A neutrosophic set-based computational model for a time-dependent decision-support system with multi-attribute criteria

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Abstract

We present a neutrosophic set-based model for a time-dependent decision-support system (DSS) with multi-attribute criteria decision-making. We describe such DSS as one that includes multiple conflicting objectives having strategies spanning over several time-periods. In this paper, we utilize the concept of neutrosophic sets and some of its operations to present a computational model that captures decision trees with various imprecise preferences for a time-dependent DSS. Given a time-dependent DSS with (a countable number of) N objectives spanning over time-periods ranging from $t = t_0$ to $t = t_n$, we are able to use a set of m attributes, denoted by variables $a_1, \ldots, a_m$, where each variable a_k ($k = 1, \ldots, m$), for each $t \in [t_0, t_n]$, is described by a triplet variable $x_k(\tau_{k_t}, i_{k_t}, f_{k_t})$, with the terms τ_{k_t}, i_{k_t}, and f_{k_t} defined as degrees of truthfulness membership, indeterminacy membership, and falsity membership for attribute a_k at time t, respectively. We then define a set of m time-dependent vectors of imprecise consequences S_q corresponding to a set of strategies, denoted by $S = \{s_k(t)\}$, derived from the membership of each attribute a_k. For each time t, we normalize the set of imprecise consequences to define the weighted values for each attribute. We conclude with an interpretation and a sensitivity analysis of the results to account for the influence of the decision-maker in the model.

Keywords: Neutrosophic set, neutrosophic logic, single-valued neutrosophic set, geometric operator score, set-theoretic model theory, decision theory.
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1. Introduction

1.1. Reflecting on decision-support systems computational modeling literature

The challenges in decision-making about complex problems that are rapidly changing and not easily specified in advance has led to the development of model-driven decision-support systems (DSS)[1]. Model-driven DSS can be described as complex systems in which a set of specific required data (or attributes) are carefully studied and analyzed to develop multiple sets of strategies, which allows the decision-maker to select the most efficient set of strategies that achieves a predefined set of objectives. That said, model-driven DSS have evolved over the past three decades from simple model-oriented systems to advanced multi-function entities[2]. This stems from the continuous evolution of the necessary computational modeling required to solve most of these challenges. As a result, the growing need for decision-support in such problems has shaped most DSS models to be robustly designed specifically to[3 4 5]:

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- facilitate dynamic decision processes through the use of a series of attributes to reach a set of strategies.
  - Interdependency between each of the attributes and their consequent strategies is accounted for and supported.
  - Each strategy is then capable of affecting future attributes and/or strategies at some later time.
- support rather than just automate decision-making.
  - Some DSS models will make decisions based on predefined rules in the environment where the DSS operates. However, in the most complex situations, many of those rules may not be as relevant as originally thought of. Thus, the best DSS models are also capable of presenting the decision-maker with relevant information about each strategy and how it aligns with the corresponding objective(s). In the end, the decision-maker will select the best set of strategies.
- be able to respond quickly to the changing needs of decision-makers.
  - DSS models must support a body of knowledge for the DSS[4 5]; which is best described as allowing a record-keeping capability that can present information on an ad hoc basis in both standardized and customized reports, a capability for selecting a desired subset of stored strategies for either presentation or for deriving new strategies, and must be designed to interact directly with a decision-maker in such a way that the user has a flexible choice and sequence of knowledge-management activities[5 6].

In light of those stated specifications above, efficient model-driven DSS serve as critical tools for the decision-maker, in that they facilitate him/her with the ability to select the best strategies based on any given set of attributes in the present, foresee consequences in the future, and present ad hoc information on each strategy's alignment with each of the given objectives at any given time. Moreover, in more complex situations, such as having multiple conflicting objectives spanning over several time-periods, ranging from t_0 to t_n, and a predefined set of attributes (and the weights and uncertainty surrounding them) that are allowed to change over time; model-driven DSS have been developed to help identify optimal strategies by using *interactive multi-objective simulated annealing*[1] or *imprecise multi-attribute additive modeling*[7]. In both cases, each attribute is assigned an absolute weight value so that at each point in time, the decision-maker is able to evaluate its importance or preference over other attributes at that point in time. In contrast, each attribute can also be set to a different weight value at each time t, ($t \in [t_0, t_n]$), as more data become available[8]. Either way, it becomes very difficult to assess each attribute (whether by the decision-maker's preference; the attribute's importance; or the attribute's relevance, at time t) when presented with a sufficiently large amount of attributes or time units within the $[t_0, t_n]$ interval. Consequently, since most multi-attribute DSS models use a finite large amount of attributes (time-dependent or not), we approach this problem using a neutrosophic set-based computational model, as described in more detail in section 2. In our model, rather than restricting each individual attribute to a weighting value, we first define three categories of membership (truthfulness τ, indeterminacy i, and falsity f); then, we assign each attribute a time-dependent triplet variable $x(\tau_t, i_t, f_t)$, meaning whether that attribute holds true, indeterminate, or false at time t; and a strategy is then developed based on the membership of that attribute at time t. This allows us to only need to know what the membership of the attribute is at time t, and therefore affords us the ability to bypass the need to evaluate each attribute individually by preference, importance, or relevance, at each time t, as mentioned above. Such model is a neutrosophic set-based model, derived from the neutrosophic theory. The next subsection (Subsection 1.2) highlights the main properties of neutrosophic logic and derived neutrosophic sets deemed necessary for the development of our model.

1.2. Reflecting on neutrosophic sets literature

Neutrosophic sets derive from neutrosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra[9]. Since decision-making involves the analysis of a finite set of alternatives described in terms of evaluative criteria, neutrosophic sets can be useful in the development of DSS models. As a result, we define the general concepts and operations on neutrosophic sets.

1.2.1. General concepts of neutrosophic sets

Definition 1.1. ***(Neutrosophic set)***[10] *Let A be a subset of a universe of discourse $\mathbb{U}$. Each element $x \in \mathbb{U}$ has degrees of truthful membership, indeterminacy membership, and falsity membership in A, which are subsets of the hyperreal interval $]^{-}0, 1^{+}[$. The notation $x(\tau, i, f) \in A$ means that*

- *the degree of truthfulness of x in A is the neutrosophic component τ;*
- *the degree of indeterminacy of x in A is the neutrosophic component i;*
- *the degree of falsity of x in A is the neutrosophic component f.*

A is called neutrosophic set, whereas τ, i, f are called neutrosophic components of the element x with respect to A.

From Definition 1.1, the *superior sum* of the scalar neutrosophic components, namely n_{sup}, is defined as $n_{sup} = \sup(\tau) + \sup(i) + \sup(f)$ and may be as high as 3 or 3^{+}, while the *inferior sum* of the components is defined as $n_{inf} = \inf(\tau) + \inf(i) + \inf(f)$, which may be as low as $^{-}0$ or 0. The notion of neutrosophic set was introduced by Florentin Smarandache in 1995 as a generalization of intuitionistic fuzzy set[9,10] when $n_{sup} = 1$, of intuitionistic set[11] when $n_{sup} < 1$, and of paraconsistent set[12] when $n_{sup} > 1$. The main distinctions between neutrosophic sets and intuitionistic fuzzy sets are the facts that (a) in neutrosophic sets, n_{sup} does not necessarily equal to 1, and can be any number in the range $]^{-}0, 3^{+}[$ in order to allow the characterization of incomplete or paraconsistent information; and (b) in neutrosophic sets, one uses the non-standard interval $]^{-}0, 1^{+}[$ for the neutrosophic components in order to differentiate between absolute membership (denoted by 1^{+}) and relative membership (denoted by 1), while the standard interval $[0, 1]$ is used in intuitionistic fuzzy sets[9].

