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Article

Algebras and Smarandache Types

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Abstract: If an algebra of type A contains a subalgebra which is also an algebra of type B, then it is a Smarandache B-type A-algebra provided the subalgebra of type B contains at least two elements. This generalizes the notion of Smarandache group, where the group of order ≥ 2 is a subsemigroup of a semigroup. In this paper we investigate a number of such pairings and we deduce a number of conclusions as a consequence.

Keywords: Smarandache algebra, point algebra, p-derived algebra.

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1. Introduction

Generally, in any human field, a Smarandache structure on a set A means a weak structure W on A such that there exists a proper subset B of A which is embedded with a strong structure S. In [6], W. B. Vasantha Kandasamy studied the concept of Smarandache groupoids, subgroupoids, ideals of groupoids, semi-normal subgroupoids, Smarandache Bol groupoids and strong Bol groupoids and obtained many interesting results about them. Smarandace semigroups are very important for the study of congruences, and it was studied by R. Padilla ([5]). It will be very interesting to study the Smarandache structure in BCI-algebras. In [2], Y. B. Jun discussed the Smarandache structure in BCI-algebras. If an algebra of type A contains a subalgebra which is also an algebra of type B, then it is a Smarandache B-type A-algebra provided the subalgebra of type B contains at least two elements. This generalizes the notion of Smarandache group, where the group of order ≥ 2 is a subsemigroup of a semigroup. In this paper we investigate a number of such pairings and we deduce a number of conclusions as a consequence.

2. Several algebras

Suppose that $(X,*)$ is a binary system, i.e., $X \times X \to X,\ (x,y) \to x * y$, is the product mapping. If $p \in X$ is a point selected as reference point, then $(X,*,p)$ is a *point algebra*.

Example 1. *Let $(G,\cdot)$ be a group. Then the identity element e is usually taken to be the special point p selected, written as $(G,\cdot,e)$. We take all algebras dealt with below to be pointed algebras, without real loss of generality. Often, for simplicity's sake we shall write $p = 0$, not intending 0 to have the usual meaning. Thus $(G,\cdot,e)$ becomes $(G,*,0)$ unless it is important to distinguish the algebras $(X,*,0)$ which contains the subalgebra $(Y,*)$ with its own distinguished point p to produce $(Y,*,p)$.*

Example 2. *(i) Let $(X,*,p)$ be a semigroup with a selected point p and let $(Y,*,e)$ be a subsemigroup of X with a selected point e. Then $p \neq e$.*
(ii) Let $(R,+,\cdot)$ be a ring with identity. Then $(R,+,0)$ is an abelian group. Let U be a group of units of $(R,+,0)$. Then $(U,\cdot,1)$ has a different pole p from $(R,+,0)$

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In terms of list of axioms to be used to describe the various algebra types we note the following section of axioms:

(1) $x * x = 0$ for all $x \in X$.
(2) $x * 0 = x$ for all $x \in X$.
(3) $0 * x = x$ for all $x \in X$.
(4) $x * y = y * x$ for all $x, y \in X$.
(5) $x * y = y * x = 0 \Leftrightarrow x = y$ for all $x, y \in X$.
(6) $x * y = y * x = 0 \Rightarrow x = y$ for all $x, y \in X$.
(7) $x * y = y * x \Rightarrow x = y$ for all $x, y \in X$.
(8) $0 * x = 0$ for all $x \in X$.
(9) $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$.
(10) $(x * y) * z = x * (z * y)$ for all $x, y, z \in X$.
(11) $(x * y) * z = x * (z * (0 * y))$ for all $x, y, z \in X$.
(12) $(x * y) * z = (x * z) * (y * z)$ for all $x, y, z \in X$.
(13) $(x * y) * (0 * y) = x$ for all $x, y \in X$.
(14) $x * (y * z) = (x * y) * z$ for all $x, y, z \in X$.
(15) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.
(16) $((x * y) * (x * z)) * (x * y) = 0$ for all $x, y, z \in X$.
(17) $\forall x \in X \exists y \in X$ with $x * y = 0$.
(18) $\forall x \in X \exists y \in X$ with $y * x = 0$.

