INTUITIONISTIC BIPOLAR NEUTROSOPHIC SET AND ITS APPLICATION TO INTUITIONISTIC BIPOLAR NEUTROSOPHIC GRAPHS

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ABSTRACT. This manuscript is devoted to study a new concept of intuitionistic bipolar neutrosophic set with the operations like union, intersection and complement. Also, an application to intuitionistic bipolar neutrosophic graph with examples are developed. Further, we presented the Cartesian product, cross product, lexicographic product and strong product with suitable examples.

1. INTRODUCTION

The neutrosophic set has three independent parts, namely truth-membership degree, indeterminacy-membership degree and falsity-membership degree provided the sum of these values lies between 0 and 3; therefore, it is applied to many different areas, such as algebra [21,22] and decision-making problems (see [26] and references therein). Author Smarandache [25] remarks the difference between neutrosophic set and logic, and intuitionistic fuzzy set and logic. Interval neutrosophic sets with applications in BCK/BCI-algebra and KU-algebras are developed in [1,2,18,22,24]. Single valued neutrosophic graphs with their degree, order and size are established in [12,13]. Intuitionistic fuzzy set is initiated by Atanassov as a significant generalization of fuzzy set. Intuitionistic fuzzy sets are very useful while representing a problem by a linguistic variable, given in terms of a membership function only, seems too complicated. Recently intuitionistic fuzzy sets have been applied to many fields such as logic programming, medical diagnosis, decision making problems etc. On the other hand, bipolar fuzzy sets are extension of fuzzy sets whose membership degree ranges from [−1, 1]. The membership degree (0, 1] represents that an object satisfies a certain property whereas the membership degree [−1, 0) represents that the element satisfies the implicit counter-property. The positive information indicates that the consideration to be possible and negative information indicates that the consideration is granted to be impossible. Application to decision making of bipolar neutrosophic sets and bipolar neutrosophic graph structures are studied in [3,4], respectively. Neutrosophic bipolar vague sets and its application to graph theory are analysed in [19,20]. Similarity
measures of bipolar neutrosophic sets and its application to decision making are established in [26]. In [10, 15], intuitionistic neutrosophic sets and its relations are discussed. Furthermore, intuitionistic neutrosophic graph structures are extensively studied in [6, 7]. Motivated by these works, we established intuitionistic bipolar neutrosophic set and its application to intuitionistic bipolar neutrosophic graphs.

The major contribution of this work as follows:

- Newly introduced intuitionistic bipolar neutrosophic set with the operations like union, intersection and complement.
- Its application to Intuitionistic Bipolar Neutrosophic Graph (IBNG) with example are developed. Also neutrosophic bipolar vague subgraph, induced subgraph, strong and complete IBNG are established.
- Further we presented the Cartesian product, cross product, lexicographic product and strong product with suitable examples. The obtained results give the generalization of above mentioned works.

2. Preliminaries

Definition 2.1. [12] Let $X$ be a space of points (objects), with a generic element in $X$ denoted by $x$. A Single Valued Neutrosophic Set (SVNS) $A$ in $X$ is characterized by truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$ and falsity-membership function $F_A(x)$. For each point $x$ in $X$, $T_A(x), F_A(x), I_A(x) \in [0, 1]$,

$$A = \{ (x, T_A(x), F_A(x), I_A(x)) \mid x \in X \} \text{ and } 0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Definition 2.2. [13] A neutrosophic graph is defined as a pair $G^* = (V, E)$ where (i) $V = \{v_1, v_2, ..., v_n\}$ such that $T_1 : V \to [0, 1], I_1 : V \to [0, 1]$ and $F_1 : V \to [0, 1]$ denote the degree of truth-membership function, indeterminacy function and falsity-membership function, respectively and

$$0 \leq T_1(u) + I_1(u) + F_1(u) \leq 3, \text{ for } u \in V.$$

(ii) $E \subseteq V \times V$ where $T_2 : E \to [0, 1], I_2 : E \to [0, 1]$ and $F_2 : E \to [0, 1]$ are such that

$$T_2(uv) \leq \min\{T_1(u), T_1(v)\}, I_2(uv) \leq \min\{I_1(u), I_1(v)\}, F_2(uv) \leq \max\{F_1(u), F_1(v)\} \text{ and } 0 \leq T_2(uv) + I_2(uv) + F_2(uv) \leq 3, \forall uv \in E.$$

