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Progress in Nonlinear Dynamics and Chaos

Determinant Theory for Fuzzy Neutrosophic Soft Matrices

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Abstract. We study determinant theory for fuzzy neutrosophic soft square matrices, its properties and also we prove that $det(Aadj(A)) = det(A) = det(adj(A)A)$.

Keywords: Adjoint, determinant, fuzzy neutrosophic soft square matrix, determinant of fuzzy neutrosophic soft square matrix.

AMS Mathematics Subject Classification (2010): 10M20

1. Introduction

The complexity of problems in economics, engineering, environmental sciences and social sciences which cannot be solved by the well known methods of classical Mathematics pose a great difficulty in today's practical world (as various types of uncertainties are presented in these problems) . To handle situation like these, many tools have been suggested. Some of them are probability theory, fuzzy set theory [18], rough set theory [11], etc.

The traditional fuzzy set is characterized by the membership value or the grade of membership value. Sometimes it may be very difficult to assign the membership value for fuzzy sets. Interval-valued fuzzy sets were proposed as a natural extension of fuzzy sets and the interval valued fuzzy sets were proposed independently by Zadeh [19] to ascertain the uncertainty of grade of membership value. In current scenario of practical problems in expert systems, belief system, information fusion and so on, we must consider the truth membership as well as the falsity-membership for proper description of an object in imprecise and doubtful environment. Neither the fuzzy sets nor the interval valued fuzzy sets is appropriate for such a situation.

Intuitionistic fuzzy set initiated by Atanassov [3] is appropriate for such a situation. The intuitionistic fuzzy sets can only handle the incomplete information considering both the truth membership (or simply membership) and falsity-membership (or non membership) values. It does not handle the indeterminate and inconsistent information which exist in belief system. The soft set theory, an utterly new theory for modeling ambiguity and uncertainties was first coined by Molodstov [9] in the year 1999.

Soft set theory research is carried out as a new trend and it shows much appreciable development well received by the users of the field.

Fuzzy matrices play crucial role in Science and Technology. Sometimes the issues cannot be solved by classical matrix theory when they occur in an uncertain environment and this failure is inevitable. Thomason [14] initiated the fuzzy matrices to represent fuzzy relation in a system based on fuzzy set theory and discussed about the convergence of power of fuzzy matrix. In 1995, Smarandache introduced the concept of neutrosophy. In neutrosophic logic, each proposition is approximated to have the percentage of truth in a subset T, the percentage of indeterminancy in a subset I and the percentage of falsity in a subset F, so that this neutrosophic logic is called an extension of fuzzy logic. In fact this mathematical tool is used to handle problems like imprecision, indeterminancy and inconsistency of data etc.

Maji et al. [5], initiated the concept of fuzzy soft set with some properties regarding fuzzy soft union, intersection, complement of fuzzy soft set. Moreover Maji et al. [6,10] extended soft sets to intuitionistic fuzzy soft sets and neutrosophic soft sets and the concept of neutrosophic set was introduced by Smarandache [12] which is a generalization of fuzzy logic and several related systems.

Yang and Ji [17], introduced a matrix representation of fuzzy soft set and applied it in decision making problems. Bora et al. [8] introduced the intuitionistic fuzzy soft matrices and applied in the application of a Medical diagnosis.

Sumathi and Arokiarani [13] introduced new operation on fuzzy neutrosophic soft matrices. Dhar et al. [7] have also defined neutrosophic fuzzy matrices and studied square neutrosophic fuzzy matrices. Uma et al. [15,16], introduced two types of fuzzy neutrosophic soft matrices and have discussed determinant and adjoint of fuzzy neutrosophic soft matrices. Kim et al. [4], introduced the concept of determinant theory for fuzzy matrices.

In this paper, some elementary properties of determinant theory for fuzzy neutrosophic soft square matrices have been established and some theorems including $det(A(adjA)) = det(A) = det(adj(A)A)$. where $det(A)$ denotes the determinant of A and $adj(A)$ denotes the adjoint matrix of A.

2. Preliminaries

Definition 2.1. [12] A neutrosophic set A on the universe of discourse X is defined as

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle, x \in X\},$$

where $T, I, F : X \to\ ]^-0,1^+[$ and $^-0 \le T_A(x) + I_A(x) + F_A(x) \le 3^+$ (1)

From philosophical point of view the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0,1^+[$. But in real life application especially in scientific and Engineering problems it is difficult to use neutrosophic set with value from real standard or non-standard subset of $]^-0,1^+[$. Hence we consider the neutrosophic set which takes the value from the subset of [0,1] .Therefore we can rewrite the equation (1) as $0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$

In short an element $\tilde{a}$ in the neutrosophic set A, can be written as $\tilde{a} = \langle a^T, a^I, a^F \rangle$, where a^T denotes degree of truth, a^I denotes degree of indeterminacy, a^F denotes degree of falsity such that $0 \le a^T + a^I + a^F \le 3$.

Example 2.2. Assume that the universe of discourse $X = \{x_1, x_2, x_3\}$, where x_1, x_2, and x_3 characterizes the quality, reliability, and the price of the objects. It may be further assumed that the values of $\{x_1, x_2, x_3\}$ are in [0,1] and they are obtained from some investigations of some experts. The experts may impose their opinion in three components viz; the degree of goodness, the degree of indeterminacy and the degree of poorness to explain the characteristics of the objects. Suppose A is a Neutrosophic Set (NS) of X, such that $A = \{\langle x_1, 0.4, 0.5, 0.3\rangle, \langle x_2, 0.7, 0.2, 0.4\rangle, \langle x_3, 0.8, 0.3, 0.4\rangle\}$, where for x_1 the degree of goodness of quality is 0.4 , degree of indeterminacy of quality is 0.5 and degree of falsity of quality is 0.3, etc.

Definition 2.3. [9] Let U be an initial universe set and E be a set of parameters. Let P(U) denotes the power set of U. Consider a nonempty set A, $A \subset E$. A pair (F,A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

Definition 2.4. [1] Let U be an initial universe set and E be a set of parameters. Consider a non empty set A, $A \subset E$. Let $P(U)$ denotes the set of all fuzzy neutrosophic sets of U. The collection (F, A) is termed to be the Fuzzy Neutrosophic Soft Set (FNSS) over U, Where F is a mapping given by $F : A \to P(U)$. Hereafter we simply consider A as FNSS over U instead of (F, A).

Definition 2.5. [2] Let $U = \{c_1, c_2, \ldots c_m\}$ be the universal set and E be the set of parameters given by $E = \{e_1, e_2, \ldots e_n\}$. Let $A \subseteq E$. A pair (F, A) be a FNSS over U. Then the subset of $U \times E$ is defined by $R_A = \{(u, e); e \in A, u \in F_A(e)\}$ which is called a relation form of (F_A, E). The membership function, indeterminacy membership function and non membership function are written by $T_{R_A} : U \times E \to [0,1]$, $I_{R_A} : U \times E \to [0,1]$ and $F_{R_A} : U \times E \to [0,1]$ where $T_{R_A}(u,e) \in [0,1], I_{R_A}(u,e) \in [0,1]$ and $F_{R_A}(u,e) \in [0,1]$ are the membership value, indeterminacy value and non membership value respectively of $u \in U$ for each $e \in E$.
If $[(T_{ij}, I_{ij}, F_{ij})] = [(T_{ij}(u_i, e_j), I_{ij}(u_i, e_j), F_{ij}(u_i, e_j)]$ we define a matrix

$$\left[\langle T_{ij}, I_{ij}, F_{ij}\rangle\right]_{m\times n} = \begin{bmatrix} \langle T_{11}, I_{11}, F_{11}\rangle & \langle T_{12}, I_{12}, F_{12}\rangle & \cdots & \langle T_{1n}, I_{1n}, F_{1n}\rangle \\ \langle T_{21}, I_{21}, F_{21}\rangle & \langle T_{22}, I_{22}, F_{22}\rangle & \cdots & \langle T_{2n}, I_{2n}, F_{2n}\rangle \\ \vdots & \vdots & \vdots & \\ \langle T_{m1}, I_{m1}, F_{m1}\rangle & \langle T_{m2}, I_{m2}, F_{m2}\rangle & \cdots & \langle T_{mn}, I_{mn}, F_{mn}\rangle \end{bmatrix}$$

This is called an $m\times n$ FNSM of the FNSS (F_A, E) over U.

Definition 2.6. [15] Let $U=\{c_1,c_2...c_m\}$ be the universal set and E be the set of parameters given by $E=\{e_1,e_2,...e_n\}$. Let $A\subseteq E$. A pair (F,A) be a fuzzy neutrosophic soft set. Then fuzzy neutrosophic soft set (F,A) in a matrix form as $A_{m\times n}=(a_{ij})_{m\times n}$ or $A=(a_{ij}), i=1,2,...m, j=1,2,...n$ where

$$(a_{ij})=\begin{cases}(T(c_i,e_j), I(c_i,e_j), F(c_i,e_j)) & \text{if } e_j\in A\\ \langle 0,0,1\rangle & \text{if } e_j\notin A\end{cases}$$

where $T_j(c_i)$ represent the membership of $c_i, I_j(c_i)$ represent the indeterminacy of c_i and $F_j(c_i)$ represent the non-membership of c_i in the FNSS (F,A). If we replace the identity element $\langle 0,0,1\rangle$ by $\langle 0,1,1\rangle$ in the above form we get FNSM of type-II.

Let $\mathcal{F}_{m\times n}$ denotes FNSM of order $m\times n$ and $\mathcal{F}_n$ denotes FNSM of order $n\times n$.

