\mathrm{r=1}}^{\mathrm{n}} (2 - \vartheta^1_{\mathcal{L}_r})^{\mathrm{w}_r} + \prod_{\mathrm{r=1}}^{\mathrm{n}} (\vartheta^1_{\mathcal{L}_r})^{\mathrm{w}_r}}, \right.$$

$$\frac{2\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{n}(2-\vartheta_{\mathcal{L}_r}^{2})^{w_r}+\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}},\ldots,\frac{2\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{p})^{w_r}}{\prod_{r=1}^{n}(2-\vartheta_{\mathcal{L}_r}^{p})^{w_r}+\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{p})^{w_r}}\Bigg)\Bigg\rangle \tag{11}$$

Proof When n=2, then ITFMNWHM$(\mathcal{L}_1,\mathcal{L}_2)$ is calculated as follows:

$$\text{NVNTNWHM}(\mathcal{L}_1,\mathcal{L}_2)=\frac{1}{\sum_{r=1}^{2}\frac{w_r}{\mathcal{L}_r}}\quad\frac{1}{\frac{w_1}{\mathcal{L}_1}+\frac{w_2}{\mathcal{L}_2}}$$

$$=$$

$$=\frac{1}{\frac{w_1}{\langle[a_1,b_1,c_1,d_1];\left(\mu_{\mathcal{L}_1}^{1},\mu_{\mathcal{L}_1}^{2},\ldots,\mu_{\mathcal{L}_1}^{P}\right),\left(\vartheta_{\mathcal{L}_1}^{1},\vartheta_{\mathcal{L}_1}^{2},\ldots,\vartheta_{\mathcal{L}_1}^{P}\right)\rangle}+\frac{w_2}{\langle[a_2,b_2,c_2,d_2];\left(\mu_{\mathcal{L}_2}^{1},\mu_{\mathcal{L}_2}^{2},\ldots,\mu_{\mathcal{L}_2}^{P}\right),\left(\vartheta_{\mathcal{L}_2}^{1},\vartheta_{\mathcal{L}_2}^{2},\ldots,\vartheta_{\mathcal{L}_2}^{P}\right)\rangle}}$$

$$=\frac{1}{w_1\frac{1}{\left\langle\left[\frac{1}{d_1},\frac{1}{c_1},\frac{1}{b_1},\frac{1}{a_1}\right];\left(\mu_{\mathcal{L}_1}^{1},\mu_{\mathcal{L}_1}^{2},\ldots,\mu_{\mathcal{L}_1}^{P}\right),\left(\vartheta_{\mathcal{L}_1}^{1},\vartheta_{\mathcal{L}_1}^{2},\ldots,\vartheta_{\mathcal{L}_1}^{P}\right)\right\rangle}+w_2\frac{1}{\left\langle\left[\frac{1}{d_2},\frac{1}{c_2},\frac{1}{b_2},\frac{1}{a_2}\right];\left(\mu_{\mathcal{L}_2}^{1},\mu_{\mathcal{L}_2}^{2},\ldots,\mu_{\mathcal{L}_2}^{P}\right),\left(\vartheta_{\mathcal{L}_2}^{1},\vartheta_{\mathcal{L}_2}^{2},\ldots,\vartheta_{\mathcal{L}_2}^{P}\right)\right\rangle}}$$

$$=\frac{1}{\frac{1}{\left\langle\left[\frac{w_1}{d_1},\frac{w_1}{c_1},\frac{w_1}{b_1},\frac{w_1}{a_1}\right];\left(\frac{\left(1+\mu_{\mathcal{L}_1}^{1}\right)^{w_1}-\left(1-\mu_{\mathcal{L}_1}^{1}\right)^{w_1}}{\left(1+\mu_{\mathcal{L}_1}^{1}\right)^{w_1}+\left(1-\mu_{\mathcal{L}_1}^{1}\right)^{w_1}},\frac{\left(1+\mu_{\mathcal{L}_1}^{2}\right)^{w_1}-\left(1-\mu_{\mathcal{L}_1}^{2}\right)^{w_1}}{\left(1+\mu_{\mathcal{L}_1}^{2}\right)^{w_1}+\left(1-\mu_{\mathcal{L}_1}^{2}\right)^{w_1}}\right),\left(\frac{2\left(\vartheta_{\mathcal{L}_1}^{1}\right)^{w_1}}{\left(2-\vartheta_{\mathcal{L}_1}^{1}\right)^{w_1}+\left(\vartheta_{\mathcal{L}_1}^{1}\right)^{w_1}},\frac{2\left(\vartheta_{\mathcal{L}_1}^{2}\right)^{w_2}}{\left(2-\vartheta_{\mathcal{L}_1}^{2}\right)^{w_2}+\left(\vartheta_{\mathcal{L}_1}^{2}\right)^{w_2}}\right)\right\rangle}+\frac{1}{\left\langle\left[\frac{w_2}{d_2},\frac{w_2}{c_2},\frac{w_2}{b_2},\frac{w_2}{a_2}\right];\left(\frac{\left(1+\mu_{\mathcal{L}_1}^{1}\right)^{w_1}-\left(1-\mu_{\mathcal{L}_1}^{1}\right)^{w_1}}{\left(1+\mu_{\mathcal{L}_1}^{1}\right)^{w_1}+\left(1-\mu_{\mathcal{L}_1}^{1}\right)^{w_1}},\frac{\left(1+\mu_{\mathcal{L}_1}^{2}\right)^{w_1}-\left(1-\mu_{\mathcal{L}_1}^{2}\right)^{w_1}}{\left(1+\mu_{\mathcal{L}_1}^{2}\right)^{w_1}+\left(1-\mu_{\mathcal{L}_1}^{2}\right)^{w_1}}\right),\left(\frac{2\left(\vartheta_{\mathcal{L}_1}^{1}\right)^{w_1}}{\left(2-\vartheta_{\mathcal{L}_1}^{1}\right)^{w_1}+\left(\vartheta_{\mathcal{L}_1}^{1}\right)^{w_1}},\frac{2\left(\vartheta_{\mathcal{L}_1}^{2}\right)^{w_2}}{\left(2-\vartheta_{\mathcal{L}_1}^{2}\right)^{w_2}+\left(\vartheta_{\mathcal{L}_1}^{2}\right)^{w_2}}\right)\right\rangle}}$$

$$=1\Bigg/\left\langle\left[\sum_{r=1}^{2}\frac{w_r}{d_r},\sum_{r=1}^{2}\frac{w_r}{c_r},\sum_{r=1}^{2}\frac{w_r}{b_r},\sum_{r=1}^{2}\frac{w_r}{a_r}\right];\left(\frac{\prod_{r=1}^{2}(1+\mu_{\mathcal{L}_r}^{1})^{w_r}-\prod_{r=1}^{2}(1-\mu_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{2}(1+\mu_{\mathcal{L}_r}^{1})^{w_r}+\prod_{r=1}^{2}(1-\mu_{\mathcal{L}_r}^{1})^{w_r}}\right.\right.$$

$$\left.\frac{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^{2})^{w_r}-\prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^{2})^{w_r}+\prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^{2})^{w_r}}\right),$$

$$\left.\left(\frac{2\prod_{r=1}^{2}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{2}(2-\vartheta_{\mathcal{L}_r}^{1})^{w_r}+\prod_{r=1}^{2}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}},\frac{2\prod_{r=1}^{2}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{2}(2-\vartheta_{\mathcal{L}_r}^{2})^{w_r}+\prod_{r=1}^{2}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}}\right)\right\rangle$$

$$= 1\Bigg/\left\langle\left[\frac{1}{\sum_{r=1}^{2}\frac{w_r}{\mathrm{a}_r}},\frac{1}{\sum_{r=1}^{2}\frac{w_r}{\mathrm{b}_r}},\frac{1}{\sum_{r=1}^{2}\frac{w_r}{\mathrm{c}_r}},\frac{1}{\sum_{r=1}^{2}\frac{w_r}{\mathrm{d}_r}}\right];\left(\frac{\prod_{\mathrm{r}=1}^{2}(1+\mu_{\mathcal{L}_r}^{1})^{w_r}-\prod_{\mathrm{r}=1}^{2}(1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{2}(1+\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{2}(1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}\right.\right.$$

$$\left.\frac{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^{2})^{w_r}-\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}\right),$$

$$\left.\left(\frac{2\prod_{\mathrm{r}=1}^{2}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{\mathrm{r}=1}^{2}(2-\vartheta_{\mathcal{L}_r}^{1})^{w_r}+\prod_{\mathrm{r}=1}^{2}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}},\frac{2\prod_{\mathrm{r}=1}^{2}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{\mathrm{r}=1}^{2}(2-\vartheta_{\mathcal{L}_r}^{2})^{w_r}+\prod_{\mathrm{r}=1}^{2}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}}\right)\right\rangle$$

Suppose that Equation 12 holds for $n = k$, i.e.,

$$\mathrm{NVNTNWHM}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots,\mathcal{L}_k) = \frac{1}{\frac{w_1}{\mathcal{L}_1}+\frac{w_2}{\mathcal{L}_2}+\cdots+\frac{w_k}{\mathcal{L}_k}}$$

$$= \left\langle\left[\frac{1}{\sum_{r=1}^{k}\frac{w_r}{\mathrm{a}_r}},\frac{1}{\sum_{r=1}^{k}\frac{w_r}{\mathrm{b}_r}},\frac{1}{\sum_{r=1}^{k}\frac{w_r}{\mathrm{c}_r}},\frac{1}{\sum_{r=1}^{k}\frac{w_r}{\mathrm{d}_r}}\right];\left(\frac{\prod_{\mathrm{r}=1}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{1})^{w_r}-\prod_{\mathrm{r}=1}^{k}(1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}\right.\right.$$

$$\left.\frac{\prod_{\mathrm{r}=1}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{2})^{w_r}-\prod_{\mathrm{r}=1}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}},\dots,\frac{\prod_{\mathrm{r}=1}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{w_r}-\prod_{\mathrm{r}=1}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}\right)$$

$$,\left(\frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{\mathrm{r}=1}^{\mathrm{k}}(2-\vartheta_{\mathcal{L}_r}^{1})^{w_r}+\prod_{\mathrm{r}=1}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}},\right.$$

$$\left.\left.\frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{\mathrm{r}=1}^{\mathrm{k}}(2-\vartheta_{\mathcal{L}_r}^{2})^{w_r}+\prod_{\mathrm{r}=1}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}},\dots,\frac{2\prod_{\mathrm{r}=1}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{w_r}}{\prod_{\mathrm{r}=1}^{\mathrm{k}}(2-\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{w_r}+\prod_{\mathrm{r}=1}^{\mathrm{k}}(\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{w_r}}\right)\right\rangle$$

For $n = k + 1$, using above expression and operational laws, we have

$$\mathrm{NVNTNWHM}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots,\mathcal{L}_k,\mathcal{L}_{k+1}) =$$

$$= \left\langle\left[\frac{1}{\sum_{r=1}^{k}\frac{w_r}{\mathrm{a}_r}},\frac{1}{\sum_{r=1}^{k}\frac{w_r}{\mathrm{b}_r}},\frac{1}{\sum_{r=1}^{k}\frac{w_r}{\mathrm{c}_r}},\frac{1}{\sum_{r=1}^{k}\frac{w_r}{\mathrm{d}_r}}\right];\left(\frac{\prod_{\mathrm{r}=1}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{1})^{w_r}-\prod_{\mathrm{r}=1}^{k}(1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}\right.\right.$$

$$\left.\frac{\prod_{\mathrm{r}=1}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{2})^{w_r}-\prod_{\mathrm{r}=1}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}},\dots,\frac{\prod_{\mathrm{r}=1}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{w_r}-\prod_{\mathrm{r}=1}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{k}}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}\right)$$

$$,\left(\frac{2\prod_{r=1}^{k}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{k}(2-\vartheta_{\mathcal{L}_r}^{1})^{w_r}+\prod_{r=1}^{k}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}},\right.$$

$$\left.\frac{2\prod_{r=1}^{k}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{k}(2-\vartheta_{\mathcal{L}_r}^{2})^{w_r}+\prod_{r=1}^{k}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}},\dots,\frac{2\prod_{r=1}^{k}(\vartheta_{\mathcal{L}_r}^{p})^{w_r}}{\prod_{r=1}^{k}(2-\vartheta_{\mathcal{L}_r}^{p})^{w_r}+\prod_{r=1}^{k}(\vartheta_{\mathcal{L}_r}^{p})^{w_r}}\right)\Bigg\rangle$$

$$=\Bigg\langle\left[\frac{1}{\frac{w_{k+1}}{a_{k+1}}},\frac{1}{\frac{w_{k+1}}{b_{k+1}}},\frac{1}{\frac{w_{k+1}}{c_{k+1}}},\frac{1}{\frac{w_{k+1}}{d_{k+1}}}\right];\left(\frac{(1+\mu_{\mathcal{L}_{k+1}}^{1})^{w_{k+1}}-(1-\mu_{\mathcal{L}_{k+1}}^{1})^{w_{k+1}}}{(1+\mu_{\mathcal{L}_{k+1}}^{1})^{w_{k+1}}+(1-\mu_{\mathcal{L}_{k+1}}^{1})^{w_{k+1}}}\right.$$

$$\left.\frac{(1+\mu_{\mathcal{L}_{k+1}}^{2})^{w_{k+1}}-(1-\mu_{\mathcal{L}_{k+1}}^{2})^{w_{k+1}}}{(1+\mu_{\mathcal{L}_{k+1}}^{2})^{w_{k+1}}+(1-\mu_{\mathcal{L}_{k+1}}^{2})^{w_{k+1}}},\dots,\frac{(1+\mu_{\mathcal{L}_{k+1}}^{p})^{w_{k+1}}-(1-\mu_{\mathcal{L}_{k+1}}^{p})^{w_{k+1}}}{(1+\mu_{\mathcal{L}_{k+1}}^{p})^{w_{k+1}}+(1-\mu_{\mathcal{L}_{k+1}}^{p})^{w_{k+1}}}\right),$$

$$\left(\frac{2(\vartheta_{\mathcal{L}_{k+1}}^{1})^{w_{k+1}}}{(2-\vartheta_{\mathcal{L}_{k+1}}^{1})^{w_{k+1}}+(\vartheta_{\mathcal{L}_{k+1}}^{1})^{w_{k+1}}},\right.$$

$$\left.\frac{2(\vartheta_{\mathcal{L}_{k+1}}^{2})^{w_{k+1}}}{(2-\vartheta_{\mathcal{L}_{k+1}}^{2})^{w_{k+1}}+(\vartheta_{\mathcal{L}_{k+1}}^{2})^{w_{k+1}}},\dots,\frac{2(\vartheta_{\mathcal{L}_{k+1}}^{p})^{w_{k+1}}}{(2-\vartheta_{\mathcal{L}_{k+1}}^{p})^{w_{k+1}}+(\vartheta_{\mathcal{L}_{k+1}}^{p})^{w_{k+1}}}\right)\Bigg\rangle$$

$$=\Bigg\langle\left[\frac{1}{\sum_{r=1}^{k+1}\frac{w_r}{a_r}},\frac{1}{\sum_{r=1}^{k+1}\frac{w_r}{b_r}},\frac{1}{\sum_{r=1}^{k+1}\frac{w_r}{c_r}},\frac{1}{\sum_{r=1}^{k+1}\frac{w_r}{d_r}}\right];\left(\frac{\prod_{r=1}^{k+1}(1+\mu_{\mathcal{L}_r}^{1})^{w_r}-\prod_{r=1}^{k+1}(1-\mu_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{k+1}(1+\mu_{\mathcal{L}_r}^{1})^{w_r}+\prod_{r=1}^{k+1}(1-\mu_{\mathcal{L}_r}^{1})^{w_r}}\right.$$

$$\left.\frac{\prod_{r=1}^{k+1}(1+\mu_{\mathcal{L}_r}^{2})^{w_r}-\prod_{r=1}^{k+1}(1-\mu_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{k+1}(1+\mu_{\mathcal{L}_r}^{2})^{w_r}+\prod_{r=1}^{k+1}(1-\mu_{\mathcal{L}_r}^{2})^{w_r}},\dots,\frac{\prod_{r=1}^{k+1}(1+\mu_{\mathcal{L}_r}^{p})^{w_r}-\prod_{r=1}^{k+1}(1-\mu_{\mathcal{L}_r}^{p})^{w_r}}{\prod_{r=1}^{k+1}(1+\mu_{\mathcal{L}_r}^{p})^{w_r}+\prod_{r=1}^{k+1}(1-\mu_{\mathcal{L}_r}^{p})^{w_r}}\right)$$

$$,\left(\frac{2\prod_{r=1}^{k+1}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{k+1}(2-\vartheta_{\mathcal{L}_r}^{1})^{w_r}+\prod_{r=1}^{k+1}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}},\right.$$

$$\left.\frac{2\prod_{r=1}^{k+1}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{k+1}(2-\vartheta_{\mathcal{L}_r}^{2})^{w_r}+\prod_{r=1}^{k+1}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}},\dots,\frac{2\prod_{r=1}^{k+1}(\vartheta_{\mathcal{L}_r}^{p})^{w_r}}{\prod_{r=1}^{k+1}(2-\vartheta_{\mathcal{L}_r}^{p})^{w_r}+\prod_{r=1}^{k+1}(\vartheta_{\mathcal{L}_r}^{p})^{w_r}}\right)\Bigg\rangle$$

So, the proof is complete.

Next, it can be easily shown that the proposed operator has the following properties.

Theorem 3.3 (Idempotency)

Let $\mathcal{L}_r=\langle[a_r,b_r,c_r,d_r];(\mu_{\mathcal{L}_r}^{1},\mu_{\mathcal{L}_r}^{2},\dots,\mu_{\mathcal{L}_r}^{p}),(\vartheta_{\mathcal{L}_r}^{1},\vartheta_{\mathcal{L}_r}^{2},\dots,\vartheta_{\mathcal{L}_r}^{p})\rangle$ be a collection of ITFM-numbers for $r=1,2,3,\dots,n$. If $\mathcal{L}_n=\mathcal{L}$ for all r that is all are identical then,

$$\text{ITFMNWHM}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots,\mathcal{L}_n)=\mathcal{L}. \tag{12}$$

Proof We know that

$$\mathrm{ITFMNWHM}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots,\mathcal{L}_n)=\frac{1}{\frac{w_1}{\mathcal{L}_1}+\frac{w_2}{\mathcal{L}_2}+\cdots+\frac{w_n}{\mathcal{L}_n}}$$

$$=\left\langle\left[\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{a}_r}},\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{b}_r}},\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{c}_r}},\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{d}_r}}\right];\left(\frac{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^{1})^{\mathrm{w}_r}-\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}\right.\right.$$

$$\left.\frac{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^{2})^{\mathrm{w}_r}-\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}},\dots,\frac{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_r}-\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}\right),$$

$$\left(\frac{2\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^{1})^{\mathrm{w}_r}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(2-\vartheta_{\mathcal{L}_r}^{1})^{\mathrm{w}_r}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^{1})^{\mathrm{w}_r}},\right.$$

$$\left.\left.\frac{2\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^{2})^{\mathrm{w}_r}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(2-\vartheta_{\mathcal{L}_r}^{2})^{\mathrm{w}_r}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^{2})^{\mathrm{w}_r}},\dots,\frac{2\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_r}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(2-\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_r}+\prod_{\mathrm{r}=1}^{\mathrm{n}}(\vartheta_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w}_r}}\right)\right\rangle$$

$$=\left\langle\left[\frac{1}{\frac{\sum_{r=1}^{n}w_r}{\mathrm{a}}},\frac{1}{\frac{\sum_{r=1}^{n}w_r}{\mathrm{b}}},\frac{1}{\frac{\sum_{r=1}^{n}w_r}{\mathrm{c}}},\frac{1}{\frac{\sum_{r=1}^{n}w_r}{\mathrm{d}}}\right];\left(\frac{(1+\mu_{\mathcal{L}}^{1})^{\sum_{r=1}^{n}w_r}-(1-\mu_{\mathcal{L}}^{1})^{\sum_{r=1}^{n}w_r}}{(1+\mu_{\mathcal{L}}^{1})^{\sum_{r=1}^{n}w_r}+(1-\mu_{\mathcal{L}}^{1})^{\sum_{r=1}^{n}w_r}}\right.\right.$$

$$\left.\frac{(1+\mu_{\mathcal{L}}^{2})^{\sum_{r=1}^{n}w_r}-(1-\mu_{\mathcal{L}}^{2})^{\sum_{r=1}^{n}w_r}}{(1+\mu_{\mathcal{L}}^{2})^{\sum_{r=1}^{n}w_r}+(1-\mu_{\mathcal{L}}^{2})^{\sum_{r=1}^{n}w_r}},\dots,\frac{(1+\mu_{\mathcal{L}}^{p})^{\sum_{r=1}^{n}w_r}-(1-\mu_{\mathcal{L}}^{p})^{\sum_{r=1}^{n}w_r}}{(1+\mu_{\mathcal{L}}^{p})^{\sum_{r=1}^{n}w_r}+(1-\mu_{\mathcal{L}}^{p})^{\sum_{r=1}^{n}w_r}}\right),$$

$$\left(\frac{2(\vartheta_{\mathcal{L}}^{1})^{\sum_{r=1}^{n}w_r}}{(2-\vartheta_{\mathcal{L}}^{1})^{\sum_{r=1}^{n}w_r}+(\vartheta_{\mathcal{L}}^{1})^{\sum_{r=1}^{n}w_r}},\right.$$

$$\left.\left.\frac{2(\vartheta_{\mathcal{L}}^{2})^{\sum_{r=1}^{n}w_r}}{(2-\vartheta_{\mathcal{L}}^{2})^{\sum_{r=1}^{n}w_r}+(\vartheta_{\mathcal{L}}^{2})^{\sum_{r=1}^{n}w_r}},\dots,\frac{2(\vartheta_{\mathcal{L}}^{p})^{\sum_{r=1}^{n}w_r}}{(2-\vartheta_{\mathcal{L}}^{p})^{\sum_{r=1}^{n}w_r}+(\vartheta_{\mathcal{L}}^{p})^{\sum_{r=1}^{n}w_r}}\right)\right\rangle$$

$$=\left\langle\left[\frac{1}{\frac{1}{\mathrm{a}}},\frac{1}{\frac{1}{\mathrm{b}}},\frac{1}{\frac{1}{\mathrm{c}}},\frac{1}{\frac{1}{\mathrm{d}}}\right];\left(\frac{(1+\mu_{\mathcal{L}}^{1})-(1-\mu_{\mathcal{L}}^{1})}{(1+\mu_{\mathcal{L}}^{1})+(1-\mu_{\mathcal{L}}^{1})},\frac{(1+\mu_{\mathcal{L}}^{2})-(1-\mu_{\mathcal{L}}^{2})}{(1+\mu_{\mathcal{L}}^{2})+(1-\mu_{\mathcal{L}}^{2})},\dots,\frac{(1+\mu_{\mathcal{L}}^{p})-(1-\mu_{\mathcal{L}}^{p})}{(1+\mu_{\mathcal{L}}^{p})+(1-\mu_{\mathcal{L}}^{p})}\right.\right.$$

$$\left.\left.\left(\frac{2(\vartheta_{\mathcal{L}}^{1})}{(2-\vartheta_{\mathcal{L}}^{1})+(\vartheta_{\mathcal{L}}^{1})},\frac{2(\vartheta_{\mathcal{L}}^{2})}{(2-\vartheta_{\mathcal{L}}^{2})+(\vartheta_{\mathcal{L}}^{2})},\dots,\frac{2(\vartheta_{\mathcal{L}}^{p})}{(2-\vartheta_{\mathcal{L}}^{p})+(\vartheta_{\mathcal{L}}^{p})}\right)\right\rangle=\mathcal{L}.$$

Theorem 3.4 (Monotoniticy property):