Definition 1.2. ***(Complement)***[9] *Given a neutrosophic set A, for all $x \in \mathbb{U}$ such that $x(\tau_A, i_A, f_A) \in A$, the complement of A, denoted c_A, is defined by*

$$\tau_{c_A} = 1 - \tau_A; \tag{1}$$
$$i_{c_A} = 1 - i_A; \tag{2}$$
$$f_{c_A} = 1 - f_A. \tag{3}$$

Definition 1.3. ***(Containment)*** *Given neutrosophic sets A_1 and A_2, for all $x \in \mathbb{U}$ such that $x(\tau_{A1}, i_{A1}, f_{A1}) \in A_1$ and $x(\tau_{A2}, i_{A2}, f_{A2}) \in A_2$, $A_1 \subseteq A_2$ if and only if*

$$\inf(\tau_{A_1}) \leq \inf(\tau_{A_2}), \sup(\tau_{A_1}) \leq \sup(\tau_{A_2}); \tag{4}$$
$$\inf(f_{A_1}) \geq \inf(f_{A_2}), \sup(f_{A_1}) \geq \sup(f_{A_2}). \tag{5}$$

Definition 1.4. ***(Union)*** *Given neutrosophic sets A_1 and A_2, for all $x \in \mathbb{U}$ such that $x(\tau_{A1}, i_{A1}, f_{A1}) \in A_1$ and $x(\tau_{A2}, i_{A2}, f_{A2}) \in A_2$, the neutrosophic components of x with respect to the union $A_3 = A_1 \cup A_2$ are defined as*

$$\tau_{A_3} = \tau_{A_1} + \tau_{A_2} - \tau_{A_1} \times \tau_{A_2}; \tag{6}$$

$$i_{A_3} = i_{A_1} + i_{A_2} - i_{A_1} \times i_{A_2}; \tag{7}$$

$$f_{A_3} = f_{A_1} + f_{A_2} - f_{A_1} \times f_{A_2}. \tag{8}$$

Definition 1.5. ***(Intersection)*** *Given neutrosophic sets A_1 and A_2, for all $x \in \mathbb{U}$ such that $x(\tau_{A1}, i_{A1}, f_{A1}) \in A_1$ and $x(\tau_{A2}, i_{A2}, f_{A2}) \in A_2$, the neutrosophic components of x with respect to the intersection $A_3 = A_1 \cap A_2$ are defined as*

$$\tau_{A_3} = \tau_{A_1} \times \tau_{A_2}; \tag{9}$$

$$i_{A_3} = i_{A_1} \times i_{A_2}; \tag{10}$$

$$f_{A_3} = f_{A_1} \times f_{A_2}. \tag{11}$$

Definition 1.6. ***(Single-valued neutrosophic set)***[9 13] *Let $u \subset \mathbb{U}$ be a space of points (or objects), given a neutrosophic set A in u, for all $x \in u$ such that $x(\tau, i, f) \in A$, A is a single-valued neutrosophic set (SVNS) if and only if τ, i, and $f \in [0, 1]$.*

From Definition 1.6, there is no restriction on the n_{inf} and n_{sup} values, which may be as low as 0 and as high as 3, respectively. Moreover, if u is continuous, for all $x \in u$, the SVNS A can be written as

$$A = \left\{ \int_u \langle x : \tau_A, i_A, f_A \rangle, x \in u \right\} \tag{12}$$

Otherwise, if u is discrete, again, for all $x \in u$, A become

$$A = \left\{ \sum_{j=1}^{N} \langle x_j : \tau_A, i_A, f_A \rangle, x_j \in u \right\} \tag{13}$$

For simplicity, we use the simple notation for a SVNS A using Definition 1.6: $A = \{\langle x : \tau, i, f \rangle, x \in u\}$.

1.2.2. Operations on single-valued neutrosophic sets

The following definitions highlight the set-theoretic operations on SVNSs.

Definition 1.7. [13] *Given a SVNS $A = \{\langle x : \tau_A, i_A, f_A \rangle, x \in u\}$; then,*

1. *the complement of A is given by*

$$c_A = \{\langle x : f_A, 1 - i_A, \tau_A \rangle, x \in u\}. \tag{14}$$

2. *for $\lambda > 0$, we have*

$$\lambda \times A = \{\langle x : 1 - (1 - \tau_A)^\lambda, i_A^\lambda, f_A^\lambda \rangle, x \in u\}; \tag{15}$$

$$A^\lambda = \{\langle x : \tau_A^\lambda, 1 - (1 - i_A)^\lambda, 1 - (1 - f_A)^\lambda \rangle, x \in u\}. \tag{16}$$

Definition 1.8. [9 13 14] *Given 2 SVNSs $A_1 = \{\langle x : \tau_{A_1}, i_{A_1}, f_{A_1} \rangle, x \in u\}$ and $A_2 = \{\langle x : \tau_{A_2}, i_{A_2}, f_{A_2} \rangle, x \in u\}$; then,*

1. *$A_1 \subseteq A_2$ if and only if*

$$\tau_{A_1} \leq \tau_{A_2}, i_{A_1} \geq i_{A_2}, f_{A_1} \geq f_{A_2}. \tag{17}$$

2. *$A_1 = A_2$ if and only if*

$$\tau_{A_1} = \tau_{A_2}, i_{A_1} = i_{A_2}, f_{A_1} = f_{A_2}. \tag{18}$$

3. *$A_3 = A_1 \cup A_2$ is defined by*

$$A_3 = \{\langle x : \max(\tau_{A_1}, \tau_{A_2}), \min(i_{A_1}, i_{A_2}), \min(f_{A_1}, f_{A_2})\rangle, x \in u\}. \tag{19}$$

4. *$A_3 = A_1 \cap A_2$ is defined by*

$$A_3 = \{\langle x : \min(\tau_{A_1}, \tau_{A_2}), \max(i_{A_1}, i_{A_2}), \max(f_{A_1}, f_{A_2})\rangle, x \in u\}. \tag{20}$$

5. *$A_3 = A_1 + A_2$ is defined by*

$$A_3 = \{\langle x : \tau_{A_1} + \tau_{A_2} - \tau_{A_1}\tau_{A_2}, i_{A_1}i_{A_2}, f_{A_1}f_{A_2}\rangle, x \in u\}. \tag{21}$$

6. *$A_3 = A_1 \times A_2$ is defined by*

$$A_3 = \{\langle x : \tau_{A_1}\tau_{A_2}, i_{A_1} + i_{A_2} - i_{A_1}i_{A_2}, f_{A_1} + f_{A_2} - f_{A_1}f_{A_2}\rangle, x \in u\}. \tag{22}$$

Definition 1.9. [13] *Given a SVNS $A = \{\langle x : \tau_A, i_A, f_A\rangle, x \in u\}$, then the score function $\sigma : u \mapsto [-1, 1]$, accuracy function $\alpha : u \mapsto [-1, 1]$, and certainty function $\upsilon : u \mapsto [0, 1]$ of A are defined as*

$$\sigma(A) = \frac{2 + \tau_A - i_A - f_A}{3}; \tag{23}$$

$$\alpha(A) = \tau_A - f_A; \tag{24}$$

$$\upsilon(A) = \tau_A. \tag{25}$$

As an extension[13] of Definition 1.9, using the 2 SVNSs A_1 and A_2 from Definition 1.8, if $\sigma(A_1) > \sigma(A_2)$, then $A_1 > A_2$. Moreover, if $\sigma(A_1) = \sigma(A_2)$ and $\alpha(A_1) > \alpha(A_2)$, then $A_1 > A_2$. Furthermore, assuming that $\sigma(A_1) = \sigma(A_2)$ and $\alpha(A_1) = \alpha(A_2)$, if $\upsilon(A_1) > \upsilon(A_2)$, then $A_1 > A_2$. Conversely, if $\sigma(A_1) = \sigma(A_2)$, $\alpha(A_1) = \alpha(A_2)$, and $\upsilon(A_1) = \upsilon(A_2)$, then $A_1 = A_2$.

Remark 1.1. *For a zero set, denoted as $0_{\mathbb{N}} = \{\langle x : 0, 1, 1\rangle, x \in u\}$, $\sigma(0_{\mathbb{N}}) = 0$, $\alpha(0_{\mathbb{N}}) = -1$, and $\upsilon(0_{\mathbb{N}}) = 0$.*

Proof. We hold this proof to be self-evident. □

Definition 1.10. ***(Truth- and falsity-favorite)***[9] *Given a SVNS $A_1 = \{\langle x : \tau_{A_1}, i_{A_1}, f_{A_1}\rangle, x \in u\}$, then the SVNS A_2, whose neutrosophic components are related to A_1 is a*

1. *truth-favorite of A_1 and is denoted by $A_2 = \Delta A_1$ if and only if*

$$A_2 = \{\langle x : \min(\tau_{A_1} + i_{A_1}, 1), 0, f_{A_1}\rangle, x \in u\}. \tag{26}$$

2. *falsity-favorite of A_1 and is denoted by $A_2 = \nabla A_1$ if and only if*

$$A_2 = \{\langle x : \tau_{A_1}, 0, \min(i_{A_1} + f_{A_1}, 1)\rangle, x \in u\}. \tag{27}$$

Remark 1.2. *The complement of a zero set is denoted as* $c_{0_{\mathbb{N}}} = \{\langle x : 1, 0, 0\rangle, x \in u\}$ *implies that* $0_{\mathbb{N}} \subset c_{0_{\mathbb{N}}}$. *Furthermore,* $c_{0_{\mathbb{N}}} = c_{\nabla_{O_{\mathbb{N}}}}$, *i.e.,* $c_{O_{\mathbb{N}}}$ *is the complement of the false-favorite of* $0_{\mathbb{N}}$, *with* $\sigma(c_{0_{\mathbb{N}}}) = 1$, $\alpha(c_{0_{\mathbb{N}}}) = 1$, *and* $\upsilon(c_{0_{\mathbb{N}}}) = 1$.

Proof. We hold this proof to be self-evident. □

Remark 1.3. *Given a SVNS* $A = \{\langle x : \tau_A, i_A, f_A\rangle, x \in u\}$, *then* $0_{\mathbb{N}} \subseteq A$.

