Generalized closed sets via neutrosophic topological spaces

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Abstract
In this paper, we have introduced the notion of generalized closed sets in Neutrosophic topological spaces and studied some of their basic properties.

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1. Introduction
In 1970, Levine [9] introduced the concept of generalized closed sets as a weaker form of closed sets in topological spaces. Zadeh [15] introduced the notion of fuzzy sets in the year 1965. In fuzzy set theory, the membership of an element to a fuzzy set is a single value between 0 and 1. The concept of fuzzy topological spaces have been introduced and developed by Chang [2]. In 1983, Atanassov [1] introduced the concept of intuitionistic fuzzy set which was generalization of fuzzy set. In intuitionistic fuzzy set theory, the elements have the degree membership and non-membership value between 0 and 1. Later, In 1997 Coker [4] introduced the concept of intuitionistic fuzzy topological spaces, by using the notion of the intuitionistic fuzzy set.

Florentin Smarandache [5] introduced the concept of Neutrosophic set. Neutrosophic set is classified into three independent functions namely, membership function, indeterminacy function and non membership function that are independently related. In 2012, Salama, Alblowi [11] introduced the concept of Neutrosophic topology. Neutrosophic topological spaces are very natural generalizations of fuzzy topological spaces, allow more general functions to be members of fuzzy topology.

In 2014, Salama, Smarandache and Valeri [10] introduced the concept of Neutrosophic closed sets and Neutrosophic continuous functions. Salama, Alblowi [11] introduced the concept of generalized Neutrosophic set and generalized Neutrosophic topological spaces. A generalized Neutrosophic set $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle : x \in X \}$ can be identified as an ordered triple $\langle \mu_A, \sigma_A, \gamma_A \rangle$ on $X$, where the triple function satisfy the condition $\mu_A(x) \cap \sigma_A(x) \cap \gamma_A(x) \leq 0.5$.


2. Preliminaries
In this section, we recollect some relevant basic preliminaries about Neutrosophic sets and its operations.

Definition 2.1. [10] Let $X$ be a non-empty fixed set. A Neutrosophic set [NS for short] $A$ is an object having the form $A = \{\langle x, \mu_A(x), \sigma_A(x), \gamma_A(x) \rangle : x \in X \}$ where $\mu_A(x), \sigma_A(x)$ and
Remark 2.2. [10] A Neutrosophic set
\( A = \{x, \mu_A(x), \sigma_A(x), \gamma_A(x) : x \in X\} \) can be identified to an ordered triple
\( A = (\mu_A(x), \sigma_A(x), \gamma_A(x)) \) in \([-1, 1]^3\) on \( X \).

Remark 2.3. [10] For the sake of simplicity, we shall use the symbol \( A = \{\mu_A, \sigma_A, \gamma_A\} \) for the neutrosophic set \( A = \{x, \mu_A(x), \sigma_A(x), \gamma_A(x) \in X \} \).

Since our main purpose is to construct the tools for developing Neutrosophic and Neutrosophic topology, we must introduce the Neutrosophic sets \( 0_N \) and \( 1_N \) in \( X \) as follows:

\[ 0_N \text{ may be defined as:} \]
\[ (0)_{0_N} = \{x, 0, 0, 0 : x \in X\} \]
\[ (02)_{0_N} = \{x, 0, 1, 1 : x \in X\} \]
\[ (03)_{0_N} = \{x, 0, 1, 0 : x \in X\} \]
\[ (04)_{0_N} = \{x, 0, 0, 0 : x \in X\} \]

\[ 1_N \text{ may be defined as:} \]
\[ (11)_{1_N} = \{x, 1, 0, 0 : x \in X\} \]
\[ (12)_{1_N} = \{x, 1, 1, 0 : x \in X\} \]
\[ (13)_{1_N} = \{x, 1, 1, 0 : x \in X\} \]
\[ (14)_{1_N} = \{x, 1, 1, 1 : x \in X\} \]

