Commutative neutrosophic quadruple ideals of neutrosophic quadruple $BCK$-algebras

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Abstract

Commutative neutrosophic quadruple ideals and $BCK$-algebras are discussed, and related properties are investigated. Conditions for the neutrosophic quadruple $BCK$-algebra to be commutative are considered. Given subsets $A$ and $B$ of a neutrosophic quadruple $BCK$-algebra, conditions for the set $NQ(A, B)$ to be a commutative ideal of a neutrosophic quadruple $BCK$-algebra are provided.

1 Introduction

The neutrosophic set which is developed by Smarandache ([17], [18] and [19]) is a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. Neutrosophic algebraic structures in $BCK/BCI$-algebras are discussed in the papers [3], [8], [9], [10], [11], [13], [16] and [21]. Smarandache [20] considered an entry (i.e., a number, an idea, an object etc.) which is represented by a known part ($a$) and an unknown part ($bT, cI, dF$) where $T, I, F$ have their usual neutrosophic logic meanings and $a, b, c, d$ are real or complex numbers, and then he introduced the concept of neutrosophic quadruple numbers. Neutrosophic quadruple algebraic structures and hyperstructures are discussed in [11, 12]. Jun et al. [12] used neutrosophic quadruple numbers based on a set, and constructed neutrosophic quadruple $BCK/BCI$-algebras. They investigated several properties, and considered ideal and positive implicative ideal in neutrosophic quadruple $BCK$-algebra, and closed ideal in neutrosophic quadruple $BCI$-algebra. Given subsets $A$ and $B$ of a neutrosophic quadruple $BCK/BCI$-algebra, they

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considered sets \( NQ(A, B) \) which consists of neutrosophic quadruple \( BCK/BCI \)-numbers with a condition. They provided conditions for the set \( NQ(A, B) \) to be a (positive implicative) ideal of a neutrosophic quadruple \( BCK \)-algebra, and the set \( NQ(A, B) \) to be a (closed) ideal of a neutrosophic quadruple \( BCI \)-algebra. They gave an example to show that the set \( \{0\} \) is not a positive implicative ideal in a neutrosophic quadruple \( BCK \)-algebra, and then they considered conditions for the set \( \{0\} \) to be a positive implicative ideal in a neutrosophic quadruple \( BCK \)-algebra.

In this paper, we discuss a commutative neutrosophic quadruple ideal and \( BCK \)-algebra and investigate several properties. We consider conditions for the neutrosophic quadruple \( BCK \)-algebra to be commutative. Given subsets \( A \) and \( B \) of a neutrosophic quadruple \( BCK \)-algebra, we give conditions for the set \( NQ(A, B) \) to be a commutative ideal of a neutrosophic quadruple \( BCK \)-algebra.

2 Preliminaries

A \( BCK/BCI \)-algebra is an important class of logical algebras introduced by K. Iséki (see [6] and [7]) and was extensively investigated by several researchers.

By a \( BCI \)-algebra, we mean a set \( X \) with a special element 0 and a binary operation \( * \) that satisfies the following conditions:

(I) \((\forall x, y, z \in X) (((x * y) * (x * z)) * (z * y) = 0), \)

(II) \((\forall x, y \in X) ((x * (x * y)) * y = 0), \)

(III) \((\forall x \in X) (x * x = 0), \)

(IV) \((\forall x, y \in X) (x * y = 0, y * x = 0 \Rightarrow x = y). \)

If a \( BCI \)-algebra \( X \) satisfies the following identity:

(V) \((\forall x \in X) (0 * x = 0), \)

then \( X \) is called a \( BCK \)-algebra. Any \( BCK/BCI \)-algebra \( X \) satisfies the following conditions:

\[
(\forall x \in X) (x * 0 = x), \\
(\forall x, y, z \in X) (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x), \\
(\forall x, y, z \in X) ((x * y) * z = (x * z) * y), \\
(\forall x, y, z \in X) ((x * z) * (y * z) \leq x * y)
\]

where \( x \leq y \) if and only if \( x * y = 0. \)

A \( BCK \)-algebra \( X \) is said to be commutative if the following assertion is valid.

\[
(\forall x, y \in X) (x * (x * y) = y * (y * x)).
\]

A subset \( I \) of a \( BCK/BCI \)-algebra \( X \) is called an ideal of \( X \) if it satisfies:

\[
0 \in I, \\
(\forall x \in X) (\forall y \in I) (x * y \in I \Rightarrow x \in I).
\]

A subset \( I \) of a \( BCK \)-algebra \( X \) is called a commutative ideal of \( X \) if it satisfies (6) and

\[
(\forall x, y \in X)(\forall z \in I) ((x * y) * z \in I \Rightarrow x * (y * (y * x)) \in I).
\]
Observe that every commutative ideal is an ideal, but the converse is not true (see [14]).