Let $\mathcal{L}_r=\langle[\mathrm{a_r},\mathrm{b_r},\mathrm{c_r},\mathrm{d_r}];(\mu_{\mathcal{L}_r}^{1},\mu_{\mathcal{L}_r}^{2},\dots,\mu_{\mathcal{L}_r}^{\mathrm{p}}),(\vartheta_{\mathcal{L}_r}^{1},\vartheta_{\mathcal{L}_r}^{2},\dots,\vartheta_{\mathcal{L}_r}^{\mathrm{p}})\rangle$ and

$$\mathcal{L}_r'=\langle[\mathrm{a}_r',\mathrm{b}_r',\mathrm{c}_r',\mathrm{d}_r'];\left(\left(\mu_{\mathcal{L}_r}'\right)^1,\left(\mu_{\mathcal{L}_r}'\right)^2,\dots,\left(\mu_{\mathcal{L}_r}'\right)^p\right),\left(\left(\vartheta_{\mathcal{L}_r}'\right)^1,\left(\vartheta_{\mathcal{L}_r}'\right)^2,\dots,\left(\vartheta_{\mathcal{L}_r}'\right)^p\right)\rangle$$

be two collection of ITFM-numbers. If $\mathrm{a_r}\le\mathrm{a}_r'$, $\mathrm{b_r}\le\mathrm{b}_r'$, $\mathrm{c_r}\le\mathrm{c}_r'$, $\mathrm{d_r}\le\mathrm{d}_r'$, $\mu_{\mathcal{L}_r}^{1}\le\left(\mu_{\mathcal{L}_r}'\right)^1$, $\mu_{\mathcal{L}_r}^{2}\le\left(\mu_{\mathcal{L}_r}'\right)^2,\dots,\mu_{\mathcal{L}_r}^{\mathrm{p}}\le\left(\mu_{\mathcal{L}_r}'\right)^p$ and $\vartheta_{\mathcal{L}_r}^{1}\ge\left(\vartheta_{\mathcal{L}_r}'\right)^1,\vartheta_{\mathcal{L}_r}^{2}\ge\left(\vartheta_{\mathcal{L}_r}'\right)^2,\dots,\vartheta_{\mathcal{L}_r}^{\mathrm{p}}\ge\left(\vartheta_{\mathcal{L}_r}'\right)^p$ then

$$\mathrm{ITFMNWHM}^{\varphi}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots,\mathcal{L}_n)\le \mathrm{ITFMNWHM}^{\varphi}(\mathcal{L}_1',\mathcal{L}_2',\mathcal{L}_3',\dots,\mathcal{L}_n'). \quad (13)$$

Proof. Since $\mathrm{a_r}\le \mathrm{a}_r'$ and $\varphi_{\mathrm{r}}\ge 0$ for all r.

$$\frac{1}{\mathrm{a_r}}\ge\frac{1}{\mathrm{a}_r'}\Rightarrow\frac{\varphi_{\mathrm{r}}}{\mathrm{a_r}}\ge\frac{\varphi_{\mathrm{r}}}{\mathrm{a}_r'}\Rightarrow\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{a_r}}\ge\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{a}_r'}\Rightarrow\ \frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{a_r}}}\ \le\frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{a}_r'}}$$

the other calculations are calculated as follows:

$$\frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{b_r}}}\ \le\frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{b}_r'}},\frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{c_r}}}\ \le\frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{c}_r'}},\frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{d_r}}}\ \le\frac{1}{\sum_{r=1}^{n}\frac{\varphi_{\mathrm{r}}}{\mathrm{d}_r'}}$$

$$\mu_{\mathcal{L}_r}^{1}\le\left(\mu_{\mathcal{L}_r}'\right)^{1}\Rightarrow-\mu_{\mathcal{L}_r}^{1}\ge-\left(\mu_{\mathcal{L}_r}'\right)^{1}\Rightarrow\left(1-\mu_{\mathcal{L}_r}^{1}\right)\ge\left(1-\left(\mu_{\mathcal{L}_r}'\right)^{1}\right)\Rightarrow\left(1-\mu_{\mathcal{L}_r}^{1}\right)^{\varphi_{\mathrm{r}}}$$
$$\ge\left(1-\left(\mu_{\mathcal{L}_r}'\right)^{1}\right)^{\varphi_{\mathrm{r}}}$$

$$\Rightarrow\prod_{r=1}^{n}\left(1-\mu_{\mathcal{L}_r}^{1}\right)^{\varphi_{\mathrm{r}}}\ge\prod_{r=1}^{n}\left(1-\left(\mu_{\mathcal{L}_r}'\right)^{1}\right)^{\varphi_{\mathrm{r}}}\Rightarrow 1-\prod_{r=1}^{n}\left(1-\mu_{\mathcal{L}_r}^{1}\right)^{\varphi_{\mathrm{r}}}$$
$$\le 1-\prod_{r=1}^{n}\left(1-\left(\mu_{\mathcal{L}_r}'\right)^{1}\right)^{\varphi_{\mathrm{r}}},$$

similarly

$$\vartheta_{\mathcal{L}_r}^{1}\ge\left(\vartheta_{\mathcal{L}_r}'\right)^{1}\Rightarrow\left(\vartheta_{\mathcal{L}_r}^{1}\right)^{\varphi_{\mathrm{r}}}\ge\left(\left(\vartheta_{\mathcal{L}_r}'\right)^{1}\right)^{\varphi_{\mathrm{r}}}\Rightarrow\prod_{r=1}^{n}\left(\vartheta_{\mathcal{L}_r}^{1}\right)^{\varphi_{\mathrm{r}}}\ge\prod_{r=1}^{n}\left(\left(\vartheta_{\mathcal{L}_r}'\right)^{1}\right)^{\varphi_{\mathrm{r}}}$$

therefore

$$\mathrm{ITFMNWHM}^{\varphi}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots,\mathcal{L}_n)\le \mathrm{ITFMNWHM}^{\varphi}(\mathcal{L}_1',\mathcal{L}_2',\mathcal{L}_3',\dots,\mathcal{L}_n').$$

Theorem 3.5 (Commutativity Property):

Let $\mathcal{L}_r=\langle[\mathrm{a_r},\mathrm{b_r},\mathrm{c_r},\mathrm{d_r}];\left(\mu_{\mathcal{L}_r}^{1},\mu_{\mathcal{L}_r}^{2},\dots,\mu_{\mathcal{L}_r}^{\mathrm{P}}\right),\left(\vartheta_{\mathcal{L}_r}^{1},\vartheta_{\mathcal{L}_r}^{2},\dots,\vartheta_{\mathcal{L}_r}^{\mathrm{P}}\right)\rangle$

be a collection of positive ITFM-numbers and $w=(w_1,w_2,w_3,\dots,w_n)^T$ be an associated weight vector where $w_r\in[0,1]$, $\sum_{r=1}^{n}w_r=1$.

$$\mathrm{ITFMNWHM}^{\varphi}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots,\mathcal{L}_n)=\mathrm{ITFMNWHM}^{\varphi}(\mathcal{L}_1',\mathcal{L}_2',\mathcal{L}_3',\dots,\mathcal{L}_n'). \quad (14)$$

where $\mathcal{L}_n'$ is any permutation of $\mathcal{L}_n$ for $r=1,2,3,\dots,n$.

Proof. We get from Equation (11). Since $\mathcal{L}_n'$ is any permutation of $\mathcal{L}_n$. Therefore

$$\mathrm{ITFMNWHM}^{\varphi}(\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3,\dots,\mathcal{L}_n)=\frac{1}{\frac{w_1}{\mathcal{L}_1}+\frac{w_2}{\mathcal{L}_2}+\cdots+\frac{w_n}{\mathcal{L}_n}}$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{n} \frac{w_r}{a_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{b_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{c_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{d_r}} \right]; \left(\frac{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^{1})^{w_r} - \prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^{1})^{w_r} + \prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^{1})^{w_r}} \right.\right.$$

$$\left. \frac{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^{2})^{w_r} - \prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^{2})^{w_r} + \prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^{2})^{w_r}}, \dots, \frac{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^{p})^{w_r} - \prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^{p})^{w_r}}{\prod_{r=1}^{n}(1+\mu_{\mathcal{L}_r}^{p})^{w_r} + \prod_{r=1}^{n}(1-\mu_{\mathcal{L}_r}^{p})^{w_r}} \right)$$

$$, \left(\frac{2\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{n}(2-\vartheta_{\mathcal{L}_r}^{1})^{w_r} + \prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{1})^{w_r}}, \right.$$

$$\left.\left. \frac{2\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{n}(2-\vartheta_{\mathcal{L}_r}^{2})^{w_r} + \prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{2})^{w_r}}, \dots, \frac{2\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{p})^{w_r}}{\prod_{r=1}^{n}(2-\vartheta_{\mathcal{L}_r}^{p})^{w_r} + \prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{p})^{w_r}} \right) \right\rangle$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{n} \frac{w_r}{a'_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{b'_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{c'_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{d'_r}} \right]; \left(\frac{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{1})^{w_r} - \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{1})^{w_r} + \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{1})^{w_r}} \right.\right.$$

$$\left. \frac{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{2})^{w_r} - \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{2})^{w_r} + \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{2})^{w_r}}, \dots, \frac{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{p})^{w_r} - \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{p})^{w_r}}{\prod_{r=1}^{n}(1+(\mu')_{\mathcal{L}_r}^{p})^{w_r} + \prod_{r=1}^{n}(1-(\mu')_{\mathcal{L}_r}^{p})^{w_r}} \right)$$

$$, \left(\frac{2\prod_{r=1}^{n}((\vartheta')_{\mathcal{L}_r}^{1})^{w_r}}{\prod_{r=1}^{n}(2-(\vartheta')_{\mathcal{L}_r}^{1})^{w_r} + \prod_{r=1}^{n}((\vartheta')_{\mathcal{L}_r}^{1})^{w_r}}, \right.$$

$$\left.\left. \frac{2\prod_{r=1}^{n}((\vartheta')_{\mathcal{L}_r}^{2})^{w_r}}{\prod_{r=1}^{n}(2-(\vartheta')_{\mathcal{L}_r}^{2})^{w_r} + \prod_{r=1}^{n}((\vartheta')_{\mathcal{L}_r}^{2})^{w_r}}, \dots, \frac{2\prod_{r=1}^{n}(\vartheta_{\mathcal{L}_r}^{p})^{w_r}}{\prod_{r=1}^{n}(2-(\vartheta')_{\mathcal{L}_r}^{p})^{w_r} + \prod_{r=1}^{n}((\vartheta')_{\mathcal{L}_r}^{p})^{w_r}} \right) \right\rangle$$

$$= \frac{1}{\frac{w_1}{\mathcal{L}'_1} + \frac{w_2}{\mathcal{L}'_2} + \cdots + \frac{w_n}{\mathcal{L}'_n}}$$

$$= \mathrm{ITFMNWHM}^{\varphi}(\mathcal{L}'_1, \mathcal{L}'_2, \mathcal{L}'_3, \dots, \mathcal{L}'_n).$$

Hence the proof completed.

Theorem 3.6 (Boundedness Property):

Let $\mathcal{L}_r = \langle [a_r, b_r, c_r, d_r]; (\mu_{\mathcal{L}_r}^{1}, \mu_{\mathcal{L}_r}^{2}, \dots, \mu_{\mathcal{L}_r}^{P}), (\vartheta_{\mathcal{L}_r}^{1}, \vartheta_{\mathcal{L}_r}^{2}, \dots, \vartheta_{\mathcal{L}_r}^{P}) \rangle$

be a collection of positive ITFM-numbers and let,

$$\mathcal{L}_r^{+} = \left\langle \left[\max_r\{a_r\}, \max_r\{b_r\}, \max_r\{c_r\}, \max_r\{d_r\} \right]; \left(\max_r\{\mu_{\mathcal{L}_r}^{1}\}, \max_r\{\mu_{\mathcal{L}_r}^{2}\}, \dots, \max_r\{\mu_{\mathcal{L}_r}^{p}\} \right), \right.$$

$$\left. \left(\min_r\{\vartheta_{\mathcal{L}_r}^{1}\}, \min_r\{\vartheta_{\mathcal{L}_r}^{2}\}, \dots, \min_r\{\vartheta_{\mathcal{L}_r}^{p}\} \right) \right\rangle$$

$$\mathcal{L}_r^{-} = \left\langle \left[\min_r\{a_r\}, \min_r\{b_r\}, \min_r\{c_r\}, \min_r\{d_r\} \right]; \left(\min_r\{\mu_{\mathcal{L}_r}^{1}\}, \min_r\{\mu_{\mathcal{L}_r}^{2}\}, \dots, \min_r\{\mu_{\mathcal{L}_r}^{p}\} \right), \right.$$

$$\left(\max_r\{\vartheta^1_{\mathcal{L}_r}\}, \max_r\{\vartheta^2_{\mathcal{L}_r}\}, \dots, \max_r\left\{\vartheta^{\mathrm{p}}_{\mathcal{L}_r}\right\}\right)\right\rangle$$

Then,

$$\mathcal{L}_r^- \leq \mathrm{ITFMNWHM}^{\varphi}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n) \leq \mathcal{L}_r^+. \quad (15)$$

Proof. Since $\min_r\{\mathrm{a_r}\} \leq \mathrm{a_r} \leq \max_r\{\mathrm{a_r}\}, \forall r$

$$\frac{\varphi_\mathrm{r}}{\min_r\{\mathrm{a_r}\}} \geq \frac{\varphi_\mathrm{r}}{\mathrm{a_r}} \geq \frac{\varphi_\mathrm{r}}{\max_r\{\mathrm{a_r}\}} \Rightarrow \sum_{r=1}^{n}\frac{\varphi_\mathrm{r}}{\min_r\{\mathrm{a_r}\}} \geq \sum_{r=1}^{n}\frac{\varphi_\mathrm{r}}{\mathrm{a_r}} \geq \sum_{r=1}^{n}\frac{\varphi_\mathrm{r}}{\max_r\{\mathrm{a_r}\}}$$

$$\Rightarrow \frac{1}{\sum_{r=1}^{n}\frac{\varphi_\mathrm{r}}{\min_r\{\mathrm{a_r}\}}} \leq \frac{1}{\sum_{r=1}^{n}\frac{\varphi_\mathrm{r}}{\mathrm{a_r}}} \leq \frac{1}{\sum_{r=1}^{n}\frac{\varphi_\mathrm{r}}{\max_r\{\mathrm{a_r}\}}}$$

In the same way,

$$\Rightarrow \frac{1}{\sum_{r=1}^{n}\frac{\varphi_\mathrm{r}}{\min_r\{\mathrm{b_r}\}}} \leq \frac{1}{\sum_{r=1}^{n}\frac{\varphi_\mathrm{r}}{\mathrm{b_r}}} \leq \frac{1}{\sum_{r=1}^{n}\frac{\varphi_\mathrm{r}}{\max_r\{\mathrm{b_r}\}}}$$

$$\Rightarrow \frac{1}{\sum_{r=1}^{n}\frac{\varphi_\mathrm{r}}{\min_r\{\mathrm{c_r}\}}} \leq \frac{1}{\sum_{r=1}^{n}\frac{\varphi_\mathrm{r}}{\mathrm{c_r}}} \leq \frac{1}{\sum_{r=1}^{n}\frac{\varphi_\mathrm{r}}{\max_r\{\mathrm{c_r}\}}}$$

$$\Rightarrow \frac{1}{\sum_{r=1}^{n}\frac{\varphi_\mathrm{r}}{\min_r\{\mathrm{d_r}\}}} \leq \frac{1}{\sum_{r=1}^{n}\frac{\varphi_\mathrm{r}}{\mathrm{d_r}}} \leq \frac{1}{\sum_{r=1}^{n}\frac{\varphi_\mathrm{r}}{\max_r\{\mathrm{d_r}\}}}$$

and,

$$\min_r\{\mu_\mathrm{r}\} \leq \mu_\mathrm{r} \leq \max_r\{\mu_\mathrm{r}\} \Rightarrow \left(1 - \min_r\{\mu_\mathrm{r}\}\right) \geq (1-\mu_\mathrm{r}) \geq \left(1 - \max_r\{\mu_\mathrm{r}\}\right)$$

$$\Rightarrow \left(1 - \min_r\{\mu_\mathrm{r}\}\right)^{\varphi_\mathrm{r}} \geq (1-\mu_\mathrm{r})^{\varphi_\mathrm{r}} \geq \left(1 - \max_r\{\mu_\mathrm{r}\}\right)^{\varphi_\mathrm{r}}, \varphi_\mathrm{r} \geq 0, \forall \mathrm{r}.$$

$$= \prod_{r=1}^{n}\left(1 - \min_r\{\mu_\mathrm{r}\}\right)^{\varphi_\mathrm{r}} \geq \prod_{r=1}^{n}(1-\mu_\mathrm{r})^{\varphi_\mathrm{r}} \geq \prod_{r=1}^{n}\left(1 - \max_r\{\mu_\mathrm{r}\}\right)^{\varphi_\mathrm{r}}$$

$$= 1 - \prod_{r=1}^{n}\left(1 - \min_r\{\mu_\mathrm{r}\}\right)^{\varphi_\mathrm{r}} \leq 1 - \prod_{r=1}^{n}(1-\mu_\mathrm{r})^{\varphi_\mathrm{r}} \leq 1 - \prod_{r=1}^{n}\left(1 - \max_r\{\mu_\mathrm{r}\}\right)^{\varphi_\mathrm{r}}$$

Again,

$$\min_r\{\vartheta_r\} \le \vartheta_r \le \max_r\{\vartheta_r\} \Rightarrow \left(\min_r\{\vartheta_r\}\right)^{\varphi_r} \le (\vartheta_r)^{\varphi_r} \le \left(\max_r\{\vartheta_r\}\right)^{\varphi_r}, \varphi_r \ge 0, \forall r$$

$$= \prod_{r=1}^{n}\left(\min_r\{\vartheta_r\}\right)^{\varphi_r} \le \prod_{r=1}^{n}(\vartheta_r)^{\varphi_r} \le \prod_{r=1}^{n}\left(\max_r\{\vartheta_r\}\right)^{\varphi_r}.$$

Definition 3.2 Let $\mathcal{L}_r = \langle [a_r, b_r, c_r, d_r]; \left(\mu^1_{\mathcal{L}_r}, \mu^2_{\mathcal{L}_r}, \ldots, \mu^P_{\mathcal{L}_r}\right), \left(\vartheta^1_{\mathcal{L}_r}, \vartheta^2_{\mathcal{L}_r}, \ldots, \vartheta^P_{\mathcal{L}_r}\right)\rangle$ be a collection of positive ITFM-number, then

$$S(\mathcal{L}_r) = \frac{1}{4p}[a+b+c+d] \times \left(2p + \sum_{r=1}^{p}\mu^P_{\mathcal{L}_r} - \sum_{r=1}^{p}\vartheta^P_{\mathcal{L}_r}\right)$$

and

$$A(\mathcal{L}_r) = \frac{1}{4p}[a+b+c+d] \times \left(2p + \sum_{r=1}^{p}\mu^P_{\mathcal{L}_r} + \sum_{r=1}^{p}\vartheta^P_{\mathcal{L}_r}\right)$$

is called the score and accuracy degrees of $\mathcal{L}_r$, respectively.

Example 3.2 : Let $\mathcal{L} = \langle [1,2,6,9]; (0.2,0.6,0.4), (0.3,0.5,0.4)\rangle$ be NVNT-number then,

$$S(\mathcal{L}) = \frac{1}{4.3}[1+2+6+9] \times \left(6 + (0.2+0.6+0.4) - (0.3+0.5+0.4)\right) = 9$$

$$A(\mathcal{L}) = \frac{1}{4.3}[1+2+6+9] \times \left(6 + (0.2+0.6+0.4) + (0.3+0.5+0.4)\right) = 12.6$$

Definition 3.4 Let $\mathcal{L}^1_r$ and $\mathcal{L}^2_r$ be two ITFM-numbers;

a. If $S(\mathcal{L}^1_r) < S(\mathcal{L}^2_r)$, then $\mathcal{L}^1_r$ is smaller than $\mathcal{L}^2_r$, denoted by $\mathcal{L}^1_r < \mathcal{L}^2_r$.

b. If $S(\mathcal{L}^1_r) = S(\mathcal{L}^2_r)$;

   i. If $A(\mathcal{L}^1_r) < A(\mathcal{L}^2_r)$, then $\mathcal{L}^1_r$ is smaller than $\mathcal{L}^2_r$, denoted by $\mathcal{L}^1_r < \mathcal{L}^2_r$.

   ii. If $A(\mathcal{L}^1_r) = A(\mathcal{L}^2_r)$, then $\mathcal{L}^1_r$ and $\mathcal{L}^2_r$ are the same, denoted by $\mathcal{L}^1_r = \mathcal{L}^2_r$.

4. An algorithm for proposed work

In this section, we shall present a multi-criteria decision-making problem with normalized ITFM-numbers under neutrosophic information using ITFMNWHM operator.

Assume that $U = \{U_1, U_2, \ldots, U_m\}$ be the set of altenatives and $C = \{c_1, c_2, \ldots, c_n\}$ be the set of criterias;

$$(\mathrm{U_{kj}})_{mxn} = \begin{pmatrix} \mathrm{U_{11}} & \mathrm{U_{12}} & \cdots & \mathrm{U_{1n}} \\ \mathrm{U_{21}} & \mathrm{U_{22}} & \cdots & \mathrm{U_{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \mathrm{U_{m1}} & \mathrm{U_{m2}} & \cdots & \mathrm{U_{mn}} \end{pmatrix}$$

such that

$\mathrm{U_{kj}} = \langle [\mathrm{a_{kj}}, \mathrm{b_{kj}}, \mathrm{c_{kj}}, \mathrm{d_{kj}}], (\mu_{kj}^1, \mu_{kj}^2, \mu_{kj}^3, \dots, \mu_{kj}^p), (\vartheta_{kj}^1, \vartheta_{kj}^2, \vartheta_{kj}^3, \dots, \vartheta_{kj}^p) \rangle$, (k=1,2,…,m) and (j=1,2,…,n).

It is carried out the following algorithm to get best choice:

Step 1: Identify and determine the criterias and alternatives and then construct decision matrices,

$$(\mathrm{U_{kj}})_{mxn}, \text{ (k=1,2,…,m; j=1,2,…,n).}$$

Step 2: Get preferable for $\mathrm{U_1}, \mathrm{U_2}, \dots, \mathrm{U_m}$ based on F_i $(i = 1,2,3, \dots, m)$ to aggregate the normalized ITFM-numbers $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n$ as;

$$F_i = \mathrm{ITFMNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n).$$

Step 3: Calculate score value whose formula is given in Definition 3.2 for each F_i to rank alternatives.

Step 4: Rank all score value of F_i according to descending order.

5. Application of the proposed method

In this section, an explanatory example is given to view the strength of the presented work. Architecture means the design of structures. It means designing and shaping structures in a way. It requires great imagination. Then it should be transferred to paper. At this stage, there may be some difficulties, and in terms of time and design, it will be difficult to put the design literally on paper. So it would be better to use computer-aided programs. The Deniz architecture firm wants to choose the computer aided programs for drawing the entrance gate of the AVM. Therefore, there are five computer programs indicated u_i (i=1,2,3,4,5) are available. For this computer aided programs have a criteria set $C = \{c_1 = \text{RAM}; c_2 = \text{SSD}, c_2 = \text{price}\}$.Using the computer data, the proposed algorithm will select the best computer aided program for the Ezgi architecture firm. In addition, is computed using proposed method as follows:

Algorithm:

Step 1: The evaluation matrix$(U_{kj})_{5x3}$ is given by an expert as;

$$(U_{kj})_{5x3} = \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix} \begin{pmatrix} \langle[0.12,0.25,0.41,0.69];(0.3,0.5,0.7,0.8),(0.6,0.3,0.5,0.6)\rangle & \langle[0.18,0.32,0.38,0.43];(0.5,0.3,0.4,0.6),(0.4,0.6,0.5,0.7)\rangle & \langle[0.18,0.27,0.50,0.85];(0.2,0.7,0.8,0.9),(0.2,0.5,0.6,0.4)\rangle \\ \langle[0.33,0.35,0.36,0.45];(0.4,0.2,0.3,0.5),(0.7,0.5,0.6,0.8)\rangle & \langle[0.11,0.15,0.18,0.23];(0.3,0.7,0.9,0.9),(0.3,0.4,0.7,0.5)\rangle & \langle[0.11,0.23,0.38,0.63];(0.3,0.8,0.9,0.7),(0.1,0.4,0.8,0.6)\rangle \\ \langle[0.56,0.62,0.69,0.76];(0.7,0.6,0.4,0.8),(0.4,0.3,0.4,0.5)\rangle & \langle[0.45,0.66,0.72,0.75];(0.6,0.8,0.9,0.8),(0.2,0.3,0.6,0.6)\rangle & \langle[0.45,0.53,0.63,0.83];(0.1,0.6,0.9,0.5),(0.1,0.3,0.5,0.8)\rangle \\ \langle[0.13,0.29,0.46,0.99];(0.8,0.7,0.5,0.6),(0.1,0.5,0.7,0.7)\rangle & \langle[0.07,0.15,0.27,0.37];(0.3,0.9,0.8,0.4),(0.1,0.1,0.4,0.3)\rangle & \langle[0.07,0.73,0.83,0.93];(0.4,0.5,0.7,0.6),(0.3,0.6,0.7,0.2)\rangle \\ \langle[0.11,0.21,0.43,0.78];(0.7,0.8,0.9,0.9),(0.1,0.7,0.8,0.4)\rangle & \langle[0.08,0.13,0.19,0.69];(0.2,0.5,0.7,0.9),(0.6,0.7,0.8,0.8)\rangle & \langle[0.08,0.43,0.68,0.74];(0.5,0.7,0.8,0.3),(0.4,0.2,0.9,0.7)\rangle \end{pmatrix}$$

Step 2: Calculated for $u_1, u_2, \ldots, u_m$ based on F_i $(i = 1,2,3,\ldots,m)$ to aggregate the normalized ITFM-numbers $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \ldots, \mathcal{L}_n$ follow as;

$F_1 = \langle[0.160,0.280,0.429,0.625];(0.001,0.004,0.01,0.03),(0.4,0.53,0.58,0.58)\rangle$

$F_2 = \langle[0.132,0.209,0.271,0.383];(0.001,0.006,0.02,0.02),(0.3,0.48,0.73,0.64)\rangle$

$F_3 = \langle[0.473,0.582,0.592,0.783];(0.002,0.017,0.03,0.02),(0.25,0.37,0.55,0.68)\rangle$

$F_4 = \langle[0.079,0.267,0.431,0.614];(0.004,0.022,0.02,0.007),(0.23,0.38,0.62,0.38)\rangle$

$F_5 = \langle[0.086,0.208,0.332,0.731];(0.003,0.017,0.04,0.02),(0.4,0.48,0.85,0.67)\rangle$

Step 3: The calculated score value whose formula is given in Definition 3.2 for each F to rank alternatives;

$$S(F_1) = \frac{1}{4.4}[0.16 + 0.28 + 0.429 + 0.625] \times \big(8 + (0.001 + 0.004 + 0.01 + 0.003) - (0.4 + 0.53 + 0.58 + 0.58)\big) = 0.557$$

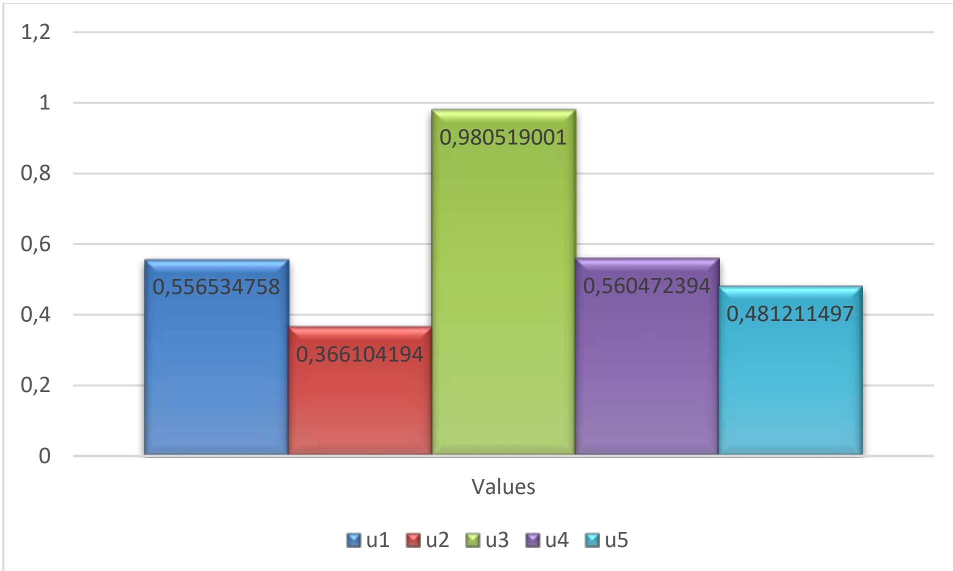
Similar to

$$S(F_2) = 0.366,\ S(F_3) = 0.981,\ S(F_4) = 0.560,\ S(F_5) = 0.481$$

Step 4: Based on the score values $S(F_i)(i = 1,2,\ldots,5)$ the ranking of alternatives $u_k(k = 1,2,\ldots,5)$ are shown in Figure 1 and given as;

$$u_3 > u_4 > u_1 > u_5 > u_2.$$

Finally the best alternative is u_3.