Definition 2.7. [15] [Type-I]

Let $A=(\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle), B=(\langle b_{ij}^T, b_{ij}^I, b_{ij}^F\rangle)\in\mathcal{F}_{m\times n}$ the component wise addition and component wise multiplication is defined as

1. $A\oplus B=\ (sup\{a_{ij}^T, b_{ij}^T\},\ sup\{a_{ij}^I, b_{ij}^I\},\ inf\ \{a_{ij}^F, b_{ij}^F\})$.
2. $A\odot B=(inf\{a_{ij}^T, b_{ij}^T\},\ inf\{a_{ij}^I, b_{ij}^I\},\ sup\{a_{ij}^F, b_{ij}^F\})$.

Definition 2.8. [15] Let $A\in\mathcal{F}_{m\times n}, B\in\mathcal{F}_{n\times p}$, the composition of A and B is defined as

$$A\circ B=\left(\sum_{k=1}^{n}(a_{ik}^T\wedge b_{kj}^T),\ \sum_{k=1}^{n}(a_{ik}^I\wedge b_{kj}^I),\ \prod_{k=1}^{n}(a_{ik}^F\vee b_{kj}^F)\right)$$

equivalently we can write the same as

$$=\ \left(\bigvee_{k=1}^{n}(a_{ik}^T\wedge b_{kj}^T),\ \bigvee_{k=1}^{n}(a_{ik}^I\wedge b_{kj}^I),\ \bigwedge_{k=1}^{n}\ (a_{ik}^F\vee b_{kj}^F)\right)$$

The product $A\circ B$ is defined if and only if the number of columns of A is same as the number of rows of B. A and B are said to be conformable for multiplication. We shall use AB instead of $A\circ B$.

Definition 2.9.[15][Type-II] Let $A=(\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle), B=(\langle b_{ij}^T, b_{ij}^I, b_{ij}^F\rangle)\in\mathcal{F}_{m\times n}$, the component wise addition and component wise multiplication is defined as

$A \oplus B = (\langle sup\{a_{ij}^T, b_{ij}^T\},\ inf\{a_{ij}^I, b_{ij}^I\},\ inf\{a_{ij}^F, b_{ij}^F\}\rangle).$

$A \odot B = (\langle inf\{a_{ij}^T, b_{ij}^T\},\ sup\{a_{ij}^I, b_{ij}^I\},\ sup\{a_{ij}^F, b_{ij}^F\}\rangle).$

Analogous to FNSM of type-I, we can define FNSM of type -II in the following way

Definition 2.10. [15] Let $A = (\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle) = (a_{ij}) \in \mathcal{F}_{m\times n}$ and $B = (\langle b_{ij}^T, b_{ij}^I, b_{ij}^F\rangle) = (b_{ij}) \in \mathcal{F}_{n\times p}$ the product of A and B is defined as

$$A * B = \left(\sum_{k=1}^{n} \left\langle a_{ik}^T \wedge b_{kj}^T \right\rangle,\ \prod_{k=1}^{n} \left\langle a_{ik}^I \vee b_{kj}^I \right\rangle,\ \prod_{k=1}^{n} \left\langle a_{ik}^F \vee b_{kj}^F \right\rangle \right)$$

equivalently we can write the same as

$$= \left(\overset{n}{\underset{k=1}{\vee}} \left\langle a_{ik}^T \wedge b_{kj}^T \right\rangle, \overset{n}{\underset{k=1}{\wedge}} \left\langle a_{ik}^I \wedge b_{kj}^I \right\rangle, \overset{n}{\underset{k=1}{\wedge}} \left\langle a_{ik}^F \vee b_{kj}^F \right\rangle \right).$$

The product $A * B$ is defined if and only if the number of columns of A is same as the number of rows of B. A and B are said to be conformable for multiplication.

Definition 2.11. [16] The determinant $|A|$ of $n \times n$ FNSM $A = (\langle a_{ij}^T, a_{ij}^I . a_{ij}^F\rangle)$ is defined as follows

$$|A| = \langle \underset{\sigma\in S_n}{\vee} a_{1\sigma(1)}^T \wedge ... \wedge a_{n\sigma(n)}^T, \underset{\sigma\in S_n}{\vee} a_{1\sigma(1)}^I \wedge ... \wedge a_{n\sigma(n)}^I, \underset{\sigma\in S_n}{\wedge} a_{1\sigma(1)}^F \vee ... \vee a_{n\sigma(n)}^F \rangle$$

where S_n denotes the symmetric group of all permutations of the indices $(1, 2, ...n)$.

Definition 2.12. [16] The adjoint of an $n \times n$ FNSM A denoted by adj A, is defined as follows $b_{ij} = |A_{ji}|$ is the determinant of the $(n-1)\times(n-1)$ FNSM formed by deleting row j and column i from A and $B = adjA$.

Remark 2.13. We can write the element b_{ij} of $adjA = B = (b_{ij})$ as follows:

$b_{ij} = \sum_{\pi\in S_{n_j n_i}} \prod_{t\in n_j} \left\langle a_{t\pi(t)}^T, a_{t\pi(t)}^I, a_{t\pi(t)}^F \right\rangle$ Where $n_j = \{1, 2, 3.....n\} \setminus \{j\}$ and $S_{n_j n_i}$ is the set of all permutation of set n_j over the set n_i.

3. Properties of the fuzzy neutrosophic soft square matrices (FNSSM)

1. The value of the determinant remains unchanged when any two rows or columns are interchanged.
2. The values of the determinant of FNSSM remain unchanged when rows and columns are interchanged.
3. If A and B be two FNSSMs then the following property will hold
   $det(AB) \neq detA\, detB.$
4. If the elements of any row (or column) of a determinant are added to the corresponding elements of another row (or column), the value of the determinant thus obtained is equal to the value of the original determinant.

Theorem 3.1. Let $A = (a_{ij}^T, a_{ij}^I, a_{ij}^F) \in$ FNSSM$_n$.
Let $A_k = (\langle a_{k1}^T, a_{k1}^I, a_{k1}^F\rangle, \langle a_{k2}^T, a_{k2}^I, a_{k2}^F\rangle, ..., \langle a_{kn}^T, a_{kn}^I, a_{kn}^F\rangle)$ be the k-th row of A. We assume that $a_1 = \langle a_{ki}^T, a_{ki}^I, a_{ki}^F\rangle$ for all $i \in 1, 2, ..., n$ and $a_{pq} \geq a_1$ for all $p, q \in 1, 2, ..., n.$ Then $det(A) = a_1$.

Theorem 3.2. Let $A \in FNSSM_n$ then

(i) $det(A) = |A| = \sum_{t=1}^{n} \langle a_{it}^T, a_{it}^I, a_{it}^F\rangle A_{it}, i \in \{1, 2, ..., n\}$.

(ii) $det(A) = \sum_{e<f} \begin{vmatrix} \langle a_{1e}^T, a_{1e}^I, a_{1e}^F\rangle & \langle a_{1f}^T, a_{1f}^I, a_{1f}^F\rangle \\ \langle a_{2e}^T, a_{2e}^I, a_{2e}^F\rangle & \langle a_{2f}^T, a_{2f}^I, a_{2f}^F\rangle \end{vmatrix} A\begin{pmatrix} 1 & 2 \\ e & f \end{pmatrix}$, where the summation is taken over all e and f in {1,2,...,n} such that $e < f$.

Definition 3.3. Let $A = (a_{ij}^T, a_{ij}^I, a_{ij}^F)_n \in FNSSM_n$, and let B be a matrix from A by striking out e_1, row e_2,..., row e_k and column g_1, column g_2, ..., column g_k. we define $A\begin{pmatrix} e_1 & e_2 & ...e_k \\ g_1 & g_2 & ...g_k \end{pmatrix} = det(H)$.

Theorem 3.4.

$$det(A) = \sum_{g_1<g_2<...<g_k} det \begin{pmatrix} \langle a_1^T g_1, a_1^I g_1, a_1^F g_1\rangle & ...\langle a_1^T g_k, a_1^T g_k, a_1^T g_k\rangle \\ \langle a_2^T g_1, a_2^I g_2, a_2^F g_2\rangle & ...\langle a_2^T g_k, a_2^I g_k, a_2^F g_k\rangle \\ \vdots \quad \vdots \, \vdots & \\ \langle a_k^T g_1, a_k^I g_1, a_1^k g_1\rangle & ...\langle a_k^T g_k, a_k^I g_k, a_k^F g_k\rangle \end{pmatrix}$$

$A\begin{pmatrix} 1 & 2 & ...k \\ g_1 & g_2 & ...g_k \end{pmatrix}$ where the summation is taken over all $g_1, g_2, ..., g_k \in \{1, 2, ...n\}$, such that $g_1 < g_2 < ... < g_k$.

