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Neutrosophic subalgebras of several types in BCK/BCI-algebras

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Abstract. Given $\Phi, \Psi \in \{\in, q, \in \vee q\}$, the notion of ($\Phi$, Ψ)-neutrosophic subalgebras of a BCK/BCI-algebra are introduced, and related properties are investigated. Characterizations of an $(\in, \in)$-neutrosophic subalgebra and an $(\in, \in \vee q)$-neutrosophic subalgebra are provided. Given special sets, so called neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in \vee q$-subsets, conditions for the neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in \vee q$-subsets to be subalgebras are discussed. Conditions for a neutrosophic set to be a (q, $\in \vee q$)-neutrosophic subalgebra are considered.

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1. Introduction

The concept of neutrosophic set (NS) developed by Smarandache [5, 6, 7] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various part. For further particulars I refer readers to the site http://fs.gallup.unm.edu/neutrosophy.htm. Agboola et al. [1] studied neutrosophic ideals of neutrosophic BCI-algebras. Agboola et al. [2] also introduced the concept of neutrosophic BCI/BCK-algebras, and presented elementary properties of neutrosophic BCI/BCK-algebras.

In this paper, we introduce the notion of (Φ, Ψ)-neutrosophic subalgebra of a BCK/BCI-algebra X for $\Phi, \Psi \in \{\in, q, \in \vee q\}$, and investigate related properties.

We provide characterizations of an $(\in, \in)$-neutrosophic subalgebra and an $(\in, \in\vee q)$-neutrosophic subalgebra. Given special sets, so called neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in\vee q$-subsets, we provide conditions for the neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in\vee q$-subsets to be subalgebras. We consider conditions for a neutrosophic set to be a $(q, \in\vee q)$-neutrosophic subalgebra.

2. Preliminaries

By a *BCI-algebra* we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms:

(a1) $((x * y) * (x * z)) * (z * y) = 0$,
(a2) $(x * (x * y)) * y = 0$,
(a3) $x * x = 0$,
(a4) $x * y = y * x = 0 \ \Rightarrow\ x = y$,

for all $x, y, z \in X$. If a BCI-algebra X satisfies the axiom

(a5) $0 * x = 0$ for all $x \in X$,

then we say that X is a *BCK-algebra*. A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

We refer the reader to the books [3] and [4] for further information regarding BCK/BCI-algebras.

Let X be a non-empty set. A neutrosophic set (NS) in X (see [6]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

3. Neutrosophic subalgebras of several types

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets:

$$T_{\in}(A; \alpha) := \{x \in X \mid A_T(x) \geq \alpha\},$$
$$I_{\in}(A; \beta) := \{x \in X \mid A_I(x) \geq \beta\},$$
$$F_{\in}(A; \gamma) := \{x \in X \mid A_F(x) \leq \gamma\},$$
$$T_q(A; \alpha) := \{x \in X \mid A_T(x) + \alpha > 1\},$$
$$I_q(A; \beta) := \{x \in X \mid A_I(x) + \beta > 1\},$$
$$F_q(A; \gamma) := \{x \in X \mid A_F(x) + \gamma < 1\},$$
$$T_{\in\vee q}(A; \alpha) := \{x \in X \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha > 1\},$$
$$I_{\in\vee q}(A; \beta) := \{x \in X \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta > 1\},$$
$$F_{\in\vee q}(A; \gamma) := \{x \in X \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1\}.$$

We say $T_\in(A;\alpha)$, $I_\in(A;\beta)$ and $F_\in(A;\gamma)$ are *neutrosophic* $\in$*-subsets*; $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are *neutrosophic q-subsets*; and $T_{\in\vee q}(A;\alpha)$, $I_{\in\vee q}(A;\beta)$ and $F_{\in\vee q}(A;\gamma)$ are *neutrosophic* $\in\vee q$*-subsets*. For $\Phi \in \{\in, q, \in\vee q\}$, the element of $T_\Phi(A;\alpha)$ (resp., $I_\Phi(A;\beta)$ and $F_\Phi(A;\gamma)$) is called a *neutrosophic* T_Φ*-point* (resp., *neutrosophic* I_Φ*-point* and *neutrosophic* F_Φ*-point*) with value α (resp., β and γ). It is clear that

$$T_{\in\vee q}(A;\alpha) = T_\in(A;\alpha) \cup T_q(A;\alpha), \tag{3.1}$$

$$I_{\in\vee q}(A;\beta) = I_\in(A;\beta) \cup I_q(A;\beta), \tag{3.2}$$

$$F_{\in\vee q}(A;\gamma) = F_\in(A;\gamma) \cup F_q(A;\gamma). \tag{3.3}$$

Proposition 3.1. *For any neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we have*

$$\alpha \in [0, 0.5] \ \Rightarrow\ T_{\in\vee q}(A;\alpha) = T_\in(A;\alpha), \tag{3.4}$$

$$\beta \in [0, 0.5] \ \Rightarrow\ I_{\in\vee q}(A;\beta) = I_\in(A;\beta), \tag{3.5}$$

$$\gamma \in [0.5, 1] \ \Rightarrow\ F_{\in\vee q}(A;\gamma) = F_\in(A;\gamma), \tag{3.6}$$

$$\alpha \in (0.5, 1] \ \Rightarrow\ T_{\in\vee q}(A;\alpha) = T_q(A;\alpha), \tag{3.7}$$

$$\beta \in (0.5, 1] \ \Rightarrow\ I_{\in\vee q}(A;\beta) = I_q(A;\beta), \tag{3.8}$$

$$\gamma \in [0, 0.5) \ \Rightarrow\ F_{\in\vee q}(A;\gamma) = F_q(A;\gamma). \tag{3.9}$$