We refer the reader to the books [5, 14] for further information regarding BCK/BCI-algebras, and to the site “http://fs.gallup.unm.edu/neutrosophy.htm” for further information regarding neutrosophic set theory.

3 Commutative neutrosophic quadruple BCK-algebras

In this section, we define commutative neutrosophic quadruple BCK-algebra under Theorem 3.3 and consider some properties of commutative neutrosophic quadruple BCK-algebra. Also, we investigate relation between commutative neutrosophic quadruple BCK-algebra and lattices.

Definition 3.1 ([12]). Let \( X \) be a set. A neutrosophic quadruple \( X \)-number is an ordered quadruple \( (a, xT, yI, zF) \) where \( a, x, y, z \in X \) and \( T, I, F \) have their usual neutrosophic logic meanings.

The set of all neutrosophic quadruple \( X \)-numbers is denoted by \( NQ(X) \), that is,

\[
NQ(X) := \{(a, xT, yI, zF) \mid a, x, y, z \in X\},
\]

and it is called the neutrosophic quadruple set based on \( X \). If \( X \) is a BCK/BCI-algebra, a neutrosophic quadruple \( X \)-number is called a neutrosophic quadruple BCK/BCI-number and we say that \( NQ(X) \) is the neutrosophic quadruple BCK/BCI-set.

Let \( X \) be a BCK/BCI-algebra. We define a binary operation \( \odot \) on \( NQ(X) \) by

\[
(a, xT, yI, zF) \odot (b, uT, vI, wF) = (a \ast b, (x \ast u)T, (y \ast v)I, (z \ast w)F)
\]

for all \( (a, xT, yI, zF), (b, uT, vI, wF) \in NQ(X) \). Given \( a_1, a_2, a_3, a_4 \in X \), the neutrosophic quadruple BCK/BCI-number \( (a_1, a_2T, a_3I, a_4F) \) is denoted by \( \bar{a} \), that is,

\[
\bar{a} = (a_1, a_2T, a_3I, a_4F),
\]

and the zero neutrosophic quadruple BCK/BCI-number \( (0T, 0I, 0F) \) is denoted by \( \bar{0} \), that is,

\[
\bar{0} = (0T, 0I, 0F).
\]

We define an order relation “\( \ll \)” and the equality “\( = \)” on \( NQ(X) \) as follows:

\[
\bar{x} \ll \bar{y} \Leftrightarrow x_i \leq y_i \text{ for } i = 1, 2, 3, 4,
\]

\[
\bar{x} = \bar{y} \Leftrightarrow x_i = y_i \text{ for } i = 1, 2, 3, 4,
\]

for all \( \bar{x}, \bar{y} \in NQ(X) \). It is easy to verify that “\( \ll \)” is a partial order on \( NQ(X) \).

Lemma 3.2 ([12]). If \( X \) is a BCK/BCI-algebra, then \( (NQ(X); \odot, \bar{0}) \) is a BCK/BCI-algebra, which is called a neutrosophic quadruple BCK/BCI-algebra.

Theorem 3.3. The neutrosophic quadruple BCK-set \( NQ(X) \) based on a commutative BCK-algebra \( X \) is a commutative BCK-algebra, which is called a commutative neutrosophic quadruple BCK-algebra.

Proof. Let \( X \) be a commutative BCK-algebra. Then \( X \) is a BCK-algebra, and so \( (NQ(X); \odot, \bar{0}) \) is a BCK-algebra by Lemma 3.2. Let \( \bar{x}, \bar{y} \in NQ(X) \). Then

\[
x_i \ast (x_i \ast y_i) = y_i \ast (y_i \ast x_i)
\]

for all \( i = 1, 2, 3, 4 \) since \( x_i, y_i \in X \) and \( X \) is a commutative BCK-algebra. Hence \( \bar{x} \odot (\bar{x} \odot \bar{y}) = \bar{y} \odot (\bar{y} \odot \bar{x}) \), and therefore \( NQ(X) \) based on a commutative BCK-algebra \( X \) is a commutative BCK-algebra. \( \square \)
Theorem 3.3 is illustrated by the following example.

