RESEARCH ON SMARANDACHE PROBLEMS IN NUMBER THEORY
(Collected papers)

Edited by

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Xi’an, P. R. China

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This book is dedicated to the memory of Florentin Smarandache, who listed many new and unsolved problems in number theory.

Contents

Preface

Arithmetic is where numbers run across your mind looking for the answer.
Arithmetic is like numbers spinning in your head faster and faster until you blow up with the answer.
KABOOM!!
Then you sit back down and begin the next problem.
(Alexander Nathanson)

Number theory is an ancient subject, but we still cannot answer many simplest and most natural questions about the integers. Some old problems have been solved, but more arise. All the research for these ancient or new problems implicated and are still promoting the development of number theory and mathematics.

American-Romanian number theorist Florentin Smarandache introduced hundreds of interest sequences and arithmetical functions, and presented many problems and conjectures in his life. In 1991, he published a book named *Only problems, Not solutions!*. He presented 105 unsolved arithmetical problems and conjectures about these functions and sequences in it. Already many researchers studied these sequences and functions from his book, and obtained important results.

This book, *Research on Smarandache Problems in Number Theory (Collected papers),* contains 41 research papers involving the Smarandache sequences, functions, or problems and conjectures on them.
All these papers are original. Some of them treat the mean value or hybrid mean value of Smarandache type functions, like the famous Smarandache function, Smarandache ceil function, or Smarandache primitive function. Others treat the mean value of some famous number theoretic functions acting on the Smarandache sequences, like k-th root sequence, k-th complement sequence, or factorial part sequence, etc. There are papers that study the convergent property of some infinite series involving the Smarandache type sequences. Some of these sequences have been first investigated too. In addition, new sequences as additive complement sequences are first studied in several papers of this book.

Most authors of these papers are my students. After this chance, I hope they will be more interested in the mysterious integer and number theory!

All the papers are supported by the N. S. F. of P. R. China (10271093). So I would like to thank the Department of Mathematical and Physical Sciences of N. S. F. C.

I would also like to thank my students Xu Zhefeng and Zhang Xiaobeng for their careful typeset and design works. My special gratitude is due to all contributors of this book for their great help to the publication of their papers and their detailed comments and corrections.

More future papers by my students will focus on the Smarandache notions, such as sequences, functions, constants, numbers, continued fractions, infinite products, series, etc. in number theory!

August 10, 2004

Zhang Wenpeng

AN ARITHMETIC FUNCTION AND THE PRIMITIVE NUMBER OF POWER P

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Abstract For any fixed prime p, we define

$$S_p(n) = m, \quad if \quad p^n \mid m! \quad and \quad p^n \dagger (m-1)!,$$
$$a_p(n) = m, \quad if \quad p^m \mid n \quad and \quad p^{m+1} \dagger n.$$

The main purpose of this paper is to study the mean value properties of $a_p(S_p(n))$, and give an interesting asymptotic formula for it.

Keywords: Primitive number; Mean value; Asymptotic formula.

§1. Introduction

Let p be a prime, n be any positive integer, we define two arithmetic functions as following:

$$S_p(n) = m, \quad if \quad p^n \mid m! \quad and \quad p^n \dagger (m-1)!,$$
$$a_p(n) = m, \quad if \quad p^m \mid n \quad and \quad p^{m+1} \dagger n.$$

In problem 49 and 68 of reference [1], Professor F.Smarandache asked us to study the properties of these two arithmetic functions. About these problems, many scholars showed great interests in them (See references [2], [3]). But it seems that no one knows the relationship between these two arithmetic functions before. In this paper, we shall use the elementary methods to study the mean value properties of $a_p(S_p(n))$, and give an interesting asymptotic formula for it. That is , we shall prove the following conclusion:

Theorem. *For any fixed prime p and any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n \leq x} a_p(S_p(n)) = \frac{p+1}{(p-1)^2} x + O\left(\ln^3 x\right).$$

Taking $p = 2, 3$ in the theorem, we may immediately obtain the following

Corollary. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} a_2(S_2(n)) = 3x + O\left(\ln^3 x\right),$$
$$\sum_{n\leq x} a_3(S_3(n)) = x + O\left(\ln^3 x\right).$$

§2. One simple lemma

To complete the proof of the theorem, we need the following simple lemma:
Lemma. *For any fixed prime p and real number $x \geq 1$, we have*

$$\sum_{\alpha\leq x} \frac{\alpha^2}{p^\alpha} = \frac{p^2+p}{(p-1)^3} + O\left(\frac{x^2}{p^x}\right).$$

Proof. First we come to calculate

$$u = \sum_{\alpha\leq n} \alpha^2/p^\alpha.$$

Note that the identities

$$\begin{aligned} u\left(1-\frac{1}{p}\right) &= \sum_{\alpha=1}^{n}\frac{\alpha^2}{p^\alpha} - \sum_{\alpha=1}^{n}\frac{\alpha^2}{p^{\alpha+1}} \\ &= \frac{1}{p} - \frac{n^2}{p^{n+1}} + \sum_{\alpha=1}^{n-1}\frac{(\alpha+1)^2-\alpha^2}{p^{\alpha+1}} \\ &= \frac{1}{p} - \frac{n^2}{p^{n+1}} + \sum_{\alpha=1}^{n-1}\frac{2\alpha+1}{p^{\alpha+1}}, \end{aligned}$$

and

$$\begin{aligned} u\left(1-\frac{1}{p}\right)^2 &= \left(\frac{1}{p} - \frac{n^2}{p^{n+1}} + \sum_{\alpha=1}^{n-1}\frac{2\alpha+1}{p^{\alpha+1}}\right)\left(1-\frac{1}{p}\right) \\ &= \frac{1}{p} - \frac{1}{p^2} + \frac{n^2-n^2p}{p^{n+2}} + \sum_{\alpha=1}^{n-1}\frac{2\alpha+1}{p^{\alpha+1}} - \sum_{\alpha=2}^{n}\frac{2\alpha-1}{p^{\alpha+1}} \\ &= \frac{1}{p} + \frac{2}{p^2} + \frac{n^2-n^2p}{p^{n+2}} + \sum_{\alpha=2}^{n-1}\frac{2}{p^{\alpha+1}} + \frac{n^2-n^2p-(2n-1)p}{p^{n+2}} \\ &= \frac{1}{p} + \frac{2(p^{n-1}-1)}{p^{n+1}-p^n} + \frac{n^2-(n^2+2n-1)p}{p^{n+2}}. \end{aligned}$$

So we have

$$u = \left(\frac{1}{p} + \frac{2(p^{n-1}-1)}{p^{n+1}-p^n} + \frac{n^2-(n^2+2n-1)p}{p^{n+2}}\right)\frac{1}{\left(1-\frac{1}{p}\right)^2}$$

$$= \frac{p}{(p-1)^2} + \frac{2(p^{n-1}-1)}{p^{n-2}(p-1)^3} + \frac{n^2-(n^2+2n-1)p}{p^n(p-1)^2}.$$

Then we can immediately obtain

$$\begin{aligned}\sum_{\alpha \le x} \frac{\alpha^2}{p^\alpha} &= \frac{p}{(p-1)^2} + \frac{2p}{(p-1)^3} + O\left(\frac{x^2}{p^x}\right)\\ &= \frac{p^2+p}{(p-1)^3} + O\left(\frac{x^2}{p^x}\right).\end{aligned}$$

This completes the proof of the Lemma.

§3. Proof of the Theorem

In this section, we shall use the above Lemma to complete the proof of the Theorem. From the definition of $S_p(n)$ and $a_p(n)$, we may immediately get

$$\begin{aligned}&\sum_{n \le x} a_p(S_p(n))\\ =& \sum_{\substack{m \le x\\ p^\alpha \| m}} \alpha^2 = \sum_{\substack{p^\alpha m \le x\\ (p,m)=1}} \alpha^2 = \sum_{p^\alpha \le x} \alpha^2 \sum_{\substack{m \le x/p^\alpha\\ (p,m)=1}} 1\\ =& \sum_{p^\alpha \le x} \alpha^2 \sum_{m \le x/p^\alpha} \sum_{d|(m,p)} \mu(d)\\ =& \sum_{p^\alpha \le x} \alpha^2 \sum_{d|p} \mu(d) \sum_{t \le x/p^\alpha} 1\\ =& \sum_{p^\alpha \le x} \alpha^2 \left(\sum_{t \le x/p^\alpha} 1 - \sum_{t \le x/p^{\alpha+1}} 1 \right)\\ =& \sum_{p^\alpha \le x} \alpha^2 \left(\frac{x}{p^\alpha} - \frac{x}{p^{\alpha+1}} + O(1) \right)\\ =& x \left(\sum_{\alpha \le \ln x/\ln p} \frac{\alpha^2}{p^\alpha} - \sum_{\alpha \le \ln x/\ln p} \frac{\alpha^2}{p^{\alpha+1}} \right) + O\left(\sum_{\alpha \le \ln x/\ln p} \alpha^2 \right)\\ =& \left(1 - \frac{1}{p}\right) x \sum_{\alpha \le \ln x/\ln p} \frac{\alpha^2}{p^\alpha} + O\left(\ln^3 x\right)\\ =& \left(1 - \frac{1}{p}\right) \left(\frac{p^2+p}{(p-1)^3} + O\left(\frac{\ln^2 x}{x}\right)\right) x + O\left(\ln^3 x\right)\\ =& \frac{p+1}{(p-1)^2} x + O\left(\ln^3 x\right).\end{aligned}$$

This completes the proof of the Theorem.

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ON THE PRIMITIVE NUMBERS OF POWER P AND K-POWER ROOTS

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Abstract Let p be a prime, n be any positive integer, $S_p(n)$ denotes the smallest integer $m \in N^+$, where $p^n|m!$. In this paper, we study the mean value properties of $S_p(a_n)$, where a_n is the superior integer part of k-power roots, and give an interesting asymptotic formula for it.

Keywords: Primitive numbers of power p; k-power roots; Asymptotic formula.

§1. Introduction and results

Let p be a prime, n be any positive integer, $S_p(n)$ denotes the smallest integer such that $S_p(n)!$ is divisible by p^n. For example, $S_3(1) = 3$, $S_3(2) = 6$, $S_3(3) = 9$, $S_3(4) = 9$, $\cdots\cdots$. In problem 49 of book [1], Professor F. Smarandache ask us to study the properties of the sequence $\{S_p(n)\}$. About this problem, Professor Zhang and Liu in [2] have studied it and obtained an interesting asymptotic formula. That is, for any fixed prime p and any positive integer n,

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p} \cdot \ln n\right).$$

For any fixed positive intger k, let a_n denotes the superior integer part of k-power roots, that is, $a_1 = 1, \cdots, a_{2^k-1} = 1, a_{2^k} = 2, \cdots$. In problem 80 of book [1], Professor F. Smarandache ask us to study the properties of the sequence a_n. About this problem, the author of [3] have studied it and obtained an interesting asymptotic formula. That is, for any real number $x \geq 1$,

$$\sum_{n \leq x} \Omega(a_n) = \frac{1}{k} x \ln \ln x + \frac{1}{k}(A - \ln k) \cdot x + O\left(\frac{x}{\ln x}\right),$$

where $\Omega(n)$ denotes the numbers of all prime divisor of n, A be a computable constant.

In this paper, we will use the elementary method to study the asymptotic properties of $S_p(a_n)$ in the following form:

$$\sum_{n\leq x}\frac{1}{p}\left|S_p(a_{n+1})-S_p(a_n)\right|,$$

where x be a positive real number, and give an interesting asymptotic formula for it. In fact, we shall prove the following result:

Theorem. *For any real number $x \geq 2$, let p be a prime and n be any positive integer. Then we have the asymptotic formula*

$$\sum_{n\leq x}\frac{1}{p}\left|S_p(a_{n+1})-S_p(a_n)\right| = x^{\frac{1}{k}}\cdot\left(1-\frac{1}{p}\right)+O_k\left(\frac{\ln x}{\ln p}\right),$$

where O_k denotes the O-constant depending only on parameter k.

§2. Proof of the Theorem

In this section, we shall complete the proof of the theorem. First we need following one simple Lemma. That is,

Lemma. *Let p be a prime and n be any positive integer, then we have*

$$|S_p(n+1)-S_p(n)| = \begin{cases} p, & \textit{if } p^n \| m!; \\ 0, & \textit{otherwise}, \end{cases}$$

where $S_p(n)=m$, $p^n \parallel m!$ denotes that $p^n|m!$ and $p^{n+1}\dagger m!$.

Proof. Now we will discuss it in two cases.

(i) Let $S_p(n)=m$, if $p^n \parallel m!$, then we have $p^n|m!$ and $p^{n+1}\dagger m!$. From the definition of $S_p(n)$ we have $p^{n+1}\dagger(m+1)!$, $p^{n+1}\dagger(m+2)!$, $\cdots$, $p^{n+1}\dagger(m+p-1)!$ and $p^{n+1}|(m+p)!$, so $S_p(n+1)=m+p$, then we get

$$|S_p(n+1)-S_p(n)| = p. \tag{1}$$

(ii) Let $S_p(n)=m$, if $p^n|m!$ and $p^{n+1}|m!$, then we have $S_p(n+1)=m$, so

$$|S_p(n+1)-S_p(n)| = 0. \tag{2}$$

Combining (1) and (2), we can easily get

$$|S_p(n+1)-S_p(n)| = \begin{cases} p, & \text{if } p^n \parallel m!; \\ 0, & \text{otherwise.} \end{cases}$$

This completes the proof of Lemma.

Now we use above Lemma to complete the proof of Theorem. For any real number $x \geq 2$, let M be a fixed positive integer such that $M^k \leq x < (M+1)^k$, then from the definition of $S_p(n)$ and the Lemma we have

$$\begin{aligned}
& \sum_{n\le x} \frac{1}{p}\left|S_p(a_{n+1})-S_p(a_n)\right| \qquad (3)\\
= & \sum_{t=1}^{M-1}\left(\sum_{t^k\le n\le (t+1)^k-1} \frac{1}{p}\left|S_p(a_{n+1})-S_p(a_n)\right|\right)\\
& + \sum_{M^k\le n\le x} \frac{1}{p}\left|S_p(a_{n+1})-S_p(a_n)\right|\\
= & \sum_{t=1}^{M-1} \frac{1}{p}\left|S_p(t+1)-S_p(t)\right| + \sum_{M^k\le n\le x} \frac{1}{p}\left|S_p(a_{n+1})-S_p(a_n)\right|\\
= & \sum_{t=1}^{M-1} \frac{1}{p}\left|S_p(t+1)-S_p(t)\right|\\
= & \sum_{\substack{t\le x^{\frac{1}{k}}\\ p^t\| m!}} 1 + O(1). \qquad (4)
\end{aligned}$$

where $S_p(t) = m$. Note that if $p^t \parallel m!$, then we have (see reference [4], Theorem 1.7.2)

$$\begin{aligned}
t &= \sum_{i=1}^{\infty}\left[\frac{m}{p^i}\right] = \sum_{i\le \log_p m}\left[\frac{m}{p^i}\right]\\
&= m\cdot \sum_{i\le \log_p m}\frac{1}{p^i} + O\left(\log_p m\right)\\
&= \frac{m}{p-1} + O\left(\frac{\ln m}{\ln p}\right). \qquad (5)
\end{aligned}$$

From (4), we can deduce that

$$m = (p-1)t + O\left(\frac{p\ln t}{\ln p}\right). \qquad (6)$$

So that

$$1 \le m \le (p-1)\cdot x^{\frac{1}{k}} + O_k\left(\frac{p\ln x}{\ln p}\right), \quad \text{if} \quad 1\le t\le x^{\frac{1}{k}}.$$

Note that for any fixed positive integer t, if there has one m such that $p^t \parallel m!$, then $p^t \parallel (m+1)!$, $p^t \parallel (m+2)!$, $\cdots$, $p^t \parallel (m+p-1)!$. Hence there have p times of m such that $t = \sum\limits_{i=1}^{\infty}\left[\frac{m}{p^i}\right]$ in the interval

$$1 \le m \le (p-1)\cdot x^{\frac{1}{k}} + O_k\left(\frac{p\ln x}{\ln p}\right).$$

Then from this and (3), we have

$$\begin{aligned}
& \sum_{n\leq x} \frac{1}{p} \left| S_p(a_{n+1}) - S_p(a_n) \right| \\
= & \sum_{\substack{t\leq x^{\frac{1}{k}} \\ p^t \| m!}} 1 + O(1) \\
= & \frac{1}{p}\left((p-1)\cdot x^{\frac{1}{k}} + O_k\left(\frac{p\ln x}{\ln p}\right)\right) + O(1) \\
= & x^{\frac{1}{k}} \cdot \left(1 - \frac{1}{p}\right) + O_k\left(\frac{\ln x}{\ln p}\right).
\end{aligned}$$

This completes the proof of Theorem.

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MEAN VALUE ON THE PSEUDO-SMARANDACHE SQUAREFREE FUNCTION

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Abstract For any positive integer n, the pseudo-Smarandache squarefree function $ZW(n)$ is defined as the least positive integer m such that m^n is divisible by n. In this paper, we study the mean value of $ZW(n)$, and give a few asymptotic formulae.

Keywords: Pseudo-Smarandache squarefree function; Mean value; Asymptotic formula.

§1. Introduction

According to [1], the pseudo-Smarandache squarefree function $ZW(n)$ is defined as the least positive integer m such that m^n is divisible by n. It is obvious that $ZW(1) = 1$. For $n > 1$, Maohua Le [1] obtained that

$$ZW(n) = p_1 p_2 \cdots p_k, \tag{1}$$

where $p_1, p_2, \cdots, p_k$ are distinct prime divisors of n. Also he showed that

$$\sum_{n=1}^{\infty} \frac{1}{(ZW(n))^a}, \quad a \in R, \quad a > 0$$

is divergence.

In this paper, we study the mean value of $ZW(n)$, and give a few asymptotic formulae. That is, we shall prove the following:

Theorem 1. *For any real numbers α, s with $s - \alpha > 1$ and $\alpha > 0$, we have*

$$\sum_{n=1}^{\infty} \frac{ZW^{\alpha}(n)}{n^s} = \frac{\zeta(s)\zeta(s-\alpha)}{\zeta(2s-2\alpha)} \prod_p \left[1 - \frac{1}{p^s + p^{\alpha}}\right],$$

where $\zeta(s)$ is the Riemann zeta function, $\prod_p$ denotes the product over all prime numbers.

Theorem 2. *For any real numbers $\alpha > 0$ and $x \geq 1$, we have*

$$\sum_{n\leq x} ZW^\alpha(n) = \frac{\zeta(\alpha+1)x^{\alpha+1}}{\zeta(2)(\alpha+1)} \prod_p \left[1 - \frac{1}{p^\alpha(p+1)}\right] + O\left(x^{\alpha+\frac{1}{2}+\epsilon}\right).$$

Noting that $\sum_{n\leq x} ZW^0(n) = x + O(1)$ and $\lim_{\alpha\longrightarrow 0^+} \alpha\zeta(\alpha+1) = 1$, so from Theorem 2 we immediately have the limit

$$\lim_{\alpha\to 0^+} \frac{1}{\alpha} \prod_p \left(1 - \frac{1}{p^\alpha(p+1)}\right) = \zeta(2).$$

§2. Proof of the theorems

Now we prove the theorems. For any real numbers α, s with $s - \alpha > 1$ and $\alpha > 0$, let

$$f(s) = \sum_{n=1}^{\infty} \frac{ZW^\alpha(n)}{n^s}.$$

From (1) and the Euler product formula [2] we have

$$\begin{aligned} f(s) &= \prod_p \left[1 + \frac{p^\alpha}{p^s} + \frac{p^\alpha}{p^{2s}} + \cdots\right] = \prod_p \left[1 + \frac{\frac{1}{p^{s-\alpha}}}{1 - \frac{1}{p^s}}\right] \\ &= \prod_p \left[\left(\frac{1 + \frac{1}{p^{s-\alpha}}}{1 - \frac{1}{p^s}}\right)\left(1 - \frac{1}{p^s + p^\alpha}\right)\right] \\ &= \frac{\zeta(s)\zeta(s-\alpha)}{\zeta(2s-2\alpha)} \prod_p \left[1 - \frac{1}{p^s + p^\alpha}\right]. \end{aligned}$$

This proves Theorem 1.

For any real numbers $\alpha > 0$ and $x \geq 1$, it is obvious that

$$|ZW^\alpha(n)| \leq n^\alpha \quad \text{and} \quad \left|\sum_{n=1}^{\infty} \frac{ZW^\alpha(n)}{n^\sigma}\right| < \frac{1}{\sigma - \alpha},$$

where σ is the real part of s. So by Perron formula [3] we can get

$$\begin{aligned} \sum_{n\leq x} \frac{ZW^\alpha(n)}{n^{s_0}} &= \frac{1}{2\pi i} \int_{b-iT}^{b+iT} f(s+s_0) \frac{x^s}{s} \mathrm{d}s + O\left(\frac{x^b B(b+\sigma_0)}{T}\right) \\ &\quad + O\left(x^{1-\sigma_0} H(2x) \min\left(1, \frac{\log x}{T}\right)\right) \\ &\quad + O\left(x^{-\sigma_0} H(N) \min\left(1, \frac{x}{||x||}\right)\right), \end{aligned}$$

where N is the nearest integer to x, and $||x|| = |x - N|$. Taking $s_0 = 0$, $b = \alpha + \frac{3}{2}$ and $T > 2$ in the above, then we have

$$\sum_{n \le x} ZW^{\alpha}(n) = \frac{1}{2\pi i} \int_{\alpha+\frac{3}{2}-iT}^{\alpha+\frac{3}{2}+iT} f(s) \frac{x^s}{s} \mathrm{d}s + O\left(\frac{x^{\alpha+\frac{3}{2}}}{T}\right).$$

Now we move the integral line from $\alpha + \frac{3}{2} \pm iT$ to $\alpha + \frac{1}{2} - iT$. This time, the function

$$f(s) \frac{x^s}{s}$$

have a simple pole point at $s = \alpha + 1$ with residue

$$\frac{\zeta(\alpha+1)x^{\alpha+1}}{\zeta(2)(\alpha+1)} \prod_p \left[1 - \frac{1}{p^{\alpha}(p+1)}\right].$$

Now taking $T = x$, then we have

$$\begin{aligned}
\sum_{n \le x} ZW^{\alpha}(n) &= \frac{\zeta(\alpha+1)x^{\alpha+1}}{\zeta(2)(\alpha+1)} \prod_p \left[1 - \frac{1}{p^{\alpha}(p+1)}\right] \\
&\quad + \frac{1}{2\pi i} \int_{\alpha+\frac{1}{2}-ix}^{\alpha+\frac{1}{2}+ix} f(s) \frac{x^s}{s} \mathrm{d}s + O\left(x^{\alpha+\frac{1}{2}+\epsilon}\right) \\
&= \frac{\zeta(\alpha+1)x^{\alpha+1}}{\zeta(2)(\alpha+1)} \prod_p \left[1 - \frac{1}{p^{\alpha}(p+1)}\right] \\
&\quad + O\left(\int_{-x}^{x} \left| f\left(\alpha + \frac{1}{2} + \epsilon + ix\right)\right| \frac{x^{\alpha+\frac{1}{2}+\epsilon}}{(1+|t|)} \mathrm{d}t\right) + O\left(x^{\alpha+\frac{1}{2}+\epsilon}\right) \\
&= \frac{\zeta(\alpha+1)x^{\alpha+1}}{\zeta(2)(\alpha+1)} \prod_p \left[1 - \frac{1}{p^{\alpha}(p+1)}\right] + O\left(x^{\alpha+\frac{1}{2}+\epsilon}\right).
\end{aligned}$$

This completes the proof of Theorem 2.

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ON THE ADDITIVE K-TH POWER COMPLEMENTS

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Abstract In this paper, similar to the Smarandache k-th power complements, we defined the additive k-th power complements. Using the elementary method, we study the mean value properties of the additive square complements, and give some interesting asymptotic formulae.

Keywords: Additive k-th power complements; Mean value; Asymptotic formula.

§1. Introduction

For any positive integer n, the Smarandache k-th power complements $b_k(n)$ is the smallest positive integer such that $nb_k(n)$ is a complete k-th power, see problem 29 of [1]. Similar to the Smarandache k-th power complements, we define the additive k-th power complements $a_k(n)$ as follows: $a_k(n)$ is the smallest nonnegative integer such that $a_k(n) + n$ is a complete k-th power. For example, if $k = 2$, we have the additive square complements sequence $\{a_2(n)\}$ $(n = 1, 2, \cdots)$ as follows: $0, 2, 1, 0, 4, 3, 2, 1, 0, 6, 5, 4, 3, 2, 1, 0, 7, \cdots$. In this paper, we stdudy the mean value properties of $a_k(n)$ and $d(a_k(n))$, where $d(n)$ is the Dirichlet divisor function, and give several interesting asymptotic formulae. That is, we shall prove the following conclusion:

Theorem 1. *For any real number $x \geq 3$, we have the asymptotic formula*

$$\sum_{n\leq x} a_k(n) = \frac{k^2}{4k-2}x^{2-\frac{1}{k}} + O\left(x^{2-\frac{2}{k}}\right).$$

Theorem 2. *For any real number $x \geq 3$, we have the asymptotic formula*

$$\sum_{n\leq x} d(a_k(n)) = \left(1-\frac{1}{k}\right)x\ln x + \left(2\gamma + \ln k - 2 + \frac{1}{k}\right)x + O\left(x^{1-\frac{1}{k}}\ln x\right),$$

where γ is the Euler constant.

§2. Some lemmas

Before the proof of the theorems, some lemmas will be usefull.

Lemma 1. *For any real number $x \geq 3$, we have the asymptotic formula:*

$$\sum_{n\leq x} d(x) = x\ln x + (2\gamma - 1)x + O\left(x^{\frac{1}{2}}\right),$$

where γ is the Euler constant.

Proof. See reference [2].

Lemma 2. *For any real number $x \geq 3$ and any nonnegative arithmetical function $f(n)$ with $f(0) = 0$, we have the asymptotic formula:*

$$\sum_{n\leq x} f(a_k(n)) = \sum_{t=1}^{\left[x^{\frac{1}{k}}\right]-1} \sum_{n\leq g(t)} f(n) + O\left(\sum_{n\leq g\left(\left[x^{\frac{1}{k}}\right]\right)} f(n)\right),$$

where $[x]$ denotes the greatest integer less than or equal to x and $g(t) = \sum_{i=1}^{k-1} \begin{pmatrix} i \\ k \end{pmatrix} t^i$.

Proof. For any real number $x \geq 1$, let M be a fixed positive integer such that

$$M^k \leq x < (M+1)^k.$$

Noting that if n pass through the integers in the interval $\left[t^k, (t+1)^k\right)$, then $a_k(n)$ pass through the integers in the inteval $\left[0, (t+1)^k - t^k - 1\right]$ and $f(0) = 0$, we can write

$$\begin{aligned}
\sum_{n\leq x} f(a_k(n)) &= \sum_{t=1}^{M-1} \sum_{t^k\leq n<(t+1)^k} f(a_k(n)) + \sum_{M^k\leq n\leq x} f(a_k(n)) \\
&= \sum_{t=1}^{M-1} \sum_{n\leq g(t)} f(n) + \sum_{g(M)+M^k-x\leq n<g(M)} f(n),
\end{aligned}$$

where $g(t) = \sum_{i=1}^{k-1} \begin{pmatrix} i \\ k \end{pmatrix} t^i$. Since $M = \left[x^{\frac{1}{k}}\right]$, so we have

$$\sum_{n\leq x} f(a_k(n)) = \sum_{t=1}^{\left[x^{\frac{1}{k}}\right]-1} \sum_{n\leq g(t)} f(n) + O\left(\sum_{n\leq g\left(\left[x^{\frac{1}{k}}\right]\right)} f(n)\right).$$

This proves Lemma 2.

Note: This Lemma is very usefull. Because if we have the mean value formula of $f(n)$, then from this lemma, we can easily get the mean value formula of $\sum_{n\leq x} f(a_k(n))$.

§3. Proof of the theorems

In this section, we will complete the proof of the theorems. First we prove Theorem 1. From Lemma 1 and the Euler summation formula (See [3]), let $f(n) = n$, we have

$$\begin{aligned}\sum_{n\le x} a_k(n) &= \sum_{t=1}^{\left[x^{\frac{1}{k}}\right]-1} \sum_{n\le g(t)} n + O\left(\sum_{n\le g\left(\left[x^{\frac{1}{k}}\right]\right)} n\right)\\ &= \frac{1}{2}\sum_{t=1}^{\left[x^{\frac{1}{k}}\right]-1} k^2 t^{2k-2} + O\left(x^{2-\frac{2}{k}}\right)\\ &= \frac{k^2}{4k-2}x^{2-\frac{1}{k}} + O\left(x^{2-\frac{2}{k}}\right).\end{aligned}$$

This proves Theorem 1.

Now we prove Theorem 2. From Lemma 1 and Lemma 2, we have

$$\begin{aligned}&\sum_{n\le x} d(a_k(n))\\ &= \sum_{t=1}^{\left[x^{\frac{1}{k}}\right]-1} \sum_{n\le g(t)} d(n) + O\left(\sum_{n\le g\left(\left[x^{\frac{1}{k}}\right]\right)} d(n)\right)\\ &= \sum_{t=1}^{\left[x^{\frac{1}{k}}\right]-1} \left(kt^{k-1}\left(\ln kt^{k-1} + \ln\left(1+O\left(\frac{1}{t}\right)\right)\right)\right.\\ &\qquad \left. +(2\gamma-1)kt^{k-1}\right) + O\left(x^{1-\frac{1}{k}}\ln x\right)\\ &= \sum_{t=1}^{\left[x^{\frac{1}{k}}\right]-1} \left(k(k-1)t^{k-1}\ln t + (2\gamma+\ln k-1)kt^{k-1}\right.\\ &\qquad \left. +O(t^{k-2})\right) + O\left(x^{1-\frac{1}{k}}\ln x\right)\\ &= k(k-1)\sum_{t=1}^{\left[x^{\frac{1}{k}}\right]-1} t^{k-1}\ln t + (2\gamma+\ln k-1)k\sum_{t=1}^{\left[x^{\frac{1}{k}}\right]-1} t^{k-1}\\ &\qquad +O\left(x^{1-\frac{1}{k}}\ln x\right).\end{aligned}$$

Then from the Euler summation formula, we can easily get

$$\sum_{n\le x} d(a_k(n)) = \left(1-\frac{1}{k}\right)x\ln x + \left(2\gamma+\ln k-2+\frac{1}{k}\right)x + O\left(x^{1-\frac{1}{k}}\ln x\right).$$

This completes the proof of the theorems.

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ON THE SMARANDACHE PSEUDO-MULTIPLES OF 5 SEQUENCE

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Abstract The main purpose of this paper is to study the mean value properties of the Smarandache pseudo-multiples of 5 number sequence, and give an interesting asymptotic formula for it.

Keywords: Pseudo-multiples of 5 numbers; Mean value; Asymptotic formula.

§1. Introduction

A number is a pseudo-multiple of 5 if some permutation of its digits is a multiple of 5, including the identity permutation. For example: $0, 5, 10, 15, 20, 25, 30, 35, 40, 50, 51, 52, \cdots$ are pseudo-multiple of 5 numbers. Let A denotes the set of all the pseudo-multiple of 5 numbers. In reference [1], Professor F. Smarandache asked us to study the properties of the pseudo-multiple of 5 sequence. About this problems, it seems that none had studied it, at least we have not seen such a paper before. In this paper, we use the elementary method to study the mean value properties of this sequence, and obtain an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n \in A \\ n \leq x}} f(n) = \sum_{n \leq x} f(n) + O\left(M x^{\frac{\ln 8}{\ln 10}}\right),$$

where $M = \max_{1 \leq n \leq x} \{|f(n)|\}$. Taking $f(n) = d(n)$, $\Omega(n)$ as the Dirichlet divisor function and the function of the number of prime factors respectively, then we have the following:

Corollary 1. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n \in A \\ n \leq x}} d(n) = x \ln x + (2\gamma - 1)x + O\left(x^{\frac{\ln 8}{\ln 10} + \epsilon}\right),$$

where γ is the Euler constant, ϵ is any fixed positive number.

Corollary 2. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n \in A \\ n \leq x}} \Omega(n) = x \ln \ln x + Bx + O\left(\frac{x}{\ln x}\right),$$

where B is a computable constant.

§2. Proof of the Theorem

Now we completes the proof of the Theorem. First let $10^k \leq x < 10^{k+1}$ $(k \geq 1)$, then $k \leq \log x < k+1$. According to the definition of set A, we know that the largest number of digits $(\leq x)$ not attribute set A is 8^{k+1}. In fact, in these numbers, there are 8 one digit, they are $1, 2, 3, 4, 6, 7, 8, 9$; There are 8^2 two digits; The number of the k digits are 8^k. So the largest number of digits $(\leq x)$ not attribute set A is $8 + 8^2 + \cdots + 8^k = \frac{8}{7}(8^k - 1) \leq 8^{k+1}$. Since

$$8^k \leq 8^{\log x} = \left(8^{\log_8 x}\right)^{\frac{1}{\log_8 10}} = (x)^{\frac{1}{\log_8 10}} = x^{\frac{\ln 8}{\ln 10}}.$$

So we have,

$$8^k = O\left(x^{\frac{\ln 8}{\ln 10}}\right).$$

Next, let M denotes the upper bounds of $|f(n)|$ $(n \leq x)$, then

$$\sum_{\substack{n \notin A \\ n \leq x}} f(n) = O\left(M x^{\frac{\ln 8}{\ln 10}}\right).$$

Finally, we have

$$\begin{aligned}
\sum_{\substack{n \in A \\ n \leq x}} f(n) &= \sum_{n \leq x} f(n) - \sum_{\substack{n \notin A \\ n \leq x}} f(n) \\
&= \sum_{n \leq x} f(n) + O\left(M x^{\frac{\ln 8}{\ln 10}}\right).
\end{aligned}$$

This proves the Theorem.

Now the Corollary 1 follows from the Theorem, the asymptotic formula

$$\sum_{n \leq x} d(n) = x \ln x + (2\gamma - 1)x + O\left(x^{\frac{1}{2}}\right)$$

(see [2]), and the estimate $d(n) \ll x^{\epsilon}$ (for all $1 \leq n \leq x$). And then, the Corollary 2 follows from the Theorem, the asymptotic formula

$$\sum_{n \leq x} \Omega(n) = x \ln \ln x + Bx + O\left(\frac{x}{\log x}\right)$$

(See [3]), and the estimate $\Omega(n) \ll x^{\epsilon}$ (for all $1 \leq n \leq x$).

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AN ARITHMETIC FUNCTION AND THE DIVISOR PRODUCT SEQUENCES

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Abstract Let n be any positive integer, $P_d(n)$ denotes the product of all positive divisors of n. Let p be a prime, $a_p(n)$ denotes the largest exponent (of power p) such that divisible by n. In this paper, we study the asymptotic properties of the mean value of $a_p(P_d(n))$, and give an interesting asymptotic formula for it.

Keywords: Divisor product sequences; Mean value; Asymptotic formula.

