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Extended Nonstandard Neutrosophic Logic, Set, and Probability Based on Extended Nonstandard Analysis

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Abstract: We extend for the second time the nonstandard analysis by adding the left monad closed to the right, and right monad closed to the left, while besides the pierced binad (we introduced in 1998) we add now the unpierced binad—all these in order to close the newly extended nonstandard space under nonstandard addition, nonstandard subtraction, nonstandard multiplication, nonstandard division, and nonstandard power operations. Then, we extend the Nonstandard Neutrosophic Logic, Nonstandard Neutrosophic Set, and Nonstandard Probability on this Extended Nonstandard Analysis space, and we prove that it is a nonstandard neutrosophic lattice of first type (endowed with a nonstandard neutrosophic partial order) as well as a nonstandard neutrosophic lattice of second type (as algebraic structure, endowed with two binary neutrosophic laws: inf_N and sup_N). Many theorems, new terms introduced, better notations for monads and binads, and examples of nonstandard neutrosophic operations are given.

Keywords: nonstandard analysis; extended nonstandard analysis; open and closed monads to the left/right; pierced and unpierced binads; MoBiNad set; infinitesimals; infinities; nonstandard reals; standard reals; nonstandard neutrosophic lattices of first type (as poset) and second type (as algebraic structure), nonstandard neutrosophic logic; extended nonstandard neutrosophic logic; nonstandard arithmetic operations; nonstandard unit interval; nonstandard neutrosophic infimum; nonstandard neutrosophic supremum

1. Short Introduction

In order to more accurately situate and fit the neutrosophic logic into the framework of extended nonstandard analysis [1–3], we present the nonstandard neutrosophic inequalities, nonstandard neutrosophic equality, nonstandard neutrosophic infimum and supremum, and nonstandard neutrosophic intervals, including the cases when the neutrosophic logic standard and nonstandard components T, I, F get values outside of the classical unit interval [0, 1], and a brief evolution of neutrosophic operators [4].

2. Theoretical Reason for the Nonstandard Form of Neutrosophic Logic

The only reason we have added the nonstandard form to neutrosophic logic (and similarly to neutrosophic set and probability) was in order to make a distinction between Relative Truth (which is truth in some Worlds, according to Leibniz) and Absolute Truth (which is truth in all possible Words, according to Leibniz as well) that occur in philosophy.

Another possible reason may be when the neutrosophic degrees of truth, indeterminacy, or falsehood are infinitesimally determined, for example a value infinitesimally bigger than 0.8 (or 0.8^+), or infinitesimally smaller than 0.8 (or $^-0.8$). But these can easily be overcome by roughly using interval neutrosophic values, for example (0.80, 0.81) and (0.79, 0.80), respectively.

3. Why the Sum of Neutrosophic Components Is Up to 3

We was more prudent when we presented the sum of single valued standard neutrosophic components [5–9], saying

$$\text{Let } T, I, F \text{ be single valued numbers, } T, I, F \in [0, 1], \text{ such that } 0 \leq T + I + F \leq 3. \quad (1)$$

The sum of the single-valued neutrosophic components, $T + I + F$ is up to 3 since they are considered completely (100%) independent of each other. But if the two components T and F are completely (100%) dependent, then $T + F \leq 1$ (as in fuzzy and intuitionistic fuzzy logics), and let us assume the neutrosophic middle component I is completely (100%) independent from T and F, then $I \leq 1$, whence $T + F + I \leq 1 + 1 = 2$.

But the degree of dependence/independence [10] between T, I, F all together, or taken two by two, may be, in general, any number between 0 and 1.

4. Neutrosophic Components outside the Unit Interval [0, 1]

Thinking out of box, inspired from the real world, was the first intent, i.e., allowing neutrosophic components (truth/indeterminacy/falsehood) values be outside of the classical (standard) unit real interval [0, 1] used in all previous (Boolean, multivalued, etc.) logics if needed in applications, so neutrosophic component values < 0 and > 1 had to occurs due to the Relative/Absolute stuff, with

$$^{-}0 <_N 0 \text{ and } 1^{+} >_N 1 \quad (2)$$

Later on, in 2007, I found plenty of cases and real applications in Standard Neutrosophic Logic and Set (therefore, not using the Nonstandard Neutrosophic Logic, Set, and Probability), and it was thus possible the extension of the neutrosophic set to Neutrosophic Overset (when some neutrosophic component is > 1), and to Neutrosophic Underset (when some neutrosophic component is < 0), and to Neutrosophic Offset (when some neutrosophic components are off the interval [0, 1], i.e., some neutrosophic component > 1 and some neutrosophic component < 0). Then, similar extensions to Neutrosophic Over/Under/Off Logic, Measure, Probability, Statistics, etc., [11–14], extending the unit interval [0, 1] to

$$[\Psi, \Omega], \text{ with } \Psi \leq 0 < 1 \leq \Omega, \quad (3)$$

where Ψ, Ω are standard (or nonstandard) real numbers.

5. Refined Neutrosophic Logic, Set, and Probability

We wanted to get the neutrosophic logic as general as possible [15], extending all previous logics (Boolean, fuzzy, intuitionistic fuzzy logic, intuitionistic logic, paraconsistent logic, and dialethism), and to have it able to deal with all kind of logical propositions (including paradoxes, nonsensical propositions, etc.).

That is why in 2013 we extended the Neutrosophic Logic to Refined Neutrosophic Logic / Set / Probability (from generalizations of 2-valued Boolean logic to fuzzy logic, also from the Kleene's and Lukasiewicz's and Bochvar's 3-symbol valued logics or Belnap's 4-symbol valued logic, to the most general n-symbol or n-numerical valued refined neutrosophic logic, for any integer $n \geq 1$), the largest ever so far, when some or all neutrosophic components T, I, F were split/refined into neutrosophic subcomponents $T_1, T_2, \ldots; I_1, I_2, \ldots; F_1, F_2, \ldots$, which were deduced from our everyday life [16].

6. From Paradoxism Movement to Neutrosophy Branch of Philosophy and then to Neutrosophic Logic

We started first from Paradoxism (that we founded in the 1980s in Romania as a movement based on antitheses, antinomies, paradoxes, contradictions in literature, arts, and sciences), then we introduced the Neutrosophy (as generalization of Dialectics of Hegel and Marx, which is actually the ancient YinYang Chinese philosophy), neutrosophy is a branch of philosophy studying the dynamics of triads, inspired from our everyday life, triads that have the form

$$\langle A\rangle, \text{its opposite } \langle antiA\rangle, \text{and their neutrals } \langle neutA\rangle, \tag{4}$$

where <*A*> is any item or entity [17]. (Of course, we take into consideration only those triads that make sense in our real and scientific world.)

The Relative Truth neutrosophic value was marked as *1*, while the Absolute Truth neutrosophic value was marked as 1^+ (a tinny bigger than the Relative Truth's value): $1^+ >_N 1$, where $>_N$ is a neutrosophic inequality, meaning 1^+ is neutrosophically bigger than *1*.

Similarly for Relative Falsehood/Indeterminacy (which is falsehood/indeterminacy in some Worlds) and Absolute Falsehood/Indeterminacy (which is falsehood/indeterminacy in all possible worlds).

7. Introduction to Nonstandard Analysis

An *infinitesimal* (or infinitesimal number) (ε) is a number ε, such that $|\varepsilon| < 1/n$, for any non-null positive integer n. An infinitesimal is close to zero, and so small that it cannot be measured.

The infinitesimal is a number smaller, in absolute value, than anything positive nonzero.

Infinitesimals are used in calculus.

An *infinite* (or infinite number) (ω) is a number greater than anything:

$$1 + 1 + 1 + \ldots + 1 \text{ (for any finite number terms)} \tag{5}$$

The infinites are reciprocals of infinitesimals.

The set of *hyperreals* (or *nonstandard reals*), denoted as R^*, is the extension of set of the real numbers, denoted as R, and it comprises the infinitesimals and the infinites, that may be represented on the *hyperreal number line*:

$$1/\varepsilon = \omega/1. \tag{6}$$

The set of hyperreals satisfies the *transfer principle*, which states that the statements of first order in R are valid in R^* as well.

A *monad* (*halo*) of an element $a \in R^*$, denoted by $\mu(a)$, is a subset of numbers infinitesimally close to a.

8. First Extension of Nonstandard Analysis

Let us denote by R_+^* the set of positive nonzero hyperreal numbers.

We consider the left monad and right monad, and the (*pierced*) *binad* that we have introduced as extension in 1998 [5]:

Left Monad {that we denote, for simplicity, by $({}^-a)$ or only ${}^-a$} is defined as:

$$\mu({}^-a) = ({}^-a) = {}^-a = \overset{-}{a} = \{a - x,\ x \in R_+^* \mid x \text{ is } infinitesimal\}. \tag{7}$$

Right Monad {that we denote, for simplicity, by (a^+) or only by a^+} is defined as:

$$\mu(a^+) = (a^+) = a^+ = \overset{+}{a} = \{a + x,\ x \in R_+^* \mid x \text{ is } infinitesimal\}. \tag{8}$$

Pierced Binad {that we denote, for simplicity, by $({}^-a^+)$ or only ${}^-a^+$} is defined as:

$$\begin{aligned}\mu({}^-a^+) = ({}^-a^+) = {}^-a^+ = \overset{-+}{a} &= \{a - x,\ x \in R_+^* \mid x \text{ is } infinitesimal\} \cup \{a + x,\ x \in R_+^* \mid x \text{ is } infinitesimal\} \\ &= \{a \pm x,\ x \in R_+^* \mid x \text{ is } infinitesimal\}.\end{aligned} \tag{9}$$

The left monad, right monad, and the pierced binad are subsets of R^*.

9. Second Extension of Nonstandard Analysis

For the necessity of doing calculations that will be used in nonstandard neutrosophic logic in order to calculate the nonstandard neutrosophic logic operators (conjunction, disjunction, negation, implication, and equivalence) and in order to have the Nonstandard Real MoBiNad Set closed under arithmetic operations, we extend, for the time being, the left monad to the Left Monad Closed to the Right, the right monad to the Right Monad Closed to the Left, and the Pierced Binad to the Unpierced Binad, defined as follows [18–21].