Proof. If $\alpha \in [0, 0.5]$, then $1-\alpha \in [0.5, 1]$ and $\alpha \le 1-\alpha$. It is clear that $T_\in(A;\alpha) \subseteq T_{\in\vee q}(A;\alpha)$ by (3.1). If $x \notin T_\in(A;\alpha)$, then $A_T(x) < \alpha \le 1-\alpha$, i.e., $x \notin T_q(A;\alpha)$. Hence $x \notin T_{\in\vee q}(A;\alpha)$, and so $T_{\in\vee q}(A;\alpha) \subseteq T_\in(A;\alpha)$. Thus (3.4) is valid. Similarly, we have the result (3.5). If $\gamma \in [0.5, 1]$, then $1-\gamma \in [0, 0.5]$ and $\gamma \ge 1-\gamma$. It is clear that $F_\in(A;\gamma) \subseteq F_{\in\vee q}(A;\gamma)$ by (3.3). Let $z \in F_{\in\vee q}(A;\gamma)$. Then $z \in F_\in(A;\gamma)$ or $z \in F_q(A;\gamma)$. If $z \notin F_\in(A;\gamma)$, then $A_F(z) > \gamma \ge 1-\gamma$, i.e., $A_F(z)+\gamma > 1$. Thus $z \notin F_q(A;\gamma)$, and so $z \notin F_{\in\vee q}(A;\gamma)$. This is a contradiction. Hence $z \in F_\in(A;\gamma)$, and therefore $F_{\in\vee q}(A;\gamma) \subseteq F_\in(A;\gamma)$. Let $\beta \in (0.5, 1]$. Then $\beta > 1-\beta$. Note that $I_q(A;\beta) \subseteq I_{\in\vee q}(A;\beta)$ by (3.2). Let $y \in I_{\in\vee q}(A;\beta)$. Then $y \in I_\in(A;\beta)$ or $y \in I_q(A;\beta)$. If $y \notin I_q(A;\beta)$, then $A_I(y)+\beta \le 1$ and so $A_I(y) \le 1-\beta < \beta$, i.e., $y \notin I_\in(A;\beta)$. Thus $y \notin I_{\in\vee q}(A;\beta)$, a contradiction. Hence $y \in I_q(A;\beta)$. Therefore $I_{\in\vee q}(A;\beta) \subseteq I_q(A;\beta)$. This shows that (3.8) is true. The result (3.7) is proved by the similar way. Let $\gamma \in [0, 0.5)$ and $z \in F_{\in\vee q}(A;\gamma)$. Then $1-\gamma > \gamma$ and $z \in F_\in(A;\gamma)$ or $z \in F_q(A;\gamma)$. If $z \notin F_q(A;\gamma)$, then $A_F(z)+\gamma \ge 1$ and so $A_F(z) \ge 1-\gamma > \gamma$, i.e., $z \notin F_\in(A;\gamma)$. Thus $z \notin F_{\in\vee q}(A;\gamma)$, which is a contradiction. Hence $F_{\in\vee q}(A;\gamma) \subseteq F_q(A;\gamma)$. The reverse inclusion is by (3.3). □

Definition 3.2. Given $\Phi, \Psi \in \{\in, q, \in\vee q\}$, a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is called a $(\Phi,\ \Psi)$*-neutrosophic subalgebra* of X if the following assertions are valid.

$$\begin{aligned} &x \in T_\Phi(A;\alpha_x),\ y \in T_\Phi(A;\alpha_y) \ \Rightarrow\ x*y \in T_\Psi(A;\alpha_x \wedge \alpha_y), \\ &x \in I_\Phi(A;\beta_x),\ y \in I_\Phi(A;\beta_y) \ \Rightarrow\ x*y \in I_\Psi(A;\beta_x \wedge \beta_y), \\ &x \in F_\Phi(A;\gamma_x),\ y \in F_\Phi(A;\gamma_y) \ \Rightarrow\ x*y \in F_\Psi(A;\gamma_x \vee \gamma_y) \end{aligned} \tag{3.10}$$

for all $x, y \in X$, α_x, α_y, $\beta_x, \beta_y, \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

Theorem 3.3. *A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in, \in)$-neutrosophic subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X)\left(\begin{array}{l} A_T(x*y) \geq A_T(x) \wedge A_T(y) \\ A_I(x*y) \geq A_I(x) \wedge A_I(y) \\ A_F(x*y) \leq A_F(x) \vee A_F(y) \end{array}\right). \tag{3.11}$$

Proof. Assume that $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of X. If there exist $x, y \in X$ such that $A_T(x*y) < A_T(x) \wedge A_T(y)$, then

$$A_T(x*y) < \alpha_t \leq A_T(x) \wedge A_T(y)$$

for some $\alpha_t \in (0, 1]$. It follows that $x, y \in T_\in(A; \alpha_t)$ but $x*y \notin T_\in(A; \alpha_t)$. Hence $A_T(x*y) \geq A_T(x) \wedge A_T(y)$ for all $x, y \in X$. Similarly, we show that

$$A_I(x*y) \geq A_I(x) \wedge A_I(y)$$

for all $x, y \in X$. Suppose that there exist $a, b \in X$ and $\gamma_f \in [0, 1]$ be such that $A_F(a*b) > \gamma_f \geq A_F(a) \vee A_F(b)$. Then $a, b \in F_\in(A; \gamma_f)$ and $a*b \notin F_\in(A; \gamma_f)$, which is a contradiction. Therefore $A_F(x*y) \leq A_F(x) \vee A_F(y)$ for all $x, y \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X which satisfies the condition (3.11). Let $x, y \in X$ be such that $x \in T_\in(A; \alpha_x)$ and $y \in T_\in(A; \alpha_y)$. Then $A_T(x) \geq \alpha_x$ and $A_T(y) \geq \alpha_y$, which imply that $A_T(x*y) \geq A_T(x) \wedge A_T(y) \geq \alpha_x \wedge \alpha_y$, that is, $x*y \in T_\in(A; \alpha_x \wedge \alpha_y)$. Similarly, if $x \in I_\in(A; \beta_x)$ and $y \in I_\in(A; \beta_y)$ then $x*y \in I_\in(A; \beta_x \wedge \beta_y)$. Now, let $x \in F_\in(A; \gamma_x)$ and $y \in F_\in(A; \gamma_y)$ for $x, y \in X$. Then $A_F(x) \leq \gamma_x$ and $A_F(y) \leq \gamma_y$, and so $A_F(x*y) \leq A_F(x) \vee A_F(y) \leq \gamma_x \vee \gamma_y$. Hence $x*y \in F_\in(A; \gamma_x \vee \gamma_y)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of X. □

Theorem 3.4. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subalgebra of a BCK/BCI-algebra X, then neutrosophic q-subsets $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$ whenever they are nonempty.*