Definition 2.3. [16] A bipolar neutrosophic set $A$ in $X$ is defined as an object of the form

$$A = \{ (x, T^P(x), I^P(x), F^P(x), T^N(x), I^N(x), F^N(x)) \mid x \in X \},$$

where $T^P, I^P, F^P : X \to [0, 1]$ and $T^N, I^N, F^N : X \to [-1, 0]$. The Positive membership degree $T^P(x), I^P(x), F^P(x)$ denote the truth membership, indeterminate membership and false membership of an element $x \in X$ corresponding to a bipolar neutrosophic set $A$ and the negative membership degree $T^N(x), I^N(x), F^N(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ to some implicit counter-property corresponding to a bipolar neutrosophic set $A$.

Definition 2.4. [16] Let $X$ be a non-empty set. Then we call $A = \{ (x, T^P(x), I^P(x), F^P(x), T^N(x), I^N(x), F^N(x)) \mid x \in X \}$ a bipolar single valued neutrosophic relation on $X$ such that $T^P_A(x, y) \in [0, 1], I^P_A(x, y) \in [0, 1], F^P_A(x, y) \in [0, 1]$ and $T^N_A(x, y) \in [-1, 0], I^N_A(x, y) \in [-1, 0], F^N_A(x, y) \in [-1, 0]$.

Definition 2.5. [3, 4] Let $A = (T^P_A, I^P_A, F^P_A, T^N_A, I^N_A, F^N_A)$ and $B = (T^P_B, I^P_B, F^P_B, T^N_B, I^N_B, F^N_B)$ be bipolar single valued neutrosophic set on $X$. If $B = (T^P_B, I^P_B, F^P_B, T^N_B, I^N_B, F^N_B)$ is a
bipolar single valued neutrosophic relation on \( A = (T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N) \) then
\[
T_B^P(xy) \leq \min(T_A^P(x), T_A^P(y)), \quad T_B^N(xy) \geq \max(T_A^N(x), T_A^N(y)), \\
I_B^P(xy) \geq \max(I_A^P(x), I_A^P(y)), \quad I_B^N(xy) \leq \min(I_A^N(x), I_A^N(y)), \\
F_B^P(xy) \geq \max(F_A^P(x), F_A^P(y)), \quad F_B^N(xy) \leq \min(F_A^N(x), F_A^N(y)).
\]

A bipolar single valued neutrosophic relation \( B \) on \( X \) is called symmetric if \( T_B^P(xy) = T_B^P(yx), I_B^P(xy) = I_B^P(yx), F_B^P(xy) = F_B^P(yx) \) and \( T_B^N(xy) = T_B^N(yx), I_B^N(xy) = I_B^N(yx), F_B^N(xy) = F_B^N(yx) \) for all \( x, y \in X \).

**Definition 2.6.** A bipolar single-valued neutrosophic graph on a nonempty set \( X \) is a pair \( G = (C, D) \), where \( C \) is a bipolar single-valued neutrosophic set on \( X \) and \( D \) is a bipolar single-valued neutrosophic relation in \( X \) such that

(i) \( T_D^P(xy) \leq \min(T_C^P(x), T_C^P(y)), I_D^P(xy) \leq \min(I_C^P(x), I_C^P(y)), F_D^P(xy) \leq \max(F_C^P(x), F_C^P(y)), \)

(ii) \( T_D^N(xy) \geq \max(T_C^N(x), T_C^N(y)), I_D^N(xy) \geq \max(I_C^N(x), I_C^N(y)), F_D^N(xy) \geq \min(F_C^N(x), F_C^N(y)). \)

for all \( x, y \in X \).

**Definition 2.7.** An element \( x \) of \( X \) is called significant with respect to neutrosophic set \( A \) of \( X \) if the degree of truth-membership or indeterminacy-membership or falsity membership value, i.e \( T_A(x) \) or \( I_A(x) \) or \( F_A(x) \) is \( \geq 0.5 \). Otherwise, we call it insignificant. Also, for neutrosophic set the truth-membership, indeterminacy-membership and falsity membership all can not be significant.