An algebra $(X, *, 0)$ is called a *group* if it satisfies (2),(3), (14), (17), and (18). An algebra $(X, *)$ is called a *semigroup* if it satisfies (14). An algebra $(X, *, 0)$ is called a *semigroup with identity* if it satisfies (2),(3), and (14). An algebra $(X, *, 0)$ is called a *B-algebra* if it satisfies (1),(2), and (11). An algebra $(X, *, 0)$ is called a *BG-algebra* if it satisfies (1),(2), and (13). An algebra $(X, *, 0)$ is called a *BH-algebra* if it satisfies (1), (2), and (6). An algebra $(X, *, 0)$ is called a *Q-algebra* if it satisfies (1), (2), and (9). An algebra $(X, *, 0)$ is called a *$\hat{Q}$-algebra* if it satisfies (1), (2), and (10). An algebra $(X, *, 0)$ is called a *d-algebra* if it satisfies (1), (5), and (8). An algebra $(X, *, 0)$ is called a *BCK-algebra* if it satisfies (1), (5), (8), (15), and (16). An algebra $(X, *, 0)$ is called a *gBCK-algebra* if it satisfies (1), (2), (9) and (12). An algebra $(X, *, 0)$ is called an *abelian group* if it satisfies (2), (3), (4), (14), (17), and (18). An algebra $(X, *, 0)$ is called an *abelian semigroup* if it satisfies (4) and (14).

3. Smarandache types

Let $(X, *)$ be a binary algebra. Then $(X, *)$ is a Smarandache-type P-algebra if it contains a subalgebra $(Y, *)$, where Y is non-trivial, i.e, $|Y| \geq 2$, or Y contains at least two distinct elements, and $(Y, *)$ is itself of type P. Thus we have Smarandache-type semigroup (the type P-algebra is a semigroup), Smarandache-type groups (the type P-algebra is a group), Smarandache-type abelian groups (the type P-algebra is an abelian group). If an algebra of type A contains a subalgebra which is also an algebra of type B, then it is a a Smarandache B-type A-algebra provided the subalgebra of type B contains at least two elements.

Theorem 1. *If $(X, *, 0)$ is an abelian d-algebra, then it is a trivial algebra.*

Proof. Suppose that $(X, *, 0)$ is an abelian d-algebra. By (8), we have $0 = 0 * x$ for all $x \in X$. Since X is abelian, $0 * x = x * 0$. Hence $0 * x = x * 0 = 0$. By (5), we obtain $x = 0$. Hence $X = \{0\}$. □

Theorem 2. *If $(X, *, 0)$ is a semigroup and d-algebra, then it is a trivial algebra.*

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Proof. For any $x \in X$, we have

$$\begin{aligned} x * 0 =& x * (0 * (x * 0)) \\ =& (x * 0) * (x * 0) = 0. \end{aligned}$$

By (8), we have $0 * x = 0$. By (5), we have $x = 0$. Hence $X = \{0\}$. □

Definition 1. *Let $(X, *, p)$ be a pointed algebra. Define $x \cdot y := x * (p * y)$, $\forall x, y \in X$. Then the algebra $(X, \cdot, p)$ is called a p-derived algebra.*

Example 3. *(i) Let $(X, *, e)$ be a group. Then the e-derived algebra $(X, \cdot, e)$ has $x \cdot y = x * (e * y) = x * y$, and $(X, *, e) = (X, \cdot, e)$.*
(ii) If $(X, *, 0)$ is a d-algebra, then $x \cdot y = x * (0 * y) = x * 0$.
(iii) If $(X, *, 0)$ is a BCK-algebra, then $x \cdot y = x * (0 * y) = x * 0 = x$. Hence $(X, \cdot, 0)$ is a left semigroup with a selected point 0.
(iv) If $(X, *, p)$ is a left semi-group with a selected point p, then $x \cdot y = x * (p * y) = x * p = x$, i.e., $x \cdot y = x$ and $(X, *, p) = (X, \cdot, p)$.

Definition 2. *A B-algebra is called a Smarandache group-type B-algebra if it contains a subalgebra $(Y, *, e)$ which is a group, where Y is non-trivial, $|Y| \geq 2$.*

Notice that $e * e = 0$ in the group, while $e * e = 0$ in the B-algebra, so that $e = 0$, i.e., the poles coincide naturally. Moreover $x * x = 0 = e$ for all $x \in X$. Hence every element in the group has order 2 and thus $(Y, *, e = 0)$ is a Boolean group. Hence Smarandache group-type B-algebra is equal to Smarandache Boolean group-type B-algebra.