Proof: Let $S(g_1, g_2, ..., g_k) = \{\sigma : \{1, 2, ..., k\} \to \{g_1, g_2, ..., g_k\}\, \sigma$ is a bijection}. Then

$$det(A) = \sum_{\sigma \in S_n} \langle a_{1\sigma(1)}^T, a_{1\sigma(1)}^I, a_{1\sigma(1)}^F\rangle ... \langle a_{n\sigma(n)}^T, a_{n\sigma(n)}^I, a_{n\sigma(n)}^F\rangle$$

$$= \sum_{g_1<g_2<...<g_k} (\sum\nolimits_{\sigma\{1,2,...,k\} \in S(g_1, g_2, ..., g_k}) \langle a_{1\sigma(1)}^T, a_{1\sigma(1)}^I, a_{1\sigma(1)}^F\rangle ... \langle a_{n\sigma(n)}^T, a_{n\sigma(n)}^I, a_{n\sigma(n)}^F\rangle)$$

$$= \sum_{g_1<g_2<\ldots<g_k} \left(\sum_{\sigma^{\cdot}\{1,2,\ldots,k\}\in S(g_1,g_2,\ldots,g_k)}\right)(\langle a^T_{1\sigma^{\cdot}(1)}, a^I_{1\sigma^{\cdot}(1)}, a^F_{1\sigma^{\cdot}(1)}\rangle \ldots \langle a^T_{n\sigma^{\cdot}(n)}, a^I_{n\sigma^{\cdot}(n)}, a^F_{n\sigma^{\cdot}(n)}\rangle)$$

$$A\begin{pmatrix} 1 & 2 & \ldots & k \\ g_1 & g_2 & \ldots & g_k \end{pmatrix} = \sum_{g_1<g_2<\ldots<g_k} det$$

$$\begin{pmatrix} \langle a_1^T g_1, a_1^I g_1, a_1^F g_1\rangle & \langle a_1^T g_2, a_1^I g_2, a_1^F g_2\rangle & \ldots & \langle a_1^T g_k, a_1^I g_k, a_1^F g_k\rangle \\ \langle a_2^T g_1, a_2^I g_1, a_2^F g_1\rangle & \langle a_2^T g_2, a_2^I g_2, a_2^F g_2\rangle & \ldots & \langle a_2^T g_k, a_2^I g_k, a_2^F g_k\rangle \\ \ldots & \ldots & \ldots & \ldots \\ \langle a_k^T g_1, a_k^I g_1, a_k^F g_1\rangle & \langle a_k^T g_2, a_k^I g_2, a_k^F g_2\rangle & \ldots & \langle a_k^T g_k, a_k^I g_k, a_k^F g_k\rangle \end{pmatrix} \quad A\begin{pmatrix} 1 & 2 & \ldots & k \\ g_1 & g_2 & \ldots & g_k \end{pmatrix}$$

Hence the pooof.

Lemma 3.5.

Let $A = \begin{pmatrix} \langle a^T, a^I, a^F\rangle & \langle b^T, b^I, b^F\rangle \\ \langle c^T, c^I, c^F\rangle & \langle d^T, d^I, d^F\rangle \end{pmatrix}$ be a FNSSM.

Then $det\begin{pmatrix} \langle a^T, a^I, a^F\rangle & \langle b^T, b^I, b^F\rangle \\ \langle a^T, a^I, a^F\rangle & \langle b^T, b^I, b^F\rangle \end{pmatrix}$ det$\begin{pmatrix} \langle c^T, c^I, c^F\rangle & \langle d^T, d^I, d^F\rangle \\ \langle c^T, c^I, c^F\rangle & \langle d^T, d^I, d^F\rangle \end{pmatrix}$

$$= \begin{vmatrix} \langle a^T, a^I, a^F\rangle & \langle b^T, b^I, b^F\rangle \\ \langle a^T, a^I, a^F\rangle & \langle b^T, b^I, b^F\rangle \end{vmatrix} \begin{vmatrix} \langle c^T, c^I, c^F\rangle & \langle d^T, d^I, d^F\rangle \\ \langle c^T, c^I, c^F\rangle & \langle d^T, d^I, d^F\rangle \end{vmatrix} \leq det(A)$$

Proof:

We see that $det\begin{pmatrix} \langle a^T, a^I, a^F\rangle & \langle b^T, b^I, b^F\rangle \\ \langle a^T, a^I, a^F\rangle & \langle b^T, b^I, b^F\rangle \end{pmatrix} \quad det\begin{pmatrix} \langle c^T, c^I, c^F\rangle & \langle d^T, d^I, d^F\rangle \\ \langle c^T, c^I, c^F\rangle & \langle d^T, d^I, d^F\rangle \end{pmatrix}$

$$= \langle a^T, a^I, a^F\rangle\langle b^T, b^I, b^F\rangle\langle c^T, c^I, c^F\rangle\langle d^T, d^I, d^F\rangle$$
$$\leq \langle a^T, a^I, a^F\rangle\langle d^T, d^I, d^F\rangle + \langle b^T, b^I, b^F\rangle\langle c^T, c^I, c^F\rangle$$
$$= det(A).$$

Notation: Let $A \in FNSSM_n$ $.Let\ A(e \Rightarrow f)$ be the matrix obtained from A by replacing row f of A by row e of A.

Theorem 3.6. Let $A \in FNSSM_n$. Then

$(i) det(A(2 \Rightarrow 1)) det(A(1 \Rightarrow 2)) \leq det(A).$

$(ii) det(A(2 \Rightarrow 1)) det(A(3 \Rightarrow 2)) \leq det(A).$

$(iii) det(A(q \Rightarrow p))) det(A(p \Rightarrow k)) \leq det(A).$

Proof: To prove

$(i) det(A(2 \Rightarrow 1)) det(A(1 \Rightarrow 2))$

$$= \begin{vmatrix} \langle a_{11}^T, a_{11}^I, a_{11}^F \rangle & \langle a_{12}^T, a_{12}^I, a_{12}^F \rangle \\ \langle a_{11}^T, a_{11}^I, a_{11}^F \rangle & \langle a_{12}^T, a_{12}^I, a_{12}^F \rangle \end{vmatrix} \; A\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} + \ldots + \begin{vmatrix} \langle a_{1e}^T, a_{1e}^I, a_{1e}^F \rangle & \langle a_{1f}^T, a_{1f}^I, a_{1f}^F \rangle \\ \langle a_{1e}^T, a_{1e}^I, a_{1e}^F \rangle & \langle a_{1f}^T, a_{1f}^I, a_{1f}^F \rangle \end{vmatrix}_{e<f}$$

$$A\begin{pmatrix} 1 & 2 \\ e & f \end{pmatrix} + \ldots +$$

$$\begin{vmatrix} \langle a_{1n-1}^T, a_{1n-1}^I, a_{1n-1}^F \rangle & \langle a_{1n}^T, a_{1n}^I, a_{1n}^F \rangle \\ \langle a_{1n-1}^T, a_{1n-1}^I, a_{1n-1}^F \rangle & \langle a_{1n}^T, a_{1n}^I, a_{1n}^F \rangle \end{vmatrix} \mathrm{A}\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix}. \quad \begin{vmatrix} \langle a_{21}^T, a_{21}^I, a_{21}^F \rangle & \langle a_{22}^T, a_{22}^I, a_{22}^F \rangle \\ \langle a_{21}^T, a_{21}^I, a_{21}^F \rangle & \langle a_{22}^T, a_{22}^I, a_{22}^F \rangle \end{vmatrix} \mathrm{A}$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} + \ldots +$$

$$\begin{vmatrix} \langle a_{2e}^T, a_{2e}^I, a_{2e}^F \rangle & \langle a_{2f}^T, a_{2f}^I, a_{2f}^F \rangle \\ \langle a_{1e}^T, a_{1e}^I, a_{1e}^F \rangle & \langle a_{1f}^T, a_{1f}^I, a_{1f}^F \rangle \end{vmatrix}_{e<f} \mathrm{A}\begin{pmatrix} 1 & 2 \\ e & f \end{pmatrix} + \ldots +$$

$$\begin{vmatrix} \langle a_{2n-1}^T, a_{2n-1}^I, a_{2n-1}^F \rangle & \langle a_{2n}^T, a_{2n}^I, a_{2n}^F \rangle \\ \langle a_{2n-1}^T, a_{2n-1}^I, a_{2n-1}^F \rangle & \langle a_{2n}^T, a_{2n}^I, a_{2n}^F \rangle \end{vmatrix} \begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix} A$$

$$= \sum_{e<f} \begin{vmatrix} \langle a_{1e}^T, a_{1e}^I, a_{1e}^F \rangle & \langle a_{1f}^T, a_{1f}^I, a_{1f}^F \rangle \\ \langle a_{1e}^T, a_{1e}^I, a_{1e}^F \rangle & \langle a_{1f}^T, a_{1f}^I, a_{1f}^F \rangle \end{vmatrix} \mathrm{A}\begin{pmatrix} 1 & 2 \\ e & f \end{pmatrix} \sum_{g<h} \begin{vmatrix} \langle a_{2g}^T, a_{2g}^I, a_{2g}^F \rangle & \langle a_{2h}^T, a_{2h}^I, a_{2h}^F \rangle \\ \langle a_{2g}^T, a_{2g}^I, a_{2g}^F \rangle & \langle a_{2h}^T, a_{2h}^I, a_{2h}^F \rangle \end{vmatrix} \mathrm{A}$$

$$\begin{pmatrix} 1 & 2 \\ g & h \end{pmatrix}$$

$$\leq \sum_{\substack{e<f \\ g<h}} \begin{vmatrix} \langle a_{1e}^T, a_{1e}^I, a_{1e}^F \rangle & \langle a_{1f}^T, a_{1f}^I, a_{1f}^F \rangle \\ \langle a_{2g}^T, a_{2g}^I, a_{2g}^F \rangle & \langle a_{2h}^T, a_{2h}^I, a_{2h}^F \rangle \end{vmatrix} \mathrm{A}\begin{pmatrix} 1 & 2 \\ e & f \end{pmatrix} \mathrm{A}\begin{pmatrix} 1 & 2 \\ g & h \end{pmatrix} \qquad \text{(by Lemma 3.5)}$$