**Example 3.4.** Let \( X = \{0, 1\} \) be a set with the binary operation \( * \) which is given in Table 1.

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Then \((X, *, 0)\) is a commutative BCK-algebra (see [14]), and the neutrosophic quadruple BCK-set \(NQ(X)\) is given as follows:

\[
NQ(X) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}, \tilde{8}, \tilde{9}, \tilde{10}, \tilde{11}, \tilde{12}, \tilde{13}, \tilde{14}, \tilde{15}\}
\]

where
- \(\tilde{0} = (0, 0T, 0I, 0F)\), \(\tilde{1} = (0, 0T, 0I, 1F)\), \(\tilde{2} = (0, 0T, 1I, 0F)\), \(\tilde{3} = (0, 0T, 1I, 1F)\),
- \(\tilde{4} = (1, 0T, 0I, 0F)\), \(\tilde{5} = (1, 0T, 1I, 0F)\), \(\tilde{6} = (1, 0T, 1I, 0F)\), \(\tilde{7} = (1, 0T, 1I, 1F)\),
- \(\tilde{8} = (1, 0T, 1I, 0F)\), \(\tilde{9} = (1, 0T, 1I, 0F)\), \(\tilde{10} = (1, 0T, 1I, 0F)\), \(\tilde{11} = (1, 0T, 1I, 1F)\),
- \(\tilde{12} = (1, 1T, 1I, 0F)\), \(\tilde{13} = (1, 1T, 1I, 0F)\), \(\tilde{14} = (1, 1T, 1I, 0F)\), \(\tilde{15} = (1, 1T, 1I, 1F)\).

Then \((NQ(X), \circ, 0)\) is a commutative BCK-algebra in which the operation \(\circ\) is given by Table 2.

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**Proposition 3.5.** The neutrosophic quadruple BCK-set \(NQ(X)\) based on a commutative BCK-
algebra $X$ satisfies the following assertions.

\begin{align}
(\forall \tilde{x}, \tilde{y}, \tilde{z} \in NQ(X))(\tilde{x} \ll \tilde{z}, \tilde{z} \circ \tilde{y} \ll \tilde{z} \circ \tilde{x} \Rightarrow \tilde{x} \ll \tilde{y}). \\
(\forall \tilde{x}, \tilde{y}, \tilde{z} \in NQ(X))(\tilde{x} \ll \tilde{z}, \tilde{z} \circ \tilde{y} \ll \tilde{z} \circ \tilde{x} \Rightarrow \tilde{x} \ll \tilde{y}). \\
(\forall \tilde{x}, \tilde{y}, \tilde{z} \in NQ(X))(\tilde{x} \ll \tilde{y} \Rightarrow \tilde{y} \circ (\tilde{y} \circ \tilde{x}) = \tilde{x}). \\
(\forall \tilde{x}, \tilde{y} \in NQ(X))(\tilde{x} \circ (\tilde{x} \circ \tilde{y}) = \tilde{y} \circ (\tilde{y} \circ (\tilde{x} \circ \tilde{y}))).
\end{align}

Proof. Assume that $\tilde{x} \ll \tilde{z}$ and $\tilde{z} \circ \tilde{y} \ll \tilde{z} \circ \tilde{x}$ for all $\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)$. Then $\tilde{x} \circ \tilde{z} = \tilde{y}$ and $(\tilde{z} \circ \tilde{y}) \circ (\tilde{z} \circ \tilde{x}) = \tilde{0}$. Since $NQ(X)$ is commutative, we have

$$\tilde{x} \circ \tilde{y} = (\tilde{x} \circ \tilde{0}) \circ \tilde{y} = (\tilde{x} \circ (\tilde{x} \circ \tilde{z})) \circ \tilde{y} = (\tilde{z} \circ (\tilde{z} \circ \tilde{x})) \circ \tilde{y} = (\tilde{z} \circ \tilde{y}) \circ (\tilde{z} \circ \tilde{x}) = \tilde{0},$$

that is, $\tilde{x} \ll \tilde{y}$. Condition (10) is clear by the condition (9). Suppose that $\tilde{x} \ll \tilde{y}$ for all $\tilde{x}, \tilde{y} \in NQ(X)$. Note that $\tilde{y} \circ (\tilde{y} \circ \tilde{x}) \ll \tilde{y}$ and $\tilde{y} \circ (\tilde{y} \circ (\tilde{y} \circ \tilde{x})) \ll \tilde{y} \circ \tilde{x}$ for all $\tilde{x}, \tilde{y} \in NQ(X)$. It follows from the condition (10) that $\tilde{x} \ll \tilde{y} \circ (\tilde{y} \circ \tilde{x})$. Obviously, $\tilde{y} \circ (\tilde{y} \circ \tilde{x}) \ll \tilde{x}$, and so $\tilde{y} \circ (\tilde{y} \circ \tilde{x}) = \tilde{x}$. Condition (12) follows directly from the condition (11). □

Theorem 3.6. The neutrosophic quadruple $BCK$-set $NQ(X)$ based on a commutative $BCK$-algebra $X$ is a lower semilattice with respect to the order “$\ll$”.

