\( \alpha \) Generalized Closed Sets in Neutrosophic Topological Spaces

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Abstract: In this paper a new concept of neutrosophic closed sets called neutrosophic \( \alpha \) generalized closed sets is introduced and their properties are thoroughly studied and analyzed. Some new interesting theorems based on the newly introduced set are presented.

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1. Introduction

The concept of neutrosophic sets was first introduced by Florentin Smarandache [3] in 1999 which is a generalization of intuitionistic fuzzy sets by Atanassov [1]. A. A. Salama and S. A. Albrowi [6] introduced the concept of neutrosophic topological spaces after Coker [2] introduced intuitionistic fuzzy topological spaces. Further the basic sets like semi open sets, pre open sets, \( \alpha \) open sets and semi-\( \alpha \) open sets are introduced in neutrosophic topological spaces and their properties are studied by various authors [4,5]. The purpose of this paper is to introduce and analyze a new concept of neutrosophic closed sets called neutrosophic \( \alpha \) generalized closed sets.

2. Preliminaries:

Here in this paper the neutrosophic topological space is denoted by \((X, \tau)\). Also the neutrosophic interior, neutrosophic closure of a neutrosophic set \( A \) are denoted by \( \text{NI}(A) \) and \( \text{NC}(A) \). The complement of a neutrosophic set \( A \) is denoted by \( C(A) \) and the empty and whole sets are denoted by \( \emptyset \) and \( X \) respectively.

Definition 2.1: Let \( X \) be a non-empty fixed set. A neutrosophic set (NS) \( A \) is an object having the form \( A = \{ (x, \mu_A(x), \sigma_A(x), \nu_A(x)) : x \in X \} \) where \( \mu_A(x), \sigma_A(x) \) and \( \nu_A(x) \) represent the degree of membership, degree of indeterminancy and the degree of non-membership respectively of each element \( x \in X \) to the set \( A \).

A Neutrosophic set \( A = \{ (\mu_A(x), \sigma_A(x), \nu_A(x)) : x \in X \} \) can be identified as an ordered triple \( \langle \mu_A, \sigma_A, \nu_A \rangle \) in \([0, 1] \) on \( X \).

Definition 2.2: Let \( A = \{ (\mu_A, \sigma_A, \nu_A) \} \) be a NS on \( X \), then the complement \( C(A) \) may be defined as

1. \( C(A) = \{ (1-\mu_A(x), 1-\sigma_A(x), \nu_A(x)) : x \in X \} \)
2. \( C(A) = \{ (\mu_A(x), \nu_A(x), \sigma_A(x)) : x \in X \} \)
3. \( C(A) = \{ (\nu_A(x), 1-\sigma_A(x), \mu_A(x)) : x \in X \} \)

Note that for any two neutrosophic sets \( A \) and \( B \),

4. \( C(A \cup B) = C(A) \cap C(B) \)
5. \( C(A \cap B) = C(A) \cup C(B) \)

Definition 2.3: For any two neutrosophic sets \( A = \{ (\mu_A(x), \sigma_A(x), \nu_A(x)) : x \in X \} \) and \( B = \{ (\mu_B(x), \sigma_B(x), \nu_B(x)) : x \in X \} \) we may have

1. \( A \subseteq B \Rightarrow \mu_A(x) \leq \mu_B(x), \sigma_A(x) \leq \sigma_B(x) \) and \( \nu_A(x) \geq \nu_B(x) \) \( \forall x \in X \)
2. \( A \subseteq B \Rightarrow \nu_A(x) \leq \nu_B(x), \sigma_A(x) \geq \sigma_B(x) \) and \( \nu_A(x) \geq \nu_B(x) \) \( \forall x \in X \)
3. \( A \cap B = \{ (\mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \vee \nu_B(x)) \} \)
4. \( A \cap B = \{ (\mu_A(x) \wedge \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \nu_A(x) \vee \nu_B(x)) \} \)
5. \( A \cup B = \{ (\mu_A(x) \vee \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \wedge \nu_B(x)) \} \)
6. \( A \cup B = \{ (\mu_A(x) \vee \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \wedge \nu_B(x)) \} \)