Proof. This remark is a generalization of Remark 1.2. If A is a zero-set, then $A = 0_{\mathbb{N}}$, which also implies that $0_{\mathbb{N}} \subseteq A$. Otherwise, for all $x \in u$, it is clear that $0 \leq \tau_A$, $1 \geq i_A$, and $1 \geq f_A$ for all $\tau_A, i_A, f_A \in [0, 1]$ (See Definition 1.8). □

2. Developing our DSS computational model

2.1. Defining objectives and attributes

We developed our model to support a finite number of predefined objectives over a time-period ranging from t_0 to t_n. The model supports the rearrangement of the objectives into an objective tree with n objective levels matching the time interval $[t_0, t_n]$, as seen in figure 1. Previous studies have been able to determine that an objective's importance can change over time based on how the decision-maker perceives each objective and its alignment with the strategies developed for each attribute [1]; thus, the model also need to take this finding into account. As a result, the model is designed to support a finite number m time-dependent attributes, denoted, $a_1, \ldots, a_m$, spanning over each $t \in [t_0, t_n]$ (Figure 1). That is, each a_k, with $k \in [1, m]$, is a $(n + 1)$-tuple, thus

$$a_k = \{a_k(t_0), \ldots, a_k(t_n)\}. \tag{28}$$

For our model, each attribute a_k's value at time t is determined based on the neutrosophic values assigned to that attribute. That is, we bypass the need to determine the importance of attribute a_k at time t beforehand. The model solely asks, once known, whether a_k is true, indeterminate, or false at time t. Those values then lead to determining a few factors (including the consequences of strategies needed to achieve the set of objectives) about each attribute and will be readily available to the decision-maker. He or she will then assess the attribute based on his/her perception of value for that attribute. Last, this model does not manipulate or make changes to the objectives; any manipulation/assessment other than rearrangement into an objective tree is left to the decision-maker. Thus, the first definition for our model is as follows:

Definition 2.1. *Given a non-empty set* N *of predefined objectives and* m *time-dependent attributes, all spanning over* $[t_0, t_n]$*; each attribute* a_k *is a SVNS, and is defined by*

$$a_k = \Big\{\langle x : \tau_{k_t}, i_{k_t}, f_{k_t}\rangle, x \in [1, m] \times [t_0, t_n]\Big\}, \tag{29}$$

where $\tau_{k_t}, i_{k_t}, f_{k_t} \in [0, 1]$, $m \in \mathbb{N}^*$, $k \in [1, m]$, *and* $t \in [t_0, t_n]$.

Remark 2.1. *For* $k \in [1, m]$, a_k *is a non-zero set.*

Proof. We prove that the following conditions are met:

1. For $k \in [1, m]$, $0_{\mathbb{N}} \subseteq a_k$, which we proved in Remark 1.3.

2. Let $x_{\text{val}} \in u$, where $u = [1, m] \times [t_0, t_n]$, for which we have 2 non-empty SVNSs $a_{\text{val}_1} = \{\langle x_{\text{val}} : \tau_{\text{val}_1}, i_{\text{val}_1}, f_{\text{val}_1} \rangle, x_{\text{val}} \in u\} \subset a_k$ and $a_{\text{val}_2} = \{\langle x_{\text{val}} : \tau_{\text{val}_2}, i_{\text{val}_2}, f_{\text{val}_2} \rangle, x_{\text{val}} \in u\} \subset a_k$, if there exist $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 \times a_{\text{val}_1} + \lambda_2 \times a_{\text{val}_2} \subset a_k$, then a_k is non-zero set. From Definition 1.8, we know that

$$\lambda_1 \times a_{\text{val}_1} = \left\{ \langle x_{\text{val}} : 1 - (1 - \tau_{\text{val}_1})^{\lambda_1}, i_{\text{val}_1}^{\lambda_1}, f_{\text{val}_1}^{\lambda_1} \rangle, x_{\text{val}} \in u \right\}; \tag{30}$$

$$\lambda_2 \times a_{\text{val}_2} = \left\{ \langle x_{\text{val}} : 1 - (1 - \tau_{\text{val}_2})^{\lambda_2}, i_{\text{val}_2}^{\lambda_2}, f_{\text{val}_2}^{\lambda_2} \rangle, x_{\text{val}} \in u \right\}. \tag{31}$$

Thus, let $a_{\text{val}} = \lambda_1 \times a_{\text{val}_1} + \lambda_2 \times a_{\text{val}_2}$, we have

$$a_{\text{val}} = \left\{ \langle x_{\text{val}} : 1 - (1 - \tau_{\text{val}_1})^{\lambda_1} (1 - \tau_{\text{val}_2})^{\lambda_2}, i_{\text{val}_1}^{\lambda_1} i_{\text{val}_2}^{\lambda_2}, f_{\text{val}_1}^{\lambda_1} f_{\text{val}_2}^{\lambda_2} \rangle, x_{\text{val}} \in u \right\} \tag{32}$$

We know that $\tau_{\text{val}_1}, i_{\text{val}_1}, f_{\text{val}_1}, \tau_{\text{val}_2}, i_{\text{val}_2}, f_{\text{val}_2} \in [0, 1]$, thus for any $\lambda_1, \lambda_2 > 0$, $0 \leq i_{\text{val}_1}^{\lambda_1} i_{\text{val}_2}^{\lambda_2} \leq 1$ and $0 \leq f_{\text{val}_1}^{\lambda_1} f_{\text{val}_2}^{\lambda_2} \leq 1$. Furthermore, $1 - \tau_{\text{val}_1} \leq 1$ implies that $(1 - \tau_{\text{val}_1})^{\lambda_1} \leq 1$. The same applies for $(1 - \tau_{\text{val}_2})^{\lambda_2}$. As a result, $0 \leq 1 - (1 - \tau_{\text{val}_1})^{\lambda_1} (1 - \tau_{\text{val}_2})^{\lambda_2} \leq 1$ for any $\lambda_1, \lambda_2 > 0$. Now, since with $x_{\text{val}} \in u$, and $a_{\text{val}_1}, a_{\text{val}_2} \subset a_k$, then $a_{\text{val}} \subset a_k$ for all $x_{\text{val}} \in u$.

3. $a_{\text{val}_1} \cap a_{\text{val}_2} \subset a_k$. if $a_{\text{val}_1} \cap a_{\text{val}_2} = 0_{\mathbb{N}}$, then condition 1 applies. Otherwise,

$$a_{\text{val}_1} \cap a_{\text{val}_2} = \left\{ \langle x_{\text{val}} : \min(\tau_{\text{val}_1}, \tau_{\text{val}_2}), \max(i_{\text{val}_1}, i_{\text{val}_2}), \max(f_{\text{val}_1}, f_{\text{val}_2}) \rangle, x_{\text{val}} \in u \right\}. \tag{33}$$

It becomes trivial that $a_{\text{val}_1} \cap a_{\text{val}_2} \subset a_k$, being that $a_{\text{val}_1}, a_{\text{val}_2}$ are non-empty sets.

All 3 conditions being met leads to the conclusion that a_k is a non-zero set. □

Having stated both Definition 2.1 and Remark 2.1, it becomes possible to rank between attributes using the score function, compare attributes using the accuracy function, and determine the likelihood of an attribute using the certainty function. In doing so, we are also able to derive strategies from attributes. The next subsection introduces a few more definitions and remarks.

2.2. *Defining consequences of strategies*

Definition 2.2. *Given a non-empty set N of predefined objectives and m time-dependent attributes represented by the SVNS $a_k = \{\langle x : \tau_{k_t}, i_{k_t}, f_{k_t} \rangle, x \in [1, m] \times [t_0, t_n]\}$, let S be the set of available strategies derived from a_k, the imprecise consequences of such strategies, denoted S_q, is a stream defined by a vector of intervals, and can be written as*

$$S_q = \{s_{q_1}(t), \ldots, s_{q_m}(t)\}; \tag{34}$$

with each $s_{q_k}(t)$, $k \in [1, m]$, being defined by

$$s_{q_k}(t) \in \left[s_k^L(t), s_k^U(t) \right], \tag{35}$$

where $s_k^L(t)$ and $s_k^U(t)$ are, respectively, the lower and upper endpoints of the imprecise consequence for attribute a_k at time $t \in [t_0, t_n]$.

This model is based on the assumption that $S_q \in S$. We also assume that there is a continuous distribution between $s_k^L(t)$ and $s_k^U(t)$ endpoints.