Left Monad Closed to the Right

$$\mu\left(\overset{-0}{a}\right) = \left(\overset{-0}{a}\right) = \overset{-0}{a} = \{a - x \mid x = 0,\ or\ x \in R_+^*\ and\ x\ is\ infinitesimal\} = \mu({}^{-}a) \cup \{a\} = ({}^{-}a)\ 0\cup \{a\} = {}^{-}a \cup \{a\}. \tag{10}$$

Right Monad Closed to the Left

$$\mu\left(\overset{0+}{a}\right) = \left(\overset{0+}{a}\right) = \overset{0+}{a} = \{a + x \mid x = 0,\ or\ x \in R_+^*\ and\ x\ is\ infinitesimal\} = \mu(a^{+}) \cup \{a\} = (a^{+})\ 0\cup \{a\} = a^{+} \cup \{a\}. \tag{11}$$

Unpierced Binad

$$\mu\left(\overset{-0+}{a}\right) = \left(\overset{-0+}{a}\right) = \overset{-0+}{a} = \{a - x \big| x \in R_+^* andxisinfinitesimal\} \cup \{a + x \big| x \in R_+^* and\ xisinfinitesimal\} \cup \{a\} = \{a \pm x \big| x = 0, orx \in R_+^* andxisinfinitesimal\} = \mu({}^{-}a^{+}) \cup \{a\} = ({}^{-}a^{+}) \cup \{a\} = {}^{-}a^{+} \cup \{a\} \tag{12}$$

The element $\{a\}$ has been included into the left monad, right monad, and pierced binad respectively.

10. Nonstandard Neutrosophic Function

In order to be able to define equalities and inequalities in the sets of monads, and in the sets of binads, we construct a *nonstandard neutrosophic function* that approximates the monads and binads to tiny open (or half open and half closed respectively) standard real intervals as below. It is called 'neutrosophic' since it deals with indeterminacy: unclear, vague monads and binads, and the function approximates them with some tiny real subsets.

Taking an arbitrary infinitesimal

$$\varepsilon_1 > 0, \text{ and writing } {}^{-}a = a - \varepsilon_1,\ a^{+} = a + \varepsilon_1, \text{ and } {}^{-}a^{+} = a \pm \varepsilon_1, \tag{13}$$

or taking an arbitrary infinitesimal $\varepsilon_2 \geq 0$, and writing

$$\overset{-0}{a} = (a - \varepsilon_2, a],\ \overset{0+}{a} = [a, a + \varepsilon_2),\ \overset{-0+}{a} = (a - \varepsilon_2, a + \varepsilon_2) \tag{14}$$

We meant to actually pick up a representative from each class of the monads and of the binads.

Representations of the monads and binads by intervals is not quite accurate from a classical point of view, but it is an approximation that helps in finding a partial order and computing nonstandard arithmetic operations on the elements of the nonstandard set NR_{MB}.

Let ε be a generic positive infinitesimal, while a be a generic standard real number.

Let $P(R)$ be the power set of the real number set R.

$$\mu_N : NR_{MB} \rightarrow P(R) \tag{15}$$

For any $a \in R$, the set of real numbers, one has

$$\mu_N((^-a)) =_N (a - \varepsilon, a), \tag{16}$$

$$\mu_N((a^+)) =_N (a, a + \varepsilon), \tag{17}$$

$$\mu_N((^-a^+)) =_N (a - \varepsilon, a) \cup (a, a + \varepsilon), \tag{18}$$

$$\mu_N\left(\left(\overset{-0}{a}\right)\right) =_N (a - \varepsilon, a], \tag{19}$$

$$\mu_N\left(\left(\overset{0+}{a}\right)\right) =_N \left[a, a + \varepsilon\right), \tag{20}$$

$$\mu_N\left(\left(\overset{-0+}{a}\right)\right) =_N (a - \varepsilon, a + \varepsilon), \tag{21}$$

$$\mu_N\left(\left(\overset{0}{a}\right)\right) =_N \mu_N(a) =_N a = [a, a], \tag{22}$$

in order to set it as real interval too.

11. General Notations for Monads and Binads

Let $a \in R$ be a standard real number. We use the following general notation for monads and binads.

$$\overset{m}{a} \in \left\{a, \overset{-}{a}, \overset{-0}{a}, \overset{+}{a}, \overset{0+}{a}, \overset{-+}{a}, \overset{-0+}{a}\right\} \text{ and by convention } \overset{0}{a} = a; \tag{23}$$

or

$$m \in \{\, , ^-, ^{-0}, ^+, ^{+0}, ^{-+}, ^{-0+}\} = \{^0, ^-, ^{-0}, ^+, ^{+0}, ^{-+}, ^{-0+}\}; \tag{24}$$

therefore "m" above a standard real number "a" may mean anything: a standard real number (^{0}, or nothing above), a left monad ($^-$), a left monad closed to the right ($^{-0}$), a right monad ($^+$), a right monad closed to the left ($^{0+}$), a pierced binad ($^{-+}$), or a unpierced binad ($^{-0+}$), respectively.

The notations of monad's and binad's diacritics above (not laterally) the number a as

$$\overset{-}{a}, \overset{-0}{a}, \overset{+}{a}, \overset{0+}{a}, \overset{-+}{a}, \overset{-0+}{a} \tag{25}$$

are the best, since they also are designed to avoid confusion for the case when the real number a is negative.

For example, if $a = {}^-2$, then the corresponding monads and binads are respectively represented as:

$$\overset{-}{{}^-2}, \overset{-0}{{}^-2}, \overset{+}{{}^-2}, \overset{0+}{{}^-2}, \overset{-+}{{}^-2}, \overset{-0+}{{}^-2} \tag{26}$$

Classical and Neutrosophic Notations

Classical notations on the set of real numbers:

$$<, \leq, >, \geq, \wedge, \vee, \rightarrow, \leftrightarrow, \cap, \cup, \subset, \supset, \subseteq, \supseteq, =, \in, \\ +, -, \times, \div, \hat{}, * \tag{27}$$

Operations with real subsets:

$$\circledast \tag{28}$$

Neutrosophic notations on nonstandard sets (that involve indeterminacies, approximations, and vague boundaries):

$$<_N, \leq_N, >_N, \geq_N, \wedge_N, \vee_N, \rightarrow_N, \leftrightarrow_N, \cap_N, \cup_N, \subset_N, \supset_N, \subseteq_N, \supseteq_N, =_N, \in_N +_N, -_N, \times_N, \div_N, \hat{}_N, *_N \tag{29}$$

12. Neutrosophic Strict Inequalities

We recall the neutrosophic strict inequality which is needed for the inequalities of nonstandard numbers.

Let α and β be elements in a partially ordered set M.

We have defined the neutrosophic strict inequality

$$\alpha >_N \beta \tag{30}$$

and read as

"α is neutrosophically greater than β"

if α in general is greater than β, or α is approximately greater than β, or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) *or subject to some contradiction* (situation when α is smaller than or equal to β) *α is greater than β.*

It means that in most of the cases, on the set M, *α is greater than β.*

And similarly for the opposite neutrosophic strict inequality

$$\alpha <_N \beta \tag{31}$$

13. Neutrosophic Equality

We have defined the neutrosophic inequality

$$\alpha =_N \beta \tag{32}$$

and read as

"α is neutrosophically equal to β"

if α in general is equal to β, or α is approximately equal to β, or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) *or subject to some contradiction* (situation when α is not equal to β) *α is equal to β.*

It means that in most of the cases, on the set M, *α is equal to β.*

14. Neutrosophic (Nonstrict) Inequalities

Combining the neutrosophic strict inequalities with neutrosophic equality, we get the $\geq_N$ and $\leq_N$ neutrosophic inequalities.

Let α and β be elements in a partially ordered set M.

The neutrosophic (nonstrict) inequality

$$\alpha \geq_N \beta \tag{33}$$

and read as

"α is neutrosophically greater than or equal to β"

if α in general is greater than or equal to β, or α is approximately greater than or equal to β, or subject to some indeterminacy (unknown or unclear ordering relationship between α and β) or subject to some contradiction (situation when α is smaller than β) α is greater than or equal to β.

It means that in most of the cases, on the set M, *α is greater than or equal to β.*

And similarly for the opposite neutrosophic (nonstrict) inequality

$$\alpha \leq_N \beta. \tag{34}$$

15. Neutrosophically Ordered Set

Let M be a set. $(M, <_N)$ is called a neutrosophically ordered set if

$$\forall \alpha, \beta \in M, \text{onehas : either} \alpha <_N \beta, \text{or} \alpha =_N \beta, \text{or} \alpha >_N \beta. \tag{35}$$

16. Neutrosophic Infimum and Neutrosophic Supremum

As an extension of the classical infimum and classical supremum, and using the neutrosophic inequalities and neutrosophic equalities, we define the neutrosophic infimum (denoted as inf_N) and the neutrosophic supremum (denoted as sup_N).

Neutrosophic Infimum.

Let $(S, <_N)$ be a set that is neutrosophically partially ordered, and M a subset of S.

The neutrosophic infimum of M, denoted as $inf_N(M)$ is the neutrosophically greatest element in S that is neutrosophically less than or equal to all elements of M:

Neutrosophic Supremum.

Let $(S, <_N)$ be a set that is neutrosophically partially ordered and M a subset of S.

The neutrosophic supremum of M, denoted as $sup_N(M)$ is the neutrosophically smallest element in S that is neutrosophically greater than or equal to all elements of M.

17. Definition of Nonstandard Real MoBiNad Set

Let $\mathbb{R}$ be the set of standard real numbers, and $\mathbb{R}^*$ be the set of hyper-reals (or nonstandard reals) that consists of infinitesimals and infinites.