Proof. Let $x, y \in T_q(A; \alpha)$. Then $A_T(x) + \alpha > 1$ and $A_T(y) + \alpha > 1$. It follows that

$$\begin{aligned} A_T(x*y) + \alpha &\geq (A_T(x) \wedge A_T(y)) + \alpha \\ &= (A_T(x) + \alpha) \wedge (A_T(y) + \alpha) > 1 \end{aligned}$$

and so that $x*y \in T_q(A; \alpha)$. Hence $T_q(A; \alpha)$ is a subalgebra of X. Similarly, we can prove that $I_q(A; \beta)$ is a subalgebra of X. Now let $x, y \in F_q(A; \gamma)$. Then $A_F(x) + \gamma < 1$ and $A_F(y) + \gamma < 1$, which imply that

$$\begin{aligned} A_F(x*y) + \gamma &\leq (A_F(x) \vee A_F(y)) + \gamma \\ &= (A_F(x) + \alpha) \vee (A_F(y) + \alpha) < 1. \end{aligned}$$

Hence $x*y \in F_q(A; \gamma)$ and $F_q(A; \gamma)$ is a subalgebra of X. □

Theorem 3.5. *If $A = (A_T, A_I, A_F)$ is a $(q, \in\vee q)$-neutrosophic subalgebra of a BCK/BCI-algebra X, then neutrosophic q-subsets $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0, 5)$ whenever they are nonempty.*

Proof. Let $x, y \in T_q(A;\alpha)$. Then $x * y \in T_{\in\vee q}(A;\alpha)$, and so $x * y \in T_{\in}(A;\alpha)$ or $x*y \in T_q(A;\alpha)$. If $x*y \in T_{\in}(A;\alpha)$, then $A_T(x*y) \geq \alpha > 1-\alpha$ since $\alpha > 0.5$. Hence $x * y \in T_q(A;\alpha)$. Therefore $T_q(A;\alpha)$ is a subalgebra of X. Similarly, we prove that $I_q(A;\beta)$ is a subalgebra of X. Let $x, y \in F_q(A;\gamma)$. Then $x * y \in F_{\in\vee q}(A;\gamma)$, and so $x * y \in F_{\in}(A;\gamma)$ or $x * y \in F_q(A;\gamma)$. If $x * y \in F_{\in}(A;\gamma)$, then $A_F(x * y) \leq \gamma < 1 - \gamma$ since $\gamma \in [0, 0, 5)$. Hence $x * y \in F_q(A;\gamma)$, and therefore $F_q(A;\gamma)$ is a subalgebra of X. □

We provide characterizations of an $(\in, \in\vee q)$-neutrosophic subalgebra.

Theorem 3.6. *A neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in, \in\vee q)$-neutrosophic subalgebra of X if and only if it satisfies:*

$$(\forall x, y \in X) \begin{pmatrix} A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\} \\ A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y).0.5\} \\ A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \end{pmatrix}. \tag{3.12}$$

Proof. Suppose that $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subalgebra of X and let $x, y \in X$. If $A_T(x) \wedge A_T(y) < 0.5$, then $A_T(x * y) \geq A_T(x) \wedge A_T(y)$. For, assume that $A_T(x * y) < A_T(x) \wedge A_T(y)$ and choose α_t such that

$$A_T(x * y) < \alpha_t < A_T(x) \wedge A_T(y).$$

Then $x \in T_{\in}(A;\alpha_t)$ and $y \in T_{\in}(A;\alpha_t)$ but $x * y \notin T_{\in}(A;\alpha_t)$. Also $A_T(x * y) + \alpha_t < 1$, i.e., $x * y \notin T_q(A;\alpha_t)$. Thus $x * y \notin T_{\in\vee q}(A;\alpha_t)$, a contradiction. Therefore $A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\}$ whenever $A_T(x) \wedge A_T(y) < 0.5$. Now suppose that $A_T(x) \wedge A_T(y) \geq 0.5$. Then $x \in T_{\in}(A; 0.5)$ and $y \in T_{\in}(A; 0.5)$, which imply that $x*y \in T_{\in\vee q}(A; 0.5)$. Hence $A_T(x*y) \geq 0.5$. Otherwise, $A_T(x*y)+0.5 < 0.5+0.5 = 1$, a contradiction. Consequently, $A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\}$ for all $x, y \in X$. Similarly, we know that $A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), 0.5\}$ for all $x, y \in X$. Suppose that $A_F(x) \vee A_F(y) > 0.5$. If $A_F(x*y) > A_F(x) \vee A_F(y) := \gamma_f$, then $x, y \in F_{\in}(A;\gamma_f)$, $x * y \notin F_{\in}(A;\gamma_f)$ and $A_F(x * y) + \gamma_f > 2\gamma_f > 1$, i.e., $x * y \notin F_q(A;\gamma_f)$. This is a contradiction. Hence $A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\}$ whenever $A_F(x) \vee A_F(y) > 0.5$. Now, assume that $A_F(x) \vee A_F(y) \leq 0.5$. Then $x, y \in F_{\in}(A; 0.5)$ and so $x*y \in F_{\in\vee q}(A; 0.5)$. Thus $A_F(x*y) \leq 0.5$ or $A_F(x*y)+0.5 < 1$. If $A_F(x*y) > 0.5$, then $A_F(x * y) + 0.5 > 0.5 + 0.5 = 1$, a contradiction. Thus $A_F(x * y) \leq 0.5$, and so $A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\}$ whenever $A_F(x) \vee A_F(y) \leq 0.5$. Therefore $A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\}$ for all $x, y \in X$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X which satisfies the condition (3.12). Let $x, y \in X$ and α_x, α_y, β_x, β_y, $\gamma_x, \gamma_y \in [0, 1]$. If $x \in T_{\in}(A;\alpha_x)$ and $y \in T_{\in}(A;\alpha_y)$, then $A_T(x) \geq \alpha_x$ and $A_T(y) \geq \alpha_y$. If $A_T(x * y) < \alpha_x \wedge \alpha_y$, then $A_T(x) \wedge A_T(y) \geq 0.5$. Otherwise, we have

$$A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\} = A_T(x) \wedge A_T(y) \geq \alpha_x \wedge \alpha_y,$$

a contradiction. It follows that

$$A_T(x * y) + \alpha_x \wedge \alpha_y > 2A_T(x * y) \geq 2\bigwedge\{A_T(x), A_T(y), 0.5\} = 1$$

and so that $x * y \in T_q(A;\alpha_x \wedge \alpha_y) \subseteq T_{\in\vee q}(A;\alpha_x \wedge \alpha_y)$. Similarly, if $x \in I_{\in}(A;\beta_x)$ and $y \in I_{\in}(A;\beta_y)$, then $x * y \in I_{\in\vee q}(A;\beta_x \wedge \beta_y)$. Now, let $x \in F_{\in}(A;\gamma_x)$ and