Definition 2.5. [10] Let \( A = \{\mu_A, \sigma_A, \gamma_A\} \) be a NS on \( X \), then the complement of the set \( A \) [\( C(A) \) for short] may be defined as three kinds of complements:

\( (C1) C(A) = \{x, 1 - \mu_A(x), 1 - \sigma_A(x), 1 - \gamma_A(x) : x \in X\} \)
\( (C2) C(A) = \{x, \gamma_A(x), \sigma_A(x), \mu_A(x) : x \in X\} \)
\( (C3) C(A) = \{x, 1 - \mu_A(x), 1 - \sigma_A(x), 1 - \gamma_A(x) : x \in X\} \)

Definition 2.6. [10] Let \( X \) be a non-empty set, and neutrosophic sets \( A \) and \( B \) in the form \( A = \{x, \mu_A(x), \sigma_A(x), \gamma_A(x) : x \in X\} \) and \( B = \{x, \mu_B(x), \sigma_B(x), \gamma_B(x) : x \in X\} \). Then we may consider two possible definitions for subsets \( (A \subseteq B) \):

\( (A \subseteq B) \iff \mu_A(x) \leq \mu_B(x), \sigma_A(x) \leq \sigma_B(x), \gamma_A(x) \geq \gamma_B(x) \forall x \in X \)
\( (A \subseteq B) \iff \mu_A(x) \leq \mu_B(x), \sigma_A(x) \leq \sigma_B(x), \gamma_A(x) \geq \gamma_B(x) \forall x \in X \)

Proposition 2.7. [10] For any Neutrosophic set \( A \), the following conditions holds:

\[ 0_N \subseteq A, 0_N \subseteq 0_N \]
\[ A \subseteq 1_N, 1_N \subseteq 1_N \]

Definition 2.8. [10] Let \( X \) be a non-empty set, and \( A = \{x, \mu_A(x), \sigma_A(x), \gamma_A(x) \in X\} \),
\( B = \{x, \mu_B(x), \sigma_B(x), \gamma_B(x) \in X\} \) are \( NS \). Then \( A \cap B \) may be defined as:
\( (I) A \cap B = \{x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \gamma_A(x) \vee \gamma_B(x)\} \)
\( (2) A \cap B = \{x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \gamma_A(x) \vee \gamma_B(x)\} \)
\( A \cup B \) may be defined as:
\( (U) A \cup B = \{x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(x), \gamma_A(x) \vee \gamma_B(x)\} \)
\( (V) A \cup B = \{x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \gamma_A(x) \vee \gamma_B(x)\} \)

We can easily generalize the operations of intersection and union in Definition 2.8 to arbitrary family of \( NS \) as follows:

Definition 2.9. [10] Let \( \{A_j : j \in J\} \) be a arbitrary family of \( NS \) in \( X \), then
\( \bigcap A_j \) may be defined as:
\( \bigcap A_j = \{x, \mu_{\bigwedge_j \mu_A_j}(x), \sigma_{\bigwedge_j \sigma_A_j}(x), \gamma_{\bigwedge_j \gamma_A_j}(x)\} \)
\( \bigcup A_j = \{x, \mu_{\bigvee_j \mu_A_j}(x), \sigma_{\bigvee_j \sigma_A_j}(x), \gamma_{\bigvee_j \gamma_A_j}(x)\} \)
\( \bigcup A_j = \{x, \mu_{\bigwedge_j \mu_A_j}(x), \sigma_{\bigwedge_j \sigma_A_j}(x), \gamma_{\bigvee_j \gamma_A_j}(x)\} \)

Proposition 2.10. [10] For all \( A \) and \( B \) are two neutrosophic sets then the following conditions are true:
\( C(A \cap B) = C(A) \cap C(B); C(A \cup B) = C(A) \cap C(B) \).