Proof. For any $\tilde{x}, \tilde{y} \in NQ(X)$, let $\tilde{y} \circ (\tilde{y} \circ \tilde{x}) = \tilde{x} \wedge \tilde{y}$. Then $\tilde{x} \wedge \tilde{y} \ll \tilde{x}$ and $\tilde{x} \wedge \tilde{y} \ll \tilde{y}$. Let $\tilde{a} \in NQ(X)$ such that $\tilde{a} \ll \tilde{x}$ and $\tilde{a} \ll \tilde{y}$. Then

$$\tilde{a} = \tilde{a} \circ \tilde{0} = \tilde{a} \circ (\tilde{a} \circ \tilde{x}) = \tilde{x} \circ (\tilde{x} \circ \tilde{a}).$$

Similarly, we have $\tilde{a} = \tilde{y} \circ (\tilde{y} \circ \tilde{a})$. Thus

$$\tilde{a} = \tilde{x} \circ (\tilde{x} \circ \tilde{a}) = \tilde{x} \circ (\tilde{x} \circ (\tilde{y} \circ (\tilde{y} \circ \tilde{a}))) \ll \tilde{x} \circ (\tilde{x} \circ \tilde{y}) = \tilde{y} \circ (\tilde{y} \circ \tilde{x}) = \tilde{x} \wedge \tilde{y}.$$

Hence $\tilde{x} \wedge \tilde{y}$ is the greatest lower bound, and therefore $(NQ(X), \ll)$ is a lower semilattice. □

Given a neutrosophic quadruple $BCK$-algebra $NQ(X)$, we consider the following set.

$$\Omega(\tilde{a}) := \{\tilde{x} \in NQ(X) \mid \tilde{x} \ll \tilde{a}\}. \quad (13)$$

Proposition 3.7. Every neutrosophic quadruple $BCK$-set $NQ(X)$ based on a commutative $BCK$-algebra $X$ satisfies the following identity.

$$\forall \tilde{a}, \tilde{b} \in NQ(X))(\Omega(\tilde{a}) \cap \Omega(\tilde{b}) = \Omega(\tilde{a} \wedge \tilde{b})) \quad (14)$$

where $\tilde{a} \wedge \tilde{b} = \tilde{b} \circ (\tilde{b} \circ \tilde{a})$.

Proof. Let $\tilde{x} \in \Omega(\tilde{a}) \cap \Omega(\tilde{b})$. Then $\tilde{x} \ll \tilde{a}$ and $\tilde{x} \ll \tilde{b}$, and so $\tilde{x} \ll \tilde{a} \wedge \tilde{b}$. Thus $\tilde{x} \in \Omega(\tilde{a} \wedge \tilde{b})$, which shows that $\Omega(\tilde{a}) \cap \Omega(\tilde{b}) \subseteq \Omega(\tilde{a} \wedge \tilde{b})$. If $\tilde{x} \in \Omega(\tilde{a} \wedge \tilde{b})$, then $\tilde{x} \ll \tilde{a} \wedge \tilde{b}$. Hence $\tilde{x} \ll \tilde{a}$ and $\tilde{x} \ll \tilde{b}$, and thus $\tilde{x} \in \Omega(\tilde{a}) \cap \Omega(\tilde{b})$. This completes the proof. □

We consider conditions for a neutrosophic quadruple $BCK$-algebra $NQ(X)$ to be commutative.

Lemma 3.8. If a neutrosophic quadruple $BCK$-algebra $NQ(X)$ satisfies the condition (11), then it is commutative.
Proof. Assume that \( NQ(X) \) is a neutrosophic quadruple \( BCK \)-algebra which satisfies the condition (11). Note that \( \tilde{y} \circ (\tilde{y} \circ \tilde{x}) \ll \tilde{x} \) for all \( \tilde{x}, \tilde{y} \in NQ(X) \). It follows from the condition (11) that

\[
\tilde{y} \circ (\tilde{y} \circ \tilde{x}) = \tilde{x} \circ (\tilde{x} \circ (\tilde{y} \circ (\tilde{y} \circ \tilde{x}))).
\]