$y \in F_{\in}(A; \gamma_y)$. Then $A_F(x) \leq \gamma_x$ and $A_F(y) \leq \gamma_y$. If $A_F(x * y) > \gamma_x \vee \gamma_y$, then $A_F(x) \vee A_F(y) \leq 0.5$ because if not, then

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \leq A_F(x) \vee A_F(y) \leq \gamma_x \vee \gamma_y,$$

which is a contradiction. Hence

$$A_F(x * y) + \gamma_x \vee \gamma_y < 2A_F(x * y) \leq 2\bigvee\{A_F(x), A_F(y), 0.5\} = 1,$$

and so $x * y \in F_q(A; \gamma_x \vee \gamma_y) \subseteq F_{\in\vee\, q}(A; \gamma_x \vee \gamma_y)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q)$-neutrosophic subalgebra of X. □

Theorem 3.7. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q)$-neutrosophic subalgebra of a BCK/BCI-algebra X, then neutrosophic q-subsets $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$ whenever they are nonempty.*

Proof. Assume that $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are nonempty for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$. Let $x, y \in T_q(A; \alpha)$. Then $A_T(x) + \alpha > 1$ and $A_T(y) + \alpha > 1$. It follows from Theorem 3.6 that

$$\begin{aligned} A_T(x * y) + \alpha &\geq \bigwedge\{A_T(x), A_T(y), 0.5\} + \alpha \\ &= \bigwedge\{A_T(x) + \alpha, A_T(y) + \alpha, 0.5 + \alpha\} \\ &> 1, \end{aligned}$$

that is, $x * y \in T_q(A; \alpha)$. Hence $T_q(A; \alpha)$ is a subalgebra of X. By the similar way, we can induce that $I_q(A; \beta)$ is a subalgebra of X. Now, let $x, y \in F_q(A; \gamma)$. Then $A_F(x) + \gamma < 1$ and $A_F(y) + \gamma < 1$. Using Theorem 3.6, we have

$$\begin{aligned} A_F(x * y) + \gamma &\leq \bigvee\{A_F(x), A_F(y), 0.5\} + \gamma \\ &= \bigvee\{A_F(x) + \gamma, A_F(y) + \gamma, 0.5 + \gamma\} \\ &< 1, \end{aligned}$$

and so $x * y \in F_q(A; \gamma)$. Therefore $F_q(A; \gamma)$ is a subalgebra of X. □

Theorem 3.8. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X, if the nonempty neutrosophic $\in\vee\, q$-subsets $T_{\in\vee\, q}(A; \alpha)$, $I_{\in\vee\, q}(A; \beta)$ and $F_{\in\vee\, q}(A; \gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, then $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q)$-neutrosophic subalgebra of X.*

Proof. Let $T_{\in\vee\, q}(A; \alpha)$ be a subalgebra of X and assume that

$$A_T(x * y) < \bigwedge\{A_T(x), A_T(y), 0.5\}$$

for some $x, y \in X$. Then there exists $\alpha \in (0, 0.5]$ such that

$$A_T(x * y) < \alpha \leq \bigwedge\{A_T(x), A_T(y), 0.5\}.$$

It follows that $x, y \in T_{\in}(A; \alpha) \subseteq T_{\in\vee\, q}(A; \alpha)$, and so that $x * y \in T_{\in\vee\, q}(A; \alpha)$. Hence $A_T(x * y) \geq \alpha$ or $A_T(x * y) + \alpha > 1$. This is a contradiction, and so

$$A_T(x * y) \geq \bigwedge\{A_T(x), A_T(y), 0.5\}$$

for all $x, y \in X$. Similarly, we show that

$$A_I(x * y) \geq \bigwedge\{A_I(x), A_I(y), 0.5\}$$

for all $x, y \in X$. Now let $F_{\in\vee\, q}(A;\gamma)$ be a subalgebra of X and assume that

$$A_F(x * y) > \bigvee\{A_F(x), A_F(y), 0.5\}$$

for some $x, y \in X$. Then

$$A_F(x * y) > \gamma \geq \bigvee\{A_F(x), A_F(y), 0.5\}, \tag{3.13}$$

for some $\gamma \in [0.5, 1)$, which implies that $x, y \in F_{\in}(A;\gamma) \subseteq F_{\in\vee\, q}(A;\gamma)$. Thus $x*y \in F_{\in\vee\, q}(A;\gamma)$. From (3.13), we have $x*y \notin F_{\in}(A;\gamma)$ and $A_F(x*y)+\gamma > 2\gamma \geq 1$, i.e., $x * y \notin F_q(A;\gamma)$. This is a contradiction, and hence

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\}$$

for all $x, y \in X$. Using Theorem 3.6, we know that $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q)$-neutrosophic subalgebra of X. □

Theorem 3.9. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q)$-neutrosophic subalgebra of a BCK/BCI-algebra X, then nonempty neutrosophic $\in\vee\, q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$.*

Proof. Assume that $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are nonempty for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$. Let $x, y \in I_{\in\vee\, q}(A;\beta)$. Then

$$x \in I_{\in}(A;\beta) \text{ or } \; x \in I_q(A;\beta),$$

and

$$y \in I_{\in}(A;\beta) \text{ or } \; y \in I_q(A;\beta).$$

Hence we have the following four cases:

(i) $x \in I_{\in}(A;\beta)$ and $y \in I_{\in}(A;\beta)$,
(ii) $x \in I_{\in}(A;\beta)$ and $y \in I_q(A;\beta)$,
(iii) $x \in I_q(A;\beta)$ and $y \in I_{\in}(A;\beta)$,
(iv) $x \in I_q(A;\beta)$ and $y \in I_q(A;\beta)$.