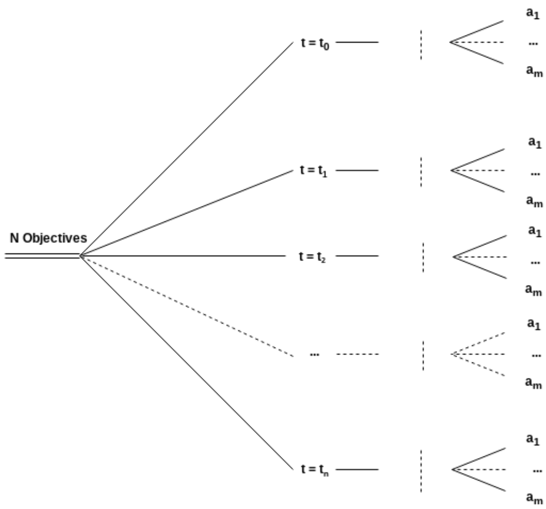

Figure 1: *Objective tree including time-periods.*

Definition 2.3. *Given a non-empty set N of predefined objectives and m time-dependent attributes represented by the SVNS $a_k = \{\langle x : \tau_{k_t}, i_{k_t}, f_{k_t} \rangle, x \in [1, m] \times [t_0, t_n]\}$ with a derived set S of available strategies, at time t, each s_{q_k} interval is determined by the following conditions:*

1. *If $i_{k_t} = 0$, then*

$$s_{q_k}(t) = s_k^L(t) = s_k^U(t) = \tau_{k_t}. \tag{36}$$

2. *If $i_{k_t} > 0$, then*

$$s_k^L(t) = \min\left(\frac{2 + \tau_{k_t} - i^*_{k_t} - f_{k_t}}{3}\right), \tag{37}$$

$$s_k^U(t) = \max\left(\frac{2 + \tau_{k_t} - i^*_{k_t} - f_{k_t}}{3}\right), \tag{38}$$

and

$$s_k^L(t) \leq s_{q_k}(t) \leq s_k^U(t), \tag{39}$$

where $i^*_{k_t} = \{i_{k_t}, 1 - i_{k_t}\}$.

In Definition 2.3, we put an emphasis on the indeterminacy of attribute a_k at time t. This stems from the fact that once we are able to precisely characterize attribute a_k at time t, i.e., $i_{k_t} = 0$, then we only care about its certainty ($\upsilon[a_k(t)] = \tau_{k_t}$). This is also applicable when $\tau_{k_t} = 0$, leading to $s_k^L(t) = s_k^U(t) = 0$, and both endpoints take the lowest possible value of 0. Conversely, if $i_{k_t} = 0$ and $\tau_{k_t} = 1$, then $s_k^L(t) = s_k^U(t) = 1$, and both endpoints take the highest possible value of 1. Now, if we have imprecise knowledge of the nature of attribute a_k at time t, i.e., $i_{k_t} > 0$, then its indeterminacy becomes prevalent, and is used in determining both endpoints of the imprecise consequence for that particular attribute. We do so by creating a masked indeterminacy value $i^*_{k_t}$ that includes both the value of i_{k_t} and its complement $1 - i_{k_t}$, and substituting them in the score function of $a_k(t)$, as seen in equations (37) and (38).

A special case is where $i_{k_t} = 1 - i_{k_t}$, meaning $i_{k_t} = 0.5$, then $s_k^L(t) = s_k^U(t) = \sigma[a_k(t)]$. As a result, each $s_{q_k}(t)$ is an interval with distinct endpoints as expected, except for cases where $i_{k_t} = 0$ or $i_{k_t} = 0.5$, in which they are equal. We interpret this result as (a) having precise knowledge of attribute $a_k(t)$ leads to a single strategy consequence that is solely based on the certainty of attribute a_k at time t; and (b) having an imprecise (or, indeterminate) knowledge of a_{k_t} at about 50% leads to a single strategy consequence that takes all neutrosophic components into consideration.

It is worth pointing that, when $i_{k_t} \neq 0$, we have an interval of imprecise consequence values as seen in equation (35). This entails that we need some creative way to generate the values between $s_k^L(t)$ and $s_k^U(t)$ and estimate the best $s_{q_k}(t)$ value(s) from the interval for decision-making purposes. Obviously, this model is intended to empower the decision-maker with a weighting tool in which attributes' impacts on the current and future strategies are accounted for. By presenting each potential strategy as a direct consequence of a particular attribute in the form of an interval, it is critical to predict which value(s) from the interval are most likely, based on i_{k_t}. Thus, we proceed with the following analysis:
First, it is clear that $0 \leq s_{q_k}(t) \leq 1$ regardless of the value of i_{k_t}. Second, let $\bar{i_{k_t}} = 1 - i_{k_t}$, if $i_{k_t} < 0.5$, we agree that $\bar{i_{k_t}} > i_{k_t}$, then,

$$s_k^L(t) = \frac{2 + \tau_{k_t} - \bar{i_{k_t}} - f_{k_t}}{3}. \tag{40}$$

Conversely, with $i_{k_t} > 0.5$, we have $\bar{i_{k_t}} < i_{k_t}$, and

$$s_k^U(t) = \frac{2 + \tau_{k_t} - \bar{i_{k_t}} - f_{k_t}}{3}. \tag{41}$$

Consequently, for all $i_{k_t} \neq 0$, it is safe to say that

$$\frac{2 + \tau_{k_t} - \max(\bar{i_{k_t}}, i_{k_t}) - f_{k_t}}{3} \leq s_{q_k}(t) \leq \frac{2 + \tau_{k_t} - \min(\bar{i_{k_t}}, i_{k_t}) - f_{k_t}}{3}. \tag{42}$$

Naturally, it is a more favorable scenario to have $\bar{i_{k_t}} > i_{k_t}$. We denote such scenario as *favorable indeterminacy*. This stems from the trivial fact that precise knowledge about an attribute at time t is achieved only when $i_{k_t} \to 0$. We also denote the opposite scenario, i.e., $\bar{i_{k_t}} < i_{k_t}$, as *unfavorable indeterminacy*. Therefore, the smaller the indeterminacy, the larger the impact of the attribute's certainty at time t. Moreover, for $i_{k_t} \neq 0.5$, it is clear that there exists a $\zeta_t > 0$ such that $| \ i_{k_t} - \bar{i_{k_t}} \ | = \zeta_t$. Then, we provide the following definition and remarks.

Definition 2.4. *Given a non-empty set N of predefined objectives and m time-dependent attributes represented by the SVNS $a_k = \{\langle x : \tau_{k_t}, i_{k_t}, f_{k_t} \rangle, x \in [1, m] \times [t_0, t_n]\}$ with a derived set S of available strategies, at time t. The SVNS $a_{k_r} = \{\langle x : \tau_{k_t}, 1 - i_{k_t}, f_{k_t} \rangle, x \in [1, m] \times [t_0, t_n]\}$ is called the reverse indeterminate SVNS of a_k, and the imprecise consequence interval $s_{k_r}^L(t) \leq s_{q_{k_r}}(t) \leq s_{k_r}^U(t)$ is the reverse imprecise consequence interval of a_k at time t.*

Remark 2.2. *Given a_k and a_{k_r}, let $\bar{i_{k_t}} = 1 - i_{k_t}$ and $i_{k_t} > 0$.*

1. *If $i_{k_t} = \bar{i_{k_t}}$, then $a_k = a_{k_r}$, and*
$$s_{q_{k_r}}(t) = s_{q_k}(t) = \sigma[a_k(t)]. \tag{43}$$

2. *If $i_{k_t} > \bar{i_{k_t}}$, then $s_k^L(t) \leq s_{q_{k_r}}(t) \leq s_{k_r}^U(t)$, where*
$$s_k^L(t) = \frac{2 + \tau_{k_t} - i_{k_t} - f_{k_t}}{3}, \tag{44}$$
$$s_{k_r}^U(t) = \frac{2 + \tau_{k_t} - \bar{i_{k_t}} - f_{k_t}}{3}. \tag{45}$$

3. *If* $i_{k_t} < \bar{i_{k_t}}$, *then* $s^L_{k_r}(t) \leq s_{q_k}(t) \leq s^U_k(t)$, *where*

$$s^L_{k_r}(t) = \frac{2 + \tau_{k_t} - \bar{i_{k_t}} - f_{k_t}}{3}, \tag{46}$$

$$s^U_k(t) = \frac{2 + \tau_{k_t} - i_{k_t} - f_{k_t}}{3}. \tag{47}$$

Proof. We hold these proofs to be self-evident. □

Remark 2.3. *Let* $k, l \in [1, m]$ *such that* $k \neq l$*; at time t, assuming that* $[s^L_k(t), s^U_k(t)] \subseteq [s^L_l(t), s^U_l(t)]$*; if* $i_{k_t} = \min(i_{k_t}, \bar{i_{k_t}})$ *and* $i_{l_t} = \min(i_{l_t}, \bar{i_{l_t}})$*, then* $a_k \leq a_l$*.*

Proof. We are aware that each interval is a continuous distribution, but we also know that $[s^L_k(t), s^U_k(t)] \subseteq [s^L_l(t), s^U_l(t)]$ is equivalent to $s^L_l(t) \leq s^L_k(t) \leq s^U_k(t) \leq s^U_l(t)$. That said, from equation (42), we agree that

$$\frac{2 + \tau_{l_t} - \max(\bar{i_{l_t}}, i_{l_t}) - f_{l_t}}{3} \leq \frac{2 + \tau_{k_t} - \max(\bar{i_{k_t}}, i_{k_t}) - f_{k_t}}{3}; \tag{48}$$

$$\frac{2 + \tau_{k_t} - \min(\bar{i_{k_t}}, i_{k_t}) - f_{k_t}}{3} \leq \frac{2 + \tau_{l_t} - \min(\bar{i_{l_t}}, i_{l_t}) - f_{l_t}}{3}. \tag{49}$$

Then, we can deduce that

$$\mid i_{k_t} - \bar{i_{k_t}} \mid \leq \mid i_{l_t} - \bar{i_{l_t}} \mid . \tag{50}$$

From Remark 2.2, we know that if

1. $i_{k_t} > \bar{i_{k_t}}$, then $s^L_k(t) \leq s_{q_{k_r}}(t) \leq s^U_{k_r}(t)$;

2. $i_{k_t} < \bar{i_{k_t}}$, then $s^L_{k_r}(t) \leq s_{q_k}(t) \leq s^U_k(t)$.