§1. Introduction

Let n be any positive integer, $P_d(n)$ denotes the product of all positive divisors of n. That is, $P_d(n) = \prod\limits_{p} d$. For example, $P_d(1) = 1, P_d(2) = 2, P_d(3) = 3, P_d(4) = 8, \cdots$. Let p be a prime, $a_p(n)$ denotes the largest exponent (of power p) such that $p^{a_p(n)} \mid n$. In problem 25 and 68 of reference [1], Professor F.Smarandache asked us to study the properties of these two arithmetic functions. About these problems, many scholars showed great interests in them (see references [2],[3]). But it seems that no one knows the relationship between these two arithmetic functions before. In this paper, we shall use the elementary methods to study the mean value properties of $a_p(S_p(n))$, and give an interesting asymptotic formula for it. That is , we shall prove the following conclusion:

Theorem. *Let p be a prime, then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} a_p(P_d(n)) = \frac{x\ln x}{p(p-1)} + \frac{(p-1)^3(2\gamma-1) - \left(2p^4+4p^3+p^2-2p+1\right)\ln p}{p(p-1)^4}x + O(x^{\frac{1}{2}+\epsilon}),$$

where γ is the Euler constant, and ϵ denotes any fixed positive number.

Taking $p = 2, 3$ in the theorem, we may immediately obtain the following

Corollary. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} a_2(P_d(n)) = \frac{1}{2}x\ln x + \frac{2\gamma - 65\ln 2 - 1}{2}x + O(x^{\frac{1}{2}+\epsilon});$$

$$\sum_{n\leq x} a_3(P_d(n)) = \frac{1}{6}x\ln x + \frac{8\gamma - 137\ln 3 - 4}{24}x + O(x^{\frac{1}{2}+\epsilon}).$$

§2. Some lemmas

To complete the proof of the theorem, we need the following simple lemmas:

Lemma 1. *For any positive integer n, we have the identity*

$$P_d(n) = n^{\frac{d(n)}{2}},$$

where $d(n)$ is the divisor function.

Proof. This formula can be immediately got from Lemma 1 of [2].

Lemma 2. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n\leq x\\(n,m)=1}} d(n) = x\left(\ln x + 2\gamma - 1 + 2\sum_{p|m}\frac{\ln p}{p-1}\right)\prod_{p|m}\left(1-\frac{1}{p}\right)^2 + O(x^{\frac{1}{2}+\epsilon}),$$

where $\prod_p$ denotes the product over all primes, γ is the Euler constant, and ϵ denotes any fixed positive number.

Proof. Let $T = x^{1/2}$, $A(s) = \prod_p \left(1-\frac{1}{p^s}\right)^2$. Then by the Perron formula (See Theorem 2 of reference [4]), we may obtain

$$\sum_{\substack{n\leq x\\(n,m)=1}} d(n) = \frac{1}{2i\pi}\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT}\zeta^2(s)A(s)\frac{x^s}{s}ds + O(x^{\frac{1}{2}+\epsilon}),$$

where $\zeta(s)$ is the Riemann-zeta function.

Moving the integral line from $\frac{3}{2} \pm iT$ to $\frac{1}{2} \pm iT$. This time, the function

$$f(s) = \zeta^2(s)A(s)\frac{x^s}{s}$$

has a second order pole point at $s = 1$ with residue

$$x\left(\ln x + 2\gamma - 1 + 2\sum_{p|m}\frac{\ln p}{p-1}\right)\prod_{p|m}\left(1-\frac{1}{p}\right)^2.$$

So we have

$$\frac{1}{2i\pi}\left(\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT}+\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}+\int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}\right)\zeta^2(s)A(s)\frac{x^s}{s}ds$$
$$=x\left(\ln x+2\gamma-1+2\sum_{p|m}\frac{\ln p}{p-1}\right)\prod_{p|m}\left(1-\frac{1}{p}\right)^2.$$

Note that

$$\frac{1}{2i\pi}\left(\int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT}+\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}+\int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}\right)\zeta^2(s)A(s)\frac{x^s}{s}ds$$
$$\ll x^{\frac{1}{2}+\epsilon}.$$

From the above we can immediately get the asymptotic formula:

$$\sum_{\substack{n\le x\\(n,m)=1}}d(n)=x\left(\ln x+2\gamma-1+2\sum_{p|m}\frac{\ln p}{p-1}\right)\prod_{p|m}\left(1-\frac{1}{p}\right)^2+O(x^{\frac{1}{2}+\epsilon}).$$

This completes the proof of Lemma 2.

Lemma 3. *Let p be a prime, then for any real number $x\ge 1$, we have the following asymptotic formulae*

$$\sum_{\alpha\le x}\frac{\alpha}{p^\alpha}=\frac{p}{(p-1)^2}+O\left(\frac{x}{p^x}\right), \tag{1}$$

$$\sum_{\alpha\le x}\frac{\alpha^2}{p^\alpha}=\frac{p^2+p}{(p-1)^3}+O\left(\frac{x^2}{p^x}\right), \tag{2}$$

$$\sum_{\alpha\le x}\frac{\alpha^3}{p^\alpha}=\frac{p^3+4p^2+p}{(p-1)^4}+O\left(\frac{x^3}{p^x}\right). \tag{3}$$

Proof. We only prove formula (2) and (3). First we come to calculate

$$f=\sum_{\alpha\le n}\alpha^2/p^\alpha.$$

Note that the identities

$$\begin{aligned} f\left(1-\frac{1}{p}\right) &= \sum_{\alpha=1}^{n}\frac{\alpha^2}{p^\alpha}-\sum_{\alpha=1}^{n}\frac{\alpha^2}{p^{\alpha+1}}\\ &= \frac{1}{p}-\frac{n^2}{p^{n+1}}+\sum_{\alpha=1}^{n-1}\frac{(\alpha+1)^2-\alpha^2}{p^{\alpha+1}}\\ &= \frac{1}{p}-\frac{n^2}{p^{n+1}}+\sum_{\alpha=1}^{n-1}\frac{2\alpha+1}{p^{\alpha+1}} \end{aligned}$$

and

$$
\begin{aligned}
& f\left(1-\frac{1}{p}\right)^2 \\
= & \left(\frac{1}{p}-\frac{n^2}{p^{n+1}}+\sum_{\alpha=1}^{n-1}\frac{2\alpha+1}{p^{\alpha+1}}\right)\left(1-\frac{1}{p}\right) \\
= & \frac{1}{p}-\frac{1}{p^2}+\frac{n^2-n^2p}{p^{n+2}}+\sum_{\alpha=1}^{n-1}\frac{2\alpha+1}{p^{\alpha+1}}-\sum_{\alpha=2}^{n}\frac{2\alpha-1}{p^{\alpha+1}} \\
= & \frac{1}{p}+\frac{2}{p^2}+\frac{n^2-n^2p}{p^{n+2}}+\sum_{\alpha=2}^{n-1}\frac{2}{p^{\alpha+1}}+\frac{n^2-n^2p-(2n-1)p}{p^{n+2}} \\
= & \frac{1}{p}+\frac{2(p^{n-1}-1)}{p^{n+1}-p^n}+\frac{n^2-(n^2+2n-1)p}{p^{n+2}}.
\end{aligned}
$$

So we have

$$
\begin{aligned}
f &= \left(\frac{1}{p}+\frac{2(p^{n-1}-1)}{p^{n+1}-p^n}+\frac{n^2-(n^2+2n-1)p}{p^{n+2}}\right)\frac{1}{\left(1-\frac{1}{p}\right)^2} \\
&= \frac{p}{(p-1)^2}+\frac{2(p^{n-1}-1)}{p^{n-2}(p-1)^3}+\frac{n^2-(n^2+2n-1)p}{p^n(p-1)^2}.
\end{aligned}
$$

Then we can immediately obtain

$$
\begin{aligned}
\sum_{\alpha\le x}\frac{\alpha^2}{p^\alpha} &= \frac{p}{(p-1)^2}+\frac{2p}{(p-1)^3}+O\left(\frac{x^2}{p^x}\right) \\
&= \frac{p^2+p}{(p-1)^3}+O\left(\frac{x^2}{p^x}\right).
\end{aligned}
$$

This proves formula (2).

Now we come to prove formula (3). Let

$$
g=\sum_{\alpha\le n}\alpha^3/p^\alpha.
$$

Note that the identities

$$
\begin{aligned}
& g\left(1-\frac{1}{p}\right) \\
= & \sum_{\alpha=1}^{n}\frac{\alpha^3}{p^\alpha}-\sum_{\alpha=1}^{n}\frac{\alpha^3}{p^{\alpha+1}} \\
= & \frac{1}{p}-\frac{n^3}{p^{n+1}}+\sum_{\alpha=1}^{n-1}\frac{(\alpha+1)^3-\alpha^3}{p^{\alpha+1}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{p} - \frac{n^3}{p^{n+1}} + \sum_{\alpha=2}^{n} \frac{3\alpha^2 - 3\alpha + 1}{p^{\alpha+1}} \\
&= \frac{1}{p} - \frac{n^3}{p^{n+1}} + \frac{p^{n-1} - 1}{p^{n+1} - p^n} - \left(\frac{2}{p^2} - \frac{n}{p^{n+1}} + \frac{p^{n-2} - 1}{p^{n+1} - p^n} \right) \frac{3}{1 - \frac{1}{p}} \\
&\quad + \left(\frac{4}{p^2} - \frac{n^2}{p^{n+1}} + \left(\frac{5}{p^3} - \frac{2n-1}{p^{n+1}} + \frac{2(p^{n-3} - 1)}{p^{n+1} - p^n} \right) \frac{1}{1 - \frac{1}{p}} \right) \frac{3}{1 - \frac{1}{p}} \\
&= \frac{p^2 + 4p + 10}{p(p-1)^2} + \frac{3n - 3n^2 + p^{n-1} - 1 + (3 - 6n)p}{p^n(p-1)} \\
&\quad - \frac{n^3}{p^{n+1}} + \frac{6p(p^{n-3} - 1) - 3(p^{n-2} - 1)(p - 1)}{p^{n-1}(p-1)^3}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\sum_{\alpha \le x} \frac{\alpha^3}{p^\alpha} &= \left(\frac{p^2 + 4p + 10}{p(p-1)^2} + \frac{1}{p(p-1)} + \frac{9 - 3p}{p(p-1)^3} \right) \frac{p}{p-1} + O\left(\frac{x^3}{p^x} \right) \\
&= \frac{p^3 + 4p^2 + p}{(p-1)^4} + O\left(\frac{x^3}{p^x} \right).
\end{aligned}
$$

This completes the proof of Lemma 3.

§3. Proof of the Theorem

In this section, we shall use the above lemmas to complete the proof of the Theorem. From the definition of $P_d(n)$ and $a_p(n)$, we may immediately get

$$
\begin{aligned}
&\sum_{n \le x} a_p(P_d(n)) \\
&= \sum_{\substack{p^\alpha l \le x \\ (p,l)=1}} \frac{(\alpha+1)\alpha}{2} d(l) = \sum_{p^\alpha \le x} \frac{(\alpha+1)\alpha}{2} \sum_{\substack{l \le x/p^\alpha \\ (p,l)=1}} d(l) \\
&= \sum_{\alpha \le \ln x/\ln p} \frac{(\alpha+1)\alpha}{2} \left(\frac{x}{p^\alpha} \left(\ln \frac{x}{p^\alpha} + 2\gamma - 1 + \frac{2\ln p}{p-1} \right) \left(1 - \frac{1}{p} \right)^2 \right) \\
&\quad + O\left(x^{\frac{1}{2}+\epsilon} \right) \\
&= \frac{x}{2} \left(1 - \frac{1}{p} \right)^2 \left(\ln x + 2\gamma - 1 + \frac{2\ln p}{p-1} \right) \sum_{\alpha \le \ln x/\ln p} \frac{(\alpha+1)\alpha}{p^\alpha} \\
&\quad - \frac{x \ln p}{2} \sum_{\alpha \le \ln x/\ln p} \frac{(\alpha+1)\alpha^2}{p^\alpha} + O\left(x^{\frac{1}{2}+\epsilon} \right) \\
&= \frac{x}{2} \left(1 - \frac{1}{p} \right)^2 \left(\ln x + 2\gamma - 1 + \frac{2\ln p}{p-1} \right)
\end{aligned}
$$

$$\times\left(\frac{p^2+p}{(p-1)^3}+\frac{p}{(p-1)^2}+O\left(\frac{\ln^2 x}{x}\right)\right)$$
$$-\frac{x\ln p}{2}\left(\frac{p^3+4p^2+p}{(p-1)^4}+\frac{p^2+p}{(p-1)^3}+O\left(\frac{\ln^3 x}{x}\right)\right)+O\left(x^{\frac{1}{2}+\epsilon}\right)$$
$$= \frac{x\ln x}{p(p-1)}+\frac{(p-1)^3(2\gamma-1)-(2p^4+4p^3+p^2-2p+1)\ln p}{p(p-1)^4}x$$
$$+O\left(x^{\frac{1}{2}+\epsilon}\right).$$

This completes the proof of the Theorem.

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THE SMARANDACHE IRRATIONAL ROOT SIEVE SEQUENCES

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Abstract In this paper, we use the analytic method to study the mean value properties of the irrational root sieve sequence, and give an interesting asymptotic formula for it.

Keywords: Smarandache irrational root sieve; Mean value; Asymptotic formula.

§1. Introduction

According to reference [1], the definition of Smarandache irrational root sieve is: from the set of natural numbers (except 0 and 1):
-take off all powers of 2^k, $k \geq 2$;
-take off all powers of 3^k, $k \geq 2$;
-take off all powers of 5^k, $k \geq 2$;
-take off all powers of 6^k, $k \geq 2$;
-take off all powers of 7^k, $k \geq 2$;
-take off all powers of 10^k, $k \geq 2$;
$\cdots$ and so on (take off all k-powers, $k \geq 2$). For example: $2, 3, 5, 6, 7, 10, 11, 12, 13, 14, 15, 17, 18, 19 \cdots$ are all irrational root sieve sequence. Let A denotes the set of all the irrational root sieve. In reference [1], Professor F. Smarandache asked us to study the properties of the irrational root sieve sequence. About this problem, it seems that none had studied it, at least we have not seen such a paper before. In this paper, we study the mean value of the irrational root sieve sequence, and give an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem. *Let $d(n)$ denote the divisor function. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\begin{aligned}
&\sum_{\substack{n\in A\\ n\leq x}} d(n)\\
= &\left(x - \frac{3}{4\pi^2}\sqrt{x}\ln x + A_1 x^{\frac{1}{3}}\ln^2 x + A_2 x^{\frac{1}{3}}\ln x + A_3 x^{\frac{1}{3}} + A_4\sqrt{x}\right)\ln x\\
+ &(2\gamma - 1)x + A_5\sqrt{x} + A_6 x^{\frac{1}{3}} + O\left(x^{\frac{139}{429}+\epsilon}\right),
\end{aligned}$$

where ϵ denotes any fixed positive number, γ is the Euler constant, $A_1, A_2, A_3, A_4, A_5, A_6$ are the computable constants.

§2. Some Lemmas

To complete the proof of the theorem, we need the following lemmas:

Lemma 1. *For any real number $x \geq 1$, we have the asymptotic formula:*

$$\sum_{n\leq x} d(n) = x\ln x + (2\gamma - 1)x + O\left(x^{\frac{139}{429}+\epsilon}\right),$$

where ϵ denotes any fixed positive number and γ is the Euler constant.

Proof. This result may be immediately got from [2].

Lemma 2. *For any real number $x \geq 1$, we have two asymptotic formulae*

$$\sum_{n\leq\sqrt{x}} d(n^2) = \frac{3\sqrt{x}\ln^2 x}{4\pi^2} + \frac{B_1}{2}\sqrt{x}\ln x + B_2\sqrt{x} + O\left(x^{\frac{1}{4}+\epsilon}\right);$$

$$\sum_{n\leq x^{\frac{1}{3}}} d(n^3) = \frac{6C_0 x^{\frac{1}{3}}\ln^3 x}{27\pi^4} + \frac{C_1}{9}x^{\frac{1}{3}}\ln^2 x + \frac{C_2}{3}x^{\frac{1}{3}}\ln x + C_3 x^{\frac{1}{3}} + O\left(x^{\frac{1}{6}+\epsilon}\right),$$

where $B_1, B_2, C_0, C_1, C_2, C_3$ are computable constants.

Proof. Let

$$f(s) = \sum_{n=1}^{\infty}\frac{d(n^2)}{n^s},$$

$\mathrm{Re}(s) > 1$. Then from the Euler product formula [3] and the multiplicative property of $d(n)$ we have

$$\begin{aligned}
f(s) &= \prod_p\left(1 + \frac{3}{p^s} + \frac{5}{p^{2s}} + \frac{7}{p^{3s}} + \cdots\right)\\
&= \prod_p\left(1 - \frac{1}{p^s}\right)^{-1}\left(1 + \frac{2}{p^s} + \frac{2}{p^{2s}} + \frac{2}{p^{3s}} + \cdots\right)\\
&= \prod_p\left(\left(1 - \frac{1}{p^s}\right)^{-1} + \frac{2}{p^s}\left(1 - \frac{1}{p^s}\right)^{-2}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \prod_p \left(1-\frac{1}{p^s}\right)^{-2}\left(1+\frac{1}{p^s}\right) \\
&= \frac{\zeta^3(s)}{\zeta(2s)}.
\end{aligned}
$$

where $\zeta(s)$ is the Riemann zeta-function. By Perron formula [2] with $s_0 = 0$, $T = x^{\frac{1}{2}}$ and $b = \frac{3}{2}$, we have

$$\sum_{n\le x} d(n^2) = \frac{1}{2\pi i}\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} \frac{\zeta^3(s)}{\zeta(2s)}\frac{x^s}{s}ds + O\left(x^{\frac{1}{2}+\epsilon}\right).$$

To estimate the main term

$$\frac{1}{2\pi i}\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} \frac{\zeta^3(s)}{\zeta(2s)}\frac{x^s}{s}ds,$$

we move the integral line from $s = \frac{3}{2} \pm iT$ to $s = \frac{1}{2} \pm iT$. This time, the function

$$f(s) = \frac{\zeta^3(s)}{\zeta(2s)}\frac{x^s}{s}$$

has a three order pole point at $s = 1$ with residue

$$\lim_{s\to 1}\frac{1}{2!}\left((s-1)^3\frac{\zeta^3(s)}{\zeta(2s)}\frac{x^s}{s}\right)^{(2)} = \frac{3}{\pi^2}x\ln^2 x + B_1 x\ln x + B_2 x,$$

where B_1, B_2 are the computable constants.

Note that

$$\frac{1}{2i\pi}\left(\int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT} + \int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}\right)\frac{\zeta^3(s)}{\zeta(2s)}\frac{x^s}{s}ds \ll x^{\frac{1}{2}+\epsilon}.$$

From above we may immediately get the asymptotic formula:

$$\sum_{n\le x} d(n^2) = \frac{3x\ln^2 x}{\pi^2} + B_1 x\ln x + B_2 x + O\left(x^{\frac{1}{2}+\epsilon}\right).$$

That is,

$$\sum_{n\le\sqrt{x}} d(n^2) = \frac{3\sqrt{x}\ln^2 x}{4\pi^2} + \frac{B_1}{2}\sqrt{x}\ln x + B_2\sqrt{x} + O\left(x^{\frac{1}{4}+\epsilon}\right).$$

This proves the first formula of Lemma 2.

Similarly, we can deduce the second asymptotic formula of Lemma 2. In fact let

$$g(s) = \sum_{n=1}^{\infty}\frac{d(n^3)}{n^s},$$

$\mathrm{Re}(s) > 1$. Then from the Euler product formula [3] and the multiplicative property of $d(n)$ we have

$$\begin{aligned} g(s) &= \prod_p \left(1 + \frac{4}{p^s} + \frac{7}{p^{2s}} + \frac{10}{p^{3s}} + \cdots\right) \\ &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 + \frac{3}{p^s} + \frac{3}{p^{2s}} + \frac{3}{p^{3s}} + \cdots\right) \\ &= \prod_p \left(\left(1 - \frac{1}{p^s}\right)^{-1} + \frac{3}{p^s}\left(1 - \frac{1}{p^s}\right)^{-2}\right) \\ &= \prod_p \left(1 - \frac{1}{p^s}\right)^{-2} \left(1 + \frac{2}{p^s}\right) \\ &= \frac{\zeta^4(s)}{\zeta^2(2s)} \prod_p \left(1 - \frac{1}{(p^s+1)^2}\right). \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function. Then by Perron formula [2] and the method of proving the first asymptotic formula of Lemma 2 we may immediately get

$$\sum_{n\le x} d(n^3) = \frac{6}{\pi^4} C_0 x \ln^3 x + C_1 x \ln^2 x + C_2 x \ln x + C_3 x + O\left(x^{\frac{1}{2}+\epsilon}\right).$$

That is,

$$\sum_{n\le x^{\frac{1}{3}}} d(n^3) = \frac{6C_0 x^{\frac{1}{3}} \ln^3 x}{27\pi^4} + \frac{C_1}{9} x^{\frac{1}{3}} \ln^2 x + \frac{C_2}{3} x^{\frac{1}{3}} \ln x + C_3 x^{\frac{1}{3}} + O\left(x^{\frac{1}{6}+\epsilon}\right),$$

This proves the Lemma 2.

§3. Proof of the Theorem

Now we completes the proof of the Theorem. According to the definition of the set A and the result of Lemma 1 and Lemma 2 , we have

$$\begin{aligned} &\sum_{\substack{n\in A \\ n\le x}} d(n) \\ = &\sum_{n\le x} d(n) - \sum_{n\le\sqrt{x}} d(n^2) - \sum_{n\le x^{\frac{1}{3}}} d(n^3) + O\left(\sum_{4\le k\le \frac{\ln x}{\ln 2}} \sum_{n\le x^{\frac{1}{k}}} d(n^k)\right) \\ = &\sum_{n\le x} d(n) - \sum_{n\le\sqrt{x}} d(n^2) - \sum_{n\le x^{\frac{1}{3}}} d(n^3) + O\left(\sum_{4\le k\le \frac{\ln x}{\ln 2}} x^{\frac{1}{k}+\epsilon}\right) \end{aligned}$$

$$
\begin{aligned}
= & \left(x - \frac{3\sqrt{x}\ln x}{4\pi^2} + A_1 x^{\frac{1}{3}} \ln^2 x + A_2 x^{\frac{1}{3}} \ln x + A_3 x^{\frac{1}{3}} + A_4\sqrt{x}\right) \ln x \\
& + (2\gamma - 1)x + A_5\sqrt{x} + A_6 x^{\frac{1}{3}} + O\left(x^{\frac{139}{429}+\epsilon}\right),
\end{aligned}
$$

where A_i $(i = 1, 2, \cdots, 6)$ are computable constants.

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A NUMBER THEORETIC FUNCTION AND ITS MEAN VALUE

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Abstract Let p be a prime, $e_p(n)$ denote the largest exponent of power p which divides n. In this paper, we study the properties of this sequence $e_p(n)$, and give an interesting asymptotic formula for $\sum_{n\leq x} e_p^m(n)$.

Keywords: Asymptotic formula; Largest exponent; Mean value.

§1. Introduction

Let p be a prime, $e_p(n)$ denote the largest exponent of power p which divides n. In problem 68 of [1], Professor F.Smarandach asked us to study the properties of the sequence $e_p(n)$. This problem is closely related to the factorization of $n!$. In this paper, we use elementary methods to study the asymptotic properties of the mean value $\sum_{n\leq x} e_p^m(n)$, and give an interesting asymptotic formula for it. That is, we will prove the following:

Theorem. *Let p be a prime, $m \geq 0$ be an integer. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} e_p^m(n) = \frac{p-1}{p} a_p(m)x + O(\log^{m+1} x),$$

where $a_p(m)$ is a computable constant.

Taking $m = 1, 2, 3$ in the theorem, we may immediately obtain the following

Corollary. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} e_p(n) = \frac{1}{p-1}x + O(\log^2 x);$$
$$\sum_{n\leq x} e_p^2(n) = \frac{p+1}{(p-1)^2}x + O(\log^3 x);$$
$$\sum_{n\leq x} e_p^3(n) = \frac{p^2+4p+1}{(p-1)^3}x + O(\log^4 x).$$

§2. Proof of the Theorem

In this section, we complete the proof of the theorem. In fact, from the definition of $e_p(n)$ we have

$$\begin{aligned}
\sum_{n\le x} e_p^m(n) &= \sum_{p^\alpha\le x}\sum_{\substack{p^\alpha u\le x\\(u,p)=1}}\alpha^m = \sum_{\alpha\le\frac{\log x}{\log p}}\alpha^m\sum_{\substack{u\le\frac{x}{p^\alpha}\\(u,p)=1}}1\\
&= \sum_{\alpha\le\frac{\log x}{\log p}}\alpha^m\left(\frac{p-1}{p}\frac{x}{p^\alpha}+O(1)\right)\\
&= \frac{p-1}{p}x\sum_{\alpha\le\frac{\log x}{\log p}}\frac{\alpha^m}{p^\alpha}+O\left(\sum_{\alpha\le\frac{\log x}{\log p}}\alpha^m\right).
\end{aligned}$$

Let

$$a_p(m)=\sum_{n=1}^{\infty}\frac{n^m}{p^n},$$

then $a_p(m)$ is a computable constant. Obviously we have

$$\begin{aligned}
&\sum_{\alpha\le\frac{\log x}{\log p}}\frac{\alpha^m}{p^\alpha}=\sum_{n=1}^{\infty}\frac{n^m}{p^n}-\sum_{\alpha>\frac{\log x}{\log p}}\frac{\alpha^m}{p^\alpha}\\
&= a_p(m)+O\left(\frac{1}{p^{\left[\frac{\log x}{\log p}\right]}}\left(\sum_{u=1}^{\infty}\frac{\left(\frac{\log x}{\log p}+u\right)^m}{p^n}\right)\right)\\
&= a_p(m)+O\left(x^{-1}\left(\left(\frac{\log x}{\log p}\right)^m\sum_{u=1}^{\infty}\frac{1}{p^u}\right.\right.\\
&\qquad\left.\left.+C_m^1\left(\frac{\log x}{\log p}\right)^{m-1}\sum_{u=1}^{\infty}\frac{u}{p^u}+\cdots\right)\right)\\
&= a_p(m)+O\left(x^{-1}\log^{m+1}x\right) \qquad (1)
\end{aligned}$$

and

$$O\left(\sum_{\alpha\le\frac{\log x}{logp}}\alpha^m\right)=O\left(\log^{m+1}x\right). \qquad (2)$$

From (1) and (2) we have

$$\sum_{n\le x}e_p^m(n) = \frac{p-1}{p}x\left(a_p(m)+O\left(x^{-1}\log^m x\right)\right)+O\left(\log^{m+1}x\right)$$

$$= \frac{p-1}{p} a_p(m) x + O\left(\log^{m+1} x\right).$$

This completes the proof of the theorem. As to $a_p(m)$, it is easy to show that

$$a_p(0) = \sum_{n=1}^{\infty} \frac{1}{p^n} = \frac{1}{p-1},$$

and

$$\begin{aligned} p \cdot a_p(m) - 1 &= \sum_{n=1}^{\infty} \frac{(n+1)^m}{p^n} \\ &= \sum_{n=1}^{\infty} \frac{n^m}{p^n} + C_m^1 \sum_{n=1}^{\infty} \frac{n^{m-1}}{p^n} + \cdots + C_m^{m-1} \sum_{n=1}^{\infty} \frac{n}{p^n} + \sum_{n=1}^{\infty} \frac{1}{p^n} \\ &= a_p(m) + C_m^1 a_p(m-1) + \cdots + C_m^{m-1} a_p(1) + a_p(0), \end{aligned}$$

so we have

$$a_p(m) = \frac{1}{p-1} \left(C_m^1 a_p(m-1) + \cdots + C_m^{m-1} a_p(1) + a_p(0) + 1 \right).$$

From this formula, we can easily compute the first several $a_p(m)$:

$$a_p(1) = \frac{p}{(p-1)^2}, a_p(2) = \frac{p^2+p}{(p-1)^3}, a_p(3) = \frac{p^3+4p^2+p}{(p-1)^4}, \cdots$$

Then use the Theorem, we can get the Corollary.

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ON THE PRIMITIVE NUMBERS OF POWER P AND ITS TRIANGLE INEQUALITY

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Abstract The main purpose of this paper is using the elementary method to study the properties of $S_p(n)$, and give a triangle inequality for it.

Keywords: Primitive numbers; Arithmetical property; triangle inequality.

§1. Introduction

Let p be a prime, n be any fixed positive integer, $S_p(n)$ denote the smallest positive integer such that $S_p(n)!$ is divisible by p^n. For example, $S_3(1) = 3$, $S_3(2) = 6$, $S_3(3) = 9$, $S_3(4) = 9$, $S_3(5) = 12$, $\cdots$. In problem 49 of book [1], Professor F. Smarandache asks us to study the properties of the sequence $S_p(n)$. About this problem, some asymptotic properties of this sequence have been studied by many scholar. In this paper, we use the elementary methods to study the arithmetical properties of $S_p(n)$, and give a triangle inequality for it. That is, we shall prove the following:

Theorem 1. *Let p be an odd prime, m_i be positive integer. Then we have the triangle inequality*

$$S_p\left(\sum_{i=1}^{k} m_i\right) \leq \sum_{i=1}^{k} S_p(m_i).$$

Theorem 2. *There are infinite integers m_i $(i = 1, 2, \cdots, k.)$ satisfying*

$$S_p\left(\sum_{i=1}^{k} m_i\right) = \sum_{i=1}^{k} S_p(m_i).$$

§2. Proof of the theorems

In this section, we complete the proof of the theorems. First we prove theorem 1. From the definition of $S_p(n)$, we know that $p^{m_i} \mid S_p(m_i)!$, $p^{m_j} \mid$

$S_p(m_j)!$ $(i \neq j)$. From this we can easily obtain:

$$p^{m_i}p^{m_j} = p^{m_i+m_j} \mid S_p(m_i)!S_p(m_j)! \mid (S_p(m_i) + S_p(m_j))!. \tag{1}$$

But from the definition of $S_p(n)$, we know that $S_p(n)!$ is the smallest positive integer that is divisible by p^n. That is

$$p^{m_i+m_j} \mid S_p(m_i + m_j)!. \tag{2}$$

From (1), (2) we can immediately get

$$S_p(m_i + m_j) \leq S_p(m_i) + S_p(m_j).$$

Now the theorem 1 follows from this inequality and the induction.

Next we complete the proof of theorem 2. For any positive integers m_i with $m_i \neq m_j$ $(1 \leq i, j \leq k)$, we let $\alpha = \alpha(p, n)$ satisfy $p^{\alpha} \| n!$. Then

$$\alpha = \alpha(p, n) = \sum_{j=1}^{\infty} \left[\frac{n}{p^j} \right].$$

For convenient, we let $k_i = \frac{p^{m_i}-1}{p-1}$. Since

$$\sum_{r=1}^{\infty} \left[\frac{p^{m_i}}{p^r} \right] = p^{m_i-1} + p^{m_i-2} + \cdots + 1 = \frac{p^{m_i}-1}{p-1} = k_i.$$

So we have

$$S_p(k_i) = p^{m_i}, \qquad i = 1, 2, \cdots, k. \tag{3}$$

On the other hand,

$$\sum_{r=1}^{\infty} \left[\frac{\sum\limits_{i=1}^{k} p^{m_i}}{p^r} \right] = \sum_{i=1}^{k} \frac{p^{m_i}-1}{p-1} = \sum_{i=1}^{k} k_i.$$

So

$$S_p\left(\sum_{i=1}^{k} k_i \right) = \sum_{i=1}^{k} p^{m_i}. \tag{4}$$

Combining (3) and (4) we may immediately obtain

$$S_p\left(\sum_{i=1}^{k} k_i \right) = \sum_{i=1}^{k} S_p(k_i).$$

This completes the proof of Theorem 2.

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THE ADDITIVE ANALOGUE OF SMARANDACHE SIMPLE FUNCTION

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Abstract The main purpose of this paper is to study the asymptotic properties of $S_p(x)$, and give two interesting asymptotic formulae for it.

Keywords: Smarandache-simple function; Additive Analogue; Asymptotic formula.

§1. Introduction and results

For any positive n, the Smarandache function $S(n)$ is defined as the smallest $m \in N^+$, where $n|m!$. For a fixed prime p, the Smarandache-simple function $S_p(n)$ is defined as the smallest $m \in N^+$, where $p^n|m!$. In reference [2], Jozsef Sandor introduced the additive analogue of the Smarandache-simple function $S_p(x)$ as follows:

$$S_p(x) = \min\{m \in N : p^x \leq m!\},\ x \in (1, \propto),$$

and

$$S_p^*(x) = \max\{m \in N : m! \leq p^x\},\ x \in [1, \propto),$$

which is defined on a subset of real numbers. It is clear that $S_p(x) = m$ if $x \in ((m-1)!, m!]$ for $m \geq 2$ (for $m = 1$ it is not defined, as 0!=1!), therefore this function is defined for $x \geq 1$. About the properties of $S(n)$, many people had studied it before (See [2], [3]). But for the asymptotic formula of $S_p(x)$, it seems that no one have studied it before. The main purpose of this paper is to study the asymptotic properties of $S_p(x)$, and obtain an interesting asymptotic formula for it. That is, we shall prove the following:

Theorem 1. *For any real number $x \geq 2$, we have the asymptotic formula*

$$S_p(x) = \frac{x \ln p}{\ln x} + O\left(\frac{x \ln\ln x}{\ln^2 x}\right).$$

Obviously, we have

$$S_p(x) = \begin{cases} S_p^*(x) + 1, & if \quad x \in (m!, (m+1)!) \quad (m \geq 1) \\ S_p^*(x), & if \quad x = (m+1)! \quad (m \geq 1) \end{cases}$$

Therefore, we can easily get the following:

Theorem 2. *For any real number $x \geq 2$, we have the asymptotic formula*

$$S_p^*(x) = \frac{x \ln p}{\ln x} + O\left(\frac{x \ln \ln x}{\ln^2 x}\right).$$

§2. Proof of the theorem

In this section, we complete the proof of the theorem 1. In fact, from the definition of $S_p(x)$, we have $(m-1)! < p^x \leq m!$ and taking the logistic computation in the two sides of the inequality, we get

$$\sum_{i=1}^{m-1} \ln i < x \ln p \leq \sum_{i=1}^{m} \ln i. \tag{1}$$

Then using the Euler's summation formula we have

$$\begin{aligned} \sum_{i=1}^{m} \ln i &= \int_1^m \ln t dt + \int_1^m (t-[t])(\ln t)' \ln t dt \\ &= m \ln m - m + O(\ln m) \end{aligned} \tag{2}$$

and

$$\begin{aligned} \sum_{i=1}^{m-1} \ln i &= \int_1^{m-1} \ln t dt + \int_1^{m-1} (t-[t])(\ln t)' dt \\ &= m \ln m - m + O(\ln m). \end{aligned} \tag{3}$$

Combining (1),(2) and (3), we can easily deduce that

$$x \ln p = m \ln m - m + O(\ln m). \tag{4}$$

So

$$m = \frac{x \ln p}{\ln m - 1} + O(1). \tag{5}$$

Similarly, we continue taking logistic computation in two sides of (5), then we also have

$$\ln m = \ln x + O(\ln \ln m). \tag{6}$$

and

$$\ln \ln m = O(\ln \ln x). \tag{7}$$

Hence, by (5), (6) and (7) we have

$$\begin{aligned}
S_p(x) &= \frac{x \ln p}{\ln x + O(\ln\ln m) - 1} + O(1) \\
&= \frac{x \ln p}{\ln x} + x \ln p \left(\frac{O(\ln\ln m)}{\ln x(\ln x + O(\ln\ln m) - 1)} \right) \\
&= \frac{x \ln p}{\ln x} + O\left(\frac{x \ln\ln x}{\ln^2 x} \right).
\end{aligned}$$

This completes the proof of Theorem 1.