The first case implies that $x * y \in I_{\in\vee\, q}(A;\beta)$. For the second case, $y \in I_q(A;\beta)$ induces $A_I(y) > 1-\beta \geq \beta$, that is, $y \in I_{\in}(A;\beta)$. Thus $x*y \in I_{\in\vee\, q}(A;\beta)$. Similarly, the third case implies $x*y \in I_{\in\vee\, q}(A;\beta)$. The last case induces $A_I(x) > 1-\beta \geq \beta$ and $A_I(y) > 1-\beta \geq \beta$, that is, $x \in I_{\in}(A;\beta)$ and $y \in I_{\in}(A;\beta)$. Hence $x*y \in I_{\in\vee\, q}(A;\beta)$. Therefore $I_{\in\vee\, q}(A;\beta)$ is a subalgebra of X for all $\beta \in (0, 0.5]$. By the similar way, we show that $T_{\in\vee\, q}(A;\alpha)$ is a subalgebra of X for all $\alpha \in (0, 0.5]$. Let $x, y \in F_{\in\vee\, q}(A;\gamma)$. Then

$$A_F(x) \leq \gamma \text{ or } \; A_F(x) + \gamma < 1,$$

and

$$A_F(y) \leq \gamma \text{ or } \; A_F(y) + \gamma < 1.$$

If $A_F(x) \leq \gamma$ and $A_F(y) \leq \gamma$, then

$$A_F(x * y) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \leq \bigvee\{\gamma, 0.5\} = \gamma$$

by Theorem 3.6, and so $x*y \in F_{\in}(A;\gamma) \subseteq F_{\in\vee q}(A;\gamma)$. If $A_F(x) \leq \gamma$ and $A_F(y)+\gamma < 1$, then

$$A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \leq \bigvee\{\gamma, 1-\gamma, 0.5\} = \gamma$$

by Theorem 3.6. Thus $x*y \in F_{\in}(A;\gamma) \subseteq F_{\in\vee q}(A;\gamma)$. Similarly, if $A_F(x)+\gamma < 1$ and $A_F(y) \leq \gamma$, then $x*y \in F_{\in\vee q}(A;\gamma)$. Finally, assume that $A_F(x)+\gamma < 1$ and $A_F(y)+\gamma < 1$. Then

$$A_F(x*y) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \leq \bigvee\{1-\gamma, 0.5\} = 0.5 < \gamma$$

by Theorem 3.6. Hence $x*y \in F_{\in}(A;\gamma) \subseteq F_{\in\vee q}(A;\gamma)$. Consequently, $F_{\in\vee q}(A;\gamma)$ is a subalgebra of X for all $\gamma \in [0.5, 1)$. □

Theorem 3.10. *If $A = (A_T, A_I, A_F)$ is a $(q, \in\vee q)$-neutrosophic subalgebra of a BCK/BCI-algebra X, then nonempty neutrosophic $\in\vee q$-subsets $T_{\in\vee q}(A;\alpha)$, $I_{\in\vee q}(A;\beta)$ and $F_{\in\vee q}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$.*

Proof. Assume that $T_{\in\vee q}(A;\alpha)$, $I_{\in\vee q}(A;\beta)$ and $F_{\in\vee q}(A;\gamma)$ are nonempty for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$. Let $x, y \in T_{\in\vee q}(A;\alpha)$. Then

$$x \in T_{\in}(A;\alpha) \text{ or } x \in T_q(A;\alpha).$$

and

$$y \in T_{\in}(A;\alpha) \text{ or } y \in T_q(A;\alpha).$$

If $x \in T_q(A;\alpha)$ and $y \in T_q(A;\alpha)$, then obviously $x*y \in T_{\in\vee q}(A;\alpha)$. Suppose that $x \in T_{\in}(A;\alpha)$ and $y \in T_q(A;\alpha)$. Then $A_T(x)+\alpha \geq 2\alpha > 1$, i.e., $x \in T_q(A;\alpha)$. It follows that $x*y \in T_{\in\vee q}(A;\alpha)$. Similarly, if $x \in T_q(A;\alpha)$ and $y \in T_{\in}(A;\alpha)$, then $x*y \in T_{\in\vee q}(A;\alpha)$. Now, let $x, y \in F_{\in\vee q}(A;\gamma)$. Then

$$x \in F_{\in}(A;\gamma) \text{ or } x \in F_q(A;\gamma),$$

and

$$y \in F_{\in}(A;\gamma) \text{ or } y \in F_q(A;\gamma).$$

If $x \in F_q(A;\gamma)$ and $y \in F_q(A;\gamma)$, then clearly $x*y \in F_{\in\vee q}(A;\gamma)$. If $x \in F_{\in}(A;\gamma)$ and $y \in F_q(A;\gamma)$, then $A_F(x)+\gamma \leq 2\gamma < 1$, i.e., $x \in F_q(A;\gamma)$. It follows that $x*y \in F_{\in\vee q}(A;\gamma)$. Similarly, if $x \in F_q(A;\gamma)$ and $y \in F_{\in}(A;\gamma)$, then $x*y \in F_{\in\vee q}(A;\gamma)$. Finally, assume that $x \in F_{\in}(A;\gamma)$ and $y \in F_{\in}(A;\gamma)$. Then $A_F(x)+\gamma \leq 2\gamma < 1$ and $A_F(y)+\gamma \leq 2\gamma < 1$, that is, $x \in F_q(A;\gamma)$ and $y \in F_q(A;\gamma)$. Therefore $x*y \in F_{\in\vee q}(A;\gamma)$. Consequently, $T_{\in\vee q}(A;\alpha)$, $I_{\in\vee q}(A;\beta)$ and $F_{\in\vee q}(A;\gamma)$ are subalgebras of X for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$. □

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set X, we consider:

$$X_0^1 := \{x \in X \mid A_T(x) > 0,\ A_I(x) > 0,\ A_F(x) < 1\}.$$

Theorem 3.11. *If a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in, \in)$-neutrosophic subalgebra of X, then the set X_0^1 is a subalgebra of X.*

Proof. Let $x, y \in X_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$ and $A_F(y) < 1$. Suppose that $A_T(x*y) = 0$. Note that $x \in T_\in(A; A_T(x))$ and $y \in T_\in(A; A_T(y))$. But $x * y \notin T_\in(A; A_T(x) \wedge A_T(y))$ because $A_T(x * y) = 0 < A_T(x) \wedge A_T(y)$. This is a contradiction, and thus $A_T(x * y) > 0$. By the similar way, we show that $A_I(x * y) > 0$. Note that $x \in F_\in(A; A_F(x))$ and $y \in F_\in(A; A_F(y))$. If $A_F(x * y) = 1$, then $A_F(x * y) = 1 > A_F(x) \vee A_F(y)$, and so $x * y \notin F_\in(A; A_F(x) \vee A_F(y))$. This is impossible. Hence $x * y \in X_0^1$, and therefore X_0^1 is a subalgebra of X. □

