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Applications of Neutrosophic $\mathcal{N}$-Structures in n-Ary Groupoids

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Abstract. This paper includes the notions of neutrosophic n-ary $\mathcal{N}$-subgroupoids of n-ary groupoids and some properties.

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1. Introduction

In 1965, the degree of membership/truth (t) and the fuzzy set were introduced by Zadeh [12]. Atanassov [1] introduced the degree of nonmembership/falsehood (f) and defined the intuitionistic fuzzy set in 1986. Neutrosophy, means knowledge of neutral, is a branch of philosophy introduced as a theory of generalization of dialectic in 1995 by Smarandache. He proposed the term neutrosophic because neutrosophic originally comes from neutrosophy. In 1999, he introduced the concept of neutrosophic logics [9] and introduced the degree of indeterminancy/neuterality (i) and proposed the neutrosophic set on three components

$$(\text{t, i, f}) = (\text{truth, indeterminacy, falsehood}).$$

Jun et al. [11] introduced a negative-valued function and defined $\mathcal{N}$-structures in 2009. Khan et al. [4] investigated the notion of neutrosophic $\mathcal{N}$-structures and their applications in semigroups in 2017. Jun et al. [10, 11] considered neutrosophic $\mathcal{N}$-structures applied to BCK/BCI-algebras. Song et al. [8] proposed neutrosophic commutative $\mathcal{N}$-ideals in BCK-algebras in 2017. Rangsuk et al. [6] discussed neutrosophic $\mathcal{N}$-structures and their applications in UP-algebras.

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Algebraic systems with one n-ary operation, for $n > 2$, have been widely investigated (see, e.g., [2, 3, 5, 7]). Algebraic n-ary systems have been applied in several fields of mathematics.

The purpose of this paper is to investigate the extension of neutrosophic $\mathcal{N}$-structures in semigroups [4] to n-ary groupoids. Some basic notations and definitions will be presented in section 2. In section 3, we extend the results of neutrosophic $\mathcal{N}$-structures and their applications in semigroups to n-ary groupoids. Section 4 contains a brief summary of this paper.

2. Preliminaries

The aim of this section is to review some notations and definitions of n-ary groupoids and neutrosophic $\mathcal{N}$-structures which can be also found in [2–4].

2.1. n-ary groupoids

Definition 1. *Let S be a nonempty set. The set S together with an n-ary operation $f : S^n \to S$, where $n \geq 2$, is called an n-ary groupoid and is denoted by (S, f).*

According to the general convention used in the theory of n-ary groupoids, the sequence of elements $x_i, x_{i+1}, \ldots, x_j$ is denoted by x_i^j. In the case $j < i$, it is the empty symbol. If $x_{i+1} = x_{i+2} = \ldots = x_{i+t} = x$, then we write $x^{(t)}$ instead of x_{i+1}^{i+t}. In this convention, $f(x_1, x_2, \ldots, x_n) = f(x_1^n)$, and

$$f(x_1, \ldots, x_i, \underbrace{x, \ldots, x}_{t}, x_{i+t+1}, \ldots, x_n) = f(x_1^i, x^{(t)}, x_{i+t+1}^n).$$

Definition 2. *A nonempty subset T of an n-ary groupoids (S, f) is an n-ary subgroupoid of S if (T, f) is an n-ary groupoid, i.e., if it is closed under the operation f.*

2.2. Neutrosophic $\mathcal{N}$-structures

Definition 3. *A neutrosophic $\mathcal{N}$-structure over X is defined to be the structure*

$$X_N := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{(T_N(x), I_N(x), F_N(x))} \mid x \in X \right\}$$

where T_N, I_N and F_N are $\mathcal{N}$-functions on X which are called the truth membership function, the indeterminacy membership function and the falsity membership function on X, respectively.

Definition 4. *Let $X_N = \dfrac{X}{(T_N, I_N, F_N)}$ and $X_M = \dfrac{X}{(T_M, I_M, F_M)}$ be neutrosophic $\mathcal{N}$-structures over X.*

(1) *X_N is a neutrosophic $\mathcal{N}$-substructure of X_M over X, denoted by $X_N \subseteq X_M$, if it satisfies the conditions*

$$T_N(x) \geq T_M(x),\ I_N(x) \leq I_M(x),\ F_N(x) \geq F_M(x)$$

for all $x \in X$.

We have that $X_N \subseteq X_M$ and $X_M \subseteq X_N$ if and only if $X_N = X_M$.

(2) *The union of X_N and X_M, denoted it briefly by $X_{N\cup M}$, is defined to be a neutrosophic $\mathcal{N}$-structure*

$$X_{N\cup M} = \frac{X}{(T_{N\cup M}, I_{N\cup M}, F_{N\cup M})}$$

where

$$T_{N\cup M}(x) = \bigwedge \{T_N(x), T_M(x)\},$$
$$I_{N\cup M}(x) = \bigvee \{I_N(x), I_M(x)\},$$
$$F_{N\cup M}(x) = \bigwedge \{F_N(x), F_M(x)\}.$$

(3) *The intersection of X_N and X_M, written it simply as $X_{N\cap M}$, is defined to be a neutrosophic $\mathcal{N}$-structure*

$$X_{N\cap M} = \frac{X}{(T_{N\cap M}, I_{N\cap M}, F_{N\cap M})}$$

where

$$T_{N\cap M}(x) = \bigvee \{T_N(x), T_M(x)\},$$
$$I_{N\cap M}(x) = \bigwedge \{I_N(x), I_M(x)\},$$
$$F_{N\cap M}(x) = \bigvee \{F_N(x), F_M(x)\}.$$

Definition 5. *Let $X_N := \frac{X}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}$-structure over X. The complement of X_N, denoted by X_{N^c}, is defined to be a neutrosophic $\mathcal{N}$-structure*

$$X_{N^c} := \frac{X}{(T_{N^c}, I_{N^c}, F_{N^c})}$$

over X, where

$$T_{N^c}(x) = -1 - T_N(x),\ I_{N^c}(x) = -1 - I_N(x),\ F_{N^c}(x) = -1 - F_N(x)$$

for all $x \in X$.