3.

Figure 1 The ranking of alternatives u_k $(k = 1,2,\ldots,5)$

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Chapter Ten

LSTM with Different Parameters for Bitcoin dataset

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ABSTRACT

Long Short-Term Memory (LSTM) is an exclusive form of Recurrent Neural Network (RNN) purpose to overcome exploding or vanishing gradient problems in traditional RNN. In this paper, to reach a lower lost result of the Bitcoin price prediction using LSTM, we want to set the hyperparameters. In this study, our aim is to detect dominant hyperparameters and their values to speed up the optimization process. This problem is a multivariate time series problem. Hyperparameters and their values are applied to the data set, and the obtained values are categorized and presented. The best working parameter set is applied by continuing with the best parameters obtained at each step, and prediction results are obtained with the predict function

Keywords: Machine learning, LSTM, Bitcoin.

INTRODUCTION

Deep learning has promising results in many areas. It produces the best results in speech recognition [1], object recognition [2], financial forecasts [3] and many more. It is also observed that the areas where deep learning is applied an enormous amount of the dataset, the number of model parameters, and the optimization of the parameters can significantly increase the accuracy of the predictions [4], [5]. Complex algorithms such as machine learning algorithms and especially Deep learning produce different results depending on the hyperparameters chosen and the combinations of values they take [6]. Multilayer neural networks are not specific to the problems they are studying, so their methods may need a lot of adaptation [7]. Given the excessive parameterization of LSTM, generalization performance is largely based on the ability to regularize models sufficiently;

for this reason, hyperparameter optimization is necessary [8]. LSTM unit consists of a cell, an entry gate, an exit gate, and a forgetting gate. The cell remembers the values randomly, the three gates coordinate with the information flows entering and leaving the cell. When solving problems in artificial neural networks, there are hyperparameters that play an important role in the solution, apart from the internal parameters used by the algorithms. These hyperparameters vary depending on the data set. In LSTM, hyperparameters and their values contribute to the solution [9]. LSTM is considered as a solution to the exploding gradient problem caused by very large or very small weights. In addition, overfitting is prevented by dropout, which is one of the layer parameter values. There are basically two approaches to hyperparameter adjustment: manual and automatic. This paper examined the manual approaches.

LITERATURE REVIEW

Which hyperparameters to use with which values requires expertise since they are specific to the problems and data sets. In the literature, there are efforts to obtain better results by using different hyperparameters, as well as studies to strengthen the prediction through different spaced data sets in time series problems such as Bitcoin price. In reference [3], which used data sets from different time intervals, better results were obtained from the second data set, even though there was less data in the minute data set. They used the dataset after converting them from the stock exchange data at 1-minute intervals to the 1-day interval trading exchange data on Theil-Sen Regression, Huber Regression LSTM, Gated Recurrent Unit (GRU). LSTM is second only to GRU and shows the best accuracy result with Mean Squared Error (MSE) [10].

In addition, because the dataset with 1-minute interval has more space and losses, [3] and [11] created the second dataset in the references to include 30, 60, 120, and 180 minutes of data and used it in the network. Looking at the references [3] and [11], they found that when making Bitcoin price estimates, the use of hourly, minute datasets instead of daily data results in better profits thanks to the increasing frequency of the dataset. However, since the Bitcoin prediction they [10] found the best result in the literature with a single-layered daily dataset. As a result, daily prices in this report were used in LSTM. They [12] investigated in detail a manually defined subset of possible hyperparameters with grid search. In some cases, a single poorly selected hyperparameter, such as a very large learning rate or a very large dropout rate, will prevent the model from learning effectively. This causes most of the trials to fail. For this reason, if we will use a grid search to avoid this, we must first calculate the possibilities with grid search after manually observing the effect of hyperparameters on performance.

In this paper, we assessed hyperparameters like go_backward, activation function, optimizer, batch size, learning rate, and epochs.

LSTM (LONG SHORT TERM MEMORY)

LSTM can figure out great numbers of time series datasets unanswerable by feed-forward networks using fixed-size time Windows [13]. Time series data frequently have periodical patterns, where the observations increase and decrease over long periods. LSTM networks can be used to remember long-short-term correlation in data. LSTM networks have been proven to model temporary sequences and their long-term dependencies more accurately than the traditional RNN model [13]. We might need five or ten, operating parallel, in what we will call a “layer". This concept of a layer is discussed below. Each LSTM cell consists of three gates. The first gate, it stands for the forget gate, which lets the network to take out data transmitted by the previous cell. The second gate stands for the input gate, which processes entry information at a certain time. The last gate combines the information on the first gate and the second gate to feed the after cell of the LSTM network with a bit of new information.

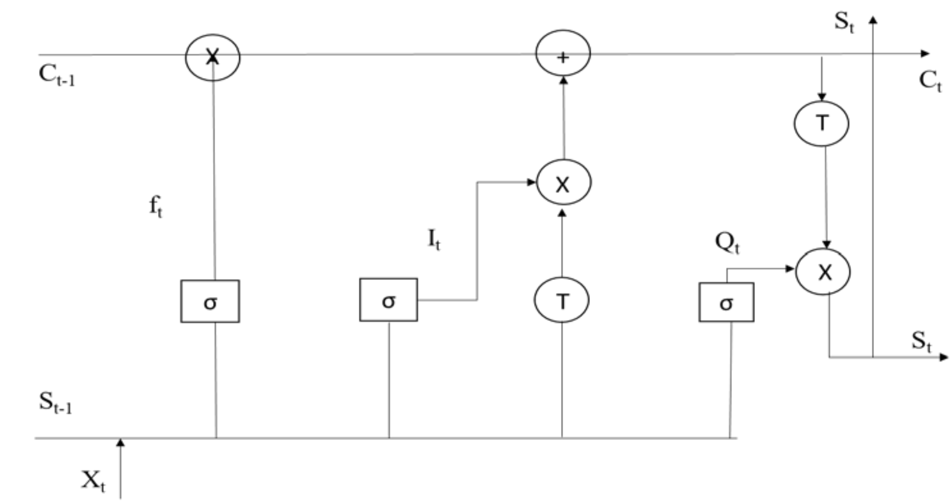

Figure 1. LSTM [13]

$C_t \rightarrow$ *Cell state*, $C_{t-1} \rightarrow$ *Previous cell state*

$H_t \rightarrow$ *Hidden state*, $H_{t-1} \rightarrow$ *Previous hidden state*

$X_t \rightarrow$ *Input*, $f_t \rightarrow$ *forget gate*, $I_t \rightarrow$ *Input gate*, $O_t \rightarrow$ *Output gate*

LSTM takes three parameters at each step, C_{t-1}, H_{t-1} and X_t, finally producing C_t and H_t results.

MATERIAL AND METHODS

This section will describe the methods we used to optimize LSTM hyperparameters coded with Python (version 3.6.5) in Spyder (version 3.2.8). We were used manual approaches with the Keras library (version 2.2.4). There are three backend applications in Keras: The TensorFlow backend, the Theano backend, and the Cognitive Toolkit (CNTK) backend. This paper used TensorFlow (Version 1.9.0) backend. There are many parameters that we can use LSTM with Keras. If we want to create the LSTM network, it can be used as a sequential () or functional Application Programming Interface (API).

Go backward: It can be 'True' or 'False'. If it is 'True', the input sequence processes from backward, return reversed backward of sequence.

Dropout, recurrent dropout: It takes values between 0 and 1. It is a regularization technique for reducing overfitting in neural networks. Rate is a parameter of dropout and controls the dropout intensity in the neural network.

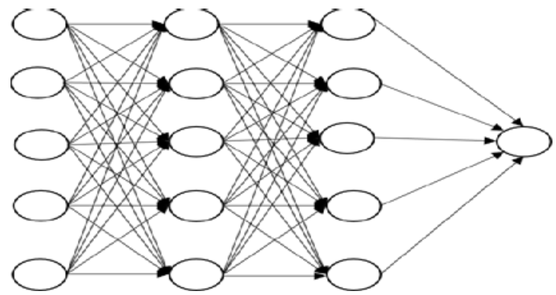

Figure 2. Neural Network without Dropout

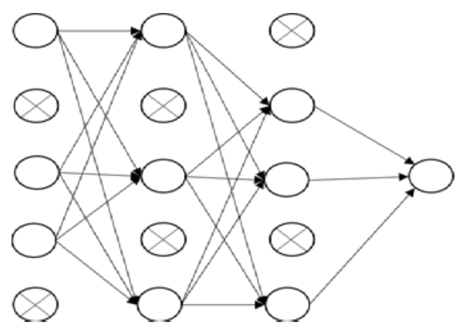

Figure 3. Neural Network with Dropout

The neural network structure before applying dropout in Fig. 2. After applying dropout in Fig.3, the network becomes sparse. When used dropout, it should utilize some adjustment to the hyperparameters [15], [14]. Based on the reference [15], they recommend increasing network size, learning rate, and momentum. Based on the reference [14], they are recommendations that change the learning rate, weight decay, momentum, max-norm, number of units in a layer, among others. They are recommendations that increase network size, learning rate, number of units in a layer, and momentum. Also, add max-norm regularization and change weight decay. Based on the reference [16], they found that deep LSTM's significantly outperformed shallow LSTMs.

RESULTS AND DISCUSSION

Using a great number of evaluations, we can identify architecture and parameter selections to improve performance in many use cases. The contribution of this report is an in-depth analysis to identify parameters that are very important and less important for optimizing hyperparameters. As a result of these tests were carried out to see the contribution of daily and hourly data sets to Bitcoin price prediction. The daily data set consists of 2211 lines from 27 December 2013 to 14 January 2020, while the hourly data set consists of 21029 records from 18 August 2017 to 14 January 2020 23:00. It was observed that the MSE and Root Mean Squared Error (RMSE) values of the daily data set gave better results than the hourly data set. These results are in line with the results of reference [10], and with the result of 0.000001 we have achieved a better result than both GRU 0.00002 and LSTM 0.000431.

Experiments with the go-backyard parameter, as can be seen in Table 1, show improvements in the result when it is appropriate for the dataset, but the effect decreases with increasing epoch.

Table 1. The effects epoch and go-backward on the RMSE and MSE.

Epoch	Go_backward	MSE	RMSE
1000	False	0,000221	0,015
1000	True	0,000284	0,017
100	False	0,000421	0,021
100	True	0,000489	0,022
1	True	0,048001	0,219
1	False	0,058856	0,243

In deep learning algorithms, the number of layers and complexity increases and as a result, the weights need to be updated many times. However, there is no specific number for this update process. It is increased as long as the result obtained improves and the increase is stopped at the optimum result [17]. As can be seen in Table 2, epoch increases the prediction result and decrease the loss.

Table 2. The effects of the epoch on the MSE and RMSE

Epoch	Batch size	Activation Function	Optimizer	MSE	RMSE
1000	100	tanh	adam	0,000221	0,015
100	100	tanh	adam	0,000369	0,019

Batch represents the amount of data that deep learning algorithms will process independently in each iteration [18]. In the experiments where the effect of batch size on learning was examined while the other parameters were constant, it had a positive effect on the results, as seen in Table 3.

Table 3. The effects of the batch size on the MSE and RMSE

Epoch	Batch size	Activation Function	Optimizer	MSE	RMSE

100	200	tanh	adam	0,000271	0,016
100	100	tanh	adam	0,000369	0,019
100	110	tanh	adam	0,000386	0,02
100	30	tanh	adam	0,000389	0,02
100	90	tanh	adam	0,000446	0,021

There are many activation functions,and these are linear, tanh, sigmoid, softmax, Relu, softplus, softsign, selu, elu, and exponential. Based on the reference [19], they conclude that the sigmoid suffers from the disappearing gradient, so there is almost no sign flow from the neuron to its weight. Sigmoid is also not centered to zero, consequently, the gradient can be high or low. On the contrary, the tanh output is zero-centered for this reason in practice is all the time preferred for sigmoid. In the experiments, we performed with activation functions such as tanh, relu, sigmoid, and softmax. The best results were obtained with tanh, although the results were close to each other, as seen in Table 4.

Table 4. The effects of the activation function on the MSE and RMSE

Activation function	MSE	RMSE
tanh	0.000424	0.020601
relu	0.000439	0.020963
sigmoid	0.000616	0.024817
softmax	0.000734	0.027101

The optimizer is an important hyperparameter of the LSTM. Stochastic gradient descent (SGD), RMSprop, Adam, Adadelta, Adagrad, Adamax, Nadam, and Ftrl are optimizers [7]. Since the best results in the activation function are obtained with tanh, Table 5 shows the results obtained from the experiments using tanh for the optimizer.

Table 5. The effects of the activation function and optimizers on the MSE

Optimizers	Activation function	MSE
adagrad	tanh	0.000385
adamax	tanh	0.000393
adam	tanh	0.000401
rmsprop	tanh	0.000406
sgd	tanh	0.000439
adadelta	tanh	0.109229

We chose SGD from SGD and Adadelta, which produced the worst results in our trials on optimizers. We selected the tanh activation function from the results of our trials regarding activation functions because it is the function that produces the best result both in SGD and in general. Besides, after achieving a good MSE and RMSE value with learning rate = 0.2 and epoch= 100, the improvement in MSE and RMSE values when the epoch= 1000 is almost one hundred thousand. That is, it is almost unaffected by the epoch. However, it is limited to knowing when and to what value it will change the learning speed [20]. Finally, the predicted and actual values obtained with the best parameter set are visualized in Figure 1.

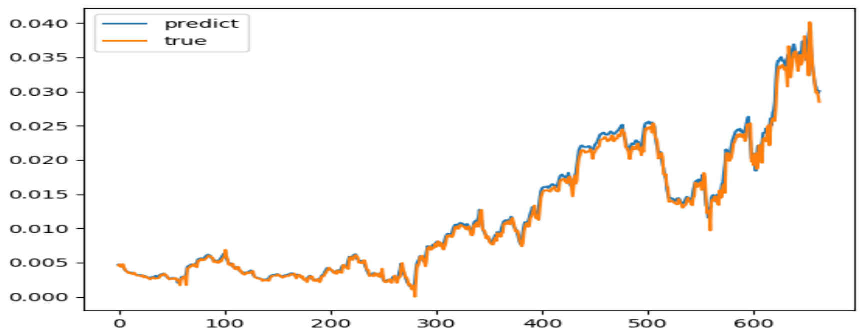

Figure 1. LSTM results.

CONCLUSIONS

In this paper, the LSTM structure and the results obtained from the values of the hyperparameters are discussed. Our aim is speed up the optimization process and decrease loss, for this we examined LSTM with hyperparameters and their values. As a result, it was observed that the proposed hyperparameters and values improved learning and consequently reduced the loss. Thanks to our pre-existing intuition about their roles/effects, we managed to achieve the best result when determining the parameters and making changes to the parameters.

Abbreviations

API: Application Programming Interface

CNTK: Cognitive Toolkit

GRU: Gated Recurrent Unit

LSTM: Long Short-Term Memory

MSE: Mean Squared Error

RMSE: Root Mean Squared Error

RNN: Recurrent Neural Network

SGD: Stochastic gradient descent

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Chapter Eleven

Some harmonic aggragation operators with trapezoidal fuzzy multi-numbers: Application of Law

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ABSTRACT

Different frameworks can be chosen to solve multicriteria decision making (MCDM) problems emerging in business, cyber environment, economy, health care, engineering and other areas. Uncertainty, vagueness and non-rigid boundaries of the initial information are frequently noticed when dealing with the practicalities of the MCDM tasks. Trapezoidal fuzzy multi-numbers express abundant and flexible information in a suitable manner and are very useful to depict the decision information in the procedure of decision making. In this chapter, trapezoidal fuzzy multi-numbers weighted harmonic mean (TFMNWHM) is developed to aggregate the decision information. The desirable properties of this operator are presented in detail. Further, we develop an approach to multi-citeria decision-making (MCDM) problem on the basis of the proposed developed aggregation operator. And then, we developed a score function for trapezoidal fuzzy multi-numbers. Finally, the effectiveness and applicability of our proposed MCDM model, as well as comparison analysis with other approaches are illustrated with a practical example.

Keywords: Fuzzy sets, Fuzzy multi-sets, trapezoidal fuzzy multi-numbers, harmonic aggragation operators, multi-criteria decision-making.

1.Introduction

With the revolutionary developments in the last quarter century in the field of technology, mankind has reached a standard of living and style that he could not even imag-ine. The enlargement of the possibilities in the information environment of the newly developing technology is the biggest problem for the lawyers in defining the terms. Computers have shrunk, mobile phones have become almost computers, and the living space where these two devices cannot be taken or used has almost disappeared. These develop-ments have also caused some important problems in the field of law, new concepts have emerged, the definitions of these concepts have begun to be discussed and to have legal consequences. However, following the developments in this field and putting forth ap-propriate definitions, reconciliation in the international arena and regulation in the na-tional field is the difficulties faced by the lawyers. So, solving fuzzy phenomena and un-certain events in real life is necessary as science and technology advance.

To address such problems, Zadeh [1] pioneered the concept of "fuzzy set" theory, which allows ambiguity to be described using mathematical models. Soon after the definition of fuzzy set, the set has been successfully applied in engineering, game theory, multi-agent systems, control systems, decision-making and so on. In the fuzzy sets, an element in a universe has a membership value in [0, 1]; however, the membership value is inadequate for providing complete information in some problems as there are situations where each element has different membership values. For this reason, a different generalization of fuzzy sets, namely multi-fuzzy sets, has been introduced. Yager [2] first proposed multi-fuzzy sets as a generalization of multisets and fuzzy sets. An element of a multi-fuzzy set may possess more-than-one membership value in [0, 1] (or there may be repeated occurrences of an element). Some Works on the multi-sets have been undertaken by Sebastian and Ramakrishnan [3], Syropoulos [4], Maturo [5], Miyamoto [6, 7] and so on. Recently, research on fuzzy numbers, with the universe of discourse as the real line, has studied.

Over the course of the past few decades, there has been a growing interest in the strategies for constructing novel aggeration operators to merge information. Harmonic mean operator is the one of the basic operators. Because of their effectiveness and numerous benefits, aggeration operators have developed into an essential component of the decision-making process. The harmonic mean is also used to reduce the influence on the average of elements in a data array that has very high values than others. It is very usable when there are anomalous alternative preferences made by decision makers. In most cases, these aggeration operators are predicated on a variety of operational rules that are designed to combine a limited number of neutrosophic numbers into a single neutrosophic number. In the literature, there few fuzzy harmonic operators developed by some researchers Aydın et al. [8], Shit et al. [9], Zhao et al. [10] and Xu [11]. They have also been widely studied in the field of uncertainties [13-43].

In order to use the concept of fuzzy multi sets to define an uncertain quantity or a quantity difficult to quantify, in Ulucay et al. [12] the authors put forward the concept of trapezoidal fuzzy multi-numbers (TFM-numbers). They developed some harmonic aggregation operators of TFM-numbers.

2.Preliminary

Definition 2.1[1]Let X be a non-empty set. A fuzzy set F on X is defined as:
$F = \{\langle x, \mu_F(\text{x})\rangle : \text{x} \in \text{X}\}$ where $\mu_F : X \to [0,1]$ for x $\in X$.