Theorem 3.12. *If a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is an $(\in, q)$-neutrosophic subalgebra of X, then the set X_0^1 is a subalgebra of X.*

Proof. Let $x, y \in X_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$ and $A_F(y) < 1$. If $A_T(x * y) = 0$, then

$$A_T(x * y) + A_T(x) \wedge A_T(y) = A_T(x) \wedge A_T(y) \leq 1.$$

Hence $x*y \notin T_q(A; A_T(x) \wedge A_T(y))$, which is a contradiction since $x \in T_\in(A; A_T(x))$ and $y \in T_\in(A; A_T(y))$. Thus $A_T(x*y) > 0$. Similarly, we get $A_I(x*y) > 0$. Assume that $A_F(x * y) = 1$. Then

$$A_F(x * y) + A_F(x) \vee A_F(y) = 1 + A_F(x) \vee A_F(y) \geq 1,$$

that is, $x * y \notin F_q(A; A_F(x) \vee A_F(y))$. This is a contradiction because of $x \in F_\in(A; A_F(x))$ and $y \in F_\in(A; A_F(y))$. Hence $A_F(x*y) < 1$. Consequently, $x*y \in X_0^1$ and X_0^1 is a subalgebra of X. □

Theorem 3.13. *If a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is a $(q, \in)$-neutrosophic subalgebra of X, then the set X_0^1 is a subalgebra of X.*

Proof. Let $x, y \in X_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$ and $A_F(y) < 1$. It follows that $A_T(x)+1 > 1$, $A_T(y)+1 > 1$, $A_I(x)+1 > 1$, $A_I(y)+1 > 1$, $A_F(x)+0 < 1$ and $A_F(y)+0 < 1$. Hence $x, y \in T_q(A; 1) \cap I_q(A; 1) \cap F_q(A; 0)$. If $A_T(x*y) = 0$ or $A_I(x*y) = 0$, then $A_T(x*y) < 1 = 1 \wedge 1$ or $A_I(x*y) < 1 = 1 \wedge 1$. Thus $x*y \notin T_q(A; 1 \wedge 1)$ or $x*y \notin I_q(A; 1 \wedge 1)$, a contradiction. Hence $A_T(x*y) > 0$ and $A_I(x*y) > 0$. If $A_F(x*y) = 1$, then $x*y \notin F_q(A; 0 \vee 0)$ which is a contradiction. Thus $A_F(x * y) < 1$. Therefore $x * y \in X_0^1$ and the proof is complete. □

Theorem 3.14. *If a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is a (q, q)-neutrosophic subalgebra of X, then the set X_0^1 is a subalgebra of X.*

Proof. Let $x, y \in X_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$ and $A_F(y) < 1$. Hence $A_T(x)+1 > 1$, $A_T(y)+1 > 1$, $A_I(x)+1 > 1$, $A_I(y)+1 > 1$, $A_F(x) + 0 < 1$ and $A_F(y) + 0 < 1$. Hence $x, y \in T_q(A; 1) \cap I_q(A; 1) \cap F_q(A; 0)$. If $A_T(x * y) = 0$ or $A_I(x * y) = 0$, then

$$A_T(x * y) + 1 \wedge 1 = 0 + 1 = 1$$

or

$$A_I(x * y) + 1 \wedge 1 = 0 + 1 = 1,$$

and so $x * y \notin T_q(A; 1 \wedge 1)$ or $x * y \notin I_q(A; 1 \wedge 1)$. This is impossible, and thus $A_T(x * y) > 0$ and $A_I(x * y) > 0$. If $A_F(x * y) = 1$, then $A_F(x * y) + 0 \vee 0 = 1$, that is, $x * y \notin F_q(A; 0 \vee 0)$. This is a contradiction, and so $A_F(x * y) < 1$. Therefore $x * y \in X_0^1$ and the proof is complete. □

Theorem 3.15. *If a neutrosophic set $A = (A_T, A_I, A_F)$ in a BCK/BCI-algebra X is a (q, q)-neutrosophic subalgebra of X, then $A = (A_T, A_I, A_F)$ is neutrosophic constant on X_0^1, that is, A_T, A_I and A_F are constants on X_0^1.*

Proof. Assume that A_T is not constant on X_0^1. Then there exist $y \in X_0^1$ such that $\alpha_y = A_T(y) \neq A_T(0) = \alpha_0$. Then either $\alpha_y > \alpha_0$ or $\alpha_y < \alpha_0$. Suppose $\alpha_y < \alpha_0$ and choose $\alpha_1, \alpha_2 \in (0, 1]$ such that $1 - \alpha_0 < \alpha_1 \leq 1 - \alpha_y < \alpha_2$. Then $A_T(0) + \alpha_1 = \alpha_0 + \alpha_1 > 1$ and $A_T(y) + \alpha_2 = \alpha_y + \alpha_2 > 1$, which imply that $0 \in T_q(A; \alpha_1)$ and $y \in T_q(A; \alpha_2)$. Since

$$A_T(y * 0) + \alpha_1 \wedge \alpha_2 = A_T(y) + \alpha_1 = \alpha_y + \alpha_1 \leq 1,$$

we get $y * 0 \notin T_q(A; \alpha_1 \wedge \alpha_2)$, which is a contradiction. Next assume that $\alpha_y > \alpha_0$. Then $A_T(y) + (1 - \alpha_0) = \alpha_y + 1 - \alpha_0 > 1$ and so $y \in T_q(A; 1 - \alpha_0)$. Since

$$A_T(y * y) + (1 - \alpha_0) = A_T(0) + 1 - \alpha_0 = \alpha_0 + 1 - \alpha_0 = 1,$$

we have $y * y \notin T_q(A; (1 - \alpha_0) \wedge (1 - \alpha_0))$. This is impossible. Therefore A_T is constant on X_0^1. Similarly, A_I is constant on X_0^1. Finally, suppose that A_F is not constant on X_0^1. Then $\gamma_y = A_F(y) \neq A_F(0) = \gamma_0$ for some $y \in X_0^1$, and we have two cases:

$$\text{(i) } \gamma_y < \gamma_0 \text{ and (ii) } \gamma_y > \gamma_0.$$

The first case implies that $A_F(y) + 1 - \gamma_0 = \gamma_y + 1 - \gamma_0 < 1$, that is, $y \in F_q(A; 1 - \gamma_0)$. Hence $y * y \in F_q(A; (1 - \gamma_0) \vee (1 - \gamma_0))$, i.e., $0 \in F_q(A; 1 - \gamma_0)$, which is a contradiction since $A_F(0) + 1 - \gamma_0 = 1$. For the second case, there exist $\gamma_1, \gamma_2 \in (0, 1)$ such that