Definition 2.11. [10] A Neutrosophic topology [\( NT \) for short] is a non-empty set \( X \) is a family \( \tau_N \) of neutrosophic subsets in \( X \) satisfying the following axioms:
\( NT_1) \emptyset, X \in \tau_N \)
\( NT_2) \bigcap_{i \in I} \tau_N \in \tau_N \) for any \( G_i \in \tau_N \)
\( NT_3) \bigcup_{i \in I} \tau_N \in \tau_N \)

Throughout this paper, the pair of \( (X, \tau_N) \) is called a neutrosophic topological space [\( NTS \) for short]. The elements of \( \tau_N \) are called neutrosophic open set [\( NOS \) for short].

A Neutrosophic set \( F \) is Neutrosophic closed if and only if \( C(F) \) is neutrosophic open.

Example 2.12. [10] Any fuzzy topological space \( (X, \tau_0) \) in the sense of Chang is obviously a \( NTS \) in the form \( \tau_N = \{A : \mu_A \in \tau_0\} \) wherever we identify a fuzzy set \( A \) whose membership function is \( \mu_A \) with its counterpart.

The following is an example of Neutrosophic topological space.

Example 2.13. [10] Let \( x = \{X\} \) and \( A = \{x, 0.5, 0.5, 0.4 : x \in X\} \)
\( B = \{x, 0.4, 0.6, 0.8 : x \in X\} \)
\( C = \{x, 0.5, 0.6, 0.4 : x \in X\} \)
\( D = \{x, 0.4, 0.5, 0.8 : x \in X\} \) Then the family \( \tau_N = \{\emptyset, A, B, C, D\} \) is called a neutrosophic topological space on \( X \).

Definition 2.14. [10] The complement of \( A \) [\( C(A) \) for short] of \( NOS \) is called a neutrosophic closed set [\( NCS \) for short] in \( X \).

Now, we define Neutrosophic closure and Neutrosophic interior operations in Neutrosophic topological spaces:

Definition 2.15. [10] Let \( (X, \tau_N) \) be \( NTS \) and \( A = \{x, \mu_A(x), \sigma_A(x), \gamma_A(x) \} \) be a \( NOS \) in \( X \). Then the neutrosophic closure and neutrosophic interior of \( A \) are defined by
\( NCl(A) = \cap \{k : K \text{ is a NCS in } X \text{ and } A \subseteq K\} \)
\( NInt(A) = \{G : G \text{ is a NOS in } X \text{ and } G \subseteq A\} \)
It can be also shown that \( NCl(A) \) is \( NCS \) and \( NInt(A) \) is a \( NOS \) in \( X \).
Proposition 2.16. [10] For any Neutrosophic set $A$ in $(X, \tau_N)$ we have

a. $NCl(C(A)) = C(NInt(A))$

b. $NInt(C(A)) = C(NCl(A))$

Proposition 2.17. [10] Let $(X, \tau_N)$ be a NTS and $A, B$ be two neutrosophic sets in $X$. Then the following properties are holds:

a) $NInt(A) \subseteq A$

b) $A \subseteq NCl(A)$

c) $A \subseteq B \Rightarrow NInt(A) \subseteq NInt(B)$

d) $A \subseteq B \Rightarrow NCl(A) \subseteq NCl(B)$

e) $NInt(NInt(A)) = NInt(A)$

f) $NCl(NCl(A)) = NCl(A)$

g) $NInt(A \cap B) = NInt(A) \cap NInt(B)$

h) $NCl(A \cup B) = NCl(A) \cup NCl(B)$

i) $NInt(0_N) = 0_N$

j) $NInt(1_N) = 1_N$

k) $NCl(0_N) = 0_N$

l) $NCl(1_N) = 1_N$

m) $A \subseteq B \Rightarrow C(A) \subseteq C(B)$

n) $NCl(A \cap B) \subseteq NCl(A) \cap NCl(B)$

o) $NInt(A \cup B) \subseteq NInt(A) \cup NInt(B)$

Definition 2.18. [14] Let $A = \{ (x, \mu_A(x), \sigma_A(x), \gamma_A(x)) : x \in X \}$ be a neutrosophic open sets and $B = \{ (x, \mu_B(x), \sigma_B(x), \gamma_B(x)) : x \in X \}$ be a neutrosophic set on a neutrosophic topological space $(X, \tau_N)$ then

a. $A$ is called neutrosophic regular open if $A = NInt(NCl(A))$.

b. If $B \in NCS(X)$ then $B$ is called neutrosophic regular closed if $A = NCl(NInt(A))$.