We now introduce symbols $\mathcal{J}_1, \mathcal{J}_2, A\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and $\mathcal{J}$. Define

$$\mathcal{J}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{vmatrix} \langle a_{1e}^T, a_{1e}^I, a_{1e}^F \rangle & \langle a_{1f}^T, a_{1f}^I, a_{1f}^F \rangle \\ \langle a_{2g}^T, a_{2g}^I, a_{2g}^F \rangle & \langle a_{2h}^T, a_{2h}^I, a_{2h}^F \rangle \end{vmatrix} \mathrm{A}\begin{pmatrix} 1 & 2 \\ e & f \end{pmatrix} \mathrm{A}\begin{pmatrix} 1 & 2 \\ g & h \end{pmatrix}$$

$$\mathcal{J}_1 = \sum_{(e,f)=(g,h)} J\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \sum_{e<f} \mathcal{J}\begin{pmatrix} e & f \\ e & f \end{pmatrix},$$

$\mathcal{J}_2 = \sum_{(e,f)\neq(g,h)} J\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ and $J = J_1 + J_2$. Then we see that

$$\mathcal{J}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = \begin{vmatrix} \langle a_{11}^T, a_{11}^I, a_{11}^F \rangle & \langle a_{12}^T, a_{12}^I, a_{12}^F \rangle \\ \langle a_{21}^T, a_{21}^I, a_{21}^F \rangle & \langle a_{22}^T, a_{22}^I, a_{22}^F \rangle \end{vmatrix} \mathrm{A}\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix},$$

$\mathcal{J}_1 = det(A)$ by Theorem 3.2(ii) and

$det(A(2 \Rightarrow)) det(A(1 \Rightarrow 2)) \leq \mathcal{J} = \mathcal{J}_1 + \mathcal{J}_2 = det(A) + \mathcal{J}_2.$

We show that $\mathcal{J}_2 \leq det(A)$.

There are two cases to be considered.

Case 1. We consider $a = \mathcal{J}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, a term of $\mathcal{J}_2$.

Let $a_1 = \langle a_{11}^T, a_{11}^I, a_{11}^F \rangle \langle a_{23}^T, a_{23}^I, a_{23}^F \rangle$ A $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ A $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ and

$a_2 = (\langle a_{12}^T, a_{12}^I, a_{12}^F \rangle \langle a_{21}^T, a_{21}^I, a_{21}^F \rangle$ A $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ A $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$

Then $a = a_1 + a_2$,

$$a_1 \leq \begin{vmatrix} \langle a_{11}^T, a_{11}^I, a_{11}^F \rangle & \langle a_{13}^T, a_{13}^I, a_{13}^F \rangle \\ \langle a_{21}^T, a_{21}^I, a_{21}^F \rangle & \langle a_{23}^T, a_{23}^I, a_{23}^F \rangle \end{vmatrix} \text{A} \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \leq det(A),$$

$$a_2 \leq \begin{vmatrix} \langle a_{11}^T, a_{11}^I, a_{11}^F \rangle & \langle a_{12}^T, a_{12}^I, a_{12}^F \rangle \\ \langle a_{21}^T, a_{21}^I, a_{21}^F \rangle & \langle a_{22}^T, a_{22}^I, a_{22}^F \rangle \end{vmatrix} \text{A} \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \leq det(A),$$

and $\mathcal{J}\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \leq det(A)$,

Case 2. We take $\mathcal{J}\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix}$

Let $b_1 = \langle a_{11}^T, a_{11}^I, a_{11}^F \rangle \langle a_{2n}^T, a_{2n}^I, a_{2n}^F \rangle$ A $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ A $\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix}$ and

$b_2 = \langle a_{12}^T, a_{12}^I, a_{12}^F \rangle \langle a_{2n-1}^T, a_{2n-1}^I, a_{2n-1}^F \rangle$ A $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ A $\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix}$.

Then $\mathcal{J}\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix} = b_1 + b_2$. To show that $b_1 \leq det(A)$ and $b_2 = det(A)$, we observe all coordinates of the elements a_{ij} involved in $A\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and $A\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix}$.

The coordinates of the elements a_{ij} involved in these determinants are all coordinates of the elements of the $k-th$ row A_k of A, for $k \geq 3$. Therefore, if we let $b = a_{3n-1}a_{4n-2}...a_{k+2n-k}...a_{nn-2}$, then we see that $b_1 \leq (\langle a_{11}^T, a_{11}^I, a_{11}^F \rangle \langle a_{2n}^T, a_{2n}^I, a_{2n}^F \rangle) c \leq det(A)$. For b_2, let $c = a_{3n}a_{4n-2}a_{5n-3}...a_{n-13}a_{2n-1}$, then we see that $b_2 \leq (\langle a_{12}^T, a_{12}^I, a_{12}^F \rangle \langle a_{2n-1}^T, a_{2n-1}^I, a_{2n-1}^F \rangle) c \leq det(A)$. For any $\mathcal{J}\begin{pmatrix} e & f \\ g & h \end{pmatrix}_{(e,f)\neq(g,h)}$,we apply either the case 1 or the case 2 and we can deduce that

$\mathcal{J}\begin{pmatrix} e & f \\ g & h \end{pmatrix}_{(e,f)\neq(g,h)} \leq det(A).$

Thus (i) holds.

(ii). First we consider

$$\begin{vmatrix} \langle b_{11}^T, b_{11}^I, b_{11}^F \rangle & \langle b_{12}^T, b_{12}^I, b_{12}^F \rangle \\ \langle a_{31}^T, a_{31}^I, a_{31}^F \rangle & \langle a_{32}^T, a_{32}^I, a_{32}^F \rangle \end{vmatrix} \begin{vmatrix} \langle b_{21}^T, b_{21}^I, b_{21}^F \rangle & \langle b_{22}^T, b_{22}^I, b_{22}^F \rangle \\ \langle b_{21}^T, b_{21}^I, b_{21}^F \rangle & \langle b_{22}^T, b_{22}^I, b_{22}^F \rangle \end{vmatrix}$$

$$\leq \begin{vmatrix} \langle b_{21}^T, b_{21}^I, b_{21}^F \rangle & \langle b_{32}^T, b_{32}^I, b_{32}^F \rangle \\ \langle a_{31}^T, a_{31}^I, a_{31}^F \rangle & \langle a_{32}^T, a_{32}^I, a_{32}^F \rangle \end{vmatrix}.$$

We introduce a symbol

$$K\begin{pmatrix} g & h \\ e & f \end{pmatrix} = \begin{vmatrix} \langle a_{2g}^T, a_{2g}^I, a_{2g}^F \rangle & \langle a_{2h}^T, a_{2h}^I, a_{2h}^F \rangle \\ \langle a_{3e}^T, a_{3e}^I, a_{3e}^F \rangle & \langle a_{3f}^T, a_{3f}^I, a_{3f}^F \rangle \end{vmatrix} A\begin{pmatrix} 2 & 3 \\ g & h \end{pmatrix} \mathrm{A} \begin{pmatrix} 2 & 3 \\ e & f \end{pmatrix}.$$

Then we can see that

$det(A(2 \Rightarrow 1))det(A(3 \Rightarrow 2))$

$$= \sum_{\substack{e<f \\ g<h}} \begin{vmatrix} \langle a_{1e}^T, a_{1e}^I, a_{1e}^F \rangle & \langle a_{1f}^T, a_{1f}^I, a_{1f}^F \rangle \\ \langle a_{3e}^T, a_{3e}^I, a_{3e}^F \rangle & \langle a_{3f}^T, a_{3f}^I, a_{3f}^F \rangle \end{vmatrix} \mathrm{A}\begin{pmatrix} 2 & 3 \\ e & f \end{pmatrix} \begin{vmatrix} \langle a_{2g}^T, a_{2g}^I, a_{2g}^F \rangle & \langle a_{2h}^T, a_{2h}^I, a_{2h}^F \rangle \\ \langle a_{2g}^T, a_{2g}^I, a_{2g}^F \rangle & \langle a_{2h}^T, a_{2h}^I, a_{2h}^F \rangle \end{vmatrix} \mathrm{A} \begin{pmatrix} 2 & 3 \\ g & h \end{pmatrix}$$

$$\leq \sum_{\substack{g<h \\ e<f}} \begin{vmatrix} \langle a_{2g}^T, a_{2g}^I, a_{2g}^F \rangle & \langle a_{2h}^T, a_{2h}^I, a_{2h}^F \rangle \\ \langle a_{3e}^T, a_{3e}^I, a_{3e}^F \rangle & \langle a_{3f}^T, a_{3f}^I, a_{3f}^F \rangle \end{vmatrix} \mathrm{A} \begin{pmatrix} 2 & 3 \\ g & h \end{pmatrix} \mathrm{A} \begin{pmatrix} 2 & 3 \\ e & f \end{pmatrix}$$

$$= \sum_{\substack{g<h \\ e<f}} A\begin{pmatrix} g & h \\ e & f \end{pmatrix}$$

$$= \sum_{(g,h)=(e,f)} K\begin{pmatrix} g & h \\ e & f \end{pmatrix} + \sum_{(g,h)\neq(e,f)} K\begin{pmatrix} g & h \\ e & f \end{pmatrix}.$$

Next we prove that $K\begin{pmatrix} g & h \\ e & f \end{pmatrix}_{(g,h)\neq(e,f)} \leq det(A).$

For this we consider two cases.