$$1 - \gamma_0 > \gamma_1 > 1 - \gamma_y > \gamma_2.$$

Then $A_F(y) + \gamma_2 = \gamma_y + \gamma_2 < 1$, i.e., $y \in F_q(A; \gamma_2)$, and $A_F(0) + \gamma_1 = \gamma_0 + \gamma_1 < 1$, i.e., $0 \in F_q(A; \gamma_1)$. It follows that $y * 0 \in F_q(A; \gamma_1 \vee \gamma_2)$. But

$$A_F(y * 0) + \gamma_1 \vee \gamma_2 = A_F(y) + \gamma_1 = \gamma_y + \gamma_1 > 1,$$

and so $y * 0 \notin F_q(A; \gamma_1 \vee \gamma_2)$. This is a contradiction. Therefore A_F is constant on X_0^1. This completes the proof. □

We provide conditions for a neutrosophic set to be a $(q, \in\vee\, q)$-neutrosophic subalgebra.

Theorem 3.16. *For a subalgebra S of a BCK/BCI-algebra X, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in X such that*

$$(\forall x \in S)\,(A_T(x) \geq 0.5, A_I(x) \geq 0.5, A_F(x) \leq 0.5)\,, \tag{3.14}$$

$$(\forall x \in X \setminus S)\,(A_T(x) = 0, A_I(x) = 0, A_F(x) = 1)\,. \tag{3.15}$$

Then $A = (A_T, A_I, A_F)$ is a $(q, \in\vee\, q)$-neutrosophic subalgebra of X.

Proof. Assume that $x \in I_q(A; \beta_x)$ and $y \in I_q(A; \beta_y)$ for all $x, y \in X$ and $\beta_x, \beta_y \in [0, 1]$. Then $A_I(x) + \beta_x > 1$ and $A_I(y) + \beta_y > 1$. If $x * y \notin S$, then $x \in X \setminus S$ or $y \in X \setminus S$ since S is a subalgebra of X. Hence $A_I(x) = 0$ or $A_I(y) = 0$, which imply that $\beta_x > 1$ or $\beta_y > 1$. This is a contradiction, and so $x * y \in S$. If $\beta_x \wedge \beta_y > 0.5$, then $A_I(x*y)+\beta_x\wedge\beta_y>1$, i.e., $x*y\in I_q(A;\beta_x\wedge\beta_y)$. If $\beta_x\wedge\beta_y\leq 0.5$, then $A_I(x*y)\geq 0.5\geq\beta_x\wedge\beta_y$, i.e., $x*y\in I_\in(A;\beta_x\wedge\beta_y)$. Hence $x*y\in I_{\in\vee\, q}(A;\beta_x\wedge\beta_y)$. Similarly, if $x\in T_q(A;\alpha_x)$ and $y\in T_q(A;\alpha_y)$ for all $x,y\in X$ and $\alpha_x,\alpha_y\in[0,1]$, then $x*y\in T_{\in\vee\, q}(A;\alpha_x\wedge\alpha_y)$. Now let $x,y\in X$ and $\gamma_x,\gamma_y\in[0,1]$ be such that $x\in F_q(A;\gamma_x)$ and $y\in F_q(A;\gamma_y)$. Then $A_F(x)+\gamma_x<1$ and $A_F(y)+\gamma_y<1$. It follows that $x*y\in S$. In fact, if not then $x\in X\setminus S$ or $y\in X\setminus S$ since S is a subalgebra of X. Hence $A_F(x)=1$ or $A_F(y)=1$, which imply that $\gamma_x<0$ or $\gamma_y<0$. This is a contradiction, and so $x*y\in S$. If $\gamma_x\vee\gamma_y\geq 0.5$, then $A_F(x*y)\leq 0.5\leq\gamma_x\vee\gamma_y$, that is, $x*y\in F_\in(A;\gamma_x\vee\gamma_y)$. If $\gamma_x\vee\gamma_y<0.5$, then $A_F(x*y)+\gamma_x\vee\gamma_y<1$, that is, $x*y\in F_q(A;\gamma_x\vee\gamma_y)$. Hence $x*y\in F_{\in\vee\, q}(A;\gamma_x\vee\gamma_y)$, and consequently $A=(A_T,A_I,A_F)$ is a $(q,\in\vee\, q)$-neutrosophic subalgebra of X. □

Combining Theorems 3.5 and 3.16, we have the following corollary.

Corollary 3.17. *For a subalgebra S of X, if $A=(A_T,A_I,A_F)$ is a neutrosophic set in X satisfying conditions (3.14) and (3.15), then $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subalgebras of X for all $\alpha,\beta\in(0.5,1]$ and $\gamma\in[0,0,5)$ whenever they are nonempty.*

Theorem 3.18. *Let $A=(A_T,A_I,A_F)$ be a $(q,\in\vee\, q)$-neutrosophic subalgebra of X in which A_T, A_I and A_F are not constant on X_0^1. Then there exist $x,y,z\in X$ such that $A_T(x)\geq 0.5$, $A_I(y)\geq 0.5$ and $A_F(z)\leq 0.5$. In particular, $A_T(x)\geq 0.5$, $A_I(y)\geq 0.5$ and $A_F(z)\leq 0.5$ for all $x,y,z\in X_0^1$.*