Hence

\[
(\tilde{y} \circ (\tilde{y} \circ \tilde{x})) \circ (\tilde{x} \circ (\tilde{x} \circ \tilde{y})) = (\tilde{x} \circ (\tilde{x} \circ (\tilde{y} \circ (\tilde{y} \circ \tilde{x})))) \circ (\tilde{x} \circ (\tilde{x} \circ \tilde{y})) = (\tilde{x} \circ \tilde{y}) \circ (\tilde{x} \circ (\tilde{y} \circ (\tilde{y} \circ \tilde{x})))
\]

\[
\ll (\tilde{y} \circ (\tilde{y} \circ \tilde{x})) \circ \tilde{y} = \tilde{0}
\]

for all \( \tilde{x}, \tilde{y} \in NQ(X) \). Similarly, we get that \( (\tilde{x} \circ (\tilde{x} \circ \tilde{y})) \circ (\tilde{y} \circ (\tilde{y} \circ \tilde{x})) = \tilde{0} \) by changing the role of \( \tilde{x} \) and \( \tilde{y} \). Therefore \( \tilde{x} \circ (\tilde{x} \circ \tilde{y}) = \tilde{y} \circ (\tilde{y} \circ \tilde{x}) \) and so \( NQ(X) \) is commutative.

\( \square \)

Theorem 3.9. If a neutrosophic quadruple \( BCK \)-algebra \( NQ(X) \) satisfies the condition (12), then it is commutative.

Proof. Assume that \( NQ(X) \) is a neutrosophic quadruple \( BCK \)-algebra which satisfies the condition (12). Let \( \tilde{x}, \tilde{y} \in NQ(X) \) such that \( \tilde{x} \ll \tilde{y} \). Then

\[
\tilde{y} \circ (\tilde{y} \circ \tilde{x}) = \tilde{y} \circ (\tilde{x} \circ (\tilde{x} \circ \tilde{y})) = \tilde{x} \circ (\tilde{x} \circ \tilde{y}) = \tilde{x} \circ \tilde{0} = \tilde{x},
\]

and so \( NQ(X) \) is commutative by Lemma 3.8.

\( \square \)

Lemma 3.10. A neutrosophic quadruple \( BCK \)-algebra \( NQ(X) \) is commutative if and only if the following assertion is valid.

\[
(\forall \tilde{x}, \tilde{y} \in NQ(X)) (\tilde{y} \circ (\tilde{y} \circ \tilde{x}) \ll (\tilde{x} \circ (\tilde{x} \circ \tilde{y}))).
\]

(15)

Proof. It is straightforward.

\( \square \)

Theorem 3.11. If a neutrosophic quadruple \( BCK \)-algebra \( NQ(X) \) satisfies the condition (14), then it is commutative.

Proof. Let \( NQ(X) \) be a neutrosophic quadruple \( BCK \)-algebra which satisfies the condition (14). Let \( \tilde{x} \land \tilde{y} := \tilde{y} \circ (\tilde{y} \circ \tilde{x}) \) for all \( \tilde{x}, \tilde{y} \in NQ(X) \). Then

\[
\Omega(\tilde{x} \land \tilde{y}) = \Omega(\tilde{x}) \cap \Omega(\tilde{y}) = \Omega(\tilde{y}) \cap \Omega(\tilde{x}) = \Omega(\tilde{y} \land \tilde{x})
\]

for all \( \tilde{x}, \tilde{y} \in NQ(X) \), and thus \( \tilde{x} \land \tilde{y} \in \Omega(\tilde{y} \land \tilde{x}) \). Hence \( \tilde{x} \land \tilde{y} \ll \tilde{y} \land \tilde{x} \), that is, \( \tilde{y} \circ (\tilde{y} \circ \tilde{x}) \ll \tilde{x} \circ (\tilde{x} \circ \tilde{y}) \). It follows from Lemma 3.10 that \( NQ(X) \) is a commutative neutrosophic quadruple \( BCK \)-algebra.

\( \square \)

Theorem 3.12. Given a nonempty set \( X \), if a neutrosophic quadruple set \( NQ(X) \) satisfies the following assertions

\[
(\forall \tilde{x} \in NQ(X)) (\tilde{x} \circ \tilde{0} = \tilde{x}, \ \tilde{x} \circ \tilde{x} = \tilde{0}),
\]

(16)

\[
(\tilde{x}, \tilde{y}, \tilde{z} \in NQ(X)) ((\tilde{x} \circ \tilde{y}) \circ \tilde{z} = (\tilde{x} \circ \tilde{z}) \circ \tilde{y}),
\]

(17)

\[
(\tilde{x}, \tilde{y} \in NQ(X)) (\tilde{x} \land \tilde{y} = \tilde{y} \land \tilde{x})
\]