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ON THE K-POWER COMPLEMENT SEQUENCE

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Abstract The main purpose of this paper is using analytic method to study the asymptotic properties of k-power complement sequence, and give several interesting asymptotic formulae.

Keywords: k-power complement sequence; asymptotic formula; mean value.

§1. Introdution

For any positive integer $n \geq 2$, let $b_k(n)$ denotes k-power complement sequence. That is, $b_k(n)$ denotes the smallest integer such that $nb_k(n)$ be a perfect k-power. In problem 29 of reference [1], professor F.Smarandache asked us to study the properties of this sequence. About this problem, some people had studied it before, see references [4]and [5]. The main purpose of this paper is using the analytic method to study the asymptotic properties of k-power complement sequence, and obtain several interesting asymptotic formulae. That is, we shall prove the following :

Theorem. *Let $d(n)$ denote the Dirichlet divisor function, then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n \leq x} d(nb_k(n)) = x(A_0 \ln^k x + A_1 \ln^{k-1} x + \cdots + A_k) + O(x^{\frac{1}{2}+\varepsilon}),$$

where A_0, A_1,$\cdots$, A_k are computable constants, ε is any fixed positive number.

From this theorem, we may immediately deduce the following

Corollary 1. *Let $a(n)$ be the square complement sequence, then for any real number $x \geq 1$, we have*

$$\sum_{n \leq x} d(na(n)) = x(A \ln^2 x + B \ln x + C) + O(x^{\frac{1}{2}+\varepsilon}),$$

where A, B, C are computable constants.

Corollary 2. *Let $b(n)$ be the cubic complement sequence, then for any real number $x \geq 1$, we have*

$$\sum_{n\leq x} d(nb(n)) = x(B_1 \ln^3 x + C_1 \ln^2 x + D_1 \ln x + E_1) + O(x^{\frac{1}{2}+\varepsilon}),$$

where B_1, C_1, D_1 and E_1 are computable constants.

§2. Proof of the Theorem

In this section, we shall complete the proof of the Theorem. Let

$$f(s) = \sum_{n=1}^{\infty} \frac{d(nb_k(n))}{n^s}.$$

From the definition of $b_k(n)$, the properties of the divisor function and the Euler product formula [2], we have

$$\begin{aligned}
f(s) &= \prod_p \left(1 + \frac{d(pb_k(p))}{p^s} + \frac{d(p^2 b_k(p^2))}{p^{2s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{d(p^k)}{p^s} + \cdots + \frac{d(p^k)}{p^{ks}} + \frac{d(p^{2k})}{p^{(k+1)s}} + \cdots + \frac{d(p^{2k})}{p^{2ks}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{k+1}{p^s} + \cdots + \frac{k+1}{p^{ks}} + \frac{2k+1}{p^{(k+1)s}} + \cdots + \frac{2k+1}{p^{2ks}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{1}{1-\frac{1}{p^{ks}}} \times \frac{1}{p^s} \times \frac{1-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}} \times \left(1 + \frac{k}{1-\frac{1}{p^{ks}}}\right)\right) \\
&= \zeta(s) \prod_p \left(1 + \frac{k}{p^s} + \frac{k}{p^s\,(p^{ks}-1)}\right) \\
&= \frac{\zeta^{k+1}(s)}{\zeta^k(2s)} \prod_p \left(1 + \frac{kp^{ks}}{p^s(p^s+1)^k(p^{ks}-1)} - \sum_{2\leq i\leq k} \frac{\binom{k}{i} p^{ks}}{p^{si}(p^s+1)^k}\right),
\end{aligned}$$

where $\zeta(s)$ is the Riemann Zeta function. Obviously, we have

$$|d(nb_k(n))| \leq n, \qquad \left|\sum_{n=1}^{\infty} \frac{d(nb_k(n))}{n^\sigma}\right| \leq \frac{1}{\sigma - 1},$$

where σ is the real part of s. Therefore by Perron formula [3]

$$\sum_{n=1}^{\infty} \frac{d(nb_k(n))}{n^{s_0}}$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s+s_0)\frac{x^s}{s}ds + O\left(\frac{x^b B(b+\sigma_0)}{T}\right)\\
&\quad + O\left(x^{1-\sigma_0}H(2x)\min\{1,\frac{\log x}{T}\}\right) + O\left(x^{-\sigma_0}H(N)\min\{1,\frac{x}{\|x\|}\}\right),
\end{aligned}
$$

where N be the nearest integer to x, $\|x\| = |x-N|$. Taking $s_0 = 0$, $b = \frac{3}{2}$, $T = x$, $H(x) = x$, $B(\sigma) = \frac{1}{\sigma-1}$, then we have

$$
\sum_{n\le x} d(nb_k(n)) = \frac{1}{2\pi i}\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} \frac{\zeta^{k+1}(s)}{\zeta^k(2s)} R(s)\frac{x^s}{s}ds + O(x^{\frac{1}{2}+\varepsilon}),
$$

where

$$
R(s) = \prod_p \left(1 + \frac{kp^{ks}}{p^s(p^s+1)^k(p^{ks}-1)} - \sum_{2\le i\le k} \frac{\binom{k}{i} p^{ks}}{p^{si}(p^s+1)^k}\right).
$$

To estimate the main term

$$
\frac{1}{2\pi i}\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} \frac{\zeta^{k+1}(s)}{\zeta^k(2s)} R(s)\frac{x^s}{s}ds,
$$

we move the integer line from $s = 3/2 \pm iT$ to $s = 1/2 \pm iT$, then the function

$$
\frac{\zeta^{k+1}(s)}{\zeta^k(2s)} R(s)\frac{x^s}{s}
$$

have one $k+1$ order pole point at $s = 1$ with residue

$$
\begin{aligned}
& \lim_{s\to 1}\frac{1}{k!}\left((s-1)^{k+1}\zeta^{k+1}(s)\frac{R(s)x^s}{\zeta^k(2s)s}\right)^{(k)}\\
=\ & \lim_{s\to 1}\frac{1}{k!}\left(\binom{k}{0}\left((s-1)^{k+1}\zeta^{k+1}(s)\right)^{(k)}\frac{R(s)x^s}{\zeta^k(2s)s}\right.\\
&\quad + \binom{k}{1}\left((s-1)^{k+1}\zeta^{k+1}(s)\right)^{(k-1)}\left(\frac{R(s)x^s}{\zeta^k(2s)s}\right)' + \cdots\\
&\quad \left. + \binom{k}{k}(s-1)^{k+1}\zeta^{k+1}(s)\left(\frac{R(s)x^s}{\zeta^k(2s)s}\right)^{(k)}\right)\\
=\ & x(A_0\ln^k x + A_1\ln^{k-1}x + \cdots + A_k),
\end{aligned}
$$

where $A_0, A_1, \cdots, A_k$ are computable constants. So we can obtain

$$
\begin{aligned}
& \frac{1}{2\pi i}\left(\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} + \int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT} + \int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}\right)\frac{\zeta^{k+1}(s)x^s}{\zeta^k(2s)s}R(s)ds\\
=\ & x(A_0\ln^k x + A_1\ln^{k-1}x + \cdots + A_k).
\end{aligned}
$$

Note that the estimates

$$\left|\frac{1}{2\pi i}\int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT}\frac{\zeta^{k+1}(s)x^s}{\zeta^k(2s)s}R(s)ds\right| \ll x^{\frac{1}{2}+\varepsilon},$$

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}\frac{\zeta^{k+1}(s)x^s}{\zeta^k(2s)s}R(s)ds\right| \ll x^{\frac{1}{2}+\varepsilon},$$

and

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}\frac{\zeta^{k+1}(s)x^s}{\zeta^k(2s)s}R(s)ds\right| \ll x^{\frac{1}{2}+\varepsilon}.$$

Therefore we get

$$\sum_{n\leq x} d(nb_k(n)) = x(A_0\ln^k x + A_1\ln^{k-1} x + \cdots + A_k) + O(x^{\frac{1}{2}+\varepsilon}).$$

This completes the proof of the Theorem .

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ON THE INFERIOR AND SUPERIOR FACTORIAL PART SEQUENCES

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Abstract The main purpose of this paper is using the elementary method to study the convergent property of an infinite series involving the inferior and superior factorial part sequences, and give an sufficient condition of the convergent property of the series.

Keywords: Inferior factorial part; Superior factorial part; Infinite series.

§1. Introduction

For any positive integer n, the inferior factorial part denoted by $a(n)$ is the largest factorial less than or equal to n. It is the sequence: $1, 2, 2, 2, 2, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 6, 24, 24, 24, 24, 24, 24, 24, \cdots$. On the other hand, the superior factorial part denoted by $b(n)$ is the smallest factorial greater than or equal to n. It is the sequence as follows: $1, 2, 6, 6, 6, 6, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 24, 120, 120, 120, \cdots$. In the 42th problem (see [1]) of his famous book $Only\ problems$, $Not\ solutions$, Professor F.Smarandache asked us to study these sequences. About this problem, it seems no one had studied it before. In this paper, we studied two infinite series involving $a(n)$ and $b(n)$ as follows:

$$I = \sum_{n=1}^{\infty} \frac{1}{a^{\alpha}(n)}, \qquad S = \sum_{n=1}^{\infty} \frac{1}{b^{\alpha}(n)},$$

and given some sufficient conditions of the convergent property of them. That is, we shall prove the following

Theorem. *Let α be any positive real number. Then the infinite series I and S are convergent if $\alpha > 1$, divergent if $\alpha \leq 1$.*

Especially, when $\alpha = 2$, we have the following

Corollary. *We have the identity*

$$\sum_{n=1}^{\infty} \frac{1}{a^2(n)} = e.$$

§2. Proof of the theorem

In this section, we will complete the proof of the theorem. Let $a(n) = m!$. It is easy to see that if $m! \leq n < (m+1)!$, then $a(n) = m!$. So the same number $m!$ repeated $(m+1)! - m!$ times in the sequence $\{a(n)\}$ $(n = 1, 2, \cdots)$. Hence, we can write

$$I = \sum_{n=1}^{\infty} \frac{1}{a^{\alpha}(n)} = \sum_{m=1}^{\infty} \frac{(m+1)! - m!}{(m!)^{\alpha}} = \sum_{m=1}^{\infty} \frac{m \cdot m!}{(m!)^{\alpha}} = \sum_{m=1}^{\infty} \frac{m}{(m!)^{\alpha-1}}.$$

It is clear that if $\alpha > 1$ then I is convergent, if $\alpha \leq 1$ then I is divergent. Using the same method, we can also get the sufficient condition of the convergent property of S. Especially, when $\alpha = 2$, from the knowledge of mathematical analysis (see [2]), we have

$$\sum_{n=1}^{\infty} \frac{1}{a^2(n)} = \sum_{m=1}^{\infty} \frac{1}{(m-1)!} = \sum_{m=0}^{\infty} \frac{1}{m!} = e.$$

This completes the proof of the theorem.

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A NUMBER THEORETIC FUNCTION AND ITS MEAN VALUE

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Abstract Let p and q are primes, $e_q(n)$ denotes the largest exponent of power q which divides n. In this paper, we study the properties of this sequence $p^{e_q(n)}$, and give an interesting asymptotic formula for $\sum\limits_{n\leq x} p^{e_q(n)}$.

Keywords: Asymptotic formula; Largest exponent; Mean value.

§1. Introduction

Let p and q are two primes, $e_q(n)$ denotes the largest exponent of power q which divides n. It is obvious that $e_q = k$ if q^k divides n but $q^{(k+1)}$ does not. In problem 68 of [1], Professor F.Smarandache asked us to study the properties of the sequence $e_q(n)$. In this paper, we use elementary methods to study the asymptotic properties of the mean value $\sum\limits_{n\leq x} p^{e_q(n)}$, where p is a prime such that $q \geq p$, and give an interesting asymptotic formula for it. That is, we will prove the following :

Theorem. *Let p and q are primes with $q \geq p$. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} p^{e_q(n)} = \begin{cases} \frac{q-1}{q-p}x + O\left(x^{\frac{1}{2}+\epsilon}\right), & \textit{if} \quad q > p\,; \\ \frac{p-1}{p\ln p}x\ln x + \left(\frac{p-1}{p\ln p}(\gamma-1) + \frac{p+1}{2p}\right)x + O\left(x^{\frac{1}{2}+\epsilon}\right), & \textit{if} \quad q = p\,; \end{cases}$$

where ϵ is any fixed positive number, γ is the Euler constant.

§2. Proof of the Theorem

In this section, we will complete the proof of the theorem. Let

$$f(s) = \sum_{n=1}^{\infty} \frac{p^{e_q(n)}}{n^s}.$$

For any positive integer n, it is clear that $e_q(n)$ is a addtive function. So $p^{e_q(n)}$ is a multiplicative function. Then from the definition of $e_q(n)$ and the Euler product formula (See Theorem 11.6 of [3]), we can write

$$\begin{aligned}
f(s) &= \sum_{n=1}^{\infty}\frac{p^{e_q(n)}}{n^s} = \prod_{p_1}\left(\sum_{m=0}^{\infty}\frac{p^{e_q(p_1^m)}}{p_1^{ms}}\right)\\
&= \prod_{p_1}\left(1+\frac{p^{e_q(p_1)}}{p_1^s}+\frac{p^{e_q(p_1^2)}}{p_1^{2s}}+\cdots\right)\\
&= \left(1+\frac{p}{q^s}+\frac{p^2}{q^{2s}}+\cdots\right)\prod_{p_1\neq q}\left(1+\frac{1}{p_1^s}+\frac{1}{p_1^{2s}}+\cdots\right)\\
&= \zeta(s)\frac{q^s-1}{q^s-p}.
\end{aligned}$$

By Perron formula (See reference [2]), let $s_0=0$, $b=2$, $T=x^{3/2}$. Then we have

$$\sum_{n\leq x}p^{e_q(n)} = \frac{1}{2\pi i}\int_{2-iT}^{2+iT}\zeta(s)R(s)\frac{x^s}{s}ds + O(x^{1/2+\epsilon}),$$

where $R(s)=\frac{q^s-1}{q^s-p}$ and ϵ is any fixed positive number.

Now we estimate the main term

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\zeta(s)R(s)\frac{x^s}{s}ds,$$

we move the integral line from $2\pm iT$ to $1/2\pm iT$.

If $q>p$, then function

$$\zeta(s)R(s)\frac{x^s}{s}$$

have a simple pole point at $s=1$, so we have

$$\frac{1}{2\pi i}\left(\int_{2-iT}^{2+iT}+\int_{2+iT}^{1/2+iT}+\int_{1/2+iT}^{1/2-iT}+\int_{1/2-iT}^{2-iT}\right)\zeta(s)R(s)\frac{x^s}{s}ds = R(1)x.$$

Taking $T=x^{\frac{3}{2}}$, we have

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\left(\int_{2+iT}^{\frac{1}{2}+iT}+\int_{\frac{1}{2}-iT}^{2-iT}\right)\zeta(s)R(s)\frac{x^s}{s}ds\right|\\
&\ll \int_{\frac{1}{2}}^{2}\left|\zeta(\sigma+iT)R(s)\frac{x^2}{T}\right|d\sigma\\
&\ll \frac{x^{2+\epsilon}}{T} = x^{\frac{1}{2}+\epsilon};
\end{aligned}$$

And we can easy get the estimate

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}\zeta(s)R(s)\frac{x^s}{s}ds\right| \ll \int_0^T \left|\zeta(\frac{1}{2}+it)R(s)\frac{x^{\frac{1}{2}}}{t}\right| dt \ll x^{\frac{1}{2}+\epsilon};$$

Note that

$$R(1) = \frac{q-1}{q-p},$$

we may immediately obtain the asymptotic formula

$$\sum_{n\le x} p^{e_q(n)} = \frac{q-1}{q-p}x + O(x^{\frac{1}{2}+\epsilon}),$$

if $q > p$.

If $q = p$, then the function

$$\zeta(s)R(s)\frac{x^s}{s}$$

have a second order pole point at $s = 1$. Let $\text{Res}\Big(\zeta(s)R(s)\frac{x^s}{s}\Big)$ denotes its residue, so we have

$$\begin{aligned}
& Res\left(\zeta(s)R(s)\frac{x^s}{s}\right) \\
= & \lim_{s\to 1}\left(\frac{p^s-1}{p^s-p}\zeta(s)(s-1)^2\frac{x^s}{s}\right)' \\
= & \lim_{s\to 1}\left(\left(\frac{p^s-1}{p^s-p}(s-1)\frac{x^s}{s}\right)'\zeta(s)(s-1)\right. \\
& \left. + \left(\frac{p^s-1}{p^s-p}(s-1)\frac{x^s}{s}\right)(\zeta(s)(s-1))'\right)
\end{aligned}$$

Noting that

$$\lim_{s\to 1}\left(\frac{p^s-1}{p^s-p}(s-1)\right)' = \frac{p+1}{2p},$$
$$\lim_{s\to 1}\zeta(s)(s-1) = 1$$

and

$$\lim_{s\to 1}(\zeta(s)(s-1))' = \gamma,$$

we may immediately get

$$Res\left(\zeta(s)R(s)\frac{x^s}{s}\right) = \frac{p-1}{p\ln p}x\ln x + \left(\frac{p-1}{p\ln p}(\gamma-1) + \frac{p+1}{2p}\right)x.$$

So we have the asymptotic formula

$$\sum_{n\le x} p^{e_q(n)} = \frac{(p-1)}{p\ln p}x\ln x + \left(\frac{p-1}{p\ln p}(\gamma-1) + \frac{p+1}{2p}\right)x + O(x^{\frac{1}{2}+\epsilon}),$$

if $q = p$. This completes the proof of Theorem.

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ON THE GENERALIZED CONSTRUCTIVE SET

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Abstract In this paper, we use the elementary methods to study the convergent properties of the series

$$\sum_{n=1}^{+\infty} \frac{1}{a_n^{\alpha}},$$

where a_n is a generalized constructive number, and α is any positive number.

Keywords: Generalized constructive set; Series; Convergent properties.

§1. Introduction

The generalized constructive set S is defined as: numbers formed by digits $d_1, d_2, \cdots, d_m$ only, all d_i being different each other, $1 \leq m \leq 9$. That is to say,

(1) $d_1, d_2, \cdots, d_m$ belong to S;

(2) If a, b belong to S, then $\overline{ab}$ belongs to S too;

(3) Only elements obtained by rules (1) and (2) applied a finite number of times belong to S.

For example, the constructive set (of digits 1, 2) is :$1, 2, 11, 12, 21, 22, 111, 112, 121, 122, 211, 212, 221, 222, 1111, 1112, 1121, \cdots$. And the constructive set (of digits 1, 2, 3) is : $1, 2, 3, 11, 12, 13, 21, 22, 23, 31, 32, 33, 111, 112, 113, 121, 122, 123, 211, 212, 213, 221, 222, 223, \cdots$. In problem 6, 7 and 8 of reference [1], Professor F.Smarandache asked us to study the properties of this sequence. About this problem, it seems that no one had studied it before. For convenience, we denote this sequence by $\{a_n\}$. In this paper, we shall use the elementary methods to study the convergent properties of the series

$$\sum_{n=1}^{+\infty} \frac{1}{a_n^{\alpha}},$$

where α is any positive number. That is , we shall prove the following conclusion:

Theorem 1. *The series* $\sum_{n=1}^{+\infty} \frac{1}{a_n^\alpha}$ *is convergent if* $\alpha > \log m$*, and divergent if* $\alpha \leq \log m$.

Especially, let

$$S_2 = 1 + \frac{1}{2} + \frac{1}{11} + \frac{1}{12} + \frac{1}{21} + \frac{1}{22} + \cdots,$$

and

$$S_3 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{21} + \frac{1}{22} + \frac{1}{23} + \frac{1}{31} + \frac{1}{32} + \frac{1}{33} + \cdots.$$

Then we have the following

Theorem 2. *The series* S_2 *and* S_3 *are convergent, and the estimate*

$$\frac{10264}{5775} < S_2 < \frac{8627}{4620} \qquad and \qquad \frac{314568337}{155719200} < S_3 < \frac{10532147}{4449120}$$

hold.

§2. Proof of the Theorems

In this section, we shall complete the proof of the Theorems. First we prove Theorem 1. Note that for $k = 1, 2, 3, \cdots$, there are m^k numbers of k digits in the generalized constructive sequence, so we have

$$\begin{aligned}
\sum_{n=1}^{+\infty} \frac{1}{a_n^\alpha} &< \sum_{k=1}^{+\infty} \frac{m^k}{(10^{k-1})^\alpha} \\
&= m \sum_{k=1}^{+\infty} \frac{m^{k-1}}{10^{(k-1)\alpha}} \\
&= \frac{m}{1 - \frac{m}{10^\alpha}} \\
&= \frac{m \cdot 10^\alpha}{10^\alpha - m},
\end{aligned}$$

where we have used the fact that the series

$$\sum_{k=1}^{+\infty} \frac{m^{k-1}}{10^{(k-1)\alpha}}$$

is convergent only if its common proportion $\frac{m}{10^\alpha} < 1$, that is $\alpha > \log m$. This completes the proof of the Theorem 1.

Now we come to prove Theorem 2.

$$S_2 = 1 + \frac{1}{2} + \frac{1}{11} + \frac{1}{12} + \frac{1}{21} + \frac{1}{22} + \cdots$$

$$\begin{aligned}
&< 1+\frac{1}{2}+\frac{1}{11}+\frac{1}{12}+\frac{1}{21}+\frac{1}{22}+\sum_{k=3}^{+\infty}\frac{2^k}{10^{k-1}}\\
&= 1+\frac{1}{2}+\frac{1}{11}+\frac{1}{12}+\frac{1}{21}+\frac{1}{22}+\frac{2^3}{10^2}\sum_{k=3}^{+\infty}\frac{2^{k-3}}{10^{k-2}}\\
&= 1+\frac{1}{2}+\frac{1}{11}+\frac{1}{12}+\frac{1}{21}+\frac{1}{22}+\frac{2}{25}\times\frac{1}{1-\frac{1}{5}}\\
&= \frac{8627}{4620}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
S_2 &> 1+\frac{1}{2}+\frac{1}{11}+\frac{1}{12}+\frac{1}{21}+\frac{1}{22}+\sum_{k=3}^{+\infty}\frac{2^k}{10^{k}}\\
&= 1+\frac{1}{2}+\frac{1}{11}+\frac{1}{12}+\frac{1}{21}+\frac{1}{22}+\frac{2^3}{10^3}\sum_{k=3}^{+\infty}\frac{2^{k-3}}{10^{k-3}}\\
&= 1+\frac{1}{2}+\frac{1}{11}+\frac{1}{12}+\frac{1}{21}+\frac{1}{22}+\frac{1}{125}\times\frac{1}{1-\frac{1}{5}}\\
&= \frac{10264}{5775}.
\end{aligned}$$

This proves the first part of Theorem 2.

Similarly, we can prove the second part of Theorem 2. This completes the proof of the Theorems.

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ON THE INFERIOR AND SUPERIOR PRIME PART

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Abstract For any positive integer n, inferior prime part function $p_p(n)$ is defined as the largest prime number less than or equal to n, and superior prime part function $P_p(n)$ is denoted as the smallest prime number greater than or equal to n. The main purpose of this paper is using the important works of D.R.Heath Brown to study the mean value of $p_p(n)$ and $P_p(n)$, and give two sharp asymptotic formulae.

Keywords: Inferior prime part; Superior prime part; Mean value.

§1. Introduction

For any positive integer n, inferior prime part function $p_p(n)$ is defined as the largest prime number less than or equal to n, and superior prime part function $P_p(n)$ is denoted as the smallest prime number greater than or equal to n. In problem 39 of [1], F.Smarandache asked us to study these sequences.

For each integer n, the prime additive complement $b(n)$ is defined as the smallest nonnegative integer such that $n + b(n)$ is a prime. In problem 44 of [1], F.Smarandache also advised us to study this sequence.

It is interesting that there exist some relationships among $p_p(n)$, $P_p(n)$ and $b(n)$. In this paper, we use the important works of D.R.Heath Brown to study the mean value properties of $p_p(n)$ and $P_p(n)$, and give two sharp asymptotic formulae. That is, we shall prove the following:

Theorem 1. *For any real number $x \geq 1$, we have*

$$\sum_{n\leq x} p_p(n) = \frac{x^2}{2} + O\left(x^{\frac{23}{18}+\epsilon}\right),$$

where ϵ is any fixed positive number.

Theorem 2. *For any real number $x \geq 1$, we have*

$$\sum_{n\leq x} P_p(n) = \frac{x^2}{2} + O\left(x^{\frac{23}{18}+\epsilon}\right).$$

§2. Proof of the theorems

To complete the proof of the theorems, we need the following:

Lemma 1. *Let $b(n)$ be the prime additive complement, then we have the estimate:*

$$\sum_{n\leq x} b(n) \ll x^{\frac{23}{18}+\epsilon}.$$

Proof. Denote p_n as the n-th prime, then from the definition of $b(n)$ we have

$$\begin{aligned}\sum_{n\leq x} b(n) &= \sum_{1\leq i\leq \pi(x)} \sum_{p_i<n\leq p_{i+1}} b(n) \\ &\leq \sum_{1\leq i\leq \pi(x)} \sum_{p_i<n\leq p_{i+1}} (p_{i+1}-p_i) \\ &\leq \sum_{1\leq i\leq \pi(x)} (p_{i+1}-p_i)^2 . \qquad (1)\end{aligned}$$

By [2] we can get

$$\sum_{1\leq i\leq \pi(x)} (p_{i+1}-p_i)^2 \ll x^{\frac{23}{18}+\epsilon}, \qquad (2)$$

so by (1) and (2) we immediately have

$$\sum_{n\leq x} b(n) \ll x^{\frac{23}{18}+\epsilon}.$$

This proves Lemma 1.

Now we prove the theorems. For any real number $x\geq 1$, from the definition of $p_p(n)$ we have

$$\sum_{n\leq x} p_p(n) = \sum_{1\leq i\leq \pi(x)} (p_{i+1}-p_i)p_i. \qquad (3)$$

On the other hand,

$$\begin{aligned}\sum_{n\leq x}(n+b(n)) &= \sum_{1\leq i\leq \pi(x)} \sum_{p_i<n\leq p_{i+1}} (n+b(n)) \\ &= \sum_{1\leq i\leq \pi(x)} (p_{i+1}-p_i)p_{i+1} \\ &= \sum_{1\leq i\leq \pi(x)} (p_{i+1}-p_i)^2 + \sum_{1\leq i\leq \pi(x)} (p_{i+1}-p_i)p_i. \qquad (4)\end{aligned}$$

Then from Lemma 1, (2), (3) and (4) we have

$$\sum_{n\leq x} p_p(n) = \sum_{n\leq x}(n+b(n)) - \sum_{1\leq i\leq \pi(x)} (p_{i+1}-p_i)^2$$

$$= \sum_{n\leq x} n + \sum_{n\leq x} b(n) - \sum_{1\leq i\leq \pi(x)} (p_{i+1}-p_i)^2 = \frac{x^2}{2} + O\left(x^{\frac{23}{18}+\epsilon}\right).$$

This proves Theorem 1.

Similarly, from the definition of $P_p(n)$, (4) and Lemma 1 we can get

$$\begin{aligned}\sum_{n\leq x} P_p(n) &= 6+\sum_{1\leq i\leq \pi(x)}(p_{i+1}-p_i)p_{i+1}\\ &= 6+\sum_{n\leq x}(n+b(n))=\frac{x^2}{2}+O\left(x^{\frac{23}{18}+\epsilon}\right).\end{aligned}$$

This completes the proof of Theorem 2.

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IDENTITIES ON THE K-POWER COMPLEMENTS

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Abstract The main purpose of this paper is to calculate the value of the series

$$\sum_{n=1}^{+\infty} \frac{1}{(na_k(n))^s},$$

where $a_k(n)$ is the k-power complement number of any positive integer n, and s is a complex number with $\mathrm{Re}(s) \geq 1$. Several interesting identities are given.

Keywords: k-power complement number; Identities; Riemann-zeta function.

§1. Introduction

For any given natural number $k \geq 2$ and any positive integer n, we call $a_k(n)$ a k-power complement number, if $a_k(n)$ denotes the smallest integer such that $n \cdot a_k(n)$ is a perfect k-power. Especially, we call $a_2(n), a_3(n), a_4(n)$ a square complement number, cubic complement number, and quartic complement number, respectively. In reference [1], Professor F. Smarandache asked us to study the properties of the k-power complement number sequence. Yet we still know very little about it. In this paper, we shall use the analytic method to calculate the value of the series

$$\sum_{n=1}^{+\infty} \frac{1}{(na_k(n))^s},$$

where s is a complex number with $\mathrm{Re}(s) \geq 1$, and obtain several interesting identities. That is, we shall prove the following several theorems:

Theorem 1. *For any complex number s with $Re(s) \geq 1$, we have the identity*

$$\sum_{n=1}^{+\infty} \frac{1}{(na_2(n))^s} = \frac{\zeta^2(2s)}{\zeta(4s)},$$

where $\zeta(s)$ is the Riemann-zeta function.

Theorem 2. *For any complex number s with $Re(s) \geq 1$, we have*

$$\sum_{n=1}^{+\infty} \frac{1}{(na_3(n))^s} = \frac{\zeta^2(3s)}{\zeta(6s)} \prod_p \left(1 + \frac{1}{p^{3s}+1}\right),$$

where $\prod\limits_{p}$ *denotes the product over all primes.*

Theorem 3. *For any complex number s with $Re(s) \geq 1$, we have*

$$\sum_{n=1}^{+\infty} \frac{1}{(na_4(n))^s} = \frac{\zeta^2(4s)}{\zeta(8s)} \prod_p \left(1 + \frac{1}{p^{4s}+1}\right)\left(1 + \frac{1}{p^{4s}+2}\right).$$

Taking $s = 1, 2$ in the theorems, and note that the fact

$$\zeta(2) = \frac{\pi^2}{6}, \quad \zeta(4) = \frac{\pi^4}{90}, \quad \zeta(6) = \frac{\pi^6}{945}, \quad \zeta(8) = \frac{\pi^8}{9450},$$
$$\zeta(12) = \frac{691\pi^{12}}{638512875}, \qquad \zeta(16) = \frac{3617\pi^{16}}{325641566250}.$$

We may immediately obtain the following two corollaries

Corollary 1.

$$\sum_{n=1}^{+\infty} \frac{1}{na_2(n)} = \frac{5}{2};$$

$$\sum_{n=1}^{+\infty} \frac{1}{na_3(n)} = \frac{\zeta^2(3)}{\zeta(6)} \prod_p \left(1 + \frac{1}{p^3+1}\right);$$

$$\sum_{n=1}^{+\infty} \frac{1}{na_4(n)} = \frac{7}{6} \prod_p \left(1 + \frac{1}{p^4+1}\right)\left(1 + \frac{1}{p^4+2}\right).$$

Corollary 2.

$$\sum_{n=1}^{+\infty} \frac{1}{(na_2(n))^2} = \frac{7}{6};$$

$$\sum_{n=1}^{+\infty} \frac{1}{(na_3(n))^2} = \frac{715}{691} \prod_p \left(1 + \frac{1}{p^6+1}\right);$$

$$\sum_{n=1}^{+\infty} \frac{1}{(na_4(n))^2} = \frac{7293}{7234} \prod_p \left(1 + \frac{1}{p^8+1}\right)\left(1 + \frac{1}{p^8+2}\right).$$

§2. Proof of the Theorems

In this section, we shall complete the proof of the Theorems. For any positive integer n, we can write it as $n = l^2 m$, where m is a square-free number (that is, $p \dagger m$ implies $p^2 \dagger m$). Then from the definition of $a_2(n)$ we have

$$\begin{aligned}
\sum_{n=1}^{+\infty} \frac{1}{(na_2(n))^s} &= \sum_{l=1}^{+\infty}\sum_{m=1}^{+\infty} \frac{|\mu(m)|}{(l^2 m \cdot m)^s} = \sum_{l=1}^{+\infty}\sum_{m=1}^{+\infty} \frac{|\mu(m)|}{l^{2s} m^{2s}} \\
&= \zeta(2s) \prod_p \left(1 + \frac{1}{p^{2s}}\right) = \frac{\zeta^2(2s)}{\zeta(4s)},
\end{aligned}$$

where $\mu(n)$ denotes the Mӧbius function. This completes the proof of Theorem 1.

Now we come to prove Theorem 2. For any positive integer n, we can write it as $n = l^3m^2r$, where $(m,r) = 1$, and rm is a square-free number. Then from the definition of $a_3(n)$, we have

$$\begin{aligned}
\sum_{n=1}^{+\infty} \frac{1}{(na_3(n))^s} &= \sum_{l=1}^{+\infty}\sum_{\substack{m=1\\(m,r)=1}}^{+\infty}\sum_{r=1}^{+\infty} \frac{|\mu(m)|\,|\mu(r)|}{(l^3m^2r\cdot mr^2)^s} \\
&= \zeta(3s)\sum_{m=1}^{+\infty}\frac{|\mu(m)|}{m^{3s}}\sum_{\substack{r=1\\(m,r)=1}}^{+\infty}\frac{|\mu(r)|}{r^{3s}} \\
&= \zeta(3s)\sum_{m=1}^{+\infty}\frac{|\mu(m)|}{m^{3s}}\prod_{p\dagger m}\left(1+\frac{1}{p^{3s}}\right) \\
&= \frac{\zeta^2(3s)}{\zeta(6s)}\sum_{m=1}^{+\infty}\frac{|\mu(m)|}{m^{3s}}\prod_{p|m}\frac{1}{\left(1+\frac{1}{p^{3s}}\right)} \\
&= \frac{\zeta^2(3s)}{\zeta(6s)}\prod_{p}\left(1+\frac{1}{p^{3s}\left(1+\frac{1}{p^{3s}}\right)}\right) \\
&= \frac{\zeta^2(3s)}{\zeta(6s)}\prod_{p}\left(1+\frac{1}{p^{3s}+1}\right).
\end{aligned}$$

This completes the proof of Theorem 2.

Now we come to prove Theorem 3. For any positive integer n, we can write it as $n = l^4m^3r^2t$, where $(m,r) = 1, (mr,t) = 1$ and mrt is a square-free number. Then from the definition of $a_4(n)$, we have

$$\begin{aligned}
&\sum_{n=1}^{+\infty} \frac{1}{(na_4(n))^s} \\
= &\sum_{l=1}^{+\infty}\sum_{\substack{m=1\\(m,r)=1\\(mr,t)=1}}^{+\infty}\sum_{r=1}^{+\infty}\sum_{t=1}^{+\infty} \frac{|\mu(m)|\,|\mu(r)|\,|\mu(t)|}{(l^4m^3r^2t\cdot mr^2t^3)^s} \\
= &\ \zeta(4s)\sum_{\substack{m=1\\(m,r)=1}}^{+\infty}\sum_{r=1}^{+\infty}\frac{|\mu(m)|\,|\mu(r)|}{m^{4s}r^{4s}}\sum_{\substack{t=1\\(mr,t)=1}}^{+\infty}\frac{|\mu(t)|}{t^{4s}} \\
= &\ \frac{\zeta^2(4s)}{\zeta(8s)}\sum_{\substack{m=1\\(m,r)=1}}^{+\infty}\sum_{r=1}^{+\infty}\frac{|\mu(m)|\,|\mu(r)|}{m^{4s}r^{4s}}\prod_{p|mr}\frac{1}{\left(1+\frac{1}{p^{4s}}\right)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\zeta^2(4s)}{\zeta(8s)} \sum_{\substack{m=1 \\ (m,r)=1}}^{+\infty} \sum_{r=1}^{+\infty} \frac{|\mu(m)|\,|\mu(r)|}{m^{4s}r^{4s}} \prod_{p|m} \frac{1}{\left(1+\frac{1}{p^{4s}}\right)} \prod_{p|r} \frac{1}{\left(1+\frac{1}{p^{4s}}\right)} \\
&= \frac{\zeta^2(4s)}{\zeta(8s)} \sum_{m=1}^{+\infty} \frac{|\mu(m)|}{m^{4s}} \prod_{p|m} \frac{1}{\left(1+\frac{1}{p^{4s}}\right)} \prod_{p\dagger m} \left(1+\frac{1}{p^{4s}+1}\right) \\
&= \frac{\zeta^2(4s)}{\zeta(8s)} \sum_{m=1}^{+\infty} \frac{|\mu(m)|}{m^{4s}} \prod_{p|m} \frac{1}{\left(1+\frac{1}{p^{4s}}\right)} \frac{\prod\limits_{p} \left(1+\frac{1}{p^{4s}+1}\right)}{\prod\limits_{p|m} \left(1+\frac{1}{p^{4s}+1}\right)} \\
&= \frac{\zeta^2(4s)}{\zeta(8s)} \prod_{p} \left(1+\frac{1}{p^{4s}+1}\right)\left(1+\frac{1}{p^{4s}+2}\right).
\end{aligned}
$$

This completes the proof of Theorem 3.