The Nonstandard Real MoBiNad Set is now defined for the first time as follows

$$NR_{MB} =_N \left\{ \begin{array}{c} \varepsilon,\ \omega,\ a, ({}^{-}a)\ , \left({}^{-}a^{0}\right),\ (a^{+}), \left({}^{0}a^{+}\right), ({}^{-}a^{+}), \left({}^{-}a^{0+}\right) |\text{where } \varepsilon \text{ are infinitesimals,} \\ \text{with } \varepsilon \in \mathbb{R}^*;\ \omega = 1/\varepsilon \text{ are infinites, with } \omega \in \mathbb{R}^*;\text{ and } a \text{ are real numbers, with } a \in \mathbb{R} \end{array} \right\} \tag{36}$$

Therefore

$$NR_{MB} =_N \mathbb{R}^* \cup \mathbb{R} \cup \mu({}^{-}\mathbb{R}) \cup \mu\left({}^{-}\mathbb{R}^{0}\right) \cup \mu\left(\mathbb{R}^{+}\right) \cup \mu\left({}^{0}\mathbb{R}^{+}\right) \cup \mu\left({}^{-}\mathbb{R}^{+}\right) \cup \mu\left({}^{-}\mathbb{R}^{0\ +}\right), \tag{37}$$

where

$\mu({}^{-}\mathbb{R})$ is the set of all real left monads,
$\mu\left({}^{-}\mathbb{R}^{0}\right)$ is the set of all real left monads closed to the right,
$\mu(\mathbb{R}^{+})$ is the set of all real right monads,
$\mu\left({}^{0}\mathbb{R}^{+}\right)$ is the set of all real right monads closed to the left,
$\mu({}^{-}\mathbb{R}^{+})$ is the set of all real pierced binads,
and $\mu\left({}^{-}\mathbb{R}^{0\ +}\right)$ is the set of all real unpierced binads.

Also,

$$NR_{MB} =_N \left\{ \varepsilon,\ \omega,\ \overset{m}{a} \middle| \begin{array}{c} \text{where } \varepsilon,\ \omega \in \mathbb{R}^*, \varepsilon \text{ are infinitesimals, } \omega = \frac{1}{\varepsilon} \text{ are infinities;} \\ a \in \mathbb{R}; \text{and } m \in \left\{ ,^{-} ,^{-\ 0} ,^{+} ,^{+\ 0} ,^{-\ +} ,^{-\ 0\ +} \right\} \end{array} \right\} \tag{38}$$

NR_{MB} is closed under addition, subtraction, multiplication, division (except division by $\overset{m}{a}$, with a = 0 and $m \in \left\{ ,^{-} ,^{-\ 0} ,^{+} ,^{0+} ,^{-\ +} ,^{-\ 0\ +} \right\}$), and power

$\{\left(\overset{m_1}{a}\right)^{\left(\overset{m_2}{b}\right)}$ with: either $a > 0$, or $a = 0$ and $m \in \left\{ ,^{+} ,^{0+} \right\}$ and $b > 0$, or $a < 0$ but $b = \frac{p}{r}$ (irreducible fraction) and p, r are integers with r an odd positive integer $r \in \{1, 3, 5, \ldots\}\}$.

These mobinad (nonstandard) above operations are reduced to set operations, using Set Analysis and Neutrosophic Analysis (both introduced by the author [22] (page 11), which are generalizations of Interval Analysis), and they deal with sets that have indeterminacies.

18. Etymology of MoBiNad

MoBiNad comes from **mo***nad* + **bi***nad*, introduced now for the first time.

19. Definition of Nonstandard Complex MoBiNad Set

The Nonstandard Complex MoBiNad Set, introduced here for the first time, is defined as

$$NC_{MB} =_N \left\{\alpha + \beta i \,|\, \text{where } i = \sqrt{-1};\ \alpha,\ \beta \in NR_{MB}\right\} \tag{39}$$

20. Definition of Nonstandard Neutrosophic Real MoBiNad Set

The Nonstandard Neutrosophic Real MoBiNad Set, introduced now for the first time, is defined as

$$NNR_{MB} =_N \left\{\alpha + \beta I \,|\, \text{where } I = \text{literal indeterminacy},\ I^2 = I;\ \alpha,\ \beta \in NR_{MB}\right\}. \tag{40}$$

21. Definition of Nonstandard Neutrosophic Complex MoBiNad Set

The Nonstandard Neutrosophic Complex MoBiNad Set, introduced now for the first time, is defined as

$$NNC_{MB} =_N \left\{\alpha + \beta I \,|\, \text{where } I = \text{literal indeterminacy},\ I^2 = I;\ \alpha,\ \beta \in NC_{MB}\right\} \tag{41}$$

22. Properties of the Nonstandard Neutrosophic Real Mobinad Set

Since in nonstandard neutrosophic logic we use only the nonstandard neutrosophic real mobinad set, we study some properties of it.

Theorem 1. *The nonstandard real mobinad set* (NR_{MB}, $\leq_N$)*, endowed with the nonstandard neutrosophic inequality is a lattice of first type [as partially ordered set (poset)].*

Proof. The set NR_{MB} is partially ordered, because (except the two-element subsets of the form $\left\{a, \overset{-+}{a}\right\}$, and $\left\{a, \overset{-0+}{a}\right\}$, with $a \in \mathbb{R}$, beetwen which there is no order) all other elements are ordered:

If $a < b$, where $a,\ b \in \mathbb{R}$, then: $\overset{m_1}{a} <_N \overset{m_2}{b}$, for any monads or binads

$$m_1,\ m_2 \in_N \left\{ ,^{-} ,^{-0} ,^{+} ,^{0+} ,^{-+} ,^{-0+} \right\}. \tag{42}$$

If $a = b$, one has:

$$^{-}a <_N a, \tag{43}$$

$$a^{-} <_N a^{+} \tag{44}$$

$$a <_N a^{+} \tag{45}$$

$$^{-}a \leq_N {}^{-}a^{+}, \tag{46}$$

$$^{-}a \leq_N {}^{-}a^{+}, \tag{47}$$

and there is no neutrosophic ordering relationship between a and $^{-}a^{+}$,

$$\text{nor between } a \text{ and } \overset{-0+}{a} \text{ (that is why } \leq_N \text{ on } NR_{MB} \text{ is } a \text{ partial ordering set).} \tag{48}$$

$$\text{If } a > b, \text{ then}: \overset{m_1}{a} >_N \overset{m_2}{b}, \text{ for any monads or binads } m_1,\ m_2. \quad (49)$$

□

Any two-element set $\{\alpha, \beta\} \subset_N NR_{MB}$ has a neutrosophic nonstandard infimum (meet, or greatest lower bound) that we denote by $\inf_N$, and a neutrosophic nonstandard supremum (joint, or least upper bound) that we denote by $\sup_N$, where both

$$\inf_N\{\alpha, \beta\} \text{ and } \sup_N\{\alpha, \beta\} \in NR_{MB}. \quad (50)$$

For the nonordered elements a and ${}^-a^+$:

$$\inf_N\left\{a, {}^-a^+\right\} =_N {}^-a \in_N NR_{MB}, \quad (51)$$

$$\sup_N\left\{a, {}^-a^+\right\} =_N a^+ \in_N NR_{MB} \quad (52)$$

And similarly for nonordered elements a and ${}^-a^{0\,+}$:

$$\inf_N\left\{a, {}^-a^{0\,+}\right\} =_N {}^-a \in_N NR_{MB}, \quad (53)$$

$$\sup_N\left\{a, {}^-a^{0\,+}\right\} =_N a^+ \in_N NR_{MB}. \quad (54)$$

Dealing with monads and binads which neutrosophically are real subsets with indeterminate borders, and similarly $a = [a, a]$ can be treated as a subset, we may compute $\inf_N$ and $\sup_N$ of each of them.

$$\inf_N({}^-a) =_N {}^-a \text{ and } \sup_N({}^-a) =_N {}^-a \quad (55)$$

$$\inf_N\left(a^+\right) =_N a^+ \text{ and } \sup_N\left(a^+\right) =_N a^+; \quad (56)$$

$$\inf_N\left({}^-a^+\right) =_N {}^-a \text{ and } \sup_N\left({}^-a^+\right) =_N a^+; \quad (57)$$

$$\inf_N\left({}^-a^{0\,+}\right) =_N {}^-a \text{ and } \sup_N\left({}^-a^{0\,+}\right) =_N a^+. \quad (58)$$

Also,

$$\inf_N(a) =_N a \text{ and } \sup_N(a) =_N a. \quad (59)$$

If $a < b$, then $\overset{m_1}{a} <_N \overset{m_2}{b}$, hence

$$\inf_N\left\{\overset{m_1}{a}, \overset{m_2}{b}\right\} =_N \inf_N\left(\overset{m_1}{a}\right) \text{ and } \sup_N\left\{\overset{m_1}{a}, \overset{m_2}{b}\right\} =_N \sup_N \overset{m_2}{b}, \quad (60)$$

which are computed as above.

Similarly, if

$$a > b, \text{ with } \overset{m_1}{a} <_N \overset{m_2}{b}. \quad (61)$$

If $a = b$, then: $\inf_N\left\{\overset{m_1}{a}, \overset{m_2}{a}\right\} =_N$ the neutrosophically smallest ($<_N$) element among

$$\inf_N\left\{\overset{m_1}{a}\right\} \text{ and } \inf_N\left\{\overset{m_2}{a}\right\}. \quad (62)$$

While $\sup_N\left\{\overset{m_1}{a}, \overset{m_2}{a}\right\} =_N$ the neutrosophically greatest ($>_N$) element among

$$\sup_N\left\{\overset{m_1}{a}\right\} \text{ and } \sup_N\left\{\overset{m_2}{a}\right\}. \quad (63)$$

Examples:

$$\inf_N\left({}^-a,\ a^+\right) =_N {}^-a \text{ and } \sup_N\left({}^-a,\ a^+\right) =_N a^+; \tag{64}$$

$$\inf_N\left({}^-a,{}^-a^+\right) =_N {}^-a \text{ and } \sup_N\left({}^-a,{}^-a^+\right) =_N a^+; \tag{65}$$

$$\inf_N\left({}^-a^+,\ a^+\right) =_N {}^-a \text{ and } \sup_N\left({}^-a^+,\ a^+\right) =_N a^+. \tag{66}$$

Therefore, $(NR_{MB},\ \leq_N)$ is a nonstandard real mobinad lattice of first type (as partially ordered set).