Doing the same for i_{l_t} and $\bar{i_{l_t}}$; when $i_{k_t} = \min(i_{k_t}, \bar{i_{k_t}})$ and $i_{l_t} = \min(i_{l_t}, \bar{i_{l_t}})$, we can see that $s^U_k(t) \leq s^U_l(t)$. As a result, $a_k \leq a_l$ (See the extension of Definition 1.9 in page 5). □

For each a_k, since there is a continuous distribution between $s^L_k(t)$ and $s^U_k(t)$ endpoints, then the probability that a particular value between $s^L_k(t)$ and $s^U_k(t)$ endpoints is assumed is 0; which is fine, as the end goal here is not to handpick a value from that interval. By contrast, assuming that the continuous distribution between $s^L_k(t)$ and $s^U_k(t)$ endpoints is uniform, if $\bar{s_{q_t}}$ is the distribution mean, then we can imply that, for any values $\epsilon, y > 0$,

$$\Pr[\bar{s_{q_t}} - \epsilon \leq y \leq \bar{s_{q_t}}] = \Pr[\bar{s_{q_t}} \leq y \leq \bar{s_{q_t}} + \epsilon]. \tag{51}$$

Thus, it is easy to see that the probability density function is

$$f(y) = \begin{cases} \dfrac{1}{s^U_k(t) - s^L_k(t)}, & \text{if } s^L_k(t) \leq y \leq s^U_k(t); \\ \\ 0, & \text{otherwise}; \end{cases} \tag{52}$$

and that any value within the $[\min(i_{k_t}, \bar{i_{k_t}}), \max(i_{k_t}, \bar{i_{k_t}})]$ will yield equally probable real y values, with the condition that $s^L_k(t) \leq y \leq s^U_k(t)$.

Now, we try a more complex distribution. Assuming that the continuous distribution is normal, using the *central limit theorem* or *CLT*[15], we know that the density of the sum of two or more independent variables within the $s_k^L(t) \leq y \leq s_k^U(t)$ interval is the convolution of their densities[16]. That is, as we add more independent variables to the sum, the density of the sum tends to converge towards the normal density. If $\bar{s_{q_t}}$ is the distribution mean, let n_s be the number of real values in $[s_k^L(t), s_k^U(t)]$, we expect $n_s \to \infty$. Thus, the classical CLT states that as n_s gets sufficiently large, the distribution gets close to the normal distribution with mean $\bar{s_{q_t}}$ and variance δ^2. As a result, within each close set $[s_k^L(t), s_k^U(t)]$, we want $s_{q_k}(t)$ to be as close to the mean $\bar{s_{q_t}}$ as possible. In the end, each attribute a_k has a consequence of strategy that is either presented as a single value within [0, 1] or as a continuous distribution in $[s_k^L(t), s_k^U(t)]$, depending on whether $i_{k_t} = 0$ or not. This gives the decision-maker the freedom to (a) develop a problem-solving approach on approximation and conduct sensitivity analysis as needed, or (b) perform discounting on each attribute over time once more data about each attribute become available and perform further sensitivity (or any other) analysis as needed. Since this model solely relies on current data based on the neutrosophic values of each attribute, we use an example in which we apply (a) and leave (b) for further discussions on this topic.

3. Computing

3.1. Model example input and computation

For our model example, we use an objective tree over time that consists of $N = 100$ objectives, $n = 10$ objective-levels, and 10 time-periods. We are also given 10 attributes, denoted by the time-dependent set $a = \{a_1, a_2, \ldots, a_{10}\}$, as seen in Table 1. We denote the time-period as a $(n+1)$-tuple, i.e., $t = \{0, 1, \ldots, 10\}$, in order to account for the initial knowledge or characteristics of attributes at time $t = 0$. At time $t = 0$, it is expected that the decision-maker's knowledge of each attribute and its relevance to objectives is precise, and therefore there are no indeterminacy and $i_k = 0$, with $k \in [1, 10]$. It is also assumed that each attribute a_k is a SVNS, therefore each τ_{k_t}, i_{k_t}, and f_{k_t} are in [0, 1]. Also, each a_k is a 11-tuple in the form of equations (28) and (29). All simulations are run using the input values and the values recorded in Tables 1 and 2. Neutrosophic components for each a_k at times $t = 1$ through $t = 10$ are recorded in Table 2.

Neutrosophic Component	**Attributes** ($a_k(t = 0)$)									
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
τ_k	0.581	0.74	0.149	0.258	0.97	0.515	0.565	0.144	0.925	0.634
i_k	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
f_k	0.419	0.26	0.851	0.742	0.03	0.485	0.435	0.856	0.075	0.366

Table 1: Initial neutrosophic values for all attributes, at time $t = 0$.

Neutrosophic Component	**Attributes** ($a_k(t > 0)$)									
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
	$a_k(t = 1)$									
τ_k	0.913	0.636	0.624	0.374	0.504	0.989	0.471	0.009	0.35	0.477
i_k	0.434	0.861	0.749	0.829	0.001	0.29	0.981	0.867	0.755	0.512
f_k	0.087	0.364	0.376	0.626	0.496	0.011	0.529	0.991	0.65	0.523
	$a_k(t = 2)$									
τ_k	0.553	0.77	0.519	0.293	0.246	0.701	0.532	0.276	0.094	0.903
i_k	0.966	0.126	0.951	0.797	0.544	0.373	0.782	0.596	0.482	0.033

f_k	0.447	0.23	0.481	0.707	0.754	0.299	0.468	0.724	0.906	0.097
	$a_k(t = 3)$									
τ_k	0.223	0.256	0.014	0.898	0.026	0.994	0.851	0.887	0.704	0.257
i_k	0.965	0.619	0.537	0.066	0.22	0.111	0.06	0.125	0.252	0.535
f_k	0.777	0.744	0.986	0.102	0.974	0.006	0.149	0.113	0.296	0.743
	$a_k(t = 4)$									
τ_k	0.227	0.505	0.254	0.872	0.116	0.634	0.274	0.614	0.904	0.1
i_k	0.4	0.548	0.012	0.39	0.843	0.129	0.558	0.305	0.198	0.574
f_k	0.773	0.495	0.746	0.128	0.884	0.366	0.726	0.386	0.096	0.9
	$a_k(t = 5)$									
τ_k	0.345	0.103	0.481	0.036	0.01	0.692	0.479	0.85	0.495	0.187
i_k	0.048	0.66	0.083	0.333	0.306	0.065	0.568	0.354	0.349	0.379
f_k	0.655	0.897	0.519	0.964	0.99	0.308	0.521	0.15	0.505	0.813
	$a_k(t = 6)$									
τ_k	0.421	0.236	0.379	0.072	0.582	0.598	0.794	0.837	0.553	0.041
i_k	0.115	0.947	0.764	0.556	0.805	0.948	0.426	0.043	0.408	0.503
f_k	0.579	0.764	0.621	0.928	0.418	0.402	0.206	0.163	0.447	0.959
	$a_k(t = 7)$									
τ_k	0.431	0.498	0.436	0.27	0.235	0.004	0.468	0.334	0.564	0.568
i_k	0.197	0.046	0.041	0.591	0.143	0.081	0.875	0.682	0.014	0.159
f_k	0.569	0.502	0.564	0.73	0.765	0.996	0.532	0.666	0.436	0.432
	$a_k(t = 8)$									
τ_k	0.767	0.571	0.761	0.344	0.032	0.168	0.239	0.807	0.359	0.051
i_k	0.265	0.164	0.436	0.68	0.054	0.778	0.514	0.228	0.855	0.846
f_k	0.233	0.429	0.239	0.656	0.968	0.832	0.761	0.193	0.641	0.949
	$a_k(t = 9)$									
τ_k	0.005	0.318	0.816	0.064	0.286	0.337	0.622	0.457	0.09	0.554
i_k	0.028	0.749	0.314	0.722	0.915	0.475	0.687	0.37	0.297	0.473
f_k	0.995	0.682	0.184	0.936	0.714	0.663	0.378	0.543	0.91	0.446
	$a_k(t = 10)$									
τ_k	0.738	0.788	0.46	0.017	0.401	0.304	0.657	0.921	0.367	0.925
i_k	0.447	0.399	0.973	0.962	0.626	0.191	0.379	0.201	0.957	0.162
f_k	0.262	0.212	0.54	0.983	0.599	0.696	0.343	0.079	0.633	0.075

Table 2: Neutrosophic values for all attributes, at $1 \leq t \leq 10$.