(18)

where \( \tilde{x} \land \tilde{y} = \tilde{y} \circ (\tilde{y} \circ \tilde{x}) \), then it is a commutative neutrosophic quadruple \( BCK \)-algebra.
Therefore, it follows from (19) and (16) that

\[
(\tilde{x} \circ (\tilde{x} \circ \tilde{y})) \circ \tilde{y} = (\tilde{x} \circ \tilde{y}) \circ (\tilde{x} \circ \tilde{y}) = \tilde{0}.
\]

Assume that \(\tilde{x} \circ \tilde{y} = \tilde{0}\) and \(\tilde{y} \circ \tilde{x} = \tilde{0}\). Then

\[
\tilde{x} = \tilde{x} \circ \tilde{0} = \tilde{x} \circ (\tilde{x} \circ \tilde{y}) = \tilde{y} \wedge \tilde{x} = \tilde{y} \wedge \tilde{y} = (\tilde{y} \circ \tilde{x}) = \tilde{x} \circ \tilde{0} = \tilde{y}.
\]

Using (17) and (18), we have

\[
(\tilde{x} \circ \tilde{y}) \circ (\tilde{x} \circ \tilde{z}) = (\tilde{x} \circ (\tilde{x} \circ \tilde{z})) \circ \tilde{y} = (= (\tilde{z} \circ \tilde{x}) \circ \tilde{y} = (\tilde{z} \circ \tilde{y}) \circ (\tilde{x} \circ \tilde{z}).
\]

(19)

If we take \(\tilde{y} = \tilde{x}\) and \(\tilde{z} = \tilde{0}\) in (19), then

\[
\tilde{0} \circ \tilde{x} = (\tilde{x} \circ \tilde{x}) \circ (\tilde{x} \circ \tilde{0}) = (\tilde{0} \circ \tilde{x}) \circ (\tilde{0} \circ \tilde{x}) = \tilde{0}.
\]

It follows from (19) and (16) that

\[
((\tilde{x} \circ \tilde{y}) \circ (\tilde{x} \circ \tilde{z})) \circ (\tilde{z} \circ \tilde{y}) = ((\tilde{z} \circ \tilde{x}) \circ (\tilde{z} \circ \tilde{x})) \circ (\tilde{z} \circ \tilde{y}) \circ \tilde{0}) = (\tilde{0} \circ (\tilde{z} \circ \tilde{x})) \circ (\tilde{0} \circ (\tilde{z} \circ \tilde{y})) = \tilde{0} \circ \tilde{0} = \tilde{0}.
\]

Therefore \((NQ(X), \circ, \tilde{0})\) is a commutative neutrosophic quadruple \(BCK\)-algebra.

Given subsets \(A\) and \(B\) of a \(BCK\)-algebra \(X\), consider the set

\[
NQ(A, B) := \{(a, xT, yI, zF) \in NQ(X) \mid a, x \in A; y, z \in B\}.
\]

**Theorem 3.13.** If \(A\) and \(B\) are commutative ideals of a \(BCK\)-algebra \(X\), then the set \(NQ(A, B)\) is a commutative ideal of \(NQ(X)\), which is called a commutative neutrosophic quadruple ideal.

**Proof.** Assume that \(A\) and \(B\) are commutative ideals of a \(BCK\)-algebra \(X\). Obviously, \(\tilde{0} \in NQ(A, B)\). Let \(\tilde{x} = (x_1, x_2T, x_3I, x_4F), \tilde{y} = (y_1, y_2T, y_3I, y_4F)\) and \(\tilde{z} = (z_1, z_2T, z_3I, z_4F)\) be elements of \(NQ(X)\) such that \(\tilde{z} \in NQ(A, B)\) and \((\tilde{x} \circ \tilde{y}) \circ \tilde{z} \in NQ(A, B)\). Then

\[
(\tilde{x} \circ \tilde{y}) \circ \tilde{z} = ((x_1 \ast y_1) \ast z_1, ((x_2 \ast y_2) \ast z_2)T, ((x_3 \ast y_3) \ast z_3)I, ((x_4 \ast y_4) \ast z_4)F) \in NQ(A, B),
\]

and so \((x_1 \ast y_1) \ast z_1 \in A, (x_2 \ast y_2) \ast z_2 \in A, (x_3 \ast y_3) \ast z_3 \in B\) and \((x_4 \ast y_4) \ast z_4 \in B\). Since \(\tilde{z} \in NQ(A, B)\), we have \(z_1, z_2 \in A\) and \(z_3, z_4 \in B\). Since \(A\) and \(B\) are commutative ideals of \(X\), it follows that \(x_1 \ast (y_1 \ast (y_1 \ast x_1)) \in A, x_2 \ast (y_2 \ast (y_2 \ast x_2)) \in A, x_3 \ast (y_3 \ast (y_3 \ast x_3)) \in B\) and \(x_4 \ast (y_4 \ast (y_4 \ast x_4)) \in B\). Hence