Using our method, we can also obtain the identities for the series

$$\sum_{n=1}^{+\infty} \frac{1}{(na_k(n))^s},$$

where $k \geq 5$.

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ON THE ASYMPTOTIC PROPERTY OF DIVISOR FUNCTION FOR ADDITIVE COMPLEMENTS

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Abstract For any positive integer n, let $a(n)$ is the additive square complements of n. That is, $a(n)$ is the smallest non-negative integer such that $n + a(n)$ is a perfect square number. In this paper, we study the mean value properties of $a(n)$ with the divisor function $d(n)$, and give an interesting mean value formula for $d(n + a(n))$.

Keywords: Additive square complements; Divisor function; Asymptotic formula.

§1. Introduction and results

For any positive integer n, the square complements $a_2(n)$ is defined as the smallest positive integer k such that nk is a perfect square. For example, $a_2(1) = 1$, $a_2(2) = 2$, $a_2(3) = 3$, $a_2(4) = 1$, $a_2(5) = 5$, $a_2(6) = 6$, $a_2(7) = 7$, $a_2(8) = 2, \cdots$. In problem 27 of [1], Professor F. Smarandache ask us to study the properties of $\{a_2(n)\}$. About this problem, some authors had studied it, and obtained some interesting results. For example, the authors [2] used the elementary method to study the mean value properties of $a_2(n)$ and $\frac{1}{a_2(n)}$. Zhang H.L. and Wang Y. in [3] studied the mean value of $\tau(a_2(n))$, and obtained an asymptotic formula by the analytic method.

Similarly, we will define the additive square complements as follows: for any positive integer n, the smallest non-negative integer k is called the additive square complements of n if $n + k$ is a perfect square number. Let

$$a(n) = \min\{k|n + k = m^2, k \geq 0, m \in N^+\},$$

then $a(1) = 0$, $a(2) = 2$, $a(3) = 1$, $a(4) = 0$, $a(5) = 4, \cdots$. In this paper, we will use the analytic methods to study the asymptotic properties of divisor function for this sequence in the following form: $\sum\limits_{n\leq x} d(n+a(n))$, where $x \geq 2$

be a real number, $d(n)$ be the divisor function, and give an sharper asymptotic formula for it. That is, we shall prove the following:

Theorem. *For any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x} d(n+a(n)) = \frac{3}{4\pi^2} x\ln^2 x + A_1 x\ln x + A_2 x + O(x^{\frac{3}{4}+\epsilon}),$$

where A_1 and A_2 are computable constants, ϵ is any fixed positive number.

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. First we need following:

Lemma. *For any real number $x > 1$, we have*

$$\sum_{n\leq x} d(n^2) = \frac{3}{\pi^2}\cdot x\ln^2 x + B_1 x\ln x + B_2 x + O\left(x^{\frac{1}{2}+\epsilon}\right),$$

where B_1 and B_2 are computable constants, ϵ is any fixed positive number.

Proof. Let $s = \sigma + it$ be a complex number and $f(s) = \sum\limits_{n=1}^{\infty} \frac{d(n^2)}{n^s}$. Note that $d(n^2) \ll n^{\epsilon}$, so it is clear that $f(s)$ is a Dirichlet series absolutely convergent for Re$(s)>1$, by the Euler Product formula [4] and the definition of $d(n)$ we get

$$\begin{aligned} f(s) &= \prod_p \left(1+\frac{d(p^2)}{p^s}+\frac{d(p^4)}{p^{2s}}+\cdots+\frac{d(p^{2n})}{p^{ns}}+\cdots\right) \\ &= \prod_p \left(1+\frac{3}{p^s}+\frac{5}{p^{2s}}+\cdots+\frac{2n+1}{p^{ns}}+\cdots\right) \\ &= \zeta^2(s)\prod_p\left(1+\frac{1}{p^s}\right) \\ &= \frac{\zeta^3(s)}{\zeta(2s)}, \end{aligned} \tag{1}$$

where $\zeta(s)$ is the Riemann zeta-function and $\prod\limits_p$ denotes the product over all primes.

From (1) and Perron's formula [5], we have

$$\sum_{n\leq x} d(n^2) = \frac{1}{2\pi i}\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} \frac{\zeta^3(s)}{\zeta(2s)}\cdot\frac{x^s}{s}ds + O\left(\frac{x^{\frac{3}{2}+\epsilon}}{T}\right). \tag{2}$$

Now we move the integral line in (2) from $s = \frac{3}{2} \pm iT$ to $s = \frac{1}{2} \pm iT$. This time, the function $\frac{\zeta^3(s)}{\zeta(2s)}\cdot\frac{x^s}{s}$ have a third order pole point at $s=1$ with residue

$$\frac{3}{\pi^2}\cdot x\ln^2 x + B_1 x\ln x + B_2 x, \tag{3}$$

where B_1 and B_2 are computable constants. Hence, we have

$$\begin{aligned}&\frac{1}{2\pi i}\left(\int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}+\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT}+\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}\right)\frac{\zeta^3(s)}{\zeta(2s)}\cdot\frac{x^s}{s}ds\\&=\frac{3}{\pi^2}\cdot x\ln^2 x+B_1x\ln x+B_2x.\end{aligned}\tag{4}$$

We can easily get the estimate

$$\left|\frac{1}{2\pi i}\left(\int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT}+\int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}\right)\frac{\zeta^3(s)}{\zeta(2s)}\cdot\frac{x^s}{s}ds\right|\ll\frac{x^{\frac{3}{2}+\epsilon}}{T}\tag{5}$$

and

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}\frac{\zeta^3(s)}{\zeta(2s)}\cdot\frac{x^s}{s}ds\right|\ll x^{\frac{1}{2}+\epsilon}.\tag{6}$$

Taking $T=x$, combining (2), (4), (5) and (6) we deduce that

$$\sum_{n\le x}d(n^2)=\frac{3}{\pi^2}\cdot x\ln^2 x+B_1x\ln x+B_2x+O\left(x^{\frac{1}{2}+\epsilon}\right).$$

This completes the proof of Lemma.

Now we use above Lemma to complete the proof of Theorem. For any real number $x\ge 2$, let M be a fixed positive number such that

$$M^2\le x<(M+1)^2.\tag{7}$$

Then from the definition of $a(n)$, we have

$$\begin{aligned}&\sum_{n\le x}d(n+a(n))\\
=&\sum_{1\le m\le M-1}\left(\sum_{m^2\le n<(m+1)^2}d(n+a(n))\right)+\sum_{M^2\le n\le x}d(n+a(n))\\
=&\sum_{1\le m\le M}\left(\sum_{m^2\le n<(m+1)^2}d(n+a(n))\right)+O(x^{\frac{1}{2}+\epsilon})\\
=&\sum_{1\le m\le M}\left(\sum_{m^2\le n<(m+1)^2}d((m+1)^2)\right)+O(x^{\frac{1}{2}+\epsilon})\\
=&\sum_{1\le m\le M}2md((m+1)^2)+O(x^{\frac{1}{2}+\epsilon})\\
=&\ 2\cdot\sum_{1\le m\le M}md(m^2)+O(x^{\frac{1}{2}+\epsilon}).\end{aligned}\tag{8}$$

Let $A(x) = \sum\limits_{n \le x} d(n^2)$, then by Able's identity and Lemma, we can easily deduce that

$$\begin{aligned}
&\sum_{1 \le m \le M} md(m^2) \\
= \;& MA(M) - \int_1^M A(t)d(t) + O(M^{1+\epsilon}) \\
= \;& M\left(\frac{3}{\pi^2} \cdot M \ln^2 M + B_1 M \ln M + B_2 M\right) \\
& - \int_1^M \left(\frac{3}{\pi^2} \cdot t \ln^2 t + B_1 t \ln t + B_2 t\right) dt + O\left(M^{\frac{3}{2}+\epsilon}\right) \\
= \;& \frac{1}{2} \cdot \frac{3}{\pi^2} \cdot M^2 \ln^2 M + C_1 M^2 \ln M + C_2 M^2 + +O\left(M^{\frac{3}{2}+\epsilon}\right), \quad (9)
\end{aligned}$$

where C_1 and C_2 are computable constants.

Note that $0 \le x - M^2 \ll \sqrt{x}$ and $\ln^2 x = 4 \ln^2 M + O(x^{-\frac{1}{2}+\epsilon})$, then from (8) and (9) we get

$$\sum_{n \le x} d(n + a(n)) = \frac{3}{4\pi^2} x \ln^2 x + A_1 x \ln x + A_2 x + O(x^{\frac{3}{4}+\epsilon}).$$

This completes the proof of Theorem.

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MEAN VALUE ON TWO SMARANDACHE-TYPE MULTIPLICATIVE FUNCTIONS

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Abstract In this paper, we study the mean value of two Smarandache-type multiplicative functions, and give a few asymptotic formulae.

Keywords: Smarandache-type multiplicative functions; Mean value; Asymptotic formula.

§1. Introduction

In [1], Henry Bottomley considered eleven particular families of interrelated multiplicative functions, which are listed in Smarandache's problems.

In this paper we study the mean value of two Smarandache-type multiplicative functions. One is $C_m(n)$, which is defined as the m-th root of largest m-th power dividing n. The other function $J_m(n)$ is denoted as m-th root of smallest m-th power divisible by n. We will give a few asymptotic formulae on these two functions. That is, we shall prove the following:

Theorem 1. *For any integer $m \geq 3$ and real number $x \geq 1$, we have*

$$\sum_{n \leq x} C_m(n) = \frac{\zeta(m-1)}{\zeta(m)} x + O\left(x^{\frac{1}{2}+\epsilon}\right).$$

Theorem 2. *For any integer $m \geq 1$ and real number $x \geq 1$, we have*

$$\sum_{n \leq x} J_m(n) = \frac{x^2}{2\zeta(2)} \prod_p \left[1 + \frac{\frac{1}{p^{2m}} + \frac{1}{p^3} - \frac{1}{p^{2m+1}} - \frac{1}{p^{2m+2}}}{\left(1+\frac{1}{p}\right)\left(1-\frac{1}{p^2}\right)\left(1-\frac{1}{p^{2m-1}}\right)}\right] + O\left(x^{\frac{3}{2}+\epsilon}\right).$$

§2. Proof of the theorems

Now we prove the theorems. Let

$$f(s)=\sum_{n=1}^{\infty}\frac{C_m(n)}{n^s}.$$

Noting that

$$C_m(p^\alpha)=p^k, \quad \text{if } mk\le\alpha<m(k+1).$$

Then from the Euler product formula [2] we have

$$\begin{aligned} f(s) &= \prod_p\left[\sum_{\alpha=0}^{\infty}\frac{C_m(p^\alpha)}{p^{\alpha s}}\right] \\ &= \prod_p\left[\sum_{k=0}^{\infty}\sum_{\beta=0}^{m-1}\frac{C_m(p^{mk+\beta})}{p^{(mk+\beta)s}}\right] \\ &= \prod_p\left[\sum_{k=0}^{\infty}\sum_{\beta=0}^{m-1}\frac{p^k}{p^{(mk+\beta)s}}\right] \\ &= \prod_p\left[\frac{1-\frac{1}{p^{ms}}}{\left(1-\frac{1}{p^s}\right)\left(1-\frac{1}{p^{ms-1}}\right)}\right]=\frac{\zeta(s)\zeta(ms-1)}{\zeta(ms)}. \end{aligned}$$

So by Perron formula [3] we can get

$$\begin{aligned} \sum_{n\le x}\frac{C_m(n)}{n^{s_0}} &= \frac{1}{2\pi i}\int_{b-iT}^{b+iT}f(s+s_0)\frac{x^s}{s}\mathrm{d}s+O\left(\frac{x^bB(b+\sigma_0)}{T}\right) \\ &+O\left(x^{1-\sigma_0}H(2x)\min\left(1,\frac{\log x}{T}\right)\right)+O\left(x^{-\sigma_0}H(N)\min\left(1,\frac{x}{||x||}\right)\right), \end{aligned}$$

where N is the nearest integer to x, and $||x||=|x-N|$. Taking $s_0=0$, $b=\frac{3}{2}$ and $T>2$ in above, then we have

$$\sum_{n\le x}ZW^\alpha(n)=\frac{1}{2\pi i}\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT}f(s)\frac{x^s}{s}\mathrm{d}s+O\left(\frac{x^{\frac{3}{2}}}{T}\right).$$

Now we move the integral line from $\frac{3}{2}\pm iT$ to $\frac{1}{2}-iT$. This time, the function

$$f(s)\frac{x^s}{s}$$

have a simple pole point at $s=1$ with residue

$$\frac{\zeta(m-1)}{\zeta(m)}x$$

Now taking $T = x$, then we have

$$\begin{aligned}
\sum_{n\leq x} C_m(n) &= \frac{\zeta(m-1)}{\zeta(m)}x + \frac{1}{2\pi i}\int_{\frac{1}{2}-ix}^{\frac{1}{2}+ix} f(s)\frac{x^s}{s}\mathrm{d}s + O\left(x^{\frac{1}{2}+\epsilon}\right) \\
&= \frac{\zeta(m-1)}{\zeta(m)}x + O\left(\int_{-x}^{x}\left|f\left(\frac{1}{2}+\epsilon+ix\right)\right|\frac{x^{\frac{1}{2}+\epsilon}}{(1+|t|)}\mathrm{d}t\right) + O\left(x^{\frac{1}{2}+\epsilon}\right) \\
&= \frac{\zeta(m-1)}{\zeta(m)}x + O\left(x^{\alpha+\frac{1}{2}+\epsilon}\right).
\end{aligned}$$

This proves Theorem 1.

For any integer $m \geq 1$ and real number $x \geq 1$, let

$$g(s) = \sum_{n=1}^{\infty}\frac{J_m(n)}{n^s}.$$

Noting that

$$J_m(p^\alpha) = p^{k+1}, \quad \text{if } mk < \alpha \leq m(k+1).$$

Then from the Euler product formula [2] we have

$$\begin{aligned}
g(s) &= \prod_p\left[1+\sum_{\alpha=1}^{\infty}\frac{J_m(p^\alpha)}{p^{\alpha s}}\right] \\
&= \prod_p\left[1+\sum_{k=0}^{\infty}\sum_{\beta=1}^{m}\frac{J_m(p^{mk+\beta})}{p^{(mk+\beta)s}}\right] \\
&= \prod_p\left[1+\sum_{k=0}^{\infty}\sum_{\beta=1}^{m}\frac{p^{k+1}}{p^{(mk+\beta)s}}\right] \\
&= \prod_p\left[1+\frac{1-\frac{1}{p^{ms}}}{p^{s-1}\left(1-\frac{1}{p^s}\right)\left(1-\frac{1}{p^{ms-1}}\right)}\right] \\
&= \prod_p\left[1+\frac{1}{p^{s-1}}+\frac{\frac{1}{p^{(m+1)s-2}}+\frac{1}{p^{2s-1}}-\frac{1}{p^{(m+1)s-1}}-\frac{1}{p^{(m+2)s-2}}}{\left(1-\frac{1}{p^s}\right)\left(1-\frac{1}{p^{ms-1}}\right)}\right] \\
&= \frac{\zeta(s-1)}{\zeta(2s-2)}\prod_p\left[1+\frac{\frac{1}{p^{(m+1)s-2}}+\frac{1}{p^{2s-1}}-\frac{1}{p^{(m+1)s-1}}-\frac{1}{p^{(m+2)s-2}}}{\left(1+\frac{1}{p^{s-1}}\right)\left(1-\frac{1}{p^s}\right)\left(1-\frac{1}{p^{ms-1}}\right)}\right].
\end{aligned}$$

So by Perron formula [3] and the methods of proving Theorem 1 we can easily get Theorem 2.

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ON THE SMARANDACHE CEIL FUNCTION AND THE NUMBER OF PRIME FACTORS

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Abstract The main purpose of this paper is using the elementary method to study the mean value properties of the compound function involving Ω and Smarandache ceil function, and give an interesting asymptotic formula.

Keywords: Smarandache ceil function; Asymptotic formula; Mean value

§1. Introduction

For a fixed positive integer k and any positive integer n, the Smarandache ceil function $S_k(n)$ is defined as follows:

$$S_k(n) = \min\{m \in N : n | m^k\}.$$

This was introduced by Professor F.Smarandache. About this function, many scholar studied its properties, see [1] and [2]. In [1], Ibstedt presented the following properties:

$$(\forall a, b \in N)\ (a, b) = 1 \Rightarrow S_k(a \cdot b) = S_k(a) \cdot S_k(b),$$

and $S_k(p^\alpha) = p^{\lceil \frac{\alpha}{k} \rceil}$, where p is a prime and $\lceil x \rceil$ denotes the least integer greater than x. That is, $S_k(n)$ is a multiplicative function. Therefore, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the prime decomposition of n, then the following identity is obviously:

$$S_k(n) = S_k(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = p_1^{\lceil \frac{\alpha_1}{k} \rceil} p_2^{\lceil \frac{\alpha_2}{k} \rceil} \cdots p_r^{\lceil \frac{\alpha_r}{k} \rceil}. \quad (1)$$

The arithmetic function Ω is defined as follows:

$$\Omega(n) = \Omega(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = \alpha_1 + \alpha_2 + \cdots + \alpha_r.$$

In this paper, we use the elementary method to study the mean value properties of the compound function involving Ω and $S_k(n)$, and give an interesting asymptotic formula. That is, we shall prove the following:

Theorem. *Let k be a given positive integer. Then for any real number $x \geq 3$, we have the asymptotic formula:*

$$\sum_{n\leq x} \Omega(S_k(n)) = x \ln\ln x + Ax + O\left(\frac{x}{\ln x}\right),$$

where $A = \gamma + \sum_p \left(\ln\left(1-\frac{1}{p}\right) + \frac{1}{p}\right)$, *$\gamma$ is the Euler constant and* $\sum_p$ *denotes the sum over all the primes.*

§2. Some simple lemma

Before the proof of the theorem, a simple lemma will be useful.

Lemma 1. *Let* $\omega(n) = \omega(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = r$ *. Then for any real number $x \geq 3$, we have the asymptotic formula:*

$$\sum_{n\leq x} \omega(n) = x \ln\ln x + Ax + O\left(\frac{x}{\ln x}\right),$$

where $A = \gamma + \sum_p \left(\ln\left(1-\frac{1}{p}\right) + \frac{1}{p}\right)$, *$\gamma$ is the Euler constant.*

Proof. See reference [3].

Lemma 2. *For any real number $x \geq 3$, we have the asymptotic formula:*

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}}} \Omega(n) \ll x^{\frac{1}{k+1}+\epsilon},$$

where $\mathcal{A}$ denotes the set of $k+1$-full numbers, ϵ is any fixed positive integer.

Proof. First we define arithmetic function $a(n)$ as follows:

$$a(n) = \begin{cases} 1, & \text{if } n \text{ is a } k+1\text{-full number ;} \\ 0, & \text{otherwise .} \end{cases}$$

Now from Euler product formula, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{a(n)}{n^s} &= \prod_p \left(1 + \frac{1}{p^{(k+1)s}} + \frac{1}{p^{(k+2)s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{1}{p^{(k+1)s}} \frac{1}{1-\frac{1}{p^s}}\right) \\
&= \prod_p \left(1 + \frac{1}{p^{(k+1)s}}\right) \left(\frac{p^{(k+1)s}}{p^{(k+1)s}+1} + \frac{p^s}{(p^{(k+1)s}+1)(p^s-1)}\right) \\
&= \frac{\zeta((k+1)s)}{\zeta(2(k+1)s)} \prod_p \left(1 + \frac{1}{(p^{(k+1)s}+1)(p^s-1)}\right),
\end{aligned}$$

where $\zeta(s)$ is the Riemann-zeta function. By Perron formula (see [4]), we can obtain

$$\sum_{n\leq x} a(n) = \sum_{\substack{n\leq x\\ n\in\mathcal{A}}} 1$$
$$= \frac{6(k+1)x^{\frac{1}{k+1}}}{\pi^2}\prod_p\left(1+\frac{1}{(p+1)(p^{\frac{1}{k+1}}-1)}\right)+O\left(x^{\frac{1}{2(k+1)}+\epsilon}\right).$$

So we have

$$\sum_{\substack{n\leq x\\ n\in\mathcal{A}}} \Omega(n) \ll x^{\frac{1}{k+1}+\epsilon}.$$

This proves Lemma 2.

§2. Proof of the theorem

In this section, we will complete the proof of the theorem. Let

$$n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}.$$

From (1) and the completely additive property of function Ω , we can write

$$\Omega(S_k(n)) = \Omega\left(p_1^{\lceil\frac{\alpha_1}{k}\rceil}p_2^{\lceil\frac{\alpha_2}{k}\rceil}\cdots p_r^{\lceil\frac{\alpha_r}{k}\rceil}\right) = \sum_{i=1}^{r}\left\lceil\frac{\alpha_i}{k}\right\rceil. \tag{2}$$

It is clear that $\lceil\frac{\alpha_i}{k}\rceil \geq 1$, so we get

$$\sum_{i=1}^{r}\left\lceil\frac{\alpha_i}{k}\right\rceil \geq \sum_{i=1}^{r} 1 = \omega(n). \tag{3}$$

On the other hand, if there have some prime p_i such that $p_i^{k+1} \mid n$, then $\lceil\frac{\alpha_i}{k}\rceil \geq 2$. Let $n = n_1n_2$, where $(n_1, n_2) = 1$ and n_1 is a $k+1$-full number. That is, if $p \mid n_1$ then $p^{k+1} \mid n_1$. Now we can easily get the following inequality:

$$\sum_{i=1}^{r}\left\lceil\frac{\alpha_i}{k}\right\rceil \leq \omega(n) + \Omega(n_1). \tag{4}$$

From (3) and (4), we can write

$$\omega(n) \leq \sum_{i=1}^{r}\left\lceil\frac{\alpha_i}{k}\right\rceil \leq \omega(n) + \Omega(n_1).$$

So we have

$$\sum_{n\leq x}\omega(n) \leq \sum_{n\leq x}\Omega(S_k(n)) = \sum_{n\leq x}\sum_{i=1}^{r}\left\lceil\frac{\alpha_i}{k}\right\rceil \leq \sum_{n\leq x}\omega(n) + \sum_{\substack{n\leq x\\ n\in\mathcal{A}}}\Omega(n), \tag{5}$$

where $\mathcal{A}$ denotes the set of $k+1$-full numbers. Now combining Lemma 1, Lemma 2 and (5), we have

$$\sum_{n\leq x}\Omega(S_k(n)) = x\ln\ln x + Ax + O\left(\frac{x}{\ln x}\right).$$

This completes the proof of the theorem.

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ON THE MEAN VALUE OF AN ARITHMETICAL FUNCTION

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Abstract Let p be a prime, $e_p(n)$ denote the largest exponent of power p which divides n. In this paper, we study the mean value of $\sum\limits_{n^m \leq x} \left((n+1)^m - n^m\right) e_p(n)$, and give an asymptotic formula for it.

Keywords: Asymptotic formula; Mean Value; Largest exponent.

§1. Introduction

Let p be a prime, $e_p(n)$ denote the largest exponent of power p which divides n. In problem 68 of [1], Professor F.Smarandach asked us to study the properties of the sequence $e_p(n)$. In this paper, we use elementary methods to study the asymptotic properties of the mean value $\sum\limits_{n^m \leq x} \left((n+1)^m - n^m\right) e_p(n)$, and give an asymptotic formula for it. That is, we will prove the following:

Theorem. *Let p be a prime, $m \geq 1$ be an integer, then for any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n^m \leq x} \left((n+1)^m - n^m\right) e_p(n) = \frac{1}{p-1} \frac{m}{m+1} x + O\left(x^{1-\frac{1}{m}}\right)$$

§2. Proof of the theorem

In this section, we complete the proof of the theorem. In fact, from the definition of $e_p(n)$ we have

$$\begin{aligned} & \sum_{n^m \leq x} \left((n+1)^m - n^m\right) e_p(n) \\ = & \sum_{p^\alpha \leq x^{\frac{1}{m}}} \sum_{\substack{p^\alpha u \leq x^{\frac{1}{m}} \\ (u,p)=1}} \left((p^\alpha u + 1)^m + (p^\alpha u)^m\right) \alpha \end{aligned}$$

$$\begin{aligned}
&= \sum_{\alpha \le \frac{1}{m}\frac{\log x}{\log p}} \alpha \sum_{\substack{u \le \frac{x^{\frac{1}{m}}}{p^\alpha} \\ (u,p)=1}} \left(C_m^1 p^{\alpha(m-1)} u^{m-1} + C_m^2 p^{\alpha(m-2)} u^{m-2} + \cdots \right) \\
&= \sum_{\alpha \le \frac{1}{m}\frac{\log x}{\log p}} \alpha \sum_{u \le \frac{x^{\frac{1}{m}}}{p^\alpha}} \left(C_m^1 p^{\alpha(m-1)} u^{m-1} + C_m^2 p^{\alpha(m-2)} u^{m-2} + \cdots \right) - \\
&\quad \sum_{\alpha \le \frac{1}{m}\frac{\log x}{\log p}} \alpha \sum_{u \le \frac{x^{\frac{1}{m}}}{p^{\alpha+1}}} \left(C_m^1 p^{\alpha(m-1)} (up)^{m-1} + C_m^2 p^{\alpha(m-2)} (up)^{m-2} + \cdots \right) \\
&= \sum_{\alpha \le \frac{1}{m}\frac{\log x}{\log p}} \alpha \left(\frac{m p^{\alpha(m-1)} (\frac{x^{\frac{1}{m}}}{p^\alpha})^m \left(1 - \frac{1}{p}\right)}{m+1} + O\left(x^{\frac{m-1}{m}} p^{-\alpha} \right) \right) \\
&= \sum_{\alpha \le \frac{1}{m}\frac{\log x}{\log p}} \alpha \left(\frac{mx}{m+1} \frac{1}{p^\alpha} \left(1 - \frac{1}{p}\right) + O\left(x^{1-\frac{1}{m}} p^{-\alpha} \right) \right) \\
&= \frac{mx}{m+1} \left(1 - \frac{1}{p}\right) \sum_{\alpha \le \frac{1}{m}\frac{\log x}{\log p}} \frac{\alpha}{p^\alpha} + O\left(x^{1-\frac{1}{m}} \sum_{\alpha \le \frac{1}{m}\frac{\log x}{\log p}} \frac{\alpha}{p^\alpha} \right)
\end{aligned}$$

and

$$\begin{aligned}
\sum_{\alpha \le \frac{1}{m}\frac{\log x}{\log p}} \frac{\alpha}{p^\alpha} &= \sum_{n=1}^{\infty} \frac{n}{p^n} - \sum_{\alpha > \frac{1}{m}\frac{\log x}{\log p}} \frac{\alpha}{p^\alpha} \\
&= \frac{p}{(p-1)^2} - \frac{1}{p^{[\frac{1}{m}\frac{\log x}{\log p}]}} \left(\sum_{n=1}^{\infty} \frac{\frac{1}{m}\frac{\log x}{\log p} + n}{p^n} \right) \\
&= \frac{p}{(p-1)^2} - \frac{1}{x} O\left(\left(\frac{1}{m}\frac{\log x}{\log p} \right)^2 \right) \\
&= \frac{p}{(p-1)^2} + O\left(x^{-1} \log^2 x \right).
\end{aligned}$$

So we have

$$\begin{aligned}
&\sum_{n^m \le x} \left((n+1)^m - n^m \right) e_p(n) \\
= \; & \frac{mx}{m+1} \left(1 - \frac{1}{p}\right) \left(\frac{p}{(p-1)^2} + O\left(x^{-1} \log^2 x \right) \right) \\
& + O\left(x^{1-\frac{1}{m}} \right) \left(\frac{p}{(p-1)^2} + O\left(x^{-1} \log^2 x \right) \right)
\end{aligned}$$

$$= \frac{1}{p-1}\frac{m}{m+1}x + O\left(x^{1-\frac{1}{m}}\right).$$

This completes the proof of the theorem.

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TWO ASYMPTOTIC FORMULAE ON THE DIVISOR PRODUCT SEQUENCES

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Abstract In this paper, we study the asymptotic property of the divisor product sequences, and obtain two interesting asymptotic formulae.

Keywords: Divisor products; Proper divisor products; Asymptotic formula.

§1. Introduction

A natural number n is called a divisor product of n if it is the product of all positive divisors of n, we denote it by $a(n)$. For example, $a(1) = 1, a(2) = 2, a(3) = 3, a(4) = 8, a(5) = 5, \cdots$; Similarly, n is called a proper divisor product of n if it is the product of all positive divisors of n except n, we denote it by $b(n)$. For example, $b(1) = b(2) = b(3) = 1, b(4) = 2, b(5) = 1, b(6) = 6, \cdots$. In reference [1], Professor F. Smarandache asked us to study the properties of these two sequences. About these problems, it seems that none had studied them before. In this paper, we shall use the analytic methods to study the asymptotic properties of these sequences, and give two interesting asymptotic formulae. That is, we shall prove the following two theorems.

Theorem 1. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} \ln a(n) = \frac{1}{2} x \ln^2 x + (C-1)\, x \ln x - (C-1)\, x + O(x^{\frac{1}{2}} \ln x).$$

where C is the Euler constant.

Theorem 2. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} \ln b(n) = x \ln^2 x + (C-2)\, x \ln x - (C-2)\, x + O(x^{\frac{1}{2}} \ln x).$$

§2. Some lemmas

To complete the proof of the theorems, we need the following simple lemmas.

Lemma 1. *For any natural number n, we have the following identities*

$$a(n) = n^{\frac{d(n)}{2}}, \qquad b(n) = n^{\frac{d(n)-1}{2}},$$

where $d(n)$ is the divisor function.

Proof. From the definition of $a(n)$ we know that

$$a(n) = \prod_{d|n} d = \prod_{d|n} \frac{n}{d}.$$

So from this formula we have

$$a^2(n) = \prod_{d|n} d \times \prod_{d|n} \frac{n}{d} = \prod_{d|n} n = n^{d(n)}. \tag{1}$$

where $d(n) = \sum_{d|n} 1$. From (1) we may immediately obtain $a(n) = n^{\frac{d(n)}{2}}$ and

$$b(n) = \prod_{d|n, d<n} d = \frac{\prod_{d|n} d}{n} = n^{\frac{d(n)}{2}-1}.$$

This completes the proof of Lemma 1.

Lemma 2. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n \leq x} d(n) = x \ln x + 2(\gamma - 1)x + O(x^{\frac{1}{2}}),$$

where γ is the Euler constant.

Proof. (See reference [2]).

Lemma 3. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n \leq x} d(n) \ln n = x \ln^2 x + 2(\gamma - 1)x \ln x - 2(\gamma - 1)x + O(x^{\frac{1}{2}} \ln x).$$

Proof. Let $A(x) = \sum_{n \leq x} d(n)$, then by Abel's identity (see Theorem 4.2 of [2]) and Lemma 2 we have

$$\begin{aligned}
& \sum_{n \leq x} d(n) \ln n \\
= \ & A(x) \ln x - A(1) \ln 1 - \int_1^x \frac{A(t)}{t} dt \\
= \ & \ln x \left(x \ln x + 2(\gamma - 1)x + O(x^{\frac{1}{2}}) \right) \\
& - \int_1^x \frac{t \ln t + 2(\gamma - 1)t + O(t^{\frac{1}{2}})}{t} dt \\
= \ & x \ln^2 x + (2\gamma - 1)x \ln x - (t \ln t - t + (2\gamma - 1)t) |_1^x + O(x^{\frac{1}{2}} \ln x) \\
= \ & x \ln^2 x + 2(\gamma - 1)x \ln x - 2(\gamma - 1)x + O(x^{\frac{1}{2}} \ln x).
\end{aligned}$$

This completes the proof of Lemma 3.

§3. Proof of the Theorems

In this section, we shall complete the proof of the Theorems. First we come to prove Theorem 1. From Lemma 1 and Lemma 3, we have

$$\begin{aligned}\sum_{n\le x}\ln a(n) &= \sum_{n\le x}\ln n^{\frac{d(n)}{2}} = \frac{1}{2}\sum_{n\le x}d(n)\ln n\\ &= \frac{1}{2}x\ln^2 x + (\gamma-1)\,x\ln x - (\gamma-1)\,x + O(x^{\frac{1}{2}}\ln x).\end{aligned}$$

This completes the proof of Theorem 2.

Similarly, we can also prove Theorem 2. From Lemma 1, we have

$$\begin{aligned}\sum_{n\le x}\ln b(n) &= \sum_{n\le x}\ln n^{\frac{d(n)-1}{2}} = \frac{1}{2}\sum_{n\le x}d(n)\ln n - \frac{1}{2}\sum_{n\le x}\ln n\\ &= \frac{1}{2}\sum_{n\le x}d(n)\ln n - \frac{1}{2}\ln\,[x]!.\end{aligned}$$

Note that

$$\ln[x]! = x\ln x - x + O(\ln x) \tag{2}$$

(see reference [2]). Then by Lemma 3 and (2), we can easily obtain

$$\sum_{n\le x}\ln b(n) = x\ln^2 x + (\gamma-2)\,x\ln x - (\gamma-2)\,x + O(x^{\frac{1}{2}}\ln x).$$

This completes the proof of Theorem 2.

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ON THE SMARANDACHE PSEUDO-EVEN NUMBER SEQUENCE

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Abstract The main purpose of this paper is to study the mean value properties of the Smarandache pseudo-even number sequence and pseudo-odd sequence, and give some interesting asymptotic formulae for them.

Keywords: Pseudo-odd numbers; Pseudo-even numbers; Asymptotic formula.

§1. Introduction

A number is called pseudo-even number if some permutation of its digits is a even number, including the identity permutation. For example: $0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 21, \cdots$ are pseudo-even numbers. Let A denotes the set of all the pseudo-even numbers. Similarly, we can define the pseudo-odd number. That is, a number is called pseudo-odd number if some permutation of its digits is an odd number, such as $1, 3, 5, 7, 9, 10, 11, 12, 13, \cdots$ are pseudo-odd numbers. Let B denote the set of all the pseudo-odd numbers.