Definition 2.2[2] t-norms are associative, monotonic and commutative two valued functions t that map from $[0,1]\times[0,1]$ into [0,1]. These properties are formulated with the following conditions:

1. $t(0,0) = 0$ and $t(\mu_{x_1}(\text{x}),1) = t(1,\mu_{x_1}(\text{x})) = \mu_{x_1}(\text{x})$
2. If $\mu_{x_1}(\text{x}) \le \mu_{x_3}(\text{x})$ and $\mu_{x_2}(\text{x}) \le \mu_{x_4}(\text{x})$ then $t(\mu_{x_1}(\text{x}),\mu_{x_2}(\text{x})) \le \text{t}(\mu_{x_3}(\text{x}),\mu_{x_4}(\text{x}))$,
3. $t(\mu_{x_1}(\text{x}),\mu_{x_2}(\text{x})) = t(\mu_{x_2}(\text{x}),\mu_{x_1}(\text{x}))$,
4. $t(\mu_{x_1}(\text{x}),\text{t}(\mu_{x_2}(\text{x}),\mu_{x_3}(\text{x}))) = t(t(\mu_{x_1}(\text{x}),\mu_{x_2}(\text{x})),\mu_{x_3}(\text{x}))$

Definition 2.3[2] s -norms are associative, monotonic and commutative two placed functions s which map from $[0,1]\times[0,1]$ into [0,1]. These properties are formulated with the following conditions:

1. $s(1,1) = 1$ and $s(\mu_{x_1}(\text{x}),0) = s\left(0,\mu_{x_1}(\text{x})\right) = \mu_{x_1}(\text{x})$,
2. If $\mu_{x_1}(\text{x}) \le \mu_{x_3}(\text{x})$ and $\mu_{x_2}(\text{x}) \le \mu_{x_4}(\text{x})$, then $s(\mu_{x_1}(\text{x}),\mu_{x_2}(\text{x})) \le s(\mu_{x_3}(\text{x}),\mu_{x_4}(\text{x}))$,
3. $s(\mu_{x_1}(\text{x}),\mu_{x_2}(\text{x})) = s(\mu_{x_2}(\text{x}),\mu_{x_1}(\text{x}))$,
4. $s(\mu_{x_1}(\text{x}),s(\mu_{x_2}(\text{x}),\mu_{x_3}(\text{x}))) = s(s(\mu_{x_1}(\text{x}),\mu_{x_2}(\text{x})),\mu_{x_3}(\text{x}))$.

t-norm and t-conorm is related in a sense of logical duality. Typical dual pairs of non-parametrized t-norm and t-conorm are compiled below:

1. Drastic product: $t_w(\mu_{x_1}(\text{x}),\mu_{x_2}(\text{x})) = \begin{cases} \min\{\mu_{x_1}(\text{x}),\mu_{x_2}(\text{x})\}, \max\{\mu_{x_1}(\text{x}),\mu_{x_2}(\text{x})\} = 1 \\ 0 \qquad otherwise \end{cases}$

2. Drastic sum: $s_w(\mu_{x_1}(\text{x}),\mu_{x_2}(\text{x})) = \begin{cases} \max\{\mu_{x_1}(\text{x}),\mu_{x_2}(\text{x})\}, \min\{\mu_{x_1}(\text{x}),\mu_{x_2}(\text{x})\} = 0 \\ 1 \qquad otherwise \end{cases}$

3. Bounded product:
$$t_1(\mu_{x_1}(\text{x}),\mu_{x_2}(\text{x})) = \max\left\{0,\mu_{x_1}(\text{x}) + \mu_{x_2}(\text{x}) - 1\right\}$$

4. Bounded sum:
$$s_1(\mu_{x_1}(\text{x}),\mu_{x_2}(\text{x})) = \min\left\{1,\mu_{x_1}(\text{x}) + \mu_{x_2}(\text{x})\right\}$$

5. Einstein product:
$$t_{1.5}(\mu_{x_1}(\text{x}),\mu_{x_2}(\text{x})) = \frac{\mu_{x_1}(\text{x}).\mu_{x_2}(\text{x})}{2 - [\mu_{x_1}(\text{x}) + \mu_{x_2}(\text{x}) - \mu_{x_1}(\text{x}).\mu_{x_2}(\text{x})]}$$

6. Einstein sum:
$$s_{1.5}(\mu_{x_1}(\text{x}),\mu_{x_2}(\text{x})) = \frac{\mu_{x_1}(\text{x}) + \mu_{x_2}(\text{x})}{1 + \mu_{x_1}(\text{x}).\mu_{x_2}(\text{x})}$$

7. Algebraic product:

$$t_2(\mu_{x_1}(\mathrm{x}),\mu_{x_2}(\mathrm{x}))=\mu_{x_1}(\mathrm{x}).\mu_{x_2}(\mathrm{x})$$

8. Algebraic sum:

$$s_2(\mu_{x_1}(\mathrm{x}),\mu_{x_2}(\mathrm{x}))=\mu_{x_1}(\mathrm{x})+\mu_{x_2}(\mathrm{x})-\mu_{x_1}(\mathrm{x}).\mu_{x_2}(\mathrm{x})$$

9. Hamacher product:

$$t_{2.5}(\mu_{x_1}(\mathrm{x}),\mu_{x_2}(\mathrm{x}))=\frac{\mu_{x_1}(\mathrm{x}).\mu_{x_2}(\mathrm{x})}{\mu_{x_1}(\mathrm{x})+\mu_{x_2}(\mathrm{x})-\mu_{x_1}(\mathrm{x}).\mu_{x_2}(\mathrm{x})}$$

10. Hamacher Sum:

$$s_{2.5}(\mu_{x_1}(\mathrm{x}),\mu_{x_2}(\mathrm{x}))=\frac{\mu_{x_1}(\mathrm{x})+\mu_{x_2}(\mathrm{x})-2.\mu_{x_1}(\mathrm{x}).\mu_{x_2}(\mathrm{x})}{1-\mu_{x_1}(\mathrm{x}).\mu_{x_2}(\mathrm{x})}$$

11. Minimum:

$$t_3(\mu_{x_1}(\mathrm{x}),\mu_{x_2}(\mathrm{x}))=\min\left\{\mu_{x_1}(\mathrm{x}),\mu_{x_2}(\mathrm{x})\right\}$$

12. Maximum:

$$s_3(\mu_{x_1}(\mathrm{x}),\mu_{x_2}(\mathrm{x}))=\max\left\{\mu_{x_1}(\mathrm{x}),\mu_{x_2}(\mathrm{x})\right\}$$

Definition 2.4 [3] Let X be a non-empty set. A multi-fuzzy set G on X is defined as $G=\left\{\left\langle x,\mu_G^1(\mathrm{x}),\mu_G^2(\mathrm{x}),...,\mu_G^i(\mathrm{x}),...\right\rangle : x\in X\right\}$ where $\mu_G^i : X\to[0,1]$ for all i $\in\left\{1,2,...,p\right\}$ and $x\in X$

Definition 2.5 [12] Let $\eta_A^i\in[0,1]$ (i $\in\left\{1,2,...,p\right\}$) and $a,b,c,d\in R$ such that $a\le b\le c\le d$. Then, a trapezoidal fuzzy multi-number (TFM number) $\tilde{a}=\left\langle\left(a,b,c,d\right);\eta_A^1,\eta_A^2,...,\eta_A^p\right\rangle$ is a special fuzzy multi-set on the real number set R, whose membership functions are defined as

$$\mu_A^i(\mathrm{x})=\begin{cases}(\mathrm{x}-a_1)\eta_A^i/(b_1-a_1) & a_1\le x\le b_1\\ \eta_A^i & b_1\le x\le c_1\\ (d_1-\mathrm{x})\eta_A^i/(d_1-c_1) & c_1\le x\le d_1\\ 0 & otherwise\end{cases}$$

Note that the set of all TFM-number on R will be denoted by Λ.

Definition 2.6[12] Let $A=\left\langle\left(a_1,b_1,c_1,d_1\right);\eta_A^1,\eta_A^2,...,\eta_A^p\right\rangle$, $B=\left\langle\left(a_2,b_2,c_2,d_2\right);\eta_B^1,\eta_B^2,...,\eta_B^p\right\rangle$ $\in\Lambda$ and $\gamma\ne 0$ be any real number. Then,

1. $A+B=\left\langle\left(a_1+a_2,b_1+b_2,c_1+c_2,d_1+d_2\right);s(\eta_A^1,\eta_B^1),s(\eta_A^2,\eta_B^2),...,s(\eta_A^P,\eta_B^P)\right\rangle$
2. $A-B=\left\langle\left(a_1-d_2,b_1-c_2,c_1-b_2,d_1-a_2\right);s(\eta_A^1,\eta_B^1),s(\eta_A^2,\eta_B^2),...,s(\eta_A^P,\eta_B^P)\right\rangle$

3.
$$A.B=\begin{cases}\left\langle (a_1a_2,b_1b_2,c_1c_2,d_1d_2);\ t(\eta_A^1,\eta_B^1),t(\eta_A^2,\eta_B^2),...,t(\eta_A^p,\eta_B^p)\right\rangle & (\mathrm{d}_1>0,\mathrm{d}_2>0)\\ \left\langle (a_1d_2,b_1c_2,c_1b_2,d_1a_2);\ t(\eta_A^1,\eta_B^1),t(\eta_A^2,\eta_B^2),...,t(\eta_A^p,\eta_B^p)\right\rangle & (\mathrm{d}_1<0,\mathrm{d}_2>0)\\ \left\langle (d_1d_2,\mathrm{c}_1c_2,b_1b_2,a_1a_2);\ t(\eta_A^1,\eta_B^1),t(\eta_A^2,\eta_B^2),...,t(\eta_A^p,\eta_B^p)\right\rangle & (\mathrm{d}_1<0,\mathrm{d}_2<0)\end{cases}$$

4.
$$A/B=\begin{cases}\left\langle (a_1/d_2,b_1/c_2,c_1/b_2,d_1/a_2);\ t(\eta_A^1,\eta_B^1),t(\eta_A^2,\eta_B^2),...,t(\eta_A^p,\eta_B^p)\right\rangle & (d_1>0,d_2>0)\\ \left\langle (d_1/d_2,c_1/c_2,b_1/b_2,a_1/a_2);\ t(\eta_A^1,\eta_B^1),t(\eta_A^2,\eta_B^2),...,t(\eta_A^p,\eta_B^p)\right\rangle & (d_1<0,d_2>0)\\ \left\langle (d_1/a_2,c_1/b_2,b_1/c_2,a_1/d_2);\ t(\eta_A^1,\eta_B^1),t(\eta_A^2,\eta_B^2),...,t(\eta_A^p,\eta_B^p)\right\rangle & (d_1<0,d_2<0)\end{cases}$$

5. $\gamma A=\left\langle (\gamma a_1,\gamma b_1,\gamma c_1,\gamma d_1);1-(1-\eta_A^1)^\gamma,1-(1-\eta_A^2)^\gamma,...,1-(1-\eta_A^p)^\gamma\right\rangle (\gamma\geq 0)$

6. $A^\gamma=\left\langle (a_1^\gamma,b_1^\gamma,c_1^\gamma,d_1^\gamma);(\eta_A^1)^\gamma,(\eta_A^2)^\gamma,...,(\eta_A^p)^\gamma\right\rangle (\gamma\geq 0)$

Definition 2.7[12] Let $A=\left\langle (a_1,b_1,c_1,d_1):\eta_A^1,\eta_A^2,...,\eta_A^p\right\rangle\in\Lambda$,Then, normalized TFM-number of A is given by

$$\overline{A}=\left\langle\left(\frac{a_1}{a_1+b_1+c_1+d_1},\frac{b_1}{a_1+b_1+c_1+d_1},\frac{c_1}{a_1+b_1+c_1+d_1},\frac{d_1}{a_1+b_1+c_1+d_1}\right);\eta_A^1,\eta_A^2,...,\eta_A^p\right\rangle$$

Definition 2.7 [11] Let $x_1,x_2,x_3,\dots,x_n$ be n real numbers. Then, harmonic mean operator

$$M_{harmonic}(x_1,x_2,x_3,\dots,x_n)=\frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\cdots+\frac{1}{x_n}}=\frac{n}{\sum_{j=1}^{n}\frac{1}{x_j}} \quad (1)$$

Definition 2.8 [11] Let $x_1,x_2,x_3,\dots,x_n$ be n real numbers. Then, weighted harmonic mean operator

$$M_{weighted\ harmonic}(x_1, x_2, x_3, \dots, x_n) = \frac{n}{\frac{w_1}{x_1} + \frac{w_2}{x_2} + \frac{w_3}{x_3} + \cdots + \frac{w_n}{x_n}} \quad (2)$$

$$= \frac{n}{\sum_{j=1}^{n} \frac{w_j}{x_j}}$$

where $w = (w_1, w_2, w_3, \dots, w_n)^T$ is a weight vector of x_j $(j = 1,2,3, \dots, n), w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

3. Some weight harmonic mean operators for TFM-numbers

Definition 3.1 Let $\mathcal{L}_r = \langle [\mathrm{a_r}, \mathrm{b_r}, \mathrm{c_r}, \mathrm{d_r}];\ (\mu^1_{\mathcal{L}_r}, \mu^2_{\mathcal{L}_r}, \dots, \mu^{\mathrm{p}}_{\mathcal{L}_r}) \rangle$ be a collection of TFM-numbers for $(r = 1,2,3, \dots, n)$. A mapping $f^{w}_{\mathrm{TFMNWHM}}: {\mathcal{L}_r}^n \to \mathcal{L}$ is called trapezoidal fuzzy multi-numbers weighted harmonic mean (TFMNWHM) operator if it satisfies:

$$\mathrm{TFMNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n) = \frac{1}{\sum_{r=1}^{n} \frac{w_r}{\mathcal{L}_r}} \quad (3)$$

where $w = (w_1, w_2, w_3, \dots, w_n)^T$ is the associated weight vector of $\mathcal{L}_r$ for $r = 1,2,3, \dots, n$ and

$$\sum_{r=1}^{n} w_r = 1.$$

Theorem 3.2 Let $\mathcal{L}_r = \langle [\mathrm{a_r}, \mathrm{b_r}, \mathrm{c_r}, \mathrm{d_r}];\ (\mu^1_{\mathcal{L}_r}, \mu^2_{\mathcal{L}_r}, \dots, \mu^{\mathrm{p}}_{\mathcal{L}_r}) \rangle$ be a collection of TFM-numbers for $r = 1,2,3, \dots, n$, $k = 1,2.3, \dots p$ and the associated weight vector of $\mathcal{L}_r$ is $w = (w_1, w_2, w_3, \dots, w_n)^T$ for $\sum_{r=1}^{n} w_r = 1$ then

$$\mathrm{TFMNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n) = \frac{1}{\frac{w_1}{\mathcal{L}_1} + \frac{w_2}{\mathcal{L}_2} + \cdots + \frac{w_n}{\mathcal{L}_n}}$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{n} \frac{w_r}{\mathrm{a}_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{\mathrm{b}_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{\mathrm{c}_r}}, \frac{1}{\sum_{r=1}^{n} \frac{w_r}{\mathrm{d}_r}} \right]; \left(\frac{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu^1_{\mathcal{L}_r})^{w_r} - \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu^1_{\mathcal{L}_r})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu^1_{\mathcal{L}_r})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu^1_{\mathcal{L}_r})^{\mathrm{w_r}}}, \right.\right.$$

$$\left.\left. \frac{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu^2_{\mathcal{L}_r})^{w_r} - \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu^2_{\mathcal{L}_r})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu^2_{\mathcal{L}_r})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu^2_{\mathcal{L}_r})^{\mathrm{w_r}}}, \dots, \frac{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu^{\mathrm{p}}_{\mathcal{L}_r})^{w_r} - \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu^{\mathrm{p}}_{\mathcal{L}_r})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}} (1 + \mu^{\mathrm{p}}_{\mathcal{L}_r})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}} (1 - \mu^{\mathrm{p}}_{\mathcal{L}_r})^{\mathrm{w_r}}} \right) \right\rangle \quad (4)$$

Proof When n=2, then TFMNWHM$(\mathcal{L}_1, \mathcal{L}_2)$ is calculated as follows:

$$\text{TFMNWHM}(\mathcal{L}_1, \mathcal{L}_2) = \frac{1}{\sum_{r=1}^{2} \frac{w_r}{\mathcal{L}_r}} = \frac{1}{\frac{w_1}{\mathcal{L}_1} + \frac{w_2}{\mathcal{L}_2}}$$

$$= \frac{1}{\frac{w_1}{\langle [\mathrm{a}_1, \mathrm{b}_1, \mathrm{c}_1, \mathrm{d}_1]; \left(\mu_{\mathcal{L}_1}^1, \mu_{\mathcal{L}_1}^2, \ldots, \mu_{\mathcal{L}_1}^{\mathrm{P}}\right)\rangle} + \frac{w_2}{\langle [\mathrm{a}_2, \mathrm{b}_2, \mathrm{c}_2, \mathrm{d}_2]; \left(\mu_{\mathcal{L}_2}^1, \mu_{\mathcal{L}_2}^2, \ldots, \mu_{\mathcal{L}_2}^{\mathrm{P}}\right)\rangle}}$$

$$= \frac{1}{w_1 \frac{1}{\left\langle \left[\frac{1}{\mathrm{d}_1}, \frac{1}{\mathrm{c}_1}, \frac{1}{\mathrm{b}_1}, \frac{1}{\mathrm{a}_1}\right]; \left(\mu_{\mathcal{L}_1}^1, \mu_{\mathcal{L}_1}^2, \ldots, \mu_{\mathcal{L}_1}^{\mathrm{P}}\right)\right\rangle} + w_2 \frac{1}{\left\langle \left[\frac{1}{\mathrm{d}_2}, \frac{1}{\mathrm{c}_2}, \frac{1}{\mathrm{b}_2}, \frac{1}{\mathrm{a}_2}\right]; \left(\mu_{\mathcal{L}_2}^1, \mu_{\mathcal{L}_2}^2, \ldots, \mu_{\mathcal{L}_2}^{\mathrm{P}}\right)\right\rangle}}$$

$$= \frac{1}{\frac{1}{\left\langle \left[\frac{w_1}{\mathrm{d}_1}, \frac{w_1}{\mathrm{c}_1}, \frac{w_1}{\mathrm{b}_1}, \frac{w_1}{\mathrm{a}_1}\right]; \left(\frac{\left(1+\mu_{\mathcal{L}_1}^1\right)^{w_1} - \left(1-\mu_{\mathcal{L}_1}^1\right)^{w_1}}{\left(1+\mu_{\mathcal{L}_1}^1\right)^{w_1} + \left(1-\mu_{\mathcal{L}_1}^1\right)^{w_1}}, \frac{\left(1+\mu_{\mathcal{L}_1}^2\right)^{w_1} - \left(1-\mu_{\mathcal{L}_1}^2\right)^{w_1}}{\left(1+\mu_{\mathcal{L}_1}^2\right)^{w_1} + \left(1-\mu_{\mathcal{L}_1}^2\right)^{w_1}}\right)\right\rangle} + \frac{1}{\left\langle \left[\frac{w_2}{\mathrm{d}_2}, \frac{w_2}{\mathrm{c}_2}, \frac{w_2}{\mathrm{b}_2}, \frac{w_2}{\mathrm{a}_2}\right];, \left(\frac{\left(1+\mu_{\mathcal{L}_1}^1\right)^{w_1} - \left(1-\mu_{\mathcal{L}_1}^1\right)^{w_1}}{\left(1+\mu_{\mathcal{L}_1}^1\right)^{w_1} + \left(1-\mu_{\mathcal{L}_1}^1\right)^{w_1}}, \frac{\left(1+\mu_{\mathcal{L}_1}^2\right)^{w_1} - \left(1-\mu_{\mathcal{L}_1}^2\right)^{w_1}}{\left(1+\mu_{\mathcal{L}_1}^2\right)^{w_1} + \left(1-\mu_{\mathcal{L}_1}^2\right)^{w_1}}\right)\right\rangle}}$$

$$= 1 \Big/ \left\langle \left[\sum_{r=1}^{2} \frac{w_r}{\mathrm{d}_r}, \sum_{r=1}^{2} \frac{w_r}{\mathrm{c}_r}, \sum_{r=1}^{2} \frac{w_r}{\mathrm{b}_r}, \sum_{r=1}^{2} \frac{w_r}{\mathrm{a}_r}\right]; \left(\frac{\prod_{\mathrm{r}=1}^{2}(1+\mu_{\mathcal{L}_r}^1)^{w_r} - \prod_{\mathrm{r}=1}^{2}(1-\mu_{\mathcal{L}_r}^1)^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{2}(1+\mu_{\mathcal{L}_r}^1)^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{2}(1-\mu_{\mathcal{L}_r}^1)^{\mathrm{w_r}}}, \frac{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^2)^{w_r} - \prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^2)^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^2)^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^2)^{\mathrm{w_r}}}\right)\right\rangle$$

$$= 1 \Big/ \left\langle \left[\frac{1}{\sum_{r=1}^{2} \frac{w_r}{\mathrm{a}_r}}, \frac{1}{\sum_{r=1}^{2} \frac{w_r}{\mathrm{b}_r}}, \frac{1}{\sum_{r=1}^{2} \frac{w_r}{\mathrm{c}_r}}, \frac{1}{\sum_{r=1}^{2} \frac{w_r}{\mathrm{d}_r}}\right]; \left(\frac{\prod_{\mathrm{r}=1}^{2}(1+\mu_{\mathcal{L}_r}^1)^{\mathrm{w_r}} - \prod_{\mathrm{r}=1}^{2}(1-\mu_{\mathcal{L}_r}^1)^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{2}(1+\mu_{\mathcal{L}_r}^1)^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{2}(1-\mu_{\mathcal{L}_r}^1)^{\mathrm{w_r}}}, \frac{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^2)^{w_r} - \prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^2)^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{n}}(1+\mu_{\mathcal{L}_r}^2)^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{n}}(1-\mu_{\mathcal{L}_r}^2)^{\mathrm{w_r}}}\right)\right\rangle$$

Suppose that Equation 4 holds for $n = k$, i.e.,

$$\text{TFMNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_k) = \frac{1}{\frac{w_1}{\mathcal{L}_1} + \frac{w_2}{\mathcal{L}_2} + \dots + \frac{w_k}{\mathcal{L}_k}}$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{k} \frac{w_r}{\mathrm{a}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{w_r}{\mathrm{b}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{w_r}{\mathrm{c}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{w_r}{\mathrm{d}_r}} \right]; \left(\frac{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{1})^{w_r} - \prod_{\mathrm{r}=1}^{k} (1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}} \right. \right.$$

$$\left. \left. \frac{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{2})^{w_r} - \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}, \dots, \frac{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{w_r} - \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}} \right) \right\rangle$$

For $n = k + 1$, using above expression and operational laws, we have

$$\text{TFMNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_k, \mathcal{L}_{k+1}) =$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{k} \frac{w_r}{\mathrm{a}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{w_r}{\mathrm{b}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{w_r}{\mathrm{c}_r}}, \frac{1}{\sum_{r=1}^{k} \frac{w_r}{\mathrm{d}_r}} \right]; \left(\frac{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{1})^{w_r} - \prod_{\mathrm{r}=1}^{k} (1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}} \right. \right.$$

$$\left. \left. \frac{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{2})^{w_r} - \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}, \dots, \frac{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{w_r} - \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}} (1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}} (1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}} \right) \right\rangle$$

$$= \left\langle \left[\frac{1}{\frac{w_{k+1}}{\mathrm{a}_{k+1}}}, \frac{1}{\frac{w_{k+1}}{\mathrm{b}_{k+1}}}, \frac{1}{\frac{w_{k+1}}{\mathrm{c}_{k+1}}}, \frac{1}{\frac{w_{k+1}}{\mathrm{d}_{k+1}}} \right]; \left(\frac{(1+\mu_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w}_{k+1}} - (1-\mu_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w}_{k+1}}}{(1+\mu_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w}_{k+1}} + (1-\mu_{\mathcal{L}_{k+1}}^{1})^{\mathrm{w}_{k+1}}} \right. \right.$$

$$\left. \left. \frac{(1+\mu_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w}_{k+1}} - (1-\mu_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w}_{k+1}}}{(1+\mu_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w}_{k+1}} + (1-\mu_{\mathcal{L}_{k+1}}^{2})^{\mathrm{w}_{k+1}}}, \dots, \frac{(1+\mu_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w}_{k+1}} - (1-\mu_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w}_{k+1}}}{(1+\mu_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w}_{k+1}} + (1-\mu_{\mathcal{L}_{k+1}}^{\mathrm{p}})^{\mathrm{w}_{k+1}}} \right) \right\rangle$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{k+1} \frac{w_r}{\mathrm{a}_r}}, \frac{1}{\sum_{r=1}^{k+1} \frac{w_r}{\mathrm{b}_r}}, \frac{1}{\sum_{r=1}^{k+1} \frac{w_r}{\mathrm{c}_r}}, \frac{1}{\sum_{r=1}^{k+1} \frac{w_r}{\mathrm{d}_r}} \right]; \left(\frac{\prod_{\mathrm{r}=1}^{\mathrm{k}+1} (1+\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}} - \prod_{\mathrm{r}=1}^{k+1} (1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}+1} (1+\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}+1} (1-\mu_{\mathcal{L}_r}^{1})^{\mathrm{w_r}}} \right. \right.$$

$$\left. \left. \frac{\prod_{\mathrm{r}=1}^{\mathrm{k}+1} (1+\mu_{\mathcal{L}_r}^{2})^{w_r} - \prod_{\mathrm{r}=1}^{\mathrm{k}+1} (1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}+1} (1+\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}+1} (1-\mu_{\mathcal{L}_r}^{2})^{\mathrm{w_r}}}, \dots, \frac{\prod_{\mathrm{r}=1}^{\mathrm{k}+1} (1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{w_r} - \prod_{\mathrm{r}=1}^{\mathrm{k}+1} (1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}}{\prod_{\mathrm{r}=1}^{\mathrm{k}+1} (1+\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}} + \prod_{\mathrm{r}=1}^{\mathrm{k}+1} (1-\mu_{\mathcal{L}_r}^{\mathrm{p}})^{\mathrm{w_r}}} \right) \right\rangle$$

So, the proof is complete.

Next, it can be easily shown that the proposed operator has the following properties.