Consequence

If we remove all pierced and unpierced binads from NR_{MB} and we denote the new set by $NR_M = \left\{\varepsilon,\ \omega,\ a,{}^-a,{}^-a^0,\ a^+,{}^0a^+, \text{ where } \varepsilon \text{ are infinitesimals, } \omega \text{ are infinites, and } a \in \mathbb{R}\right\}$ we obtain a totally neutrosophically ordered set.

Theorem 2. *Any finite non-empty subset L of* $(NR_{MB}, \leq_N)$ *is also a sublattice of first type.*

Proof. It is a consequence of any classical lattice of first order (as partially ordered set). □

Theorem 3. $(NR_{MB},\ \leq_N)$ *is bounded neither to the left nor to the right, since it does not have a minimum (bottom, or least element), or a maximum (top, or greatest element).*

Proof. Straightforward, since NR_{MB} includes the set of real number $R = (-\infty,\ +\infty)$ which is clearly unbounded to the left and right-hand sides. □

Theorem 4. $(NR_{MB},\ inf_N, sup_N)$*, where* inf_N *and* sup_N *are two binary operations, dual to each other, defined before as a lattice of second type (as an algebraic structure).*

Proof. We have to show that the two laws inf_N and sup_N are commutative, associative, and verify the absorption laws.

Let $\alpha,\ \beta,\ \gamma \in NR_{MB}$ be two arbitrary elements.

Commutativity Laws

(i)

$$\inf_N\{\alpha,\ \beta\} =_N \inf_N\{\beta,\ \alpha\} \tag{67}$$

(ii)

$$\sup_N\{\alpha,\ \beta\} =_N \sup_N\{\beta,\ \alpha\} \tag{68}$$

Their proofs are straightforward.

Associativity Laws

(i)

$$\inf_N\{\alpha,\ \inf_N\{\beta,\gamma\}\} =_N \inf_N\{\inf_N\{\alpha,\ \beta\},\ \gamma\}. \tag{69}$$

□

Proof.

$$\inf_N\{\alpha,\ \inf_N\{\beta,\gamma\}\} =_N \inf_N\{\alpha,\ \beta,\ \gamma\}, \tag{70}$$

and

$$\inf_N\{\inf_N\{\alpha,\ \beta\},\ \gamma\} =_N \inf_N\{\alpha,\ \beta,\ \gamma\}, \tag{71}$$

where we have extended the binary operation $\inf_N$ to a trinary operation $\inf_N$.

(ii)

$$\sup_N\left\{\alpha,\ \sup_N\{\beta,\gamma\}\right\} =_N \sup_N\left\{\sup_N\{\alpha,\ \beta\},\ \gamma\right\} \tag{72}$$

□

Proof.

$$\sup_N\left\{\alpha,\ \sup_N\{\beta,\gamma\}\right\} =_N \sup_N\{\alpha,\ \beta,\ \gamma\},\tag{73}$$

and

$$\sup_N\left\{\alpha,\ \sup_N\{\beta,\gamma\}\right\} =_N \sup_N\{\alpha,\ \beta,\ \gamma\},\tag{74}$$

where similarly we have extended the binary operation $\sup_N$ to a trinary operation $\sup_N$.

Absorption Laws (as peculiar axioms to the theory of lattice)

(i) We need to prove that

$$\inf_N\left\{\alpha,\ \sup_N\{\alpha,\ \beta\}\right\} =_N \alpha.\tag{75}$$

Let $\alpha \leq_N \beta$, then

$$\inf_N\left\{\alpha,\ \sup_N\{\alpha,\ \beta\}\right\} =_N \inf_N\{\alpha,\alpha\} =_N \alpha.\tag{76}$$

Let $\alpha >_N \beta$, then

$$\inf_N\left\{\alpha,\ \sup_N\{\alpha,\ \beta\}\right\} =_N \inf_N\{\alpha,\alpha\} =_N \alpha.\tag{77}$$

(ii) Now, we need to prove that

$$\sup_N\{\alpha,\ \inf_N\{\alpha,\ \beta\}\} =_N \alpha.\tag{78}$$

Let $\alpha \leq_N \beta$, then

$$\sup_N\{\alpha,\ \inf_N\{\alpha,\ \beta\}\} =_N \sup_N\{\alpha,\ \alpha\} =_N \alpha.\tag{79}$$

Let $\alpha >_N \beta$, then

$$\sup_N\{\alpha,\ \inf_N\{\alpha,\ \beta\}\} =_N \sup_N\{\alpha,\beta\} =_N \alpha.\tag{80}$$

Consequence

The binary operations $\inf_N$ and $\sup_N$ also satisfy the idempotent laws:

$$\inf_N\{\alpha,\ \alpha\} =_N \alpha,\tag{81}$$

$$\sup_N\{\alpha,\ \alpha\} =_N \alpha.\tag{82}$$

□

Proof. The axioms of idempotency follow directly from the axioms of absorption proved above. □

Thus, we have proved that (NR_{MB}, inf_N, sup_N) is a lattice of second type (as algebraic structure).

23. Definition of General Nonstandard Real MoBiNad Interval

Let $a,\ b \in \mathbb{R}$, with

$$-\infty < a \leq b < \infty,\tag{83}$$

$$]^-a,\ b^+[_{MB} = \left\{x \in NR_{MB},{}^-a \leq_N x \leq_N b^+\right\}.\tag{84}$$

As particular edge cases:

$$]^-a,\ a^+[_{MB} =_N \left\{^-a,\ a,{}^-a^+,a^+\right\},\tag{85}$$

a discrete nonstandard real set of cardinality 4.

$$]^-a,\ ^-a[_{MB} =_N \{^-a\};\tag{86}$$

$$]a^+,\ a^+[_{MB} =_N \left\{a^+\right\}\tag{87}$$

$$]a,\ a^+[_{MB} =_N \left\{a,\ a^+\right\}\tag{88}$$

$$]^-a,\ a[_{MB} =_N \{^-a,\ a\}\tag{89}$$

$$]^{-}a,\ ^{-}a^{+}[_{MB} =_N \left\{ ^{-}a, ^{-}a^{+},\ a^{+} \right\}, \tag{90}$$

where $a \notin]^{-}a, ^{-}a^{+}[_{MB}$ since $a \nleq_N {}^{-}a^{+}$ (there is no relation of order between a and $^{-}a^{+}$);

$$]^{-}a^{+},\ a^{+}[_{MB} =_N \left\{ ^{-}a^{+},\ a^{+} \right\}. \tag{91}$$

Theorem 5.

$$\left( ]^{-}a,\ b^{+}[,\ \leq_N \right) \textit{ is a nonstandard real mobinad sublattice of first type (poset).} \tag{92}$$

Proof. Straightforward since $]^{-}a,\ b^{+}[$ is a sublattice of the lattice of first type NR_{MB}. □

Theorem 6.

$$\left( ]^{-}a,\ b^{+}[,\ \inf_N,\ \sup_N, ^{-}a,\ b^{+} \right) \textit{ is a nonstandard bounded real mobinad sublattice of second type (as algebraic structure).} \tag{93}$$

Proof. $]^{-}a,\ b^{+}[_{MB}$ as a nonstandard subset of NR_{MB} is also a poset, and for any two-element subset

$$\{\alpha,\ \beta\} \subset_N]^{-}0,\ 1^{+}[_{MB} \tag{94}$$

one obviously has the triple neutrosophic nonstandard inequality:

$$^{-}a \leq_N \inf_N\{\alpha,\ \beta\} \leq_N \sup_N\{\alpha,\ \beta\} \leq_N b^{+} \tag{95}$$

hence $( ]^{-}a,\ b^{+}[_{MB} \leq_N)$ is a nonstandard real mobinad sublattice of first type (poset), or sublattice of NR_{MB}.

Further on, $]^{-}a,\ b^{+}[$, endowed with two binary operations $\inf_N$ and $\sup_N$, is also a sublattice of the lattice NR_{MB}, since the lattice axioms (Commutative Laws, Associative Laws, Absorption Laws, and Idempotent Laws) are clearly verified on $]^{-}a,\ b^{+}[$.

The nonstandard neutrosophic modinad Identity Join Element (Bottom) is ^{-}a, and the nonstandard neutrosophic modinad Identity Meet Element (Top) is b^{+}, or

$$\inf_N]^{-}a,\ b^{+}[=_N {}^{-} \text{ and } \sup_N]^{-}a,\ b^{+}[=_N b^{+}. \tag{96}$$

The sublattice Identity Laws are verified below.

$$\text{Let } \alpha \in_N]^{-}a,\ b^{+}[, \text{ whence } ^{-}a \leq_N \alpha \leq_N b^{+}. \tag{97}$$

Then:

$$\inf_N\left\{\alpha,\ b^{+}\right\} =_N \alpha, \text{ and } \sup_N\{\alpha, ^{-}a\} =_N \alpha. \tag{98}$$

□

24. Definition of Nonstandard Real MoBiNad Unit Interval

$$\begin{gathered}]^{-}0,\ 1^{+}[_{MB} =_N \{x \in NR_{MB}, ^{-}0 \leq_N x \leq_N 1^{+}\} \\ =_N \left\{ \begin{array}{c} \varepsilon, a, \overset{-}{a}, \overset{-0}{a}, \overset{+}{a}, \overset{0+}{a}, \overset{-+}{a}, \overset{-0+}{a} \,\middle|\, \textit{where } \varepsilon \textit{ are infinitesimals}, \\ \varepsilon \in \mathbb{R}^{*}, \textit{ with } \varepsilon > 0, \textit{ and } a \in [0,\ 1] \end{array} \right\} \end{gathered} \tag{99}$$

This is an extension of the previous definition (1998) [5] of nonstandard unit interval

$$]^{-}0,\ 1^{+}[=_N ({}^{-}0) \cup [0,\ 1] \cup \left(1^{+}\right) \tag{100}$$

Associated to the first published definitions of neutrosophic set, logic, and probability was used. One has

$$]^{-}0,\ 1^{+}[\subset_N\ ]^{-}0,\ 1^{+}[_{MB} \tag{101}$$

where the index $_{MB}$ means: all monads and binads included in $]^{-}0,\ 1^{+}[$, for example,

$$({}^{-}0.2),\ ({}^{-}0.3^{0}),\ (0.5^{+}),\ ({}^{-}0.7^{+}),\ ({}^{-}0.8^{0+})\ \text{etc.} \tag{102}$$

or, using the top diacritics notation, respectively,

$$\overset{-}{0.2}, \overset{-0}{0.3}, \overset{+}{0.5}, \overset{-+}{0.7}, \overset{-0+}{0.8}\ \text{etc.} \tag{103}$$

Theorem 7. *The Nonstandard Real MoBiNad Unit Interval $]^{-}0,\ 1^{+}[_{MB}$ is a partially ordered set (poset) with respect to $\leq_N$, and any of its two elements have an inf_N and sup_N hence $]^{-}0,\ 1^{+}[_{MB}$ is a nonstandard neutrosophic lattice of first type (as poset).*

Proof. Straightforward. □

Theorem 8. *The Nonstandard Real MoBiNad Unit Interval $]^{-}0,\ 1^{+}[_{MB}$, endowed with two binary operations inf_N and sup_N, is also a nonstandard neutrosophic lattice of second type (as an algebraic structure).*

Proof. Replace $a = 0$ and $b = 1$ into the general nonstandard real mobinad interval $]^{-}a,\ b^{+}[$. □

25. Definition of Extended General Neutrosophic Logic

We extend and present in a clearer way our 1995 definition (published in 1998) of neutrosophic logic.