In reference [1], Professor F. Smarandache asked us to study the properties of the pseudo-even number sequence and pseudo-odd number sequence. About these problems, it seems that none had studied them before. In this paper, we use the elementary method to study the mean value properties of these two sequences, and obtain some asymptotic formulae for them. That is, we shall prove the following:

Theorem 1. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{\substack{n \in A \\ n \leq x}} f(n) = \sum_{n \leq x} f(n) + O\left(M x^{\frac{\ln 5}{\ln 10}}\right),$$

where $M = \max_{1 \leq n \leq x} \{|f(n)|\}$.

Theorem 2. *For any real number $x \geq 1$, we have*

$$\sum_{\substack{n \in B \\ n \leq x}} f(n) = \sum_{n \leq x} f(n) + O\left(M x^{\frac{\ln 5}{\ln 10}}\right).$$

Corollary. *For any real number $x \geq 1$, let d(n) denote the divisor function, then we have the asymptotic formulae*

$$\sum_{\substack{n \in A \\ n \leq x}} d(n) = x \ln x + (2\gamma - 1)x + O\left(x^{\frac{\ln 5}{\ln 10} + \epsilon}\right)$$

and

$$\sum_{\substack{n \in B \\ n \leq x}} d(n) = x \ln x + (2\gamma - 1)x + O\left(x^{\frac{\ln 5}{\ln 10} + \epsilon}\right),$$

where γ is the Euler constant.

§2. Proof of the Theorems

Now we completes the proof of the Theorems. First we prove Theorem 1. Let $10^k \leq x < 10^{k+1}$ $(k \geq 1)$, then $k \leq \log x < k+1$. According to the definition of set A, we know that the largest number of digits ($\leq x$) not attribute set A is 5^{k+1}. In fact, in these numbers, there are 5 one digit, they are $0, 2, 4, 6, 8$; There are 5^2 two digits; The number of the k digits are 5^k. So the largest number of digits ($\leq x$) not attribute set A is $5 + 5^2 + \cdots + 5^k = \frac{5}{4}(5^k - 1) \leq 5^{k+1}$. Since

$$5^k \leq 5^{\log x} = \left(5^{\log_5 x}\right)^{\frac{1}{\log_5 10}} = (x)^{\frac{1}{\log_5 10}} = x^{\frac{\ln 5}{\ln 10}}.$$

So we have,

$$5^k = O\left(x^{\frac{\ln 5}{\ln 10}}\right).$$

Next, let M denotes the upper bounds of $|f(n)|$ $(n \leq x)$, then

$$\sum_{\substack{n \notin A \\ n \leq x}} f(n) = O\left(M x^{\frac{\ln 5}{\ln 10}}\right).$$

Finally, we have

$$\begin{aligned} \sum_{\substack{n \in A \\ n \leq x}} f(n) &= \sum_{n \leq x} f(n) - \sum_{\substack{n \notin A \\ n \leq x}} f(n) \\ &= \sum_{n \leq x} f(n) + O\left(M x^{\frac{\ln 5}{\ln 10}}\right). \end{aligned}$$

This proves the Theorem 1.

Use the same method, we may immediately get:

$$\sum_{\substack{n\in B\\ n\le x}} f(n) = \sum_{n\le x} f(n) + O\left(Mx^{\frac{\ln 5}{\ln 10}}\right).$$

This completes the proof of the Theorems.

Now the corollary follows from Theorem 1 and 2, the asymptotic formula

$$\sum_{n\le x} d(n) = x\ln x + (2\gamma - 1)x + O\left(x^{\frac{1}{2}}\right)$$

(see [2]), and the estimate $d(n) \ll x^{\epsilon}$ (for all $1 \le n \le x$).

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ON THE MEAN VALUE OF AN ARITHMETICAL FUNCTION

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Abstract Let p be a prime, $e_p(n)$ denote the largest exponent of power p which divides n. In this paper, we use elementary and analytic methods to study the asymptotic properties of $\sum_{n\leq x} e_p(n)\phi(n)$, and give an interesting asymptotic formula for it.

Keywords: Asymptotic formula; Largest exponent; Perron formula.

§1. Introduction

Let p be a prime, $e_p(n)$ denote the largest exponent of power p which divides n. In problem 68 of [1], Professor F.Smarandache asked us to study the properties of the sequence $e_p(n)$. About this problem, it seems that none had studied it, at least we have not seen related papers before. In this paper, we use elementary and analytic methods to study the asymptotic properties of the mean value $\sum_{n\leq x} e_p(n)\phi(n)$ ($\phi(n)$ is the Euler totient function), and give an interesting asymptotic formula for it. That is, we will prove the following:

Theorem. *Let p be a prime, $\phi(n)$ is the Euler totient function. Then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} e_p(n)\phi(n) = \frac{3p}{(p^2-1)\pi^2}x^2 + O\left(x^{\frac{3}{2}+\epsilon}\right).$$

§2. Some lemmas

To complete the proof of the theorem, we need the following:

Lemma 1. Let p be a given prime. Then for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{\substack{n\leq x\\(n,p)=1}} \phi(n) = \frac{3p}{(p+1)\pi^2}x^2 + O\left(x^{\frac{3}{2}+\epsilon}\right).$$

Proof. Let

$$f(s) = \sum_{\substack{n=1\\(u,p)=1}}^{\infty} \frac{\phi(n)}{n^s},$$

$Re(s) > 1$. Then from the Euler product formula [3] and the multiplicative property of $\phi(n)$, we have

$$\begin{aligned}
\sum_{\substack{n=1\\(n,p)=1}}^{\infty} \frac{\phi(n)}{n^s} &= \prod_{q\neq p} \sum_{m=0}^{\infty} \frac{\phi(q^m)}{q^{ms}} \\
&= \prod_{q\neq p} \left(1 + \frac{q-1}{q^s} + \frac{q^2-q}{q^{2s}} + \frac{q^3-q^2}{q^{3s}} + \cdots\right) \\
&= \prod_{q\neq p} \left(1 + \frac{1-\frac{1}{q}}{q^{s-1}} \left(1 + \frac{1}{q^{s-1}} + \frac{1}{q^{2(s-1)}} + \cdots\right)\right) \\
&= \prod_{q\neq p} \left(1 + \frac{1-\frac{1}{q}}{q^{s-1}} \frac{q^{s-1}}{q^{s-1}-1}\right) \\
&= \frac{\zeta(s-1)}{\zeta(s)} \frac{p^s-p}{p^s-1},
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function. By Perron formula [2] with $s_0 = 0$, $T = x$ and $b = \frac{5}{2}$, we have

$$\sum_{\substack{n\leq x\\(n,p)=1}} \frac{\phi(n)}{n^s} = \frac{1}{2\pi i} \int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT} \frac{\zeta(s-1)}{\zeta(s)} \frac{p^s-p}{p^s-1} \frac{x^s}{s} ds + O\left(\frac{x^{\frac{5}{2}}}{T}\right).$$

To estimate the main term

$$\frac{1}{2\pi i} \int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT} \frac{\zeta(s-1)}{\zeta(s)} \frac{p^s-p}{p^s-1} \frac{x^s}{s} ds,$$

we move the integral line from $s = \frac{5}{2} \pm iT$ to $s = \frac{3}{2} \pm iT$. This time, the function

$$f(s) = \frac{\zeta(s-1)}{\zeta(s)} \frac{p^s-p}{p^s-1} \frac{x^s}{s}$$

has a simple pole point at $s = 2$, and the residue is $\frac{3px^2}{(p+1)\pi^2}$. So we have

$$\frac{1}{2i\pi} \left(\int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT} + \int_{\frac{5}{2}+iT}^{\frac{3}{2}+iT} + \int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT} + \int_{\frac{3}{2}-iT}^{\frac{5}{2}-iT} \right) \frac{\zeta(s-1)}{\zeta(s)} \frac{p^s-p}{p^s-1} \frac{x^s}{s} = \frac{3px^2}{(p+1)\pi^2}.$$

Note that

$$\frac{1}{2i\pi}\left(\int_{\frac{5}{2}+iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT}+\int_{\frac{3}{2}-iT}^{\frac{5}{2}-iT}\right)\frac{\zeta(s-1)}{\zeta(s)}\frac{p^s-p}{p^s-1}\frac{x^s}{s}\ll x^{\frac{3}{2}+\epsilon}.$$

From above we may immediately get the asymptotic formula:

$$\sum_{\substack{n\le x\\(n,p)=1}}\phi(n)=\frac{3px^2}{(p+1)\pi^2}+O\left(x^{\frac{3}{2}+\epsilon}\right).$$

This completes the proof of the Lemma 1.

Lemma 2. *Let α is an any fixed integer, and p is a prime. Then for any real number $x\ge 1$, we have the asymptotic formula*

$$\sum_{\alpha\le\frac{\log x}{\log p}}\frac{\alpha}{p^\alpha}=\frac{p}{(p-1)^2}+O\left(x^{-1}\log x\right);$$

$$\sum_{\alpha\le\frac{\log x}{\log p}}\frac{\alpha}{p^{\frac{\alpha}{2}}}=\frac{p^{\frac{1}{2}}}{\left(p^{\frac{1}{2}}-1\right)^2}+O\left(x^{-\frac{1}{2}}\log x\right).$$

Proof. From the properties of geometrical series, we have

$$\begin{aligned}\sum_{\alpha\le\frac{\log x}{\log p}}\frac{\alpha}{p^\alpha} &= \sum_{t=1}^{\infty}\frac{t}{p^t}-\sum_{\alpha>\frac{\log x}{\log p}}\frac{\alpha}{p^\alpha}\\ &= \sum_{t=1}^{\infty}\frac{t}{p^t}-\frac{1}{p^{[\frac{\log x}{\log p}]}}\sum_{t=1}^{\infty}\frac{[\frac{\log x}{\log p}]+t}{p^t}\\ &= \sum_{t=1}^{\infty}\frac{t}{p^t}+O\left(x^{-1}\left(\frac{[\frac{\log x}{\log p}]}{p-1}+\sum_{t=1}^{\infty}\frac{t}{p^t}\right)\right)\\ &= \frac{p}{(p-1)^2}+O\left(x^{-1}\log x\right),\end{aligned}$$

and

$$\begin{aligned}\sum_{\alpha\le\frac{\log x}{\log p}}\frac{\alpha}{p^{\frac{\alpha}{2}}} &= \sum_{t=1}^{\infty}\frac{t}{p^{\frac{t}{2}}}-\sum_{\alpha>\frac{\log x}{\log p}}\frac{\alpha}{p^{\frac{\alpha}{2}}}\\ &= \sum_{t=1}^{\infty}\frac{t}{p^{\frac{t}{2}}}-\frac{1}{p^{\frac{1}{2}[\frac{\log x}{\log p}]}}\sum_{t=1}^{\infty}\frac{[\frac{\log x}{\log p}]+t}{p^{\frac{t}{2}}}\\ &= \sum_{t=1}^{\infty}\frac{t}{p^{\frac{1}{2}}}+O\left(x^{-\frac{1}{2}}\left(\frac{[\frac{\log x}{\log p}]}{p^{\frac{1}{2}}-1}+\sum_{t=1}^{\infty}\frac{t}{p^{\frac{t}{2}}}\right)\right)\\ &= \frac{p^{\frac{1}{2}}}{\left(p^{\frac{1}{2}}-1\right)^2}+O\left(x^{-\frac{1}{2}}\log x\right).\end{aligned}$$

This completes the proof of the Lemma 2.

§3. Proof of the Theorem

In this section, we complete the proof of the theorem.

$$
\begin{aligned}
&\sum_{n\le x} e_p(n)\phi(n)\\
=\ & \sum_{p^\alpha\le x}\sum_{\substack{p^\alpha u\le x\\(u,p)=1}}\alpha\phi(p^\alpha u)=\sum_{p^\alpha\le x}\alpha\phi(p^\alpha)\sum_{\substack{u\le\frac{x}{p^\alpha}\\(u,p)=1}}\phi(u)\\
=\ & \frac{p-1}{p}\sum_{\alpha\le\frac{\log x}{\log p}}\alpha p^\alpha\left(\frac{3p}{(p+1)\pi^2}\left(\frac{x}{p^\alpha}\right)^2+O\left(\left(\frac{x}{p^\alpha}\right)^{\frac{3}{2}+\epsilon}\right)\right)\\
=\ & \frac{3(p-1)}{(p+1)\pi^2}x^2\sum_{\alpha\le\frac{\log x}{\log p}}\frac{\alpha}{p^\alpha}+O\left(x^{\frac{3}{2}+\epsilon}\sum_{\alpha\le\frac{\log x}{\log p}}\frac{\alpha}{p^{\frac{\alpha}{2}}}\right)\\
=\ & \frac{3(p-1)}{(p+1)\pi^2}x^2\left(\frac{p}{(p-1)^2}+O\left(x^{-1}\log x\right)\right)\\
&+O\left(x^{\frac{3}{2}+\epsilon}\left(\frac{p^{\frac{1}{2}}}{\left(p^{\frac{1}{2}}-1\right)^2}+O\left(x^{-\frac{1}{2}}\log x\right)\right)\right)\\
=\ & \frac{3p}{(p^2-1)\pi^2}x^2+O\left(x^{\frac{3}{2}+\epsilon}\right).
\end{aligned}
$$

This completes the proof of the Theorem.

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AN ARITHMETICAL FUNCTION AND ITS CUBIC COMPLEMENTS

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Abstract The main purpose of this paper is using the elementary method to study the mean value properties of the $\overline{S}_2(a(n))$, and give an interesting asymptotic formula.

Keywords: Cubic complements; Mean value; Asymptotic formula.

§1. Introduction

Let n be an positive integer, if $a(n)$ is the smallest integer such that $na(n)$ is a cubic number, then we call $a(n)$ is the cubic complements of n. For any positive integer n and any fixed positive integer k, we define the arithmetical function $\overline{S}_k(n)$ as follows:

$$\overline{S}_k(n) = \max\{x \in N \mid x^k \mid n\}.$$

Obviously, that $\overline{S}_k(n)$ is a multiplicative function. In this paper, we use the elementary method to study the mean value properties of $\overline{S}_2(a(n))$, and give a sharp asymptotic formula for it. That is, we shall prove the following:

Theorem. *For any real number $x \geq 3$, we have the asymptotic formula*

$$\sum_{n \leq x} \overline{S}_2(a(n)) = \frac{x^2\pi^4}{315} \prod_p \left(1 + \frac{1}{p^4 + p^3}\right) + O\left(x^{\frac{3}{2}+\epsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, $\prod_p$ denotes the product over all prime p, and ϵ be any fixed positive integer.

§2. A Lemma

To complete the proof of the theorem, we need the following famous Perron formula [1]:

Lemma. *Suppose that the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$, $s = \sigma + it$, converge absolutely for $\sigma > \beta$, and that there exist a positive λ and a positive* *increasing function $A(s)$ such that*

$$\sum_{n=1}^{\infty} |a(n)|\, n^{-\sigma} \ll (\sigma-\beta)^{-1}, \sigma \to \beta + 0$$

and

$$a(n) \ll A(n), n = 1, 2, \cdots .$$

Then for any $b > 0$, $b + \sigma > \beta$, and x not to be an integer, we have

$$\begin{aligned}\sum_{n\le x} a(n)n^{-s_0} &= \frac{1}{2\pi i}\int_{b-iT}^{b+iT} f(s_0+\omega)\frac{x^\omega}{\omega}d\omega + O\left(\frac{x^b}{T(b+\sigma-\beta)^\lambda}\right) \\ &+O\left(\frac{A(2x)x^{1-\sigma}\log x}{T \,||\, x \,||}\right),\end{aligned}$$

where $||\ x\ ||$ is the nearest integer to x.

§3. Proof of the theorem

In this section, we complete the proof of the theorem. Let

$$f(s) = \sum_{n=1}^{\infty} \frac{\overline{S}_2(a(n))}{n^s},$$

$\mathrm{Re}(s) > 1$. Then by the Euler product formula [2] and the multiplicative property of $\overline{S}_2(n)$ we have

$$\begin{aligned} f(s) &= \prod_p \left(1 + \frac{\overline{S}_2(a(p))}{p^s} + \frac{\overline{S}_2(a(p^2))}{p^{2s}} + \frac{\overline{S}_2(a(p^3))}{p^{3s}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{\overline{S}_2(p^2)}{p^s} + \frac{\overline{S}_2(p)}{p^{2s}} + \frac{\overline{S}_2(1)}{p^{3s}} + \frac{\overline{S}_2(p^2)}{p^{4s}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{p}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \frac{p}{p^{4s}} + \cdots\right) \\ &= \prod_p \left(1 + p\frac{\frac{1}{p^s}}{1-\frac{1}{p^{3s}}} + \frac{\frac{1}{p^{2s}}}{1-\frac{1}{p^{3s}}} + \frac{\frac{1}{p^{3s}}}{1-\frac{1}{p^{3s}}}\right) \\ &= \zeta(3s)\prod_p \left(1 + \frac{1}{p^{s-1}} + \frac{1}{p^{2s}}\right) \\ &= \frac{\zeta(3s)}{\zeta(2s-2)}\zeta(s-1)\prod_p \left(1 + \frac{1}{p^{2s}+p^{s+1}}\right), \end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function. So by Perron formula, with $s_0 = 0$, $t = x^{\frac{3}{2}}$, $b = 3$, we have

$$\sum_{n\le x} \overline{S}_2(n) = \frac{1}{2i\pi}\int_{3-iT}^{3+iT} \frac{\zeta(s-1)\zeta(3s)}{\zeta(2s-2)}R(s)\frac{x^s}{s}ds + O(x^{\frac{3}{2}+\varepsilon}),$$

where

$$R(s)=\prod_p\left(1+\frac{1}{p^{2s}+p^{s+1}}\right).$$

To estimate the main term

$$\frac{1}{2i\pi}\int_{3-iT}^{3+iT}\frac{\zeta(3s)\zeta(s-1)}{\zeta(2s-2)}R(s)\frac{x^s}{s}ds$$

we move the integral line from $s=3\pm iT$ to $s=\frac{3}{2}\pm iT$. This time, the function

$$f(s)=\frac{\zeta(3s)\zeta(s-1)x^s}{\zeta(2s-2)s}R(s)$$

has a simple pole point at $s=2$ with residue $\frac{x^2}{\zeta(2)}\zeta(6)R(2)$. So we have

$$\frac{1}{2i\pi}\left(\int_{3-iT}^{3+iT}+\int_{3+iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT}+\int_{\frac{3}{2}-iT}^{3-iT}\right)\frac{\zeta(3s)\zeta(s-1)x^s}{\zeta(2s-2)s}R(s)ds$$
$$=\frac{x^2}{2\zeta(2)}\zeta(6)\prod_p\left(1+\frac{1}{p^4+p^3}\right)$$

Noting that

$$\frac{1}{2i\pi}\left(\int_{3+iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT}+\int_{\frac{3}{2}-iT}^{3-iT}\right)\frac{\zeta(3s)\zeta(s-1)x^s}{\zeta(2s-2)s}R(s)ds\ll x^{\frac{3}{2}+\epsilon},$$

and

$$\zeta(2)=\frac{\pi^2}{6},\qquad \zeta(6)=\frac{\pi^6}{945},$$

we may immediately get the asymptotic formula:

$$\sum_{n\le x}\overline{S}_2(a(n))=\frac{x^2\pi^4}{315}\prod_p\left(1+\frac{1}{p^4+p^3}\right)+O\left(x^{\frac{3}{2}+\epsilon}\right).$$

This completes the proof of the theorem.

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ON THE SYMMETRIC SEQUENCE AND ITS SOME PROPERTIES

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Abstract For any positive integer n, let $\{a_n\}$ denotes the symmetric sequence. In this paper, we study the asymptotic properties of $\{a_n\}$, and give an interesting identity for it.

Keywords: Symmetric sequence; Series ; Identity.

§1. Introduction

For any positive integer n, we define the symmetric sequence $\{a_n\}$ as follows: $a_1 = 1, a_2 = 11, a_3 = 121, a_4 = 1221, a_5 = 12321, a_6 = 123321, \cdots, a_{2k-1} = 123\cdots(k-1)k(k-1)\cdots 321, a_{2k} = 123\cdots(k-1)kk(k-1)\cdots 321, \cdots$. In problem 17 of [1], Professor F.Smarandache asks us to study the properties of the sequence $\{a_n\}$. About this problem, Professor Zhang Wenpeng [2] gave an interesting asymptotic formula for it. In this paper, we define $A(a_n)$ as follows: $A(a_1) = 1, A(a_2) = 2, A(a_3) = 4, \cdots, A(a_{2k-1}) = 1+2+\cdots+k-1+k+k-1+\cdots+1, A(a_{2k}) = (1+\cdots+k-1+k+k+k-1+\cdots+1), \cdots$. We shall use elementary method to study the properties of sequence $A(a_n)$, and obtain an interesting identity involving $A(a_n)$. That is, we shall prove the following:

Theorem. *Let $A(a_n)$ as the definition of the above. Then we have*

$$\sum_{n=1}^{\infty} \frac{1}{A(a_n)} = \frac{\pi^2}{6} + 1.$$

§2. Proof of the Theorem

In this section, we complete the proof of the theorem. From the definition of $A(a_n)$, we know that

$$A(a_k) = A(a_{k-1}) + \left[\frac{k-1}{2}\right] + 1.$$

So by this formula we have

$$\sum_{k=1}^{n} A(a_k) = \sum_{k=1}^{n-1} A(a_{k-1}) + \sum_{k=1}^{n} \left[\frac{k-1}{2}\right] + n. \tag{1}$$

From (1) we may immediately get

$$A(a_n) = \sum_{k=1}^{n} \left[\frac{k-1}{2}\right] + n = \left[\frac{(n-1)^2}{4}\right] + n. \tag{2}$$

In the following, we separate the summation in the Theorem into two parts. For the first part, if $n = 2k+1$, we have

$$A(a_n) = \left[\frac{[(2k+1)-1]^2}{4}\right] + 2k + 1 = (k+1)^2. \tag{3}$$

For the second part, if $n = 2k$, we have

$$A(a_n) = \left[\frac{(2k-1)^2}{4}\right] + 2k = k^2 + k. \tag{4}$$

Combining (2), (3), (4) and note that $\zeta(2) = \frac{\pi^2}{6}$ we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{A(a_n)} &= \frac{1}{A(a_1)} + \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} + \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \\
&= \sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{k=1}^{\infty} \frac{1}{k(k+1)} \\
&= \zeta(2) + \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) \\
&= \frac{\pi^2}{6} + 1.
\end{aligned}$$

This completes the proof of the Theorem.

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THE ADDITIVE ANALOGUE OF SMARANDACHE FUNCTION

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Abstract In this paper, we use the elementary method to study the mean value properties of the additive analogue Smarandache function acting on the integral part of the k-th root sequence, and give two interesting asymptotic formulae.

Keywords: Smarandche function; Additive analogue; Divisor function; Mean value.

§1. Introduction

Let $n \geq 2$ be a positive integer, $a(n)$ denotes the integral part of the k-th root sequence, we can express it as $a(n) = \left[n^{\frac{1}{k}}\right]$. In paper [2], Jozsef Sandor defined the following analogue of the Smarandache function:

$$S_1(x) = \min\{m \in N : x \leq m!\}, x \in (1, \infty),$$

which is defined on a subset of real numbers. In this paper, we study the mean value properties of the additive analogue Smarandache function acting on the floor of the k-th root sequence, and obtain two interesting asymptotic formulae. That is, we shall prove the following:

Theorem 1. *For any real number $x \geq 2$ and integer $k \geq 2$, we have the asymptotic formula*

$$\sum_{n \leq x} S_1(a(n)) = \frac{x \log x}{k \log\log x^{\frac{1}{k}}} + O\left(\frac{x(\log x)(\log\log\log x^{\frac{1}{k}})}{(\log\log x^{\frac{1}{k}})^2}\right).$$

Theorem 2. *For any real number $x \geq 2$, we have the estimate*

$$\sum_{n \leq x} d(n) S_1(n) = \frac{x \log^2 x}{\log\log x}\left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right),$$

where $d(n)$ be the divisor function.

§2. Proof of the Theorems

In this section, we shall complete the proof of the Theorems. Firstly, we need following:

Lemma 1. For any real number $x \geq 2$, we have the mean value formula

$$\sum_{n \leq x} S_1(n) = \frac{x \log x}{\log \log x} + O\left(\frac{x(\log x)(\log \log \log x)}{(\log \log x)^2}\right).$$

Proof. From the definition of S_1, we know that if $(m-1)! < n \leq m!$, then $S_1(n) = m$. For $(m-1)! < n \leq m!$, by taking the logistic computation in the two sides, we have

$$\sum_{i=1}^{m-1} \log i < \log n \leq \sum_{i=1}^{m} \log i.$$

Using the Euler's summation formula, we get

$$\sum_{i=1}^{m} \log i = m \log m - m + O(\log m) = \sum_{i=1}^{m-1} \log i.$$

So

$$\log n = m \log m - m + O(\log m),$$

then we can obtain

$$m = \frac{\log n}{\log m - 1} + O(1),$$

we continue taking the logistic computation in two sides, then

$$m = \frac{\log n}{\log \log n} + O\left(\frac{(\log n)(\log \log \log n)}{(\log \log n)^2}\right).$$

Using the Euler's formula, we have the estimate

$$\begin{aligned}
\sum_{n \leq x} S_1(n) &= \sum_{n \leq x} \sum_{(m-1)! < n \leq m!} m \\
&= \sum_{n \leq x} \left(\frac{\log n}{\log \log n} + O\left(\frac{(\log n)(\log \log \log n)}{(\log \log n)^2}\right)\right) \\
&= \sum_{n \leq x} \frac{\log n}{\log \log n} + O\left(\frac{x(\log x)(\log \log \log x)}{(\log \log x)^2}\right) \\
&= \frac{x \log x}{\log \log x} + O\left(\frac{x(\log x)(\log \log \log x)}{(\log \log x)^2}\right).
\end{aligned}$$

This proves Lemma 1.

Lemma 2. For any real number $x \geq 2$, we have the estimate

$$\sum_{n \leq x} d(n) = x \log x + (2\gamma - 1)x + O(\sqrt{x}),$$

where γ is the Euler's constant.

Proof. See reference [3].

Now we use the above Lemmas to complete the proof of the Theorems. First we prove Theorem 1, from the definition of $a(n)$, we have

$$\begin{aligned}
&\sum_{n\le x} S_1(a(n)) = \sum_{n\le x} S_1\left(\left[n^{\frac{1}{k}}\right]\right)\\
= &\sum_{1^k\le i<2^k} S_1(1) + \sum_{2^k\le i<3^k} S_1(2) + \cdots +\\
&+ \sum_{N^k\le i\le x<(N+1)^k} S_1(N) + O(N^{k-1+\varepsilon})\\
= &\sum_{1\le j\le N} \left((j+1)^k - j^k\right) S(j) + O(N^{k-1+\varepsilon})
\end{aligned}$$

Let

$$A(x) = \sum_{n\le x} S_1(n) = \frac{x\log x}{\log\log x} + O\left(\frac{x(\log x)(\log\log\log x)}{(\log\log x)^2}\right),$$

and $f(j) = (j+1)^k - j^k$, suppose $N^k \le x < (N+1)^k$, then by Abel's identity we can get

$$\begin{aligned}
&\sum_{n\le x} S_1(a(n))\\
= \;& A(N)f(N) - A(2)f(2) - \int_2^N A(t)f'(t)dt + O(N^{k-1+\varepsilon})\\
= \;& \left(\frac{N\log N}{\log\log N} + O\left(\frac{N(\log N)(\log\log\log N)}{(\log\log N)^2}\right)\right)\left((N+1)^k - N^k\right)\\
&-\int_2^N \left(\frac{t\log t}{\log\log t} + O\left(\frac{t(\log t)(\log\log\log t)}{(\log\log t)^2}\right)\right)\left((t+1)^k - t^k\right)' dt\\
= \;& \frac{N^k\log N}{\log\log N} + O\left(\frac{N^k(\log N)(\log\log\log N)}{(\log\log N)^2}\right)\\
= \;& \frac{x\log x}{k\log\log x^{\frac{1}{k}}} + O\left(\frac{x(\log x)(\log\log\log x^{\frac{1}{k}})}{(\log\log x^{\frac{1}{k}})^2}\right).
\end{aligned}$$

This completes the proof of the Theorem 1.

Now we prove Theorem 2. From the process of proof Lemma 1 and applying Lemma 2 we have

$$\begin{aligned}
&\sum_{n\le x} d(n)S_1(n)\\
= &\sum_{n\le x}\sum_{(m-1)!<n\le m!} d(n)m
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n\le x} d(n)\left(\frac{\log n}{\log\log n} + O\left(\frac{(\log n)(\log\log\log n)}{(\log\log n)^2}\right)\right) \\
&= \sum_{n\le x} d(n)\frac{\log n}{\log\log n} + O\left(\sum_{n\le x} d(n)\frac{(\log n)(\log\log\log n)}{(\log\log n)^2}\right)
\end{aligned}
$$

Let

$$A(x) = \sum_{n\le x} d(n) = x\log x + (2\gamma - 1)x + O(\sqrt{x}),$$

and

$$f_1(t) = \frac{\log t}{\log\log t}, \quad f_2(t) = \frac{(\log t)(\log\log\log t)}{(\log\log t)^2}.$$

From Abel's identity, we can obtain

$$
\begin{aligned}
& \sum_{n\le x} d(n)S_1(n) \\
&= A(x)f_1(x) - A(2)f_1(2) - \int_2^x A(t)f_1'(t)dt + \\
&\quad +O\left(A(x)f_2(x) - A(2)f_2(2) - \int_2^x A(t)f_2'(t)dt\right) \\
&= \frac{x\log^2 x}{\log\log x} + O\left(\frac{x\log^2 x\log\log\log x}{(\log\log x)^2}\right) \\
&= \frac{x\log^2 x}{\log\log x}\left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right).
\end{aligned}
$$

This completes the proof of Theorem 2.

References

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AN ASYMPTOTIC FORMULA ON SMARANDACHE CEIL FUNCTION

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Abstract The main purpose of this paper is using the elementary method to study the asymptotic properties of the Smarandache ceil function acting on factorial number, and give an interesting asymptotic formula.

Keywords: Smarandache ceil function; Factorial number; Asymptotic formula.

§1. Introduction

For a fixed positive integer k and any positive integer n, the Smarandache ceil function $S_k(n)$ is defined as follows:

$$S_k(n) = \min\{m \in N : n|m^k\}.$$

This was introduced by Professor F.Smarandache. In [1], Ibstedt presented that $S_k(n)$ is a multiplicative function. That is,

$$(\forall a, b \in N)\ (a, b) = 1 \Rightarrow S_k(a \cdot b) = S_k(a) \cdot S_k(b).$$

It is easily to show $S_k(p^\alpha) = p^{\lceil \frac{\alpha}{k} \rceil}$, where p is a prime and $\lceil x \rceil$ denotes the least integer greater than x. So, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the prime decomposition of n, then the followinig identity is obviously:

$$S_k(n) = S_k(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = p_1^{\lceil \frac{\alpha_1}{k} \rceil} p_2^{\lceil \frac{\alpha_2}{k} \rceil} \cdots p_r^{\lceil \frac{\alpha_r}{k} \rceil}. \qquad (1)$$

In this paper, we used the elementary method to study the value distribution properties of $S_k(n!)$, and given an interesting asymptotic formula. That is, we shall prove the following

Theorem. *Let k be a given positive integer. Then for any integer $n \geq 3$, we have the asymptotic formula:*

$$\Omega(S_k(n!)) = \frac{n}{k}(\ln \ln n + C) + O\left(\frac{n}{\ln n}\right),$$

where C is a computable constant.

§2. Some simple lemmas

Before the proof of the theorem, some simple lemmas will be useful.

Lemma 1. *Let n be any positive integer, we have the asymptotic formula:*

$$\sum_{p\le n}\frac{1}{\ln p}=\frac{n}{\ln^2 n}+O\left(\frac{n}{\ln^3 n}\right), \tag{2}$$

where p denotes primes.

Proof. From Abel's identity (see [2]), we have

$$\sum_{p\le n}\frac{1}{\ln p}=\pi(n)\frac{1}{\ln n}+\int_2^n \pi(t)\frac{1}{t\ln^2 t}dt,$$

where $\pi(n)$ denotes the number of the primes up to n. Noting that

$$\pi(n)=\frac{n}{\ln n}+O\left(\frac{n}{\ln^2 n}\right),$$

we can get

$$\sum_{p\le n}\frac{1}{\ln p}=\frac{n}{\ln^2 n}+O\left(\frac{n}{\ln^3 n}\right).$$

This proves Lemma 1.

Lemma 2. *Let m be any positive integer, we have the asymptotic formula:*

$$\sum_{p\le m}\frac{1}{p}=\ln\ln n+A+O\left(\frac{1}{\ln n}\right), \tag{3}$$

where A is a computable constant.

Proof. See reference [3].

§3. Proof of the theorem

In this section, we will complete the proof of the theorem. Let

$$n!=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}.$$

From (1) and the completely additive property of function Ω , we can write

$$\Omega(S_k(n!))=\Omega\left(p_1^{\lceil\frac{\alpha_1}{k}\rceil}p_2^{\lceil\frac{\alpha_2}{k}\rceil}\cdots p_r^{\lceil\frac{\alpha_r}{k}\rceil}\right)=\sum_{i=1}^{r}\left\lceil\frac{\alpha_i}{k}\right\rceil. \tag{4}$$

It is clear that

$$\alpha_i=\sum_{j=1}^{\infty}\left[\frac{n}{p_i^j}\right],\quad i=1,2,\cdots,r.$$

Noting that if $p^j > n$ then $\left[\frac{n}{p^j}\right] = 0$, from Lemma 1 we can write

$$\begin{aligned}
\Omega(S_k(n!)) &= \sum_{p\leq n}\left\lceil \frac{1}{k}\sum_{j\leq\frac{\ln n}{\ln p}}\left[\frac{n}{p_i^j}\right]\right\rceil \\
&= \sum_{p\leq n}\left\lceil \frac{1}{k}\sum_{j=1}^{\left[\frac{\ln n}{\ln p}\right]}\frac{n}{p^j} + O\left(\frac{\ln n}{\ln p}\right)\right\rceil \\
&= \sum_{p\leq n}\left(\frac{1}{k}\sum_{j=1}^{\left[\frac{\ln n}{\ln p}\right]}\frac{n}{p^j} + O\left(\frac{\ln n}{\ln p}\right)\right) + O\left(\frac{n}{\ln n}\right) \\
&= \frac{n}{k}\left(\sum_{p\leq n}\frac{1}{p-1} - \sum_{p\leq n}\frac{1}{p^{\left[\frac{\ln n}{\ln p}\right]}(p-1)}\right) + O\left(\frac{n}{\ln n}\right) \\
&= \frac{n}{k}\left(\sum_{p\leq n}\frac{1}{p-1}\right) + O\left(\frac{n}{\ln n}\right) \\
&= \frac{n}{k}\left(\sum_{p\leq n}\frac{1}{p} + \sum_{p\leq n}\frac{1}{p(p-1)}\right) + O\left(\frac{n}{\ln n}\right). \qquad (5)
\end{aligned}$$

Noting that

$$\sum_{p\leq n}\frac{1}{p(p-1)} = \sum_{p}\frac{1}{p(p-1)} - \sum_{p>n}\frac{1}{p(p-1)} = B + O\left(\frac{1}{n}\right), \qquad (6)$$

where $B = \sum_{p}\frac{1}{p(p-1)}$ is a constant. Combining (5), (6) and Lemma 2, we can get

$$\Omega(S_k(n!)) = \frac{n}{k}(\ln\ln n + C) + O\left(\frac{n}{\ln n}\right).$$

This completes the proof of the theorem.