Definition 2.19. [14] A neutrosophic set $A$ in a neutrosophic topological space $(X, \tau_N)$ is called

1. Neutrosophic semi-open set (NSOS) if $A \subseteq NCI(NInt(A))$.

2. Neutrosophic pre-open set (NPOS) if $A \subseteq NInt(NCI(A))$.

3. Neutrosophic α-open set (NαOS) if $A \subseteq NInt(NCl(NInt(A)))$.

4. Neutrosophic β-open set (NβOS) if $A \subseteq NCI(NInt(NCl(A)))$.

An (NSs) $A$ is called neutrosophic semi-closed set, neutrosophic α-closed set, Neutrosophic pre-closed set and Neutrosophic regular closed set respectively (NSCS, NαCS, NPOS and NRCS, resp.), if the complement of $A$ is a NSOS, NαOS, NPOS and NROS respectively.

Definition 2.20. [8] Let $A$ be a subset of a neutrosophic spaces $(X, \tau_N)$ is called neutrosophic generalized semi closed (NGs-closed) if neutrosophic semi $- cl(A) \subseteq G$, whenever $A \subseteq G$ and $G$ is NOS.

3. Neutrosophic generalized closed sets

In this section, we introduce the new concept namely Neutrosophic generalized closed sets in Neutrosophic topological spaces.

Definition 3.1. Let $(X, \tau_N)$ be a neutrosophic topological space. A subset $A$ of $(X, \tau_N)$ is called Neutrosophic generalized closed set (Ngc-closed) if $NCl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is Neutrosophic open set (NOS). Complement of neutrosophic-g-closed set is called the neutrosophic-g-open set.

Example 3.2. Let $X = \{a, b, c\}$ with $\tau_N = \{0_N, 1_N, A, B\}$ where

$A = \{(0, 0.5, 0.4), (0.7, 0.5, 0.5), (0.4, 0.5, 0.5)\}$,

$B = \{(0.3, 0.4, 0.4), (0.4, 0.5, 0.5), (0.3, 0.4, 0.6)\}$.

Then $(X, \tau_N)$ is a neutrosophic topological space. The closed sets of $(X, \tau_N)$ are

$A' = \{(0.4, 0.5, 0.5), (0.5, 0.5, 0.7), (0.5, 0.5, 0.4)\}$,

$B' = \{(0.4, 0.6, 0.3), (0.5, 0.5, 0.4), (0.6, 0.6, 0.3)\}$.

Consider the Neutrosophic set $C = \{(0.4, 0.6, 0.5), (0.4, 0.3, 0.5), (0.5, 0.6, 0.4)\}$ in $(X, \tau_N)$. Here $C$ is Neutrosophic-g-closed set in $(X, \tau_N)$.

Theorem 3.3. Every Neutrosophic closed set is a Neutrosophic generalized closed set in $(X, \tau_N)$.

Proof. Let $A \subseteq G$, where $G$ Neutrosophic open set in in $(X, \tau_N)$. Since $A$ is Neutrosophic closed set, $NCl(A) \subseteq A$ [Since $A = NCl(A)$]. Therefore $NCl(A) \subseteq A \subseteq G$. Hence $A$ is a neutrosophic-g-closed set in $(X, \tau_N)$.

Remark 3.4. The converse of the above theorem need not be true as seen in the following example.

Example 3.5. Let $X = \{a, b\}$ with $\tau_N = \{0_N, 1_N, A, B\}$ and where

$A = \{(0.4, 0.5, 0.5), (0.2, 0.4, 0.6)\}$,

$B = \{(0.7, 0.5, 0.3), (0.3, 0.4, 0.5)\}$.