Theorem 3.3 (Idempotency)

Let $\mathcal{L}_r = \langle [\mathrm{a_r}, \mathrm{b_r}, \mathrm{c_r}, \mathrm{d_r}];\ \left(\mu^1_{\mathcal{L}_r}, \mu^2_{\mathcal{L}_r}, \dots, \mu^{\mathrm{p}}_{\mathcal{L}_r}\right)\rangle$ be a collection of TFM-numbers for $r = 1,2,3,\dots,n$. If $\mathcal{L}_n = \mathcal{L}$ for all r that is all are identical then,

$$\mathrm{TFMNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n) = \mathcal{L}. \tag{5}$$

Proof We know that

$$\mathrm{TFMNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n) = \frac{1}{\frac{w_1}{\mathcal{L}_1} + \frac{w_2}{\mathcal{L}_2} + \dots + \frac{w_n}{\mathcal{L}_n}}$$

$$= \left\langle \left[\frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{a}_r}}, \frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{b}_r}}, \frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{c}_r}}, \frac{1}{\sum_{r=1}^{n}\frac{w_r}{\mathrm{d}_r}}\right]; \left(\frac{\prod_{\mathrm{r=1}}^{\mathrm{n}}(1+\mu^1_{\mathcal{L}_r})^{\mathrm{w}_r} - \prod_{\mathrm{r=1}}^{\mathrm{n}}(1-\mu^1_{\mathcal{L}_r})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}}(1+\mu^1_{\mathcal{L}_r})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}}(1-\mu^1_{\mathcal{L}_r})^{\mathrm{w_r}}}\right.\right.$$

$$\left.\left.\frac{\prod_{\mathrm{r=1}}^{\mathrm{n}}(1+\mu^2_{\mathcal{L}_r})^{\mathrm{w}_r} - \prod_{\mathrm{r=1}}^{\mathrm{n}}(1-\mu^2_{\mathcal{L}_r})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}}(1+\mu^2_{\mathcal{L}_r})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}}(1-\mu^2_{\mathcal{L}_r})^{\mathrm{w_r}}}, \dots, \frac{\prod_{\mathrm{r=1}}^{\mathrm{n}}(1+\mu^{\mathrm{p}}_{\mathcal{L}_r})^{\mathrm{w}_r} - \prod_{\mathrm{r=1}}^{\mathrm{n}}(1-\mu^{\mathrm{p}}_{\mathcal{L}_r})^{\mathrm{w_r}}}{\prod_{\mathrm{r=1}}^{\mathrm{n}}(1+\mu^{\mathrm{p}}_{\mathcal{L}_r})^{\mathrm{w_r}} + \prod_{\mathrm{r=1}}^{\mathrm{n}}(1-\mu^{\mathrm{p}}_{\mathcal{L}_r})^{\mathrm{w_r}}}\right)\right\rangle$$

$$= \left\langle \left[\frac{1}{\frac{\sum_{r=1}^{n} w_r}{\mathrm{a}}}, \frac{1}{\frac{\sum_{r=1}^{n} w_r}{\mathrm{b}}}, \frac{1}{\frac{\sum_{r=1}^{n} w_r}{\mathrm{c}}}, \frac{1}{\frac{\sum_{r=1}^{n} w_r}{\mathrm{d}}}\right]; \left(\frac{(1+\mu^1_{\mathcal{L}})^{\sum_{r=1}^{n} w_r} - (1-\mu^1_{\mathcal{L}})^{\sum_{r=1}^{n} w_r}}{(1+\mu^1_{\mathcal{L}})^{\sum_{r=1}^{n} w_r} + (1-\mu^1_{\mathcal{L}})^{\sum_{r=1}^{n} w_r}}\right.\right.$$

$$\left.\left.\frac{(1+\mu^2_{\mathcal{L}})^{\sum_{r=1}^{n} w_r} - (1-\mu^2_{\mathcal{L}})^{\sum_{r=1}^{n} w_r}}{(1+\mu^2_{\mathcal{L}})^{\sum_{r=1}^{n} w_r} + (1-\mu^2_{\mathcal{L}})^{\sum_{r=1}^{n} w_r}}, \dots, \frac{(1+\mu^p_{\mathcal{L}})^{\sum_{r=1}^{n} w_r} - (1-\mu^p_{\mathcal{L}})^{\sum_{r=1}^{n} w_r}}{(1+\mu^p_{\mathcal{L}})^{\sum_{r=1}^{n} w_r} + (1-\mu^p_{\mathcal{L}})^{\sum_{r=1}^{n} w_r}}\right)\right\rangle$$

$$= \left\langle \left[\frac{1}{\frac{1}{\mathrm{a}}}, \frac{1}{\frac{1}{\mathrm{b}}}, \frac{1}{\frac{1}{\mathrm{c}}}, \frac{1}{\frac{1}{\mathrm{d}}}\right]; \left(\frac{(1+\mu^1_{\mathcal{L}}) - (1-\mu^1_{\mathcal{L}})}{(1+\mu^1_{\mathcal{L}}) + (1-\mu^1_{\mathcal{L}})}, \frac{(1+\mu^2_{\mathcal{L}}) - (1-\mu^2_{\mathcal{L}})}{(1+\mu^2_{\mathcal{L}}) + (1-\mu^2_{\mathcal{L}})}, \dots, \frac{(1+\mu^p_{\mathcal{L}}) - (1-\mu^p_{\mathcal{L}})}{(1+\mu^p_{\mathcal{L}}) + (1-\mu^p_{\mathcal{L}})}\right)\right\rangle$$

$$= \mathcal{L}.$$

Theorem 3.4 (Monotoniticy property):

Let $\mathcal{L}_r = \langle [\mathrm{a_r}, \mathrm{b_r}, \mathrm{c_r}, \mathrm{d_r}];\ \left(\mu^1_{\mathcal{L}_r}, \mu^2_{\mathcal{L}_r}, \dots, \mu^{\mathrm{P}}_{\mathcal{L}_r}\right)\rangle$ and

$$\mathcal{L}'_r = \langle [\mathrm{a}'_r, \mathrm{b}'_r, \mathrm{c}'_r, \mathrm{d}'_r];, \left(\left(\mu'_{\mathcal{L}_r}\right)^1, \left(\mu'_{\mathcal{L}_r}\right)^2, \dots, \left(\mu'_{\mathcal{L}_r}\right)^p\right)\rangle$$

be two collection of TFM-numbers. If $\mathrm{a_r} \le \mathrm{a}'_r$, $\mathrm{b_r} \le \mathrm{b}'_r$, $\mathrm{c_r} \le \mathrm{c}'_r$, $\mathrm{d_r} \le \mathrm{d}'_r$ *and* $\mu^1_{\mathcal{L}_r} \le \left(\mu'_{\mathcal{L}_r}\right)^1$,

$\mu^2_{\mathcal{L}_r} \le \left(\mu'_{\mathcal{L}_r}\right)^2, \dots, \mu^{\mathrm{p}}_{\mathcal{L}_r} \le \left(\mu'_{\mathcal{L}_r}\right)^p$ then

$$\mathrm{TFMNWHM}^{\varphi}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n) \le \mathrm{TFMNWHM}^{\varphi}(\mathcal{L}'_1, \mathcal{L}'_2, \mathcal{L}'_3, \dots, \mathcal{L}'_n). \tag{6}$$

Theorem 3.5 (Commutativity Property):

Let $\mathcal{L}_r = \langle[\mathrm{a_r}, \mathrm{b_r}, \mathrm{c_r}, \mathrm{d_r}];\ (\mu^1_{\mathcal{L}_r}, \mu^2_{\mathcal{L}_r}, \dots, \mu^{\mathrm{p}}_{\mathcal{L}_r})\rangle$ be a collection of positive TFM-numbers and $w = (w_1, w_2, w_3, \dots, w_n)^T$ be an associated weight vector where $w_r \in [0,1]$, $\sum_{r=1}^{n} w_r = 1$.

$$\mathrm{TFMNWHM}^{\varphi}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n) = \mathrm{TFMNWHM}^{\varphi}(\mathcal{L}'_1, \mathcal{L}'_2, \mathcal{L}'_3, \dots, \mathcal{L}'_n). \tag{7}$$

where $\mathcal{L}'_n$ is any permutation of $\mathcal{L}_n$ for $r = 1,2,3, \dots, n$.

Theorem 3.6 (Boundedness Property):

Let $\mathcal{L}_r = \langle[\mathrm{a_r}, \mathrm{b_r}, \mathrm{c_r}, \mathrm{d_r}];\ (\mu^1_{\mathcal{L}_r}, \mu^2_{\mathcal{L}_r}, \dots, \mu^{\mathrm{p}}_{\mathcal{L}_r})\rangle$ be a collection of positive TFM-numbers and let,

$$\mathcal{L}^+_r = \left\langle\left[\max_r\{\mathrm{a_r}\}, \max_r\{\mathrm{b_r}\}, \max_r\{c_r\}, \max_r\{\mathrm{d_r}\}\right]; \left(\max_r\{\mu^1_{\mathcal{L}_r}\}, \max_r\{\mu^2_{\mathcal{L}_r}\}, \dots, \max_r\left\{\mu^{\mathrm{p}}_{\mathcal{L}_r}\right\}\right)\right\rangle$$

$$\mathcal{L}^-_r = \left\langle\left[\min_r\{\mathrm{a_r}\}, \min_r\{\mathrm{b_r}\}, \min_r\{c_r\}, \min_r\{\mathrm{d_r}\}\right]; \left(\min_r\{\mu^1_{\mathcal{L}_r}\}, \min_r\{\mu^2_{\mathcal{L}_r}\}, \dots, \min_r\left\{\mu^{\mathrm{p}}_{\mathcal{L}_r}\right\}\right)\right\rangle$$

then,

$$\mathcal{L}^-_r \le \mathrm{TFMNWHM}^{\varphi}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n) \le \mathcal{L}^+_r. \tag{8}$$

Definition 3.2 Let $\mathcal{L}_r = \langle[\mathrm{a_r}, \mathrm{b_r}, \mathrm{c_r}, \mathrm{d_r}];\ (\mu^1_{\mathcal{L}_r}, \mu^2_{\mathcal{L}_r}, \dots, \mu^{\mathrm{p}}_{\mathcal{L}_r})\rangle$ be a collection of positive TFM-number, then

$$S(\mathcal{L}_r) = \frac{1}{4p}[a + b + c + d] \times \left(2p + \sum_{r=1}^{p} \mu^{\mathrm{p}}_{\mathcal{L}_r}\right).$$

Example 3.2 : Let $\mathcal{L} = \langle[3,5,6,10];, (0.4,0.7,0.9)\rangle$ be NVNT-number then,

$$S(\mathcal{L}) = \frac{1}{4.3}[3 + 5 + 6 + 10] \times \big(6 + (0.4 + 0.7 + 0.9)\big) = 16$$

Definition 3.4 Let $\mathcal{L}^1_r$ and $\mathcal{L}^2_r$ be two TFM-numbers;

c. If $S(\mathcal{L}^1_r) < S(\mathcal{L}^2_r)$, then $\mathcal{L}^1_r$ is smaller than $\mathcal{L}^2_r$, denoted by $\mathcal{L}^1_r < \mathcal{L}^2_r$.

4.An algorithm for proposed work

In this section, we shall present a multi-criteria decision-making problem with normalized TFM-numbers under uncertain information using TFMNWHM operator.

Assume that $\mathrm{U} = \{\mathrm{U_1}, \mathrm{U_2}, \dots, \mathrm{U_m}\}$ be the set of altenatives and $\mathrm{C} = \{\mathrm{c_1}, \mathrm{c_2}, \dots, \mathrm{c_n}\}$ be the set of criterias;

$$(\mathrm{U_{kj}})_{\mathrm{mxn}} = \begin{pmatrix} \mathrm{U_{11}} & \mathrm{U_{12}} & \cdots & \mathrm{U_{1n}} \\ \mathrm{U_{21}} & \mathrm{U_{22}} & \cdots & \mathrm{U_{2n}} \\ \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \vdots \\ \mathrm{U_{m1}} & \mathrm{U_{m2}} & \cdots & \mathrm{U_{mn}} \end{pmatrix}$$

such that

$\mathrm{U_{kj}} = \langle [\mathrm{a_{kj}}, \mathrm{b_{kj}}, \mathrm{c_{kj}}, \mathrm{d_{kj}}], (\mu^1_{\mathrm{kj}}, \mu^2_{\mathrm{kj}}, \mu^3_{\mathrm{kj}}, \dots, \mu^{\mathrm{p}}_{\mathrm{kj}}) \rangle$, (k=1,2,…,m) and (j=1,2,…,n).

It is carried out the following algorithm to get best choice:

Step 1: Identify and determine the criterias and alternatives and then construct decision matrices,

$$(\mathrm{U_{kj}})_{\mathrm{mxn}}, \text{ (k=1,2,…,m; j=1,2,…,n).}$$

Step 2: Get preferable for $\mathrm{U_1}, \mathrm{U_2}, \dots, \mathrm{U_m}$ based on F_i $(i = 1,2,3, \dots, m)$ to aggregate the normalized TFM-numbers $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n$ as;

$$F_i = \mathrm{TFMNWHM}(\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n).$$

Step 3: Calculate score value whose formula is given in Definition 3.2 for each F_i to rank alternatives.

Step 4: Rank all score value of F_i according to descending order.

5.Application of the proposed method

In this section, an explanatory example is given to view the strength of the presented work. The increase in cyber war threats in the world obliges states to take precautions in this regard. Although developed countries have come a long way in this regard, there are still many countries that do not take adequate steps in this regard. Especially underde-veloped and developing countries, as they are insufficient in cyber warfare, can be vul-nerable and suffer victimization in case of any cyber-attack. In order to prevent this situa-tion, a few developing countries that decided to take action have taken the models of the countries that have achieved success in this subject to examination and have decided to take the model they found suitable for them as an example. Especially developing countries wants to use proposed method when choosing a model. The models he can take to are $\mathrm{U} = \{\mathrm{u_1} = \text{USA model}, \mathrm{u_2} = \text{Russian model}, \mathrm{u_3} = \text{Türkiye model}, \mathrm{u_4} = \text{China model}, \mathrm{u_5} = \text{Holland model}\}$ and according to three criteria determined $\mathrm{C} = \{\mathrm{c_1} = \text{full protection}, \mathrm{c_2} = \text{price}, \mathrm{c_3} = \text{usefulness}\}$. Thent we try to choose and rank all alternatives $\mathrm{K_k}$ for all k=1,2,…,5 by using the following algorithm.

Algorithm:

Step 1: The evaluation matrix$(\mathrm{U_{kj}})_{5x3}$ is given by an expert as;

$$(\mathrm{U_{kj}})_{5x3} = \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{matrix} \begin{pmatrix} \langle[0.22,0.25,0.41,0.69];(0.3,0.5,0.7,0.8),\rangle & \langle[0.28,0.32,0.38,0.43];(0.4,0.6,0.5,0.7)\rangle & \langle[0.28,0.27,0.50,0.85];(0.2,0.5,0.6,0.4)\rangle \\ \langle[0.31,0.35,0.36,0.45];(0.7,0.5,0.6,0.8)\rangle & \langle[0.12,0.15,0.18,0.23];(0.3,0.4,0.7,0.5)\rangle & \langle[0.22,0.23,0.38,0.63];(0.1,0.4,0.8,0.6)\rangle \\ \langle[0.46,0.62,0.69,0.76];(0.7,0.6,0.4,0.8)\rangle & \langle[0.55,0.66,0.72,0.75];(0.6,0.8,0.9,0.8)\rangle & \langle[0.37,0.53,0.63,0.83];(0.1,0.3,0.5,0.8)\rangle \\ \langle[0.23,0.29,0.46,0.99];(0.1,0.5,0.7,0.7)\rangle & \langle[0.14,0.15,0.27,0.37];(0.1,0.1,0.4,0.3)\rangle & \langle[0.67,0.73,0.83,0.93];(0.3,0.6,0.7,0.2)\rangle \\ \langle[0.20,0.21,0.43,0.78];(0.1,0.7,0.8,0.4)\rangle & \langle[0.12,0.13,0.19,0.69];(0.6,0.7,0.8,0.8)\rangle & \langle[0.42,0.43,0.68,0.74];(0.5,0.7,0.8,0.3)\rangle \end{pmatrix}$$

Step 2: Calculated for $\mathrm{u}_1, \mathrm{u}_2, \dots, \mathrm{u_m}$ based on F_i $(i = 1,2,3,\dots,m)$ to aggregate the normalized TFM-numbers $\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n$ follow as;

$F_1 = \langle[0.262,0.280,0.429,0.625];(0.0009,0.0070,0.0110,0.0132)\rangle$

$F_2 = \langle[0.180,0.209,0.271,0.383];(0.0009,0.0034,0.0209,0.0138)\rangle$

$F_3 = \langle[0.442,0.582,0.592,0.783];(0.0020,0.0077,0.0108,0.0406)\rangle$

$F_4 = \langle[0.239,0.267,0.431,0.614];(0.0001,0.0013,0.0105,0.0019)\rangle$

$F_5 = \langle[0.195,0.208,0.332,0.731];(0.0013,0.0210,0.0406,0.0048)\rangle$

Step 3: The calculated score value whose formula is given in Definition 3.2 for each *F* to rank alternatives;

$$S(F_1) = \frac{1}{4.4}[0.262 + 0.28 + 0.429 + 0.625] \times \big(8 + (0.0009 + 0.007 + 0.011 + 0.00132)\big) = 0.8001$$

Similar to

$$S(F_2) = 0.5241,\ S(F_3) = 1.2553,\ S(F_4) = 0.7768,\ S(F_5) = 0.7392$$

Step 4: Based on the score values $\mathrm{S}(F_\mathrm{i})(\mathrm{i} = 1,2,\dots,5)$ the ranking of alternatives $\mathrm{u_k}(\mathrm{k} = 1,2,\dots,5)$ are shown in Figure 1 and given as;

$$\mathrm{u}_3 > \mathrm{u}_1 > \mathrm{u}_4 > \mathrm{u}_5 > \mathrm{u}_2.$$

Finally the best alternative is u_3.

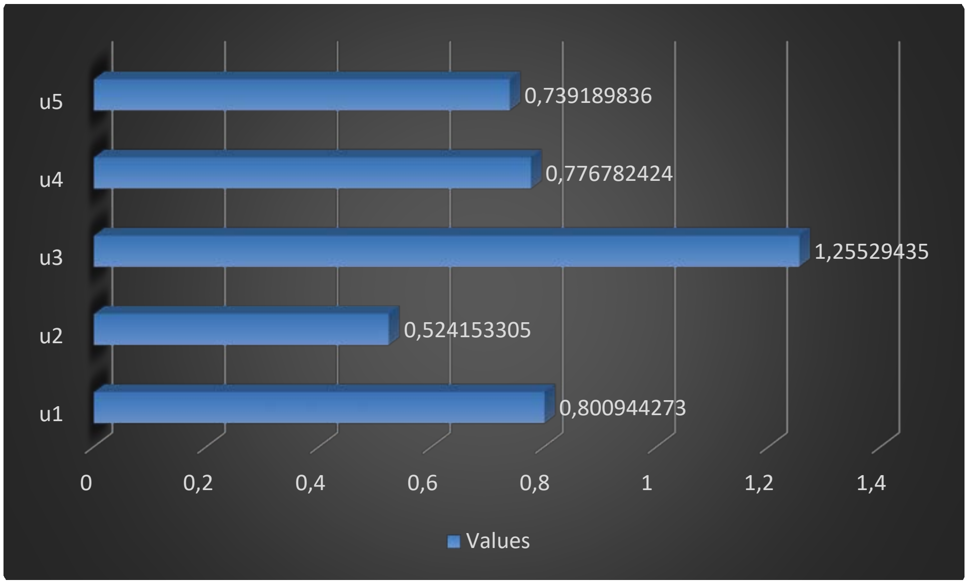

Figure 1 The ranking of alternatives u_k $(k = 1,2,\ldots,5)$

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Chapter Twelve

NEUTROSOPHIC INVENTORY MODEL WITH QUICK RETURN FOR DAMAGED MATERIALS AND PYTHON-ANALYSIS

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Abstract
The present study explores two distinct kinds of neutrosophic numbers to solve a neutrosophic control of inventory issue with an immediate return for defective items: triangular neutrosophic values and trapezoidal neutrosophic values. The triangular and trapezoidal neutrosophic figures represent the neutrosophic perfect rate(NPR), neutrosophic demand rates(NDR), and neutrosophic cost of purchase(NCP), respectively. To determine the ideal order quantity (IOQ) in neutrosophic terms, the median rule is applied. The idea for a model is presented with an example of Python analysis.

Keywords: Demand, Inventory Model, Fuzzy set, Neutrosophic, Defuzzification, Python.

1. Introduction

L. Zadeh (1965) was the first to present the idea of fuzzy sets. Since that time, numerous applications involving uncertainty have made extensive use of fuzzy sets and fuzzy logic. However, it has been shown that there are still some instances that fuzzy sets cannot account for, hence the interval-valued (Iv) fuzzy sets(FS) (Zadeh, 1975) was proposed to account for those circumstances. While fuzzy set theory is particularly effective at handling uncertainties resulting from the ambiguity or partial belongingness of an element in a set, it is unable to simulate various types of uncertainties that are present in various real-world issues, such as those that include incomplete information. Atanassov (1986) created intuitionistic fuzzy sets (IFs), a further generalisation of the fuzzy set. The studies need to give more emphasis on some important elements while working with inventory models, like deterioration. In recent years, the academic community has witnessed growing research interests in uncertainty set theory [26-56]. It is evident that depreciation is a time-dependent factor and that it also worsens with passage of time, which reduces consumer demand for the commodity. Holding costs thus have a considerable impact on the value of the amount that is stored. Products that are kept in storage gradually lose value due to depreciation. Iron, steel, toys, electronic devices, furniture, tools, jewellery, cars, sporting goods, and other durable things degrade slowly over time. Chang (2004) demonstrated how fuzzy sets theory may be used in the EOQ model that includes imperfect items of quality. The problem of receiving inventory in poor condition was looked at. For each order lot, Eroglu and Ozdemir in 2007 developed an EOQ model that accounts for certain damaged goods and backordered shortages. Wee (2007), which are released a study on an optimal inventory concept for products with imperfect cleanliness and shortfall backordering. This study made the assumption that all customers would be willing to wait for a new supply in cases of shortage. In a fuzzy inventory model developed by Ranganathan & Thirunavukarasu (2015), subpar products are returned right away. A paradigm for a non-scarce neutrosophic assessment was put out by Mullai & Broumi (2018). Smarandache was (2006) show that introduced neutrosophic set and neutrosophic logic by looking at non-standard analysis. Many research treating imprecision and uncertainty have been developed and studied[57-76]. Using neutrosophic concepts as neutrosophic ideal rate, neutrosophic market rate, and neutrosophic buy cost, this work aims to explore the inventory control issues with quick returns for defective goods before determining the neutrosophic ideal order quantity. Finally, a numerical illustration of the suggested model is offered. Food items, medications, clothing, cosmetics, and other semi-durable goods experience fast fluctuations in the deterioration rate. The study of the degradation of items in systems of inventory is also essential due to the diverse deteriorating patterns in the EOQ (economic order quantity) model. Tadikamalla developed an EOQ model using a gamma distribution to show the constant, increasing, and decreasing rates of deterioration over time. Alshanbari et al. suggested a two-parameter Weibull distribution-based inventory model for deteriorating goods. Wang and Lin developed the best replenishment technique by combining degradation, market demand, and price variations. Demand is important for inventory management because it is impossible to estimate future inventory without taking demand into account. The demand rate varies depending on the item. At the beginning of the cycle, several products experience an increase in demand. While the demand rate (DR) for some things remains constant, it increases as the product nears its end. The interest demand (ID) for baked items like bread, candy, cakes, & so on increases at the start of the cycle since consumers like freshly made goods. The expiration date causes the demand rate for products like fish, fruits and vegetables, and so on to decrease at the conclusion of the cycle. The inconsistent behavior of the systems is explained by neutrosophic numbers. Since the majority of the parameters, in reality, are unclear, neutrosophic numbers are essential in this scenario for removing uncertainty. Others, however, such as domestic commodities like milk, sugar, and other dairy products, as well as furniture and electrical equipment, have a stable demand rate throughout the cycle and a rising pace of demand at the end. Khedlekar and Sukhla developed a dynamic pricing model for logarithmic demand. Smaila and Chukwu presented a model of EOQ with quadratic patterns of demand and quasi-partial backlogs. Dutta Choudhury et al. created an inventory model employing a two-component demand. Prasad & Mukherjee developed an inventory model with time-dependent demand and stock availability. Wu developed a plan for stocks for demand patterns with a maximum lifespan under trade credits and a trapezoidal shape. Mullai and Surya developed a price-break EOQ model using triangular neutrosophic numbers to represent neutrosophic demand and purchasing cost. Mariagrazia et al. developed a supplier selection technique using uncertainty. Ge and Zhang presented an inventory model in a fuzzy, ambiguous setting. De developed an inventory model for the non-random uncertain environment using the neutrosophic fuzzy technique. The ability of machine learning algorithms to address a range of problems has long been a mystery, despite the recent ten years' worth of attention they have gotten. The vast majority of these techniques work under the assumption that the data is required to be true, complete, and unadulterated. Since the machine learning system cannot work if the learning issues are defined under a collection of unclear or inconsistent data, the data must be prepared, which makes the data science process exceedingly time-consuming and impractical. However, incomplete, inconsistent, unreliable, and confusing information is typically present in real learning problems. If we can model the learning problem as it is while utilising the flaws in the material, the data science process, which commonly switches from modelling, which is the last stage, to planning, which is the first step, can be sped up. Single-valued set neutrosophic (SVNs) are a paradigm for modelling missing information. Contrary to conventional machine learning methods, single-valued neutrosophic algorithms for learning cope with learning challenges involving complex information modelling by manipulating incomplete information. Recently, a variety of machine-learning techniques have been created to improve the performance of current learning algorithms and deal with imperfect input in practical settings.