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A HYBRID NUMBER THEORETIC FUNCTION AND ITS MEAN VALUE

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Abstract Let p and q are two primes, $e_q(n)$ denotes the largest exponent of power q which divides n. And $b(n)$ is the cubic complements. In this paper, we study the properties of this sequence $p^{e_q(b(n))}$, and give an interesting asymptotic formula for the mean value $\sum\limits_{n\leq x} p^{e_q(b(n))}$.

Keywords: Asymptotic formula; Largest exponent; Cubic Complements.

§1. Introduction

Let p and q are two primes, $e_q(n)$ denotes the largest exponent of power q which divides n. It is obvious that $e_q(n) = k$ if q^k divides n but $q^{(k+1)}$ does not. For any positive integer n, the cubic complements $b(n)$ is the smallest positive integer such that $nb(n)$ is a perfect cubic. In problem 28 and 68 of [1], Professor F.Smarandache let us to study the sequences $e_q(n)$ and $b(n)$. In this paper, we use the elementary methods to study the mean value properties of $\sum\limits_{n\leq x} p^{e_q(b(n))}$, and give an interesting asymptotic formula for it. That is, we will prove the following :

Theorem. *Let p and q are two primes, then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} p^{e_q(b(n))} = \frac{q^2+p^2q+p}{q^2+q+1}x + O(x^{\frac{1}{2}+\epsilon}),$$

where ϵ is any fixed positive number.

From this Theorem we may immediately deduce the following

Corollary. *Let q be a prime, then for any real number $x \geq 1$, we have the asymptotic formula*

$$\sum_{n\leq x} q^{e_q(b(n))} = qx + O(x^{\frac{1}{2}+\epsilon}),$$

where ϵ is any fixed positive number.

§2. Proof of the Theorem

In this section, we will complete the proof of the theorem. Let positive integer $n = u^3v^2w$, where v, w are square free numbers and $(v, w) = 1$. Then from the definition of $b(n)$, we can get $b(n) = vw^2$. For any prime p and any nonnegative integer m, we have

$$b(p^m) = \begin{cases} 1, & if \quad m = 3t \\ p^2, & if \quad m = 3t+1 \\ p, & if \quad m = 3t+2 \end{cases} \tag{1}$$

For any complex s, we define the function

$$f(s) = \sum_{n=1}^{\infty} \frac{p^{e_q(b(n))}}{n^s}.$$

It is clear that for any positive integer n, $e_q(n)$ is an additive function and $b(n)$ is a multiplicative function. So we can prove that $p^{e_q(b(n))}$ is also a multiplicative function.

If $\mathrm{Re}(s) > 1$, then from the definition of $e_q(n)$ and the formula (1), applying the Euler product formula (See Theorem 11.6 of [3]), we can get:

$$\begin{aligned} f(s) = \sum_{n=1}^{\infty} \frac{p^{e_q(b(n))}}{n^s} &= \prod_{p_1} \left(\sum_{m=0}^{\infty} \frac{p^{e_q(b(p_1^m))}}{p_1^{ms}} \right) \\ &= \prod_{p_1} \left(\sum_{t=0}^{\infty} \frac{p^{e_q(1)}}{p_1^{3ts}} + \sum_{t=0}^{\infty} \frac{p^{e_q(p_1^2)}}{p_1^{(3t+1)s}} + \sum_{t=0}^{\infty} \frac{p^{e_q(p_1)}}{p_1^{(3t+2)s}} \right) \\ &= \left(\sum_{t=0}^{\infty} \frac{1}{q^{3ts}} + \sum_{t=0}^{\infty} \frac{p^2}{q^{(3t+1)s}} + \sum_{t=0}^{\infty} \frac{p}{q^{(3t+2)s}} \right) \prod_{p_1 \neq q} \left(1 + \frac{1}{p_1^s} + \frac{1}{p_1^{2s}} + \cdots \right) \\ &= \frac{q^{3s} + p^2q^{2s} + pq^s}{q^{3s} - 1} \prod_{p_1 \neq q} \left(1 + \frac{1}{p_1^s} + \frac{1}{p_1^{2s}} + \cdots \right) \\ &= \zeta(s) \left(\frac{q^{2s} + p^2q^s + p}{q^{2s} + q^s + 1} \right). \end{aligned}$$

By Perron formula (See [2]), takinig $s_0 = 0$, $b = 2$, $T = x^{3/2}$, then we have

$$\sum_{n \leq x} p^{e_q(b(n))} = \frac{1}{2\pi i} \int_{2-iT}^{2+iT} \zeta(s) R(s) \frac{x^s}{s} ds + O(x^{1/2+\epsilon}),$$

where $R(s) = \frac{q^{2s}+p^2q^s+p}{q^{2s}+q^s+1}$ and ϵ is any fixed positive number.

Now we estimate the main term

$$\frac{1}{2\pi i} \int_{2-iT}^{2+iT} \zeta(s) R(s) \frac{x^s}{s} ds,$$

we move the integral line from $2 \pm iT$ to $1/2 \pm iT$, this time, the function

$$\zeta(s)R(s)\frac{x^s}{s}$$

have a simple pole point at $s = 1$ with the residue $R(1)x$, so we have

$$\frac{1}{2\pi i}\left(\int_{2-iT}^{2+iT} + \int_{2+iT}^{1/2+iT} + \int_{1/2+iT}^{1/2-iT} + \int_{1/2-iT}^{2-iT}\right)\zeta(s)R(s)\frac{x^s}{s}ds = R(1)x.$$

Taking $T = x^{\frac{3}{2}}$, we have

$$\begin{aligned}
&\left|\frac{1}{2\pi i}\left(\int_{2+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}-iT}^{2-iT}\right)\zeta(s)R(s)\frac{x^s}{s}ds\right| \\
&\ll \int_{\frac{1}{2}}^{2}\left|\zeta(\sigma+iT)R(s)\frac{x^2}{T}\right|d\sigma \\
&\ll \frac{x^{2+\epsilon}}{T} = x^{\frac{1}{2}+\epsilon};
\end{aligned}$$

And we can easy get the estimate

$$\left|\frac{1}{2\pi i}\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}\zeta(s)R(s)\frac{x^s}{s}ds\right| \ll \int_0^T\left|\zeta(\frac{1}{2}+it)R(s)\frac{x^{\frac{1}{2}}}{t}\right|dt \ll x^{\frac{1}{2}+\epsilon}.$$

Noting that

$$R(1) = \frac{q^2+p^2q+p}{q^2+q+1},$$

so we have the asymptotic formula

$$\sum_{n\le x} p^{e_q(b(n))} = \frac{q^2+p^2q+p}{q^2+q+1}x + O(x^{\frac{1}{2}+\epsilon}).$$

This completes the proof of the Theorem.

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ON THE SMARANDACHE PSEUDO-NUMBER

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Abstract In this paper, we study the counting problems of the Smarandache pseudo-number sequences, and give some interesting asymptotic formulae for them.

Keywords: Pseudo-odd numbers; Pseudo-even numbers; Pseudo-multiples of 5 Sequence; Asymptotic formula.

§1. Introduction

According to reference [1], a number is called pseudo-even number if some permutation of its digits is a even number, including the identity permutation. For example: $0, 2, 4, 6, 8, 10, 12,\ \ 14, 16, 18, 20, 21, \cdots$ are pseudo-even numbers. Similarly, a pseudo-odd number is defined that if some permutation of its digits is an odd number, such as $1, 3, 5, 7, 9, 10, 11, 12, 13, \cdots$ are pseudo-odd numbers. Let A and B denote the set of all the pseudo-even numbers and the pseudo-odd numbers respectively. In addition, a number is called pseudo-multiple of 5 if some permutation of the digits is a multiple of 5, including the identity permutation. For example: $0, 5, 10, 15, 20, 25, 30, 35, 40, 50, 51, \cdots$ are pseudo-multiple of 5 numbers. Let C denotes the set of all the pseudo-multiple of 5 numbers. For convenience, let $A(x)$, $B(x)$ and $C(x)$ denote the number of pseudo-even numbers, pseudo-odd numbers, and pseudo-multiple of 5 numbers that not exceeding x. That is,

$$A(x) = \sum_{\substack{n \in A \\ n \leq x}} 1; \qquad B(x) = \sum_{\substack{n \in B \\ n \leq x}} 1; \qquad C(x) = \sum_{\substack{n \in C \\ n \leq x}} 1.$$

In reference [1], Professor F. Smarandache asked us to study the properties of the pseudo-number sequence. In this paper, we use the elementary method to study the counting problem of these three sequences, and obtain three interesting asymptotic formulae for them. That is, we shall prove the following:

Theorem 1. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\ln\left(x - A(x)\right) = \frac{\ln 5}{\ln 10} \ln x + O(1).$$

Theorem 2. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\ln (x - B(x)) = \frac{\ln 5}{\ln 10} \ln x + O(1).$$

Theorem 3. *For any real number $x \geq 1$, we have the asymptotic formula*

$$\ln (x - C(x)) = \frac{\ln 8}{\ln 10} \ln x + O(1).$$

§2. A Lemma

To complete the proof of the theorems, we need the following lemma:
Lemma. *For any real number $x \geq 1$, we have the inequalities*

$$5^k \leq x - A(x) < \frac{5}{4}(5^k - 1);$$

$$5^k \leq x - B(x) < \frac{5}{4}(5^k - 1);$$

$$8^k \leq x - C(x) < \frac{8}{7}(8^k - 1),$$

where k is a positive integer such that $10^k \leq x < 10^{k+1}$.

Proof. Let $10^k \leq x < 10^{k+1}$ $(k \geq 1)$, then $k \leq \log x < k+1$. According to the definition of set A, we know that the largest number of digits ($\leq x$) not attribute set A is 5^{k+1}. That is, in these numbers, there are 5 one digit, they are $0, 2, 4, 6, 8$; There are 5^2 two digits; The number of the k digits are 5^k. So the largest number of digits ($\leq x$) not attribute set A is $5 + 5^2 + \cdots + 5^k = \frac{5}{4}(5^k - 1) \leq 5^{k+1}$. Then we get

$$5^k \leq x - A(x) < \frac{5}{4}(5^k - 1).$$

Use the same method, we may immediately get:

$$5^k \leq x - B(x) < \frac{5}{4}(5^k - 1);$$

and

$$8^k \leq x - C(x) < \frac{8}{7}(8^k - 1).$$

This proves the Lemma.

§3. Proof of the Theorems

Now we prove the Theorems. In fact from the Lemma and note that $k \leq \ln x \leq k+1$ we have

$$5^k \leq 5^{\log x} = \left(5^{\log_5 x}\right)^{\frac{1}{\log_5 10}} = (x)^{\frac{1}{\log_5 10}} = x^{\frac{\ln 5}{\ln 10}}$$

and

$$5^{k+1} \geq 5^{\log x} = \left(5^{\log_5 x}\right)^{\frac{1}{\log_5 10}} = (x)^{\frac{1}{\log_5 10}} = x^{\frac{\ln 5}{\ln 10}}.$$

Therefore

$$\frac{1}{5} \times x^{\frac{\ln 5}{\ln 10}} \leq \frac{1}{5} \times 5^{k+1} \leq x - A(x) \leq 5^{k+1} \leq 5 \times x^{\frac{\ln 5}{\ln 10}}.$$

Now taking logarithm on both sides of above, we get

$$\ln\left(x - A(x)\right) = \frac{\ln 5}{\ln 10} \ln x + O(1);$$

Use the same method, the following formula will be immediately got.

$$\ln\left(x - B(x)\right) = \frac{\ln 5}{\ln 10} \ln x + O(1);$$

$$\ln\left(x - C(x)\right) = \frac{\ln 8}{\ln 10} \ln x + O(1).$$

This completes the proof of the Theorems.

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SEVERAL ASYMPTOTIC FORMULAE ON A NEW ARITHMETICAL FUNCTION

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Abstract A new arithmetical function is introduced, and several interesting asymptotic formulae on its mean value are given.

Keywords: Arithmetical function; Mean value; Asymptotic formulae.

§1. Introduction

For any fixed positive integer n, the famous Smarandache ceil function of order k is defined as following:

$$S_k(n) = \min\{x \in N \mid n \mid x^k\}\ (\forall n \in N^*)\,.$$

For example, $S_2(1) = 1$, $S_2(2) = 2$, $S_2(3) = 3$, $S_2(4) = 2$, $S_2(5) = 5$, $S_2(6) = 6$, $S_2(7) = 7$, $S_2(8) = 4$, $S_2(9) = 3$, $\cdots$. This function was first introduced by Professor Smarandache [1], and many scholars showed great interest in it (see references [2], [3], [4]). Similarly, for any positive integer n and any fixed positive integer k, we define an arithmetical function $\overline{S}_k(n)$ as following:

$$\overline{S}_k(n) = \max\{x \in N \mid x^k \mid n\}.$$

Because

$$(\forall a, b \in N^*)(a, b) = 1,$$

so we have

$$\begin{aligned} \overline{S}_k(ab) &= \max\{x \in N \mid x^k \mid a\} \cdot \max\{x \in N \mid x^k \mid b\} \\ &= \overline{S}_k(a) \cdot \overline{S}_k(b), \end{aligned}$$

and

$$\overline{S}_k(p^\alpha) = p^{\lfloor \frac{\alpha}{k} \rfloor},$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Therefore, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ is the prime powers decomposition of n, then we have

$$\overline{S}_k(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = p^{\lfloor \frac{\alpha_1}{k} \rfloor} \cdots p^{\lfloor \frac{\alpha_r}{k} \rfloor} = \overline{S}_(p_1^{\alpha_1}) \cdots \overline{S}_(p_r^{\alpha_r}).$$

So $\overline{S}_k(n)$ is a multiplicative function. There are close relations between this function and the Smarandache ceil function [4]. In this paper, we shall use analytic methods to study the mean value properties of $\sigma_\alpha(\overline{S}_k(n))$, and give several asymptotic formulae for it. That is, we shall prove the followings:

Theorem 1. *Let $\alpha \geq 0$, $\sigma_\alpha(n) = \sum\limits_{d|n} d^\alpha$. Then for any real number $x \geq 1$ and any fixed positive integer $k \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x} \sigma_\alpha\left(\overline{S}_k(n)\right) = \begin{cases} \frac{k\zeta\left(\frac{\alpha+1}{k}\right)}{\alpha+1} x^{\frac{\alpha+1}{k}} + O\left(x^{\frac{\alpha+1}{2k}+\epsilon}\right), & \text{if} \quad \alpha > k-1, \\ \zeta(k-\alpha)x + O\left(x^{\frac{1}{2}+\epsilon}\right), & \text{if} \quad \alpha \leq k-1. \end{cases}$$

where $\zeta(s)$ is the Riemann zeta-function, and ϵ be any fixed positive number.

Theorem 2. *Let $d(n)$ denotes the divisor function. Then for any real number $x \geq 1$ and any fixed positive integer $k \geq 2$, we have*

$$\sum_{n\leq x} d\left(\overline{S}_k(n)\right) = \zeta(k)x + O\left(x^{\frac{1}{2}+\epsilon}\right).$$

Taking $k = 2, 3$ in Theorem 2, we may immediately deduce the following:

Corollary. *For any real number $x \geq 1$, we have*

$$\sum_{n\leq x} d\left(\overline{S}_2(n)\right) = \frac{\pi^2}{6}x + O\left(x^{\frac{1}{2}+\epsilon}\right);$$

$$\sum_{n\leq x} d\left(\overline{S}_3(n)\right) = \frac{\pi^4}{90}x + O\left(x^{\frac{1}{2}+\epsilon}\right).$$

§2. Proof of the Theorems

In this section, we shall complete the proof of the Theorems. First we prove Theorem 1. Let

$$f(s) = \sum_{n=1}^{\infty} \frac{\sigma_\alpha\left(\overline{S}_k(n)\right)}{n^s}.$$

From the Euler product formula [5] and the multiplicative property of $\sigma_\alpha\left(\overline{S}_k(n)\right)$ we have

$$\begin{aligned} f(s) &= \prod_p \left(1 + \frac{\sigma_\alpha\left(\overline{S}_k(p)\right)}{p^s} + \cdots + \frac{\sigma_\alpha\left(\overline{S}_k(p^k)\right)}{p^{ks}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{\sigma_\alpha(1)}{p^s} + \cdots + \frac{\sigma_\alpha(1)}{p^{(k-1)s}}\right. \\ &\qquad \left. + \frac{\sigma_\alpha(p)}{p^{ks}} + \cdots + \frac{\sigma_\alpha(p)}{p^{(2k-1)s}} + \frac{\sigma_\alpha(p^2)}{p^{2ks}} + \cdots\right) \end{aligned}$$

$$\begin{aligned}
&= \prod_p \left(\frac{1-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}} + \frac{1-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}} \left(\frac{1+p^\alpha}{p^{ks}} + \frac{1+p^\alpha+p^{2\alpha}}{p^{ks}} + \cdots \right) \right) \\
&= \zeta(s) \prod_p \left(1 + \frac{1}{p^{ks-\alpha}} + \frac{1}{p^{2(ks-\alpha)}} + \frac{1}{p^{3(ks-\alpha)}} + \cdots \right) \\
&= \zeta(s)\zeta(ks-\alpha),
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function. Obviously, we have inequality

$$|\sigma_\alpha\left(\overline{S}_k(n)\right)| \le n, \qquad \left|\sum_{n=1}^{\infty} \frac{\sigma_\alpha\left(\overline{S}_k(n)\right)}{n^\sigma}\right| < \frac{1}{\sigma - 1 - \frac{\alpha+1}{k}},$$

where $\sigma > 1 + \frac{\alpha+1}{k}$ is the real part of s. So by Perron formula [5], we have

$$\begin{aligned}
&\sum_{n\le x} \frac{\sigma_\alpha\left(\overline{S}_k(n)\right)}{n^{s_0}} \\
= \quad & \frac{1}{2i\pi}\int_{b-iT}^{b+iT} f(s+s_0)\frac{x^s}{s}ds + O\left(\frac{x^b B(b+\sigma_0)}{T}\right) \\
& +O\left(x^{1-\sigma_0}H(2x)\min(1,\frac{\log x}{T})\right) + O\left(x^{-\sigma_0}H(N)\min(1,\frac{x}{||x||})\right),
\end{aligned}$$

where N is the nearest integer to x, $\|x\| = |x-N|$. Taking

$$s_0 = 0, b = \frac{\alpha+1}{k} + \frac{1}{\ln x}, T = x^{\frac{\alpha+1}{2k}}, H(x) = x, B(\sigma) = \frac{1}{\sigma - 1 - \frac{\alpha+1}{k}},$$

we have

$$\sum_{n\le x}\sigma_\alpha\left(\overline{S}_k(n)\right) = \frac{1}{2i\pi}\int_{b-iT}^{b+iT}\zeta(s)\zeta(ks-\alpha)\frac{x^s}{s}ds + O\left(x^{\frac{\alpha+1}{2k}+\epsilon}\right).$$

Taking $a = \frac{\alpha+1}{2k} + \frac{1}{\ln x}$, to estimate the main term

$$\frac{1}{2i\pi}\int_{b-iT}^{b+iT}\zeta(s)\zeta(ks-\alpha)\frac{x^s}{s}ds,$$

we move the integral line from $s = b \pm iT$ to $s = a \pm iT$. This time, when $\alpha > k-1$ the function

$$f(s) = \zeta(s)\zeta(ks-\alpha)\frac{x^s}{s}$$

has a simple pole point at $s = \frac{\alpha+1}{k}$ with residue $\frac{k\zeta(\frac{\alpha+1}{k})}{\alpha+1}x^{\frac{\alpha+1}{k}}$. So we have

$$\begin{aligned}
&\frac{1}{2i\pi}\left(\int_{b-iT}^{b+iT} + \int_{b+iT}^{a+iT} + \int_{a+iT}^{a-iT} + \int_{a-iT}^{b-iT}\right)\zeta(s)\zeta(ks-\alpha)\frac{x^s}{s}ds \\
&= \frac{k\zeta\left(\frac{\alpha+1}{k}\right)}{\alpha+1}x^{\frac{\alpha+1}{k}}.
\end{aligned}$$

Note that

$$\frac{1}{2i\pi}\left(\int_{b+iT}^{a+iT}+\int_{a+iT}^{a-iT}+\int_{a-iT}^{b-iT}\right)\zeta(s)\zeta(ks-\alpha)\frac{x^s}{s}ds \ll x^{\frac{\alpha+1}{2k}+\epsilon}.$$

From the above we can immediately get the asymptotic formula:

$$\sum_{n\le x}\sigma_\alpha\left(\overline{S}_k(n)\right)=\frac{k\zeta\left(\frac{\alpha+1}{k}\right)}{\alpha+1}x^{\frac{\alpha+1}{k}}+O\left(x^{\frac{\alpha+1}{2k}+\epsilon}\right).$$

This proves the first part of Theorem 1.

If $0\le\alpha\le k-1$, then the function

$$f(s)=\zeta(s)\zeta(ks-\alpha)\frac{x^s}{s}$$

has a simple pole point at $s=1$ with residue $\zeta(k-\alpha)x$. Similarly, we can get the asymptotic formula:

$$\sum_{n\le x}\sigma_\alpha\left(\overline{S}_k(n)\right)=\zeta(k-\alpha)x+O\left(x^{\frac{1}{2}+\epsilon}\right).$$

This proves the second part of Theorem 1.

Taking $\alpha=0$ in Theorem 1, we can easily get the result of Theorem 2. This completes the proof of the Theorems.

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ON THE SMARANDACHE FUNCTION AND THE K-TH ROOTS OF A POSITIVE INTEGER*

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Abstract The main purpose of this paper is using the elementary method to study the mean value properties of the Smarandache function acting on k-th roots sequences, and give an interesting asymptotic formula.

Keywords: Smarandache function; k-th roots; Mean value

§1. Introduction

Let n be an positive integer, $a_k(n)$ denotes the integer part of k-th root of n, that is $a_k(n) = \left[n^{\frac{1}{k}}\right]$, where $[x]$ is the greatest integer that less than or equal to real number x. In problem 80, 81, 82 of [1], professor F.Smarandache let us to study the properties of the sequences $a_k(n)$. The famous Smarandache function $S(n)$ is defined as following:

$$S(n) = \min\{m : m \in N, n|m!\}.$$

It seems no one know the relation between this sequence and the Smaradache function before. In this paper, we study the mean value properties of the Smarandache function acting on the k-th roots sequences, and give an interesting asymptotic formula. That is , we shall prove the following conclusion:

Theorem. *For any real number $x \geq 3$, we have the asymptotic formula:*

$$\sum_{n\leq x} S(a_k(n)) = \frac{\pi^2 x^{1+\frac{1}{k}}}{6(k+1)\ln x} + O\left(\frac{x^{1+\frac{1}{k}}}{\ln^2 x}\right).$$

§2. Some Lemmas

To complete the proof of the theorem, we need some simple Lemmas. For convenience, we denotes the greatest prime divisor of n by $p(n)$.

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Lemma 1. *If* $p(n) > \sqrt{n}$, *then* $S(n) = p(n)$.

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p(n)$, so we have

$$p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} < \sqrt{n}$$

then

$$p_i^{\alpha_i} | p(n)!, \qquad i = 1, 2, \cdots, r.$$

So $n | p(n)!$, but $p(n) \dagger (p(n) - 1)!$, so $S(n) = p(n)$.

This proves Lemma 1.

Lemma 2. *Let* $x \geq 1$ *be any real number, then we have the asymptotic formula:*

$$\sum_{n \leq x} S(n) = \frac{\pi^2 x^2}{12 \ln x} + O\left(\frac{x^2}{\ln^2 x}\right).$$

Proof. It is clear that

$$\sum_{n \leq x} S(n) = \sum_{\substack{n \leq x \\ p(n) > \sqrt{n}}} S(n) + \sum_{\substack{n \leq x \\ p(n) \leq \sqrt{n}}} S(n). \tag{1}$$

From the Euler summation formula we can easily get the estimate of the second term in the right side of (1):

$$\begin{aligned} &\sum_{\substack{n \leq x \\ p(n) \leq \sqrt{n}}} S(n) \ll \sum_{n \leq x} \sqrt{n} \ln n \\ &= \int_1^x \sqrt{t} \ln t dt + \int_1^x (t - [t])(\sqrt{t} \ln t)' dt + \sqrt{x} \ln x (x - [x]) \\ &\ll x^{\frac{3}{2}} \ln x. \end{aligned} \tag{2}$$

Now we calculate the first term. From Lemma 1, we can write

$$\begin{aligned} \sum_{\substack{n \leq x \\ p(n) > \sqrt{n}}} S(n) &= \sum_{\substack{np \leq x \\ p > \sqrt{np}}} S(n) = \sum_{\substack{n \leq \sqrt{x} \\ \sqrt{x} < p \leq \frac{x}{n}}} \\ &= \sum_{n \leq \sqrt{x}} \sum_{\sqrt{x} < p \leq \frac{x}{n}} p. \end{aligned} \tag{3}$$

Let $\pi(x)$ denotes the number of the primes up to x. Noting that

$$\pi(x) = \frac{x}{\ln x} + O\left(\frac{x}{\ln^2 x}\right),$$

from the Abel's identity [2], we have

$$\sum_{\sqrt{x} < p \leq \frac{x}{n}} p = \pi\left(\frac{x}{n}\right) \frac{x}{n} - \pi(\sqrt{x})\sqrt{x} - \int_{\sqrt{x}}^{\frac{x}{n}} \pi(t) dt$$

$$= \frac{x^2}{n^2 \ln x} - \frac{1}{2}\frac{x^2}{n^2 \ln x} + O\left(\frac{x^2}{n^2 \ln^2 x}\right)$$
$$= \frac{x^2}{2n^2 \ln x} + O\left(\frac{x^2}{n^2 \ln^2 x}\right). \tag{4}$$

Because

$$\sum_{n \le \sqrt{x}} \frac{1}{n^2} = \zeta(2) + O\left(\frac{1}{x}\right), \tag{5}$$

Combining $(1), (2), (3), (4)$ and (5), we can get the result of Lemma 2.

Lemma 3. *For any positive integer k and nonnegative integer i, we have the asymptotic formula:*

$$\sum_{t \le x^{\frac{1}{k}} - 1} t^i S(t) = \frac{\pi^2 x^{\frac{i+2}{k}}}{6(i+2)k \ln x} + O\left(\frac{x^{\frac{i+2}{k}}}{\ln^2 x}\right).$$

Proof. Applying Abel's identity, combining Lemma 2, we have

$$\begin{aligned}
\sum_{t \le x^{\frac{1}{k}} - 1} t^i S(t) &= (x^{\frac{1}{k}} - 1)^i \sum_{t \le x^{\frac{1}{k}} - 1} S(t) - i \int_1^{x^{\frac{1}{k}} - 1} \left(\sum_{l \le t} S(l)\right) t^{i-1} dt \\
&= \frac{\pi^2 x^{\frac{i+2}{k}}}{12k \ln x} - \frac{i\pi^2}{12} \int_1^{x^{\frac{1}{k}} - 1} \frac{t^{i+1}}{\ln t} dt + O\left(\frac{x^{\frac{i+2}{k}}}{\ln^2 x}\right) \\
&= \frac{\pi^2 x^{\frac{i+2}{k}}}{6(i+2)k \ln x} + O\left(\frac{x^{\frac{i+2}{k}}}{\ln^2 x}\right).
\end{aligned}$$

This proves Lemma 3.

§2. Proof of the theorem

In this section, we will complete the proof of the theorem. For any real number $x \ge 1$, let M be a fixed positive integer such that

$$M^k \le x < (M+1)^k.$$

Then we can write

$$\begin{aligned}
\sum_{n \le x} S(a_k(n)) &= \sum_{t=1}^{M-1} \sum_{t^k \le n < (t+1)^k} S(a_k(n)) + \sum_{M^k \le n < x} S(a_k(n)) \\
&= \sum_{t=1}^{M-1} [(t+1)^k - t^k] S(t) + \sum_{M^k \le n < x} S(M) \\
&= \sum_{i=0}^{k-1} \left(\begin{array}{c} i \\ k \end{array}\right) \sum_{t \le x^{\frac{1}{k}} - 1} t^i S(t) + O\left(x^{\frac{1}{k}}\right)
\end{aligned}$$

Now from Lemma 3 we have

$$\begin{aligned}\sum_{n\leq x} S(a_k(n)) &= \sum_{i=0}^{k-1} \left(\begin{array}{c} i \\ k \end{array} \right) \left(\frac{\pi^2 x^{\frac{i+2}{k}}}{6(i+2)k\ln x} + O\left(\frac{x^{\frac{i+2}{k}}}{\ln^2 x} \right) \right) + O\left(x^{\frac{1}{k}} \right) \\ &= \frac{\pi^2 x^{1+\frac{1}{k}}}{6(k+1)\ln x} + O\left(\frac{x^{1+\frac{1}{k}}}{\ln^2 x} \right).\end{aligned}$$

This completes the proof of Theorem.

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ON A DUAL OF THE PSEUDO SMARANDACHE FUNCTION AND ITS ASYMPTOTIC FORMULA

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Abstract In this paper, we study the mean value of the Pseudo Smarandache function and give an asymptotic formula.

Keywords: Simple numbers; Pseudo Smarandache function; Asymptotic formula.

§1. Introduction

According to [1], a number n is called simple number if the product of its proper divisors is less than or equal to n. For example: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 21,$\cdots$ are simple numbers. Let A denote the set of all the simple numbers. In [2], Jozsef Sandor denoted the dual of the Pseudo Smarandache function by analogy by Z_* as following:

$$Z_*(n) = \max\{m \in N^* : \frac{m(m+1)}{2}|n\},$$

Remark:

$$Z_*(1) = 1;$$

and

$$Z_*(p) = \begin{cases} 2, & \text{if } p = 3 \\ 1, & \text{if } p \neq 3. \end{cases}$$

where p is an arbitrary prime.

In this paper, we study the mean value of $Z_*(n)$ and give an asymptotic formula. That is, we shall prove the following:

Theorem. *For any real number $x \geq 1$, we have*

$$\sum_{\substack{n \in A \\ n \leq x}} Z_*(n) = C_1 \frac{x^2}{\ln x} + C_2 \frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x}\right).$$

where C_1, C_2 are computable constants.

§2. Some lemmas

To complete the proof, we need the following lemmas:

Lemma 1. *Let $s \geq 1$ be an integer and p a prime. Then:*

$$Z_*(p^s) = \begin{cases} 2, & \text{if } p = 3 \\ 1, & \text{if } p \neq 3. \end{cases}$$

Proof. This formula can be immediately got from Proposition 1 of [2].

Lemma 2. *Let q be a prime such that $p = 2q - 1$ is a prime, too. Then:*

$$Z_*(pq) = p.$$

Proof. This formula can be immediately got from Proposition 2 of [2].

Lemma 3. *Let $n \in A$, then n has the form:*

$$n = p, \ or\, p^2, \ or\, p^3, \ or\, pq,$$

where p and q are distinct primes.

Proof. First we define :$p_d(n) = \prod_{d|n} d$, and $q_d(n) = \prod_{d|n, d<n} d$. According to the definition of $p_d(n)$, we have

$$(p_d(n))^2 = \prod_{d|n} n = n^{d(n)},$$

where $d(n)$ is the divisor function. That is: $d(n) = \sum_{d|n} 1$, then

$$p_d(n) = n^{\frac{d(n)}{2}};$$

and

$$q_d(n) = \frac{p_d(n)}{n} = n^{\frac{d(n)}{2}-1}. \tag{1}$$

Because n is a simple number, then we have $q_d(n) \leq n$, so from (1) we have

$$n^{\frac{d(n)}{2}-1} \leq n.$$

That is,

$$d(n) \leq 4.$$

Then Lemma 3 can be immediately proved from the definition of $d(n)$.

Lemma 4. *Let $k \geq 0$ and $x \geq 3$, p denotes a prime. Then:*

$$\sum_{p \leq x} p^k = \frac{1}{k+1} \frac{x^{k+1}}{\ln x} + \frac{1}{(k+1)^2} \frac{x^{k+1}}{\ln^2 x} + O\left(\frac{x^{k+1}}{\ln^3 x}\right).$$

Proof. Noting that $\pi(x) = \frac{x}{\ln x} + \frac{x}{\ln^2 x} + O\left(\frac{x}{\ln^3 x}\right)$, then by Abel identity we have

$$\sum_{p\le x} p^k = \pi(x)x^k - \int_1^x \pi(t)kt^{k-1}dt \tag{2}$$

$$\begin{aligned}
&= \frac{x^{k+1}}{\ln x} + \frac{x^{k+1}}{\ln^2 x} + O\left(\frac{x^{k+1}}{\ln^3 x}\right) - k\int_2^x \frac{t^k}{\ln t}dt \\
&\quad -k\int_2^x \frac{t^k}{\ln^2 t}dt + O\left(\int_2^x \frac{t^k}{\ln^3 t}dt\right) \\
&= \frac{x^{k+1}}{\ln x} + \frac{x^{k+1}}{\ln^2 x} + O\left(\frac{x^{k+1}}{\ln^3 x}\right) \\
&\quad -\frac{k}{k+1}\frac{x^{k+1}}{\ln x} - \frac{k^2+2k}{(k+1)^2}\frac{x^{k+1}}{\ln^2 x} + O\left(\int_2^x \frac{t^k}{\ln^3 t}dt\right) \\
&= \frac{1}{k+1}\frac{x^{k+1}}{\ln x} + \frac{1}{(k+1)^2}\frac{x^{k+1}}{\ln^2 x} + O\left(\frac{x^{k+1}}{\ln^3 x}\right). \qquad (3)
\end{aligned}$$

This completes the proof of the Lemma 4.

Lemma 5. *Let p and q are primes, Then:*

$$\sum_{pq\le x} p = C_1\frac{x^2}{\ln x} + C_2\frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x}\right), \tag{4}$$

where C_1, C_2 are computable constants.