\[
\tilde{x} \circ (\tilde{y} \circ (\tilde{y} \circ \tilde{x})) = (x_1 \ast (y_1 \ast (y_1 \ast x_1)), (x_2 \ast (y_2 \ast (y_2 \ast x_2)))T, (x_3 \ast (y_3 \ast (y_3 \ast x_3))I, (x_4 \ast (y_4 \ast (y_4 \ast x_4)))F) \in NQ(A, B),
\]

and therefore \(NQ(A, B)\) is a commutative ideal of \(NQ(X)\).

**Lemma 3.14** ([12]). If \(A\) and \(B\) are ideals of a \(BCK\)-algebra \(X\), then the set \(NQ(A, B)\) is an ideal of \(NQ(X)\), which is called a neutrosophic quadruple ideal.
Theorem 3.15. Let $A$ and $B$ be ideals of a BCK-algebra $X$ such that

$$(\forall x, y \in X) (x * y \in A \text{ (resp., } B) \Rightarrow x * (y * (y * x)) \in A \text{ (resp., } B)).$$

Then $NQ(A, B)$ is a commutative ideal of $NQ(X)$.

Proof. If $A$ and $B$ are ideals of a BCK-algebra $X$, then $NQ(A, B)$ is an ideal of $NQ(X)$ by Lemma 3.14. Let $\bar{x} = (x_1, x_2 T, x_3 I, x_4 F)$, $\bar{y} = (y_1, y_2 T, y_3 I, y_4 F)$ and $\bar{z} = (z_1, z_2 T, z_3 I, z_4 F)$ be elements of $NQ(X)$ such that $(\bar{x} \circ \bar{y}) \circ \bar{z} \in NQ(A, B)$ and $\bar{z} \in NQ(A, B)$. Then

$$(\bar{x} \circ \bar{y}) \circ \bar{z} = ((x_1 * y_1) * z_1, (x_2 * y_2) * z_2) T, (x_3 * y_3) * z_3 I, ((x_4 * y_4) * z_4) F \in NQ(A, B),$$

and $\bar{z} = (z_1, z_2 T, z_3 I, z_4 F) \in NQ(A, B)$. So $(x_1 * y_1) * z_1 \in A$, $(x_2 * y_2) * z_2 \in A$, $(x_3 * y_3) * z_3 \in B$, $(x_4 * y_4) * z_4 \in B$. Since $A$ and $B$ are ideals of $X$, we get that $x_1 * y_1 \in A$, $x_2 * y_2 \in A$, $x_3 * y_3 \in B$ and $x_4 * y_4 \in B$. It follows from (20) that $x_1 * (y_1 * x_1) \in A$, $x_2 * (y_2 * x_2) \in A$, $x_3 * (y_3 * x_3) \in B$ and $x_4 * (y_4 * x_4) \in B$. Hence

$$\bar{x} \circ (\bar{y} \circ (\bar{y} \circ \bar{x})) = (x_1 * (y_1 * (y_1 * x_1)), (x_2 * (y_2 * (y_2 * x_2))) T, (x_3 * (y_3 * (y_3 * x_3))) I, (x_4 * (y_4 * (y_4 * x_4))) F \in NQ(A, B).$$

Therefore $NQ(A, B)$ is a commutative ideal of $NQ(X)$.

Corollary 3.16. For any ideals $A$ and $B$ of a BCK-algebra $X$, if the set $NQ(A, B)$ satisfies

$$(\forall \bar{x}, \bar{y} \in NQ(A, B)) (\bar{x} \circ \bar{y} \in NQ(A, B) \Rightarrow \bar{x} \circ (\bar{y} \circ (\bar{y} \circ \bar{x})) \in NQ(A, B)),$$

then $NQ(A, B)$ is a commutative ideal of $NQ(X)$.

Theorem 3.17. Let $I$, $J$, $A$ and $B$ be ideals of a BCK-algebra $X$ such that $I \subseteq A$ and $J \subseteq B$. If $I$ and $J$ are commutative ideals of $X$, then the set $NQ(I, B)$ is a commutative ideal of $NQ(X)$.