Definition 2.4: A neutrosophic topology (NT) on a non-empty set \( X \) is a family of neutrosophic subsets in \( X \) satisfies the following axioms:

1. \( (NT_1) \emptyset, X \in \tau \)
2. \( (NT_2) G_1 \cap G_2 \in \tau \) for any \( G_1, G_2 \in \tau \)
3. \( (NT_3) \cup \{ G_i : i \in I \} \subset \tau \)

In this case the pair \((X, \tau)\) is a neutrosophic topological space (NTS) and any neutrosophic set in \( \tau \) is known as a neutrosophic open set (NOS) in \( X \). A neutrosophic set \( A \) is a neutrosophic closed set (NCS)
if and only if its complement $C(A)$ is a neutrosophic open set in $X$.

Here the empty set $(0_N)$ and the whole set $X$ may be defined as follows:

$$(0_1) \quad 0_N = \{ (x, 0, 0, 1) : x \in X \}$$

$$(0_2) \quad 0_N = \{ (x, 0, 1, 1) : x \in X \}$$

$$(0_3) \quad 0_N = \{ (x, 1, 0, 0) : x \in X \}$$

$$(0_4) \quad 0_N = \{ (x, 0, 0, 0) : x \in X \}$$

$$(1_1) \quad 1_N = \{ (x, 1, 0, 0) : x \in X \}$$

$$(1_2) \quad 1_N = \{ (x, 1, 0, 1) : x \in X \}$$

$$(1_3) \quad 1_N = \{ (x, 1, 1, 0) : x \in X \}$$

$$(1_4) \quad 1_N = \{ (x, 1, 1, 1) : x \in X \}$$

**Definition 2.5:** Let $(X, \tau)$ be a NTS and $A = \{ (x, \mu_A(x), \sigma_A(x), \nu_A(x)) : x \in X \}$ be a NS in $X$. Then the neutrosophic interior and the neutrosophic closure of $A$ are defined by

$$NInt(A) = \bigcup \{ G : G \text{ is a NOS in } X \text{ and } G \subseteq A \}$$

$$NCl(A) = \bigcap \{ K : K \text{ is an NCS in } X \text{ and } A \subseteq K \}$$

Note that for any NS $A$, $NCl(C(A)) = C(NInt(A))$ and $NInt(C(A)) = C(NInt(A))$.

**Definition 2.6:** A NS $A$ of a NTS $X$ is said to be

(i) a neutrosophic pre-open set (NP-OS) if $A \subseteq NInt(NCl(A))$

(ii) a neutrosophic semi-open set (NS-OS) if $A \subseteq NCl(NInt(A))$

(iii) a neutrosophic $\alpha$-open set (N$\alpha$-OS) if $A \subseteq NInt(NCl(NInt(A)))$

(iv) a neutrosophic semi-$\alpha$-open set (NS$\alpha$-OS) if $A \subseteq NCl(\alpha NInt(A))$

**Definition 2.7:** A NS $A$ of a NTS $X$ is said to be

(i) a neutrosophic pre-closed set (NP-CS) if $NCl(NInt(A)) \subseteq A$

(ii) a neutrosophic semi-closed set (NS-CS) if $NInt(NCl(A)) \subseteq A$

(iii) a neutrosophic $\alpha$-closed set (N$\alpha$-CS) if $NCl(NInt(NCl(A))) \subseteq A$

(iv) a neutrosophic semi-$\alpha$-closed set (NS$\alpha$-CS) if $NCl(\alpha NInt(A)) \subseteq A$

3. $\alpha$ generalized closed sets in neutrosophic topological spaces

In this section we introduce neutrosophic $\alpha$ closure, neutrosophic $\alpha$ interior and $\alpha$ generalized closed set and its respective open set in neutrosophic topological spaces and discuss some of their properties.