Case 1. We take $K\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$. We see that

$$K\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = (\langle a_{21}^T, a_{21}^I, a_{21}^F \rangle \langle a_{33}^T, a_{33}^I, a_{33}^F \rangle + \langle a_{22}^T, a_{22}^I, a_{22}^F \rangle \langle a_{31}^T, a_{31}^I, a_{31}^F \rangle)\, \mathrm{A} \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \mathrm{A} \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$$

$= (\langle a_{21}^T, a_{21}^I, a_{21}^F\rangle\langle a_{33}^T, a_{33}^I, a_{33}^F\rangle)\, \mathrm{A}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \mathrm{A}\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} + (\langle a_{22}^T, a_{22}^I, a_{22}^F\rangle\langle a_{31}^T, a_{31}^I, a_{31}^F\rangle)\, \mathrm{A}$

$\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \mathrm{A}\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$

$\leq \begin{vmatrix} \langle a_{21}^T, a_{21}^I, a_{21}^F\rangle & \langle a_{23}^T, a_{23}^I, a_{23}^F\rangle \\ \langle a_{31}^T, a_{31}^I, a_{31}^F\rangle & \langle a_{33}^T, a_{33}^I, a_{33}^F\rangle \end{vmatrix} \mathrm{A}\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} + \begin{vmatrix} \langle a_{21}^T, a_{21}^I, a_{21}^F\rangle & \langle a_{22}^T, a_{22}^I, a_{22}^F\rangle \\ \langle a_{31}^T, a_{31}^I, a_{31}^F\rangle & \langle a_{32}^T, a_{32}^I, a_{32}^F\rangle \end{vmatrix} \mathrm{A}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$

$\leq det(A) + det(A)$

$= det(A).$

Case 2. We consider

$$K\begin{pmatrix} n-1 & n \\ 1 & 2 \end{pmatrix} = \langle a_{2n-1}^T, a_{2n-1}^I, a_{2n-1}^F\rangle\langle a_{32}^T, a_{32}^I, a_{32}^F\rangle\, \mathrm{A}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \mathrm{A}\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix} +$$

$$\langle a_{2n}^T, a_{2n}^I, a_{2n}^F\rangle\langle a_{31}^T, a_{31}^I, a_{31}^F\rangle\, \mathrm{A}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \mathrm{A}\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix}.$$

Considering the coordinates of the elements a_{ij} involved in $A\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \mathrm{A}\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix}$, we claim that

$(\langle a_{2n-1}^T, a_{2n-1}^I, a_{2n-1}^F\rangle\langle a_{32}^T, a_{32}^I, a_{32}^F\rangle)\, \mathrm{A}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \mathrm{A}\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix} \leq det(A)$

and

$(\langle a_{2n}^T, a_{2n}^I, a_{2n}^F\rangle\langle a_{31}^T, a_{31}^I, a_{31}^F\rangle\ \mathrm{A}\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \mathrm{A}\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix} \leq det(A).$

Similarly we can prove (iii).

Theorem 3.7. Let $A = (a_{ij}^T, a_{ij}^I, a_{ij}^F)_n, B = (b_{ij}^T, b_{ij}^I, {}_{ij}^F)_n, (c_{ij}^T, c_{ij}^I, c_{ij}^F)_n \in FNSSM_n$. Then

1. If $\langle a_{ii}^T, a_{ii}^I, a_{ii}^F\rangle \geq \langle a_{ik}^T, a_{ik}^I, a_{ik}^F\rangle (k = 1, 2, ..., n)$ for all $1 \leq i \leq n,$ then $det(A) = \langle a_{11}^T, a_{11}^T, a_{11}^T\rangle\langle a_{22}^T, a_{22}^T, a_{22}^T\rangle...\langle a_{nn}^T, a_{nn}^T, a_{nn}^T\rangle.$
2. det $\begin{pmatrix} A & C \\ O & B \end{pmatrix}$ $det(A)det(B)$ where $O = (\langle 0,0,1\rangle)_n \in FNSSM_n$
3. $det(AA^T) \geq det(A).$
4. If $\langle a_{ij}^T, a_{ij}^I, a_{ij}^F\rangle \geq \langle b_{ij}^T, b_{ij}^I, b_{ij}^F\rangle$ for all i, j, then $det(A) = det(B).$

Proof:

1. We have

$\langle a_{11}^T, a_{11}^I, a_{11}^F\rangle\langle a_{22}^T, a_{22}^I, a_{22}^F\rangle...\langle a_{nn}^T, a_{nn}^I, a_{nn}^F\rangle \geq \langle a_{1\sigma(1)}^T, a_{2\sigma(2)}^I, a_{n\sigma(n)}^F\rangle$ for every $\sigma \in S_n,$ since $\langle a_{ii}^T, a_{ii}^I, a_{ii}^F\rangle \leq \langle a_{ik}^T, a_{ik}^I, a_{ik}^F\rangle (k = 1, 2, ..., n)$ for all $1 \leq i \leq n.$

Hence $det(A)=\sum_{\sigma\in S_n}\langle a^T_{1\sigma(1)},a^I_{1\sigma(1)},a^F_{1\sigma(1)}\rangle\langle a^T_{2\sigma(2)},a^I_{2\sigma(2)},a^F_{2\sigma(2)}\rangle...\langle a^T_{n\sigma(n)},a^I_{n\sigma(n)},a^F_{n\sigma(n)}\rangle$

$=\langle a^T_{11},a^I_{11},a^F_{11}\rangle\langle a^T_{22},a^I_{22},a^F_{22}\rangle...\langle a^T_{nn},a^I_{nn},a^F_{nn}\rangle$

2. Let $\begin{pmatrix} A & C \\ O & B \end{pmatrix}=(\langle d^T_{ij},d^I_{ij},d^F_{ij}\rangle)_{2n}$.

Then

$$\det\begin{pmatrix} A & C \\ O & B \end{pmatrix}=\sum_{\sigma\in S_{2n}}\langle d^T_{1\sigma(1)},d^I_{1\sigma(1)},d^F_{1\sigma(1)}\rangle...\langle d^T_{2n\sigma(2n)},d^I_{2n\sigma(2n)},d^F_{2n\sigma(2n)}\rangle$$

$$=\sum_{\sigma\in S_{2n},\sigma(i)\le n(\text{if } i\le n)}\langle d^T_{1\sigma(1)},d^I_{1\sigma(1)},d^F_{1\sigma(1)}\rangle...\langle d^T_{2n\sigma(2n)},d^I_{2n\sigma(2n)},d^F_{2n\sigma(2n)}\rangle+$$

$$\sum_{\sigma\in S_{2n},\exists k>n, if\,\sigma(k)\le n}\langle d^T_{1\sigma(1)},d^I_{1\sigma(1)},d^F_{1\sigma(1)}\rangle...\langle d^T_{2n\sigma(2n)},d^I_{2n\sigma(2n)},d^F_{2n\sigma(2n)}\rangle$$

$$=\sum_{\sigma\in S_{2n},\sigma(i)\le n(if i\le n)}\langle d^T_{1\sigma(1)},d^I_{1\sigma(1)},d^F_{1\sigma(1)}\rangle...\langle d^T_{2n\sigma(2n)},d^I_{2n\sigma(2n)},d^F_{2n\sigma(2n)}\rangle+0$$

$$=\sum_{\sigma'\in S_n}\langle d^T_{1\sigma'(1)},d^I_{1\sigma'(1)},d^F_{1\sigma'(1)}\rangle...\ \langle d^T_{n\sigma'(n)},d^I_{n\sigma'(n)},d^F_{n\sigma'(n)}\rangle det(B)$$

$$=(\sum_{\sigma\in S_n}\langle d^T_{1\sigma(1)},d^I_{1\sigma(1)},d^F_{1\sigma(1)}\rangle...\langle d^T_{n\sigma(n)},d^I_{n\sigma(n)},d^F_{n\sigma(n)}\rangle)det(B)$$

$$=det(A)det(B).$$

3. Let $AA^T=(\langle g^T_{ij},g^I_{ij},g^F_{ij}\rangle)_n$, where $\langle g^T_{ij},g^I_{ij},g^F_{ij}\rangle=\sum_{k=1}^{n}\langle a^T_{ik},a^I_{ik},a^F_{ik}\rangle\langle a^T_{kj},a^I_{kj},a^F_{kj}\rangle$.

We have, for every $\sigma\in S_n$

$\langle g^T_{11},g^I_{11},g^F_{11}\rangle\langle g^T_{22},g^I_{22},g^F_{22}\rangle...\langle g^T_{nn},g^I_{nn},g^F_{nn}\rangle$

$=(\sum_{k=1}^{n}\langle a^T_{1k},a^I_{1k},a^F_{1k}\rangle)...(\sum_{k=1}^{n}\langle a^T_{nk},a^I_{nk},a^F_{nk}\rangle)$

$\ge\langle a^T_{1\sigma(1)},a^I_{1\sigma(1)},a^F_{1\sigma(1)}\rangle...\langle a^T_{n\sigma(n)},a^I_{n\sigma(n)},a^F_{n\sigma(n)}\rangle$

Hence $det(AA^T)\ge\langle g^T_{11},g^I_{11},g^F_{11}\rangle\langle g^T_{22},f^I_{22},f^F_{22}\rangle...\langle g^T_{nn},g^I_{nn},g^F_{nn}\rangle$

$\ge\sum_{\sigma\in S_n}\langle a^T_{1\sigma(1)},a^I_{1\sigma(1)},a^F_{1\sigma(1)}\rangle...\langle a^T_{n\sigma(n)},a^I_{n\sigma(n)},a^F_{n\sigma(n)}\rangle$

$=det(A)$.

Theorem 3.8. Let $A=(a_{ij})$ be a FNSSM. Then we have the following $det(Aadj(A))=det(A)=det(adj(A)A)$.

Proof: We prove that $det(Aadj(A))=det(A)$.

We first consider n=2.