We calculate the score, accuracy, and certainty functions for each a_k for all t, as seen in Tables 3. Using Definition 2.3 and equations (36) through (38), we then determine the imprecise consequences intervals for each a_k at all t in Table 4. As expected, since the neutrosophic values used for $t = 0$ have 0 indeterminacy, the imprecise consequences intervals only takes one value, which is the value of τ_k. In this case, the term *imprecise* is an oxymoron since technically, the consequence for each a_k is precise and refers to the certainty of that a_k. At times $t > 0$, we have the specified intervals with minimums and maximums for each attributes a_k. As we said in the definition of this model, those intervals are continuous distributions and will be used as such in our analysis.

Function	**Attributes** (a_k)									
	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
	$a_k(t=0)$									
$\sigma(a_k)$	0.721	0.827	0.433	0.505	0.98	0.677	0.71	0.429	0.95	0.756
$\alpha(a_k)$	0.162	0.48	-0.702	-0.484	0.94	0.03	0.13	-0.712	0.85	0.268
$\upsilon(a_k)$	0.581	0.74	0.149	0.258	0.97	0.515	0.565	0.144	0.925	0.634
	$a_k(t=1)$									
$\sigma(a_k)$	0.797	0.47	0.5	0.306	0.669	0.896	0.32	0.05	0.315	0.481
$\alpha(a_k)$	0.826	0.272	0.248	-0.252	0.008	0.978	-0.058	-0.982	-0.3	-0.046
$\upsilon(a_k)$	0.913	0.636	0.624	0.374	0.504	0.989	0.471	0.009	0.35	0.477
	$a_k(t=2)$									
$\sigma(a_k)$	0.38	0.805	0.362	0.263	0.316	0.676	0.427	0.319	0.235	0.924
$\alpha(a_k)$	0.106	0.54	0.038	-0.414	-0.508	0.402	0.064	-0.448	-0.812	0.806
$\upsilon(a_k)$	0.553	0.77	0.519	0.293	0.246	0.701	0.532	0.276	0.094	0.903
	$a_k(t=3)$									
$\sigma(a_k)$	0.16	0.298	0.164	0.91	0.277	0.959	0.881	0.883	0.719	0.326
$\alpha(a_k)$	-0.554	-0.488	-0.972	0.796	-0.948	0.988	0.702	0.774	0.408	-0.486
$\upsilon(a_k)$	0.223	0.256	0.014	0.898	0.026	0.994	0.851	0.887	0.704	0.257
	$a_k(t=4)$									
$\sigma(a_k)$	0.351	0.487	0.499	0.785	0.13	0.713	0.33	0.641	0.87	0.209
$\alpha(a_k)$	-0.546	0.01	-0.492	0.744	-0.768	0.268	-0.452	0.228	0.808	-0.8
$\upsilon(a_k)$	0.227	0.505	0.254	0.872	0.116	0.634	0.274	0.614	0.904	0.1
	$a_k(t=5)$									
$\sigma(a_k)$	0.547	0.182	0.626	0.246	0.238	0.773	0.463	0.782	0.547	0.332
$\alpha(a_k)$	-0.31	-0.794	-0.038	-0.928	-0.98	0.384	-0.042	0.7	-0.01	-0.626
$\upsilon(a_k)$	0.345	0.103	0.481	0.036	0.01	0.692	0.479	0.85	0.495	0.187
	$a_k(t=6)$									
$\sigma(a_k)$	0.576	0.175	0.331	0.196	0.453	0.416	0.721	0.877	0.566	0.193
$\alpha(a_k)$	-0.158	-0.528	-0.242	-0.856	0.164	0.196	0.588	0.674	0.106	-0.918
$\upsilon(a_k)$	0.421	0.236	0.379	0.072	0.582	0.598	0.794	0.837	0.553	0.041
	$a_k(t=7)$									
$\sigma(a_k)$	0.555	0.65	0.61	0.316	0.442	0.309	0.354	0.329	0.705	0.659
$\alpha(a_k)$	-0.138	-0.004	-0.128	-0.46	-0.53	-0.992	-0.064	-0.332	0.128	0.136
$\upsilon(a_k)$	0.431	0.498	0.436	0.27	0.235	0.004	0.468	0.334	0.564	0.568
	$a_k(t=8)$									
$\sigma(a_k)$	0.756	0.659	0.695	0.336	0.337	0.186	0.321	0.795	0.288	0.085
$\alpha(a_k)$	0.534	0.142	0.522	-0.312	-0.936	-0.664	-0.522	0.614	-0.282	-0.898
$\upsilon(a_k)$	0.767	0.571	0.761	0.344	0.032	0.168	0.239	0.807	0.359	0.051
	$a_k(t=9)$									
$\sigma(a_k)$	0.327	0.296	0.773	0.135	0.219	0.4	0.519	0.515	0.294	0.545
$\alpha(a_k)$	-0.99	-0.364	0.632	-0.872	-0.428	-0.326	0.244	-0.086	-0.82	0.108
$\upsilon(a_k)$	0.005	0.318	0.816	0.064	0.286	0.337	0.622	0.457	0.09	0.554
	$a_k(t=10)$									

$\sigma(a_k)$	0.676	0.726	0.316	0.024	0.392	0.472	0.645	0.88	0.259	0.896
$\alpha(a_k)$	0.476	0.576	-0.08	-0.966	-0.198	-0.392	0.314	0.842	-0.266	0.85
$\upsilon(a_k)$	0.738	0.788	0.46	0.017	0.401	0.304	0.657	0.921	0.367	0.925

Table 3: Score (α), accuracy (σ), and certainty (υ) functions for all attributes' SVNSs, at time $t \in [0, 10]$.

Imprecise Consequences Set (S_q) of Strategies Set S				
s_{q_1} s_{q_6}	s_{q_2} s_{q_7}	s_{q_3} s_{q_8}	s_{q_4} s_{q_9}	s_{q_5} $s_{q_{10}}$
$t = 0$				
0.581 0.515	0.74 0.565	0.149 0.144	0.258 0.925	0.97 0.634
$t = 1$				
[0.753, 0.797] [0.756, 0.896]	[0.47, 0.711] [0.32, 0.641]	[0.5, 0.666] [0.05, 0.295]	[0.306, 0.526] [0.315, 0.485]	[0.336, 0.669] [0.481, 0.489]
$t = 2$				
[0.38, 0.691] [0.592, 0.676]	[0.555, 0.805] [0.427, 0.615]	[0.362, 0.663] [0.319, 0.383]	[0.263, 0.461] [0.223, 0.235]	[0.316, 0.345] [0.613, 0.924]
$t = 3$				
[0.16, 0.47] [0.7, 0.959]	[0.298, 0.377] [0.587, 0.881]	[0.164, 0.188] [0.633, 0.883]	[0.621, 0.91] [0.553, 0.719]	[0.091, 0.277] [0.326, 0.35]
$t = 4$				
[0.285, 0.351] [0.466, 0.713]	[0.487, 0.519] [0.33, 0.369]	[0.173, 0.499] [0.511, 0.641]	[0.711, 0.785] [0.669, 0.87]	[0.13, 0.358] [0.209, 0.258]
$t = 5$				
[0.246, 0.547] [0.483, 0.773]	[0.182, 0.289] [0.463, 0.509]	[0.348, 0.626] [0.685, 0.782]	[0.135, 0.246] [0.446, 0.547]	[0.109, 0.238] [0.251, 0.332]
$t = 6$				
[0.319, 0.576] [0.416, 0.715]	[0.175, 0.473] [0.671, 0.721]	[0.331, 0.507] [0.572, 0.877]	[0.196, 0.233] [0.505, 0.566]	[0.453, 0.656] [0.193, 0.195]
$t = 7$				
[0.353, 0.555] [0.03, 0.309]	[0.347, 0.65] [0.354, 0.604]	[0.304, 0.61] [0.329, 0.45]	[0.316, 0.377] [0.381, 0.705]	[0.204, 0.442] [0.432, 0.659]
$t = 8$				
[0.6, 0.756] [0.186, 0.371]	[0.435, 0.659] [0.321, 0.331]	[0.653, 0.695] [0.614, 0.795]	[0.336, 0.456] [0.288, 0.524]	[0.039, 0.337] [0.085, 0.316]
$t = 9$				
[0.013, 0.327] [0.383, 0.4]	[0.296, 0.462] [0.519, 0.644]	[0.649, 0.773] [0.428, 0.515]	[0.135, 0.283] [0.159, 0.294]	[0.219, 0.496] [0.527, 0.545]
$t = 10$				
[0.641, 0.676] [0.266, 0.472]	[0.658, 0.726] [0.564, 0.645]	[0.316, 0.631] [0.681, 0.88]	[0.024, 0.332] [0.259, 0.564]	[0.392, 0.476] [0.671, 0.896]

Table 4: Imprecise consequence intervals for all attributes' SVNSs, at time $t \in [0, 10]$.