Then $(X, \tau_N)$ is a neutrosophic topological space. The closed sets of $(X, \tau_N)$ are $A' = \{(0.3, 0.5, 0.4), (0.6, 0.6, 0.2)\}$, $B' = \{(0.3, 0.5, 0.7), (0.5, 0.6, 0.3)\}$.

Consider the Neutrosophic set $C = \{(0.6, 0.5, 0.6), (0.4, 0.3, 0.7)\}$ in $(X, \tau_N)$. Here $C$ is Neutrosophic-g-closed set, but $C$ is not NCS. (Since $NCl(C) \neq C$).

Theorem 3.6. If $A$ and $B$ are neutrosophic-g-closed sets in $(X, \tau_N)$ then $A \cup B$ is neutrosophic-g-closed set in $(X, \tau_N)$.

Proof. Let $A$ and $B$ are neutrosophic-g-closed sets in $(X, \tau_N)$. Then $NCl(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is NOS in $(X, \tau_N)$ and $NCl(B) \subseteq G$ whenever $B \subseteq G$ and $G$ is NOS in $X$. Since $A$ and $B$ are subsets of $G$, $A \cup B$ is a subset of $G$ and $G$ is neutrosophic open set. Then $NCl(A \cup B) = NCl(A) \cup NCl(B)$ [by proposition 2.17(h)], $NCl(A \cup B) \subseteq G$. Therefore $A \cup B$ is neutrosophic-g-closed set in $(X, \tau_N)$.

Theorem 3.7. If $A$ and $B$ are neutrosophic-g-closed sets in $(X, \tau_N)$, then $NCl(A \cap B) \subseteq NCl(A) \cap NCl(B)$.
Proof. Let $A$ and $B$ are neutrosophic-$g$-closed sets in $(X, \tau_N)$. Then $NCI(A) \subseteq G$ whenever $A \subseteq G$ and $G$ is $NOS$ in $(X, \tau_N)$ and $NCI(B) \subseteq G$ whenever $B \subseteq G$ and $G$ is $NOS$ in $(X, \tau_N)$. Since $A$ and $B$ are subsets of $G$, $A \cap B$ is a subset of $G$ and $G$ is $NOS$. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we know that, if $A \subseteq B$ then $NCI(A) \subseteq NCI(B)$ [10]. Therefore $NCI(A \cap B) \subseteq NCI(A)$ and $NCI(A \cap B) \subseteq NCI(B)$, which implies that $NCI(A \cap B) \subseteq NCI(A \cap NCI(B))$. Hence proved. 

\[ \Box \]

Remark 3.8. The intersection of two neutrosophic-$g$-closed sets need not be a neutrosophic-$g$-closed set as seen from the following example.

Example 3.9. Let $X = \{a, b, c\}$ with $\tau_N = \{0_N, 1_N, A, B, C\}$ where

\[ A = \{(0.4, 0.5, 0.4), (0.5, 0.5, 0.6), (0.7, 0.4, 0.3)\}, \]

\[ B = \{(0.3, 0.4, 0.5), (0.5, 0.4, 0.8), (0.6, 0.3, 0.4)\}, \]

\[ C = \{(0.4, 0.5, 0.4), (0.5, 0.5, 0.8), (0.7, 0.5, 0.3)\}. \]

Then $(X, \tau_N)$ is a neutrosophic topological space. The closed sets are $A' = \{(0.4, 0.5, 0.4), (0.6, 0.5, 0.5), (0.3, 0.6, 0.7)\}$, $B' = \{(0.5, 0.6, 0.3), (0.8, 0.6, 0.8), (0.4, 0.7, 0.6)\}$, $C' = \{(0.4, 0.5, 0.4), (0.8, 0.5, 0.5), (0.3, 0.5, 0.7)\}$.

Consider the neutrosophic-$g$-closed sets

\[ D = \{(0.5, 0.6, 0.7), (0.5, 0.5, 0.5), (0.6, 0.4, 0.6)\}, \]

and $E = \{(0.4, 0.3, 0.8), (0.2, 0.6, 0.7), (0.5, 0.4, 0.7)\}$, then $D \cap E = \{(0.4, 0.3, 0.8), (0.2, 0.5, 0.7), (0.5, 0.4, 0.7)\}$, is not a neutrosophic-$g$-closed set.