Let $A = \begin{pmatrix} \langle a_{11}^T, a_{11}^I, a_{11}^F \rangle & \langle a_{12}^T, a_{12}^I, a_{12}^F \rangle \\ \langle a_{21}^T, a_{21}^I, a_{21}^F \rangle & \langle a_{22}^T, a_{22}^I, a_{22}^F \rangle \end{pmatrix}.$

Then we see that

$$adj(A) = \begin{pmatrix} \langle a_{22}^T, a_{22}^I, a_{22}^F \rangle & \langle a_{12}^T, a_{12}^I, a_{12}^F \rangle \\ \langle a_{21}^T, a_{21}^I, a_{21}^F \rangle & \langle a_{11}^T, a_{11}^I, a_{11}^F \rangle \end{pmatrix}.$$

$$det(Aadj(A)) = \begin{vmatrix} det(A) & \langle a_{11}^T, a_{11}^I, a_{11}^F \rangle \langle a_{12}^T, a_{12}^I, a_{12}^F \rangle \\ \langle a_{21}^T, a_{21}^I, a_{21}^F \rangle \langle a_{22}^T, a_{22}^I, a_{22}^F \rangle & det(A) \end{vmatrix}$$

$$= det(A) + (\langle a_{11}^T, a_{11}^I, a_{11}^F \rangle \langle a_{12}^T, a_{12}^I, a_{12}^F \rangle \langle a_{21}^T, a_{21}^I, a_{21}^F \rangle \langle a_{22}^T, a_{22}^I, a_{22}^F \rangle)$$

$$\leq det(A).$$

Next consider $n > 2.$ We can see that

$$Aadj(A) = \begin{pmatrix} \sum \langle a_{1t}^T, a_{1t}^I, a_{1t}^F \rangle A_{1t} & \sum \langle a_{1t}^T, a_{1t}^I, a_{1t}^F \rangle A_{2t} & \ldots & \sum \langle a_{1t}^T, a_{1t}^I, a_{1t}^F \rangle A_{nt} \\ \sum \langle a_{2t}^T, a_{2t}^I, a_{2t}^F \rangle A_{1t} & \sum \langle a_{2t}^T, a_{2t}^I, a_{2t}^F \rangle A_{2t} & \ldots & \sum \langle a_{2t}^T, a_{2t}^I, a_{2t}^F \rangle A_{nt} \\ \ldots & \ldots & \ldots & \ldots \\ \sum \langle a_{nt}^T, a_{nt}^I, a_{3t}^F \rangle A_{1t} & \sum \langle a_{nt}^T, a_{nt}^I, a_{nt}^F \rangle A_{2t} & \ldots & \sum \langle a_{nt}^T, a_{nt}^I, a_{nt}^F \rangle A_{nt} \end{pmatrix}$$

$$= \left(\sum \langle a_{it}^T, a_{it}^I, a_{it}^F \rangle A_{jt} \right),$$

$$det(Aadj(A)) = \sum_{\pi \in S_n} (\sum \langle a_{1t}^T, a_{1t}^I, a_{1t}^F \rangle A_{\pi(1)t})(\sum \langle a_{2t}^T, a_{2t}^I, a_{2t}^F \rangle A_{\pi(2)t}) \ldots (\sum \langle a_{nt}^T, a_{nt}^I, a_{nt}^F \rangle A_{\pi(n)t}).$$

Clearly any diagonal entry of the matrix $Aadj(A)$ is equal to $det(A).$

We prove the result in the following way.

(1) Let us define

$$T_\pi = (\sum \langle a_{1t}^T, a_{1t}^I, a_{1t}^F \rangle A_{\pi(1)t})(\sum \langle a_{2t}^T, a_{2t}^I, a_{2t}^F \rangle A_{\pi(2)t}) \ldots (\sum \langle a_{nt}^T, a_{nt}^I, a_{nt}^F \rangle A_{\pi(n)t}),$$

for $\pi \in S_n$. Let e be the identity of the group S_n. If $\pi = e,$ then $T_\pi = det(A).$ Suppose that there exists $k \in \{1, 2, \ldots, n\}$ such that $\pi(k) = k.$ Then we see that

$$\sum \langle a_{kt}^T a_{kt}^I a_{kt}^F \rangle A_{\pi(k)t} = \sum \langle a_{kt}^T a_{kt}^I a_{kt}^F \rangle A_{kt}$$

$= det(A)$ and

$$\mathcal{J}_\pi = (\sum \langle a_{1t}^T, a_{1t}^I, a_{1t}^F \rangle A_{\pi(1)t})(\sum \langle a_{2t}^T, a_{2t}^I, a_{2t}^F \rangle A_{\pi(2)t}) \ldots det(A) \ldots (\sum \langle a_{nt}^T, a_{nt}^I, a_{nt}^F \rangle A_{\pi(n)t})$$

$$\leq det(A).$$

(2) Let π be a permutation in S_n. Assume that $\pi(k) \neq k$ for all $k \in \{1, 2, \ldots n\}$.

We know that every permutation π can be written as a product of disjoint cycles π_i and let $\pi = \pi_1 \pi_2 \ldots \pi_k$. We further assume that $\pi_1 = (1 \ \ 2),$ a transposition.

Then J_π has two factors, $(\sum \langle a_{1t}^T, a_{1t}^I, a_{1t}^F \rangle A_{\pi(1)t})$ and

$(\sum \langle a_{2t}^T, a_{2t}^I, a_{2t}^F \rangle A_{\pi(2)t}),$ and from these we see that

$$(\sum\langle a^T_{1t},a^I_{1t},a^F_{1t}\rangle A_{\pi(1)t})(\sum\langle a^T_{2t},a^I_{2t},a^F_{2t}\rangle A_{\pi(2)t})=(\sum\langle a^T_{1t},a^I_{1t},a^F_{1t}\rangle A_{2t})(\sum\langle a^T_{2t},a^I_{2t},a^F_{2t}\rangle A_{1t})$$

$= det(A(2\Rightarrow 1))det(A(1\Rightarrow 2))$
$\leq det(A)$ (by theorem3. 2(i))

(3) If $\pi=\pi_1\pi_2...\pi_k$ and $\pi_1(s,t)$, then we can prove that $\mathcal{J}_\pi \leq det(A)$ by an argument used in(2). Consider $\mathcal{J}_\pi$ for $\pi=\pi_1\pi_2...\pi_k$. If
$\pi_1=(k,e,f,...)$, then we see that

$$\mathcal{J}_\pi=(\sum\langle a^T_{kt},a^I_{kt},a^F_{kt}\rangle A_{\pi(k)t)})(\sum\langle a^T_{et},a^I_{et},a^F_{et}\rangle A_{\pi(e)t)})...$$

$$=(\sum\langle a^T_{kt},a^I_{kt},a^F_{kt}\rangle A_{et})(\sum\langle a^T_{et},a^I_{et},a^F_{et}\rangle A_{ft})...=det(A(e\Rightarrow k))det(A(f\Rightarrow e))...$$

From Theorem 3.6(iii), we obtain that $det(A(e\Rightarrow k))det(A(f\Rightarrow e))\leq det(A)$ and consequently that $\mathcal{J}_\pi\leq det(A)$. This proves that $det(Aadj(A))=det(A)$. Similarly, we can prove that $det(adj(A)A)=det(A)$. Hence the proof.

Theorem 3.9. Let $A,B\in FNSSM_n$. Then
(1) $det(AB)\geq det(A)det(B)$.
(2) $det(AB)\leq det(A+B)$,
where $A+B=(sup\{a^T_{ij},b^T_{ij}\},sup\{a^I_{ij},b^I_{ij}\},inf\{a^F_{ij},b^F_{ij}\})$

Proof.

$$det(AB)=det((\sum_{k=1}^n a^T_{ik}\wedge b^T_{kj},\sum_{k=1}^n a^I_{ik}\wedge b^I_{kj},\prod_{k=1}^n a^F_{ik}\vee b^F_{kj}))$$

$$=\sum_{\sigma\in S_n}\{[\sum_{k=1}^n a^T_{1k}\wedge b^T_{k\sigma(1)},\sum_{k=1}^n a^I_{1k}\wedge b^I_{k\sigma(1)},\prod_{k=1}^n a^F_{ik}\vee b^F_{k\sigma(1)})],\cdots,$$

$$[\sum_{k=1}^n a^T_{nk}\wedge b^T_{k\sigma(n)},\sum_{k=1}^n a^I_{nk}\wedge b^I_{k\sigma(n)},\prod_{k=1}^n (a^F_{nk}\vee b^F_{k\sigma(n)}]\}$$

$$=\sum_{\sigma\in S_n}(\sum_{k_1,k_2...k_n}(a^T_{1k_1}\wedge a^T_{2k_2}...\wedge a^T_{nk_n}\wedge b^T_{k_1\sigma(1)}\wedge b^T_{k_2\sigma(2)}...\wedge b^T_{k_n\sigma(n)})$$

$$(\sum_{k_1,k_2...k_n}(a^I_{1k_1}\wedge a^I_{2k_2}...\wedge a^I_{nk_n}\wedge b^I_{k_1\sigma(1)}\wedge b^I_{k_{2\sigma(2)}}...\wedge b^I_{k_{n\sigma(n)}})$$

$$\prod_{k_1,k_2...k_n}(a^F_{1k_1}\vee a^F_{2k_2}\vee ...a^F_{nk_n}\vee b^F_{k_{1\sigma(1)}}\vee b^F_{k_{2\sigma(2)}}\vee ...b^F_{k_{n\sigma(n)}})$$

$$\geq\sum_{\{k_1,k_2...k_n\}\in S_n}\langle a^T_{1k_1},a^I_{1k_1},a^F_{1k_1}\rangle...\langle a^T_{nk_n},a^I_{nk_n},a^F_{nk_n}\rangle$$