Let $\mathcal{U}$ be a universe of discourse of propositions and $P \in \mathcal{U}$ be a generic proposition.

A General Neutrosophic Logic is a multivalued logic in which each proposition P has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsehood (F), and where T, I, and F are standard real subsets or nonstandard real mobinad subsets of the nonstandard real mobinat unit interval $]^{-}0,\ 1^{+}[_{MB}$,

With

$$T,\ I,\ F \subseteq_N]^{-}0,\ 1^{+}[_{MB} \tag{104}$$

where

$${}^{-}0 \leq_N \inf_N T + \inf_N I + \inf_N F \leq_N \sup_N T + \sup_N I + \sup_N F \leq 3^{+}. \tag{105}$$

26. Definition of Standard Neutrosophic Logic

If in the above definition of general neutrosophic logic all neutrosophic components, T, I, and F are standard real subsets, included in or equal to the standard real unit interval, $T,\ I,\ F \subseteq [0,\ 1]$, where

$$0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3 \tag{106}$$

we have a standard neutrosophic logic.

27. Definition of Extended Nonstandard Neutrosophic Logic

If in the above definition of general neutrosophic logic at least one of the neutrosophic components T, I, or F is a nonstandard real mobinad subset, neutrosophically included in or equal to the nonstandard real mobinad unit interval $]^{-}0,\ 1^{+}[_{MB}$, where

$$^{-}0 \leq_N \inf_N T + \inf_N I + \inf_N F \leq_N \sup_N T + \sup_N I + \sup_N F \leq 3^{+}, \tag{107}$$

we have an extended nonstandard neutrosophic logic.

Theorem 9. *If M is a standard real set, $M \subset \mathbb{R}$, then*

$$inf_N(M) = inf(M) \text{ and } sup_N(M) = sup(M). \tag{108}$$

Proof. The neutrosophic infimum and supremum coincide with the classical infimum and supremum since there is no indeterminacy on the set M, meaning M contains no nonstandard numbers. □

28. Definition of Extended General Neutrosophic Set

We extend and present in a clearer way our 1995 definition of neutrosophic set.

Let $\mathcal{U}$ be a universe of discourse of elements and $S \in \mathcal{U}$ a subset.

A Neutrosophic Set is a set such that each element x from S has a degree of membership (T), a degree of indeterminacy (I), and a degree of nonmembership (F), where T, I, and F are standard real subsets or nonstandard real mobinad subsets, neutrosophically included in or equal to the nonstandard real mobinat unit interval

$$]^{-}0,\ 1^{+}[_{MB}, \text{with } T,\ I,\ F \subseteq_N]^{-}0,\ 1^{+}[_{MB}, \tag{109}$$

where

$$^{-}0 \leq_N \inf_N T + \inf_N I + \inf_N F \leq_N \sup_N T + \sup_N I + \sup_N F \leq 3^{+}. \tag{110}$$

29. Definition of Standard Neutrosophic Set

If in the above general definition of neutrosophic set all neutrosophic components, T, I, and F, are standard real subsets included in or equal to the classical real unit interval, $T,\ I,\ F \subseteq [0,\ 1]$, where

$$0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3, \tag{111}$$

we have a standard neutrosophic set.

30. Definition of Extended Nonstandard Neutrosophic Set

If in the above general definition of neutrosophic set at least one of the neutrosophic components T, I, or F is a nonstandard real mobinad subsets, neutrosophically included in or equal to $]^{-}0,\ 1^{+}[_{MB}$, where

$$^{-}0 \leq_N \inf_N T + \inf_N I + \inf_N F \leq_N \sup_N T + \sup_N I + \sup_N F \leq 3^{+}, \tag{112}$$

we have a nonstandard neutrosophic set.

31. Definition of Extended General Neutrosophic Probability

We extend and present in a clearer way our 1995 definition of neutrosophic probability.

Let $\mathcal{U}$ be a universe of discourse of events, and $E \in \mathcal{U}$ be an event.

A Neutrosophic Probability is a multivalued probability such that each event E has a chance of occuring (T), an indeterminate (unclear) chance of occuring or not occuring (I), and a chance of not occuring (F), and where T, I, and F are standard or nonstandard real mobinad subsets, neutrosophically included in or equal to the nonstandard real mobinat unit interval

$$]^{-}0,\ 1^{+}[_{MB},\ T,\ I,\ F \subseteq_N]^{-}0,\ 1^{+}[_{MB},\ \text{where } {}^{-}0 \leq_N \inf_N T + \inf_N I + \inf_N F \leq_N \sup_N T + \sup_N I + \sup_N F \leq 3^{+}. \tag{113}$$

32. Definition of Standard Neutrosophic Probability

If in the above general definition of neutrosophic probability all neutrosophic components, T, I, and F are standard real subsets, included in or equal to the standard unit interval $T,\ I,\ F \subseteq [0,\ 1]$, where

$$0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3, \tag{114}$$

we have a standard neutrosophic probability.

33. Definition of Extended Nonstandard Neutrosophic Probability

If in the above general definition of neutrosophic probability at least one of the neutrosophic components T, I, F is a nonstandard real mobinad subsets, neutrosophically included in or equal to $]^{-}0,\ 1^{+}[_{MB}$, where

$${}^{-}0 \leq_N \inf_N T + \inf_N I + \inf_N F \leq_N \sup_N T + \sup_N I + \sup_N F \leq 3^{+}, \tag{115}$$

we have a nonstandard neutrosophic probability.

34. Classical Operations with Real Sets

Let $A,\ B \subseteq \mathbb{R}$ be two real subsets. Let $\circledast$ and * denote any of the real subset classical operations and real number classical operations respectively: addition (+), subtraction (−), multiplication (×), division (÷), and power (^).

Then,

$$A \circledast B = \{a * b,\ \text{where } a \in A \text{ and } b \in B\} \tag{116}$$

Thus

$$A \oplus B = \{a + b | a \in A, b \in B\} \tag{117}$$

$$A \ominus B = \{a - b | a \in A, b \in B\} \tag{118}$$

$$A \otimes B = \{a \times b | a \in A, b \in B\} \tag{119}$$

$$A \oslash B = \{a \div b | a \in A; b \in B, b \neq 0\} \tag{120}$$

$$A^B = \left\{a^b \middle| a \in A, a > 0; b \in B\right\} \tag{121}$$

For the division (÷), of course, we consider $b \neq 0$. While for the power (^), we consider $a > 0$.

35. Operations on the Nonstandard Real MoBiNad Set (NR_{MB})

For all nonstandard (addition, subtraction, multiplication, division, and power) operations

$$\alpha, \beta \in_N NR_{MB}, \alpha *_N \beta =_N \mu_N(\alpha) \circledast \mu_N(\beta) \tag{122}$$

where $*_N$ is any neutrosophic arithmetic operations with neutrosophic numbers ($+_N$, $-_N$, $\times_N$, $\div_N$, $\hat{}_N$), while the corresponding $\circledast$ is an arithmetic operation with real subsets.

So, we approximate the nonstandard operations by standard operations of real subsets.

We sink the nonstandard neutrosophic real mobinad operations into the standard real subset operations, then we resurface the last ones back to the nonstandard neutrosophic real mobinad set.

Let ε_1 and ε_2 be two non-null positive infinitesimals. We present below some particular cases, all others should be deduced analogously.

Nonstandard Addition

First Method

$$({}^{-}a) + ({}^{-}b) =_N (a - \varepsilon_1,\, a) + (b - \varepsilon_2,\, b) =_N (a + b - \varepsilon_1 - \varepsilon_2,\, a + b) =_N (a + b - \varepsilon,\, a + b) =_N {}^{-}(a + b), \quad (123)$$

where we denoted $\varepsilon_1 + \varepsilon_2 = \varepsilon$ (the addition of two infinitesimals is also an infinitesimal).

Second Method

$$({}^{-}a) + ({}^{-}b) =_N (a - \varepsilon_1) + (b - \varepsilon_2) =_N (a + b - \varepsilon_1 - \varepsilon_2) =_N {}^{-}(a + b) \quad (124)$$

Adding two left monads, one also gets a left monad.