Proof. Assume that $A_T(x)<0.5$ for all $x\in X$. Since there exists $a\in X_0^1$ such that $\alpha_a=A_T(a)\neq A_T(0)=\alpha_0$, we have $\alpha_a>\alpha_0$ or $\alpha_a<\alpha_0$. If $\alpha_a>\alpha_0$, then we can choose $\delta>0.5$ such that $\alpha_0+\delta<1<\alpha_a+\delta$. It follows that $a\in T_q(A;\delta)$, $A_T(a*a)=A_T(0)=\alpha_0<\delta=\delta\wedge\delta$ and $A_T(a*a)+\delta\wedge\delta=A_T(0)+\delta=\alpha_0+\delta<1$ so that $a*a\notin T_{\in\vee\, q}(A;\delta\wedge\delta)$. This is a contradiction. Now if $\alpha_a<\alpha_0$, we can take $\delta>0.5$ such that $\alpha_a+\delta<1<\alpha_0+\delta$. Then $0\in T_q(A;\delta)$ and $a\in T_q(A;1)$, but $a*0\notin T_{\in\vee\, q}(A;1\wedge\delta)$ since $A_T(a)<0.5<\delta$ and $A_T(a)+\delta=\alpha_a+\delta<1$. This is also a contradiction. Thus $A_T(x)\geq 0.5$ for some $x\in X$. Similarly, we know that $A_I(y)\geq 0.5$ for some $y\in X$. Finally, suppose that $A_F(z)>0.5$ for all $z\in X$. Note that $\gamma_c=A_F(c)\neq A_F(0)=\gamma_0$ for some $c\in X_0^1$. It follows that $\gamma_c<\gamma_0$ or $\gamma_c>\gamma_0$. We first consider the case $\gamma_c<\gamma_0$. Then $\gamma_0+\varepsilon>1>\gamma_c+\varepsilon$ for some $\varepsilon\in[0,0.5)$, and so $c\in F_q(A;\varepsilon)$. Also $A_F(c*c)=A_F(0)=\gamma_0>\varepsilon$ and $A_F(c*c)+\varepsilon\vee\varepsilon=A_F(0)+\varepsilon=\gamma_0+\varepsilon>1$ which shows that $c*c\notin F_{\in\vee\, q}(A;\varepsilon\vee\varepsilon)$. This is impossible. Now, if $\gamma_c>\gamma_0$, then we can take $\varepsilon\in[0,0.5)$ and so that $\gamma_0+\varepsilon<1<\gamma_c+\varepsilon$. It follows that $0\in F_q(A;\varepsilon)$ and $c\in F_q(A;0)$. Since $A_F(c*0)=A_F(c)=\gamma_c>\varepsilon$ and $A_F(c*0)+\varepsilon=A_F(c)+\varepsilon=\gamma_c+\varepsilon>1$, we have $c*0\notin F_{\in\vee\, q}(A;\varepsilon)$. This is a contradiction, and therefore $A_F(z)<0.5$ for some $z\in X$. We now show that $A_T(0)\geq 0.5$, $A_I(0)\geq 0.5$ and $A_F(0)\leq 0.5$. Suppose that $A_T(0)=\alpha_0<0.5$. Since there exists $x\in X$ such that $A_T(x)=\alpha_x\geq 0.5$, it follows that $\alpha_0<\alpha_x$. Choose $\alpha_1\in[0,1]$ such that $\alpha_1>\alpha_0$ and $\alpha_0+\alpha_1<1<\alpha_x+\alpha_1$. Then $A_T(x)+\alpha_1=\alpha_x+\alpha_1>1$, and so $x\in T_q(A;\alpha_1)$. Now we have $A_T(x*x)+\alpha_1\wedge\alpha_1=A_T(0)+\alpha_1=\alpha_0+\alpha_1<1$ and $A_T(x*x)=A_T(0)=\alpha_0<\alpha_1=\alpha_1\wedge\alpha_1$. Thus $x*x\notin T_{\in\vee\, q}(A;\alpha_1\wedge\alpha_1)$, a contradiction. Hence $A_T(0)\geq 0.5$. Similarly, we have $A_I(0)\geq 0.5$. Assume that $A_F(0)=\gamma_0>0.5$. Note that $A_F(z)=\gamma_z\leq 0.5$ for some $z\in X$. Hence $\gamma_z<\gamma_0$, and so we can take $\gamma_1 \in [0, 1]$ such that $\gamma_1 < \gamma_0$ and $\gamma_0 + \gamma_1 > 1 > \gamma_z + \gamma_1$. It follows that $A_F(z) + \gamma_1 = \gamma_z + \gamma_1 < 1$, that is, $z \in F_q(A; \gamma_1)$. Also $A_F(z*z) = A_F(0) = \gamma_0 > \gamma_1 = \gamma_1 \vee \gamma_1$, i.e., $z*z \notin F_\in(A; \gamma_1 \vee \gamma_1)$, and $A_F(z*z) + \gamma_1 \vee \gamma_1 = A_F(0) + \gamma_1 = \gamma_0 + \gamma_1 > 1$, i.e., $z * z \notin F_q(A; \gamma_1 \vee \gamma_1)$. Thus $z * z \notin F_{\in\vee\, q}(A; \gamma_1 \vee \gamma_1)$, a contradiction. Hence $A_F(0) \leq 0.5$. We finally show that $A_T(x) \geq 0.5$, $A_I(y) \geq 0.5$ and $A_F(z) \leq 0.5$ for all $x, y, z \in X_0^1$. We first assume that $A_I(y) = \beta_y < 0.5$ for some $y \in X_0^1$, and take $\beta > 0$ such that $\beta_y + \beta < 0.5$. Then $A_I(y) + 1 = \beta_y + 1 > 1$ and $A_I(0) + \beta + 0.5 > 1$, which imply that $y \in I_q(A; 1)$ and $0 \in I_q(A; \beta + 0.5)$. But $y * 0 \notin I_{\in\vee\, q}(A; \beta + 0.5)$ since $A_I(y * 0) = A_I(y) < \beta + 0.5 < 1 \wedge (\beta + 0.5)$ and $A_I(y*0) + 1 \wedge (\beta + 0.5) = A_I(y) + \beta + 0.5 = \beta_y + \beta + 0.5 < 1$. This is a contradiction. Hence $A_I(y) \geq 0.5$ for all $y \in X_0^1$. Similarly, we induces $A_T(x) \geq 0.5$ for all $x \in X_0^1$. Suppose $A_F(z) = \gamma_z > 0.5$ for some $z \in X_0^1$, and take $\gamma \in (0, 0.5)$ such that $\gamma_z > 0.5 + \gamma$. Then $z \in F_q(A; 0)$ and $A_F(0) + 0.5 - \gamma \leq 1 - \gamma < 1$, i.e., $0 \in F_q(A; 0.5 - \gamma)$. But $A_F(z*0) = A_F(z) > 0.5 > 0.5 - \gamma$ and $A_F(z*0) + 0.5 - \gamma = A_F(z) + 0.5 - \gamma = \gamma_z + 0.5 - \gamma > 1$, which imply that $z * 0 \notin F_{\in\vee\, q}(A; 0.5 - \gamma)$. This is a contradiction, and the proof is complete. □

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