Theorem 3.10. If $A$ is neutrosophic-$g$-closed set in $(X, \tau_N)$ and $A \subseteq B \subseteq NCI(A)$, then $B$ is neutrosophic-$g$-closed set in $(X, \tau_N)$.

Proof. Let $B \subseteq G$ where $G$ is $NOS$ in $(X, \tau_N)$. Then $A \subseteq B$ implies $A \subseteq G$. Since $A$ is neutrosophic-$g$-closed, $NCI(A) \subseteq G$. Also $A \subseteq NCI(B)$ implies $NCI(B) \subseteq NCI(A)$. Thus $NCI(B) \subseteq G$ and so $B$ is neutrosophic-$g$-closed set in $(X, \tau_N)$. 

\[ \Box \]

Theorem 3.11. An neutrosophic-$g$-closed set $A$ is neutrosophic closed set iff $NCI(A)$-$A$ is neutrosophic closed set.

Proof. Assume that, $A$ is $NCS$, then $NCI(A) = A$ and so $NCI(A) - A = 0_N$ which is $NSC[x]$. Conversely, suppose $NCI(A)$-$A = NCS$. Then $NCI(A) - A = 0_N$, that is $NCI(A) = A$. Therefore $A$ is $NCS$. Hence proved.

\[ \Box \]

Theorem 3.12. Suppose that $A \subseteq B \subseteq X$, $B$ is an neutrosophic-$g$-closed set relative to $A$ and that $A$ is an neutrosophic-$g$-closed subset of $X$. Then $B$ is neutrosophic-$g$-closed set relative to $X$.

Proof. Let $B \subseteq G$ and suppose that $G$ is $NOS$ in $X$. Then $B \subseteq A \cap G$. Therefore $NCI(B) \subseteq A \cap G$. It follows that $A \cap NCI(B) \subseteq A \cap G$ and $A \subseteq G \cup NCI(B)$. Since $A$ is neutrosophic-$g$-closed in $X$, we have $NCI(A) \subseteq G \cup NCI(B)$. Therefore $NCI(B) \subseteq NCI(A) \subseteq G \cup NCI(B)$ and $NCI(B) \subseteq G$. Then $B$ is neutrosophic-$g$-closed relative to $B$ is an neutrosophic-$g$-closed set relative to $G$.

\[ \Box \]

Corollary 3.13. Let $A$ be a neutrosophic-$g$-closed set and suppose that $F$ is a $NCS$. Then $A \cap F$ is a neutrosophic-$g$-closed set.

Theorem 3.14. If $A$ is neutrosophic-$g$-closed set in $X$, then $A$ is neutrosophic-gs-closed set in $X$.

Proof. Let $A$ be a neutrosophic-$g$-closed set in $X$. Therefore $NCI(A) \subseteq G$ and $A \subseteq G$ whenever $G$ is $NOS$ in $X$. $NCI(A) \subseteq G$. Then $NCS(A) \subseteq NCI(A)$, $NCS(A) \subseteq G$. Therefore $A$ is neutrosophic-gs-closed in $X$.

\[ \Box \]

Remark 3.15. The converse of the above theorem need not be true as seen from the following example.

Example 3.16. Let $X = \{a\}$ with $\tau_N = \{0_N, 1_N, A, B, C, D\}$ where $A = \{(0.6, 0.7, 0.9)\}$, $B = \{(0.5, 0.4, 0.7)\}$, $C = \{(0.6, 0.7, 0.7)\}$, $D = \{(0.5, 0.4, 0.9)\}$.

Then $(X, \tau_N)$ is a neutrosophic topological space. Consider the neutrosophic set $E = \{(0.4, 0.3, 0.7)\}$. $E$ is neutrosophic-gs-closed set in $(X, \tau_N)$, but $E$ is not a neutrosophic-g-closed set $(X, \tau_N)$, since $NCI(E) \not\subseteq G$.

References