Definition 6. *Let X_N be a neutrosophic $\mathcal{N}$-structure over X and let α, β, γ be real numbers such that $\alpha, \beta, \gamma \in [-1, 0]$. Define the sets*

$$\begin{aligned} T_N^{\alpha} &= \{x \in X \mid T_N(x) \leq \alpha\}, \\ I_N^{\beta} &= \{x \in X \mid I_N(x) \geq \beta\}, \\ F_N^{\gamma} &= \{x \in X \mid F_N(x) \leq \gamma\}. \end{aligned}$$

We call a set

$$X_N(\alpha, \beta, \gamma) = \{x \in X \mid T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}$$

an (α, β, γ)-level set of X_N.

For the convenience, we note that

$$X_N(\alpha, \beta, \gamma) = T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}.$$

From now on, an n-ary groupoid X denotes the universe of discourse unless otherwise specified.

3. Main results

In this section, we will look closely at neutrosophic n-ary $\mathcal{N}$-subgroupoids, the (α, β, γ)-level set, the intersection of neutrosophic n-ary $\mathcal{N}$-subgroupoids, neutrosophic n-ary $\mathcal{N}$-subgroupoid products, ε-neutrosophic n-ary $\mathcal{N}$-subgroupoids, homomorphic preimage of the neutrosophic n-ary $\mathcal{N}$-subgroupoids and onto homomorphic image of the neutrosophic n-ary $\mathcal{N}$-subgroupoids.

Definition 7. *Let $X_N := \dfrac{X}{(T_N, I_N, F_N)}$ be a neutrosophic structure over an n-ary groupoid X. Then X_N is called a neutrosophic n-ary $\mathcal{N}$-subgoupoid of X if the following conditions are valid:*

$$\begin{aligned} T_N(f(x_1^n)) &\leq \bigvee\{T_N(x_1), \ldots, T_N(x_n)\}, \\ I_N(f(x_1^n)) &\geq \bigwedge\{I_N(x_1), \ldots, I_N(x_n)\}, \\ F_N(f(x_1^n)) &\leq \bigvee\{F_N(x_1), \ldots, F_N(x_n)\}, \end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in X$.

Theorem 1. *Let X_N be a neutrosophic n-ary $\mathcal{N}$-subgroupoid of an n-ary groupoid X and let $\alpha, \beta, \gamma \in [-1, 0]$. If the (α, β, γ)-level set of X_N is nonempty, then it is an n-ary subgroupoid of X.*

Proof. Let $x_1, \ldots, x_n \in X_N(\alpha, \beta, \gamma)$. Then

$$T_N(x_1) \le \alpha, I_N(x_1) \ge \beta, F_N(x_1) \le \gamma, \ldots, T_N(x_n) \le \alpha, I_N(x_n) \ge \beta, F_N(x_n) \le \gamma.$$

It follows that

$$T_N(f(x_1^n)) \le \bigvee\{T_N(x_1), \ldots, T_N(x_n)\} \le \alpha,$$
$$I_N(f(x_1^n)) \ge \bigwedge\{I_N(x_1), \ldots, I_N(x_n)\} \ge \beta,$$
$$F_N(f(x_1^n)) \le \bigvee\{F_N(x_1), \ldots, F_N(x_n)\} \le \gamma.$$

Therefore $f(x_1^n) \in X_N(\alpha, \beta, \gamma)$. This implies that $X_N(\alpha, \beta, \gamma)$ is an n-ary subgroupoid of X.

Theorem 2. *Let X_N be a neutrosophic $\mathcal{N}$-structure over an n-ary groupoid X. If $T_N^{\alpha}, I_N^{\beta}$ and F_N^{γ} are n-ary subgroupoids of X for all $\alpha, \beta, \gamma \in [-1, 0]$, then X_N is a neutrosophic n-ary $\mathcal{N}$-subgroupoid of X.*

Proof. We prove this theorem by contradiction. Assume that there exist $x_1, \ldots, x_n \in X$ such that $T_N(f(x_1^n)) > \bigvee\{T_N(x_1), \ldots, T_N(x_n)\}$. Then

$$T_N(f(x_1^n)) > t_\alpha \ge \bigvee\{T_N(x_1), \ldots, T_N(x_n)\}$$

for some $t_\alpha \in [-1, 0)$. Thus $x_1, \ldots, x_n \in T_N^{t_\alpha}$ but $f(x_1^n) \notin T_N^{t_\alpha}$, which is a contradiction. Thus

$$T_N(f(x_1^n)) \le \bigvee\{T_N(x_1), \ldots, T_N(x_n)\}$$

for all $x_1, \ldots, x_n \in X$.

We now assume that $I_N(f(x_1^n)) < \bigvee\{I_N(x_1), \ldots, I_N(x_n)\}$ for some $x_1, \ldots, x_n \in X$. Then

$$I_N(f(x_1^n)) < t_\beta \le \bigvee\{I_N(x_1), \ldots, I_N(x_n)\}$$

for some $t_\beta \in [-1, 0)$. Thus $x_1, \ldots, x_n \in I_N^{t_\beta}$ but $f(x_1^n) \notin I_N^{t_\beta}$. This is a contradiction. Hence

$$I_N(f(x_1^n)) \ge \bigwedge\{I_N(x_1), \ldots, I_N(x_n)\}$$

for all $x_1, \ldots, x_n \in X$.

It remains to prove that

$$F_N(f(x_1^n)) \le \bigvee\{F_N(x_1), \ldots, F_N(x_n)\}$$

for all $x_1, \ldots, x_n \in X$. Suppose contrary to our claim that there are $x_1, \ldots, x_n \in X$ such that $F_N(f(x_1^n)) > \bigvee\{F_N(x_1), \ldots, F_N(x_n)\}$. Then

$$F_N(f(x_1^n)) > t_\gamma \ge \bigvee\{F_N(x_1), \ldots, F_N(x_n)\}$$

for some $t_\gamma \in [-1, 0)$. Thus $x_1, \ldots, x_n \in F_N^{t_\gamma}$ but $f(x_1^n) \notin F_N^{t_\gamma}$, which is a contradiction.

Therefore X_N is a neutrosophic n-ary $\mathcal{N}$-subgroupoid of X.