Nonstandard Subtraction

First Method

$$\begin{aligned}({}^{-}a) - ({}^{-}b) =_N \;& (a - \varepsilon_1, a) \\ & -(b - \varepsilon_2, b) =_N (a - \varepsilon_1 - b,\, a - b + \varepsilon_2) =_N (a - b - \varepsilon_1,\, a - b \\ & +\varepsilon_2) =_N \begin{pmatrix} - & 0 & + \\ & a - b & \end{pmatrix}\end{aligned} \quad (125)$$

Second Method

$$({}^{-}a) - ({}^{-}b) =_N (a - \varepsilon_1) - (b - \varepsilon_2) =_N a - b - \varepsilon_1 + \varepsilon_2, \quad (126)$$

since ε_1 and ε_2 may be any positive infinitesimals,

$$=_N \begin{cases} -(a - b), \text{ when } \varepsilon_1 > \varepsilon_2; \\ \begin{pmatrix} 0 \\ a - b \end{pmatrix}, \text{ when } \varepsilon_1 = \varepsilon_2 \\ (a - b)^{+}, \text{ when } \varepsilon_1 < \varepsilon_2. \end{cases} =_N \begin{pmatrix} 0 \\ a - b \end{pmatrix} =_N a - b; \quad (127)$$

Subtracting two left monads, one obtains an unpierced binad (that is why the unpierced binad had to be introduced).

Nonstandard Division

Let $a, b > 0$.

$$({}^{-}a) \div ({}^{-}b) =_N (a - \varepsilon_1,\, a) \div (b - \varepsilon_2,\, b) =_N \left(\frac{a - \varepsilon_1}{b}, \frac{a}{b - \varepsilon_2}\right) \quad (128)$$

Since

$$\varepsilon_1 > 0 \text{ and } \varepsilon_2 > 0,\ \frac{a - \varepsilon_1}{b} < \frac{a}{b} \text{ and } \frac{a}{b - \varepsilon_2} > \frac{a}{b}, \quad (129)$$

while between $\frac{a-\varepsilon_1}{b}$ and $\frac{a}{b-\varepsilon_2}$ there is a continuum whence there are some infinitesimals ε_1^0 and ε_2^0 such that $\frac{a-\varepsilon_1^0}{b-\varepsilon_2^0} = \frac{a}{b}$, or $ab - b\varepsilon_1^0 = ab - a\varepsilon_2^0$, and for a given ε_1^0 there exists an

$$\varepsilon_2^0 = \varepsilon_1^0 \cdot \frac{b}{a} \quad (130)$$

Hence

$$\frac{({}^{-}a)}{({}^{-}b)} =_N \begin{pmatrix} - & 0 & + \\ & \frac{a}{b} & \end{pmatrix} \quad (131)$$

For a or/and b negative numbers, it is similar but it is needed to compute the inf_N and sup_N of the products of intervals.

Dividing two left monads, one obtains an unpierced binad.

Nonstandard Multiplication

Let $a,\ b \geq 0$.

$$\left({}^{-}a^{0}\right)\times\left({}^{-}b^{0\,+}\right) \begin{array}{l} =_N (a-\varepsilon_1,\ a] \\ \times(b-\varepsilon_2,\ b+\varepsilon_2) =_N ((a-\varepsilon_1)\cdot(b-\varepsilon_2), a\cdot(b+\varepsilon_2)) =_N \left({}^{-}ab^{0\,+}\right) \end{array} \tag{132}$$

Since

$$(a-\varepsilon_1)\cdot(b-\varepsilon_2) < a\cdot b \text{ and } a\cdot(b+\varepsilon_2) > a\cdot b. \tag{133}$$

For a or/and b negative numbers, it is similar but it is needed to compute the inf_N and sup_N of the products of intervals.

Multiplying a positive left monad closed to the right, with a positive unpierced binad, one obtains an unpierced binad.

Nonstandard Power

Let $a,\ b > 1$.

$$\left({}^{0}a^{+}\right)^{\left({}^{-}b^{0}\right)} =_N [a,\ a+\varepsilon_1)^{(b-\varepsilon_2,\ b]} =_N \left(a^{b-\varepsilon_2},\ (a+\varepsilon_1)^b\right) =_N \begin{pmatrix} - & 0 & + \\ & a^b & \end{pmatrix} \tag{134}$$

$$\text{since } a^{b-\varepsilon_1} < a^b \text{ and } (a+\varepsilon_1)^b > a^b. \tag{135}$$

Raising a right monad closed to the left to a power equal to a left monad closed to the right, for both monads above 1, the result is an unpierced binad.

Consequence

In general, when doing arithmetic operations on nonstandard real monads and binads, the result may be a different type of monad or binad.

That is why is was imperious to extend the monads to closed monads, and the pierced binad to unpierced binad, in order to have the whole nonstandard neutrosophic real mobinad set closed under arithmetic operations.

36. Conditions of Neutrosophic Nonstandard Inequalities

Let NR_{MB} be the Nonstandard Real MoBiNad. Let's endow (NR_{MB}, $<_N$) with a neutrosophic inequality.

Let $\alpha, \beta \in NR_{MB}$, where α, β may be real numbers, monads, or binads.

And let

$$\begin{array}{c} \binom{-}{a}, \binom{-0}{a}, \binom{+}{a}, \binom{0+}{a}, \binom{-+}{a}, \binom{-0+}{a} \in NR_{MB}, \text{ and} \\ \binom{-}{b}, \binom{-0}{b}, \binom{+}{b}, \binom{0+}{b}, \binom{-+}{b}, \binom{-0+}{b} \in NR_{MB}, \end{array} \tag{136}$$

be the left monads, left monads closed to the right, right monads, right monads closed to the left, and binads, and binads nor pierced of the elements (standard real numbers) a and b, respectively. Since all monads and binads are real subsets, we may treat the single real numbers

$$a = [a, a] \text{ and } b = [b, b] \text{ as real subsets too} \tag{137}$$

as real subsets too.

NR_{MB} is a set of subsets, and thus we deal with neutrosophic inequalities between subsets.

(i) If the subset α has many of its elements above all elements of the subset β,
(ii) then $\alpha >_N \beta$ (partially).
(iii) If the subset α has many of its elements below all elements of the subset β,
(iv) then $\alpha <_N \beta$ (partially).

(v) If the subset α has many of its elements equal with elements of the subset β,

(vi) then $\alpha =_{\mathrm{N}} \beta$ (partially).

If the subset α verifies (*i*) and (*iii*) with respect to subset β, then $\alpha \geq_{\mathrm{N}} \beta$.

If the subset α verifies (*ii*) and (*iii*) with respect to subset β, then $\alpha \leq_{\mathrm{N}} \beta$.

If the subset α verifies (*i*) and (*ii*) with respect to subset β, then there is no neutrosophic order (inequality) between α and β.

For example, between a and $({}^{-}a^{+})$ there is no neutrosophic order, similarly between a and $\overset{-0+}{a}$.

Similarly, if the subset α verifies (*i*), (*ii*) and (*iii*) with respect to subset β, then there is no neutrosophic order (inequality) between α and β.

37. Open Neutrosophic Research

The quantity or measure of "many of its elements" of the above (*i*), (*ii*), or (*iii*) conditions depends on each *neutrosophic application* and on its *neutrosophic experts*.

An approach would be to employ the *Neutrosophic Measure* [23,24], that handles indeterminacy, which may be adjusted and used in these cases.

In general, we do not try in purpose to validate or invalidate an existing scientific result, but to investigate how an existing scientific result behaves in a new environment (that may contain indeterminacy), or in a new application, or in a new interpretation.

38. Nonstandard Neutrosophic Inequalities

For the neutrosophic nonstandard inequalities, we propose, based on the previous six neutrosophic equalities, the following.

$$({}^{-}\mathrm{a}) <_N, \mathrm{a} <_N (a^{+}) \tag{138}$$

Since the standard real interval $(a - \varepsilon, a)$ is below a, and a is below the standard real interval $(a, a + \varepsilon)$ by using the approximation provided by the nonstandard neutrosophic function μ, or because

$$\forall x \in R_{+}^{*}, a - x < a < a + x \tag{139}$$

where x is of course a (nonzero) positive infinitesimal (the above double neutrosophic inequality actually becomes a double classical standard real inequality for each fixed positive infinitesimal).

The converse double neutrosophic inequality is also neutrosophically true:

$$(a^{+}) >_N, \mathrm{a} >_N ({}^{-}a) \tag{140}$$

Another nonstandard neutrosophic double inequality:

$$({}^{-}\mathrm{a}) \leq_N ({}^{-}a^{+}) \leq_N (a^{+}) \tag{141}$$

This double neutrosophic inequality may be justified since $({}^{-}a^{+}) = ({}^{-}a) \cup (a^{+})$ and, geometrically, on the Real Number Line, the number a is in between the subsets ${}^{-}a = (a - \varepsilon, a)$ and $a^{+} = (a, a + \varepsilon)$, so

$$({}^{-}a) \leq_N ({}^{-}a) \cup (a^{+}) \leq_N (a^{+}) \tag{142}$$

Hence the left side of the inequality's middle term coincides with the inequality first term, while the right side of the inequality middle term coincides with the third inequality term.