2. Assumptions & Notations for Neutrosophic Inventory Model With Quick Return for Damaged Materials

2.1 Notations

R= Rate of Neutrosophic-Demand (Units per Year)

S=Neutrosophic-Unit Selling Cost

C=Neutrosophic-Purchase Cost

A=Neutrosophic-Hold price

B=Neutrosophic-Hold price

F=Neutrosophic-Order price

U=Neutrosophic-Perfect Cost

Y=Neutrosophic-Deficiency Rate

W=Rate of Neutrosophic-Screening (Units per Year)

Z=Neutrosophic-Screening Unit Cost

D=Neutrosophic-Order Size

Q=Neutrosophic-Cycle Length

2.2 Assumption

- A neutrosophic lot-size of D is thereafter replenished at the start of every neutrosophic inventory cycle (NIC).
- The neutrosophic lot should be screened periodically Q to time Qe. The rates of neutrophil demand (ND) and neutrophil screenings happen at the same time, and the former is higher than the latter (e > R).
- Following examination, any product that is shown to be flawed will be promptly sent back to the supplier.
- To avoid shortages, assume that at certain points throughout the screening procedure, the wide range of excellent goods is at least equivalent to the neutrosophic demand. e≥ R/U.
- The neutrosophic EOQ paradigm provides support for all additional hypotheses.

3. Neutrosophic Inventory Model With Quick Return for Damaged Materials: The Model Description

This portion provides neutrosophic triangular method(NTM) & neutrosophic trapezoidal method(NTrM) for the neutrosophic inventory framework with quick return for damaged materials to determine the ideal order quantity.

3.1 Neutrosophic Inventories Modelling with Quick Return for Damaged Material applying a triangular approach

In this model, we assume that the triangle neutrosophic values U, R, and C correspond to the neutrosophic perfection rate, neutrosophic supply rate, and neutrosophic purchase cost.

Suppose,

$U = (U1, U2, U3)(U1', U2', U3')(U1'', U2'', U3''),$

$R = (R1, R2, R3)(R1', R2', R3')(R1", R2", R3"),$

$C = (C1, C2, C3)(C1', C2', C3')(C1", C2", C3").$

The Neutrosophic Overall Profitability (Y (D)) then looks like this:

$$Y(D) = SR - (C \otimes R) + \frac{AD}{2e}(C \otimes R) - \frac{AD}{2}(C \otimes U) - \left(\frac{B}{D} + Z\right)\left(\frac{R}{U}\right) - \frac{AD}{2e}\left(C \otimes \left(\frac{R}{U}\right)\right)$$

The formula for the Neutrosophic the total expenses are as follows:

$$\begin{aligned} Y(D) = S\big((R1 + R2 + R3) + (R1"{+}R2" + R3")\big) \\ - \Big(\big((C1 + C2 + C3) + (C1{+}C2"{+}C3)\big) \\ + \big((R1 + R2 + R3) + (R1"{+}R2 + R3")\big)\Big) \\ + \frac{AD}{2e}\Big(\big((C1 + C2 + C3) + (C1{+}C2 + C3")) \\ + \big((R1 + R2 + R3) + (R1{+}R2 + R3")\big)\Big)\Big) \\ - \frac{AD}{2}\big((C1 + C2 + C3) + (C1"{+}C2" + C3") + (U1 + U2 + U3\big) \\ + (U1"{+}U2" + U3") \\ - \left(\frac{B}{D} + Z\right)\left(\left(\frac{R1}{U1} + \frac{R2}{U2} + \frac{R3}{U3}\right) + \left(\frac{R1"}{U1"} + \frac{R2"}{U2"} + \frac{R3"}{U3"}\right)\right) \\ - \frac{AD}{2e}\left(\frac{C1R1}{U1} + \frac{C2R2}{U2} + \frac{C3R3}{U3}\right) + \left(\frac{C1"R1"}{U1"} + \frac{C2"R2"}{U2"} + \frac{C3"R3"}{U3"}\right)) \end{aligned}$$

Given by is the Defuzzified Neutrosophic Overall Cost (DNOC).

$$Y(D) = \frac{1}{8}\{S\big((R1 + R2 + R3) + (R1"\text{+R2"} + R3")\big)$$
$$- \Big(\big((C1 + C2 + C3) + (C1\text{+C2"+C3})\big)$$
$$+ \big((R1 + R2 + R3) + (R1"\text{+R2} + R3")\big)\Big)$$
$$+ \frac{AD}{2e}\Big(\big((C1 + C2 + C3) + (C1\text{+C2} + C3")\big)$$
$$+ \big((R1 + R2 + R3) + (R1\text{+R2} + R3")\big)\Big)$$
$$- \frac{AD}{2}\big((C1 + C2 + C3) + (C1"\text{+C2"} + C3") + \ (U1 + U2 + U3\big)$$
$$+ (U1"\text{+U2"} + U3")$$
$$- \left(\frac{B}{D} + Z\right)\left(\left(\frac{R1}{U1} + \frac{R2}{U2} + \frac{R3}{U3}\right) + \left(\frac{R1"}{U1"} + \frac{R2"}{U2"} + \frac{R3"}{U3"}\right)\right)$$
$$- \frac{AD}{2e}\left(\frac{C1R1}{U1} + \frac{C2R2}{U2} + \frac{C3R3}{U3}\right) + \left(\frac{C1"R1"}{U1"} + \frac{C2"R2"}{U2"} + \frac{C3"R3"}{U3"}\right))\}$$

We get,

$$R = \sqrt{\frac{B}{A}}$$

Where, $B = 2be\big((R1 + 2R2 + R3) + (R1, \text{2R2} + R3")\big)$

$$A = a\big((C1 + 2C2 + C3) + (C1" + 2C2" + C3")\big)[\big(1 - (U1 + 2U2 + U3)$$
$$+ (U1\text{+2U2} + U3)))(\text{(R1+2R2+R3)+(R1} + 2R2\text{+R3)}\big)$$
$$+ e((U1^2 + 2U2^2 + U3^2) + (U1"^2 + 2U2"^2 + U3"^2)]$$

The parameters for the neutrosophic perfection rate, neutrosophic supply rate, and neutrosophic cost of purchase are U, R, and C, respectively. The neutrosophic-holding expense per unit time and the neutrosophic-holding expense per units/per unit time, respectively, are indicated above by the letters A and B.

3.2 Neutrosophic Inventory Modelling with a Trapezoidal Method including Quick Return for Damage Materials

In this model, we assume that the trapezoidal neutrosophic values U, R, and C for the neutrosophic perfection rate, neutrosophic supply rate, and neutrosophic purchase cost.

Let, $U = (U1, U2, U3, U4)(U1', U2', U3', U4')(U1", U2", U3", U4"),$

$$R = (R1, R2, R3, R4)(R1', R2', R3', R4')(R1, \text{R2}, R3, \text{R4}),$$

$$C = (C1, C2, C3, C4)(C1', C2', C3', C4')(C1", C2", C3", C4")$$

The Neutrosophic Total Gain Y(R)is the following.

$$Y(D) = SR - (C \otimes\ R) + \frac{AD}{2e}(C \otimes\ R) - \frac{AD}{2}(C \otimes\ \ U) - \left(\frac{B}{D} + Z\right)\left(\frac{R}{U}\right) - \frac{AD}{2e}(C \otimes (R/U))$$

The formula for the Neutrosophic total expenditure (NTE) is as follows:

$$Y(D) = \sum_{i=1}^{4} SR - (C\ R) + \frac{AD}{2e}(C\ R) - \frac{AD}{2}(C\ U) - \left(\frac{B}{D} + Z\right)\left(\frac{R}{U}\right) - \frac{AD}{2e}\left(C\left(\frac{R}{U}\right)\right) +$$

$$\sum_{i=1}^{4} SR' - (C'R') + \frac{AD}{2e}(C'R') - \frac{AD}{2}(C'U') - \left(\frac{B}{D} + Z\right)\left(\frac{R'}{U'}\right) - \frac{AD}{2e}\left(C'\left(\frac{R'}{U'}\right)\right) +$$

$$\sum_{i=1}^{4} SR" - \left(C^{"}R^{"}\right) + \frac{AD}{2e}\left(C^{"}R^{"}\right) - \frac{AD}{2}\left(C^{"}U^{"}\right) - \left(\frac{B}{D} + Z\right)\left(\frac{R^{"}}{U^{"}}\right) - \frac{AD}{2e}(C"(R"/U"))$$

The Defuzzified Neutrosophic Total Cost (DNTC) is given by

$$Y(D) = \frac{1}{8}[\sum_{i=1}^{4} SR - (C\ R) + \frac{AD}{2e}(C\ R) - \frac{AD}{2}(C\ U) - \left(\frac{B}{D} + Z\right)\left(\frac{R}{U}\right) - \frac{AD}{2e}\left(C\left(\frac{R}{U}\right)\right) +$$

$$\sum_{i=1}^{4} SR" - \left(C^{"}R^{"}\right) + \frac{AD}{2e}\left(C^{"}R^{"}\right) - \frac{AD}{2}\left(C^{"}U^{"}\right) - \left(\frac{B}{D} + Z\right)\left(\frac{R^{"}}{U^{"}}\right) - \frac{AD}{2e}(C"(R"/U"))]$$

We get,

$$D = \sqrt{\frac{\sum_{i=1}^{4} 2Be(R + R")}{\sum_{i=1}^{4}\{A(C + C)[(1\text{-}(U\text{+}U"))(R + R") + e(U^2 + U^2")]\}}}$$

The Neutrosophic-Order Size for the Neutrosophic Inventory Model with Quick Returning for Damaged Materials can be found here.

4. Neutrosophic Inventory Model with Quick Return for Damaged Materials: Mathematical Example

An organisation needs to determine the EOQ. However, the business is protected and will immediately refund any damaged items. According to the corporation, the overall demand (R) would likely be 4500 units year. Additionally, the buy price (c) is about \$20 each order, & perfect rates (U) and insufficient rates (Y) are both 0.9 for each order. The holding costs (A) are estimated to be about 0.25 cents per unit, the holding costs (B) to be about 100 cents per order, the sale price (S) to be about 175200, and the screening price (Z) to be about 0.5 cents.

The mathematical calculations and tables of Neutrosophic Inventory Model are as follows:

TABLE 1 Optimal order quantity for Neutrosophic Inventory Model -Using Triangular number(TN)

Parameters/Cases	CRISP SET(CS)	FUZZY SET(FS)	INTUTIONSTIC FUZZY SET(IFS)	NEUTROSOPHIC SET(NS)
R	4500	(4300, 4500, 4600)	(4300, 4500, 4600) (4100, 4500, 4800)	(4300, 4500, 4600) (4100, 4500, 48000) (3900, 4500, 5000)
U	0.9	(0.7, 0.9, 1.0)	(0.7, 0.9, 1.0) (0.3, 0.9, 1.1)	(0.7, 0.9, 1.0) (0.3, 0.9, 1.1) (0.1, 0.9, 1.3)
C	20	(18, 20, 21)	(18, 20, 21) (16, 20, 23)	(18, 20, 21) (16, 20, 23) (14, 20, 25)
OPTIMAL ORDER QUANTITY(OOQ)	1276.49	418.864	212.83	205.637

TABLE 2 Optimal order quantity for Neutrosophic Inventory Model -Using Trapezoidal number(TrN)

Parameters/Cases	CRISP SET(CS)	FUZZY SET(FS)	INTUTIONSTIC FUZZY SET(IFS)	NEUTROSOPHIC SET(NS)
D	4500	(4300, 4400, 4600, 4700)	4300, 4400, 4600, 4700) (4100, 4200, 4800, 4900)	(4300, 4400, 4600, 4700) (4100, 4200, 4800, 4900) (3900, 4000, 5000, 5100)
Q	0.9	(0.7, 0.8, 1.0, 1.1)	(0.7, 0.8, 1.0, 1.1) (0.3, 0.6, 0.9, 1.2)	(0.7, 0.8, 1.0, 1.1) (0.3, 0.6, 0.9, 1.2) (0.1, 0.5, 1.3, 1.7)
C	20	(18, 19, 21, 22)	(18, 19, 21, 22) (16, 17, 23, 24)	(18, 19, 21, 22) (16, 17, 23, 24) (14, 15, 25, 26)
OPTIMAL ORDER QUANTITY(OOQ)	1276.49	409.64	200.165	199.918

5. Neutrosophic Inventory Model Sensitivity Analysis & Observations With Quick Return For Damaged Materials

This section analyses the best order amount for the following sets: The findings are compared graphically.

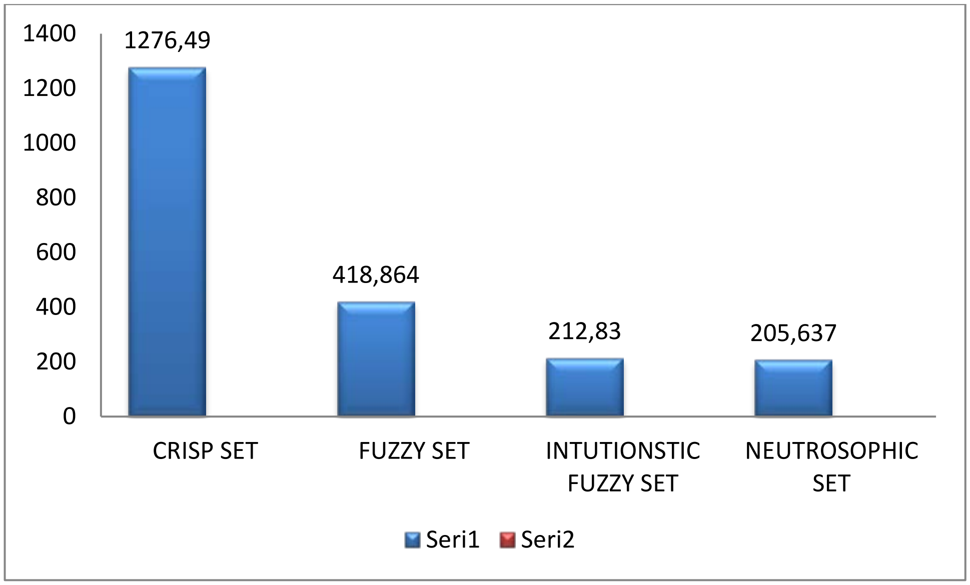

Figure 1. Sensitivity Analysis (Neutrosophic Inventory Model With Quick Return For Damaged Materials) for (CS), (FS), (IFS), and (NS) by triangular method(TM).

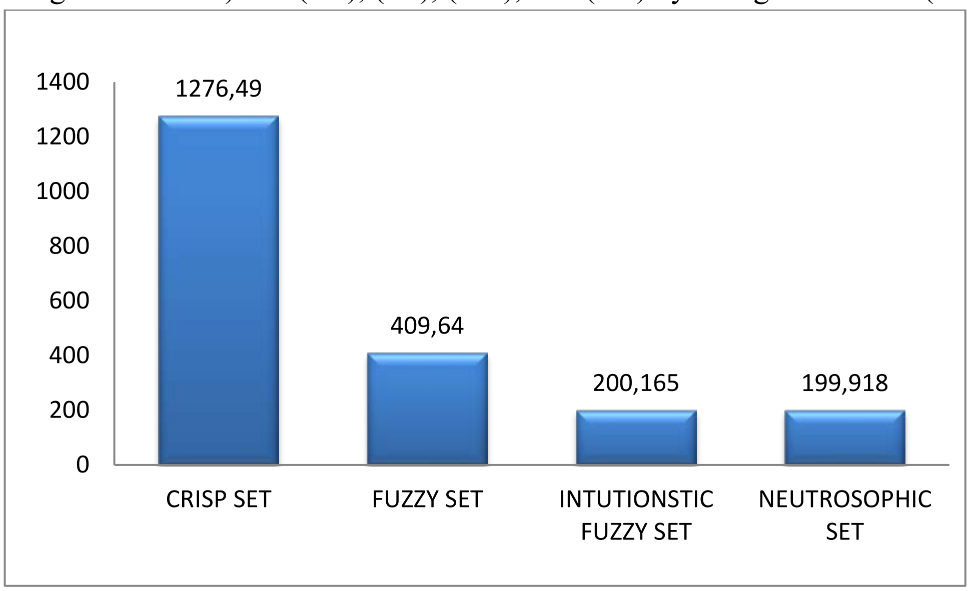

Figure 2**.** Sensitivity Analysis (Neutrosophic Inventory Model With Quick Return For Damaged Materials) for (CS), (FS), (IFS), and (NS) by trapezoidal method (TrM).

When compared to a crisp, fuzzy, and intuitionistic fuzzy set, a neutrosophic set provides the best solution for the ideal order quantity, according to the research discussed above. The trapezoidal neutrosophic method gives a better answer for the ideal order amount than the triangular neutrosophic approach happens. The neutrosophic optimal order quantity is minimum, illustrated in the Figure 1 & Figure 2.

6. Python Analysis - Neutrosophic Inventory Model with Quick Return for Damaged Materials

An informative violin-plot(VP) is superior to a straightforward box plot. In actuality, the violin-plot(VP) displays the entire distribution of the data, whereas a box plot just displays summary statistics like mean.

```
import matplotlib.pyplot as plt

import numpy as np

np.random.seed(10)

collectn_1 = np.random.normal(418, 212, 205)

collectn_2 = np.random.normal(409, 200, 199)

## combine these different collections into a list

data_to_plot = [collectn_1, collectn_2]

# Create a figure instance

fig = plt.figure()

# Create an axes instance

ax = fig.add_axes([0,0,1,1])

# Create the boxplot

bp = ax.violinplot(data_to_plot)

plt.show()
```

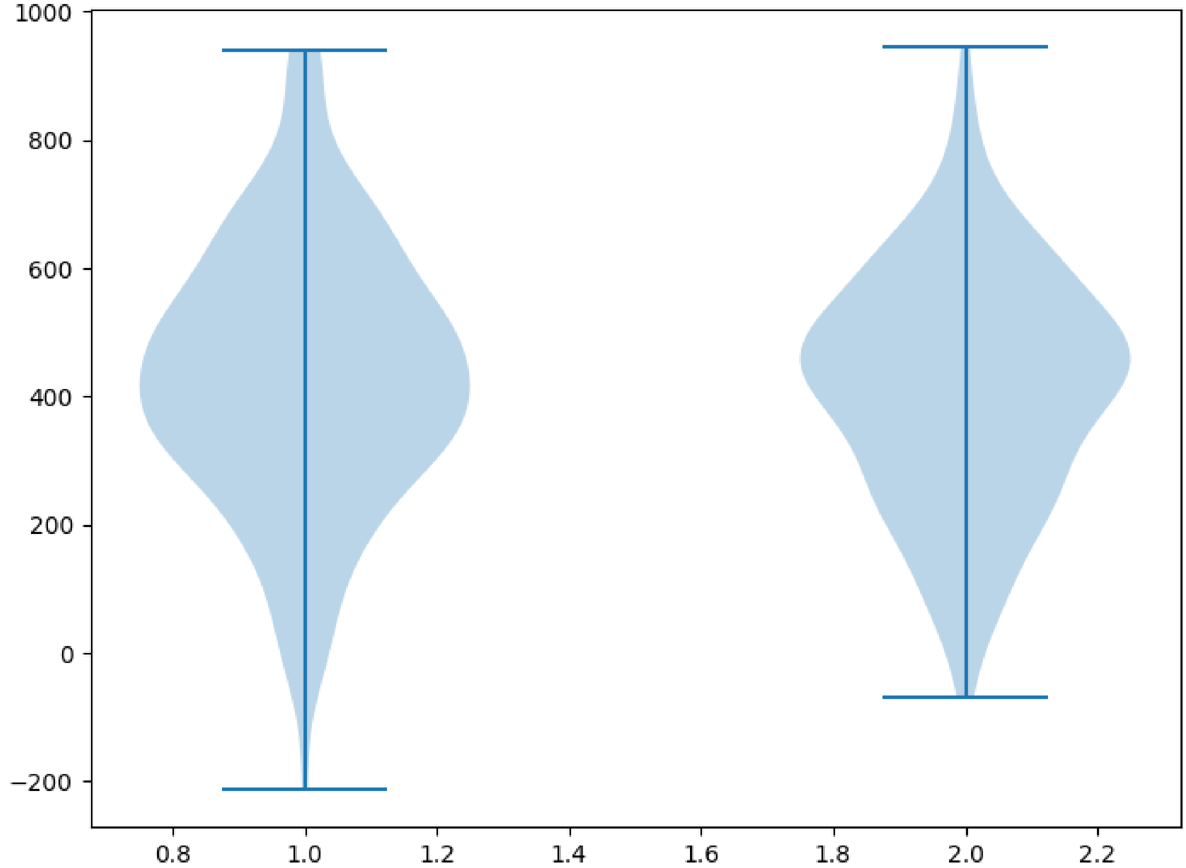

Figure 3. Python: Employ the triangular & trapezoidal methods to compare the Neutrosophic Inventory Model with Quick Return for Damaged Materials.

According to the research examined above, a neutrosophic set provides the best answer for the optimal order amount when compared to a crisp, fuzzy, and intuitionistic fuzzy set. In comparison to the triangular neutrosophic approach, the trapezoidal neutrosophic method provides a superior solution for the optimal order quantity. Figures illustrate that the neutrosophic optimal order quantity is minimal.

7. Conclusion

In neutrosophic perspectives, the study discusses the difficulty of inventory management with speedy returns for subpar products. The neutrosophic perfect rate, neutrosophic desire rate, and neutrosophic purchasing cost are calculated for the neutrosophic model using triangles and trapezoid neutrosophic numbers. The neutrosophic optimal order quantity is calculated using triangles and trapezoid neutrosophic numbers, and the problem is defuzzed using the median rule. According to the study, the trapezoid neutrosophic number provides a better answer for the ideal order amount than the triangular neutrosophic number. The trapezoidal neutrosophic method allows for both maintaining neutrosophic levels of stock and increasing total neutrosophic income.

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Chapter Thirteen

Bonferroni Geometric Mean Operator of Trapezoidal Fuzzy Multi Numbers and Its Application to Multiple Attribute Decision Making Problems

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Abstract

In the chapter, we introduce a new method for assessing solutions for multiple attribute decision-making problems that involve trapezoidal fuzzy multi numbers (TFM-numbers). To do this, we've developed a TFM-Bonferroni geometric mean operator to aggregate trapezoidal fuzzy multi numbers and examine properties of the TFM-Bonferroni geometric mean operator. Furthermore, we present an approach for multiple attribute decision making in the context of the TFM-numbers. To show the effectiveness and applicability of our method, we give a practical example under trapezoidal fuzzy multi contexts. In our concluding remarks, we introduce a comparative analysis table comparing our method with pre-existing techniques.

Keywords: Fuzzy multi set·Trapezoidal fuzzy number·Trapezoidal fuzzy multi numbers·Bonferroni geometric mean·Multiple attribute decision making

1 Introduction

By expanding the classical set under uncertain information fuzzy set theory, proposed by Zadeh [48] in 1965. After of introduction of the theory, it has found wide application areas in the literature. For instance, Marimin and Musthofa [24] delved into the implications of fuzzy logic systems within agro-industrial technology and engineering. Another notable contribution was by Bozkurt et al. [7], where they applied fuzzy logic in the legal domain, specifically for assessing the role of national human rights in safeguarding and endorsing human rights. As the field matured, specialized versions of fuzzy sets emerged, particularly focused on the real number set, $\mathbb{R}$. Dubois and Prade [16] then provided a comprehensive review of fuzzy numbers, further extending established operations on $\mathbb{R}$. Yun et al. [47] formulated generalized triangular fuzzy numbers based on Zadeh's extension principle. Rezvani [29] approached the ranking of exponential trapezoidal fuzzy numbers using variance. Further readings on trapezoidal and triangular fuzzy numbers include works by Alim et al. [2], Ban and Coroianu [5], Chen and Wang [9], Deli [11,12]. In recent years, Yager [42] proposed a novel expansion of fuzzy sets known as multi-fuzzy sets (or fuzzy bags). which is generalization of multi-sets and fuzzy sets. Then, multi-fuzzy sets are studied in [25,26,27,28,33,34]. Moreover, Uluçay et al. [38] proposed the trapezoidal fuzzy multi-numbers on the real number set $\mathbb{R}$. Then, Şahin et al. [31], Uluçay [37] and Kesen and Deli [22] conducted some works.

Introduced by Bonferroni [6] in 1950, Bonferroni operators are adept at identifying interrelations among various factors to aggegate trapezoidal fuzzy multi numbers. The operators have been the focal point of numerous research endeavors such as; Yager [43], Yu et al. [45], Zhu and Xu [50] Gong et al. [18], Garg and Arora [17], Wang et al. [39], Wang and Li [40], Deli [12] Yang and Pang [44], Abbas et al. [1], Kesen and Deli [21], Banerjee et al. [4], Ayub and Malik [3], Kakati and Borkotokey [19]. But, there hasn't been a study on Bonferroni aggregation operators based on trapezoidal fuzzy multi-numbers. Therefore, in second section, we provides foundational knowledge by defining terms such as fuzzy sets, fuzzy numbers, fuzzy multi-sets, and trapezoidal fuzzy multi-numbers. In third section, we developed a TFM-Bonferroni geometric mean operator. In fourth section, we introduce an algorithm for multi-attribute decision-making within the context of trapezoidal fuzzy multi numbers. In fifth section, an illustrative example is given to see application of the method. In sixth section, we give an analysis of the proposed approach by providing a brief comparative analysis of the methods with existing methods. Finally some conclusions are given in seventh section. The present expository chapter is a condensation of part of the dissertation prepared by Kesen [23].