Theorem 3. *Let* $X_N := \dfrac{X}{(T_N, I_N, F_N)}$ *and* $X_M := \dfrac{X}{(T_M, I_M, F_M)}$ *be two neutrosophic* n*-ary* $\mathcal{N}$*-subgroupoids over* X*. Then* $X_{N\cap M}$ *is also a neutrosophic* n*-ary* $\mathcal{N}$*-subgroupoid of* X*.*

Proof. Let $x_1, \ldots, x_n \in X$. We obtain

$$\begin{aligned}
T_{N\cap M}(f(x_1^n)) &= \bigvee \{T_N(f(x_1^n)), T_M(f(x_1^n))\} \\
&\leq \bigvee \left\{ \bigvee \{T_N(x_1), \ldots, T_N(x_n)\}, \bigvee \{T_M(x_1), \ldots, T_M(x_n)\} \right\} \\
&= \bigvee \left\{ \bigvee \{T_N(x_1), T_M(x_1)\}, \ldots, \bigvee \{T_N(x_n), T_M(x_n)\} \right\} \\
&= \bigvee \{T_{N\cap M}(x_1), \ldots, T_{N\cap M}(x_n)\}, \\
I_{N\cap M}(f(x_1^n)) &= \bigwedge \{I_N(f(x_1^n)), I_M(f(x_1^n))\} \\
&\geq \bigwedge \left\{ \bigwedge \{I_N(x_1), \ldots, I_N(x_n)\}, \bigwedge \{I_M(x_1), \ldots, I_M(x_n)\} \right\} \\
&= \bigwedge \left\{ \bigwedge \{I_N(x_1), I_M(x_1)\}, \ldots, \bigwedge \{I_N(x_n), I_M(x_n)\} \right\} \\
&= \bigwedge \{I_{N\cap M}(x_1), \ldots, I_{N\cap M}(x_n)\}, \\
F_{N\cap M}(f(x_1^n)) &= \bigvee \{F_N(f(x_1^n)), F_M(f(x_1^n))\} \\
&\leq \bigvee \left\{ \bigvee \{F_N(x_1), \ldots, F_N(x_n)\}, \bigvee \{F_M(x_1), \ldots, F_M(x_n)\} \right\} \\
&= \bigvee \left\{ \bigvee \{F_N(x_1), F_M(x_1)\}, \ldots, \bigvee \{F_N(x_n), F_M(x_n)\} \right\} \\
&= \bigvee \{F_{N\cap M}(x_1), \ldots, F_{N\cap M}(x_n)\}
\end{aligned}$$

for all $x_1, \ldots, x_n \in X$. Therefore $X_{N\cap M}$ is a neutrosophic n-ary $\mathcal{N}$-subgroupoid of X.

Corollary 1. *Let* $\{X_{N_i} \mid i \in \mathbb{N}\}$ *be a family of neutrosophic* n*-ary* $\mathcal{N}$*-subgroupoids of an* n*-ary groupoid* X*.Then* $\bigcap_{i\in\mathbb{N}} X_{N_i}$ *is also a neutrosophic* n*-ary* $\mathcal{N}$*-subgroupoid of* X*.*

For each $i \in \{1, 2, \ldots, n\}$, let $X_{N_i} := \dfrac{X}{(T_{N_i}, I_{N_i}, F_{N_i})}$ be a neutrosophic $\mathcal{N}$-structure over an n-ary groupoid (X, f). Then a neutrosophic $\mathcal{N}$-structure over X

$$\begin{aligned}
X_{N_1} \odot \ldots \odot X_{N_n} &= \frac{X}{(T_{N_1} \odot \ldots \odot T_{N_n}, I_{N_1} \odot \ldots \odot I_{N_n}, F_{N_1} \odot \ldots \odot F_{N_n})} \\
&= \left\{ \frac{x}{T_{N_1} \odot \ldots \odot T_{N_n}(x), I_{N_1} \odot \ldots \odot I_{N_n}(x), F_{N_1} \odot \ldots \odot F_{N_n}(x)} \middle| x \in X \right\}
\end{aligned}$$

is defined to be a neutrosophic $\mathcal{N}$-product of $X_{N_1}, X_{N_2}, \ldots, X_{N_n}$ where

$$T_{N_1} \odot \ldots \odot T_{N_n}(x) = \begin{cases} \bigwedge\limits_{x=f(x_1^n)} \{T_{N_1}(x_1) \vee ... \vee T_{N_n}(x_n)\}, & \text{if } x = f(x_1^n) \ \exists\, x_1, ..., x_n \in X, \\ 0, & \text{otherwise,} \end{cases}$$

$$I_{N_1} \odot \ldots \odot I_{N_n}(x) = \begin{cases} \bigvee_{x=f(x_1^n)} \{I_{N_1}(x_1) \wedge \ldots \wedge I_{N_n}(x_n)\}, & \text{if } x = f(x_1^n)\ \exists x_1, \ldots, x_n \in X, \\ -1, & \text{otherwise,} \end{cases}$$

and

$$F_{N_1} \odot \ldots \odot F_{N_n}(x) = \begin{cases} \bigwedge_{x=f(x_1^n)} \{F_{N_1}(x_1) \vee \ldots \vee F_{N_n}(x_n)\}, & \text{if } x = f(x_1^n)\ \exists x_1, \ldots, x_n \in X, \\ 0, & \text{otherwise.} \end{cases}$$

If $X_N = X_{N_1} = X_{N_2} = \ldots = X_{N_n}$, then $X_{N_1} \odot \ldots \odot X_{N_n}$ is denoted by $\odot(X_N)^{(n)}$. For any $x \in X$, the element $\dfrac{x}{\odot(T_N)^{(n)}(x), \odot(I_N)^{(n)}(x), \odot(F_N)^{(n)}(x)}$ is denoted by

$$\odot(X_N)^{(n)}(x) := \left(\odot(T_N)^{(n)}(x), \odot(I_N)^{(n)}(x), \odot(F_N)^{(n)}(x)\right).$$

Theorem 4. *A neutrosophic $\mathcal{N}$-structure X_N over X is a neutrosophic $\mathcal{N}$-subgroupoid of X if and only if $\odot(X_N)^{(n)} \subseteq X_N$.*