Example 4. *Consider the case of a group-type Q-algebra. Let $(X, *, 0)$ be a Q-algebra and the subalgebra which is a group is $(Y, *, e)$. Then $e * e = e$ in the group and $e * e = 0$ in the Q-algebra is a very common situation and it follows that $x * x = 0 = e$, whence every element in the group has order 2. Hence Smarandache group-type Q-algebras is Smarandache Boolean group-type Q-algebra.*

Theorem 3. *Let $(X, *, 0)$ be a group and let $(Y, *, 0)$ be a subalgebra (not necessary a subgroup) which is a B-algebra. Then Smarandache B-algebra -type groups are Smarandache Boolean group-type group.*

Proof. Since $(X, *, 0)$ is a group, $0 * e = e = e * e$. By the cancellation law in $(X, *, 0)$, we have $e = 0$. Now $x \in Y$ has $x * x = e$ and thus $x = x^{-1}$. $(Y, *, e)$, so that $(Y, *, e)$ has $x * y^{-1} = x * y \in Y$ for all $x, y \in Y$. Hence Y is a subgroup of X. But also for all $x \in Y$ and hence Y is a Boolean group. Thus Smarandache B-algebra-type groups are Smarandache Boolean group-type group. □

Corollary 1. *Smarandach Q-algebra-type groups is Smarandache Boolean group-type groups.*

An algebra $(X, *, 0)$ is a *BQ-algebra* if it satisfies (1), (2), (11), and (9).

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Theorem 4. *Let $(X, *, 0)$ be BQ-algebra. Let $(X, *, \cdot)$ be an algebra where $x \cdot y := x * (0 * y)$. Then $(X, \cdot, 0)$ is an abelian group.*

Proof. Since $(X, *, 0)$ is a B-algebra, $(X, \cdot, 0)$ is a group. Now $(x * (0 * y)) * (0 * z) = (x * (0 * z)) * (0 * y)$ by (9). Hence $(x \cdot y) \cdot z = (x \cdot z) \cdot y = x \cdot (z \cdot y)$ since $(X, *, 0)$ is a group. Put $x := 0$ in $(x \cdot y) \cdot z = (x \cdot z) \cdot y$. Then $(0 \cdot y) \cdot z = 0 \cdot (z \cdot y)$. Since $0 \cdot y = 0 * (0 * y) = y$, we have $y \cdot z = z \cdot y$, i.e., $(X, \cdot, 0)$ is a commutative group. □

Theorem 5. *Let $(X, \cdot, 0)$ be an abelian group. Define $x * y = x \cdot y^{-1}$ for any $x, y \in X$. Then $(X, *, 0)$ is a BQ-algebra.*

Proof. For any $x \in X$, $x * x = x \cdot x^{-1} = 0$. Thus (1) holds. By (2), we have $x * 0 = x \cdot 0^{-1} = x \cdot 0 = x$. Hence (2) holds. Using $(X, \cdot, 0)$ is a commutative group, we obtain $(x * y) * z = (x \cdot y^{-1}) \cdot z^{-1} = (x \cdot 0^{-1})y^{-1} = (x * z) * y$. (9) holds. Since $(X, \cdot, 0)$ is an abelian group, we have

$$
\begin{aligned}
x * (z * (0 * y)) =& x \cdot (z \cdot (0 \cdot y^{-1})^{-1})^{-1} \\
=& x \cdot [z \cdot ((y^{-1})^{-1} \cdot 0^{-1})]^{-1} \\
=& [z \cdot (y \cdot 0)]^{-1} \\
=& x \cdot [z \cdot y]^{-1} \\
=& x \cdot (y^{-1} \cdot z^{-1}).
\end{aligned}
$$

But $(x * y) * z = (x \cdot y^{-1}) \cdot \cdot z^{-1}$. Hence we obtain $(x * y) * z = x * (z * (0 * y))$. (11) holds. Thus $(X, *, 0)$ is a BQ-algebra. □

The interesting fact to note is that we are able to take advantage of the relationship $x \cdot y = x * (0 * y)$ to understand better what the meaning of the class of BQ-algebra is. Other such questions around in this setting as well as others. E.q., what class of B-algebras corresponds to the class of solvable groups ? Can it be considered to be of the form: BV-algebras corresponds to solvable groups where "V-algebras" is some nicely identifiable class, as the class for BQ-algebras ?