**Definition 3.1:** A NS $A$ of a NTS $X$ is said to be a neutrosophic regular closed set (NRCS) if $NCl(NInt(A)) = A$ and neutrosophic regular open set if $NInt(NCl(A)) = A$.

**Definition 3.2:** A NS $A$ of a NTS $X$ is said to be a neutrosophic $\beta$ closed set (N$\beta$CS) if $NCl(NInt(NCl(A))) \cap A$ and neutrosophic $\beta$ open set if $A \cap NCl(NInt(NCl(A)))$.

**Definition 3.3:** Let $A$ be a NS of a NTS $(X, \tau)$. Then the neutrosophic $\alpha$ interior and the neutrosophic $\alpha$ closure are defined as

$N_{\alpha}Int(A) = \bigcup \{ G : G \text{ is a } N\alpha-OS \text{ in } X \text{ and } G \subseteq A \}$

$N_{\alpha}Cl(A) = \bigcap \{ K : K \text{ is a } N\alpha-CS \text{ in } X \text{ and } A \subseteq K \}$

**Result 3.4:** Let $A$ be a NS in $X$. Then $N_{\alpha}Cl(A) = A \cup NCl(NInt(NCl(A)))$.

**Proof:** Since $N_{\alpha}Cl(A)$ is a $N\alpha-CS$, $NCl(NInt(NCl(N_{\alpha}Cl(A)))) \subseteq N_{\alpha}Cl(A)$ and $A \cup NCl(NInt(NCl(N_{\alpha}Cl(A)))) \subseteq A \cup N_{\alpha}Cl(A) = N_{\alpha}Cl(A)$ --- (i). Therefore $A \subseteq NCl(NInt(NCl(A)))$.

From (i), $N_{\alpha}Cl(A) = A \cup NCl(NInt(NCl(A)))$.

**Definition 3.5:** A NS $A$ of a NTS $X$ is said to be a neutrosophic $\alpha$ generalized closed set (N$\alpha_g$CS) if $N_{\alpha_g}Cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is a NOS in $X$. The complement $C(A)$ of a N$\alpha_g$CS $A$ is a N$\alpha_g$OS in $X$.

**Example 3.6:** Let $X = \{ a, b \}$ and $\tau = \{ 0_N, A, B, 1_N \}$ where $A = \{ x, (0.5, 0.6), (0.3, 0.2), (0.4, 0.1) \}$ and $B = \{ x, (0.4, 0.4), (0.4, 0.3), (0.5, 0.4) \}$. Then $\tau$ is a NTS. Here $\mu_A(a) = 0.5, \mu_A(b) = 0.6, \sigma_A(a) = 0.3, \sigma_A(b) = 0.2, \nu_A(a) = 0.4$ and $\nu_A(b) = 0.1$. Also $\mu_B(a) = 0.4, \mu_B(b) = 0.4, \sigma_B(a) = 0.4, \sigma_B(b) = 0.3, \nu_B(a) = 0.5$ and $\nu_B(b) = 0.4$. Let $M = \{ x, (0.5, 0.4), ((0.4, 0.4), (0.4, 0.5)) \}$ be any NS in $X$. Then $M \subseteq A$ where $A$ is a NOS in $X$. Now $N_{\alpha_g}Cl(M) = M \cup C(B) = C(B) \subseteq A$. Therefore $M$ is a N$\alpha_g$CS in $X$.

**Proposition 3.7:** Every NCS $A$ is a N$\alpha_g$CS in $X$ but not conversely in general.

**Proof:** Let $A \subseteq U$ where $U$ is a NOS in $X$. Now $N_{\alpha_g}Cl(A) = A \cup NCl(NInt(NCl(A))) \subseteq A \cup NCl(A) = A \cup A = A \subseteq U$, by hypothesis. Therefore $A$ is a N$\alpha_g$CS in $X$.
Remark 3.9: Every NS-CS and every N_{\alpha}-CS in a NTS X are independent to each other in general.