Proof. We first prove that if a neutrosophic $\mathcal{N}$-structure X_N over X is a neutrosophic n-ary $\mathcal{N}$-subgroupoid of X, then $\odot(X_N)^{(n)} \subseteq X_N$. We assume that X_N is a neutrosophic n-ary $\mathcal{N}$-subgroupoid of X and let $x \in X$. If $x \neq f(x_1^n)$ for all $x_1, \ldots, x_n \in X$, then this clearly forces $\odot(X_N)^{(n)} \subseteq X_N$. Suppose that there are $x_1, \ldots, x_n \in X$ such that $x = f(x_1^n)$, we obtain

$$\begin{aligned} \odot(T_N)^{(n)}(x) &= \bigwedge_{x=f(x_1^n)} \{T_N(x_1) \vee \ldots \vee T_N(x_n)\} \geq \bigwedge_{x=f(x_1^n)} T_N(f(x_1^n)) = T_N(x), \\ \odot(I_N)^{(n)}(x) &= \bigvee_{x=f(x_1^n)} \{I_N(x_1) \wedge \ldots \wedge I_N(x_n)\} \leq \bigvee_{x=f(x_1^n)} I_N(f(x_1^n)) = I_N(x), \\ \odot(F_N)^{(n)}(x) &= \bigwedge_{x=f(x_1^n)} \{F_N(x_1) \vee \ldots \vee F_N(x_n)\} \geq \bigwedge_{x=f(x_1^n)} F_N(f(x_1^n)) = F_N(x). \end{aligned}$$

Therefore $\odot(X_N)^{(n)} \subseteq X_N$.

Conversely, let X_N be any neutrosophic n-ary $\mathcal{N}$-subgroupoid of X such that $\odot(X_N)^{(n)} \subseteq X_N$. We only need to show that X_N is a neutrosophic n-ary $\mathcal{N}$-subgroupoid of X. Let $x_1, \ldots, x_n$ be elements of X and let $x = f(x_1^n)$. Then

$$\begin{aligned} T_N(f(x_1^n)) = T_N(x) \leq \odot(T_N)^{(n)}(x) &= \bigwedge_{x=f(x_1^n)} \{T_N(x_1) \vee \ldots \vee T_N(x_n)\} \\ &\leq T_N(x_1) \vee \ldots \vee T_N(x_n), \\ I_N(f(x_1^n)) = I_N(x) \geq \odot(I_N)^{(n)}(x) &= \bigvee_{x=f(x_1^n)} \{I_N(x_1) \wedge \ldots \wedge I_N(x_n)\} \\ &\geq I_N(x_1) \wedge \ldots \wedge I_N(x_n), \end{aligned}$$

$$F_N(f(x_1^n)) = F_N(x) \leq \odot(F_N)^{(n)}(x) = \bigwedge_{x=f(x_1^n)} \{F_N(x_1) \vee \ldots \vee F_N(x_n)\}$$
$$\leq F_N(x_1) \vee \ldots \vee F_N(x_n).$$

Therefore X_N is a neutrosophic n-ary $\mathcal{N}$-subgroupoid of X.

Theorem 5. *Let X be an n-ary groupoid with identity e and let $X_N := \dfrac{X}{(T_N, I_N, F_N)}$ be a neutrosophic n-ary $\mathcal{N}$-subgroupoid over X such that*

$$X_N(e) \geq X_N(x)$$

for all $x \in X$, that is, $T_N(e) \leq T_N(x), I_N(e) \geq I_N(x)$ and $F_N(e) \leq F_N(x)$ for all $x \in X$. If X_N is a neutrosophic n-ary $\mathcal{N}$-subgroupoid of X, then $\odot(X_N)^{(n)} = X_N$.

Proof. For any $x \in X$, we have

$$\odot(T_N)^{(n)}(x) = \bigwedge_{x=f(x_1^n)} \{T_N(x_1) \vee \ldots \vee T_N(x_n)\} \leq T_N(x) \vee T_N(e) = T_N(x),$$
$$\odot(I_N)^{(n)}(x) = \bigvee_{x=f(x_1^n)} \{I_N(x_1) \wedge \ldots \wedge I_N(x_n)\} \geq I_N(x) \wedge I_N(e) = I_N(x),$$
$$\odot(F_N)^{(n)}(x) = \bigwedge_{x=f(x_1^n)} \{F_N(x_1) \vee \ldots \vee F_N(x_n)\} \leq F_N(x) \vee F_N(e) = F_N(x).$$

This shows that $X_N \subseteq \odot(X_N)^{(n)}$. From Theorem 4, we already have $\odot(X_N)^{(n)} \subseteq X_N$. Then $\odot(X_N)^{(n)} = X_N$.

Definition 8. *A neutrosophic $\mathcal{N}$-structure X_N over X is said to be an ε-neutrosophic n-ary $\mathcal{N}$-subgroupoid of X if the conditions*

$$T_N(f(x_1^n)) \leq \bigvee \{T_N(x_1), \ldots, T_N(x_n), \varepsilon_T\},$$
$$I_N(f(x_1^n)) \geq \bigwedge \{I_N(x_1), \ldots, I_N(x_n), \varepsilon_I\},$$
$$F_N(f(x_1^n)) \leq \bigvee \{F_N(x_1), \ldots, F_N(x_n), \varepsilon_F\},$$

hold for all $x_1, \ldots, x_n \in X$ where $\varepsilon_T, \varepsilon_I, \varepsilon_F \in [-1, 0]$.

Proposition 1. *Let X_N be an ε-neutrosophic n-ary $\mathcal{N}$-subgroupoid of X. If $X_N(x) \leq (\varepsilon_T, \varepsilon_I, \varepsilon_F)$, that is, $T_N(x) \geq \varepsilon_T, I_N(x) \leq \varepsilon_I, F_N(x) \geq \varepsilon_F$ for all $x \in X$, then X_N is a neutrosophic n-ary $\mathcal{N}$-subgroupoid of X.*

Theorem 6. *Let X_N be a neutrosophic $\mathcal{N}$-structure over X and let α, β, γ be real numbers on the interval $[-1, 0]$. If X_N is an ε-neutrosophic n-ary $\mathcal{N}$-subgroupoid of X, then the (α, β, γ)-level set of X_N is an n-ary subgroupoid of X whenever $(\alpha, \beta, \gamma) \leq (\varepsilon_T, \varepsilon_I, \varepsilon_F)$, that is $\alpha \geq \varepsilon_T, \beta \leq \varepsilon_I$, and $\gamma \geq \varepsilon_F$.*

Proof. Assume that $X_N(\alpha,\beta,\gamma) \neq \emptyset$ for $\alpha,\beta,\gamma \in [-1,0]$. Let $x_1,\dots,x_n \in X_N(\alpha,\beta,\gamma)$. Then $T_N(x_1) \leq \alpha, I_N(x_1) \geq \beta, F_N(x_1) \leq \gamma, \dots, T_N(x_n) \leq \alpha, I_N(x_n) \geq \beta, F_N(x_n) \leq \gamma$. It follows that

$$\begin{aligned} T_N(f(x_1^n)) &\leq \bigvee \{T_N(x_1),\dots,T_N(x_n),\varepsilon_T\} \leq \bigvee \{\alpha,\varepsilon_T\} = \alpha, \\ I_N(f(x_1^n)) &\geq \bigwedge \{I_N(x_1),\dots,I_N(x_n),\varepsilon_I\} \geq \bigwedge \{\beta,\varepsilon_I\} = \beta, \\ F_N(f(x_1^n)) &\leq \bigvee \{F_N(x_1),\dots,F_N(x_n),\varepsilon_F\} \leq \bigvee \{\gamma,\varepsilon_F\} = \gamma. \end{aligned}$$

Hence $f(x_1^n) \in X_N(\alpha,\beta,\gamma)$. It follows that $X_N(\alpha,\beta,\gamma)$ is an n-ary subgroupoid of X.

Theorem 7. *Let X_N be a neutrosophic $\mathcal{N}$-structure over X and let α,β,γ be real numbers on the interval $[-1,0]$. If T_N^α, I_N^β and F_N^γ are n-ary subgroupoids of X for all $\varepsilon_T,\varepsilon_I,\varepsilon_F \in [-1,0]$ and $(\alpha,\beta,\gamma) \leq (\varepsilon_T,\varepsilon_I,\varepsilon_F)$, then X_N is an ε-neutrosophic n-ary $\mathcal{N}$-subgroupoid of X.*

Proof. We prove this theorem by contradiction. We begin the proof by assuming that

$$T_N(f(x_1^n)) > \bigvee \{T_N(x_1),\dots,T_N(x_n),\varepsilon_T\}.$$

for some $x_1,\dots,x_n \in X$. Then

$$T_N(f(x_1^n)) > t_\alpha \geq \bigvee \{T_N(x_1),\dots,T_N(x_n),\varepsilon_T\}$$

for some $t_\alpha \in [-1,0)$. It follows that $x_1,\dots,x_n \in T_N^{t_\alpha}$, $f(x_1^n) \notin T_N^{t_\alpha}$ and $t_\alpha \geq \varepsilon_T$. This is a contradiction since $T_N^{t_\alpha}$ is an n-ary subgroupoid of X by hypothesis. Thus

$$T_N(f(x_1^n)) \leq \bigvee \{T_N(x_1),\dots,T_N(x_n),\varepsilon_T\}$$

for all $x_1,\dots,x_n \in X$.

Suppose now that there are $x_1,\dots,x_n \in X$ such that

$$I_N(f(x_1^n)) < \bigwedge \{I_N(x_1),\dots,I_N(x_n),\varepsilon_I\}.$$

Then

$$I_N(f(x_1^n)) < t_\beta \leq \bigwedge \{I_N(x_1),\dots,I_N(x_n),\varepsilon_I\}$$

for some $t_\beta \in [-1,0)$. It follows that $x_1,\dots,x_n \in I_N^{t_\beta}$, $f(x_1^n) \notin I_N^{t_\beta}$ and $t_\beta \leq \varepsilon_I$. This contradicts to the fact that $I_N^{t_\beta}$ is an n-ary subgroupoid of X. Thus

$$I_N(f(x_1^n)) \geq \bigwedge \{I_N(x_1),\dots,I_N(x_n),\varepsilon_I\}$$

for all $x_1,\dots,x_n \in X$.

Similarly, assume that

$$F_N(f(x_1^n)) > \bigvee \{F_N(x_1),\dots,F_N(x_n),\varepsilon_F\}$$

for some $x_1, \ldots, x_n \in X$. Then

$$F_N(x_1^n) > t_\gamma \geq \bigvee \{F_N(x_1), \ldots, F_N(x_n), \varepsilon_F\}$$

for some $t_\gamma \in [-1, 0)$. It implies that $x_1, \ldots, x_n \in F_N^{t_\gamma}$, $f(x_1^n) \notin F_N^{t_\gamma}$ and $t_\gamma \geq \varepsilon_F$. This is a contradiction since $F_N^{t_\gamma}$ is an n-ary subgroupoid of X. Thus

$$F_N(f(x_1^n)) \leq \bigvee \{F_N(x_1), \ldots, F_N(x_n), \varepsilon_F\}$$

for all $x_1, \ldots, x_n \in X$. Therefore X_N is an ε-neutrosophic n-ary $\mathcal{N}$-subgroupoid of X.

Theorem 8. *Let $\varepsilon_T, \varepsilon_I, \varepsilon_F, \delta_T, \delta_I, \delta_F \in [-1, 0]$. Let X_N and X_M be an ε-neutrosophic n-ary $\mathcal{N}$-subgroupoid and a δ-neutrosophic n-ary $\mathcal{N}$-subgroupoid of X, respectively. The intersection of X_N and X_M is a ξ-neutrosophic n-ary $\mathcal{N}$-subgroupoid of X for $\xi := \varepsilon \wedge \delta$ where $(\xi_T, \xi_I, \xi_F) = (\varepsilon_T \vee \delta_T, \varepsilon_I \wedge \delta_I, \varepsilon_F \vee \delta_F)$.*

Proof. For any $x_1, \ldots, x_n \in X$, we have

$$\begin{aligned}
T_{N\cap M}(f(x_1^n)) &= \bigvee \{T_N(f(x_1^n)), T_M(f(x_1^n))\} \\
&\leq \bigvee \left\{\bigvee \{T_N(x_1), \ldots, T_N(x_n), \varepsilon_T\}, \bigvee \{T_M(x_1), \ldots, T_M(x_n), \delta_T\}\right\} \\
&\leq \bigvee \left\{\bigvee \{T_N(x_1), \ldots, T_N(x_n), \xi_T\}, \bigvee \{T_M(x_1), \ldots, T_M(x_n), \xi_T\}\right\} \\
&= \bigvee \left\{\bigvee \{T_N(x_1), T_M(x_1), \xi_T\}, \ldots, \bigvee \{T_N(x_n), T_M(x_n), \xi_T\}\right\} \\
&= \bigvee \left\{\bigvee \{T_N(x_1), T_M(x_1)\}, \ldots, \bigvee \{T_N(x_n), T_M(x_n)\}, \xi_T\right\} \\
&= \bigvee \{T_{N\cap M}(x_1), \ldots, T_{N\cap M}(x_n), \xi_T\}, \\
I_{N\cap M}(f(x_1^n)) &= \bigwedge \{I_N(f(x_1^n)), I_M(f(x_1^n))\} \\
&\geq \bigwedge \left\{\bigwedge \{I_N(x_1), \ldots, I_N(x_n), \varepsilon_I\}, \bigwedge \{I_M(x_1), \ldots, I_M(x_n), \delta_I\}\right\} \\
&\geq \bigwedge \left\{\bigwedge \{I_N(x_1), \ldots, I_N(x_n), \xi_I\}, \bigwedge \{I_M(x_1), \ldots, I_M(x_n), \xi_I\}\right\} \\
&= \bigwedge \left\{\bigwedge \{I_N(x_1), I_M(x_1), \xi_I\}, \ldots, \bigwedge \{I_N(x_n), I_M(x_n), \xi_I\}\right\} \\
&= \bigwedge \left\{\bigwedge \{I_N(x_1), I_M(x_1)\}, \ldots, \bigwedge \{I_N(x_n), I_M(x_n)\}, \xi_I\right\} \\
&= \bigwedge \{I_{N\cap M}(x_1), \ldots, I_{N\cap M}(x_n), \xi_I\}, \\
F_{N\cap M}(f(x_1^n)) &= \bigvee \{F_N(f(x_1^n)), F_M(f(x_1^n))\} \\
&\leq \bigvee \left\{\bigvee \{F_N(x_1), \ldots, F_N(x_n), \varepsilon_F\}, \bigvee \{F_M(x_1), \ldots, F_M(x_n), \delta_F\}\right\} \\
&\leq \bigvee \left\{\bigvee \{F_N(x_1), \ldots, F_N(x_n), \xi_F\}, \bigvee \{F_M(x_1), \ldots, F_M(x_n), \xi_F\}\right\} \\
&= \bigvee \left\{\bigvee \{F_N(x_1), F_M(x_1), \xi_F\}, \ldots, \bigvee \{F_N(x_n), F_M(x_n), \xi_F\}\right\}
\end{aligned}$$

$$= \bigvee \left\{ \bigvee \{F_N(x_1), F_M(x_1)\}, \ldots, \bigvee \{F_N(x_n), F_M(x_n)\}, \xi_F \right\}$$
$$= \bigvee \{F_{N\cap M}(x_1), \ldots, F_{N\cap M}(x_n), \xi_F\}.$$

Therefore $X_{N\cap M}$ is a ξ-neutrosophic n-ary $\mathcal{N}$-subgroupoid of X.

Theorem 9. *Let X_N be an ε-neutrosophic n-ary $\mathcal{N}$-subgroupoid of X. If*

$$\kappa := (\kappa_T, \kappa_I, \kappa_F) = \left(\bigvee_{x\in X} \{T_N(x)\}, \bigwedge_{x\in X} \{I_N(x)\}, \bigvee_{x\in X} \{F_N(x)\} \right)$$

then the set

$$\Omega := \{x \in X \mid T_N(x) \le \kappa_T \vee \varepsilon_T,\ I_N(x) \ge \kappa_I \wedge \varepsilon_I,\ F_N(x) \le \kappa_F \vee \varepsilon_F\}$$

is an n-ary subgroupoid of X.

Proof. Let $x_1, \ldots, x_n \in \Omega$ for any $x_1, \ldots, x_n \in X$. Then

$$T_N(x_1) \le \kappa_T \vee \varepsilon_T = \bigvee_{x_1\in X} \{T_N(x_1)\} \vee \varepsilon_T,$$
$$I_N(x_1) \ge \kappa_I \wedge \varepsilon_I = \bigwedge_{x_1\in X} \{I_N(x_1)\} \wedge \varepsilon_I,$$
$$F_N(x_1) \le \kappa_F \vee \varepsilon_F = \bigvee_{x_1\in X} \{F_N(x_1)\} \vee \varepsilon_F,$$
$$\vdots$$
$$T_N(x_n) \le \kappa_T \vee \varepsilon_T = \bigvee_{x_n\in X} \{T_N(x_n)\} \vee \varepsilon_T,$$
$$I_N(x_n) \ge \kappa_I \wedge \varepsilon_I = \bigwedge_{x_n\in X} \{I_N(x_n)\} \wedge \varepsilon_I,$$
$$F_N(x_n) \le \kappa_F \vee \varepsilon_F = \bigvee_{x_n\in X} \{F_N(x_n)\} \vee \varepsilon_F.$$

It follows that

$$T_N(f(x_1^n)) \le \bigvee \{T_N(x_1), \ldots, T_N(x_n), \varepsilon_T\}$$
$$\le \bigvee \{\kappa_T \vee \varepsilon_T, \ldots, \kappa_T \vee \varepsilon_T, \varepsilon_T\}$$
$$= \kappa_T \vee \varepsilon_T,$$
$$I_N(f(x_1^n)) \ge \bigwedge \{I_N(x_1), \ldots, I_N(x_n), \varepsilon_I\}$$
$$\ge \bigwedge \{\kappa_I \wedge \varepsilon_I, \ldots, \kappa_I \wedge \varepsilon_I, \varepsilon_I\}$$
$$= \kappa_I \wedge \varepsilon_I,$$

$$\begin{aligned}F_N(f(x_1^n)) &\leq \bigvee\{F_N(x_1),\ldots,F_N(x_n),\varepsilon_F\}\\ &\leq \bigvee\{\kappa_F\vee\varepsilon_F,\ldots,\kappa_F\vee\varepsilon_F,\varepsilon_F\}\\ &= \kappa_F\vee\varepsilon_F,\end{aligned}$$

Then $f(x_1^n)\in\Omega$. Hence Ω is an n-ary subgroupoid of X.

Let X and Y be sets, $g : X \to Y$ be a function, $Y_N := \dfrac{Y}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}$-structure over Y with $\varepsilon = (\varepsilon_T, \varepsilon_I, \varepsilon_F)$. An ε-neutrosophic $\mathcal{N}$-structure over X is defined by $X_N^\varepsilon := \dfrac{X}{(T_N^\varepsilon, I_N^\varepsilon, F_N^\varepsilon)}$ where

$$\begin{aligned}T_N^\varepsilon &: X \to [-1,0], x \mapsto \vee\{T_N(g(x)),\varepsilon_T\},\\ I_N^\varepsilon &: X \to [-1,0], x \mapsto \wedge\{I_N(g(x)),\varepsilon_I\},\\ F_N^\varepsilon &: X \to [-1,0], x \mapsto \vee\{F_N(g(x)),\varepsilon_F\}.\end{aligned}$$

Theorem 10. *Let X, Y be two n-ary groupoids and $g : X \to Y$ be a homomorphism. If a neutrosophic $\mathcal{N}$-structure $Y_N := \dfrac{Y}{(T_N, I_N, F_N)}$ over Y is an ε-neutrosophic n-ary $\mathcal{N}$-subgroupoid of Y, then $X_N^\varepsilon := \dfrac{X}{(T_N^\varepsilon, I_N^\varepsilon, F_N^\varepsilon)}$ is an ε-neutrosophic n-ary $\mathcal{N}$-subgroupoid of X.*

Proof. For any $x_1,\ldots,x_n \in X$, we have

$$\begin{aligned}T_N^\varepsilon(f(x_1^n)) &= \bigvee\{T_N(g(f(x_1^n))),\varepsilon_T\}\\ &= \bigvee\{T_N(g(x_1)\ldots g(x_n)),\varepsilon_T\}\\ &\leq \bigvee\{\bigvee\{T_N(g(x_1)),\ldots,T_N(g(x_n)),\varepsilon_T\},\varepsilon_T\}\\ &= \bigvee\{\bigvee\{T_N(g(x_1)),\varepsilon_T\},\ldots,\bigvee\{T_N(g(x_n)),\varepsilon_T\},\varepsilon_T\}\\ &= \bigvee\{T_N^\varepsilon(x_1),\ldots,T_N^\varepsilon(x_n),\varepsilon_T\},\\ I_N^\varepsilon(f(x_1^n)) &= \bigwedge\{I_N(g(f(x_1^n))),\varepsilon_I\}\\ &= \bigwedge\{I_N(g(x_1)\ldots g(x_n)),\varepsilon_I\}\\ &\geq \bigwedge\{\bigwedge\{I_N(g(x_1)),\ldots,I_N(g(x_n)),\varepsilon_I\},\varepsilon_I\}\\ &= \bigwedge\{\bigwedge\{I_N(g(x_1)),\varepsilon_I\},\ldots,\bigwedge\{I_N(g(x_n)),\varepsilon_I\},\varepsilon_I\}\\ &= \bigwedge\{I_N^\varepsilon(x_1),\ldots,I_N^\varepsilon(x_n),\varepsilon_I\},\\ F_N^\varepsilon(f(x_1^n)) &= \bigvee\{F_N(g(f(x_1^n))),\varepsilon_F\}\\ &= \bigvee\{F_N(g(x_1)\ldots g(x_n)),\varepsilon_F\}\end{aligned}$$

$$\leq \bigvee\{\bigvee\{F_N(g(x_1)),\ldots,F_N(g(x_n)),\varepsilon_F\},\varepsilon_F\}$$
$$= \bigvee\{\bigvee\{F_N(g(x_1)),\varepsilon_F\},\ldots,\bigvee\{F_N(g(x_n)),\varepsilon_F\},\varepsilon_F\}$$
$$= \bigvee\{F_N^\varepsilon(x_1),\ldots,F_N^\varepsilon(x_n),\varepsilon_F\}.$$

Therefore $X_N^\varepsilon := \dfrac{X}{(T_N^\varepsilon, I_N^\varepsilon, F_N^\varepsilon)}$ is an ε-neutrosophic n-ary $\mathcal{N}$-subgroupoid of X.

Let X, Y be two sets and $g : X \to Y$ be a function. If $Y_M := \dfrac{Y}{(T_M, I_M, F_M)}$ is a neutrosophic $\mathcal{N}$-structure over Y, then the preimage of Y_M under g is a neutrosophic $\mathcal{N}$-structure over X defined by

$$g^{-1}(Y_M) := \frac{X}{(g^{-1}(T_M), g^{-1}(I_M), g^{-1}(F_M))}$$

where $g^{-1}(T_M)(x) = T_M(g(x)), g^{-1}(I_M)(x) = I_M(g(x))$, and $g^{-1}(F_M)(x) = F_M(g(x))$ for all $x \in X$.

Theorem 11. *Let X, Y be two n-ary groupoids and $g : X \to Y$ be a homomorphism. If $Y_M := \dfrac{Y}{(T_M, I_M, F_M)}$ is a neutrosophic n-ary $\mathcal{N}$-subgroupoid of Y, then the preimage of Y_M under g,*

$$g^{-1}(Y_M) = \frac{X}{(g^{-1}(T_M), g^{-1}(I_M), g^{-1}(F_M))},$$

is a neutrosophic n-ary $\mathcal{N}$-subgroupoid of X.

Proof. For any $x_1, \ldots, x_n \in X$, we have

$$\begin{aligned}
g^{-1}(T_M)(f(x_1^n)) &= T_M(g(f(x_1^n))) = T_M(g(x_1)\ldots g(x_n)) \\
&\leq \bigvee \{T_M(g(x_1)),\ldots,T_M(g(x_n))\} \\
&= \bigvee \left\{g^{-1}(T_M)(x_1),\ldots,g^{-1}(T_M)(x_n)\right\}, \\
g^{-1}(I_M)(f(x_1^n)) &= I_M(g(f(x_1^n))) = I_M(g(x_1)\ldots g(x_n)) \\
&\geq \bigwedge \{I_M(g(x_1)),\ldots,I_M(g(x_n))\} \\
&= \bigwedge \left\{g^{-1}(I_M)(x_1),\ldots,g^{-1}(I_M)(x_n)\right\}, \\
g^{-1}(F_M)(f(x_1^n)) &= F_M(g(f(x_1^n))) = F_M(g(x_1)\ldots g(x_n)) \\
&\leq \bigvee \{F_M(g(x_1)),\ldots,F_M(g(x_n))\} \\
&= \bigvee \left\{g^{-1}(F_M)(x_1),\ldots,g^{-1}(F_M)(x_n)\right\}.
\end{aligned}$$

Therefore $g^{-1}(Y_M)$ is a neutrosophic n-ary $\mathcal{N}$-subgroupoid of X.

Let X, Y be two sets and $g : X \to Y$ be an onto function. If $X_N := \dfrac{X}{(T_N, I_N, F_N)}$ is a neutrosophic $\mathcal{N}$-structure over X, then the image of X_N under g is a neutrosophic $\mathcal{N}$-structure over Y defined by

$$g(X_N) := \frac{Y}{(g(T_N), g(I_N), g(F_N))}$$

where

$$\begin{aligned} g(T_N)(y) &= \bigwedge_{x \in g^{-1}(y)} T_N(x), \\ g(I_N)(y) &= \bigvee_{x \in g^{-1}(y)} I_N(x), \\ g(F_N)(y) &= \bigwedge_{x \in g^{-1}(y)} F_N(x). \end{aligned}$$

Theorem 12. *Let X, Y be two n-ary groupoids and let $g : X \to Y$ be an onto homomorphism. Let $X_N := \dfrac{X}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}$-structure of X such that for all $A \subseteq X$, there is $x_0 \in A$ such that*

$$T_N(x_0) = \bigwedge_{z \in A} T_N(z), \qquad I_N(x_0) = \bigvee_{z \in A} I_N(z), \qquad F_N(x_0) = \bigwedge_{z \in A} F_N(z).$$

If X_N is a neutrosophic n-ary $\mathcal{N}$-subgroupoid of X, then the image of X_N under g,

$$g(X_N) = \frac{Y}{(g(T_N), g(I_N), g(F_N))},$$

is a neutrosophic n-ary $\mathcal{N}$-subgroupoid of Y.

Proof. Let

$$g(X_N) = \frac{Y}{(g(T_N), g(I_N), g(F_N))}$$

be the image of X_N under g. Let $y_1, \ldots, y_n \in Y$. Then $g^{-1}(y_1) \neq \emptyset, \ldots, , g^{-1}(y_n) \neq \emptyset$ in X which implies that there are $x_{y_1} \in g^{-1}(y_1), \ldots, x_{y_n} \in g^{-1}(y_n)$ such that

$$\begin{aligned} &T_N(x_{y_1}) = \bigwedge_{z_1 \in g^{-1}(y_1)} T_N(z_1), \; I_N(x_{y_1}) = \bigvee_{z_1 \in g^{-1}(y_1)} I_N(z_1), \; F_N(x_{y_1}) = \bigwedge_{z_1 \in g^{-1}(y_1)} F_N(z_1), \\ &\quad \vdots \\ &T_N(x_{y_n}) = \bigwedge_{z_n \in g^{-1}(y_n)} T_N(z_n), \; I_N(x_{y_n}) = \bigvee_{z_n \in g^{-1}(y_n)} I_N(z_n), \; F_N(x_{y_n}) = \bigwedge_{z_n \in g^{-1}(y_n)} F_N(z_n). \end{aligned}$$

Hence

$$\begin{aligned}
g(T_N)(y_1^n) &= \bigwedge_{x\in g^{-1}(y_1^n)} T_N(x) \leq T_N(x_{y_1}\dots x_{y_n}) \\
&\leq \bigvee \{T_N(x_{y_1}),\dots,T_N(x_{y_n})\} \\
&= \bigvee \left\{ \bigwedge_{z_1\in g^{-1}(y_1)} T_N(z_1),\dots, \bigwedge_{z_n\in g^{-1}(y_n)} T_N(z_n) \right\} \\
&= \bigvee \{g(T_N)(y_1),\dots,g(T_N)(y_n)\}, \\
g(I_N)(y_1^n) &= \bigvee_{x\in g^{-1}(y_1^n)} I_N(x) \geq I_N(x_{y_1}\dots x_{y_n}) \\
&\geq \bigwedge \{I_N(x_{y_1}),\dots,I_N(x_{y_n})\} \\
&= \bigwedge \left\{ \bigvee_{z_1\in g^{-1}(y_1)} I_N(z_1),\dots, \bigvee_{z_n\in g^{-1}(y_n)} I_N(z_n) \right\} \\
&= \bigwedge \{g(I_N)(y_1),\dots,g(I_N)(y_n)\}, \\
g(F_N)(y_1^n) &= \bigwedge_{x\in g^{-1}(y_1^n)} F_N(x) \leq F_N(x_{y_1}\dots x_{y_n}) \\
&\leq \bigvee \{F_N(x_{y_1}),\dots,F_N(x_{y_n})\} \\
&= \bigvee \left\{ \bigwedge_{z_1\in g^{-1}(y_1)} F_N(z_1),\dots, \bigwedge_{z_n\in g^{-1}(y_n)} F_N(z_n) \right\} \\
&= \bigvee \{g(F_N)(y_1),\dots,g(F_N)(y_n)\}.
\end{aligned}$$

Hence $g(X_N)$ is a neutrosophic n-ary $\mathcal{N}$-subgroupoid of Y.

4. Conclusions

We have studied the neutrosophic $\mathcal{N}$-structure and applied it to n-ary groupoids. We also investigated the notion of neutrosophic $\mathcal{N}$-structures in n-ary groupoids and showed some properties. We have investigated the conditions for neutrosophic $\mathcal{N}$-structures to be neutrosophic n-ary $\mathcal{N}$-subgroupiods. A neutrosophic $\mathcal{N}$-product has been introduced. In addition, we have introduced neutrosophic n-ary $\mathcal{N}$-subgroupoids, ε-neutrosophic n-ary $\mathcal{N}$-subgroupoids and shown the relation between n-ary subgroupoids and neutrosophic n-ary $\mathcal{N}$-subgroupoids. Finally, we showed that the homomorphic preimage of the neutrosophic n-ary $\mathcal{N}$-subgroupoids is a neutrosophic n-ary $\mathcal{N}$-subgroupoids and the onto homomorphic image of the neutrosophic n-ary $\mathcal{N}$-subgroupoids is also a neutrosophic n-ary $\mathcal{N}$-subgroupoids.

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