we define an intuitionistic neutrosophic set by \( A = (x, T_A(x), I_A(x), F_A(x)) \), where \( \min\{T_A(x), F_A(x)\} \leq 0.5 \), \( \min\{T_A(x), I_A(x)\} \leq 0.5 \), & \( \min\{I_A(x), F_A(x)\} \leq 0.5 \), for all \( x \in X \) with the condition \( 0 \leq \{T_A(x) + I_A(x) + F_A(x)\} \leq 2 \)

**Definition 2.8.** A INS Relation (INSR) is defined as a intuitionistic subset of \( X \times Y \), having the form
\[
R = \{ (x, y), T_R(x, y), I_R(x, y), F_R(x, y) : x \in X, y \in Y \}
\]
where,
\[
T_R : X \times Y \rightarrow [0, 1], I_R : X \times Y \rightarrow [0, 1], F_R : X \times Y \rightarrow [0, 1]
\]
satisfies the conditions
(i) at least one of this \( T_R(x, y), I_R(x, y) \) and \( F_R(x, y) \) is \( \geq 0.5 \) and
(ii) \( 0 \leq \{T_A(x) + I_A(x) + F_A(x)\} \leq 2 \). The collection of all INSR on \( X \times Y \) is denoted as \( GR(X \times Y) \)

**Definition 2.9.** An intuitionistic neutrosophic graph is a pair \( G = (A, B) \) with underlying set \( V \), where \( T_A, F_A, I_A : V \rightarrow [0, 1] \) denote the truth, falsity and indeterminacy membership values of the vertices in \( V \) and \( T_B, F_B, I_B : E \subseteq V \times V \rightarrow [0, 1] \) denote the truth, falsity and indeterminacy membership values of the edges \( kl \in E \) such that
\[
(i) T_B(kl) \leq T_A(k) \wedge T_A(l), I_B(kl) \leq I_A(k) \wedge I_A(l), F_B(kl) \geq F_A(k) \wedge F_A(l)
\]
\[
(ii) T_B(kl) \wedge I_B(kl) \leq 0.5, T_B(kl) \wedge F_B(kl) \leq 0.5, I_B(kl) \wedge F_B(kl) \leq 0.5,
\]
\[
(iii) 0 \leq T_B(kl) + I_B(kl) + F_B(kl) \leq 2 \forall k, l \in V.
\]
3. INTUITIONISTIC BIPOLAR NEUTROSOPHIC SET

Definition 3.1. An element \( x \) of \( X \) is called significant with respect to neutrosophic set \( A \) of \( X \) if the degree of truth-membership or indeterminacy-membership or falsity membership value, i.e \( T_A(x) \) or \( I_A(x) \) or \( F_A(x) \) \( \geq 0.5 \). Otherwise, we call it insignificant. Also, for neutrosophic set the truth-membership, indeterminacy-membership and falsity-membership all can not be significant. we define an intuitionistic bipolar neutrosophic set by

\[
A = (x, T^P_A(x), I^P_A(x), F^P_A(x), T^N_A(x), I^N_A(x), F^N_A(x))
\]

where

\[
\min\{T^P_A, F^P_A\} \leq 0.5, \max\{T^N_A, F^N_A\} \geq -0.5, \min\{T^P_A, I^P_A\} \leq 0.5,
\]

\[
\max\{T^N_A, I^N_A\} \geq -0.5, \min\{F^P_A, I^P_A\} \leq 0.5, \max\{F^N_A, I^N_A\} \geq -0.5
\]

\[
T^P_A : X \rightarrow [0, 1], T^N_A : X \rightarrow [-1, 0], I^P_A : X \rightarrow [0, 1],
\]

\[
I^N_A : X \rightarrow [-1, 0], F^P_A : X \rightarrow [0, 1], F^N_A : X \rightarrow [-1, 0],
\]

with the conditions

\[
0 \leq T^P_A(x) + I^P_A(x) + F^P_A(x) \leq 2, -2 \geq T^N_A(x) + I^N_A(x) + F^N_A(x) \geq 0.
\]

Definition 3.2. A IBNS relation (IBNSR) is defined as an intuitionistic bipolar subset of \( X \times Y \), having the form

\[
R = \{ <(x, y), T^R(x, y), I^R(x, y), F^R(x, y), T^N_R(x, y), I^N_R(x, y), F^N_R(x, y) > : x \in X, y \in Y \}
\]

where

\[
T^R : X \times Y \rightarrow [0, 1], I^R : X \times Y \rightarrow [0, 1], F^R : X \times Y \rightarrow [0, 1],
\]

\[
T^N : X \times Y \rightarrow [-1, 0], I^N : X \times Y \rightarrow [-1, 0], F^N : X \times Y \rightarrow [-1, 0]
\]

satisfy the conditions (i) at least one of this \( T^R(x, y), I^R(x, y) \) and \( F^R(x, y) \) is \( \geq 0.5 \) at least one of this \( T^N(x, y), I^N(x, y) \) and \( F^N(x, y) \) is \( \leq -0.5 \) and