$$\sum_{\sigma\in S_n} \langle b^T_{k_1\sigma(1)}, b^I_{k_1\sigma(1)}, b^F_{k_1\sigma(1)}\rangle ... \langle b^T_{k_n\sigma(n)}, b^I_{k_n\sigma(n)}, b^F_{k_n\sigma(n)}\rangle) =$$

$$(\sum_{(k_1,k_2...k_n)\in S_n} \langle a^T_{1k_1}, a^I_{1k_1}, a^F_{1k_1}\rangle ... \langle a^T_{nk_n}, a^I_{nk_n}, a^F_{nk_n}\rangle) det(B))$$

$$= det(A) det(B)$$

(2) We know that $det(AB) = det((\sum_{k=1}^{n} a^T_{ik} \wedge b^T_{kj}, \sum_{k=1}^{n} a^I_{ik} \wedge b^I_{kj}, \prod_{k=1}^{n} a^F_{ik} + b^F_{kj}))$

$$= \sum_{\sigma\in S_n} [\sum_{k=1}^{n} a^T_{1k} \wedge b^T_{k\sigma(1)}, \sum_{k=1}^{n} a^I_{1k} \wedge b^I_{k\sigma(1)}, \prod_{k=1}^{n} a^F_{1k} \vee b^T_{k\sigma(1)} ...$$

$$\sum_{k=1}^{n} (a^T_{nk} \wedge b^T_{k\sigma(n)}), \sum_{k=1}^{n} (a^I_{nk} \wedge b^I_{k\sigma(n)}), \prod_{k=1}^{n} (a^F_{nk} \vee b^T_{k\sigma(n)})]$$

$$= \sum_{\sigma\in S_n} (\text{ì}_{t\le s, t\le n} (a^T_{1s} \vee b^T_{t\sigma(1)}), (\text{ì}_{t\le s, t\le n} (a^I_{1s} \vee b^I_{t\sigma(1)}), \text{ê}_{t\le s, t\le n} (a^F_{1s} \wedge b^F_{s\sigma(1)}) ...$$

$$(\text{ì}_{t\le s, t\le n} (a^T_{ns} \vee b^T_{t\sigma(n)}), (\text{ì}_{t\le s, t\le n} (a^I_{ns} \vee b^I_{t\sigma(n)}), \text{ê}_{t\le s, t\le n} (a^F_{ns} \wedge b^F_{s\sigma(n)})$$

$$\le \sum_{\sigma\in S_n} (a^T_{1\sigma(1)} \vee b^T_{1\sigma(1)}) \wedge (a^I_{1\sigma(1)} \vee b^I_{1\sigma(1)}) \wedge (a^F_{1\sigma(1)} \wedge b^F_{1\sigma(1)})$$

$$... \text{ì}_{t\le s, t\le n} (a^T_{n\sigma(n)} \vee b^T_{n\sigma(n)}) \wedge (a^I_{n\sigma(n)} \vee b^I_{n\sigma(n)}) \wedge (a^F_{n\sigma(n)} \wedge b^F_{n\sigma(n)})$$

$$= det((\langle a^T_{ij}, a^I_{ij}, a^F_{ij}\rangle + \langle b^T_{ij}, b^T_{ij}, b^T_{ij}\rangle)_n)$$

$$= det(A+B)$$

Corollary 3.10. Let A be a FNSSM, $A_r = (a^r_{ij}) \in FNSSM_n$ (r=1, 2, 3,...,m}. Then

(1) $det(A_1) det(A_2) ... det(A_m) \le det(\sum_{r=1}^{m} A_r)$ where $\sum_{r=1}^{m} A_r = (\sum_{r=1}^{m} a^r_{ij})_n \in FNSSM_n$.

(2) $det(A^{(r)}) = det(A),$ where $A = (a_{ij})_n \in FNSSM_n$ and $r \in N$.

Example 3.11. Consider the 4×4 matrix

$$\begin{pmatrix} \langle a^T_{11}, a^I_{11}, a^F_{11}\rangle & \langle a^T_{12}, a^I_{12}, a^F_{12}\rangle & \langle a^T_{13}, a^I_{13}, a^F_{13}\rangle & \langle a^T_{14}, a^I_{14}, a^F_{14}\rangle \\ \langle a^T_{21}, a^I_{21}, a^F_{21}\rangle & \langle a^T_{22}, a^I_{22}, a^F_{22}\rangle & \langle a^T_{23}, a^I_{23}, a^F_{23}\rangle & \langle a^T_{24}, a^I_{24}, a^F_{24}\rangle \\ \langle a^T_{31}, a^I_{31}, a^F_{31}\rangle & \langle a^T_{32}, a^I_{32}, a^F_{32}\rangle & \langle a^T_{33}, a^I_{33}, a^F_{33}\rangle & \langle a^T_{34}, a^I_{34}, a^F_{34}\rangle \\ \langle a^T_{41}, a^I_{41}, a^F_{41}\rangle & \langle a^T_{42}, a^I_{42}, a^F_{42}\rangle & \langle a^T_{43}, a^I_{43}, a^F_{43}\rangle & \langle a^T_{44}, a^I_{44}, a^F_{44}\rangle \end{pmatrix}$$

We find the determinant of the above matrix in the following method

$$= \begin{vmatrix} \langle a^T_{11}, a^I_{11}, a^F_{11}\rangle & \langle a^T_{12}, a^I_{12}, a^F_{12}\rangle \\ \langle a^T_{21}, a^I_{21}, a^F_{21}\rangle & \langle a^T_{22}, a^I_{22}, a^F_{22}\rangle \end{vmatrix}_{1<2} \begin{vmatrix} \langle a^T_{33}, a^I_{33}, a^F_{33}\rangle & \langle a^T_{34}, a^I_{34}, a^F_{34}\rangle \\ \langle a^T_{43}, a^I_{43}, a^F_{43}\rangle & \langle a^T_{44}, a^I_{44}, a^F_{44}\rangle \end{vmatrix}$$

$$+\begin{vmatrix}\langle a_{11}^T,a_{11}^I,a_{11}^F\rangle & \langle a_{13}^T,a_{13}^I,a_{13}^F\rangle\\ \langle a_{21}^T,a_{21}^I,a_{21}^F\rangle & \langle a_{23}^T,a_{23}^I,a_{23}^F\rangle\end{vmatrix}_{1<3}\begin{vmatrix}\langle a_{32}^T,a_{32}^I,a_{32}^F\rangle & \langle a_{34}^T,a_{34}^I,a_{34}^F\rangle\\ \langle a_{42}^T,a_{42}^I,a_{42}^F\rangle & \langle a_{44}^T,a_{44}^I,a_{44}^F\rangle\end{vmatrix}$$

$$+\begin{vmatrix}\langle a_{11}^T,a_{11}^I,a_{11}^F\rangle & \langle a_{14}^T,a_{14}^I,a_{14}^F\rangle\\ \langle a_{21}^T,a_{21}^I,a_{21}^F\rangle & \langle a_{24}^T,a_{24}^I,a_{24}^F\rangle\end{vmatrix}_{1<4}\begin{vmatrix}\langle a_{32}^T,a_{32}^I,a_{32}^F\rangle & \langle a_{33}^T,a_{33}^I,a_{33}^F\rangle\\ \langle a_{42}^T,a_{42}^I,a_{42}^F\rangle & \langle a_{43}^T,a_{43}^I,a_{43}^F\rangle\end{vmatrix}$$

$$+\begin{vmatrix}\langle a_{12}^T,a_{12}^I,a_{12}^F\rangle & \langle a_{13}^T,a_{13}^I,a_{13}^F\rangle\\ \langle a_{22}^T,a_{22}^I,a_{22}^F\rangle & \langle a_{23}^T,a_{23}^I,a_{23}^F\rangle\end{vmatrix}_{2<3}\begin{vmatrix}\langle a_{31}^T,a_{31}^I,a_{31}^F\rangle & \langle a_{34}^T,a_{34}^I,a_{34}^F\rangle\\ \langle a_{41}^T,a_{41}^I,a_{41}^F\rangle & \langle a_{44}^T,a_{44}^I,a_{44}^F\rangle\end{vmatrix}$$

$$+\begin{vmatrix}\langle a_{12}^T,a_{12}^I,a_{12}^F\rangle & \langle a_{14}^T,a_{14}^I,a_{14}^F\rangle\\ \langle a_{22}^T,a_{22}^I,a_{22}^F\rangle & \langle a_{24}^T,a_{24}^I,a_{24}^F\rangle\end{vmatrix}_{2<4}\begin{vmatrix}\langle a_{31}^T,a_{31}^I,a_{31}^F\rangle & \langle a_{33}^T,a_{33}^I,a_{33}^F\rangle\\ \langle a_{41}^T,a_{41}^I,a_{41}^F\rangle & \langle a_{43}^T,a_{43}^I,a_{43}^F\rangle\end{vmatrix}$$

$$+\begin{vmatrix}\langle a_{13}^T,a_{13}^I,a_{13}^F\rangle & \langle a_{14}^T,a_{14}^I,a_{14}^F\rangle\\ \langle a_{23}^T,a_{23}^I,a_{23}^F\rangle & \langle a_{24}^T,a_{24}^I,a_{24}^F\rangle\end{vmatrix}_{3<4}\begin{vmatrix}\langle a_{31}^T,a_{31}^I,a_{31}^F\rangle & \langle a_{32}^T,a_{32}^I,a_{32}^F\rangle\\ \langle a_{41}^T,a_{41}^I,a_{41}^F\rangle & \langle a_{42}^T,a_{42}^I,a_{42}^F\rangle\end{vmatrix}$$

using this method we can find the determinant of the given matrix

$$\begin{pmatrix}\langle 0.4,0.2,0.1\rangle & \langle 0.6,0.7,0.8\rangle & \langle 0.7,0.3,0.4\rangle & \langle 0.6,0.7,0.8\rangle\\ \langle 0.4,0.6,0.7\rangle & \langle 0.3,0.2,0.1\rangle & \langle 0.5,0.6,0.7\rangle & \langle 0.4,0.3,0.2\rangle\\ \langle 0.6,0.7,0.8\rangle & \langle 0.8,0.9,0.3\rangle & \langle 0.5,0.6,0.7\rangle & \langle 0.6,0.7,0.8\rangle\\ \langle 0.9,0.5,0.3\rangle & \langle 0.5,0.3,0.2\rangle & \langle 0.5,0.6,0.7\rangle & \langle 0.8,0.3,0.9\rangle\end{pmatrix}$$