Theorem 6. *If $(X, *, 0)$ is a Smarandache B-algebra-type Q-algebra, then $(X, \cdot, p)$ is a Smarandache abelian group-type 0-derived Q-algebra, where $x \cdot y := x * (0 * y)$ for any $x, y \in X$.*

Proof. Suppose that $(X, *, 0)$ is a Q-algebra and let $(Y, *, e)$ be a B-algebra and a subalgebra of X. For any $x \in Y$, $x * x = e$ and so $x * x = 0$, so that $e = 0$ in any case. Hence $(Y, *, 0)$ is a BQ-algebra. Therefore if $(X, \cdot, 0)$ is the 0-derived algebra obtained from a Q-algebra, then $(Y, \cdot, 0)$ is a abelian subgroup. Thus if $(X, \cdot, 0)$ is a Smarandache B-algebra-type Q-algebra, then $(Y, \cdot, 0)$ is a Smarandache abelian group type 0-derived Q-algebra. □

Corollary 2. *$(X, *, 0)$ is a Smarandache Q-algebra-type B-algebra, then $(X, \cdot, 0)$ is a Smarandach abelian group-type 0-derived. i.e., a Smarandache abelian group type group.*

Proof. It is similar to Theorem 6. □

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Proposition 1. *Every B-algebra is a Smarandache Q-algebra-type B-algebra.*

Proof. Let $(X, \cdot, 0)$ be a group and let $x \in X$. Then $< x >$ the group generated by x is an abelian group, $| < x > | \geq 2$ if $x \neq 0$. Hence if $(X, *, 0)$ is obtained by $x * y = x \cdot y^{-1}$, then $(X, *, 0)$ is a B-algebra, while $(< x >, *, 0)$ is a BQ-algebra. Thus if $|x| \geq 2$, then every B-algebra is a Smarandache Q-algebra-type B-algebra. □

Theorem 7. *Any non-trivial d-algebra cannot be a Smarandache group type d-algebra.*

Proof. Let $(X, *, 0)$ be a d-algebra and let $(Y, *, e)$ be a group which is a subalgebra. Then $e * e = e = 0$ and $x = e * x = 0 * x = 0$ for any $y \in Y$. Hence $|Y| = 1$. Thus any non-trivial d-algebra cannot be a Smarandache group-type d-algebra. □

Theorem 8. *Any non-trivial gBCK-algebra cannot be a Smarandache group-type gBCK-algebra.*

Proof. Let $(X, *, 0)$ be a $gBCK$-algebra and let $(Y, *, e)$ be a group which is a subalgebra. Then $e = e * e = 0$ and $x * x = 0 = e$ for any $x \in Y$. Hence $o(x) = 2$ for any $x \in Y$. Therefore $(Y, *, e = 0)$ is trivial or a Boolean group. Also, $(x * y) * z = (x * z) * (y * z)$ for any $z \in X$ and the fact that $(Y, *, e)$ is a group, requires $z = 0$ and $|Y| = 1$. Thus any non-trivial $gBCK$-algebra cannot be a Smarandache group-type $gBCK$-algebra. □

Theorem 9. *Any non-trivial group cannot be a Smarandache d-algebra-type group.*

Proof. Let $(X, *, e)$ be a group and let $(Y, *, 0)$ be a d-algebra which is a subalgebra. For any $x \in Y$, $x * x = 0, 0 * x = 0$. By cancellation law in the group, $x = 0$ for any $x \in Y$, i.e., $|Y| = 1$. Hence any non-trivial group cannot be a Smarandache d-algebra-type group. □

Theorem 10. *Any group cannot group be a Smarandache gBCK-algebra type group.*

Proof. Let $(X, *, e)$ be a group let $(Y, *, 0)$ be a $gBCK$-algebra which is a subalgebra. For any $x \in Y$, $x * 0 = x = x * e$ and so $e = 0$ by cancellation law in the group. Now consider the axiom $(x * y) * z = (x * z) * (y * z)$. Since $x * x = 0 = e$, Y consists of elements of order 2. Thus for any $x \in Y$, $x = x^{-1}$ means Y is not only a subalgebra but also a subgroup which is abelian. Hence $y = z * y$. By cancellation law and $y = e * y$, we obtain $z = e = 0$, for any $z \in Y$, i.e., $|Y| = 1$. Thus any group cannot group be a Smarandache $gBCK$-algebra type group. □

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