Proof. Noting that when $x < 1$, we have $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^m + \cdots$, then

$$\begin{aligned}
&\sum_{p\le\sqrt{x}} p \sum_{q\le x/p} 1 \\
&= \sum_{p\le\sqrt{x}} p\left(\frac{\frac{x}{p}}{(\ln x - \ln p)} + \frac{\frac{x}{p}}{(\ln x - \ln p)^2} + O\left(\frac{\frac{x}{p}}{(\ln x - \ln p)^3}\right)\right) \qquad (5) \\
&= \frac{x}{\ln x}\sum_{p\le\sqrt{x}}\left(1 + \frac{\ln p}{\ln x} + \frac{\ln^2 p}{\ln^2 x} + \cdots + \frac{\ln^m p}{\ln^m x} + \cdots\right) \\
&\quad + \frac{x}{\ln^2 x}\sum_{p\le\sqrt{x}}\left(1 + 2\frac{\ln p}{\ln x} + \cdots + m\frac{\ln^{m-1} p}{\ln^{m-1} x} + \cdots\right) \\
&\quad + O\left(\sum_{p\le\sqrt{x}} \frac{x}{\ln^3 \frac{x}{p}}\right) = B_1\frac{x^{\frac{3}{2}}}{\ln^2 x} + B_2\frac{x^{\frac{3}{2}}}{\ln^3 x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^4 x}\right), \qquad (6)
\end{aligned}$$

where B_1, B_2 are computable constants. And then,

$$\sum_{q\le\sqrt{x}} 1 \sum_{p\le x/q} p$$

$$\begin{aligned}
&= \sum_{q\le\sqrt{x}} \left(\frac{(\frac{x}{q})^2}{2(\ln x - \ln q)} + \frac{(\frac{x}{q})^2}{4(\ln x - \ln q)^2} + O\left(\frac{(\frac{x}{q})^2}{(\ln x - \ln q)^3} \right) \right) \qquad (7)\\
&= \frac{x^2}{2\ln x} \sum_{q\le\sqrt{x}} \frac{1}{q^2} \left(1 + \frac{\ln q}{\ln x} + \frac{\ln^2 q}{\ln^2 x} + \cdots + \frac{\ln^m q}{\ln^m x} + \cdots \right)\\
&\quad + \frac{x^2}{4\ln^2 x} \sum_{q\le\sqrt{x}} \frac{1}{q^2} \left(1 + 2\frac{\ln q}{\ln x} + \cdots + m\frac{\ln^{m-1} q}{\ln^{m-1} x} + \cdots \right)\\
&\quad + O\left(\sum_{q\le\sqrt{x}} \frac{x^2}{q^2 \ln^3 \frac{x}{q}} \right)\\
&= \frac{x^2}{2\ln x} \sum_{q} \frac{1}{q^2} + \frac{x^2}{\ln^2 x} \left(\frac{1}{2} \sum_{q} \frac{\ln q}{q^2} + \frac{1}{4} \sum_{q} \frac{1}{q^2} \right) + O\left(\frac{x^2}{\ln^3 x} \right). \qquad (8)
\end{aligned}$$

So from (6) and (8) we get,

$$\begin{aligned}
\sum_{pq\le x} p &= \sum_{p\le\sqrt{x}} p \sum_{q\le x/p} 1 + \sum_{q\le\sqrt{x}} 1 \sum_{p\le x/q} p - \Big(\sum_{p\le\sqrt{x}} p\Big)\Big(\sum_{q\le\sqrt{x}} 1\Big)\\
&= C_1 \frac{x^2}{\ln x} + C_2 \frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x} \right), \qquad (9)
\end{aligned}$$

where C_1, C_2 are computable constants. This proves Lemma 5.

§3. Proof of the Theorem

Now we prove the Theorem. From Lemmas 1, 2, 3, 4 and 5 we immediately get

$$\begin{aligned}
\sum_{\substack{n\in A\\ n\le x}} Z_*(n) &= \sum_{p\le x} Z_*(p) + \sum_{p^2\le x} Z_*(p^2) + \sum_{p^3\le x} Z_*(p^3) + \sum_{\substack{pq\le x\\ p\ne q}} Z_*(pq)\\
&= 3 + \sum_{p\le x} 1 + \sum_{p^2\le x} 1 + \sum_{p^3\le x} 1 + \sum_{\substack{pq\le x\\ p\ne q}} p\\
&= 3 + \sum_{p\le x} 1 + \sum_{p^2\le x} 1 + \sum_{p^3\le x} 1 + \sum_{pq\le x} p - \sum_{p^2\le x} p\\
&= C_1 \frac{x^2}{\ln x} + C_2 \frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x} \right).
\end{aligned}$$

This completes the proof of the Theorem.

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THE PRIMITIVE NUMBERS OF POWER P AND ITS ASYMPTOTIC PROPERTY

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Abstract Let p be a prime, n be any positive integer, $S_p(n)$ denotes the smallest integer $m \in N^+$, where $p^n|m!$. In this paper, we study the mean value properties of $S_p(n)$, and give an interesting asymptotic formula for it.

Keywords: Smarandache function; Primitive numbers; Asymptotic formula.

§1. Introduction and results

Let p be a prime, n be any positive integer, $S_p(n)$ denotes the smallest integer such that $S_p(n)!$ is divisible by p^n. For example, $S_3(1) = 3$, $S_3(2) = 6$, $S_3(3) = 3$, $S_3(4) = 9$, $\cdots\cdots$. In problem 49 of book [1], Professor F. Smarandache ask us to study the properties of the sequence $\{S_p(n)\}$. About this problem, Professor Zhang and Liu in [2] have studied it and obtained an interesting asymptotic formula. That is, for any fixed prime p and any positive integer n,

$$S_p(n) = (p-1)n + O\left(\frac{p}{\ln p} \cdot \ln n\right).$$

In this paper, we will use the elementary method to study the asymptotic properties of $S_p(n)$ in the following form:

$$\sum_{\substack{n \le x \\ S_p(n+1)=S_p(n)}} 1,$$

where x be a positive real number, and give an interesting asymptotic formula for it. In fact, we shall prove the following result:

Theorem. *For any real number $x \geq 2$, let p be a prime and n be any positive integer. Then we have the asymptotic formula*

$$\sum_{\substack{n\leq x\\ S_p(n+1)=S_p(n)}} 1 = \frac{x}{p} + O\left(\frac{\ln x}{\ln p}\right).$$

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. First we need following:

Lemma. *Let p be a prime and n be any positive integer, then we have*

$$|S_p(n+1) - S_p(n)| = \begin{cases} p, & \text{if } p^n \parallel m!; \\ 0, & \text{otherwise}, \end{cases}$$

where $S_p(n) = m$, $p^n \parallel m!$ denotes that $p^n | m!$ and $p^{n+1} \dagger m!$.

Proof. Now we will discuss it in two cases.

(i) Let $S_p(n) = m$, if $p^n \parallel m!$, then we have $p^n|m!$ and $p^{n+1}\dagger m!$. From the definition of $S_p(n)$ we have $p^{n+1}\dagger(m+1)!$, $p^{n+1}\dagger(m+2)!$, $\cdots$, $p^{n+1}\dagger(m+p-1)!$ and $p^{n+1}|(m+p)!$, so $S_p(n+1) = m+p$, then we get

$$|S_p(n+1) - S_p(n)| = p. \tag{1}$$

(ii) Let $S_p(n) = m$, if $p^n|m!$ and $p^{n+1}|m!$, then we have $S_p(n+1) = m$, so

$$|S_p(n+1) - S_p(n)| = 0. \tag{2}$$

Combining (1) and (2), we can easily get

$$|S_p(n+1) - S_p(n)| = \begin{cases} p, & \text{if } p^n \parallel m!; \\ 0, & \text{otherwise.} \end{cases}$$

This completes the proof of Lemma.

Now we use above Lemma to complete the proof of Theorem. For any real number $x \geq 2$, by the definition of $S_p(n)$ and Lemma we have

$$\sum_{\substack{n\leq x\\ S_p(n+1)=S_p(n)}} 1 = \sum_{\substack{n\leq x\\ p^n|m!,p^{n+1}|m!}} 1 = x - \sum_{\substack{n\leq x\\ p^n\|m!}} 1, \tag{3}$$

where $S_p(n) = m$. Note that if $p^n \parallel m!$, then we have (see reference [3], Theorem 1.7.2)

$$\begin{aligned} n &= \sum_{i=1}^{\infty}\left[\frac{m}{p^i}\right] = \sum_{i\leq \log_p m}\left[\frac{m}{p^i}\right] \\ &= m\cdot \sum_{i\leq \log_p m}\frac{1}{p^i} + O\left(\log_p m\right) \\ &= \frac{m}{p-1} + O\left(\frac{\ln m}{\ln p}\right). \end{aligned} \tag{4}$$

From (4), we can deduce that

$$m=(p-1)n+O\left(\frac{p\ln n}{\ln p}\right). \tag{5}$$

So that

$$1\le m\le (p-1)\cdot x+O\left(\frac{p\ln x}{\ln p}\right), \quad \text{if} \quad 1\le n\le x.$$

Note that for any fixed positive integer n, if there has one m such that $p^n \parallel m!$, then $p^n \parallel (m+1)!$, $p^n \parallel (m+2)!$, $\cdots$, $p^n \parallel (m+p-1)!$. Hence there have p times of m such that $n=\sum\limits_{i=1}^{\infty}\left[\frac{m}{p^i}\right]$ in the interval $1\le m\le (p-1)\cdot x+O\left(\frac{p\ln x}{\ln p}\right)$. Then we have

$$\begin{aligned}\sum_{\substack{n\le x\\ p^n||m!}} 1 &= \frac{1}{p}\left((p-1)\cdot x+O\left(\frac{p\ln x}{\ln p}\right)\right)\\ &= x\cdot\left(1-\frac{1}{p}\right)+O\left(\frac{\ln x}{\ln p}\right). \end{aligned} \tag{6}$$

Combining (3) and (6), we can easily deduce that

$$\begin{aligned}\sum_{\substack{n\le x\\ S_p(n+1)=S_p(n)}} 1 &= x-\sum_{\substack{n\le x\\ p^n||m!}} 1\\ &= x-x\cdot\left(1-\frac{1}{p}\right)+O\left(\frac{\ln x}{\ln p}\right)\\ &= \frac{x}{p}+O\left(\frac{\ln x}{\ln p}\right).\end{aligned}$$

This completes the proof of Theorem.

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SOME ASYMPTOTIC PROPERTIES INVOLVING THE SMARANDACHE CEIL FUNCTION

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Abstract For any fixed positive integer n, the Smarandache ceil function of order k is denoted by $N^* \to N$ and has the following definition:

$$S_k(n) = \min\{x \in N \mid n \mid x^k\} \, (\forall n \in N^*) \, .$$

In this paper, we study the mean value properties of a new arithmetical function $\sigma_\alpha(S_k(n))$ concerning with the Smarandache ceil function, and give several asymptotic formulae for it.

Keywords: Smarandache ceil function; Mean value; Asymptotic formulae.

§1. Introduction

For any fixed positive integer n, the famous Smarandache ceil function of order k is defined as folloing:

$$S_k(n) = \min\{x \in N \mid n \mid x^k\} \, (\forall n \in N^*) \, .$$

For example, $S_2(1) = 1$, $S_2(2) = 2$, $S_2(3) = 3$, $S_2(4) = 2$, $S_2(5) = 5$, $S_2(6) = 6$, $S_2(7) = 7$, $S_2(8) = 4$, $S_2(9) = 3$, $\cdots$. This function was first introduced by Professor Smarandache (see reference [1]), and many scholars showed great interest in it. For example, Ibstedt [2] and [3] studied this function both theoretically and computationally, and got the following conclusions:

$$(\forall a, b \in N^*)(a, b) = 1 \Rightarrow S_k(ab) = S_k(a)S_k(b),$$

$$S_k(p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}) = S(p_1^{\alpha_1}) \cdots S(p_r^{\alpha_r}).$$

While Professor Tabirca established the asymptotic density of fixed point is $\frac{6}{\pi^2}$, and found the average function of the Smarandache ceil function behaves linearly.

In this paper, we shall use the analytic methods to study the mean value properties of a new arithmetical function $\sigma_\alpha(S_k(n))$, and give two asymptotic formulae for it. That is, we shall prove the following:

Theorem 1. *Let $\alpha > 0$, $\sigma_\alpha(n) = \sum\limits_{d|n} d^\alpha$. Then for any real number $x \geq 2$, and any fixed positive integer $k \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x} \sigma_\alpha\left(S_k(n)\right) = \frac{6x^{\alpha+1}\zeta(\alpha+1)\zeta(k(\alpha+1)-\alpha)}{(\alpha+1)\pi^2}R(\alpha+1) + O\left(x^{\alpha+\frac{1}{2}+\epsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, ϵ be any fixed positive number, and

$$R(\alpha+1) = \prod_p \left(1 - \frac{1}{p^{k(\alpha+1)-\alpha} - p^{(k-1)(\alpha+1)}}\right).$$

Theorem 2. *Let $d(n)$ denotes the Dirichlet divisor function. Then for any real number $x \geq 1$, and any fixed positive integer $k \geq 2$, we have the asymptotic formula*

$$\sum_{n\leq x} d\left(S_k(n)\right) = \frac{6\zeta(k)x\ln x}{\pi^2}\prod_p \left(1 - \frac{1}{p^k + p^{k-1}}\right) + Cx + O\left(x^{\frac{1}{2}+\epsilon}\right),$$

where C is a computable constant.

Taking $k = 2$ in Theorem 2, we may immediately deduce the following:

Corollary. *For any real number $x \geq 2$, we have the asymptotic formula:*

$$\sum_{n\leq x} d\left(S_2(n)\right) = x\ln x\prod_p \left(1 - \frac{1}{p^2 + p}\right) + Cx + O\left(x^{\frac{1}{2}+\epsilon}\right).$$

§2. Proof of the Theorems

In this section, we shall complete the proof of the Theorems. First we prove Theorem 1. Let

$$f(s) = \sum_{n=1}^{\infty} \frac{\sigma_\alpha\left(S_k(n)\right)}{n^s}.$$

From the Euler product formula [5] and the multiplicative property of $\sigma_\alpha\left(S_k(n)\right)$ we have

$$\begin{aligned} f(s) &= \prod_p \left(1 + \frac{\sigma_\alpha\left(S_k(p)\right)}{p^s} + \frac{\sigma_\alpha\left(S_k(p^2)\right)}{p^{2s}} + \cdots + \frac{\sigma_\alpha\left(S_k(p^k)\right)}{p^{ks}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{\sigma_\alpha(p)}{p^s} + \cdots + \frac{\sigma_\alpha(p)}{p^{ks}} + \frac{\sigma_\alpha(p^2)}{p^{(k+1)s}} + \cdots \right. \\ &\qquad \left. + \frac{\sigma_\alpha(p^2)}{p^{2ks}} + \frac{\sigma_\alpha(p^3)}{p^{(2k+1)s}} + \cdots\right) \\ &= \prod_p \left(1 + \frac{1-\frac{1}{p^{ks}}}{1-\frac{1}{p^s}}\left(\frac{1+p^\alpha}{p^s} + \frac{1+p^\alpha+p^{2\alpha}}{p^{(k+1)s}} + \cdots\right)\right) \end{aligned}$$

$$\begin{aligned}
&= \zeta(s)\prod_p\left(1+\frac{1}{p^{s-\alpha}}\left(1+\frac{1}{p^{ks-\alpha}}+\frac{1}{p^{2(ks-\alpha)}}+\frac{1}{p^{3(ks-\alpha)}}+\cdots\right)\right)\\
&= \zeta(s)\zeta(ks-\alpha)\prod_p\left(1-\frac{1}{p^{ks-\alpha}}+\frac{1}{p^{s-\alpha}}\right)\\
&= \frac{\zeta(s)\zeta(s-\alpha)\zeta(ks-\alpha)}{\zeta(2(s-\alpha))}\prod_p\left(1-\frac{1}{p^{ks-\alpha}-p^{(k-1)s}}\right),
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function. Obviously, we have inequality

$$|\sigma_\alpha(S_k(n))| \le n, \qquad \left|\sum_{n=1}^{\infty}\frac{\sigma_\alpha(S_k(n))}{n^\sigma}\right| < \frac{1}{\sigma-1-\frac{\alpha+1}{k}},$$

where $\sigma > 1+\frac{\alpha+1}{k}$ is the real part of s. So by Perron formula (see reference [5]),

$$\begin{aligned}
&\sum_{n\le x}\frac{\sigma_\alpha(S_k(n))}{n^{s_0}}\\
&= \frac{1}{2i\pi}\int_{b-iT}^{b+iT} f(s+s_0)\frac{x^s}{s}ds + O\left(\frac{x^bB(b+\sigma_0)}{T}\right)\\
&\quad +O\left(x^{1-\sigma_0}H(2x)\min(1,\frac{\log x}{T})\right)+O\left(x^{-\sigma_0}H(N)\min(1,\frac{x}{||x||})\right),
\end{aligned}$$

where N is the nearest integer to x, $\|x\| = |x-N|$. Taking $s_0 = 0$, $b = \alpha+\frac{3}{2}$, $T = x^{\alpha+\frac{1}{2}}$, $H(x) = x$, $B(\sigma) = \frac{1}{\sigma-1-\alpha}$, we have

$$\begin{aligned}
&\sum_{n\le x}\sigma_\alpha(S_k(n))\\
&= \frac{1}{2i\pi}\int_{\alpha+\frac{3}{2}-iT}^{\alpha+\frac{3}{2}+iT}\frac{\zeta(s)\zeta(s-\alpha)\zeta(ks-\alpha)}{\zeta(2(s-\alpha))}R(s)\frac{x^s}{s}ds + O\left(x^{\alpha+\frac{1}{2}+\epsilon}\right),
\end{aligned}$$

where

$$R(s) = \prod_p\left(1-\frac{1}{p^{ks-\alpha}-p^{(k-1)s}}\right).$$

To estimate the main term

$$\frac{1}{2i\pi}\int_{\alpha+\frac{3}{2}-iT}^{\alpha+\frac{3}{2}+iT}\frac{\zeta(s)\zeta(s-\alpha)\zeta(ks-\alpha)}{\zeta(2(s-\alpha))}R(s)\frac{x^s}{s}ds,$$

we move the integral line from $s = \alpha+\frac{3}{2}\pm iT$ to $s = \alpha+\frac{1}{2}\pm iT$. This time, when $\alpha > 0$, the function

$$g(s) = \frac{\zeta(s)\zeta(s-\alpha)\zeta(ks-\alpha)}{\zeta(2(s-\alpha))}R(s)\frac{x^s}{s}$$

has a simple pole point at $s = \alpha + 1$ with residue

$$\frac{\zeta(\alpha+1)\zeta(k(\alpha+1)-\alpha)}{(\alpha+1)\zeta(2)}R(\alpha+1)x^{\alpha+1}.$$

So we have

$$\begin{aligned}
&\frac{1}{2i\pi}\left(\int_{\alpha+\frac{3}{2}-iT}^{\alpha+\frac{3}{2}+iT}+\int_{\alpha+\frac{3}{2}+iT}^{\alpha+\frac{1}{2}+iT}+\int_{\alpha+\frac{1}{2}+iT}^{\alpha+\frac{1}{2}-iT}+\int_{\alpha+\frac{1}{2}-iT}^{\alpha+\frac{3}{2}-iT}\right)\\
&\quad\frac{\zeta(s)\zeta(s-\alpha)\zeta(ks-\alpha)x^s}{\zeta(2(s-\alpha))s}R(s)ds\\
=&\quad\frac{\zeta(\alpha+1)\zeta(k(\alpha+1)-\alpha)}{(\alpha+1)\zeta(2)}R(\alpha+1)x^{\alpha+1}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\frac{1}{2i\pi}\left(\int_{\alpha+\frac{3}{2}+iT}^{\alpha+\frac{1}{2}+iT}+\int_{\alpha+\frac{1}{2}+iT}^{\alpha+\frac{1}{2}-iT}+\int_{\alpha+\frac{1}{2}-iT}^{\alpha+\frac{3}{2}-iT}\right)\\
&\quad\frac{\zeta(s)\zeta(s-\alpha)\zeta(ks-\alpha)x^s}{\zeta(2(s-\alpha))s}R(s)ds\\
&\ll x^{\alpha+\frac{1}{2}+\epsilon}
\end{aligned}$$

and $\zeta(2) = \frac{\pi^2}{6}$.

From the above we can immediately get the asymptotic formula:

$$\sum_{n\le x}\sigma_\alpha\left(S_k(n)\right) = \frac{6x^{\alpha+1}\zeta(\alpha+1)\zeta(k(\alpha+1)-\alpha)}{(\alpha+1)\pi^2}R(\alpha+1)+O\left(x^{\alpha+\frac{1}{2}+\epsilon}\right).$$

This completes the proof of Theorem 1.

If $\alpha = 0$, then the function

$$g(s) = \frac{x^s\zeta^2(s)\zeta(ks)}{s\zeta(2s)}\prod_p\left(1-\frac{1}{p^{ks}-p^{(k-1)s}}\right)$$

has a second order pole point at $s = 1$ with residue

$$\begin{aligned}
&\lim_{s\to1}[(s-1)^2g(s)]'\\
\equiv&\lim_{s\to1}\left[(s-1)^2h(s)\frac{x^s}{s}\right]'\\
=&\lim_{s\to1}\left\{\left[(s-1)^2h(s)\right]'\frac{x^s}{s}+(s-1)^2h(s)\frac{sx^s\ln x-x^s}{s^2}\right\}.
\end{aligned}$$

Note that

$$\lim_{s\to1}(s-1)^2h(s) = \frac{\zeta(k)}{\zeta(2)}\prod_p\left(1-\frac{1}{p^k+p^{k-1}}\right).$$

From the above we have

$$\sum_{n\leq x} d\left(S_k(n)\right) = \frac{6\zeta(k)x\ln x}{\pi^2}\prod_p\left(1-\frac{1}{p^k+p^{k-1}}\right)+Cx+O\left(x^{\frac{1}{2}+\epsilon}\right),$$

where C is a computable constant.

This completes the proof of Theorem 2.

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ASYMPTOTIC FORMULAE OF SMARANDACHE-TYPE MULTIPLICATIVE FUNCTIONS

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Abstract In this paper, Analytic method is used to study the mean value properties of Smarandache-Type multiplicative function $K_m(n)$ and $L_m(n)$, and give their asymptotic formula respectively.

Keywords: Smarandache-Type multiplicative function; Mean value; Asymptotic formula.

§1. Introduction

According to [1], the definition of Smarandache-Type multiplicative function $K_m(n)$ is the largest m^{th} power-free number dividing n. Another Smarandache type multiplicative function $L_m(n)$ is defined as: n divided by largest m^{th} power-free number dividing n. That is, for any positive integer n, if n has the prime power decomposition $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $K_m(n)$ and $L_m(n)$ are presented in the following

$$K_m(n) = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}, \quad L_m(n) = p_1^{\gamma_1} p_2^{\gamma_2} \cdots p_k^{\gamma_k},$$

where $\beta_i = \min(\alpha_i, m-1), \gamma_i = \max(0, \alpha_i - m + 1)$.
It is obvious to show that $K_m(n)$ and $L_m(n)$ are multiplicative functions. In this paper, we study the mean value properties of these two functions, and give their asymptotic formulae respectively. That is, we shall prove the following

Theorem 1. *Let $m \geq 2$ is a given integer, then for any real number $x \geq 1$, we have*

$$\sum_{n \leq x} K_m(n) = \frac{x^2}{2\zeta(m)} \prod_p \left(1 + \frac{1}{(p^m - 1)(p+1)}\right) + O\left(x^{\frac{3}{2}+\epsilon}\right).$$

Theorem 2. *Let $m \geq 2$ is a given integer, then for any real number $x \geq 1$, we have*

$$\sum_{n \leq x} \frac{1}{L_m(n)} = \frac{x}{\zeta(m)} \prod_p \left(1 + \frac{1}{(p^m - 1)(p+1)}\right) + O\left(x^{\frac{1}{2}+\epsilon}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

§2. Proof of the Theorem

Now we prove the Theorem 1. Let

$$f(s)=\sum_{n=1}^{\infty}\frac{K_m(n)}{n^s},$$

$\mathrm{Re}(s)>1$. From the Euler product formula [3] and the multiplicative property of $K_m(n)$ we have

$$\begin{aligned}
f(s) &= \prod_p\left(1+\frac{K_m(p)}{p^s}+\frac{K_m(p^2)}{p^{2s}}+\cdots\right)\\
&= \prod_p\left(1+\frac{p}{p^s}+\frac{p^2}{p^{2s}}+\cdots+\frac{p^{m-1}}{p^{(m-1)s}}+\frac{p^{m-1}}{p^{ms}}+\frac{p^{m-1}}{p^{(m+1)s}}+\cdots\right)\\
&= \prod_p\left(\frac{1-\frac{1}{p^{m(s-1)}}}{1-\frac{1}{p^{s-1}}}+\frac{p^{m-1}}{p^{ms}}\left(1+\frac{1}{p^s}+\frac{1}{p^{2s}}+\cdots\right)\right)\\
&= \prod_p\left(\frac{1-\frac{1}{p^{m(s-1)}}}{1-\frac{1}{p^{s-1}}}+\frac{1}{p^{m(s-1)+1}}\frac{1}{1-\frac{1}{p^s}}\right)\\
&= \frac{\zeta(s-1)}{\zeta(m(s-1))}\prod_p\left(1+\frac{p^{s-1}-1}{\left(p^{m(s-1)}-1\right)(p^s-1)}\right)
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function. By Perron formula [2], with $s_0=0$, $T=x$, $b=\frac{5}{2}$, we have

$$\sum_{n\le x}K_m(n)=\frac{1}{2\pi i}\int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT}\frac{\zeta(s-1)}{\zeta(m(s-1))}R(s)\frac{x^s}{s}ds+O\left(\frac{x^{\frac{5}{2}}}{T}\right),$$

where

$$R(s)=\prod_p\left(1+\frac{p^{s-1}-1}{\left(p^{m(s-1)}-1\right)(p^s-1)}\right).$$

To estimate the main term

$$\frac{1}{2\pi i}\int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT}\frac{\zeta(s-1)}{\zeta(m(s-1))}R(s)\frac{x^s}{s}ds,$$

we move the integral line from $s=\frac{5}{2}\pm iT$ to $s=\frac{3}{2}\pm iT$. This time, the function

$$f(s)=\frac{\zeta(s-1)}{\zeta(m(s-1))}R(s)\frac{x^s}{s}$$

has a simple pole point at $s=2$, and the residue is $\frac{x^2}{2\zeta(m)}R(2)$. So we have

$$\frac{1}{2i\pi}\left(\int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT}+\int_{\frac{5}{2}+iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT}+\int_{\frac{3}{2}-iT}^{\frac{5}{2}-iT}\right)\frac{\zeta(s-1)}{\zeta(m(s-1))}R(s)\frac{x^s}{s}ds$$

$$= \frac{x^2}{2\zeta(m)} \prod_p \left(1 + \frac{1}{(p^m-1)(p+1)}\right).$$

Note that

$$\frac{1}{2i\pi}\left(\int_{\frac{5}{2}+iT}^{\frac{3}{2}+iT} + \int_{\frac{3}{2}+iT}^{\frac{3}{2}-iT} + \int_{\frac{3}{2}-iT}^{\frac{5}{2}-iT}\right) \frac{\zeta(s-1)}{\zeta(m(s-1))} R(s) \frac{x^s}{s} ds \ll x^{\frac{3}{2}+\epsilon}$$

From above we may immediately get the asymptotic formula:

$$\sum_{n \le x} K_m(n) = \frac{x^2}{2\zeta(m)} \prod_p \left(1 + \frac{1}{(p^m-1)(p+1)}\right) + O\left(x^{\frac{3}{2}+\epsilon}\right).$$

This completes the proof of the Theorem 1.

Now we give the proof of Theorem 2. Let

$$g(s) = \sum_{n=1}^{\infty} \frac{1}{G_m(n) n^s},$$

$\mathrm{Re}(s) > 1$. From the Euler product formula [3] and the multiplicative property of $L_m(n)$ we have

$$\begin{aligned}
g(s) &= \prod_p \left(1 + \frac{1}{L_m(p)p^s} + \frac{1}{L_m(p^2)p^{2s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{(m-1)s}}\right. \\
&\qquad \left. + \frac{1}{p \cdot p^{ms}} + \frac{1}{p^2 \cdot p^{(m+1)s}} + \cdots\right) \\
&= \prod_p \left(\frac{1 - \frac{1}{p^{ms}}}{1 - \frac{1}{p^s}} + \frac{1}{p^{ms+1}}\left(1 + \frac{1}{p^{s+1}} + \frac{1}{p^{2(s+1)}} + \cdots\right)\right) \\
&= \prod_p \left(\frac{1 - \frac{1}{p^{ms}}}{1 - \frac{1}{p^s}} + \frac{1}{p^{ms+1}} \frac{1}{1 - \frac{1}{p^{s+1}}}\right) \\
&= \frac{\zeta(s)}{\zeta(ms)} \prod_p \left(1 + \frac{p^s - 1}{(p^{ms}-1)(p^{s+1}-1)}\right)
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function. By Perron formula [2], with $s_0 = 0$, $T = x$, $b = \frac{3}{2}$, we have

$$\sum_{n \le x} \frac{1}{L_m(n)} = \frac{1}{2\pi i} \int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} \frac{\zeta(s)}{\zeta(ms-)} T(s) \frac{x^s}{s} ds + O\left(\frac{x^{\frac{3}{2}}}{T}\right),$$

where

$$T(s) = \prod_p \left(1 + \frac{p^s - 1}{(p^{ms}-1)(p^{s+1}-1)}\right).$$

To estimate the main term

$$\frac{1}{2\pi i}\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT}\frac{\zeta(s-1)}{\zeta(m(s-1))}T(s)\frac{x^s}{s}ds,$$

we move the integral line from $s=\frac{3}{2}\pm iT$ to $s=\frac{1}{2}\pm iT$. This time, the function

$$g(s)=\frac{\zeta(s)}{\zeta(ms)}T(s)\frac{x^s}{s}$$

has a simple pole point at $s=1$ with residue $\frac{x}{\zeta(m)}T(1)$. So we have

$$\frac{1}{2i\pi}\left(\int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT}+\int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT}+\int_{\frac{1}{2}+iT}^{\frac{1}{2}-iT}+\int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT}\right)\frac{\zeta(s)}{\zeta(ms)}T(s)\frac{x^s}{s}ds$$
$$=\frac{x}{\zeta(m)}\prod_p\left(1+\frac{1}{(p^m-1)(p+1)}\right).$$

Note that

$$\frac{1}{2i\pi}\left(\int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT}+\int_{\frac{5}{2}+iT}^{\frac{1}{2}-iT}+\int_{\frac{5}{2}-iT}^{\frac{3}{2}-iT}\right)\frac{\zeta(s)}{\zeta(ms)}T(s)\frac{x^s}{s}ds\ll x^{\frac{1}{2}+\epsilon},$$

from above we may immediately get the asymptotic formula:

$$\sum_{n\le x}\frac{1}{L_m(n)}=\frac{x}{\zeta(m)}\prod_p\left(1+\frac{1}{(p^m-1)(p+1)}\right)+O\left(x^{\frac{1}{2}+\epsilon}\right).$$

This completes the proof of the Theorem 2.

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ON THE INTEGER PART OF THE K-TH ROOT OF A POSITIVE INTEGER*

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Abstract For any positive integer m, let $a(m)$ denotes the integer part of the k-th root of m. That is, $a(m) = \left[m^{1/k}\right]$. In this paper, we study the asymptotic properties of the sequences $\{a(m)\}$, and give two interesting asymptotic formulae.

Keywords: Integer part sequence; k-th root; Mean value; Asymptotic formulas.

§1. Introduction

For any positive integer m, let $a(m)$ denotes the integer part of the k-th root of m. That is, $a(m) = \left[m^{1/k}\right]$. For example, let $k = 3$ then $a(1) = a(2) = \cdots = a(7) = 1$, $a(8) = a(9) = \cdots = a(26) = 8$, $b(1) = b(2) = \cdots = b(7) = 8$, $b(8) = b(9) = \cdots = b(26) = 27,\cdots$. In problem 80 of reference [1], Professor F.Smarandach asked us to study the asymptotic properties of the sequence $\{a(m)\}$. About this problem, it seems that none had studied it, at least we have not seen related paper before. In this paper, we shall use the elementary method to study the asymptotic properties of this sequence, and give two interesting asymptotic formulas. For convenience, we define $\Omega(n)$ and $\omega(n)$ as following: $\Omega(n) = \alpha_1 + \alpha_2 + \ldots + \alpha_r$, $\omega(n) = r$, if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into prime powers. Then we have the following:

Theorem. *For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n \le x} \omega(a(n)) = x \ln \ln x + (A - \ln k)\, x + O\left(\frac{x}{\ln x}\right),$$

$$\sum_{n \le x} \Omega(a(n)) = x \ln \ln x + (B - \ln k)\, x + O\left(\frac{x}{\ln x}\right),$$

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where

$$A = \gamma + \sum_p \left(\ln\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right), B = A + \sum_p \frac{1}{p(p-1)}$$

are two constants.

Taking $k = 3$ on the above, we can immediately obtain the following

Corollary. *For any real number $x > 1$, we have the asymptotic formula*

$$\sum_{n \le x} \omega(a(n)) = x \ln\ln x + (A - \ln 3)\, x + O\left(\frac{x}{\ln x}\right),$$

$$\sum_{n \le x} \Omega(a(n)) = x \ln\ln x + (B - \ln 3)\, x + O\left(\frac{x}{\ln x}\right).$$

§2. Proof of the Theorems

In this section, we shall complete the proof of the Theorem. First we come to prove the first part of the Theorem. For any real number $x \ge 1$, let M be a fixed positive integer such that

$$M^k \le x < (M+1)^k.$$

Then from the definition of $a(n)$ we have

$$\begin{aligned}
\sum_{n \le x} \omega(a(n)) &= \sum_{m=1}^{M} \sum_{(m-1)^k \le n < m^k} \omega(a(n)) + \sum_{M^k \le n \le x} \omega(a(n)) \\
&= \sum_{m=1}^{M-1} \sum_{m^k \le n < (m+1)^k} \omega(m) + \sum_{M^k \le n \le x} \omega(M) \\
&= \sum_{m=1}^{M-1} \left(C_k^1 m^{k-1} + C_k^2 m^{k-2} + \cdots + 1\right) \omega(m) \\
&\quad + O\left(\sum_{M \le n \le (M+1)^k} \omega(M)\right) \\
&= k \sum_{m=1}^{M} m^{k-1}\, \omega(m) + O(M^{k-1} \ln M),
\end{aligned}$$

where we have used the estimate $\omega(n) \ll \ln n$.

Note that (see reference [2])

$$\sum_{n \le x} \omega(n) = x \ln\ln x + Ax + O\left(\frac{x}{\ln x}\right),$$

where A is a constant. Let $B(y) = \sum\limits_{m \leq y} \omega(m)$, by Abel's identity (see Theorem 4.2 of [3]) we have

$$\begin{aligned} & \sum_{m=1}^{M} m^{k-1}\, \omega(m) = M^{k-1} B(M) - \int_2^M B(y) dy \\ = \quad & M^{k-1} \left(M \ln\ln M + AM \right) - \int_2^M \left(y^{k-1} \ln\ln y + A y^{k-1} \right) dy \\ & + O\left(\frac{M^k}{\ln M} \right) \\ = \quad & M^k \ln\ln M + AM^k - \frac{k-1}{k} \left(M^k \ln\ln M + AM^k \right) + O\left(\frac{M^k}{\ln M} \right) \\ = \quad & \frac{1}{k} M^k \ln\ln M + \frac{1}{k} AM^k + O\left(\frac{M^k}{\ln M} \right). \end{aligned}$$

Therefore, we can obtain the asymptotic formula

$$\sum_{n \leq x} \omega(a(n)) = M^k \ln\ln M + AM^k + O\left(\frac{M^k}{\ln M} \right),$$

where A is a constant.

On the other hand, note that the estimates

$$0 \leq x - M^k < (M+1)^k - M^k = C_k^1 M^{k-1} + C_k^2 M^{k-2} + \cdots + 1 \ll x^{\frac{k-1}{k}}$$

and

$$\ln k + \ln\ln M \leq \ln\ln x < \ln k + \ln\ln(M+1) \leq \ln k + \ln\ln M + O\left(x^{-1/k} \right).$$

Now combining the above, we may immediately obtain the asymptotic formula

$$\sum_{n \leq x} \omega(a(n)) = x \ln\ln x + (A - \ln k)\, x + O\left(\frac{x}{\ln x} \right).$$

This proves the first part of the Theorem.

Similarly, we can prove the second part of the Theorem. This completes the proof of the Theorem.

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ON THE ADDITIVE CUBIC COMPLEMENTS

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Abstract For any positive integer n, let $b(n)$ denotes the additive cubic complements of n. That is, $b(n)$ denotes the smallest non-negative integer such that $n+b(n)$ is a perfect cubic number. In this paper, we study the mean value properties of $b(n)$ and the function $\Omega(n)$, here $\Omega(n)$ denotes the numbers of all prime divisors of n, and give a sharper asymptotic formula for the mean value of $\Omega(n+b(n))$.