Conversely, it is neutrosophically true as well:

$$(a^{+}) \geq_N ({}^{-}a) \cup (a^{+}) \geq_N ({}^{-}a) \tag{143}$$

Also,

$$\overset{-}{a} \leq_N \overset{-0}{a} \leq_N a \leq_N \overset{0+}{a} \leq_N \overset{+}{a} \text{ and } \overset{-}{a} \leq_N \overset{-+}{a} \leq_N \overset{-0+}{a} \leq_N \overset{+}{a} \tag{144}$$

Conversely, they are also neutrosophically true:

$$\overset{+}{a} \geq_N \overset{0+}{a} \geq_N a \geq_N \overset{-0}{a} \geq_N \overset{-}{a} \text{ and } \overset{+}{a} \geq_N \overset{-0+}{a} \geq_N \overset{-+}{a} \geq_N \overset{-}{a} \text{ respectively.} \tag{145}$$

If $a > b$, which is a (standard) classical real inequality, then we have the following neutrosophic nonstandard inequalities.

$$a >_N ({}^{-}b), a >_N \left(b^{+}\right), a >_N \left({}^{-}b^{+}\right), a >_N \overset{-0}{b}, a >_N \overset{0+}{b}, a >_N \overset{-0+}{b}; \tag{146}$$

$$({}^{-}a) >_N b, ({}^{-}a) >_N ({}^{-}b), ({}^{-}a) >_N \left(b^{+}\right), ({}^{-}a) >_N \left({}^{-}b^{+}\right), \overset{-}{a} >_N \overset{-0}{b}, \overset{-}{a} >_N \overset{0+}{b}, \overset{-}{a} >_N \overset{-0+}{b}; \tag{147}$$

$$\left(a^{+}\right) >_N b, \left(a^{+}\right) >_N ({}^{-}b), \left(a^{+}\right) >_N \left(b^{+}\right), \left(a^{+}\right) >_N \left({}^{-}b^{+}\right), \overset{+}{a} >_N \overset{-0}{b}, \overset{+}{a} >_N \overset{0+}{b}, \overset{+}{a} >_N \overset{-0+}{b}; \tag{148}$$

$$({}^{-}a^{+}) >_N b, ({}^{-}a^{+}) >_N ({}^{-}b), ({}^{-}a^{+}) >_N (b^{+}), ({}^{-}a^{+}) >_N ({}^{-}b^{+}), \text{ etc.} \tag{149}$$

No Ordering Relationships

For any standard real number a, there is no relationship of order between the elements a and $({}^{-}a^{+})$, or between the elements a and

$$\left(\overset{-0+}{a}\right) \tag{150}$$

Therefore, $\mathrm{NR_{MB}}$ is a neutrosophically partially order set.

$$\text{If one removes all binads from } \mathrm{NR_{MB}}\text{, then } (\mathrm{NR_{MB}}, \leq_N) \text{ is neutrosophically totally ordered.} \tag{151}$$

Theorem 10. *Using the nonstandard general notation one has:*

If $a > b$, which is a (standard) classical real inequality, then

$$\overset{m_1}{a} >_N \overset{m_2}{b} \text{ for any } m_1,\ m_2 \in \left\{ ,\ ^{-},\ ^{-0},\ ^{+},\ ^{+0},\ ^{-0},\ ^{-0+} \right\}. \tag{152}$$

Conversely, if $a < b$, which is a (standard) classical real inequality, then

$$\overset{m_1}{a} <_N \overset{m_2}{b} \text{ for any } m_1,\ m_2 \in \left\{ ,\ ^{-},\ ^{-0},\ ^{+},\ ^{+0},\ ^{-0},\ ^{-0+} \right\}. \tag{153}$$

39. Nonstandard Neutrosophic Equalities

Let a, b be standard real numbers; if $a = b$ that is a (classical) standard equality, then

$$({}^{-}a) =_N ({}^{-}b),\ (a^{+}) =_N (b^{+}),\ ({}^{-}a^{+}) =_N ({}^{-}b^{+}), \tag{154}$$

$$\left(\overset{-0}{a}\right) =_N \left(\overset{-0}{b}\right), \left(\overset{0+}{a}\right) =_N \left(\overset{0+}{b}\right), \left(\overset{-0+}{a}\right) =_N \left(\overset{-0+}{b}\right) \tag{155}$$

40. Nonstandard Neutrosophic Belongingness

On the nonstandard real set NR_{MB}, we say that

$$\overset{m}{c} \in_N \left] \overset{m_1}{a}, \overset{m_2}{b} \right[\text{ iff } \overset{m_1}{a} \leq_N \overset{m}{c} \leq_N \overset{m_2}{b}, \tag{156}$$

where

$$m_1, m_2, m \in \{, ^-, ^{-0}, ^+, ^{+0}, ^{-+}, ^{-0+}\}. \tag{157}$$

We use the previous nonstandard neutrosophic inequalities.

41. Nonstandard Hesitant Sets

Nonstandard Hesitant sets are sets of the form:

$$A = \{a_1,\ a_2,\ \ldots, a_n\},\ 2 \le n < \infty,\ A \subset_N NR_{MB}, \tag{158}$$

where at least one element $a_{i_0}, 1 \le i_0 \le n$, is an infinitesimal, a monad, or a binad (of any type); while other elements may be standard real numbers, infinitesimals, or also monads or binads (of any type).

If the neutrosophic components T, I, and F are nonstandard hesitant sets, then one has a Nonstandard Hesitant Neutrosophic Logic/Set/Probability.

42. Nonstandard Neutrosophic Strict Interval Inclusion

On the nonstandard real set NR_{MB},

$$]\overset{m_1}{a}, \overset{m_2}{b}[\subset_N]\overset{m_3}{c}, \overset{m_4}{d}[\tag{159}$$

iff

$$\overset{m_3}{c} \le_N \overset{m_1}{a} <_N \overset{m_2}{b} <_N \overset{m_4}{d} \text{ or } \overset{m_3}{c} <_N \overset{m_1}{a} <_N \overset{m_2}{b} \le_N \overset{m_4}{d} \text{ or } \overset{m_3}{c} <_N \overset{m_1}{a} <_N \overset{m_2}{b} <_N \overset{m_4}{d} \tag{160}$$

43. Nonstandard Neutrosophic (Nonstrict) Interval Inclusion

On the nonstandard real set NR_{MB},

$$]\overset{m_1}{a}, \overset{m_2}{b}[\subseteq_N]\overset{m_3}{c}, \overset{m_4}{d}[\text{ iff} \tag{161}$$

$$\overset{m_3}{c} \le_N \overset{m_1}{a} <_N \overset{m_2}{b} \le_N \overset{m_4}{d}. \tag{162}$$

44. Nonstandard Neutrosophic Strict Set Inclusion

The nonstandard set A is neutrosophically strictly included in the nonstandard set B, $A \subset_N B$, if:

$$\forall x \in_N A, x \in_N B, \text{ and } \exists y \in_N B : y \notin_N A. \tag{163}$$

45. Nonstandard Neutrosophic (Nonstrict) Set Inclusion

The nonstandard set A is neutrosophically not strictly included in the nonstandard set B,

$$A \subseteq_N B, \text{ iff:} \tag{164}$$

$$\forall x \in_N A, x \in_N B. \tag{165}$$

46. Nonstandard Neutrosophic Set Equality

The nonstandard sets A and B are neutrosophically equal,

$$A =_N B, \text{ iff:} \tag{166}$$

$$A \subseteq_N B \text{ and } B \subseteq_N A. \tag{167}$$

47. The Fuzzy, Neutrosophic, and Plithogenic Logical Connectives $\wedge$, $\vee$, $\rightarrow$

All fuzzy, intuitionistic fuzzy, and neutrosophic logic operators are *inferential approximations*, not written in stone. They are improved from application to application.

Let's denote:

$$\wedge_F,\ \wedge_N,\ \wedge_P \text{ representing respectively the fuzzy conjunction, neutrosophic conjunction, and plithogenic conjunction;} \tag{168}$$

similarly

$$\vee_F,\ \vee_N,\ \vee_P \text{ representing respectively the fuzzy disjunction, neutrosophic disjunction, and plithogenic disjunction,} \tag{169}$$

and

$$\rightarrow_F,\ \rightarrow_N,\ \rightarrow_P \text{ representing respectively the fuzzy implication, neutrosophic implication, and plithogenic implication.} \tag{170}$$

I agree that my beginning neutrosophic operators (when I applied the same *fuzzy t-norm*, or the same *fuzzy t-conorm*, to all neutrosophic components T, I, F) were less accurate than others developed later by the neutrosophic community researchers. This was pointed out in 2002 by Ashbacher [25] and confirmed in 2008 by Rivieccio [26]. They observed that if on T_1 and T_2 one applies a *fuzzy t-norm*, for their opposites F_1 and F_2, one needs to apply the *fuzzy t-conorm* (the opposite of fuzzy t-norm), and reciprocally.

About inferring I_1 and I_2, some researchers combined them in the same directions as T_1 and T_2.

Then,

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \wedge_F I_2, F_1 \vee_F F_2), \tag{171}$$

$$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \vee_F I_2, F_1 \wedge_F F_2), \tag{172}$$

$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) = (F_1 \vee_F T_2, I_1 \vee_F I_2, T_1 \wedge_F F_2). \tag{173}$$

others combined I_1 and I_2 in the same direction as F_1 and F_2 (since both I and F are negatively qualitative neutrosophic components, while F is qualitatively positive neutrosophic component), the most used one is as follows.

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \vee_F I_2, F_1 \vee_F F_2), \tag{174}$$

$$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \wedge_F I_2, F_1 \wedge_F F_2), \tag{175}$$

$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) = (F_1 \vee_F T_2, I_1 \wedge_F I_2, T_1 \wedge_F F_2). \tag{176}$$

Even more, recently, in an extension of neutrosophic set to *plithogenic set* [27] (which is a set whose each element is characterized by many attribute values), the *degrees of contradiction* $c(,)$ between the neutrosophic components T, I, and F have been defined (in order to facilitate the design of the aggregation operators), as follows:

$$c(T, F) = 1 \text{ (or 100\%, because they are totally opposite)}, \ c(T, I) = c(F, I) = 0.5 \text{ (or 50\%, because they are only half opposite).} \tag{177}$$

Then,

$$(T_1, I_1, F_1) \wedge_P (T_2, I_2, F_2) = (T_1 \wedge_F T_2, 0.5(I_1 \wedge_F I_2) + 0.5(I_1 \vee_F I_2), F_1 \vee_F F_2), \tag{178}$$

$$(T_1, I_1, F_1) \vee_P (T_2, I_2, F_2) = (T_1 \vee_F T_2, 0.5(I_1 \vee_F I_2) + 0.5(I_1 \wedge_F I_2), F_1 \wedge_F F_2), \tag{179}$$

$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) = (F_1 \vee_F T_2, 0.5(I_1 \vee_F I_2) + 0.5(I_1 \wedge_F I_2), T_1 \wedge_F F_2). \tag{180}$$