3.2. Approximation-based approach

In Definition 2.4, and remarks 2.2 and 2.3, we introduced the concept of reverse indeterminate SVNS and imprecise consequence interval for each attribute. These are critical in determining one of the two endpoints in the imprecise intervals for each a_k at time t when $i_{k_t} > \bar{i_{k_t}}$. We also determined that, for distinct $k, l \in [1, m]$, at time t, having $[s_k^L(t), s_k^U(t)] \subseteq [s_l^L(t), s_l^U(t)]$ where $i_{k_t} = \min(i_{k_t}, \bar{i_{k_t}})$ and $i_{l_t} = \min(i_{l_t}, \bar{i_{l_t}})$ leads to the conclusion that $a_k \leq a_l$ from an attribute scoring standpoint. Depending on the amount of objectives and attributes being present, drawing such conclusion can be a tough task for a fairly large amount of attributes. Thus, we describe the following approximation-based approach to help determine the most desirable attribute(s) at time t. We assume that each imprecise consequence interval is a normal distribution, and we look at the median of each distribution at time t (Table 5). Once again, at time $t = 0$, we had both endpoints being equal; therefore, for $t = 0$, we just take that τ value. The $\bar{s_{q_k}}$ values in Table 5 are then normalized to generate each attribute weight value at time t, with the sum of all a_k at time t being equal to 1 (See Table 6). If determining which attribute to prioritize at time t is indeed the goal for the decision-maker, then the $\hat{s_{q_k}}$ results in Table 6 can be used to assign attribute weights based on their imprecise consequence values. As a result, to obtain the $\hat{s_{q_k}}$ values in Table 6, at each t, with $m = 10$ and $k \in [1, m]$, we use

$$\hat{s_{q_k}} = \frac{\bar{s_{q_k}}}{\sum_{k=1}^{m} \bar{s_{q_k}}}. \tag{53}$$

Time (t)	**Imprecise Consequence Median Values** ($\bar{s_{q_k}}$) **for** a_k **at Time** t									
	$\bar{s_{q_1}}$	$\bar{s_{q_2}}$	$\bar{s_{q_3}}$	$\bar{s_{q_4}}$	$\bar{s_{q_5}}$	$\bar{s_{q_6}}$	$\bar{s_{q_7}}$	$\bar{s_{q_8}}$	$\bar{s_{q_9}}$	$\bar{s_{q_{10}}}$
$t = 0$	0.581	0.74	0.149	0.258	0.97	0.515	0.565	0.144	0.925	0.634
$t = 1$	0.775	0.59	0.583	0.416	0.503	0.826	0.481	0.172	0.4	0.485
$t = 2$	0.536	0.68	0.512	0.362	0.33	0.634	0.521	0.351	0.229	0.768
$t = 3$	0.315	0.338	0.176	0.766	0.184	0.829	0.734	0.758	0.636	0.338
$t = 4$	0.318	0.503	0.336	0.748	0.244	0.59	0.35	0.576	0.77	0.233
$t = 5$	0.396	0.236	0.487	0.19	0.174	0.628	0.486	0.734	0.497	0.292
$t = 6$	0.448	0.324	0.419	0.215	0.554	0.566	0.696	0.724	0.536	0.194
$t = 7$	0.454	0.498	0.457	0.347	0.323	0.169	0.479	0.39	0.543	0.546
$t = 8$	0.678	0.547	0.674	0.396	0.188	0.278	0.326	0.704	0.406	0.2
$t = 9$	0.17	0.379	0.711	0.209	0.358	0.392	0.582	0.472	0.226	0.536
$t = 10$	0.659	0.692	0.474	0.178	0.434	0.369	0.604	0.78	0.412	0.784

Table 5: Imprecise consequence intervals medians for at time t, for each a_k.

Time (t)	**Imprecise Consequence Weighted Values** ($\hat{s_{q_k}}$) **for** a_k **at Time** t									$\sum_{k=1}^{m} \hat{s_{q_k}}$
	$\hat{s_{q_1}}$	$\hat{s_{q_2}}$	$\hat{s_{q_3}}$	$\hat{s_{q_4}}$	$\hat{s_{q_5}}$	$\hat{s_{q_6}}$	$\hat{s_{q_7}}$	$\hat{s_{q_8}}$	$\hat{s_{q_9}}$	$\hat{s_{q_{10}}}$	
$t = 0$	0.106	0.135	0.027	0.047	0.177	0.094	0.103	0.026	0.169	0.116	1.00
$t = 1$	0.148	0.113	0.111	0.08	0.096	0.158	0.092	0.033	0.076	0.093	1.00
$t = 2$	0.109	0.138	0.104	0.074	0.067	0.129	0.106	0.071	0.047	0.156	1.00
$t = 3$	0.062	0.067	0.035	0.151	0.036	0.163	0.145	0.149	0.125	0.067	1.00
$t = 4$	0.068	0.108	0.072	0.16	0.052	0.126	0.075	0.123	0.165	0.05	1.00
$t = 5$	0.096	0.057	0.118	0.046	0.042	0.152	0.118	0.178	0.121	0.071	1.00

$t = 6$	0.096	0.069	0.09	0.046	0.118	0.121	0.149	0.155	0.115	0.041	1.00
$t = 7$	0.108	0.118	0.109	0.083	0.077	0.04	0.114	0.093	0.129	0.13	1.00
$t = 8$	0.154	0.124	0.153	0.09	0.043	0.063	0.074	0.16	0.092	0.045	1.00
$t = 9$	0.042	0.094	0.176	0.052	0.089	0.097	0.144	0.117	0.056	0.133	1.00
$t = 10$	0.122	0.128	0.088	0.033	0.081	0.069	0.112	0.145	0.076	0.146	1.00

Table 6: Imprecise consequence weight values at time t for each a_k, obtained via the normalization of the imprecise consequence values of a_k for each t.

3.3. Interpretation of results and sensitivity analysis

By corresponding each weight value in Table 6 to its relative attribute, the decision-maker can choose to prioritize directly based on these weight values. Additionally, these values can also be used to perform discounting on each attribute with the end goal being the prioritization of attributes at time t. Using discounting, however, is an extra measure for comparing attributes, for the purpose of this paper; so it has been left to the decision-maker's discretion.

At first glance, it is easy to see that at time $t = 0$, attribute a_5 would be ranked first, followed closely by attribute a_9, then by a_2, a_{10}, a_1, a_7, a_6, a_4, a_3, and a_8, respectively. Doing the same for $t = 1$, the attribute ranking is a_6, a_1, a_2, a_3, a_5, a_{10}, a_7, a_4, a_9, and a_8, respectively. For $t = 2$, we have a_{10}, a_2, a_6, a_1, a_7, a_3, a_4, a_8, a_5, and a_9, in that order. The same process is used to determine the ranking or priority for the same attributes at times $t = 3, \ldots, 10$.

The approximation-based approach using the medians of the imprecise consequence intervals in Subsection 3.2 gives us an outlet in assigning a weighted value to each attribute because it allows us to bypass the need to compare each imprecise consequence interval with another (See Remark 2.3). That approach, however, does not take into account whether attribute a_k contains a favorable or unfavorable indeterminacy at time t. This stems from the fact that the median is the closest to the halfway point between $s_k^L(t)$ and $s_k^U(t)$, therefore that approach does not take into account whether $i_{k_t} = \min(i_{k_t}, \bar{i}_{k_t})$ (favorable indeterminacy) or $i_{k_t} = \max(i_{k_t}, \bar{i}_{k_t})$ (unfavorable indeterminacy). We know that each $[s_k^L(t), s_k^U(t)]$ is a continuous distribution, so we denote $\epsilon_{k_t} = \mathrm{d}s_{q_k(t)}$ as an arbitrary infinitesimal variation from the median $\bar{s}_{q_k}(t)$ such that $\bar{s}_{q_k}(t) - \epsilon_{k_t} \leq \bar{s}_{q_k}(t) \leq \bar{s}_{q_k}(t) + \epsilon_{k_t}$, at time t. Thus, we recompute the weighted values ($\hat{s}_{q_k}$) using

$$\hat{s}_{q_k} = \begin{cases} \dfrac{\bar{s}_{q_k} + \epsilon_{k_t}}{\sum\limits_{k=1}^{m} \bar{s}_{q_k}}, & \text{if } i_{k_t} = \min(i_{k_t}, \bar{i}_{k_t}); \\ \\ \dfrac{\bar{s}_{q_k} - \epsilon_{k_t}}{\sum\limits_{k=1}^{m} \bar{s}_{q_k}}, & \text{otherwise.} \end{cases} \tag{54}$$

For simplicity, we use the same ϵ_{k_t} throughout $t = 1, \ldots, 10$. However, it is normal to envision a case where the decision-maker would choose a different ϵ_{k_t} value as t progresses on. Moreover, since at $t = 0$, there is no imprecise consequence interval for any of the attributes (See Table 4), the newly computed values only affect the previous weighted values in Table 6 from time $t = 1$ through $t = 10$. Those newly computed weighted values are then reflected in Table 7 using $\epsilon_{k_t} = 0.1$.