Example 3.10: In Example 3.6, M is a N_{\alpha}-CS but not a NS-CS as NInt(NCl(M)) = B \not\subset M.

Example 3.11: Let X = \{a, b\} and \tau = \{0_{\alpha}, A, B, C, 1_{\alpha}\}, where A = \langle x, (0.5, 0.4), (0.3, 0.2), (0.5, 0.6)\rangle, B = \langle x, (0.8, 0.7), (0.4, 0.3), (0.2, 0.3)\rangle and C = \langle x, (0.2, 0.1), (0.3, 0.2), (0.8, 0.9)\rangle. Then \tau is a NT. Let M = \langle x, (0.5, 0.3), (0.3, 0.2), (0.5, 0.7)\rangle. Then M is a NS-CS but not a N_{\alpha}-CS as M \not\subset A, B and N_{\alpha}Cl(M) = M \cup C(A) = C(A) \not\subset A.

Remark 3.12: Every NP-CS and every N_{\alpha}-CS in a NTS X are independent to each other in general.

Example 3.13: In Example 3.11, M is a NP-CS but not a N_{\alpha}-CS as seen in the respective example.

Example 3.14: Let X = \{a, b\} and \tau = \{0_{\alpha}, A, B, 1_{\alpha}\}, where A = \langle x, (0.5, 0.4), (0.3, 0.2), (0.5, 0.6)\rangle and B = \langle x, (0.4, 0.3), (0.3, 0.1), (0.6, 0.7)\rangle. Then \tau is a NT. Let M = \langle x, (0.5, 0.5), (0.2, 0.1), (0.4, 0.4)\rangle. Then M is a N_{\alpha}-CS but not a NP-CS as NCl(NInt(M)) = C(A) \not\subset M.

Proposition 3.15: Every N_{\alpha}-CS A is a N_{\alpha}-CS in X but not conversely in general.

Proof: Let A \not\subset U, where U is a NOS in X. Then N_{\alpha}Cl(A) = A \not\subset NCl(NInt(NCl(A))) \not\subset A \not\subset A \not\subset U, by hypothesis. Hence A is a N_{\alpha}-CS in X.

Example 3.16: In Example 3.6, M is a N_{\alpha}-CS in X but not a N_{\alpha}-CS as NCl(NInt(NCl(M))) = C(B) \not\subset M.

Proposition 3.17: Every NOS, N_{\alpha}-OS are N_{\alpha}-OS but not conversely in general.

Proof: Obvious.

Example 3.18: In Example 3.6, C(M) is a N_{\alpha}-OS but not a NOS, N_{\alpha}-OS in X.

Remark 3.19: Both NS-OS and NP-OS are independent to N_{\alpha}-OS in X in general.

Example 3.20: The above Remark can be proved easily from the Examples 3.10, 3.11 and 3.13, 3.14 respectively.

Proposition 3.21: The union of any two N_{\alpha}-CSs is a N_{\alpha}-CS in a NTS X.

Proof: Let A and B be any two N_{\alpha}-CSs in a NTS X. Let A \not\subset B \not\subset U where U is a NOS in X. Then A \not\subset U and B \not\subset U. Now N_{\alpha}Cl(A \not\subset B) = (A \not\subset B) \not\subset NCl(NInt(NCl(A \not\subset B))) \not\subset (A \not\subset B) \not\subset NCl(NCl(A \not\subset B)) \not\subset (A \not\subset B) \not\subset NCl(A \not\subset B) = NCl(A) \not\subset NCl(B) \not\subset U \not\subset U = U, by hypothesis. Hence A \not\subset B is a N_{\alpha}-CS in X.

Remark 3.22: The intersection of any two N_{\alpha}-CSs need not be a N_{\alpha}-CS in a NTS X.