Proof. If $I$ and $J$ are commutative ideals of $X$, then $NQ(I, J)$ is a commutative ideal of $NQ(X)$ by Theorem 3.13. Note that $NQ(A, B)$ is an ideal of $NQ(X)$ by Lemma 3.14 and $NQ(I, J) \subseteq NQ(A, B)$. Assume that $x * y \in A$ (resp., $B$) for all $x, y \in X$ and let $a := x * y$. Then

$$(x * a) * y = (x * y) * a = 0 \in I \text{ (resp., } J),$$

and so $((x * a) * y) * 0 = (x * a) * y \in I \text{ (resp., } J)$. Since $I$ and $J$ are commutative ideals of $X$ with $I \subseteq A$ and $J \subseteq B$, it follows that

$$(x * (y * (x * a))) * a = (x * a) * (y * (y * (x * a))) \in I \subseteq A \text{ (resp., } J \subseteq B),$$

thus, $x * (y * (x * a)) \in A \text{ (resp., } B)$. On the other hand,

$$((x * y * (x * a))) * (x * (y * (x * a))) \leq (y * (y * (x * a))) * (y * (y * x)) \leq (y * x) * (y * (x * a)) \leq (x * a) * x = 0 * a = 0.$$
Example 3.18. Consider a BCK-algebra $X = \{0,1,2\}$ with the binary operation $*$ which is given in Table 3.

Table 3: Cayley table for the binary operation “$*$”

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

Then the neutrosophic quadruple BCK-algebra $NQ(X)$ has 81 elements. If we take commutative ideals $A = \{0,1\}$ and $B = \{0,2\}$ of $X$, then

$$NQ(A,B) = \{(0,0T,0I,0F), (0,0T,0I,2F), (0,0T,2I,0F), (0,0T,2I,2F), (0,1T,0I,0F), (0,1T,0I,2F), (0,1T,2I,0F), (0,1T,2I,2F), (1,0T,0I,0F), (1,0T,0I,2F), (1,0T,2I,0F), (1,0T,2I,2F), (1,1T,0I,0F), (1,1T,0I,2F), (1,1T,2I,0F), (1,1T,2I,2F)\}$$

which is a commutative ideal of $NQ(X)$.

Example 3.19. Consider a BCK-algebra $X = \{0,a,b,c\}$ with the binary operation $*$ which is given in Table 4.

Table 4: Cayley table for the binary operation “$*$”

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Then $(X,*,0)$ is a commutative BCK-algebra (see [14]), and the neutrosophic quadruple BCK-set $NQ(X)$ based on $X$ has 256 elements and it is a commutative BCK-algebra by Theorem 3.3. If we take commutative ideals $A = \{0,a,b\}$ and $B = \{0,c\}$ of $X$, then the set $NQ(A,B)$ consists of
36 elements, which is a commutative ideal of $NQ(X)$ by Theorem 3.13, and it is given as follows.

$$NQ(A, B) = \{(0, 0T, 0I, 0F), (0, 0T, 0I, cF), (0, 0T, cI, 0F), (0, 0T, cI, cF),$$

$$\quad (0, aT, 0I, 0F), (0, aT, 0I, cF), (0, aT, cI, 0F), (0, aT, cI, cF),$$

$$\quad (0, bT, 0I, 0F), (0, bT, 0I, cF), (0, bT, cI, 0F), (0, bT, cI, cF),$$

$$\quad (a, 0T, 0I, 0F), (a, 0T, 0I, cF), (a, 0T, cI, 0F), (a, 0T, cI, cF),$$

$$\quad (a, aT, 0I, 0F), (a, aT, 0I, cF), (a, aT, cI, 0F), (a, aT, cI, cF),$$

$$\quad (a, bT, 0I, 0F), (a, bT, 0I, cF), (a, bT, cI, 0F), (a, bT, cI, cF),$$

$$\quad (b, 0T, 0I, 0F), (b, 0T, 0I, cF), (b, 0T, cI, 0F), (b, 0T, cI, cF),$$

$$\quad (b, aT, 0I, 0F), (b, aT, 0I, cF), (b, aT, cI, 0F), (b, aT, cI, cF),$$

$$\quad (b, bT, 0I, 0F), (b, bT, 0I, cF), (b, bT, cI, 0F), (b, bT, cI, cF)\}.$$

4 Conclusions

We have considered a commutative neutrosophic quadruple ideals and BCK-algebras are discussed, and investigated several related properties are investigated. Conditions for the neutrosophic quadruple BCK-algebra to be commutative are considered. Given subsets $A$ and $B$ of a neutrosophic quadruple BCK algebra, conditions for the set $NQ(A, B)$ to be a commutative ideal of a neutrosophic quadruple BCK-algebra are provided.

References