Keywords: Additive cubic complements; Function of prime divisors; Asymptotic formula.

§1. Introduction and results

For any positive integer n, the cubic complements $b_3(n)$ is defined as the smallest integer k such that nk is a perfect cubic number. For example, $b_3(1) = 1$, $b_3(2) = 4$, $b_3(3) = 9$, $b_3(4) = 2$, $b_3(5) = 25$, $b_3(6) = 36$, $b_3(7) = 49$, $b_3(8) = 1, \cdots$. In problem 28 of [1], Professor F. Smaradache ask us to study the properties of $\{b_3(n)\}$. About this problem, there have some authors to study and proved some interesting results. For example, Wang Y. [2] studied the asymptotic properties of $\sum\limits_{n\le x} \frac{1}{b_3(n)}$ and $\sum\limits_{n\le x} \frac{n}{b_3(n)}$, and obtained several asymptotic formulae.

Similarly, we will define the additive cubic complements as follows: for any positive integer n, the smallest non-negtive integer k is called the additive cubic complements of n if $n+k$ is a perfect cubic number. Let

$$b(n) = \min\{k|n+k = m^3, k \ge 0, m \in N^+\},$$

then $b(1) = 0$, $b(2) = 6$, $b(3) = 5$, $b(4) = 4$, $b(5) = 3$, $b(6) = 2$, $b(7) = 1$, $b(8) = 0$, $b(9) = 18, \cdots$. About this sequence, it seems that none had studied it before, at least we have not seen any results at present.

In this paper, we will use the analytic methods to study the asymptotic property of this sequence in the following form: $\sum\limits_{n\le x} \Omega(n+b(n))$, where $x \ge 2$

be a real number, $\Omega(n)$ denotes the numbers of all prime divisors of n, i.e., $\Omega(n) = \alpha_1 + \alpha_2 + \cdots + \alpha_r$ if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into prime powers, and give a sharper asymptotic formula for it. That is, we shall prove the following:

Theorem. *For any real number $x \geq 2$, we have the asymptotic formula*

$$\sum_{n \leq x} \Omega(n + b(n)) = 3x \ln \ln x + 3(A - \ln 3)x + O(\frac{x}{\ln x}),$$

where $A = \gamma + \sum\limits_p \left(\ln(1 - \frac{1}{p}) + \frac{1}{p} \right) + \sum\limits_p \frac{1}{p(p-1)}$, $\sum\limits_p$ denotes the summation over all primes, and γ be the Euler constant.

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem. First we need following:

Lemma. *For any real number $x > 1$, we have*

$$\sum_{n \leq x} \Omega(n) = x \ln \ln x + Ax + O\left(\frac{x}{\ln x} \right),$$

where $A = \gamma + \sum\limits_p \left(\ln(1 - \frac{1}{p}) + \frac{1}{p} \right) + \sum\limits_p \frac{1}{p(p-1)}$, γ be the Euler constant.

Proof. (See reference [3]).

Now we use above Lemma to complete the proof of Theorem. For any real number $x \geq 2$, let M be a fixed positive such that

$$M^3 \leq x < (M+1)^3. \tag{1}$$

For any prime p and positive intger α, note that $\Omega(p^\alpha) = \alpha p$. Then from the definition of $b(n)$, we have

$$\begin{aligned} & \sum_{n \leq x} \Omega(n + b(n)) \qquad (2) \\ = & \sum_{1 \leq t \leq M-1} \left(\sum_{t^3 \leq n < (t+1)^3} \Omega(n + b(n)) \right) + \sum_{M^3 \leq n \leq x} \Omega(n + b(n)) \\ = & \sum_{1 \leq t \leq M} \left(\sum_{t^3 \leq n < (t+1)^3} \Omega(n + b(n)) \right) + O(x^{\frac{2}{3}+\epsilon}) \\ = & \sum_{1 \leq t \leq M} \left(\sum_{t^3 \leq n < (t+1)^3} \Omega((t+1)^3) \right) + O(x^{\frac{2}{3}+\epsilon}) \\ = & \sum_{1 \leq t \leq M} (3t^2 + 3t + 1)\Omega((t+1)^3) + O(x^{\frac{2}{3}+\epsilon}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{1\le t\le M} (9t^2+3t+3)\Omega(t+1) + O(x^{\frac{2}{3}+\epsilon})\\
&= 9\cdot \sum_{1\le t\le M} (t+1)^2\Omega(t+1) + O(x^{\frac{2}{3}+\epsilon})\\
&= 9\cdot \sum_{1\le t\le M} t^2\Omega(t) + O(x^{\frac{2}{3}+\epsilon}), \qquad (3)
\end{aligned}
$$

where we have used the estimate $\Omega(n) \ll n^{\epsilon}$.

Let $A(x) = \sum_{n\le x} \Omega(n)$, then by Able's identity (see reference [4], Theorem 4.2) and Lemma, we can easily deduce that

$$
\begin{aligned}
\sum_{1\le t\le M} t^2\Omega(t) &= M^2A(M) - \int_1^M A(t)\cdot (t^2)'d(t) + O(1)\\
&= M^2\left(M\ln\ln M + AM + O\left(\frac{M}{\ln M}\right)\right)\\
&\quad - \int_1^M \left(t\ln\ln t + At + O\left(\frac{t}{\ln t}\right)\right)\cdot 2tdt\\
&= M^3\ln\ln M + AM^3 + O\left(\frac{M^3}{\ln M}\right)\\
&\quad - \int_1^M \left(2t^2\ln\ln t + 2At^2\right)dt\\
&= \frac{1}{3}M^3\ln\ln M + \frac{1}{3}AM^3 + O\left(\frac{M^3}{\ln M}\right). \qquad (4)
\end{aligned}
$$

Note that

$$
0 \le x - M^3 < (M+1)^3 - M^3 = 3M^2+3M+1 = M^2(3+\frac{3}{M}+\frac{1}{M^2}) \ll x^{\frac{2}{3}}, \qquad (5)
$$

and

$$
\ln\ln x = \ln\ln M + \ln 3 + O(1). \qquad (6)
$$

From (3), (4) and (5), we have

$$
\sum_{1\le t\le M} t^2\Omega(t) = \frac{1}{3}x\ln\ln x + \frac{1}{3}(A-\ln 3)x + O(\frac{x}{\ln x}). \qquad (7)
$$

Combining (2) and (6), we may immediately get

$$
\sum_{n\le x} \Omega(n+b(n)) = 3x\ln\ln x + 3(A-\ln 3)x + O(\frac{x}{\ln x}).
$$

This completes the proof of Theorem.

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AN ARITHMETICAL FUNCTION AND ITS HYBRID MEAN VALUE

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Abstract For any positive integer n, let $k_2(n)$ be the smallest integer such that $nk_2(n)$ is the double factorial number. The main purpose of this paper is to study the hybrid mean value of $k_2(n)$ and the Mangoldt function, and give a sharp asymptotic formula.

Keywords: Double factorial; hybrid Mean; Asymptotic formula.

§1. Introduction

For any positive integer n, let $k_2(n)$ be the smallest integer such that $nk_2(n)$ is the double factorial number. For example, $k_2(1) = 1$, $k_2(2) = 1$, $k_2(3) = 1$, $k_2(4) = 2$, $k_2(5) = 3$, $k_2(6) = 8$, $k_2(7) = 15$, $\cdots$. It seems that $k_2(n)$ relates to $k(n)$, which denotes the smallest integer such that $nk(n)$ is a factorial number [1]. In this paper, we study the hybrid mean value of $k_2(n)$ and the Mangoldt function, and give a sharp formula. That is, we shall prove the following:

Theorem. *If $x \geq 2$, then we have the asymptotic formula*

$$\sum_{n\leq x} \Lambda(n)\log(k_2(n)) = \frac{1}{2}x^2\log x + O(x^2).$$

§2. A Lemma

To complete the proof of the theorem, we need the following:

Lemma. *Let $x \geq 2$, then we have*

$$\log[x]! = x\log x - x + O(\log x),$$

where $[y]$ denotes the largest integer not exceeding y.

Proof. This is Theorem 3.15 of [2].

§3. Proof of the theorem

In this section, we complete the proof of the theorem. From the definition of $k_2(n)$ we have

$$k_2(p^\alpha) = (p^\alpha - 2)!! < (p^\alpha - 1)! < p^\alpha!$$

So from the Lemma, we obtain

$$\begin{aligned} & \sum_{n \le x} \Lambda(n) \log k_2(n) \\ = & \sum_{p \le x} \log p \log(p^\alpha - 2)!! \\ = & \sum_{p \le x} p \log^2 p + O\left(\sum_{\substack{p^\alpha \le x \\ 2 \le \alpha}} p^\alpha \log p \log p^\alpha \right). \end{aligned}$$

Let

$$a(n) = \begin{cases} 1, & \text{if } n \text{ is a prime,} \\ 0, & \text{otherwise.} \end{cases}$$

then

$$\sum_{n \le x} a(n) = \pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

By Abel's identity we have

$$\begin{aligned} \sum_{p \le x} p \log^2 p &= \sum_{n \le x} a(n) n \log^2 n \\ &= \pi(x) \cdot x \log^2 x - \int_2^x \pi(t) \left(\log^2 t + 2 \log t\right) dt \\ &= x^2 \log x + O(x^2) - \int_2^x (t \log t + O(t))\, dt \end{aligned}$$

We can easily get

$$\int_2^x t \log t dt = \frac{1}{2} x^2 \log x + O(x^2),$$

Therefore

$$\sum_{p \le x} p \log^2 p = \frac{1}{2} x^2 \log x + O(x^2).$$

Similarly we can get

$$\sum_{p \le x} \alpha \log^2 p \ll x^2.$$

But

$$\begin{aligned}
&\sum_{p^\alpha \le x} \alpha p^\alpha \log^2 p \\
\le\ & x \sum_{2 \le \alpha < \frac{\log x}{\log 2}} \alpha \sum_{p \le x^{\frac{1}{\alpha}}} \log^2 p \\
\ll\ & x \sum_{2 \le \alpha < \frac{\log x}{\log 2}} \alpha x^{\frac{1}{2}} \log x^{\frac{1}{\alpha}} \\
\ll\ & x^{\frac{3}{2}} \log x
\end{aligned}$$

So we have

$$\sum_{n \le x} \Lambda(n) \log k_2(n) = \frac{1}{2} x^2 \log x + O(x^2).$$

This completes the proof of theorem.

References

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ON THE K-TH POWER FREE SIEVE SEQUENCE

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Abstract In this paper, we use the elementary method to study the arithmetical properties of the k-th power free sieve sequence, and give some interesting identities.

Keywords: k-th power free sieve sequence; Infinite series; Divisor function

§1. Introduction

For any positive integer $k \geq 2$, one can obtains the k-th power free sieve sequence as follows: from the set of natural numbers (except 0 and 1), take off all multiples of 2^k, afterwards all multiples of 3^k, $\cdots$, and so on (take off all multiples of all k-th power primes). In problem 31 of [1], Professor F.Smarandache let us to study this sequence. Let $\mathcal{A}$ denotes the set of all numbers in the k-th power free sieve sequence. In this paper, we study the convergent property of some infinite series invovling this sequence, and give some interesting identities. That is, we shall prove the following conclusions:

Theorem 1. *Let $k \geq 2$ be any positive integer. For any real number $\alpha > 1$, we have the identity:*

$$\sum_{\substack{n=1 \\ n \in \mathcal{A}}}^{\infty} \frac{1}{n^{\alpha}} = \frac{\zeta(\alpha)}{\zeta(k\alpha)},$$

where $\zeta(s)$ denotes the Riemann-zeta function.

From this Theorem we may immediately deduce the following:

Corollary. *Let $\mathcal{B}$ be the set of all numbers in the square free sieve sequence, $\mathcal{C}$ be the set of all numbers in the cubic free sieve sequence. Then we have the identities:*

$$\sum_{\substack{n=1 \\ n \in \mathcal{B}}}^{\infty} \frac{1}{n^2} = \frac{15}{\pi^2} \qquad \textit{and} \qquad \sum_{\substack{n=1 \\ n \in \mathcal{C}}}^{\infty} \frac{1}{n^2} = \frac{315}{2\pi^4}$$

Theorem 2. *Let $k \geq 2$ be any positive integer. For any real number $\alpha > 1$, we have the identity:*

$$\sum_{\substack{n=1 \\ n\in\mathcal{A}}}^{\infty} \frac{d(n)}{n^\alpha} = \frac{\zeta^2(\alpha)}{\zeta(k\alpha)} \prod_p \left(1 - \frac{k(p^\alpha - 1)}{p^{(k+1)\alpha} - p^\alpha}\right).$$

§2. Proof of the theorems

In this section, we will complete the proof of the theorems. First, we prove Theorem 1. For any real number $\alpha > 0$, it is clear that

$$\sum_{\substack{n=1 \\ n\in\mathcal{A}}}^{\infty} \frac{1}{n^\alpha} < \sum_{n=1}^{\infty} \frac{1}{n^\alpha},$$

and $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ is convergent if $\alpha > 1$. So from the Euler product formula (See Theorem 11.6 of [2]) and the definition of the k-th power free sieve sequence, we have

$$\begin{aligned}
\sum_{\substack{n=1 \\ n\in\mathcal{A}}}^{\infty} \frac{1}{n^\alpha} &= \prod_p \left(1 + \frac{1}{p^\alpha} + \frac{1}{p^{2\alpha}} + \cdots + \frac{1}{p^{(k-1)\alpha}}\right) \\
&= \prod_p \frac{1 - \frac{1}{p^{k\alpha}}}{1 - \frac{1}{p^\alpha}} \\
&= \frac{\zeta(\alpha)}{\zeta(k\alpha)}.
\end{aligned}$$

This proves Theorem 1.

Now we prove Theorem 2. Similarly, from the Euler product formula and the definition of the k-th power free sieve sequence, we have

$$\begin{aligned}
\sum_{\substack{n=1 \\ n\in\mathcal{A}}}^{\infty} \frac{d(n)}{n^\alpha} &= \prod_p \left(1 + \frac{2}{p^\alpha} + \frac{3}{p^{2\alpha}} + \cdots + \frac{k}{p^{(k-1)\alpha}}\right) \\
&= \prod_p \left(\frac{1}{1 - \frac{1}{p^\alpha}}\right) \left(1 + \frac{1}{p^\alpha} + \frac{1}{p^{2\alpha}} + \cdots + \frac{1}{p^{(k-1)\alpha}} - \frac{k}{p^{k\alpha}}\right) \\
&= \zeta(\alpha) \prod_p \left(\frac{1 - \frac{1}{p^{k\alpha}}}{1 - \frac{1}{p^\alpha}} - \frac{k}{p^{k\alpha}}\right) \\
&= \frac{\zeta^2(\alpha)}{\zeta(k\alpha)} \prod_p \left(1 - \frac{k(p^\alpha - 1)}{p^{(k+1)\alpha} - p^\alpha}\right).
\end{aligned}$$

This completes the proof of the theorems.

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ON A NEW SMARANDACHE SEQUENCE*

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Abstract In this paper, we study the mean value of a new Smarandache sequence and give an asymptotic formula.

Keywords: Simple numbers; Smarandache sequence ; Asymptotic formula.

§1. Introduction

According to [1], a number n is called simple number if the product of its proper divisors is less than or equal to n. For example: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 17, 19, 21,$\cdots$ are simple numbers. Let A denote the set of all the simple numbers. Generally speaking, n has the form: $n = p$, or p^2, or p^3, or pq, where p and q are distinct primes. In [2], Jason Earls defined $sopfr(n)$ as a new Smarandache sequence as following: Let $sopfr(n)$ denote the sum of primes dividing n (with repetition). That is,

$$sopfr(n) = \sum_{p|n} p.$$

For example:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$sopfr(n)$	0	2	3	4	5	5	7	6	6	7	11	7	13	9	8	8	17	8	19

In this paper, we study the mean value properties of $sopfr(n)$, and give an interesting asymptotic formula. That is, we shall prove the following:

Theorem. *For any real number $x \geq 1$, we have*

$$\sum_{\substack{n\in A\\ n\leq x}} sopfr(n) = A_1 \frac{x^2}{\ln x} + A_2 \frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x}\right),$$

where A_1, A_2 are computable constants.

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§2. Some lemmas

To complete the proof of the theorem, we need the following lemmas:

Lemma 1. *Let $k \geq 0$ and $x \geq 3$, p denotes a prime. Then:*

$$\sum_{p \leq x} p^k = \frac{1}{k+1} \frac{x^{k+1}}{\ln x} + \frac{1}{(k+1)^2} \frac{x^{k+1}}{\ln^2 x} + O\left(\frac{x^{k+1}}{\ln^3 x}\right).$$

Proof. Noting that $\pi(x) = \frac{x}{\ln x} + \frac{x}{\ln^2 x} + O\left(\frac{x}{\ln^3 x}\right)$, then by Abel's identity we have

$$\begin{aligned} \sum_{p \leq x} p^k &= \pi(x)x^k - \int_1^x \pi(t)kt^{k-1}dt \qquad (1) \\ &= \frac{x^{k+1}}{\ln x} + \frac{x^{k+1}}{\ln^2 x} + O\left(\frac{x^{k+1}}{\ln^3 x}\right) \\ &\quad -k\int_2^x \frac{t^k}{\ln t}dt - k\int_2^x \frac{t^k}{\ln^2 t}dt + O\left(\int_2^x \frac{t^k}{\ln^3 t}dt\right) \\ &= \frac{x^{k+1}}{\ln x} + \frac{x^{k+1}}{\ln^2 x} + O\left(\frac{x^{k+1}}{\ln^3 x}\right) \\ &\quad -\frac{k}{k+1}\frac{x^{k+1}}{\ln x} - \frac{k^2+2k}{(k+1)^2}\frac{x^{k+1}}{\ln^2 x} + O\left(\int_2^x \frac{t^k}{\ln^3 t}dt\right) \\ &= \frac{1}{k+1}\frac{x^{k+1}}{\ln x} + \frac{1}{(k+1)^2}\frac{x^{k+1}}{\ln^2 x} + O\left(\frac{x^{k+1}}{\ln^3 x}\right). \qquad (2) \end{aligned}$$

This completes the proof of the Lemma 1.

Lemma 2. *Let p and q are primes, Then:*

$$\sum_{pq \leq x} p = C_1 \frac{x^2}{\ln x} + C_2 \frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x}\right), \qquad (3)$$

where C_1, C_2 are computable constants.

Proof. Noting that when $x < 1$, we have $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^m + \cdots$, then

$$\begin{aligned} &\sum_{p \leq \sqrt{x}} p \sum_{q \leq x/p} 1 \\ = &\sum_{p \leq \sqrt{x}} p \left(\frac{\frac{x}{p}}{(\ln x - \ln p)} + \frac{\frac{x}{p}}{(\ln x - \ln p)^2} + O\left(\frac{\frac{x}{p}}{(\ln x - \ln p)^3}\right)\right) \qquad (4) \\ = &\frac{x}{\ln x} \sum_{p \leq \sqrt{x}} \left(1 + \frac{\ln p}{\ln x} + \frac{\ln^2 p}{\ln^2 x} + \cdots + \frac{\ln^m p}{\ln^m x} + \cdots\right) \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{x}{\ln^2 x} \sum_{p \le \sqrt{x}} \left(1 + 2\frac{\ln p}{\ln x} + \cdots + m\frac{\ln^{m-1} p}{\ln^{m-1} x} + \cdots \right) \\
&+ \quad O\left(\sum_{p \le \sqrt{x}} \frac{x}{\ln^3 \frac{x}{p}} \right) = B_1 \frac{x^{\frac{3}{2}}}{\ln^2 x} + B_2 \frac{x^{\frac{3}{2}}}{\ln^3 x} + O\left(\frac{x^{\frac{3}{2}}}{\ln^4 x} \right), \qquad (5)
\end{aligned}
$$

where B_1, B_2 are computable constants.
And then,

$$
\begin{aligned}
&\sum_{q \le \sqrt{x}} 1 \sum_{p \le x/q} p \\
&= \quad \sum_{q \le \sqrt{x}} \left(\frac{(\frac{x}{q})^2}{2(\ln x - \ln q)} + \frac{(\frac{x}{q})^2}{4(\ln x - \ln q)^2} + O\left(\frac{(\frac{x}{q})^2}{(\ln x - \ln q)^3} \right) \right) \qquad (6) \\
&= \quad \frac{x^2}{2\ln x} \sum_{q \le \sqrt{x}} \frac{1}{q^2} \left(1 + \frac{\ln q}{\ln x} + \frac{\ln^2 q}{\ln^2 x} + \cdots + \frac{\ln^m q}{\ln^m x} + \cdots \right) \\
&+ \quad \frac{x^2}{4\ln^2 x} \sum_{q \le \sqrt{x}} \frac{1}{q^2} \left(1 + 2\frac{\ln q}{\ln x} + \cdots + m\frac{\ln^{m-1} q}{\ln^{m-1} x} + \cdots \right) \\
&+ \quad O\left(\sum_{q \le \sqrt{x}} \frac{x^2}{q^2 \ln^3 \frac{x}{q}} \right) \\
&= \quad \frac{x^2}{2\ln x} \sum_q \frac{1}{q^2} + \frac{x^2}{\ln^2 x} \left(\frac{1}{2} \sum_q \frac{\ln q}{q^2} + \frac{1}{4} \sum_q \frac{1}{q^2} \right) + O\left(\frac{x^2}{\ln^3 x} \right). \qquad (7)
\end{aligned}
$$

So from (5) and (7) we get,

$$
\begin{aligned}
\sum_{pq \le x} p \quad &= \quad \sum_{p \le \sqrt{x}} p \sum_{q \le x/p} 1 + \sum_{q \le \sqrt{x}} 1 \sum_{p \le x/q} p - (\sum_{p \le \sqrt{x}} p)(\sum_{q \le \sqrt{x}} 1) \\
&= \quad C_1 \frac{x^2}{\ln x} + C_2 \frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x} \right), \qquad (8)
\end{aligned}
$$

where C_1, C_2 are computable constants. This proves Lemma 2.

§3. Proof of the Theorem

Now we prove the Theorem. From Lemma 1 and Lemma 2 we may immediately get

$$
\begin{aligned}
&\sum_{\substack{n \in A \\ n \le x}} sopfr(n) \\
&= \quad \sum_{p \le x} sopfr(p) + \sum_{p^2 \le x} sopfr(p^2) + \sum_{p^3 \le x} sopfr(p^3) + \sum_{\substack{pq \le x \\ p \ne q}} sopfr(pq)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{p\le x} p + 2\sum_{p^2\le x} p + 3\sum_{p^3\le x} p + 2\sum_{\substack{pq\le x\\ p\ne q}} p \\
&= \sum_{p\le x} p + 3\sum_{p^3\le x} p + 2\sum_{pq\le x} p \\
&= \frac{x^2}{\ln x}(C_1+\frac{1}{2}) + \frac{x^2}{\ln^2 x}(C_2+\frac{1}{4}) + O\left(\frac{x^2}{\ln^3 x}\right) \\
&= A_1\frac{x^2}{\ln x} + A_2\frac{x^2}{\ln^2 x} + O\left(\frac{x^2}{\ln^3 x}\right).
\end{aligned}
$$

This completes the proof of the Theorem.

References

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ON SOME ASYMPTOTIC FORMULAE INVOLVING SMARANDACHE MULTIPLICATIVE FUNCTIONS

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Abstract In this paper, we shall use the analytic method to study the mean value properties of Smarandache-Type multiplicative function $F_m(n)$ and $G_m(n)$, and give two asymptotic formulae for them.

Keywords: Smarandache-Type multiplicative function; Mean value; Asymptotic formula.

§1. Introduction

In reference [1], the definition of Smarandache-Type multiplicative function $F_m(n)$ is the smallest m^{th} power divisible by n divided by largest m^{th} power which divides n. Another Smarandache-Type multiplicative function $G_m(n)$ is defined as m^{th} root of smallest m^{th} power divisible by n divided by largest m^{th} power which divides n. That is, for any fixed positive integer n with the normal factorization $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, $(1 \le i \le k)$ we have

$$F_m(p_i^{\alpha_i}) = \begin{cases} 1, & \text{if } \alpha_i = mk; \\ p_i^m, & \text{otherwise.} \end{cases}$$

and

$$G_m(p_i^{\alpha_i}) = \begin{cases} 1, & \text{if } \alpha_i = mk; \\ p_i, & \text{otherwise.} \end{cases}$$

It is clear that $F_m(n)$ and $G_m(n)$ are multiplicative functions. In this paper, we study the mean value properties of these two functions, and give some asymptotic formulae for them. That is, we shall prove the following:

Theorem 1. *For any real number $x \ge 1$ and integer $m \ge 2$, we have the asymptotic formula*

$$\sum_{n\le x} F_m(n) = \frac{6\zeta(m^2+m)\zeta(m+1)R(m+1)x^{m+1}}{\pi^2} + O\left(x^{m+\frac{1}{2}+\epsilon}\right).$$

where $\zeta(s)$ is the Riemann zeta-function, ϵ be any fixed positive integer, and

$$R(m+1) = \prod_p \left(1 - \frac{1}{p^{m+1}+p^m} - \frac{1}{p^{m^2}+p^{m^2-1}}\right).$$

Theorem 2. *For any real number $x \geq 1$ and integer $m \geq 2$, we have*

$$\sum_{n \leq x} G_m(n) = \zeta(2m) R(2) x^2 + O\left(x^{\frac{3}{2}+\epsilon}\right),$$

where

$$R(2) = \prod_p \left(1 - \frac{1}{p^2+p} - \frac{1}{p^{2m-1}+p^{2m-2}}\right).$$

§2. Proof of the Theorems

Now we prove the Theorem 1. Let

$$f(s) = \sum_{n=1}^{\infty} \frac{F_m(n)}{n^s},$$

$\mathrm{Re}(s) > 1$. From the Euler product formula [3] and the multiplicative property of $F_m(n)$ we have

$$\begin{aligned}
f(s) &= \prod_p \left(1 + \frac{F_m(p)}{p^s} + \frac{F_m(p^2)}{p^{2s}} + \frac{F_m(p^3)}{p^{3s}} + \cdots\right) \\
&= \prod_p \left(1 + \frac{p^m}{p^s} + \frac{p^m}{p^{2s}} + \cdots + \frac{p^m}{p^{(m-1)s}} + \frac{1}{p^{ms}}\right. \\
&\qquad \left. + \frac{p^m}{p^{(m+1)s}} + \frac{p^m}{p^{(m+2)s}} + \cdots + \frac{p^m}{p^{(2m-1)s}} + \frac{1}{p^{2ms}} + \cdots\right) \\
&= \prod_p \left(\frac{1}{1-p^{-ms}}\right) \prod_p \left(1 + \frac{1}{p^{s-m}} + \cdots + \frac{1}{p^{s(m-1)-m}}\right) \\
&= \zeta(ms) \prod_p \left(1 + p^m \left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{(m-1)s}}\right)\right) \\
&= \zeta(ms)\zeta(s) \prod_p \left(1 + \frac{1}{p^{s-m}} - \frac{1}{p^s} - \frac{1}{p^{m(s-1)}}\right) \\
&= \zeta(ms)\zeta(s) \frac{\zeta(s-m)}{\zeta(2s-2m)} \prod_p \left(1 - \frac{1}{p^s+p^m} - \frac{1}{p^{m(s-1)}+p^{ms-s}}\right),
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function. By Perron formula [2], with $s_0 = 0$, $T = x^{m+\frac{1}{2}}$, $b = m + \frac{3}{2}$, we have

$$\sum_{n \leq x} F_m(n) = \frac{1}{2\pi i} \int_{m+\frac{3}{2}-iT}^{m+\frac{3}{2}+iT} \zeta(ms)\zeta(s) \frac{\zeta(s-m)}{\zeta(2s-2m)} R(s) \frac{x^s}{s} ds + O\left(x^{m+\frac{1}{2}+\epsilon}\right),$$

where

$$R(s) = \prod_p \left(1 - \frac{1}{p^s+p^m} - \frac{1}{p^{m(s-1)}+p^{ms-s}}\right).$$

To estimate the main term

$$\frac{1}{2\pi i}\int_{m+\frac{3}{2}-iT}^{m+\frac{3}{2}+iT}\zeta(ms)\zeta(s)\frac{\zeta(s-m)}{\zeta(2s-2m)}R(s)\frac{x^s}{s}ds,$$

we move the integral line from $s = m+\frac{3}{2}\pm iT$ to $s = m+\frac{1}{2}\pm iT$. This time, the function

$$f(s) = \zeta(ms)\zeta(s)\frac{\zeta(s-m)}{\zeta(2s-2m)}\frac{x^s}{s}R(s)$$

has a simple pole point at $s = m+1$ with residue

$$\zeta(m^2+m)\zeta(m+1)\frac{1}{\zeta(2)}x^{m+1}R(m+1).$$

So we have

$$\frac{1}{2i\pi}\left(\int_{m+\frac{3}{2}-iT}^{m+\frac{3}{2}+iT}+\int_{m+\frac{3}{2}+iT}^{m+\frac{1}{2}+iT}+\int_{m+\frac{1}{2}+iT}^{m+\frac{1}{2}-iT}+\int_{m+\frac{1}{2}-iT}^{m+\frac{3}{2}-iT}\right)$$
$$\zeta(ms)\zeta(s)\frac{\zeta(s-m)}{\zeta(2s-2m)}\frac{x^s}{s}R(s)ds$$
$$=\zeta(m^2+m)\zeta(m+1)\frac{1}{\zeta(2)}x^{m+1}R(m+1).$$

Note that

$$\frac{1}{2i\pi}\left(\int_{m+\frac{3}{2}+iT}^{m+\frac{1}{2}+iT}+\int_{m+\frac{1}{2}+iT}^{m+\frac{1}{2}-iT}+\int_{m+\frac{1}{2}-iT}^{m+\frac{3}{2}-iT}\right)$$
$$\zeta(ms)\zeta(s)\frac{\zeta(s-m)}{\zeta(2s-2m)}\frac{x^s}{s}R(s)ds$$
$$\ll x^{m+\frac{1}{2}+\epsilon}$$

and $\zeta(2) = \frac{\pi^2}{6}$.

From above we may immediately get the asymptotic formula:

$$\sum_{n\leq x}F_m(n) = \frac{6\zeta(m^2+m)\zeta(m+1)x^{m+1}R(m+1)}{\pi^2}+O\left(x^{m+\frac{1}{2}+\epsilon}\right).$$

This completes the proof of the Theorem 1.

Next, we will give the proof of Theorem 2. Let

$$g(s) = \sum_{n=1}^{\infty}\frac{G_m(n)}{n^s},$$

$\mathrm{Re}(s) > 1$. From the Euler product formula [3] and the multiplicative property of $G_m(n)$ we have

$$\begin{aligned}
g(s) &= \prod_p \left(1 + \frac{G_m(p)}{p^s} + \frac{G_m(p^2)}{p^{2s}} + \frac{G_m(p^3)}{p^{3s}} + \cdots\right) \\
&= \prod_p (1 + \frac{p}{p^s} + \frac{p}{p^{2s}} + \cdots + \frac{p}{p^{(m-1)s}} + \frac{1}{p^{ms}} \\
&+ \frac{p}{p^{(m+1)s}} + \frac{p}{p^{(m+2)s}} + \cdots + \frac{p}{p^{(2m-1)s}} + \frac{1}{p^{2ms}} \\
&+ \cdots) \\
&= \prod_p \left(\frac{1}{1 - p^{-ms}}\right) \prod_p \left(1 + \frac{1}{p^{s-1}} + \cdots + \frac{1}{p^{s(m-1)-1}}\right) \\
&= \zeta(ms) \prod_p \left(1 + p\left(\frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots + \frac{1}{p^{(m-1)s}}\right)\right) \\
&= \zeta(ms)\zeta(s) \prod_p \left(1 + \frac{1}{p^{s-1}} - \frac{1}{p^s} - \frac{1}{p^{ms-1}}\right) \\
&= \zeta(ms)\zeta(s)\frac{\zeta(s-1)}{\zeta(2s-2)} \prod_p \left(1 - \frac{1}{p^s + p} - \frac{1}{p^{ms-1} + p^{ms-s}}\right),
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function. By Perron formula [2], with $s_0 = 0$, $T = x^{\frac{3}{2}}$, $b = \frac{5}{2}$, we have

$$\sum_{n \le x} G_m(n) = \frac{1}{2\pi i} \int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT} \zeta(ms)\zeta(s)\frac{\zeta(s-1)}{\zeta(2s-2)} R(s) \frac{x^s}{s} ds + O\left(x^{\frac{3}{2}+\epsilon}\right),$$

where

$$R(s) = \prod_p \left(1 - \frac{1}{p^s + p} - \frac{1}{p^{ms-1} + p^{ms-s}}\right).$$

To estimate the main term

$$\frac{1}{2\pi i} \int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT} \zeta(ms)\zeta(s)\frac{\zeta(s-1)}{\zeta(2s-2)} R(s) \frac{x^s}{s} ds,$$

we move the integral line from $s = \frac{5}{2} \pm iT$ to $s = \frac{3}{2} \pm iT$. This time, the function

$$g(s) = \zeta(ms)\zeta(s)\frac{\zeta(s-1)}{\zeta(2s-2)} \frac{x^s}{s} R(s)$$

has a simple pole point at $s = 2$ with residue

$$\zeta(2m)x^2 R(2).$$

So we have

$$\begin{aligned}&\frac{1}{2i\pi}\left(\int_{\frac{5}{2}-iT}^{\frac{5}{2}+iT}+\int_{\frac{5}{2}+iT}^{m+\frac{3}{2}+iT}+\int_{m+\frac{3}{2}+iT}^{m+\frac{3}{2}-iT}+\int_{m+\frac{3}{2}-iT}^{\frac{5}{2}-iT}\right)\\&\qquad\zeta(ms)\zeta(s)\frac{\zeta(s-1)}{\zeta(2s-2)}\frac{x^s}{s}R(s)ds\\&=\zeta(2m)x^2R(2).\end{aligned}$$

Note that

$$\begin{aligned}&\frac{1}{2i\pi}\left(\int_{\frac{5}{2}+iT}^{m+\frac{3}{2}+iT}+\int_{m+\frac{3}{2}+iT}^{m+\frac{3}{2}-iT}+\int_{m+\frac{3}{2}-iT}^{\frac{5}{2}-iT}\right)\\&\qquad\zeta(ms)\zeta(s)\frac{\zeta(s-1)}{\zeta(2s-2)}\frac{x^s}{s}R(s)ds\\&\ll x^{\frac{3}{2}+\epsilon}.\end{aligned}$$

So we may immediately get the asymptotic formula:

$$\sum_{n\leq x}G_m(n)=\zeta(2m)x^2R(2)+O\left(x^{\frac{3}{2}+\epsilon}\right).$$

This proves the Theorem 2.

References

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[3] Tom M. Apostol. Introduction to Analytic Number Theory. New York: Springer-Verlag, 1976.

The book contains 41 research papers involving the Smarandache sequences, functions, or problems and conjectures on them.
All these papers are original. Some of them treat the mean value or hybrid mean value of Smarandache type functions, like the famous Smarandache function, Smarandache ceil function, or Smarandache primitive function. Others treat the mean value of some famous number theoretic functions acting on the Smarandache sequences, like k-th root sequence, k-th complement sequence, or factorial part sequence, etc. There are papers that study the convergent property of some infinite series involving the Smarandache type sequences. Some of these sequences have been first investigated too. In addition, new sequences as additive complement sequences are first studied in several papers of this book.

(The Editor)

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