Example 3.23: Let X = \{a, b\} and \tau = \{0_{\alpha}, A, B, 1_{\alpha}\} where A = \langle x, (0.5, 0.4), (0.3, 0.2), (0.5, 0.6)\rangle and B = \langle x, (0.8, 0.7), (0.3, 0.2), (0.2, 0.3)\rangle. Then \tau is a NT. Let M = \langle x, (0.6, 0.9), (0.3, 0.2), (0.4, 0.1)\rangle and N = \langle x, (0.9, 0.7), (0.3, 0.2), (0.1, 0.3)\rangle. Then M and N are N_{\alpha}-CSs in X but M \cap N = \langle x, (0.6, 0.7), (0.3, 0.2), (0.4, 0.3)\rangle is not a N_{\alpha}-CS as M \cap N \subset B and N_{\alpha}Cl(M \cap N) = 1_{\alpha} \not\subset A.

Proposition 3.24: Let (X, \tau) be a NTS. Then for every A \subset N_{\alpha}Cl(X) and for every B \subset NS(X), A \subset B \subset N_{\alpha}Cl(A) implies B \subset N_{\alpha}Cl(X).

Proof: Let B \subset U and U be a NOS in (X, \tau). Then since A \subset U, B \subset U. By hypothesis, B \subset N_{\alpha}Cl(A). Therefore N_{\alpha}Cl(B) \subset N_{\alpha}Cl(N_{\alpha}Cl(A)) = N_{\alpha}Cl(A) \subset U, since A is an N_{\alpha}-CS in (X, \tau). Hence B \subset N_{\alpha}Cl(X).

Proposition 3.25: If A is a NOS and a N_{\alpha}-CS in (X, \tau), then A is a N_{\alpha}-CS in (X, \tau).

Proof: Since A \subset A and A is a NOS in (X, \tau), by hypothesis, N_{\alpha}Cl(A) \subset A. But A \subset N_{\alpha}Cl(A). Therefore N_{\alpha}Cl(A) = A. Hence A is a N_{\alpha}-CS in (X, \tau).

Proposition 3.26: Let (X, \tau) be a NTS. Then every NS in (X, \tau) is a N_{\alpha}-CS in (X, \tau) if and only if N_{\alpha}-O(X) = N_{\alpha}-C(X).

Proof: Necessity: Suppose that every NS in (X, \tau) is a N_{\alpha}-CS in (X, \tau). Let U \subset NO(X). Then U \subset N_{\alpha}-O(X) and by hypothesis, N_{\alpha}Cl(U) \subset U \subset N_{\alpha}Cl(U). This implies N_{\alpha}Cl(U) = U. Therefore U \subset N_{\alpha}-C(X). Hence N_{\alpha}-O(X) \subset N_{\alpha}-C(X). Let A \subset N_{\alpha}-C(X). Then C(A) \subset N_{\alpha}-O(X) \subset N_{\alpha}-C(X). That is C(A) \subset N_{\alpha}-C(X). Therefore A \subset N_{\alpha}-O(X). Hence N_{\alpha}-C(X) \subset N_{\alpha}-O(X). Thus N_{\alpha}-O(X) = N_{\alpha}-C(X).

Sufficiency: Suppose that N_{\alpha}-O(X) = N_{\alpha}-C(X). Let A \subset U and U be a NOS in (X, \tau). Then
Proposition 3.27: If $A$ is a NOS and a NCS in $(X, \tau)$, then $A$ is a NROS in $(X, \tau)$.

Proof: Let $A$ be any NOS and a NCS in $(X, \tau)$. Then $A$ is a NCS in $(X, \tau)$. Hence $B$ is a NCS in $(X, \tau)$. Now $N\text{Cl}(N\text{Int}(N\text{Cl}(A))) = N\text{Cl}(N\text{Int}(A))$ and $A$. Since $A$ is a NOS, $A = N\text{Int}(A) = N\text{Cl}(N\text{Int}(A))$. Hence $N\text{Int}(N\text{Cl}(A)) = A$ and $A$ is a NROS in $(X, \tau)$.