48. Fuzzy t-norms and Fuzzy t-conorms

The most used $\wedge_F$ (Fuzzy t-norms), and $\vee_F$ (Fuzzy t-conorms) are as follows.
Let

$$a, b \in [0, 1]. \tag{181}$$

Fuzzy t-norms (fuzzy conjunctions, or fuzzy intersections):

$$a \wedge_F b = \min\{a, b\}; \tag{182}$$

$$a \wedge_F b = ab; \tag{183}$$

$$a \wedge_F b = \max\{a + b - 1, 0\}. \tag{184}$$

Fuzzy t-conorms (fuzzy disjunctions, or fuzzy unions):

$$a \vee_F b = \max\{a, b\}; \tag{185}$$

$$a \vee_F b = a + b - ab; \tag{186}$$

$$a \vee_F b = \min\{a + b, 1\} \tag{187}$$

49. Nonstandard Neutrosophic Operators

Nonstandard Neutrosophic Conjunctions

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \vee_F I_2, F_1 \vee_F F_2) = (inf_N(T_1, T_2), sup_N(I_1, I_2), sup_N(F_1, F_2)) \tag{188}$$

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \vee_F I_2, F_1 \vee_F F_2) = (T_1 \times_N T_2, I_1 +_N I_2 -_N I_1 \times_N I_2, F_1 +_N F_2 -_N F_1 \times_N F_2) \tag{189}$$

Nonstandard Neutrosophic Disjunctions

$$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \wedge_F I_2, F_1 \wedge_F F_2) = (sup_N(T_1, T_2), inf_N(I_1, I_2), inf_N(F_1, F_2)) \tag{190}$$

$$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \wedge_F I_2, F_1 \wedge_F F_2) = (T_1 +_N T_2 -_N T_1 \times_N T_2, I_1 \times_N I_2, F_1 \times_N F_2) \tag{191}$$

Nonstandard Neutrosophic Negations

$$\neg(T_1, I_1, F_1) = (F_1, I_1, T_1) \tag{192}$$

$$\neg(T_1, I_1, F_1) = (F_1, (1^+) -_N I_1, T_1) \tag{193}$$

Nonstandard Neutrosophic Implications

$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) = (F_1 \vee_F T_2, I_1 \wedge_F I_2, T_1 \wedge_F F_2) = (F_1 +_N T_2 -_N F_1 \times_N T_2, I_1 \times_N I_2, T_1 \times_N F_2) \tag{194}$$

$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, (1^+) -_N I_1, T_1) \vee_N (T_2, I_2, F_2) = (F_1 \vee_F T_2, ((1^+) -_N I_1) \wedge_F I_2, T_1 \wedge_F F_2) = (F_1 +_N T_2 -_N F_1 \times_N T_2, ((1^+) -_N I_1) \times_N I_2, T_1 \times_N F_2) \tag{195}$$

Let $P_1(T_1, I_1, F_1)$ and $P_2(T_2, I_2, F_2)$ be two nonstandard neutrosophic logical propositions, whose nonstandard neutrosophic components are, respectively,

$$T_1, I_1, F_1, T_2, I_2, F_2 \in_N NR_{MB}. \tag{196}$$

50. Numerical Examples of Nonstandard Neutrosophic Operators

Let us take a particular numeric example, where

$$P_1 =_N (\overset{0+}{0.3}, \overset{-+}{0.2}, 0.4), P_2 =_N (\overset{-0}{0.6}, \overset{-0+}{0.1}, \overset{+}{0.5}) \tag{197}$$

are two nonstandard neutrosophic logical propositions.

We use the nonstandard arithmetic operations previously defined *Numerical Example of Nonstandard Neutrosophic Conjunction*

$$\overset{0+}{0.3} \times \overset{-0}{0.6} =_N [0.3, 0.3+\varepsilon_1) \times (0.6-\varepsilon_2, 0.6) = (0.18-0.3\varepsilon_2, 0.18+0.6\varepsilon_1) =_N \overset{-0+}{0.18} \tag{198}$$

$$\begin{aligned}
&\overset{-+}{0.2} +_N \overset{-0+}{0.1} -_N \overset{-+}{0.2} \times_N \overset{-0+}{0.1} =_N [(0.2-\varepsilon_1, 0.2) \cup (0.2, 0.2+\varepsilon_1)] + (0.1-\varepsilon_2, 0.1+\varepsilon_2) \\
&-[(0.2-\varepsilon_1, 0.2) \cup (0.2, 0.2+\varepsilon_1)] \times (0.1-\varepsilon_2, 0.1+\varepsilon_2) \\
&= [(0.3-\varepsilon_1-\varepsilon_2, 0.3+\varepsilon_2) \cup (0.3-\varepsilon_2, 0.3+\varepsilon_1+\varepsilon_2)] \\
&-[(0.2-\varepsilon_1) \times (0.1-\varepsilon_2), (0.02+0.2\varepsilon_2)] \cup [(0.02-0.2\varepsilon_2), (0.2+\varepsilon_1) \times (0.1+\varepsilon_2)] \\
&= [\overset{-0+}{0.3} \cup \overset{-0+}{0.3}] - [\overset{-0+}{0.02} \cup \overset{-0+}{0.02}] = [\overset{-0+}{0.3}] - [\overset{-0+}{0.02}] = 0.3 \overset{-0+}{-} 0.02 =_N \overset{-0+}{0.28}
\end{aligned} \tag{199}$$

$$\begin{aligned}
&0.4 +_N \overset{+}{0.5} =_N [0.4, 0.4] + (0.5, 0.5+\varepsilon_1) - [0.4, 0.4] \times (0.5, 0.5+\varepsilon_1) \\
&= (0.4+0.5, 0.4+0.5+\varepsilon_1) - (0.4 \times 0.5, 0.4 \times 0.5 + 0.4\varepsilon_1) \\
&= (0.9, 0.9+\varepsilon_1) - (0.2, 0.2+0.4\varepsilon_1) \\
&= (0.9-0.2-0.4\varepsilon_1, 0.9+\varepsilon_1-0.2) = (0.7-0.4\varepsilon_1, 0.7+\varepsilon_1) =_N \overset{-0+}{0.70}
\end{aligned} \tag{200}$$

Hence

$$P_1 \wedge P_2 =_N (\overset{-0+}{0.18}, \overset{-0+}{0.28}, \overset{-0+}{0.70}) \tag{201}$$

Numerical Example of Nonstandard Neutrosophic Disjunction

$$\begin{aligned}
&\overset{0+}{0.3} +_N \overset{-0}{0.6} - \overset{0+}{0.3} \times_N \overset{-0}{0.6} =_N \{[0.3, 0.3+\varepsilon_1) + (0.6-\varepsilon_1, 0.6]\} - \{[0.3, 0.3+\varepsilon_1) \times (0.6-\varepsilon_1, 0.6]\} \\
&= (0.9-\varepsilon_1, 0.9+\varepsilon_1) - (0.18-0.3\varepsilon_1, 0.18+0.6\varepsilon_1) = (0.72-1.6\varepsilon_1, 0.72+1.3\varepsilon_1) =_N \overset{-0+}{0.72}
\end{aligned} \tag{202}$$

$$\overset{-+}{0.2} \times_N \overset{-0+}{0.1} =_N \left(0.2 \overset{-0+}{\times} 0.1\right) =_N \overset{-0+}{0.02} \tag{203}$$

$$0.4 \times_N \overset{+}{0.5} =_N \left(0.4 \overset{+}{\times} 0.5\right) =_N \overset{+}{0.20} \tag{204}$$

Hence

$$P_1 \vee_N P_2 =_N (\overset{-0+}{0.72}, \overset{-0+}{0.02}, \overset{+}{0.20}) \tag{205}$$

Numerical Example of Nonstandard Neutrosophic Negation

$$\neg_N P_1 =_N \neg_N (\overset{0+}{0.3}, \overset{-+}{0.2}, 0.4) =_N (0.4, \overset{-+}{0.2}, \overset{0+}{0.3}) \tag{206}$$

Numerical Example of Nonstandard Neutrosophic Implication

$$(P_1 \to {}_N P_2) \Leftrightarrow {}_N(\neg_N P_1 \vee_N P_2) =_N (0.4, \overset{-+}{0.2}, \overset{0+}{0.3}) \vee_N (\overset{-0}{0.6}, \overset{-0+}{0.1}, \overset{+}{0.5}) \tag{207}$$

Afterwards,

$$0.4 +_N \overset{-0}{0.6} - 0.4 \times_N \overset{-0}{0.6} =_N \left(0.4 \overset{-0}{+} 0.6\right) -_N \left(0.4 \overset{-0}{\times} 0.6\right) =_N \overset{-0}{1.0} -_N \overset{-0}{0.24} =_N \overset{-0+}{0.76} \tag{208}$$

$$\overset{-+}{0.2} \times_N \overset{-0+}{0.1} =_N \overset{-0+}{0.02} \tag{209}$$

$$\overset{0+}{0.3} \times \overset{+}{0.5} =_N \overset{+}{0.15} \tag{210}$$

whence

$$\neg_N P_1 =_N (\overset{-0+}{0.76}, \overset{-0+}{0.02}, \overset{+}{0.15}) \tag{211}$$

Therefore, we have showed above how to do nonstandard neutrosophic arithmetic operations on some concrete examples.

51. Conclusions

In the history of mathematics, critics on nonstandard analysis, in general, have been made by Paul Halmos, Errett Bishop, Alain Connes, and others.

That's why we have extended in 1998 for the first time the monads to pierced binad, and then in 2019 for the second time we extended the left monad to left monad closed to the right, the right monad to right monad closed to the left, and the pierced binad to unpierced binad. These were necessary in order to construct a general nonstandard neutrosophic real mobinad space, which is closed under the nonstandard neutrosophic arithmetic operations (such as addition, subtraction, multiplication, division, and power), which are needed in order to be able to define the nonstandard neutrosophic operators (such as conjunction, disjunction, negation, implication, and equivalence) on this space, and to transform the newly constructed nonstandard neutrosophic real mobinad space into a lattice of first order (as partially ordered nonstandard set, under the neutrosophic inequality $\leq_N$) and a lattice of second type (as algebraic structure, endowed with two binary laws: neutrosophic infimum (*infN*) and neutrosophic supremum (sup_N)).

As a consequence of extending the nonstandard analysis, we also extended the nonstandard neutrosophic logic, set, measure, probability and statistics.

As future research it would be to introduce the nonstandard neutrosophic measure, and to find applications of extended nonstandard neutrosophic logic, set, probability into calculus, since in calculus one deals with infinitesimals and their aggregation operators, due to the tremendous number of applications of the neutrosophic theories [28].

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