Time ($t > 0$)	**Imprecise Consequence Weighted Values** ($\hat{s}_{q_k}$) **for** a_k **at Future** t	$\sum_{k=1}^{m} \hat{s}_{q_k}$

	$\hat{s_{q_1}}$	$\hat{s_{q_2}}$	$\hat{s_{q_3}}$	$\hat{s_{q_4}}$	$\hat{s_{q_5}}$	$\hat{s_{q_6}}$	$\hat{s_{q_7}}$	$\hat{s_{q_8}}$	$\hat{s_{q_9}}$	$\hat{s_{q_{10}}}$	
$t = 1$	0.167	0.094	0.092	0.06	0.115	0.177	0.073	0.014	0.057	0.074	0.923
$t = 2$	0.089	0.158	0.084	0.053	0.047	0.149	0.086	0.051	0.067	0.176	0.96
$t = 3$	0.042	0.047	0.015	0.171	0.056	0.183	0.164	0.169	0.145	0.047	1.039
$t = 4$	0.09	0.086	0.093	0.182	0.031	0.148	0.054	0.145	0.186	0.028	1.043
$t = 5$	0.12	0.033	0.142	0.07	0.067	0.177	0.094	0.202	0.145	0.095	1.145
$t = 6$	0.117	0.048	0.068	0.025	0.097	0.1	0.17	0.176	0.136	0.02	0.957
$t = 7$	0.132	0.142	0.132	0.059	0.101	0.064	0.09	0.069	0.153	0.154	1.096
$t = 8$	0.177	0.147	0.176	0.067	0.065	0.04	0.051	0.183	0.07	0.023	0.999
$t = 9$	0.067	0.069	0.201	0.027	0.064	0.122	0.119	0.142	0.081	0.158	1.05
$t = 10$	0.141	0.147	0.069	0.014	0.062	0.087	0.131	0.163	0.058	0.164	1.036

Table 7: Imprecise consequence weight values at future time t for each a_k, obtained via adding or subtracting an arbitrary $\epsilon_{k_t} = 0.1$ from the imprecise consequence interval median for each attribute a_k, depending on whether the value of i_{k_t} is less than 0.5 or not, at time $t > 0$.

Using the results in Table 7, there are no new attribute weighting or ranking for $t = 0$, as expected. For $t = 1$, we have the attributes in which $i_{k_t} = \min(i_{k_t}, \bar{i_{k_t}})$ (i.e., a_6, a_1, and a_5, respectively), then followed by those in which $i_{k_t} \neq \min(i_{k_t}, \bar{i_{k_t}})$, such as a_2, a_3, a_{10}, a_7, a_4, a_9, and a_8, in that order. For $t = 2$, we have a_{10}, a_2, a_6, a_1, a_7, a_3, a_9, a_4, a_8, and a_5, in that order. The same process is repeated to determine the rankings at times $t = 3, \ldots, 10$.

As expected, our first observation is that when $i_{k_t} = \min(i_{k_t}, \bar{i_{k_t}})$, attributes with largest τ_{k_t} and smallest i_{k_t} values yield a larger $\hat{s_{q_k}}$ than those that do not. Moreover, we can see that when two or more attributes contain $i_{k_t} \neq \min(i_{k_t}, \bar{i_{k_t}})$, priority is given to the one(s) with the largest τ_{k_t}. This new ranking aligns more with Remark 2.3 than the approximation-based approach of just using the normalized weights of the median values of the imprecise consequence intervals (Table 6). For instance, at $t = 4$, a_4 has $i_{k_t} = 0.39$ while i_{k_t} for a_9 is 0.198; we also can see that $[0.711, 0.785] \subseteq [0.669, 0.87]$, and according to both Remark 2.3 and the $\hat{s_{q_k}}$ values obtained in Table 7, we observe that a_9 would be ranked just ahead of a_4. As a result, where applicable, either Remark 2.3 or the approach in Table 7 can be used to determine with attribute has the most impactful strategy between any given set of attributes, at a specific time in the future. The challenge, in applying the approach in Table 7, is how to determine which ϵ_{k_t} is best to facilitate prioritizing attributes with favorable indeterminacy over those with unfavorable indeterminacy. The standard deviations from the imprecise consequence interval medians, at each $t > 0$, are 0.178, 0.163, 0.247, 0.189, 0.179, 0.174, 0.11, 0.189, 0.166, and 0.187, respectively. Thus, choosing $\epsilon_{k_t} = 0.1$ is a sensible pick. Any pick too small (i.e., $\epsilon_{k_t} \to 0$) would get us right around the median value, which defeats the purpose of establish some bias towards favorable indeterminacy. Any pick too large creates a significant gap between attributes with favorable indeterminacy and those with unfavorable indeterminacy. Ultimately, having the intervals available to the decision-maker empowers him/her in choosing any approximation approach that suits the end-goal of the scenario at hand.

4. Future work and discussion

We have developed a DSS computation model for decision-support scenarios where we have N objectives with m attributes spanning over n time-periods. We present each attribute in the form of a single-valued neutrosophic set and performed necessary operations to determine a specific set of strategies in which each attribute's imprecise consequence can be presented as a continuous distribution interval. We proceed with defining the model to detect when indeterminacy is favorable or unfavorable, and present approaches that help achieve that bias. We provide a computation example in which the model is used, and conduct a sensitivity analysis on the results. The example seems to have provided a clear application of the model but we are aware that certain areas still need clarification. Such areas, which can be addressed in future discussions regarding this computation model, so far include, but are not limited to:

1. Since each imprecise consequence is given in the form of a continuous distribution interval, assuming that the distribution is normal, would the normal density of each interval help establish a trend about each attribute?

2. Can the neutrosophic components of an attribute be linked with any property other than how impactful the attribute would be in the future?

References

1. Ríos-Insua S, Jiménez-Martín A, Mateos A. A time-dependent decision support system for multi-attribute decision-making. Integrated Computer-Aided Engineering. 2004 01;11. doi:10.3233/ICA-2004-11105.
2. Sauter V. Decision support systems. New York, New York: John Wiley and Sons, Inc.; 1997.
3. Alter S. Decision support systems: Current practice and continuing challenges. Reading, Massachusetts: Addison-Wesley; 1980.
4. Holsapple CW, Whinston A. Decision support systems: A knowledge-based approach. Minneapolis, Minnesota: West Publishing, Inc.; 1996.
5. Power D. What are the characteristics of a decision support system? DSS News. 2003;4(7).
6. Kock ED. Decentralising the codification of rules in a decision support expert knowledge base. University of Pretoria. 2004;.
7. Pan J, Rahman S. Multiattribute utility analysis with imprecise information: An enhanced decision support technique for the evaluation of electric generation expansion strategies. Electric Power Systems Research. 1998 06;46:101–109. doi:10.1016/S0378-7796(98)00022-4.
8. Atherton E, French S. Issues in supporting intertemporal choice. In: Karwan MH, Spronk J, Wallenius J, editors. Essays in decision making. Berlin, Heidelberg: Springer; 1997. p. 135–156. doi:https://doi.org/10.1007/978-3-642-60663-2_10.
9. Wang H, Smarandache F, Zhang Y, Sunderraman R. Single valued neutrosophic sets. vol. 10; 2010. p. 1–4.
10. Smarandache F. Neutrosophic set: a generalization of the intuitionistic fuzzy set. 2006 IEEE International Conference on Granular Computing. 2006;p. 38–42.
11. Moschovakis J. Intuitionistic logic. In: Zalta EN, editor. The Stanford Encyclopedia of Philosophy. winter 2018 ed. Metaphysics Research Lab, Stanford University; 2018. p. . Available from: https://plato.stanford.edu/archives/win2018/entries/logic-intuitionistic.
12. Priest G, Tanaka K, Weber Z. Paraconsistent logic. In: Zalta EN, editor. The Stanford Encyclopedia of Philosophy. summer 2018 ed. Metaphysics Research Lab, Stanford University; 2018. p. . Available from: https://plato.stanford.edu/archives/sum2018/entries/logic-paraconsistent.
13. Deli I, Ali M, Smarandache F. Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. 2015 International Conference on Advanced Mechatronic Systems (ICAMechS). 2015;p. 249–254. doi:http://10.1109/ICAMechS.2015.7287068.
14. Radwan N, Senousy MB, el-din Riad A. Neutrosophic AHP multi criteria decision making method applied on the selection of Learning Management System. International Journal of Advancements in Computing Technology. 2016 12;8:95–105.
15. Durrett R. Probability: theory and examples. 3rd ed. Cambridge University Press; 2004.
16. Sums of independent random variables;. Accessed: 2019-07-31. https://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/Chapter7.pdf.