2 Essential terms and operations

In this section, we proposed some basic concepts related to fuzzy sets, fuzzy numbers, fuzzy-multi sets and trapezoidal fuzzy multi-numbers which are needful for the next sections.

Definition 2.1 [49] Let X be a non-empty set. A fuzzy set F on X is defined as follows:

$$F = \{(x, \mu_F(x)) : x \in X\}$$

where $\mu_F : X \longrightarrow [0,1]$ for $x \in X$.

Definition 2.2 [28] Let X be a non-empty set. A fuzzy-multi set G on X is defined as follows:

$$G = \{(x, \mu_G^1(x),\ \mu_G^2(x), \dots, \mu_G^i(x), \dots) : x \in X\}$$

where $\mu_G^i : X \rightarrow [0, 1]$ for all i $\in$ {1, 2, ..., p} and $x \in X$.

Definition 2.3 [20] Let $\eta_A \in [0,1]$ and $a, b, c, d \in \mathbb{R}$ such that $a \le b \le c \le d$. Then, a generalized trapezoidal fuzzy number (GTF-number) $A = \langle (a, b, c, d); \eta_A \rangle$ is a special fuzzy set on the real number set $\mathbb{R}$, whose membership functions are defined as follows:

$$\mu_A(x) = \begin{cases} (x-a)\eta_A/(b-a) & a \le x < b \\ \eta_A & b \le x \le c \\ (d-x)\eta_A/(d-c) & c < x \le d \\ 0 & otherwise \end{cases}$$

Definition 2.4 [38] Let $\eta_N^s \in [0,1]$ $s \in \{1,2,\dots,p\}$ and $x_i, y_i, z_i, t_i \in \mathbb{R}$ such that $x_i \le y_i \le z_i \le t_i$. Then, trapezoidal fuzzy multi-number (TFM-number) shown by $N = \langle (x_i, y_i, z_i, t_i); \eta_N^1, \eta_N^2, \dots, \eta_N^P \rangle$ is a special fuzzy multi-set on the real numbers set $\mathbb{R}$ and its membership functions are defined as:

$$\mu_N^s(x)=\begin{cases}(x-x_i)\eta_N^s/(y_i-x_i) & x_i\le x\le y_i\\ \eta_N^s & y_i\le x\le z_i\\ (t_i-x)\eta_N^s/(t_i-z_i) & z_i\le x\le t_i\\ 0 & otherwise\end{cases}$$

Note that the set of all TFM-number on $\mathbb{R}^+$ will be denoted by $\mho(\mathbb{R}^+)$, $\{1,2,\ldots,n\}$ and $\{1,2,\ldots,m\}$ will be denoted by I_n and I_m respectively.

Definition 2.5 [38] Let $N_1=\langle(x_1,y_1,z_1,t_1);\eta_{N_1}^1,\eta_{N_1}^2,\ldots,\eta_{N_1}^P\rangle$ and $N_2=\langle(x_2,y_2,z_2,t_2);\eta_{N_2}^1,\eta_{N_2}^2,\ldots,\eta_{N_2}^P\rangle\in\mho(\mathbb{R}^+)$ and $\gamma\neq 0,\ \gamma\in\mathbb{R}$. Then,

1. $N_1+N_2=(x_1+x_2,y_1+y_2,z_1+z_2,t_1+t_2);\eta_{N_1}^1+\eta_{N_2}^1-\eta_{N_1}^1.\eta_{N_2}^1,\eta_{N_1}^2+\eta_{N_2}^2-\eta_{N_1}^2.\eta_{N_2}^2,\ldots,\eta_{N_1}^P+\eta_{N_2}^P-\eta_{N_1}^P.\eta_{N_2}^P)\rangle$
2. $N_1\times N_2=$
$$\begin{cases}\langle(x_1x_2,y_1y_2,z_1z_2,t_1t_2);\eta_{N_1}^1.\eta_{N_2}^1,\eta_{N_1}^2.\eta_{N_2}^2,\ldots\eta_{N_1}^P.\eta_{N_2}^P)\rangle & (t_1>0,t_2>0)\\ \langle(x_1t_2,y_1z_2,z_1y_2,t_1x_2);\eta_{N_1}^1.\eta_{N_2}^1,\eta_{N_1}^2.\eta_{N_2}^2,\ldots\eta_{N_1}^P.\eta_{N_2}^P)\rangle & (t_1<0,t_2>0)\\ \langle(t_1t_2,z_1z_2,y_1y_2,x_1x_2);\eta_{N_1}^1.\eta_{N_2}^1,\eta_{N_1}^2.\eta_{N_2}^2,\ldots\eta_{N_1}^P.\eta_{N_2}^P)\rangle & (t_1<0,t_2<0)\end{cases}$$
3. $\gamma N_1=\langle(\gamma x_1,\gamma y_1,\gamma z_1,\gamma t_1);1-(1-\eta_{N_1}^1)^\gamma,1-(1-\eta_{\bar N_1}^2)^\gamma,\ldots,1-(1-\eta_{N_1}^p)^\gamma\rangle(\gamma\ge 0)$
4. $N_1^\gamma=\langle(x_1^\gamma,y_1^\gamma,z_1^\gamma,t_1^\gamma);(\eta_{N_1}^1)^\gamma,(\eta_{N_1}^2)^\gamma,\ldots,(\eta_{N_1}^P)^\gamma\rangle(\gamma\ge 0)$

Definition 2.6 [21] Let $N_1=\langle(x_1,y_1,z_1,t_1);\eta_{N_1}^1,\eta_{N_1}^2,\ldots,\eta_{N_2}^P\rangle$ and $N_2=\langle(x_2,y_2,z_2,t_2);\eta_{N_2}^1,\eta_{N_2}^2,\ldots,\eta_{N_2}^P\rangle\in\mho(\mathbb{R}^+)$. Followings are right:

- If $x_1<x_2,\ y_1<y_2,\ z_1<z_2,\ t_1<t_2,\ \eta_{N_1}^1<\eta_{N_2}^1,\eta_{N_1}^2<\eta_{N_2}^2,\ldots,\eta_{N_1}^P<\eta_{N_2}^P$ then $N_1<N_2$.
- If $x_1>x_2,\ y_1>y_2,\ z_1>z_2,\ t_1>t_2,\ \eta_{N_1}^1>\eta_{N_2}^1,\eta_{N_1}^2>\eta_{N_2}^2,\ldots,\eta_{N_1}^P>\eta_{N_2}^P$ then $N_1>N_2$.
- If $x_1=x_2,\ y_1=y_2,\ z_1=z_2,\ t_1=t_2,\ \eta_{N_1}^1=\eta_{N_2}^1,\eta_{N_1}^2=\eta_{N_2}^2,\ldots,\eta_{N_1}^P=\eta_{N_2}^P$ then $N_1=N_2$.

Definition 2.7 Let $N=\langle(x_1,y_1,z_1,t_1);\eta_N^1,\eta_N^2,\ldots,\eta_N^P\rangle$ be a TFM-number. Value of N denoted $Val(N)$ based on centroid point denoted by $deff(N_i)$ is computed as;

$$Val(N)=\frac{\sum_{i=1}^P deff(N_i)}{P}$$

where

$$deff(N_i)=\frac{\int_{x_1}^{y_1}x\frac{(x-x_1)\eta_N^i}{(y_1-x_1)}dx+\int_{y_1}^{z_1}x\eta_N^i dx+\int_{z_1}^{t_1}x\frac{(t_1-x)\eta_N^i}{(t_1-z_1)}dx}{\int_{x_1}^{y_1}\frac{(x-x_1)\eta_N^i}{(y_1-x_1)}dx+\int_{y_1}^{z_1}\eta_N^i dx+\int_{z_1}^{t_1}\frac{(t_1-x)\eta_N^i}{(t_1-z_1)}dx},(i=1,2,\ldots,P)$$

Definition 2.8 [21] Let $N=\langle(x,y,z,t);\eta_N^1,\eta_N^2,\ldots,\eta_N^P\rangle$ be a TFM-number and P is number of η_N^i. Then score of N denoted $S(N)$ is defined as:

$$S(N)=\frac{t^2+z^2-x^2-y^2}{2.P}\sum_{s=1}^P\eta_N^s$$

2.1 Critic method for determining of weight of criteria

CRITIC method which was firstly introduced by Diakoulaki et al. [15] helps to decision makers to determine the weight of each criteria by means of values in the decision matrix. Its steps are given follows:

Step 1 Construct the decision matrix according to decision makers' preferences:

$$(D_{ij})_{mxn} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1x} \\ x_{21} & x_{22} & \cdots & x_{2x} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}$$

Step 2 Find normalised decision matrix as follows:

$$(\bar{D}_{ij})_{mxn} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1r} \\ r_{21} & r_{22} & \cdots & r_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ r_{m1} & r_{m2} & \cdots & r_{mn} \end{pmatrix}$$

where

$$r_{ij} = \frac{x_{ij} - x_j^{min}}{x_j^{max} - x_j^{min}} \quad (j \in I_n) \quad for \quad benefit \quad attribute$$

$$r_{ij} = \frac{x_j^{max} - x_{ij}}{x_j^{max} - x_j^{min}} \quad (j \in I_n) \quad for \quad cost \quad attribute$$

Step 3 Construct the relation-coefficient matrix as follows:

$$\rho_{jk} = \frac{\sum_{i=1}^{m} (r_{ij} - \bar{r}_j).(r_{ik} - \bar{r}_k)}{\sqrt{\sum_{i=1}^{m} (r_{ij} - \bar{r}_j)^2 . \sum_{i=1}^{m} (r_{ik} - \bar{r}_k)^2}} \qquad (j, k \in I_n)$$

Step 4 Critic method aims to get information from contrast and conflicts in the criteria. In this context, combining two concept and expressing aggregated information in *j*th criterion, c_j is computed as follows:

$$c_j = \sigma_j \sum_{k=1}^{n} (1 - \rho_{jk}) \quad (j \in I_n)$$

where

$$\sigma_j = \sqrt{\frac{\sum_{i=1}^{m} (r_{ij} - \bar{r}_j)^2}{m-1}}$$

Step 5 Computing weights of criteria:

$$w_j = \frac{c_j}{\sum_{k=1}^{n} c_j}$$

3 Bonferroni geometric mean operators on TFM numbers

Here, TFM-Bonferroni geometric mean operator to aggregate the TFM-information is developed. It is quoted/adopted and/or inspired and/or generalized from Deli [11,12], Deli and Keles [14], Yu et al. [45].

3.1 Bonferroni geometric mean operator on TFM numbers

Definition 3.1 Let $N_i = \langle (x_i, y_i, z_i, t_i); \eta^1_{N_i}, \eta^2_{N_i}, \ldots, \eta^P_{N_i} \rangle$ $(i \in I_n)$ be a TFM-numbers' collection, p and $q > 0$. Then, TFM Bonferroni geometric mean operator denoted by $TFMBGM^{(p,q)}$ is defined as:

$$TFMBGM^{(p,q)}(N_1, N_2, \ldots, N_n) = \frac{1}{p+q} \bigotimes_{i,j=1, i\neq j}^{n} ((p.N_i \oplus q.N_j))^{\frac{1}{n.(n-1)}} \quad (1)$$

or

$$TFMBGM^{(p,q)}(N_1, N_2, \ldots, N_n) = \frac{1}{p+q} \bigotimes_{i,j=1, i<j}^{n} ((p.N_i \oplus q.N_j) \otimes (p.N_j \oplus q.N_i))^{\frac{2}{n.(n-1)}} \quad (2)$$

Theorem 3.2 Let $N_i = \langle (x_i, y_i, z_i, t_i); \eta^1_{N_i}, \eta^2_{N_i}, \ldots, \eta^P_{N_i} \rangle$ $(i \in I_n)$ be a collection of TFM-numbers, p and $q > 0$. Then, aggregated value by using $TFMBGM^{(p,q)}$ operator is a TFM-number and computed as follows:

$$\begin{aligned}TFMBGM^{(p,q)}(N_1, N_2, \ldots, N_n) = \; & \langle (\frac{1}{p+q} \prod_{i,j=1,i<j}^{n} ((p.x_i + q.x_j).(p.x_j + q.x_i))^{\frac{2}{n.(n-1)}}, \\ & \frac{1}{p+q} \prod_{i,j=1,i<j}^{n} ((p.y_i + q.y_j).(p.y_j + q.y_i))^{\frac{2}{n.(n-1)}}, \\ & \frac{1}{p+q} \prod_{i,j=1,i<j}^{n} ((p.z_i + q.z_j).(p.z_j + q.z_i))^{\frac{2}{n.(n-1)}}, \\ & \frac{1}{p+q} \prod_{i,j=1,i<j}^{n} ((p.t_i + q.t_j).(p.t_j + q.t_i))^{\frac{2}{n.(n-1)}}); \\ & 1 - (1 - \prod_{i,j=1,i<j}^{n} [(1 - (1 - \eta^1_{N_i})^p.(1 - \eta^1_{N_j})^q). \\ & (1 - (1 - \eta^1_{N_j})^p.(1 - \eta^1_{N_i})^q)]^{\frac{2}{n.(n-1)}})^{\frac{1}{p+q}}, \\ & 1 - (1 - \prod_{i,j=1,i<j}^{n} [(1 - (1 - \eta^2_{N_i})^p.(1 - \eta^2_{N_j})^q). \\ & (1 - (1 - \eta^2_{N_j})^p.(1 - \eta^2_{N_i})^q)]^{\frac{2}{n.(n-1)}})^{\frac{1}{p+q}}, \ldots, \\ & 1 - (1 - \prod_{i,j=1,i<j}^{n} [(1 - (1 - \eta^P_{N_i})^p.(1 - \eta^P_{N_j})^q). \\ & (1 - \left(1 - \eta^P_{N_j}\right)^p.\left(1 - \eta^P_{N_i}\right)^q)]^{\frac{2}{n.(n-1)}})^{\frac{1}{p+q}} \rangle \quad (3)\end{aligned}$$

Proposition 3.3 Let $N_i = \langle (x_i, y_i, z_i, t_i); \eta^1_{N_i}, \eta^2_{N_i}, \ldots, \eta^P_{N_i} \rangle$ $(i \in I_n)$ and $M_i = \langle (k_i, l_i, m_i, n_i); \eta^1_{M_i}, \eta^2_{M_i}, \ldots, \eta^P_{M_i} \rangle$ $(i \in I_n)$ be two collections of TFM-numbers,

1. **(Monotonicity)** Based on Definition 2.6, if $x_i \leq k_i, y_i \leq l_i, z_i \leq m_i, t_i \leq n_i (i \in I_n)$ and $\eta^1_{N_i} \leq \eta^1_{M_i}$, $\eta^2_{N_i} \leq \eta^2_{M_i}$,...,$\eta^p_{N_i} \leq \eta^p_{M_i}$ then

$$TFMBGM^{(p,q)}(N_1, N_2, \ldots, N_n) \leq TFMBGM^{(p,q)}(M_1, M_2, \ldots, M_n)$$

2. **(Boundedness)**

$$N^- \leq TFMBGM^{(p,q)}(N_1, N_2, \ldots, N_n) \leq N^+$$

where

$$N^{+} = \langle (max\{x_i\}_{i\in I_n}, max\{y_i\}_{i\in I_n}, max\{z_i\}_{i\in I_n}, max\{t_i\}_{i\in I_n});$$

$$max\{\eta^1_{N_i}\}_{i\in I_n}, max\{\eta^2_{N_i}\}_{i\in I_n}, \ldots, max\{\eta^P_{N_i}\}_{i\in I_n}\rangle$$

and

$$N^{-} = \langle (min\{x_i\}_{i\in I_n}, min\{y_i\}_{i\in I_n}, min\{z_i\}_{i\in I_n}, min\{t_i\}_{i\in I_n});$$

$$min\{\eta^1_{N_i}\}_{i\in I_n}, min\{\eta^2_{N_i}\}_{i\in I_n}, \ldots, min\{\eta^P_{N_i}\}_{i\in I_n}\rangle$$

3. **(Commutativity)** If $(\dot{N}_1, \dot{N}_2, \ldots, \dot{N}_n)$ any permutation of $(N_1, N_2, \ldots, N_n)$, then

$$TFMBGM^{(p,q)}(N_1, N_2, \ldots, N_n) = (\frac{1}{n.(n-1)} \bigotimes_{i,j=1, i\neq j}^{n} (N_i^p \oplus N_j^q))^{\frac{1}{p+q}}$$

$$= (\frac{1}{n.(n-1)} \bigoplus_{i,j=1, i\neq j}^{n} (\dot{N}_i^{\,p} \oplus \dot{N}_j^{\,q})^{\frac{1}{p+q}}$$

$$= TFMBGM^{(p,q)}(\dot{N}_1, \dot{N}_2, \ldots, \dot{N}_n)$$

4. **(Idempotent Commutativity)** If we interchange p and q parameters, we have:

$$TFMBGM^{(p,q)}(N_1, N_2, \ldots, N_n) = \frac{1}{p+q} \bigotimes_{i,j=1, i<j}^{n} ((p.N_i \oplus q.N_j) \otimes (p.N_j \oplus q.N_i))^{\frac{2}{n.(n-1)}}$$

$$= \frac{1}{q+p} \bigotimes_{i,j=1, i<j}^{n} ((q.N_i \oplus p.N_j) \otimes (q.N_j \oplus p.N_i))^{\frac{2}{n.(n-1)}}$$

$$= TFMBGM^{(q,p)}(N_1, N_2, \ldots, N_n)$$

Next, if we change the parameters p and q of the $TFMBGM^{(p,q)}$ operator then, we can get some special cases of $TFMBGM^{(p,q)}$ as follows:

Case 1. If $q = 0$, $TFMBGM^{(p,q)}$ operator converted into a generalized TFM geometric mean operator:

$$TFMBGM^{(p,q)}(N_1, N_2, \ldots, N_n) = \frac{1}{p} \bigotimes_{i,j=1, i<j}^{n} (p.N_i \otimes p.N_j)^{\frac{2}{n.(n-1)}}$$

$$= \langle (\frac{1}{p}\prod_{i,j=1,i<j}^{n} (px_i.px_j)^{\frac{2}{n.(n-1)}}, \frac{1}{p}\prod_{i,j=1,i<j}^{n} (py_i.py_j)^{\frac{2}{n.(n-1)}},$$

$$\frac{1}{p}\prod_{i,j=1,i<j}^{n} (pz_i.pz_j)^{\frac{2}{n.(n-1)}}, \frac{1}{p}\prod_{i,j=1,i<j}^{n} (pt_i.pt_j)^{\frac{2}{n.(n-1)}});$$

$$1-(1-\prod_{i,j=1,i<j}^{n} [(1-(1-\eta^1_{N_i})^p.(1-(1-\eta^1_{N_j})^p]^{\frac{2}{n.(n-1)}})^{\frac{1}{p}},$$

$$1-(1-\prod_{i,j=1,i<j}^{n} [(1-(1-\eta^2_{N_i})^p.(1-(1-\eta^2_{N_j})^p]^{\frac{2}{n.(n-1)}})^{\frac{1}{p}}, \ldots,$$

$$1-(1-\prod_{i,j=1,i<j}^{n} [(1-(1-\eta^P_{N_i})^p.(1-(1-\eta^P_{N_j})^p]^{\frac{2}{n.(n-1)}})^{\frac{1}{p}}\rangle$$

$$= TFMBGM^{(p,0)}(N_1, N_2, \ldots, N_n) \qquad (4)$$

Case 2. If $p = 1$ and $q = 0$, $TFMBGM^{(p,q)}$ operator converted into a TFM geometric mean operator:

$$
\begin{aligned}
TFMBGM^{(p,q)}(N_1,N_2,\ldots,N_n) &= \bigotimes_{i,j=1,i<j}^{n}(N_i\otimes N_j)^{\frac{2}{n.(n-1)}}\\
&= \langle(\textstyle\prod_{i,j=1,i<j}^{n}(x_i.x_j)^{\frac{2}{n.(n-1)}},\prod_{i,j=1,i<j}^{n}(y_i.y_j)^{\frac{2}{n.(n-1)}},\\
&\textstyle\prod_{i,j=1,i<j}^{n}(z_i.z_j)^{\frac{2}{n.(n-1)}},\prod_{i,j=1,i<j}^{n}(t_i.t_j)^{\frac{2}{n.(n-1)}});\\
&1-(1-\textstyle\prod_{i,j=1,i<j}^{n}[(1-(1-\eta_{N_i}^{1}).(1-(1-\eta_{N_j}^{1})]^{\frac{2}{n.(n-1)}}),\\
&1-(1-\textstyle\prod_{i,j=1,i<j}^{n}[(1-(1-\eta_{N_i}^{2}).(1-(1-\eta_{N_j}^{2})]^{\frac{2}{n.(n-1)}}),\ldots,\\
&1-(1-\textstyle\prod_{i,j=1,i<j}^{n}[\left(1-(1-\eta_{N_i}^{P}).\left(1-\left(1-\eta_{N_j}^{P}\right)\right]^{\frac{2}{n.(n-1)}}\right)\rangle \quad (5)
\end{aligned}
$$

Case 3. If $p=2$ and $q=0$, $TFMBGM^{(p,q)}$ operator converted into TFM square geometric mean operator:

$$
\begin{aligned}
TFMBGM^{(p,q)}(N_1,N_2,\ldots,N_n) &= \frac{1}{2}\bigotimes_{i,j=1,i<j}^{n}(2.N_i\otimes 2.N_j)^{\frac{2}{n.(n-1)}}\\
&= \langle(\tfrac{1}{2}\textstyle\prod_{i,j=1,i<j}^{n}(2x_i.2x_j)^{\frac{2}{n.(n-1)}},\tfrac{1}{2}\prod_{i,j=1,i<j}^{n}(2y_i.2y_j)^{\frac{2}{n.(n-1)}},\\
&\tfrac{1}{2}\textstyle\prod_{i,j=1,i<j}^{n}(2z_i.2z_j)^{\frac{2}{n.(n-1)}},\tfrac{1}{2}\prod_{i,j=1,i<j}^{n}(2t_i.2t_j)^{\frac{2}{n.(n-1)}});\\
&1-(1-\textstyle\prod_{i,j=1,i<j}^{n}[(1-(1-\eta_{N_i}^{1})^2.(1-(1-\eta_{N_j}^{1})^2]^{\frac{2}{n.(n-1)}})^{\frac{1}{2}},\\
&1-(1-\textstyle\prod_{i,j=1,i<j}^{n}[(1-(1-\eta_{N_i}^{2})^2.(1-(1-\eta_{N_j}^{2})^2]^{\frac{2}{n.(n-1)}})^{\frac{1}{2}},\ldots,\\
&1-(1-\textstyle\prod_{i,j=1,i<j}^{n}[(1-(1-\eta_{N_i}^{P})^2.(1-(1-\eta_{N_j}^{P})^2]^{\frac{2}{n.(n-1)}})^{\frac{1}{2}},\rangle
\end{aligned} \quad (6)
$$

Case 4. If $p=q=1$, $TFMBGM^{(p,q)}$ operator converted into a TFM interrelated square geometric mean operator:

$$
\begin{aligned}
TFMBGM^{(p,q)}(N_1,N_2,\ldots,N_n) &= \frac{1}{2}\bigotimes_{i,j=1,i<j}^{n}((N_i\oplus N_j)\otimes(N_j\oplus N_i))^{\frac{2}{n.(n-1)}}\\
&= \langle(\tfrac{1}{2}\textstyle\prod_{i,j=1,i<j}^{n}((x_i+x_j)^2)^{\frac{2}{n.(n-1)}},\tfrac{1}{2}\prod_{i,j=1,i<j}^{n}((y_i+y_j)^2)^{\frac{2}{n.(n-1)}},\\
&\tfrac{1}{2}\textstyle\prod_{i,j=1,i<j}^{n}((z_i+z_j)^2)^{\frac{1}{n.(n-1)}},\tfrac{1}{2}\prod_{i,j=1,i<j}^{n}((t_i+t_j)^2)^{\frac{2}{n.(n-1)}});\\
&1-(1-\textstyle\prod_{i,j=1,i<j}^{n}[(1-(1-\eta_{N_i}^{1}).(1-\eta_{N_j}^{1})).\\
&(1-(1-\eta_{N_j}^{1}).(1-\eta_{N_i}^{1}))]^{\frac{2}{n.(n-1)}})^{\frac{1}{2}},\\
&1-(1-\textstyle\prod_{i,j=1,i<j}^{n}[(1-(1-\eta_{N_i}^{2}).(1-\eta_{N_j}^{2})).\\
&(1-(1-\eta_{N_j}^{2}).(1-\eta_{N_i}^{2}))]^{\frac{2}{n.(n-1)}})^{\frac{1}{2}},\ldots,\\
&1-(1-\textstyle\prod_{i,j=1,i<j}^{n}[(1-(1-\eta_{N_i}^{P}).(1-\eta_{N_j}^{P})).\\
&(1-(1-\eta_{N_j}^{P}).(1-\eta_{N_i}^{P}))]^{\frac{2}{n.(n-1)}})^{\frac{1}{2}}\rangle\\
&= TFMBGM^{(1,1)}(N_1,N_2,\ldots,N_n) \quad (7)
\end{aligned}
$$