Definition 3.28: A NCS $A$ in $(X, \tau)$ is a neutrosophic Q-set (NQ-S) in $X$ if $N\text{Int}(N\text{Cl}(A)) = N\text{Cl}(N\text{Int}(A))$.

Proposition 3.29: For a NOS $A$ in $(X, \tau)$, the following conditions are equivalent:

(i) $A$ is a NCS in $(X, \tau)$.
(ii) $A$ is a NCS and a NQ-S in $(X, \tau)$.

Proof: (i) $\Rightarrow$ (ii) Since $A$ is a NCS, it is a NCS in $(X, \tau)$. Now $N\text{Int}(N\text{Cl}(A)) = N\text{Int}(A) = A = N\text{Cl}(A) = N\text{Cl}(N\text{Int}(A))$, by hypothesis. Hence $A$ is a NQ-S in $(X, \tau)$.

(ii) $\Rightarrow$ (i) Since $A$ is a NOS and a NCS in $(X, \tau)$, by Theorem 3.27, $A$ is a NROS in $(X, \tau)$. Therefore $A = N\text{Int}(N\text{Cl}(A)) = N\text{Cl}(N\text{Int}(A)) = N\text{Cl}(A)$, by hypothesis. Hence $A$ is a NCS in $(X, \tau)$.

Proposition 3.30: Let $(X, \tau)$ be a NTS. Then for every $A \subseteq N_{g}O(X)$ and for every $B \subseteq NS(X)$, $N\text{Int}(A) \subseteq B \subseteq A$ implies $B \subseteq N_{g}O(X)$.

Proof: Let $B$ be any $N_{g}O$ of $X$ and $B$ be any NS of $X$. By hypothesis $N\text{Int}(A) \subseteq B \subseteq A$. Then $C(A)$ is a NCS in $X$ and $C(A) \subseteq C(B) \subseteq N\text{Cl}(C(A))$. By Theorem 3.24, $C(B)$ is a NCS in $(X, \tau)$. Therefore $B$ is a NCS in $(X, \tau)$. Hence $B \subseteq N_{g}O(X)$.

Proposition 3.31: Let $(X, \tau)$ be a NTS. Then for every $A \in NS(X)$ and for every $B \in NS-O(X)$, $B \subseteq A \subseteq N\text{Int}(N\text{Cl}(B))$ implies $A \in N_{g}O(X)$.

Proof: Let $B$ be a NS-O in $(X, \tau)$. Then $B \subseteq N\text{Int}(N\text{Cl}(B))$. By hypothesis, $A \subseteq N\text{Int}(N\text{Cl}(B))$. Therefore $A \subseteq N\text{Int}(N\text{Cl}(N\text{Int}(N\text{Cl}(B)))) \subseteq N\text{Int}(N\text{Cl}(N\text{Int}(A))))$. Therefore $A$ is a NCS and by Proposition 3.17, $A$ is a NOS in $(X, \tau)$. Hence $A \subseteq N_{g}O(X)$.

Proposition 3.32: A NOS $A$ of a NTS $(X, \tau)$ is a NCS in $(X, \tau)$ if and only if $F \subseteq N\text{Int}(A)$ whenever $F$ is a NCS in $(X, \tau)$ and $F \subseteq A$.

Proof: Necessity: Suppose $A$ is a NCS in $(X, \tau)$. Let $F$ be a NCS in $(X, \tau)$ such that $F \subseteq A$. Then $C(F)$ is a NOS and $C(A) \subseteq C(F)$. By hypothesis $C(A)$ is a NCS in $(X, \tau)$, we have $N\text{Cl}(C(A)) \subseteq C(F)$. Therefore $F \subseteq N\text{Int}(A)$.

Sufficiency: Let $U$ be a NOS in $(X, \tau)$ such that $C(A) \subseteq U$. By hypothesis, $C(U) \subseteq N\text{Int}(A)$. Therefore $N\text{Cl}(C(A)) \subseteq U$ and $C(A)$ is an NSCS in $(X, \tau)$. Hence $A$ is a NCS in $(X, \tau)$.

References: