Commutative falling neutrosophic ideals in $BCK$-algebras

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Abstract: The notions of a commutative $(\varepsilon, \varepsilon)$-neutrosophic ideal and a commutative falling neutrosophic ideal are introduced, and several properties are investigated. Characterizations of a commutative $(\varepsilon, \varepsilon)$-neutrosophic ideal and $(\varepsilon, \varepsilon)$-neutrosophic ideal are obtained. Relations between commutative $(\varepsilon, \varepsilon)$-neutrosophic ideal and $(\varepsilon, \varepsilon)$-neutrosophic ideal are discussed. Conditions for a $(\varepsilon, \varepsilon)$-neutrosophic ideal to be a commutative $(\varepsilon, \varepsilon)$-neutrosophic ideal are established. Relations between commutative $(\varepsilon, \varepsilon)$-neutrosophic ideal, falling neutrosophic ideal and commutative falling neutrosophic ideal are considered. Conditions for a falling neutrosophic ideal to be commutative are provided.

Keywords: (commutative) $(\varepsilon, \varepsilon)$-neutrosophic ideal; neutrosophic random set; neutrosophic falling shadow; (commutative) falling neutrosophic ideal.

1 Introduction

Neutrosophic set (NS) developed by Smarandache [11, 12, 13] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Neutrosophic set theory is applied to various parts which are referred to the site http://fs.gallup.unm.edu/neutrosophy.htm. Jun, Borumand Saeid and Öztürk studied neutrosophic subalgebras/ideals in $BCK/BCI$-algebras based on neutrosophic points (see [1], [6] and [10]). Goodman [2] pointed out the equivalence of a fuzzy set and a class of random sets in the study of a unified treatment of uncertainty modeled by means of combining probability and fuzzy set theory. Wang and Sanchez [16] introduced the theory of falling shadows which directly relates probability concepts with the membership function of fuzzy sets. The mathematical structure of the theory of falling shadows is formulated in [17]. Tan et al. [14, 15] established a theoretical approach to define a fuzzy inference relation and fuzzy set operations based on the theory of falling shadows. Jun and Park [7] considered a fuzzy subalgebra and a fuzzy ideal as the falling shadow of the cloud of the subalgebra and ideal. Jun et al. [8] introduced the notion of neutrosophic random set and neutrosophic falling shadow. Using these notions, they introduced the concept of falling neutrosophic subalgebra and falling neutrosophic ideal in $BCK/BCI$-algebras, and investigated related properties. They discussed relations between falling neutrosophic subalgebra and falling neutrosophic ideal, and established a characterization of falling neutrosophic ideal.

In this paper, we introduce the concepts of a commutative $(\varepsilon, \varepsilon)$-neutrosophic ideal and a commutative falling neutrosophic ideal, and investigate several properties. We obtain characterizations of a commutative $(\varepsilon, \varepsilon)$-neutrosophic ideal, and discuss relations between a commutative $(\varepsilon, \varepsilon)$-neutrosophic ideal and an $(\varepsilon, \varepsilon)$-neutrosophic ideal. We provide conditions for an $(\varepsilon, \varepsilon)$-neutrosophic ideal to be a commutative $(\varepsilon, \varepsilon)$-neutrosophic ideal, and consider relations between a commutative $(\varepsilon, \varepsilon)$-neutrosophic ideal, a falling neutrosophic ideal and a commutative falling neutrosophic ideal. We give conditions for a falling neutrosophic ideal to be commutative.

2 Preliminaries

A $BCK/BCI$-algebra is an important class of logical algebras introduced by K. Iséki (see [3] and [4]) and was extensively investigated by several researchers.

By a $BCI$-algebra, we mean a set $X$ with a special element $0$ and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X) \ ((((x * y) * (x * z)) * (z * y)) = 0)$,

(II) $(\forall x, y \in X) \ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X) \ (x * x = 0)$,

(IV) $(\forall x, y \in X) \ ((x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a $BCI$-algebra $X$ satisfies the following identity:

(V) $(\forall x \in X) \ (0 * x = 0)$,

then $X$ is called a $BCK$-algebra. Any $BCK/BCI$-algebra $X$
where \( x \leq y \) if and only if \( x \neq y = 0 \). A nonempty subset \( S \) of a BCK/BCI-algebra \( X \) is called a subalgebra of \( X \) if \( x \neq y = S \) for all \( x \neq y = S \). A subset \( I \) of a BCK/BCI-algebra \( X \) is called an ideal of \( X \) if it satisfies:

\[
\begin{align*}
0 & \in I, \\
(\forall x \in X) (\forall y \in I) (x \neq y = I \Rightarrow x \in I).
\end{align*}
\]

A subset \( I \) of a BCK-algebra \( X \) is called a commutative ideal of \( X \) if it satisfies (2.5) and

\[
(\forall x \in X) (\forall y \in I) (x \neq y = I \Rightarrow x \neq y \neq x \in I)
\]

for all \( x \neq y = X \).

Observe that every commutative ideal is an ideal, but the converse is not true (see [9]).

We refer the reader to the books [5, 9] for further information regarding BCK/BCI-algebras.

For any family \( \{a_i | i \in \Lambda\} \) of real numbers, we define

\[
\bigvee \{a_i | i \in \Lambda\} := \sup \{a_i | i \in \Lambda\}
\]

and

\[
\bigwedge \{a_i | i \in \Lambda\} := \inf \{a_i | i \in \Lambda\}.
\]

If \( \Lambda = \{1, 2\} \), we will also use \( a_1 \vee a_2 \) and \( a_1 \wedge a_2 \) instead of \( \bigvee \{a_i | i \in \Lambda\} \) and \( \bigwedge \{a_i | i \in \Lambda\} \), respectively.

Let \( X \) be a non-empty set. A neutrosophic set (NS) in \( X \) (see [12]) is a structure of the form:

\[
A := \{(x; A_T(x), A_I(x), A_F(x)) \mid x \in X\}
\]

where \( A_T : X \to [0, 1] \) is a truth membership function, \( A_I : X \to [0, 1] \) is an indeterminate membership function, and \( A_F : X \to [0, 1] \) is a false membership function. For the sake of simplicity, we shall use the symbol \( A = (A_T, A_I, A_F) \) for the neutrosophic set

\[
A := \{(x; A_T(x), A_I(x), A_F(x)) \mid x \in X\}.
\]

Given a neutrosophic set \( A = (A_T, A_I, A_F) \) in a set \( X, \alpha, \beta \in [0, 1] \) and \( \gamma \in [0, 1] \), we consider the following sets:

\[
\begin{align*}
T_c(A; \alpha) := \{x \in X \mid A_T(x) \geq \alpha\}, \\
I_c(A; \beta) := \{x \in X \mid A_I(x) \geq \beta\}, \\
F_c(A; \gamma) := \{x \in X \mid A_F(x) \leq \gamma\}.
\end{align*}
\]

We say \( T_c(A; \alpha), I_c(A; \beta) \) and \( F_c(A; \gamma) \) are neutrosophic \( \in \)-subsets.

A neutrosophic set \( A = (A_T, A_I, A_F) \) in a BCK/BCI-algebra \( X \) is called an \((\in, \in)\)-neutrosophic subalgebra of \( X \) (see [6]) if the following assertions are valid.

\[
\begin{align*}
(\forall x, y \in X) \begin{cases}
x \neq y = T_c(A; \alpha) = \{x \in X \mid A_T(x) \geq \alpha\}, \\
x \neq y = I_c(A; \beta) = \{x \in X \mid A_I(x) \geq \beta\}, \\
x \neq y = F_c(A; \gamma) = \{x \in X \mid A_F(x) \leq \gamma\}.
\end{cases}
\end{align*}
\]

for all \( \alpha, \alpha, \beta, \gamma \in [0, 1] \) and \( \gamma, \gamma \in [0, 1] \).

A neutrosophic set \( A = (A_T, A_I, A_F) \) in a BCK/BCI-algebra \( X \) is called an \((\in, \in)\)-neutrosophic ideal of \( X \) (see [10]) if the following assertions are valid.

\[
\begin{align*}
(\forall x \in X) \begin{cases}
x \neq y = T_c(A; \alpha) = \{x \in X \mid A_T(x) \geq \alpha\}, \\
x \neq y = I_c(A; \beta) = \{x \in X \mid A_I(x) \geq \beta\}, \\
x \neq y = F_c(A; \gamma) = \{x \in X \mid A_F(x) \leq \gamma\}.
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
(\forall x, y \in X) \begin{cases}
x \neq y = T_c(A; \alpha) = \{x \in X \mid A_T(x) \geq \alpha\}, \\
x \neq y = I_c(A; \beta) = \{x \in X \mid A_I(x) \geq \beta\}, \\
x \neq y = F_c(A; \gamma) = \{x \in X \mid A_F(x) \leq \gamma\}.
\end{cases}
\end{align*}
\]

for all \( \alpha, \alpha, \beta, \gamma \in [0, 1] \) and \( \gamma, \gamma \in [0, 1] \).

In what follows, let \( X \) and \( \mathcal{P}(X) \) denote a BCK/BCI-algebra and the power set of \( X \), respectively, unless otherwise specified.

For each \( x \in X \) and \( D \in \mathcal{P}(X) \), let

\[
\bar{x} := \{C \in \mathcal{P}(X) \mid x \in C\},
\]

and

\[
\bar{x} := \{C \in \mathcal{P}(X) \mid x \in D\}.
\]

An ordered pair \( (\mathcal{P}(X), \mathcal{B}) \) is said to be a hyper-measurable structure on \( X \) if \( \mathcal{B} \) is a \( \sigma \)-field in \( \mathcal{P}(X) \) and \( \bar{x} \subseteq \mathcal{B} \).

Given a probability space \((\Omega, A, P)\) and a hyper-measurable structure \( (\mathcal{P}(X), \mathcal{B}) \) on \( X \), a neutrosophic random set on \( X \) (see [8]) is defined to be a triple \( \xi := (\xi_T, \xi_I, \xi_F) \) in which \( \xi_T, \xi_I \) and \( \xi_F \) are mappings from \( \Omega \) to \( \mathcal{P}(X) \) which are \( A-B \) measurable,
that is,

\[
(\forall C \in \mathcal{B}) \left( \begin{array}{l}
\xi^{-1}_r(C) = \{\omega_T \in \Omega \mid \xi_r(\omega_T) \in C\} \in \mathcal{A} \\
\xi^{-1}_I(C) = \{\omega_I \in \Omega \mid \xi_I(\omega_I) \in C\} \in \mathcal{A} \\
\xi^{-1}_F(C) = \{\omega_F \in \Omega \mid \xi_F(\omega_F) \in C\} \in \mathcal{A}
\end{array} \right).
\]

(2.13)

Given a neutrosophic random set \(\xi := (\xi_T, \xi_I, \xi_F)\) on \(X\), consider functions:

\[
\hat{H}_T : X \to [0, 1], \ x_T \mapsto P(\omega_T \mid x_T \in \xi_T(\omega_T)),
\]

\[
\hat{H}_I : X \to [0, 1], \ x_I \mapsto P(\omega_I \mid x_I \in \xi_I(\omega_I)),
\]

\[
\hat{H}_F : X \to [0, 1], \ x_F \mapsto 1 - P(\omega_F \mid x_F \in \xi_F(\omega_F)).
\]

Then \(\hat{H} := (\hat{H}_T, \hat{H}_I, \hat{H}_F)\) is a neutrosophic set on \(X\), and we call it a neutrosophic falling shadow (see [8]) of the neutrosophic random set \(\xi := (\xi_T, \xi_I, \xi_F)\), and \(\xi' := (\xi_T', \xi_I', \xi_F')\) is called a neutrosophic cloud (see [8]) of \(H := (\hat{H}_T, \hat{H}_I, \hat{H}_F)\).

For example, consider a probability space \((\Omega, \mathcal{A}, P) = ([0, 1], A, m)\) where \(A\) is a Borel field on \([0, 1]\) and \(m\) is the usual Lebesgue measure. Let \(\hat{H} := (\hat{H}_T, \hat{H}_I, \hat{H}_F)\) be a neutrosophic set in \(X\). Then a triple \(\xi := (\xi_T, \xi_I, \xi_F)\) in which

\[
\xi_T : [0, 1] \to \mathcal{P}(X), \ \alpha \mapsto T_\xi(H; \alpha),
\]

\[
\xi_I : [0, 1] \to \mathcal{P}(X), \ \beta \mapsto I_\xi(H; \beta),
\]

\[
\xi_F : [0, 1] \to \mathcal{P}(X), \ \gamma \mapsto F_\xi(H; \gamma)
\]

is a neutrosophic random set and \(\xi := (\xi_T, \xi_I, \xi_F)\) is a neutrosophic cloud of \(H := (\hat{H}_T, \hat{H}_I, \hat{H}_F)\). We will call \(\xi := (\xi_T, \xi_I, \xi_F)\) defined above as the neutrosophic cut-cloud (see [8]) of \(H := (\hat{H}_T, \hat{H}_I, \hat{H}_F)\).

Let \((\Omega, \mathcal{A}, P)\) be a probability space and \(\xi := (\xi_T, \xi_I, \xi_F)\) be a neutrosophic random set on \(X\). If \(\xi_T(\omega_T)\), \(\xi_I(\omega_I)\) and \(\xi_F(\omega_F)\) are subalgebras (resp., ideals) of \(X\) for all \(\omega_T, \omega_I, \omega_F \in \Omega\), then the neutrosophic falling shadow \(H := (\hat{H}_T, \hat{H}_I, \hat{H}_F)\) of \(\xi := (\xi_T, \xi_I, \xi_F)\) is called a falling neutrosophic subalgebra (resp., falling neutrosophic ideal) of \(X\) (see [8]).

## 3 Commutative \((\in, \in)\)-neutrosophic ideals

**Definition 3.1.** A neutrosophic set \(A = (A_T, A_I, A_F)\) in a \(BCK\)-algebra \(X\) is called a commutative \((\in, \in)\)-neutrosophic ideal of \(X\) if it satisfies the condition (2.9) and

\[
(x \ast y) \ast z \in T_\xi(A; \alpha_x), \ z \in T_\xi(A; \alpha_y)
\]

\[
\Rightarrow x \ast (y \ast (y \ast x)) \in T_\xi(A; \alpha_x \land \alpha_y)
\]

\[
(x \ast y) \ast z \in I_\xi(A; \beta_x), \ z \in I_\xi(A; \beta_y)
\]

\[
\Rightarrow x \ast (y \ast (y \ast x)) \in I_\xi(A; \beta_x \land \beta_y)
\]

\[
(x \ast y) \ast z \in F_\xi(A; \gamma_x), \ z \in F_\xi(A; \gamma_y)
\]

\[
\Rightarrow x \ast (y \ast (y \ast x)) \in F_\xi(A; \gamma_x \lor \gamma_y)
\]

(3.1)

for all \(x, y, z \in X, \ \alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1)\) and \(\gamma_x, \gamma_y \in [0, 1]\).

**Example 3.2.** Consider a set \(X = \{0, 1, 2, 3\}\) with the binary operation \(*\) which is given in Table 1.

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

Then \((X; *, 0)\) is a \(BCK\)-algebra (see [9]). Let \(A = (A_T, A_I, A_F)\) be a neutrosophic set in \(X\) defined by Table 2

<table>
<thead>
<tr>
<th>X</th>
<th>(A_T(x))</th>
<th>(A_I(x))</th>
<th>(A_F(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.7</td>
<td>0.9</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0.3</td>
<td>0.6</td>
<td>0.8</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
<td>0.6</td>
<td>0.8</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>0.4</td>
<td>0.7</td>
</tr>
</tbody>
</table>

It is routine to verify that \(A = (A_T, A_I, A_F)\) is a commutative \((\in, \in)\)-neutrosophic ideal of \(X\).

**Theorem 3.3.** For a neutrosophic set \(A = (A_T, A_I, A_F)\) in a \(BCK\)-algebra \(X\), the following are equivalent.

1. The non-empty \(\in\)-subsets \(T_\xi(A; \alpha), I_\xi(A; \beta)\) and \(F_\xi(A; \gamma)\) are commutative ideals of \(X\) for all \(\alpha, \beta \in (0, 1)\) and \(\gamma \in [0, 1]\).

2. \(A = (A_T, A_I, A_F)\) satisfies the following assertions.

\[
(\forall x \in X) \left( \begin{array}{l}
A_T(0) \geq A_T(x) \\
A_I(0) \geq A_I(x) \\
A_F(0) \leq A_F(x)
\end{array} \right)
\]

(3.2)

and for all \(x, y, z \in X\),

\[
A_T(x \ast (y \ast (y \ast x))) \geq A_T(x \ast y \ast x)
\]

\[
A_I(x \ast (y \ast (y \ast x))) \geq A_I(x \ast y \ast x)
\]

\[
A_F(x \ast (y \ast (y \ast x))) \leq A_F(x \ast y \ast (y \ast x))
\]

(3.3)

**Proof.** Assume that the non-empty \(\in\)-subsets \(T_\xi(A; \alpha), I_\xi(A; \beta)\) and \(F_\xi(A; \gamma)\) are commutative ideals of \(X\) for all \(\alpha, \beta \in (0, 1)\) and \(\gamma \in [0, 1]\). If \(A_T(0) < A_T(a)\) for some \(a \in X\), then \(a \in T_\xi(A; A_T(a))\) and \(0 \notin T_\xi(A; A_T(a))\). This is a contradiction, and so \(A_T(0) \geq A_T(x)\) for all \(x \in X\). Similarly,
\[ A_T(0) \geq A_T(a) \text{ for all } a \in X. \text{ Suppose that } A_F(0) > A_F(a) \text{ for some } a \in X. \text{ Then } a \in F_T(E; A; \beta) \text{ and } 0 \notin F_E(A; A_F(a)). \] 
This is a contradiction, and thus \( A_F(0) \leq A_F(x) \) for all \( x \in X \). Therefore (3.2) is valid. Assume that there exist \( a, b, c \in X \) such that 
\[ A_T((a * (b * (b * a)))) < A_T((a * b) * c) \wedge A_T(c). \]
Taking \( \alpha := A_T((a * b) * c) \wedge A_T(c) \) implies that \( (a * b) * c \in T \) and \( c \in T \), but \( a * (b * (b * a)) \notin T \), which is a contradiction. Hence 
\[ A_T(x * (y * (y * x))) \geq A_T((x * y) * z) \wedge A_T(z) \]
for all \( x, y, z \in X \). By the similar way, we can verify that 
\[ A_I((x * y) * (x * y)) \geq A_I((x * y) * z) \wedge A_I(z) \]
for all \( x, y, z \in X \). Now suppose there are \( x, y, z \in X \) such that
\[ A_F((x * y) * (x * y)) > A_F((x * y) * z) \vee A_F(z) : = \gamma. \]
Then \( (x * y) * z \in F_T \) and \( z \in F_T \) but \( (x * (y * x)) \notin F_T \), a contradiction. Thus 
\[ A_F((x * y) * (x * y)) \leq A_F((x * y) * z) \vee A_F(z) \]
for all \( x, y, z \in X \).

Conversely, let \( A = (A_T, A_I, A_F) \) be a neutrosophic set in \( X \) satisfying two conditions (3.2) and (3.3). Assume that \( T \) and \( I \) are nonempty for \( \alpha, \beta \in [0, 1] \) and \( \gamma \in [0, 1] \). Let \( x \in T \) and \( a \in I \) and \( u \in F \). Then \( A_T(0) \geq A_T(x) \), \( A_I(0) \geq A_I(a) \geq 0 \), and \( A_F(0) \leq A_F(u) \leq \gamma \) by (3.2). It follows that \( 0 \notin T \), \( 0 \in I \) and \( 0 \in F \). Let \( a, b, c \in X \) be such that \( (a * b) * c \in T \) and \( c \in T \) for \( \alpha \in [0, 1] \). Then 
\[ A_T((a * (b * (b * a)))) \geq A_T((a * b) * c) \wedge A_T(c) \geq \alpha \]
by (3.3), and so \( a * (b * (b * a)) \in I \). If \( (x * y) * z \in I \) and \( z \in I \) for all \( x, y, z \in X \) and \( \beta \in [0, 1] \), then \( A_T((x * y) * (x * y)) \geq \beta \) and \( A_I(z) \geq \beta \). Hence the condition (3.3) implies that 
\[ A_I((x * (y * (x * x)))) \geq A_I((x * y) * z) \wedge A_I(z) \geq \beta, \]
that is, \( x * (y * (x * x)) \in I \). Finally, suppose that 
\[ (x * y) * z \in F_T \text{ and } z \in F_T \]
for all \( x, y, z \in X \) and \( \gamma \in [0, 1] \). Then \( A_F((x * y) * z) \leq \gamma \) and \( A_F(z) \leq \gamma \), which imply from the condition (3.3) that 
\[ A_F((x * (y * (y * x)))) \leq A_F((x * y) * z) \vee A_F(z) \leq \gamma. \]
Hence \( x * (y * (y * x)) \in F_T \). Therefore the non-empty \( \in \)-subsets \( T \), \( I \) and \( F \) are commutative ideals of \( X \) for all \( \alpha, \beta \in [0, 1] \) and \( \gamma \in [0, 1] \).

**Theorem 3.4.** Let \( A = (A_T, A_I, A_F) \) be a neutrosophic set in a BCK-algebra \( X \). Then \( A = (A_T, A_I, A_F) \) is a commutative \((\in, \varepsilon, \gamma)\)-neutrosophic ideal of \( X \) if and only if the non-empty neutrosophic \( \varepsilon \)-subsets \( T \), \( I \) and \( F \) are commutative ideals of \( X \) for all \( \alpha, \beta \in [0, 1] \) and \( \gamma \in [0, 1] \).

**Proof.** Let \( A = (A_T, A_I, A_F) \) be a commutative \((\in, \varepsilon, \gamma)\)-neutrosophic ideal of \( X \) and assume that \( T \), \( I \) and \( F \) are nonempty ideals of \( X \). Then \( x, y, z \in X \) such that \( x \in T, y \in I, z \in F \). It follows from (2.9) that \( 0 \in F \) and \( 0 \in F \) for all \( \alpha, \beta \in [0, 1] \). Let \( x, y, z \in X \) such that 
\[ (x * y) * z \in T \text{ and } z \in T \text{ and } (a * b) * c \in I \text{ and } c \in I \text{ and } (u * v) * w \in F \text{ and } w \in F \]
by (2.10). Hence the non-empty neutrosophic \( \varepsilon \)-subsets \( T \), \( I \) and \( F \) are commutative ideals of \( X \) for all \( \alpha, \beta \in [0, 1] \) and \( \gamma \in [0, 1] \).

Conversely, let \( A = (A_T, A_I, A_F) \) be a neutrosophic set in \( X \) for which \( T \), \( I \) and \( F \) are nonempty ideals of \( X \). Then \( x, y, z \in X \) and \( \alpha, \beta, \gamma \in [0, 1] \) be such that \( (x * y) * z \in T \) and \( z \in T \). Then \( (x * y) * z \in T \) and \( z \in T \) with \( \alpha = \alpha \cap \beta \). Since \( T \) is a commutative ideal of \( X \), it follows that 
\[ x * (y * (y * x)) \in T \text{ and } T \text{ and } (a * b) * c \in I \text{ and } c \in I \text{ and } (u * v) * w \in F \text{ and } w \in F \]
by (2.10). Hence the non-empty neutrosophic \( \varepsilon \)-subsets \( T \), \( I \) and \( F \) are commutative ideals of \( X \) for all \( \alpha, \beta \in [0, 1] \) and \( \gamma \in [0, 1] \).

Now, suppose that \( (x * y) * z \in F \) and \( z \in F \) for all \( x, y, z \in X \) and \( \gamma \in [0, 1] \). Then \( (x * y) * z \in F \) and \( z \in F \) where \( \gamma = \gamma \vee \gamma \). Hence 
\[ x * (y * (y * x)) \in F \text{ and } F \text{ and } (a * b) * c \in I \text{ and } c \in I \text{ and } (u * v) * w \in F \text{ and } w \in F \]
by (2.10). Hence the non-empty neutrosophic \( \varepsilon \)-subsets \( T \), \( I \) and \( F \) are commutative ideals of \( X \). Therefore \( A = (A_T, A_I, A_F) \) is a commutative \((\in, \varepsilon, \gamma)\)-neutrosophic ideal of \( X \).

**Corollary 3.5.** Let \( A = (A_T, A_I, A_F) \) be a neutrosophic set in a BCK-algebra \( X \). Then \( A = (A_T, A_I, A_F) \) is a commutative \((\in, \varepsilon, \gamma)\)-neutrosophic ideal of \( X \).
Proposition 3.6. Every commutative \((\varepsilon, \varepsilon)-\)neutrosophic ideal \(A = (A_T, A_I, A_F)\) of a \(BCK\)-algebra \(X\) satisfies:

\[
(\forall x, y \in X) \begin{align*}
\forall x, y \in T_\varepsilon(A; \alpha) &\Rightarrow x * (y * (y * x)) \in T_\varepsilon(A; \alpha) \\
\forall x, y \in I_\varepsilon(A; \beta) &\Rightarrow x * ((y * (y * x)) \in I_\varepsilon(A; \beta) \\
\forall x, y \in F_\varepsilon(A; \gamma) &\Rightarrow x * ((y * (y * x)) \in F_\varepsilon(A; \gamma) \\
\forall x, y \in A &\Rightarrow x * (y * (y * x)) \in A
\end{align*}
\]

(3.4)

for all \(\alpha, \beta \in (0, 1)\) and \(\gamma \in [0, 1]\).

Proof. It is induced by taking \(z = 0\) in (3.1).

\[\Box\]

Theorem 3.7. Every commutative \((\varepsilon, \varepsilon)-\)neutrosophic ideal of a \(BCK\)-algebra \(X\) is an \((\varepsilon, \varepsilon)-\)neutrosophic ideal of \(X\).

Proof. Let \(A = (A_T, A_I, A_F)\) be a commutative \((\varepsilon, \varepsilon)-\)neutrosophic ideal of a \(BCK\)-algebra \(X\). Assume that

\[
x * y \in T_\varepsilon(A; \alpha_x), y \in T_\varepsilon(A; \alpha_y), \\
a * b \in I_\varepsilon(A; \beta_a), b \in I_\varepsilon(A; \beta_b), \\
c * d \in F_\varepsilon(A; \gamma_c), d \in F_\varepsilon(A; \gamma_d)
\]

for all \(x, y, a, b, c, d \in X\). Using (2.1), we have

\[
(x * 0) * y = x * y \in T_\varepsilon(A; \alpha_x), \\
(a * 0) * (0 * 0) = a * b \in I_\varepsilon(A; \beta_a), \\
(c * 0) * (0 * 0) = c * d \in F_\varepsilon(A; \gamma_c).
\]

It follows from (3.1), (2.1) and (V) that

\[
x = x * 0 = x * (0 * (0 * x)) \in T_\varepsilon(A; \alpha_x \land \alpha_y), \\
a = a * 0 = a * (0 * (0 * a)) \in I_\varepsilon(A; \beta_a \land \beta_b), \\
c = c * 0 = c * (0 * (0 * c)) \in F_\varepsilon(A; \gamma_c \lor \gamma_d).
\]

Therefore \(A = (A_T, A_I, A_F)\) is an \((\varepsilon, \varepsilon)-\)neutrosophic ideal of \(X\). \[\Box\]

The converse of Theorem 3.7 is not true as seen in the following example.

Example 3.8. Consider a set \(X = \{0, 1, 2, 3, 4\}\) with the binary operation \(*\) which is given in Table 3

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</table>

Routine calculations show that \(A = (A_T, A_I, A_F)\) is an \((\varepsilon, \varepsilon)-\)neutrosophic ideal of \(X\). But it is not a commutative \((\varepsilon, \varepsilon)-\)neutrosophic ideal of \(X\) since \((2 * 3) * 0 \not\in T_\varepsilon(A; 0.6)\) and \(0 \in T_\varepsilon(A; 0.5)\) but \(2 * (3 * (3 * 2)) \not\in T_\varepsilon(A; 0.5 \land 0.6)\), \((1 * 3) * 2 \not\in I_\varepsilon(A; 0.55)\) and \(2 \in I_\varepsilon(A; 0.63)\) but \((1 * (3 * (3 * 1))) \not\in I_\varepsilon(A; 0.55 \lor 0.63)\), and/or \((2 * (3 * 2)) \not\in F_\varepsilon(A; 0.43)\) and \(0 \in F_\varepsilon(A; 0.39)\) but \((2 * (3 * (3 * 2))) \not\in F_\varepsilon(A; 0.43 \lor 0.39)\).

We provide conditions for an \((\varepsilon, \varepsilon)-\)neutrosophic ideal to be a commutative \((\varepsilon, \varepsilon)-\)neutrosophic ideal.

Theorem 3.9. Let \(A = (A_T, A_I, A_F)\) be an \((\varepsilon, \varepsilon)-\)neutrosophic ideal of a \(BCK\)-algebra \(X\) in which the condition (3.4) is valid. Then \(A = (A_T, A_I, A_F)\) is a commutative \((\varepsilon, \varepsilon)-\)neutrosophic ideal of \(X\).

Proof. Let \(A = (A_T, A_I, A_F)\) be an \((\varepsilon, \varepsilon)-\)neutrosophic ideal of \(X\) and \(x, y, z \in X\) be such that \((x * y) * z \in T_\varepsilon(A; \alpha_x)\) and \(z \in T_\varepsilon(A; \alpha_y)\) for \(\alpha_x, \alpha_y \in (0, 1)\). Then \(x * y \in T_\varepsilon(A; \alpha_x \land \alpha_y)\) since \(A = (A_T, A_I, A_F)\) is an \((\varepsilon, \varepsilon)-\)neutrosophic ideal of \(X\). It follows from (3.4) that \(x * ((y * (y * x))) \in T_\varepsilon(A; \alpha_x \land \alpha_y)\). Similarly, if \((x * y) * z \in I_\varepsilon(A; \beta_x)\) and \(z \in I_\varepsilon(A; \beta_y)\), then \((x * ((y * (y * x)))) \in I_\varepsilon(A; \beta_x \land \beta_y)\). Let \(a, b, c \in X\) and \(\gamma_x, \gamma_y, \gamma_z \in (0, 1)\) be such that \((a * b) * c \not\in F_\varepsilon(A; \gamma_a)\) and \(c \not\in F_\varepsilon(A; \gamma_a)\). Then \(a * b \in F_\varepsilon(A; \gamma_a \lor \gamma_b)\), which implies from (3.4) that \(a * (b * (b * a)) \not\in F_\varepsilon(A; \gamma_a \lor \gamma_b)\). Therefore \(A = (A_T, A_I, A_F)\) is a commutative \((\varepsilon, \varepsilon)-\)neutrosophic ideal of \(X\). \[\Box\]

Lemma 3.10. Every \((\varepsilon, \varepsilon)-\)neutrosophic ideal \(A = (A_T, A_I, A_F)\) of a \(BCK\)-algebra \(X\) satisfies:

\[
y, z \in T_\varepsilon(A; \alpha) \Rightarrow x \in T_\varepsilon(A; \alpha) \\
y, z \in I_\varepsilon(A; \beta) \Rightarrow x \in I_\varepsilon(A; \beta) \\
y, z \in F_\varepsilon(A; \gamma) \Rightarrow x \in F_\varepsilon(A; \gamma)
\]

for all \(\alpha, \beta \in [0, 1], \gamma \in [0, 1]\) and \(x, y, z \in X\) with \(x * y \leq z\).

Proof. For any \(\alpha, \beta \in [0, 1], \gamma \in [0, 1]\) and \(x, y, z \in X\) with \(x * y \leq z\), let \(y, z \in T_\varepsilon(A; \alpha)\), \(y, z \in I_\varepsilon(A; \beta)\) and \(y, z \in F_\varepsilon(A; \gamma)\). Then

\[
(x * y) * z = 0 \in T_\varepsilon(A; \alpha) \land I_\varepsilon(A; \beta) \land F_\varepsilon(A; \gamma)
\]
by (2.9). It follows from (2.10) that
\[
x \ast y \in T_e(A; \alpha) \cap I_e(A; \beta) \cap F_e(A; \gamma)
\]
and so that
\[
x \in T_e(A; \alpha) \cap I_e(A; \beta) \cap F_e(A; \gamma).
\]
Thus (3.5) is valid.

**Theorem 3.11.** In a commutative BCK-algebra, every \((\varepsilon, \varepsilon)\)-neutrosophic ideal is a commutative \((\varepsilon, \varepsilon)\)-neutrosophic ideal.

**Proof.** Let \(A = (A_T, A_I, A_F)\) be an \((\varepsilon, \varepsilon)\)-neutrosophic ideal of a commutative BCK-algebra \(X\). Let \(x, y, z \in X\) be such that
\[
(x \ast y) \ast z \in T_e(A; \alpha_x) \cap I_e(A; \beta_x) \cap F_e(A; \gamma_x)
\]
and
\[
z \in T_e(A; \alpha_y) \cap I_e(A; \beta_y) \cap F_e(A; \gamma_y)
\]
for \(\alpha_x, \alpha_y, \beta_x, \beta_y \in (0, 1]\) and \(\gamma_x, \gamma_y \in [0, 1]\). Note that
\[
((x \ast (y \ast (y \ast x))) \ast ((x \ast y) \ast z)) \ast z
\]
\[
= ((x \ast (y \ast (y \ast x))) \ast z) \ast ((x \ast y) \ast z)
\]
\[
\leq (x \ast (y \ast (y \ast x))) \ast (x \ast y)
\]
\[
= (x \ast (x \ast y)) \ast (y \ast (y \ast x))
\]
\[
= 0
\]
by (2.3), (2.4) and (III), which implies that
\[
(x \ast (y \ast (y \ast x))) \ast ((x \ast y) \ast z) \leq z.
\]
It follows from Lemma 3.10 that
\[
x \ast (y \ast (y \ast x)) \in T_e(A; \alpha_x) \cap I_e(A; \beta_x) \cap F_e(A; \gamma_x).
\]
Therefore \(A = (A_T, A_I, A_F)\) is a commutative \((\varepsilon, \varepsilon)\)-neutrosophic ideal of \(X\).

**4 Commutative falling neutrosophic ideals**

**Definition 4.1.** Let \((\Omega, A, P)\) be a probability space and let \(\xi := (\xi_T, \xi_I, \xi_F)\) be a neutrosophic random set on a BCK-algebra \(X\). Then the neutrosophic falling shadow \(\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)\) of \(\xi := (\xi_T, \xi_I, \xi_F)\) is called a commutative falling neutrosophic ideal of \(X\) if \(\xi_T(\omega_T), \xi_I(\omega_I)\) and \(\xi_F(\omega_F)\) are commutative ideals of \(X\) for all \(\omega_T, \omega_I, \omega_F \in \Omega\).

**Example 4.2.** Consider a set \(X = \{0, 1, 2, 3, 4\}\) with the binary operation \(*\) which is given in Table 5

Then \((X; *, 0)\) is a BCK-algebra (see [9]). Consider \((\Omega, A, P) := ([0, 1], A, m)\) and let \(\xi := (\xi_T, \xi_I, \xi_F)\) be a neutrosophic random set on \(X\) which is given as follows:

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</table>

Then \(\xi_T(t), \xi_I(t)\) and \(\xi_F(t)\) are commutative ideals of \(X\) for all \(t \in [0, 1]\). Hence the neutrosophic falling shadow \(\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)\) of \(\xi := (\xi_T, \xi_I, \xi_F)\) is a commutative falling neutrosophic ideal of \(X\), and it is given as follows:

\[
\tilde{H}_T(x) = \begin{cases} 
1 & \text{if } x = 0, \\
0.3 & \text{if } x \in \{1, 2\}, \\
0.4 & \text{if } x = 3, \\
0.45 & \text{if } x = 4,
\end{cases}
\]

\[
\tilde{H}_I(x) = \begin{cases} 
1 & \text{if } x \in \{0, 1, 2\}, \\
0.3 & \text{if } x = 3, \\
0.25 & \text{if } x = 4,
\end{cases}
\]

\[
\tilde{H}_F(x) = \begin{cases} 
0 & \text{if } x = 0, \\
0.5 & \text{if } x \in \{1, 2, 4\}, \\
0.3 & \text{if } x = 3.
\end{cases}
\]

**Given a probability space \((\Omega, A, P)\), let \(\tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F)\) be a neutrosophic falling shadow of a neutrosophic random set**
ξ := (ξ_T, ξ_I, ξ_F). For x ∈ X, let
\[
Ω(x; ξ_T) := \{ Ω_T ∈ Ω \mid x ∈ ξ_T(Ω_T) \},
\]
\[
Ω(x; ξ_I) := \{ Ω_I ∈ Ω \mid x ∈ ξ_I(Ω_I) \},
\]
\[
Ω(x; ξ_F) := \{ Ω_F ∈ Ω \mid x ∈ ξ_F(Ω_F) \}.
\]
Then Ω(x; ξ_T), Ω(x; ξ_I), Ω(x; ξ_F) ∈ A (see [8]).

**Proposition 4.3.** Let \( \hat{H} := (\hat{H}_T, \hat{H}_I, \hat{H}_F) \) be a neutrosophic falling shadow of the neutrosophic random set \( ξ := (ξ_T, ξ_I, ξ_F) \) on a BCK-algebra X. If \( H := (H_T, H_I, H_F) \) is a commutative falling neutrosophic ideal of X, then
\[
Ω((x * y) * z; ξ_T) \subseteq Ω((x * y) * z; ξ_T)
\]
\[
Ω((x * y) * z; ξ_I) \subseteq Ω((x * y) * z; ξ_I)
\]
and
\[
Ω((x * y) * z; ξ_F) \subseteq Ω((x * y) * z; ξ_F)
\]
for all x, y, z ∈ X.

**Proof.** Let \( \omega_T ∈ Ω((x * y) * z; ξ_T) \cap Ω(z; ξ_T) \), \( \omega_I ∈ Ω((x * y) * z; ξ_I) \cap Ω(z; ξ_I) \), \( \omega_F ∈ Ω((x * y) * z; ξ_F) \cap Ω(z; ξ_F) \). Then
\[
(x * y) * z ∈ ξ_T(Ω_T) \quad \text{and} \quad z ∈ ξ_T(Ω_T),
\]
\[
(x * y) * z ∈ ξ_I(Ω_I) \quad \text{and} \quad z ∈ ξ_I(Ω_I),
\]
\[
(x * y) * z ∈ ξ_F(Ω_F) \quad \text{and} \quad z ∈ ξ_F(Ω_F).
\]
Since \( ξ_T(Ω_T) \), \( ξ_I(Ω_I) \) and \( ξ_F(Ω_F) \) are commutative ideals of X, it follows from (2.7) that
\[
x * (y * (y * x)) ∈ ξ_T(Ω_T) \cap ξ_I(Ω_I) \cap ξ_F(Ω_F)
\]
and so that
\[
ω_T ∈ Ω(x * (y * (y * x)); ξ_T),
\]
\[
ω_I ∈ Ω(x * (y * (y * x)); ξ_I),
\]
\[
ω_F ∈ Ω(x * (y * (y * x)); ξ_F).
\]
Hence (4.1) is valid. Now let
\[
ω_T ∈ Ω(x * (y * (y * x)); ξ_T),
\]
\[
ω_I ∈ Ω(x * (y * (y * x)); ξ_I),
\]
\[
ω_F ∈ Ω(x * (y * (y * x)); ξ_F)
\]
for all x, y, z ∈ X. Then
\[
x * (y * (y * x)) ∈ ξ_T(Ω_T) \cap ξ_I(Ω_I) \cap ξ_F(Ω_F).
\]
Note that
\[
((x * y) * z) * (x * (y * (y * x))) = ((x * y) * (x * (y * (y * x)))) * z
\]
\[
≤ ((y * (y * x)) * z) * z = (y * y) * (y * x)) * z
\]
\[
= 0 * (y * x)) * z = 0 * 0 = 0,
\]
which yields
\[
((x * y) * z) * (x * (y * (y * x))) = 0 ∈ ξ_T(Ω_T) \cap ξ_I(Ω_I) \cap ξ_F(Ω_F).
\]
(4.1) Since \( ξ_T(Ω_T) \), \( ξ_I(Ω_I) \) and \( ξ_F(Ω_F) \) are commutative ideals and hence ideals of X, it follows that
\[
(x * y) * z ∈ ξ_T(Ω_T) \cap ξ_I(Ω_I) \cap ξ_F(Ω_F).
\]
Hence
\[
ω_T ∈ Ω((x * y) * z; ξ_T),
\]
\[
ω_I ∈ Ω((x * y) * z; ξ_I),
\]
\[
ω_F ∈ Ω((x * y) * z; ξ_F).
\]
Therefore (4.2) is valid.

**Given a probability space \( (Ω, A, P) \), let**
\[
\mathcal{F}(X) := \{ f \mid f : Ω → X \text{ is a mapping} \}. \tag{4.3}
\]
**Define a binary operation \( @ \) on \( \mathcal{F}(X) \) as follows:**
\[
∀ ω ∈ Ω \quad ((f @ g)(ω) = f(ω) * g(ω)) \tag{4.4}
\]
**for all \( f, g ∈ \mathcal{F}(X) \). Then \( (\mathcal{F}(X); @, θ) \) is a BCK/BCI-algebra (see [7]) where \( θ \) is given as follows:**
\[
θ : Ω → X, \quad ω → 0.
\]
**For any subset \( A \) of \( X \) and \( g_T, g_I, g_F ∈ \mathcal{F}(X) \), consider the followings:**
\[
A_T^ω := \{ ω_T ∈ Ω \mid g_T(ω_T) ∈ A \},
\]
\[
A_I^ω := \{ ω_I ∈ Ω \mid g_I(ω_I) ∈ A \},
\]
\[
A_F^ω := \{ ω_F ∈ Ω \mid g_F(ω_F) ∈ A \}
\]
and
\[
ξ_T : Ω → \mathcal{P}(\mathcal{F}(X)), \quad ξ_T := \{ g_T ∈ \mathcal{F}(X) \mid g_T(ω_T) ∈ A \},
\]
\[
ξ_I : Ω → \mathcal{P}(\mathcal{F}(X)), \quad ξ_I := \{ g_I ∈ \mathcal{F}(X) \mid g_I(ω_I) ∈ A \},
\]
\[
ξ_F : Ω → \mathcal{P}(\mathcal{F}(X)), \quad ξ_F := \{ g_F ∈ \mathcal{F}(X) \mid g_F(ω_F) ∈ A \}.
\]
Then \( A_T^ω, A_I^ω, A_F^ω ∈ A \) (see [8]).
Theorem 4.4. If $K$ is a commutative ideal of a BCK-algebra $X$, then

$$
\xi_{T}(\omega_{T}) = \{g_{T} \in F(X) \mid g_{T}(\omega_{T}) \in K\},
$$

$$
\xi_{I}(\omega_{I}) = \{g_{I} \in F(X) \mid g_{I}(\omega_{I}) \in K\},
$$

$$
\xi_{F}(\omega_{F}) = \{g_{F} \in F(X) \mid g_{F}(\omega_{F}) \in K\}
$$

are commutative ideals of $F(X)$.

Proof. Assume that $K$ is a commutative ideal of a BCK-algebra $X$. Since $\theta(\omega_{T}) = 0 \in K$, $\theta(\omega_{I}) = 0 \in K$ and $\theta(\omega_{F}) = 0 \in K$ for all $\omega_{T}, \omega_{I}, \omega_{F} \in \Omega$, we have $\theta \in \xi_{T}(\omega_{T}), \theta \in \xi_{I}(\omega_{I})$ and $\theta \in \xi_{F}(\omega_{F})$. Let $f_{T}, g_{T}, h_{T} \in F(X)$ be such that

$$(f_{T} \circ g_{T}) \circ h_{T} \in \xi_{T}(\omega_{T}) \text{ and } h_{T} \in \xi_{T}(\omega_{T}).$$

Then

$$(f_{T} \circ g_{T} \circ h_{T})(\omega_{T}) = (f_{T} \circ g_{T}) \circ h_{T}(\omega_{T}) \in K,$n

and $h_{T}(\omega_{T}) \in K$. Since $K$ is a commutative ideal of $X$, it follows from (2.7) that

$$(f_{T} \circ g_{T} \circ h_{T})(\omega_{T}) = f_{T}(\omega_{T}) \circ (g_{T}(\omega_{T}) \circ f_{T}(\omega_{T}))(\omega_{T}) \in K,$n

that is, $f_{T} \circ g_{T} \circ h_{T}(\omega_{T}) \in \xi_{T}(\omega_{T})$. Hence $\xi_{T}(\omega_{T})$ is a commutative ideal of $F(X)$. Similarly, we can verify that $\xi_{I}(\omega_{I})$ is a commutative ideal of $F(X)$. Now, let $f_{F}, g_{F}, h_{F} \in F(X)$ be such that $(f_{F} \circ g_{F}) \circ h_{F} \in \xi_{F}(\omega_{F})$ and $h_{F} \in \xi_{F}(\omega_{F})$. Then

$$(f_{F} \circ g_{F}) \circ h_{F}(\omega_{F}) = (f_{F} \circ g_{F}) \circ (h_{F}(\omega_{F}) \circ f_{F}(\omega_{F}))(\omega_{F}) \in K,$n

and $h_{F}(\omega_{F}) \in K$. Then

$$(f_{F} \circ g_{F} \circ h_{F})(\omega_{F}) = f_{F}(\omega_{F}) \circ (g_{F}(\omega_{F}) \circ f_{F}(\omega_{F}))(\omega_{F}) \in K,$n

and $f_{F}(\omega_{F}) \in K$.

Then (X, 0) is a BCK-algebra (see [9]). Consider

Table 6: Cayley table for the binary operation “*”

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<td>4</td>
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<td>0</td>
</tr>
</tbody>
</table>

Then $X \neq 0$ is a BCK-algebra and $\xi_{I}(\omega_{I})$ is a commutative ideal of $X$ as given by [9]. Then $\xi_{T}(\omega_{T})$ and $\xi_{F}(\omega_{F})$ are commutative ideals of $X$ for all $\omega_{T}, \omega_{F} \in \Omega$.

Hence $\xi_{T}(\omega_{T})$ is a commutative ideal of $F(X)$.

The converse of Theorem 4.5 is not true as seen in the following example.

Example 4.6. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 6

Then $X \neq 0$ is a BCK-algebra and $\xi_{I}(\omega_{I})$ is a commutative ideal of $X$ as given by [9]. Then $\xi_{T}(\omega_{T})$ and $\xi_{F}(\omega_{F})$ are commutative ideals of $X$ for all $\omega_{T}, \omega_{F} \in \Omega$.

Hence $\xi_{T}(\omega_{T})$ is a commutative ideal of $F(X)$.

The converse of Theorem 4.5 is not true as seen in the following example.

Example 4.6. Consider a set $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 6

$$
\xi_{T} : [0, 1] \rightarrow \mathcal{P}(X), \ x \mapsto \begin{cases} 0, & \text{if } t \in [0, 0.2), \ \\
0, & \text{if } t \in [0.2, 0.55), \ \\
0, & \text{if } t \in [0.55, 0.75), \ \\
1, & \text{if } t \in [0.75, 1], \end{cases}
$$

$$
\xi_{I} : [0, 1] \rightarrow \mathcal{P}(X), \ x \mapsto \begin{cases} 0, & \text{if } t \in [0, 0.34), \ \\
0, & \text{if } t \in [0.34, 0.66), \ \\
1, & \text{if } t \in [0.66, 0.78), \ \\
1, & \text{if } t \in [0.78, 1], \end{cases}
$$

and

$$
\xi_{F} : [0, 1] \rightarrow \mathcal{P}(X), \ x \mapsto \begin{cases} 0, & \text{if } t \in (0.87, 1], \ \\
0, & \text{if } t \in (0.76, 0.87], \ \\
0, & \text{if } t \in (0.58, 0.76], \ \\
0, & \text{if } t \in (0.33, 0.58], \ \\
X & \text{if } t \in [0, 0.33). \end{cases}
$$

Then $\xi_{T}(t), \xi_{I}(t)$ and $\xi_{F}(t)$ are commutative ideals of $X$ for all $t \in [0, 1]$. Hence the neutrosophic falling shadow $\tilde{H} := (\tilde{H}_{T}, \tilde{H}_{I}, \tilde{H}_{F})$ of $\xi := (\xi_{T}, \xi_{I}, \xi_{F})$ is a commutative falling neutrosophic ideal of $X$, and it is given as follows:

$$
\tilde{H}_{T}(x) = \begin{cases} 1 & \text{if } x = 0, \ \\
0.4 & \text{if } x = 1, \ \\
0.8 & \text{if } x = 2, \ \\
0.25 & \text{if } x = 3, \ \\
0.2 & \text{if } x = 4, \end{cases}
$$

$$
\tilde{H}_{I}(x) = \begin{cases} 1 & \text{if } x = 0, \ \\
0.68 & \text{if } x = 1, \ \\
0.22 & \text{if } x \in \{2, 3\}, \ \\
0.66 & \text{if } x = 4, \end{cases}
$$

and
\[ \tilde{H}_F(x) = \begin{cases} 
0 & \text{if } x = 0, \\
0.67 & \text{if } x \in \{1, 3\}, \\
0.31 & \text{if } x = 2, \\
0.24 & \text{if } x = 4.
\end{cases} \]

But \( \tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F) \) is not a commutative \((\varepsilon, \varepsilon)\)-neutrosophic ideal of \( X \) since
\[ (3 \ast 4) \ast 2 \in T_{\varepsilon}(\tilde{H}; 0.4) \text{ and } 2 \in T_{\varepsilon}(\tilde{H}; 0.6), \]
but \( 3 \ast (4 \ast (4 \ast 3)) = 3 \notin T_{\varepsilon}(\tilde{H}; 0.4) \).

We provide relations between a falling neutrosophic ideal and a commutative falling neutrosophic ideal.

**Theorem 4.7.** Let \((\Omega, A, P)\) be a probability space and let \( H := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F) \) be a neutrosophic falling shadow of a neutrosophic random set \( \xi := (\xi_T, \xi_I, \xi_F) \) on a \( BCK \)-algebra. If \( \tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F) \) is a commutative falling neutrosophic ideal of \( X \), then it is a falling neutrosophic ideal of \( X \).

**Proof.** Let \( \tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F) \) be a commutative falling neutrosophic ideal of a \( BCK \)-algebra \( X \). Then \( \xi_T(\omega_T), \xi_I(\omega_I) \) and \( \xi_F(\omega_F) \) are commutative ideals of \( X \) for all \( \omega_T, \omega_I, \omega_F \in \Omega \). Thus \( \xi_T(\omega_T), \xi_I(\omega_I) \) and \( \xi_F(\omega_F) \) are ideals of \( X \) for all \( \omega_T, \omega_I, \omega_F \in \Omega \). Therefore \( \tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F) \) is a falling neutrosophic ideal of \( X \).

The following example shows that the converse of Theorem 4.7 is not true in general.

**Example 4.8.** Consider a set \( X = \{0, 1, 2, 3, 4\} \) with the binary operation \( \ast \) which is given in Table 7.

| \(| \ast | \) | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 1 | 0 |
| 2 | 2 | 1 | 0 | 2 | 0 |
| 3 | 3 | 3 | 3 | 3 | 0 |
| 4 | 4 | 4 | 4 | 4 | 0 |

Then \((X, \ast, 0)\) is a \( BCK \)-algebra (see [9]). Consider \((\Omega, A, P) = ([0, 1], A, m)\) and let \( \xi := (\xi_T, \xi_I, \xi_F) \) be a neutrosophic random set on \( X \) which is given as follows:

\[
\xi_T : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} 
\{0, 3\} & \text{if } t \in [0, 0.27), \\
\{0, 1, 2, 3\} & \text{if } t \in [0.27, 0.66), \\
\{0, 1, 2, 4\} & \text{if } t \in [0.67, 1], 
\end{cases}
\]

\[
\xi_I : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} 
\{0, 3\} & \text{if } t \in [0, 0.35), \\
\{0, 1, 2, 4\} & \text{if } t \in [0.35, 1], 
\end{cases}
\]

and
\[
\xi_F : [0, 1] \to \mathcal{P}(X), \quad x \mapsto \begin{cases} 
\{0\} & \text{if } t \in (0.84, 1), \\
\{0, 3\} & \text{if } t \in (0.76, 0.84], \\
\{0, 1, 2, 4\} & \text{if } t \in (0.58, 0.76], \\
X & \text{if } t \in [0, 0.58].
\end{cases}
\]

Then \( \xi_T(t), \xi_I(t) \) and \( \xi_F(t) \) are ideals of \( X \) for all \( t \in [0, 1] \). Hence the neutrosophic falling shadow \( H := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F) \) of \( \xi := (\xi_T, \xi_I, \xi_F) \) is a falling neutrosophic ideal of \( X \). But it is not a commutative falling neutrosophic ideal of \( X \) because if \( \alpha \in [0, 0.27), \beta \in [0, 0.35) \) and \( \gamma \in (0.76, 0.84) \), then \( \xi_T(\alpha) = \{0, 3\}, \xi_I(\beta) = \{0, 3\} \) and \( \xi_F(\gamma) = \{0, 3\} \) are not commutative ideals of \( X \) respectively.

Since every ideal is commutative in a commutative \( BCK \)-algebra, we have the following theorem.

**Theorem 4.9.** Let \((\Omega, A, P)\) be a probability space and let \( H := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F) \) be a neutrosophic falling shadow of a neutrosophic random set \( \xi := (\xi_T, \xi_I, \xi_F) \) on a commutative \( BCK \)-algebra. If \( H := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F) \) is a falling neutrosophic ideal of \( X \), then it is a commutative falling neutrosophic ideal of \( X \).

**Corollary 4.10.** Let \((\Omega, A, P)\) be a probability space. For any \( BCK \)-algebra \( X \) which satisfies one of the following assertions

\[ (\forall x, y \in X)(x \leq y \Rightarrow x \ast y \ast (y \ast x)), \]
\[ (\forall x, y \in X)(x \leq y \Rightarrow x \ast y \ast (y \ast x)), \]
\[ (\forall x, y, z \in X)(x \ast (x \ast y) = y \ast (y \ast (x \ast y))), \]
\[ (\forall x, y, z \in X)(x, y \leq z, z \ast y \leq z \ast x \Rightarrow x \leq y), \]
\[ (\forall x, y, z \in X)(x, y \leq z, z \ast x \ast y \leq z \ast x \Rightarrow x \leq y), \]

let \( \tilde{H} := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F) \) be a neutrosophic falling shadow of a neutrosophic random set \( \xi := (\xi_T, \xi_I, \xi_F) \) on \( X \). If \( H := (\tilde{H}_T, \tilde{H}_I, \tilde{H}_F) \) is a falling neutrosophic ideal of \( X \), then it is a commutative falling neutrosophic ideal of \( X \).

**References**


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