Now, we give an example to illustrate the results below:

Example 3.4 Assume that we have three TFM-numbers as follows;

$$N_1 = \langle (0.1,0.4,0.5,0.6); 0.5,0.3,0.4,0.2 \rangle$$
$$N_2 = \langle (0.1,0.2,0.5,0.8); 0.9,0.6,0.3,0.5 \rangle$$
$$N_3 = \langle (0.2,0.3,0.3,0.4); 0.7,0.8,0.3,0.4 \rangle.$$

Then based on the operations in Definition 2.5 and Equation (3) for $p, q = 1$, we have

$$N_1^1 \oplus N_2^1 = \langle (0.2,0.6,1,1.4); 0.95,0.72,0.58,0.6 \rangle$$
$$N_2^1 \oplus N_1^1 = \langle (0.2,0.6,1,1.4); 0.95,0.72,0.58,0.6 \rangle$$
$$N_1^1 \oplus N_3^1 = \langle (0.3,0.7,0.8,1); 0.85,0.86,0.58,0.52 \rangle$$
$$N_3^1 \oplus N_1^1 = \langle (0.3,0.7,0.8,1); 0.85,0.86,0.58,0.52 \rangle$$
$$N_2^1 \oplus N_3^1 = \langle (0.3,0.5,0.8,1.2); 0.97,0.92,0.51,0.7 \rangle$$
$$N_3^1 \oplus N_2^1 = \langle (0.3,0.5,0.8,1.2); 0.97,0.92,0.51,0.7 \rangle$$

and then, we obtain:

$TFMBGM^{(1,1)}(N_1, N_2, N_3) = \langle (0.034,0.176,0.371,0.706); 0.612,0.440,0.168,0.201 \rangle$
In a similar way, if $p, q = 2$, from Equation (3) we have:

$TFMBGM^{(2,2)}(N_1, N_2, N_3) = \langle (0.068,0.353,0.742,1.413); 0.637,0.486,0.227,0.261 \rangle$
if $p = 1, q = 3$, from Equation (3) we have:

$TFMBGM^{(1,3)}(N_1, N_2, N_3) = \langle (0.067,0.348,0.734,1.392); 0.613,0.450,0.225,0.248 \rangle$
if $p = 3, q = 1$, from Equation (3) we have:

$TFMBGM^{(3,1)}(N_1, N_2, N_3) = \langle (0.067,0.348,0.734,1.392); 0.613,0.450,0.225,0.248 \rangle$
if $p = 10, q = 2$, from Equation (3) we have:

$TFMBGM^{(10,2)}(N_1, N_2, N_3) = \langle (0.199,1.032,2.186,4.131); 0.576,0.406,0.284,0.267 \rangle$

Definition 3.5 Let $N_i = \langle (x_i, y_i, z_i, t_i); \eta_{N_i}^1, \eta_{N_i}^2, \ldots, \eta_{N_i}^P \rangle$ $(i \in I_n)$ be a TFM-numbers' collection, $p, q > 0$ and N_i*'s* weight vector is $w = (w_1, w_2, \ldots, w_n)^T$. Here, w_i is N_i*'s* importance degree, satisfying $w_i \in [0,1]$, $(i \in I_n)$ such that $\sum_{i=1}^{n} w_i = 1$. Then, weighted trapezoidal fuzzy multi geometric Bonferroni mean denoted by $TFMBGM_w^{(p,q)}$ is defined as:

$$TFMBGM_w^{(p,q)}(N_1, N_2, \ldots, N_n) \quad = \frac{1}{p+q} \Big(\bigotimes_{i,j=1, i<j}^{n} ((p.N_i^{w_i} \oplus q.N_j^{w_j}) \otimes (p.N_j^{w_j} \oplus q.N_i^{w_i}) \Big)^{\frac{2}{n.(n-1)}}$$

Theorem 3.6 Let $N_i = \langle (x_i, y_i, z_i, t_i); \eta_{N_i}^1, \eta_{N_i}^2, \ldots, \eta_{N_i}^P \rangle$ $(i \in I_n)$ be a TFM-numbers' collection, $p, q > 0$ and N_i*'s* weight vector is $w = (w_1, w_2, \ldots, w_n)^T$. Here, w_i is N_i*'s* importance degree, satisfying $w_i \in [0,1]$, $(i \in I_n)$ such that $\sum_{i=1}^{n} w_i = 1$. Then, aggregated value by using $TFMBGM_w^{(p,q)}$ operator is a TFM-number and computed as follows:

$$TFMBGM_w^{(p,q)}(N_1, N_2, \ldots, N_n) = \langle (\frac{1}{p+q} \prod_{i,j=1,i<j}^{n} ((p.x_i^{w_i} + q.x_j^{w_j}).(p.x_j^{w_j} + q.x_i^{w_i}))^{\frac{2}{n.(n-1)}},$$
$$\frac{1}{p+q} \prod_{i,j=1,i<j}^{n} ((p.y_i^{w_i} + q.y_j^{w_j}).(p.y_j^{w_j} + q.y_i^{w_i}))^{\frac{2}{n.(n-1)}},$$
$$\frac{1}{p+q} \prod_{i,j=1,i<j}^{n} ((p.z_i^{w_i} + q.z_j^{w_j}).(p.z_j^{w_j} + q.z_i^{w_i}))^{\frac{2}{n.(n-1)}},$$
$$\frac{1}{p+q} \prod_{i,j=1,i<j}^{n} ((p.t_i^{w_i} + q.t_j^{w_j}).(p.t_j^{w_j} + q.t_i^{w_i}))^{\frac{2}{n.(n-1)}});$$
$$1 - (1 - \prod_{i,j=1,i<j}^{n} [(1 - (1 - (\eta_{N_i}^1)^{w_i})^p.(1 - (\eta_{N_j}^1)^{w_j})^q).$$
$$(1 - (1 - (\eta_{N_j}^1)^{w_j})^p.(1 - (\eta_{N_i}^1)^{w_i})^q)]^{\frac{2}{n.(n-1)}})^{\frac{1}{p+q}}, \qquad (8)$$
$$1 - (1 - \prod_{i,j=1,i<j}^{n} [(1 - (1 - (\eta_{N_i}^2)^{w_i})^p.(1 - (\eta_{N_j}^2)^{w_j})^q).$$
$$(1 - (1 - (\eta_{N_j}^2)^{w_j})^p.(1 - (\eta_{N_i}^2)^{w_i})^q)]^{\frac{2}{n.(n-1)}})^{\frac{1}{p+q}}, \ldots,$$
$$1 - (1 - \prod_{i,j=1,i<j}^{n} [(1 - (1 - (\eta_{N_i}^P)^{w_i})^p.(1 - (\eta_{N_j}^P)^{w_j})^q).$$
$$(1 - (1 - (\eta_{N_j}^P)^{w_j})^p.(1 - (\eta_{N_i}^P)^{w_i})^q)]^{\frac{2}{n.(n-1)}})^{\frac{1}{p+q}} \rangle$$

4 An approach to multi attribute making problems for TFM-numbers

In this section, based on Bonferroni geometric mean operator of generalized hesitant TFM-numbers proposed by Deli [12], we developed an algorithm to solve multi attribute making problems by using TFM-Bonferroni geometric mean operator for aggregating the trapezoidal fuzzy multi information.

Definition 4.1 [38] Let $Z = \{z_i | i \in I_m\}$ be alternatives' set, $C = \{c_j | j \in I_n\}$ set of criteria and $w = (w_1, w_2, \ldots, w_n)$ be weights' set. Here, w_j $(j \in I_n)$ is the weight of criteria c_j such that $w_j > 0$ and $\sum_{j=1}^{n} w_j = 1$. Then, the characteristic of the alternative z_i on criteria c_j is represented by the TFM-number $\overline{N}_{ij}$. All the possible values that the alternative z_i $(i \in I_m)$ satisfies the criteria c_j $(j \in I_n)$ represented in the following TFM decision matrix $(\overline{N}_{ij})_{mxn}$;

$$(\overline{N}_{ij})_{mxn} = \begin{pmatrix} \overline{N}_{11} & \overline{N}_{12} & \cdots & \overline{N}_{1n} \\ \overline{N}_{21} & \overline{N}_{22} & \cdots & \overline{N}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{N}_{m1} & \overline{N}_{m2} & \cdots & \overline{N}_{mn} \end{pmatrix}$$

Note: In next example, Table 1 [21] as follows will be used as linguistic terms table.

Linguistic terms	TFM-numbers
Definitely-low(DL)	$\langle(0.01,0.05,0.10,0.15); 0.1,0.2,0.3,0.4\rangle$
Too-Low(TL)	$\langle(0.05,0.10,0.15,0.20); 0.2,0.3,0.4,0.1\rangle$
Very-Low(VL)	$\langle(0.10,0.15,0.15,0.20); 0.2,0.4,0.5,0.3\rangle$
Low(L)	$\langle(0.10,0.20,0.20,0.30); 0.3,0.4,0.8,0.1\rangle$
Fairly-low(FL)	$\langle(0.15,0.20,0.25,0.30); 0.4,0.6,0.2,0.5\rangle$
Medium(M)	$\langle(0.25,0.30,0.35,0.40); 0.4,0.5,0.6,0.8\rangle$
Fairly-high(FH)	$\langle(0.30,0.35,0.40,0.45); 0.6,0.1,0.8,0.4\rangle$
High(H)	$\langle(0.40,0.45,0.50,0.55); 0.8,0.9,0.3,0.6\rangle$
Very-High(VH)	$\langle(0.45,0.55,0.65,0.75); 0.7,0.8,0.6,0.3\rangle$
Too-High(TH)	$\langle(0.50,0.60,0.70,0.80); 0.1,0.7,0.8,0.9\rangle$
Definitely-high(DH)	$\langle(0.70,0.80,0.90,1.00); 0.7,0.8,0.9,0.2\rangle$

Table 1: TFM-numbers of linguistic terms

Algorithm

Step 1 Present TFM decision matrix showing results of evaluation of the expert based upon the characteristic of the alternative z_i $(i \in I_m)$ satisfies the criteria c_j $(j \in I_n)$ based on linguistic terms Table 1 as;

$$(\overline{N}_{ij})_{mxn} = \begin{pmatrix} \overline{N}_{11} & \overline{N}_{12} & \cdots & \overline{N}_{1n} \\ \overline{N}_{21} & \overline{N}_{22} & \cdots & \overline{N}_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{N}_{m1} & \overline{N}_{m2} & \cdots & \overline{N}_{mn} \end{pmatrix}$$

Step 2 Find the weights of criteria as follows:

Substep 1 Construct a matrix consisting of real numbers by value of TFM-numbers obtain from defuzzification of each element of the decision matrix $(\overline{N}_{ij})_{mxn}$ by using Definition 2.7 as follows:

$$(D_{ij})_{mxn} = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1x} \\ x_{21} & x_{22} & \cdots & x_{2x} \\ \vdots & \vdots & \ddots & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{pmatrix}$$

Substep 2 Find the weights of criteria according to criteria in the decision making problem and values in $(D_{ij})_{mxn}$ matrix by using critic method given in Subsection 2.1:

$$w = (w_1, w_2, \dots, w_n)$$

Step 3 For all i $(i \in I_m)$, find the aggregation values according to Equation (8) in order to obtain the ultimate performance value corresponding to the alternative z_i $(i \in I_m)$ as;

$$\overline{N}_i = TFMBGM_w^{(p,q)}(\overline{N}_{i1}, \overline{N}_{i2}, \ldots, \overline{N}_{in})(i \in I_m)$$

Step 4 Calculate score value whose formula is given in Definition 2.8 for each $(\overline{N}_i)$ $(i \in I_m)$ and rank all the alternatives.

5 Illustrative example

Here, we give an illustrative example to show effectiveness of the proposed method and see results.

Example 5.1 Assume that a car fleet selection problem can be used as a multiple attribute decision making problem in which alternatives are car fleets to be selected by considering the attributes under consideration. A manager of a courier company aims to hire a new car fleet to speed up delivering of items they transport. After pre-assessment, five alternatives $Z = \{x_i | i \in I_5\}$ have remained to be selected. Also, there are four attributes to be considered;

1. Carbon emission level (c_1)
2. Comfort (c_2)
3. Safety (c_3)
4. Low fuel consuming (c_4)

Step 1 Evaluation results of the manager are presented in TFM decision matrix $(\overline{N}_{ij})_{5\times4}$ as;

$$(\overline{N}_{ij})_{5\times4} = \begin{pmatrix} \langle(0.10,0.15,0.15,0.20);0.2,0.4,0.5,0.3\rangle & \langle(0.15,0.20,0.25,0.30);0.4,0.6,0.2,0.5\rangle & \langle(0.30,0.35,0.40,0.45);0.6,0.1,0.8,0.4\rangle & \langle(0.45,0.55,0.65,0.75);0.7,0.8,0.6,0.3\rangle \\ \langle(0.05,0.10,0.15,0.20);0.2,0.3,0.4,0.1\rangle & \langle(0.10,0.20,0.20,0.30);0.3,0.4,0.8,0.1\rangle & \langle(0.25,0.30,0.35,0.40);0.4,0.5,0.6,0.8\rangle & \langle(0.40,0.45,0.50,0.55);0.8,0.9,0.3,0.6\rangle \\ \langle(0.70,0.80,0.90,1.00);0.7,0.8,0.9,0.2\rangle & \langle(0.50,0.60,0.70,0.80);0.1,0.7,0.8,0.9\rangle & \langle(0.10,0.20,0.20,0.30);0.3,0.4,0.8,0.1\rangle & \langle(0.01,0.05,0.10,0.15);0.1,0.2,0.3,0.4\rangle \\ \langle(0.25,0.30,0.35,0.40);0.4,0.5,0.6,0.8\rangle & \langle(0.05,0.10,0.15,0.20);0.2,0.3,0.4,0.1\rangle & \langle(0.50,0.60,0.70,0.80);0.1,0.7,0.8,0.9\rangle & \langle(0.25,0.30,0.35,0.40);0.4,0.5,0.6,0.8\rangle \\ \langle(0.50,0.60,0.70,0.80);0.1,0.7,0.8,0.9\rangle & \langle(0.70,0.80,0.90,1.00);0.7,0.8,0.9,0.2\rangle & \langle(0.40,0.45,0.50,0.55);0.8,0.9,0.3,0.6\rangle & \langle(0.10,0.20,0.20,0.30);0.3,0.4,0.8,0.1\rangle \end{pmatrix}$$

Step 2

Substep 1 Construct a matrix consisting of real numbers by defuzzification of each element of the decision matrix $(\overline{N}_{ij})_{mxn}$ by using Definition 2.7 as follows:

$$(D_{ij})_{mxn} = \begin{pmatrix} 0{,}1500 & 0{,}2250 & 0{,}3750 & 0{,}6000 \\ 0{,}1250 & 0{,}2000 & 0{,}3250 & 0{,}4750 \\ 0{,}8500 & 0{,}6500 & 0{,}2000 & 0{,}0779 \\ 0{,}3250 & 0{,}1250 & 0{,}6500 & 0{,}3250 \\ 0{,}6500 & 0{,}8500 & 0{,}4750 & 0{,}2000 \end{pmatrix}$$

Substep 2 Find the weights of criteria according to criteria in the decision making problem and values in $(D_{ij})_{mxn}$ matrix by using critic method given in Subsection 2.1:

$$w = (0.328,0.250,0.197,0.223)$$

Step 3 For all i $(i \in I_5)$, the aggregation values are computed as stated in the Equation (8) for $p = 1$ and $q = 1$ to obtain the final performance value as:

$$\overline{N}_1 = TFMBGM_w^{(1,1)}(\overline{N}_{11}, \overline{N}_{12}, \overline{N}_{13}, \overline{N}_{14}) = \langle(0.9081,1.0340,1.1009,1.2061); 0.7441,0.7383,0.7745,0.6944\rangle$$

$$\overline{N}_2 = TFMBGM_w^{(1,1)}(\overline{N}_{21}, \overline{N}_{22}, \overline{N}_{23}, \overline{N}_{24}) = \langle(0.7675,0.9405,1.0216,1.1460); 0.7075,0.7764,0.7836,0.6390\rangle$$

$$\overline{N}_3 = TFMBGM_w^{(1,1)}(\overline{N}_{31}, \overline{N}_{32}, \overline{N}_{33}, \overline{N}_{34}) = \langle(0.8962,1.1440,1.2700,1.4227); 0.6085,0.7932,0.8895,0.6741\rangle$$

$$\overline{N}_4 = TFMBGM_w^{(1,1)}(\overline{N}_{41}, \overline{N}_{42}, \overline{N}_{43}, \overline{N}_{44}) = \langle(0.9051,1.0430,1.1563,1.2564); 0.6075,0.7695,0.8266,0.8503\rangle$$

$$\overline{N}_5 = TFMBGM_w^{(1,1)}(\overline{N}_{51}, \overline{N}_{52}, \overline{N}_{53}, \overline{N}_{54}) = \langle(1.2255,1.3916,1.4665,1.5988); 0.7240,0.8792,0.8886,0.7208\rangle$$

Step 4 The scores of $\overline{N}_i$ $(i \in I_5)$ $(s(\overline{N}_i))$ are computed as:
$s(\overline{N}_1) = 0.2851$, $s(\overline{N}_2) = 0.3209$, $s(\overline{N}_3) = 0.5652$, $s(\overline{N}_4) = 0.3849$, $s(\overline{N}_5) = 0.5093$
and all the alternatives ranked as:

$$z_3 > z_5 > z_4 > z_2 > z_1$$

(p, q)	i	1	2	3	4	5	Ranking
(1.0,1.0)	$s(\overline{N}_i)$	0.2851	0.3209	0.5652	0.3849	0.5093	$z_3 > z_5 > z_4 > z_2 > z_1$
(1.0,0.5)	$s(\overline{N}_i)$	0.1547	0.1761	0.3100	0.2102	0.2804	$z_3 > z_5 > z_4 > z_2 > z_1$
(0.5,1.0)	$s(\overline{N}_i)$	0.1547	0.1761	0.3100	0.2102	0.2804	$z_3 > z_5 > z_4 > z_2 > z_1$
(2.0,2.0)	$s(\overline{N}_i)$	0.1673	0.2882	2.3014	1.5829	2.0331	$z_3 > z_5 > z_4 > z_2 > z_1$
(3.0,3.0)	$s(\overline{N}_i)$	0.6159	0.8570	5.1540	3.5727	4.5176	$z_3 > z_5 > z_4 > z_2 > z_1$
(0.5,0.6)	$s(\overline{N}_i)$	0.0804	0.0918	0.1612	0.1094	0.1484	$z_3 > z_5 > z_4 > z_2 > z_1$
(0.7,0.8)	$s(\overline{N}_i)$	0.1561	0.1770	0.3110	0.2113	0.2829	$z_3 > z_5 > z_4 > z_2 > z_1$
(0.8,0.7)	$s(\overline{N}_i)$	0.1561	0.1770	0.3110	0.2113	0.2829	$z_3 > z_5 > z_4 > z_2 > z_1$
(0.4,0.5)	$s(\overline{N}_i)$	0.0519	0.0595	0.1045	0.0709	0.0971	$z_3 > z_5 > z_4 > z_2 > z_1$
(0.5,0.4)	$s(\overline{N}_i)$	0.0519	0.0595	0.1045	0.0709	0.0971	$z_3 > z_5 > z_4 > z_2 > z_1$
(0.5,2.0)	$s(\overline{N}_i)$	0.0292	0.0338	0.0599	0.0406	0.0558	$z_3 > z_5 > z_4 > z_2 > z_1$
(2.0,0.5)	$s(\overline{N}_i)$	0.0292	0.0338	0.0599	0.0406	0.0558	$z_3 > z_5 > z_4 > z_2 > z_1$
(3.0,1.0)	$s(\overline{N}_i)$	0.1346	0.2764	2.2718	1.5499	1.9816	$z_3 > z_5 > z_4 > z_2 > z_1$
(1.0,3.0)	$s(\overline{N}_i)$	0.1346	0.2764	2.2718	1.5499	1.9816	$z_3 > z_5 > z_4 > z_2 > z_1$
(5.0,5.0)	$s(\overline{N}_i)$	0.1780	0.7713	4.1343	0.8966	2.3458	$z_3 > z_5 > z_4 > z_2 > z_1$

Table 2: Rankings for some alternatives in terms of different $TFMBGM_w^{(p,q)}$ of Example 5.1

6 Comparative study

In the Table 3 given below, we compare our proposed method with some other methods given by Kesen and Deli [21], Deli and Keles [14], Uluçay [37], Uluçay et al. [38] and Şahin et al. [31], based on Example 5.1.

Developed aggregation technique called TFM-Bonferroni geometric mean operator can be used to handle the multiple attribute decision making problems. Therefore, in order to compare the performance of the proposed method based on Example 5.1 with some existing methods in Kesen and Deli [21], Deli and Keles [14], Uluçay [37], Uluçay et al. [38] and Şahin et al. [31], a comparative study is presented and their corresponding final rankings are summarized in Table 2. From the Table 2, it is clear that the ranking order of the alternatives are generally same. Also, if we choose different values of (p, q) the ranking order of the alternatives is generally same as found the existing approaches in Kesen and Deli [21], Deli and Keles [14], Uluçay [37], Uluçay et al. [38] and Şahin et al. [31]. Thus, our proposed method can be suitably utilized to solve by aggregating the multi attribute decision making problems in addition to the other existing methods under trapezoidal fuzzy multi information. Also, in the developed method, the solutions of Example 5.1 with different values of (p, q) is shown in Table 2. As seen in the table, the results are approximately the same. So, the introduced method is flexible that contain the existing methods according to the value of (p, q) and it has more application fields than existing methods to overcome the limitations of the multi attribute decision making problems.

Method	Operator	Ranking
Method of Uluçay et al. [38]	$TFMG_w$	$z_5 > z_3 > z_4 > z_1 > z_2$
Proposed method	$TFMBGM_w^{(1,1)}$	$z_3 > z_5 > z_4 > z_2 > z_1$
Proposed method	$TFMBGM_w^{(2,2)}$	$z_3 > z_5 > z_4 > z_2 > z_1$
Method of Uluçay [37]	S_w	$z_4 > z_3 > z_1 > z_5 > z_2$
Method of Şahin et al. [31]	D_w	$z_3 > z_5 > z_1 > z_4 > z_2$
Method of Deli and Keles [14]	S_i	$z_5 > z_3 > z_4 > z_1 > z_2$
Method of Kesen and Deli [21]	$TFMBHM_w^{(1,1)}$	$z_1 > z_3 > z_5 > z_2 > z_4$

Table 3: Ranking for all alternatives according to different methods and proposed methods of Example 5.1

7 Conclusion

This study introduces a solution to the challenge of multi-attribute decision-making using trapezoidal fuzzy multi numbers (TFM-numbers). Initially, an aggregation method called TFM-Bonferroni geometric mean operator is proposed for aggregating trapezoidal fuzzy multi information. Then, properties and special cases of this technique are further explored. Furthermore, an algorithm is devised for multi-attribute decision-making within trapezoidal fuzzy multi environments. This method was then applied to a multi-criteria decision-making problem within the trapezoidal multi fuzzy context. To demonstrate the efficacy of our results, a hands-on example is provided. To conclude, In future, we plan to extend our work to TOPSIS method, VIKOR method, QUALIFLEX method, ELECTRE I method, ELECTRE II method, ELECTRE III method, defuzzification techniques, and so on.

Compliance with ethical standards

Conflict of interest: The authors declare that there is no conflict of interest with other organization or people on this article.

Human and animal rights: This article does not contain any studies with human participants or animals performed by the authors.

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In general, a system S (that may be a company, association, institution, society, country, etc.) is formed by sub-systems S_i { or $P(S)$, the powerset of S }, and each sub-system S_i is formed by sub-sub-systems S_{ij} { or $P(P(S)) = P^2(S)$ } and so on. That's why the n-th PowerSet of a Set S { defined recursively and denoted by $P^n(S) = P(P^{n-1}(S)$ } was introduced, to better describes the organization of people, beings, objects etc. in our real world.

The n-th PowerSet was used in defining the SuperHyperOperation, SuperHyperAxiom, and their corresponding Neutrosophic SuperHyperOperation, Neutrosophic SuperHyperAxiom in order to build the SuperHyperAlgebra and Neutrosophic SuperHyperAlgebra. In general, in any field of knowledge, one in fact encounters **SuperHyperStructures**, https://fs.unm.edu/SuperHyperAlgebra.pdf.

Also, six new types of topologies have been introduced in the last years (2019-2022), such as: Refined Neutrosophic Topology, Refined Neutrosophic Crisp Topology, NeutroTopology, AntiTopology, SuperHyperTopology, and Neutrosophic SuperHyperTopology, http://fs.unm.edu/TT/.

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