Solution.

$$=\begin{vmatrix}\langle 0.4,0.2,0.1\rangle\langle 0.6,0.7,0.8\rangle\\ \langle 0.4,0.6,0.7\rangle\langle 0.3,0.2,0.1\rangle\end{vmatrix}\begin{vmatrix}\langle 0.5,0.6,0.7\rangle\langle 0.6,0.7,0.8\rangle\\ \langle 0.5,0.6,0.7\rangle\langle 0.8,0.3,0.9\rangle\end{vmatrix}+$$

$$\begin{vmatrix}\langle 0.4,0.2,0.1\rangle\langle 0.7,0.3,0.4\rangle\\ \langle 0.4,0.6,0.7\rangle\langle 0.5,0.6,0.7\rangle\end{vmatrix}\begin{vmatrix}\langle 0.8,0.9,0.3\rangle\langle 0.6,0.7,0.8\rangle\\ \langle 0.5,0.3,0.2\rangle\langle 0.8,0.3,0.9\rangle\end{vmatrix}+$$

$$\begin{vmatrix}\langle 0.4,0.2,0.1\rangle\langle 0.6,0.7,0.8\rangle\\ \langle 0.4,0.6,0.7\rangle\langle 0.4,0.3,0.2\rangle\end{vmatrix}\begin{vmatrix}\langle 0.8,0.9,0.3\rangle\langle 0.5,0.6,0.7\rangle\\ \langle 0.5,0.3,0.2\rangle\langle 0.5,0.6,0.7\rangle\end{vmatrix}$$

$$+\begin{vmatrix}\langle 0.6,0.7,0.8\rangle\langle 0.7,0.3,0.4\rangle\\ \langle 0.3,0.2,0.1\rangle\langle 0.5,0.6,0.7\rangle\end{vmatrix}\begin{vmatrix}\langle 0.6,0.7,0.8\rangle\langle 0.6,0.7,0.8\rangle\\ \langle 0.9,0.5,0.3\rangle\langle 0.8,0.3,0.9\rangle\end{vmatrix}+$$

$$\begin{vmatrix}\langle 0.6,0.7,0.8\rangle\langle 0.6,0.7,0.8\rangle\\ \langle 0.3,0.2,0.1\rangle\langle 0.4,0.3,0.2\rangle\end{vmatrix}\begin{vmatrix}\langle 0.6,0.7,0.8\rangle\langle 0.5,0.6,0.7\rangle\\ \langle 0.9,0.5,0.3\rangle\langle 0.5,0.6,0.7\rangle\end{vmatrix}+$$

$$\begin{vmatrix}\langle 0.7,0.3,0.4\rangle\langle 0.6,0.7,0.8\rangle\\ \langle 0.5,0.6,0.7\rangle\langle 0.4,0.3,0.2\rangle\end{vmatrix}\begin{vmatrix}\langle 0.6,0.7,0.8\rangle\langle 0.8,0.9,0.3\rangle\\ \langle 0.9,0.5,0.3\rangle\langle 0.5,0.3,0.2\rangle\end{vmatrix}$$

$$\begin{aligned}=&[\langle 0.3,0.2,0.1\rangle\vee\langle 0.4,0.6,0.8\rangle][\langle 0.5,0.3,0.9\rangle\vee\langle 0.5,0.6,0.8\rangle]+\\ &[\langle 0.4,0.2,0.7\rangle\vee\langle 0.4,0.3,0.7\rangle][\langle 0.8,0.3,0.9\rangle\vee\langle 0.5,0.3,0.8\rangle]+\\ &[\langle 0.4,0.2,0.2\rangle\vee\langle 0.4,0.6,0.8\rangle][\langle 0.5,0.6,0.7\rangle\vee\langle 0.5,0.3,0.7\rangle]+\\ &[\langle 0.5,0.6,0.8\rangle\vee\langle 0.3,0.2,0.4\rangle][\langle 0.6,0.3,0.9\rangle\vee\langle 0.6,0.5.0.8\rangle]+\end{aligned}$$

$$[\langle 0.4,0.3,0.8\rangle \vee \langle 0.3,0.2,0.8\rangle][\langle 0.5,0.6,0.8\rangle \vee \langle 0.5,0.5,0.7\rangle]+$$
$$[\langle 0.4,0.3,0.4\rangle \vee \langle 0.5,0.6,0.8\rangle][\langle 0.5,0.3,0.8\rangle \vee \langle 0.8,0.5,0.3\rangle]$$
$$=[\langle 0.4,0.6,0.1\rangle \vee \langle 0.5,0.6,0.8\rangle][\langle 0.4,0.3,0.7\rangle \vee \langle 0.8,0.3,0.8\rangle]+$$
$$[\langle 0.4,0.6,0.2\rangle\langle 0.5,0.6,0.7\rangle]+[\langle 0.5,0.6,0.4\rangle\langle 0.6,0.5,0.8\rangle]+$$
$$[\langle 0.4,0.3,0.8\rangle\langle 0.5,0.6,0.7\rangle]+[\langle 0.5,0.6,0.4\rangle\langle 0.8,0.5,0.3\rangle]$$
$$=\langle 0.4,0.6,0.8\rangle+\langle 0.4,0.3,0.8\rangle+\langle 0.4,0.6,0.7\rangle+$$
$$\langle 0.5,0.5,0.8\rangle+\langle 0.4,0.3,0.8\rangle+\langle 0.5,0.5,0.4\rangle.$$
$$det(A)=\langle 0.5,0.6,0.4\rangle$$

4. Conclusion

In this paper, we have studied properties of determinant and adjoint of fuzzy neutrosophic soft square matrices.

REFERENCES

1. I.Arockiarani, I.R.Sumathi and M.Jency, *Fuzzy Neutrosophic soft topological spaces, IJMA*, 10 (2013) 225-238.
2. I.Arockiarani and I.R.Sumathi, A fuzzy neutrosophic soft Matrix approach in decision making, *JGRMA*, 2 (2) (2014) 14-23.
3. K.Atanassov, Intuitionistic fuzzy sets, *Fuzzy Sets and Systems*, 20 (1986) 87-96.
4. J.B.Kim, A.Baartmas and N.Sahadin, Determinant theory for fuzzy matrix, *Fuzzy Sets and Systems*, 29 (1989) 346-356.
5. P.K.Maji, R.Biswas and A.R.Roy, Fuzzy Soft set, *The Journal of Fuzzy Mathematics*, 9 (3) (2001) 589-602.
6. P.K.Maji, R.Biswas and A.R.Roy, Intuitionistic fuzzy soft sets, *Journal of Fuzzy Mathematics*, 12 (2004) 669-683.
7. M.Dhar, S.Broumi and F.Smarandache, A note on square neutrosophic fuzzy matrices, *Neutrosophic Sets and Systems,* 3 (2014) 37-41.
8. M.Bora, B.Bora and T.J.Neog and D.K.Sut, Intuitionistic fuzzy soft matrix theory and its application in medical diagnosis, *Annals of Fuzzy Mathematics and Informatics*, 7(1) (2013) 157-168.
9. D.Molodtsov, Soft set theory first results, *Computer and mathematics with applications*, 37 (1999) 19-31.
10. P.K.Maji, Neutrosophic soft set, *Annals of fuzzy mathematics and informatics*, 5 (2013) 157-168.
11. Z.Pawlak, Rough sets, *International journal of computing and information sciences*, 11 (1982) 341-356.
12. F.Smarandache, Neutrosophic set, a generalisation of the intuitionistic fuzzy set, *Inter. J. Pure Appl. Math.*, 24 (2005) 287-297.
13. I.R.Sumathi and I.Arockiarani, New operations on fuzzy neutrosophic soft matrices, *International Journal of Innovative Research and Studies*, 3 (3) (2014) 110-124.
14. M.G.Thomas, Convergence of powers of a fuzzy matrix, *Journal. Math. Annal. Appl.*, 57 (1977) 476-480.
15. R.Uma, P.Murugadas and S.Sriram, Fuzzy neutrosophic soft matrices of type-I and type-II, Communicated.

16. R.Uma, P.Murugadas and S.Sriram, Determinant and adjoint of fuzzy neutrosophic soft matrices, Communicated.
17. Y.Yang and C.Ji, Fuzzy soft matrices and their application, Part I, *LNAI* 7002; (2011) 618-627.
18. L.A.Zadeh, Fuzzy sets, *Information and control*, 8 (1965) 338-353.
19. L.A.Zadeh, The concept of a linguistic variable and its application to approximate reasoning-I, *Information Science*, 8 (1975) 199-249.
20. M.Bhowmik and M.Pal, Intuitionistic neutrosophic set, *Journal of Information and Computing Science,* 4(2) (2009) 142-152.
21. M.Bhowmik and M.Pal, Intuitionistic neutrosophic set relations and some of its properties, *Journal of Information and Computing Science,* 5(3) (2010) 183-192.