(ii) \( 0 \leq T^R(x) + I^R(x) + F^R(x) \leq 2, -2 \geq T^N(x) + I^N(x) + F^N(x) \geq 0. \)

Definition 3.3. Let \( A_1 = < x, T^P_{A_1}(x), I^P_{A_1}(x), F^P_{A_1}(x), T^N_{A_1}(x), I^N_{A_1}(x), F^N_{A_1}(x) > \) and \( A_2 = < x, T^P_{A_2}(x), I^P_{A_2}(x), F^P_{A_2}(x), T^N_{A_2}(x), I^N_{A_2}(x), F^N_{A_2}(x) > \) be two IBNSs. then \( A_1 \subset A_2 \) if any only if

\[
T^P_{A_1}(x) \leq T^P_{A_2}(x), T^N_{A_1}(x) \geq T^N_{A_2}(x),
\]

\[
I^P_{A_1}(x) \leq I^P_{A_2}(x), I^N_{A_1}(x) \geq I^N_{A_2}(x),
\]

\[
F^P_{A_1}(x) \geq F^P_{A_2}(x), F^N_{A_1}(x) \leq F^N_{A_2}(x) \forall x \in X.
\]

Definition 3.4. The union of two IBNSs \( A \) and \( B \) is also IBNS, whose truth membership, intermediate membership and false membership functions are,

\[
T^P_{(A \cup B)}(x) = \max\{T^P_A(x), T^P_B(x)\}
\]

\[
I^P_{(A \cup B)}(x) = \min\{I^P_A(x), I^P_B(x)\}
\]

\[
F^P_{(A \cup B)}(x) = \min\{F^P_A(x), F^P_B(x)\},
\]

and

\[
T^N_{(A \cup B)}(x) = \min\{T^N_A(x), T^N_B(x)\}
\]

\[
T^N_{(A \cup B)}(x) = \max\{T^N_A(x), T^N_B(x)\}
\]

\[
T^N_{(A \cup B)}(x) = \max\{T^N_A(x), T^N_B(x)\},
\]

for all \( x \in X \).
Example 3.5. Let $A = \{(x_1, 0.7, 0.3, 0.4)^P(-0.6, -0.4, -0.3)^N, (x_2, 0.5, 0.5, 0.8)^P(-0.6, -0.5, -0.4)^N\}$ and $B = \{(x_1, 0.4, 0.7, 0.4)^P(-0.4, -0.7, -0.3)^N, (x_2, 0.4, 0.3, 0.9)^P(-0.5, -0.6, -0.2)^N\}$ be two IBNSs of $X$. Then by definition of union we get, $A \cup B = \{(x_1, 0.7, 0.3, 0.3)^P(-0.6, -0.7, -0.3)^N, (x_2, 0.5, 0.3, 0.8)^P(-0.6, -0.5, -0.2)^N\}$

Definition 3.6. The intersection of two IBNSs $A$ and $B$ is also IBNS, whose truth-membership, indeterminacy-membership and falsity-membership functions are, 

$$
T_{(A \cap B)}(x) = \min\{T_A^P(x), T_B^P(x)\} \\
I_{(A \cap B)}(x) = \max\{I_A^P(x), I_B^P(x)\} \\
F_{(A \cap B)}(x) = \max\{F_A^P(x), F_B^P(x)\},
$$

and

$$
T_{(A \cap B)}(x) = \max\{T_A^N(x), T_B^N(x)\} \\
T_{(A \cap B)}(x) = \min\{T_A^N(x), T_B^N(x)\} \\
T_{(A \cap B)}(x) = \min\{T_A^N(x), T_B^N(x)\},
$$

for all $x \in X$.

Example 3.7. For above example, then by definition of intersection, we obtain $A \cap B = \{(x_1, 0.4, 0.3, 0.4)^P(-0.4, -0.4, -0.3)^N, (x_2, 0.4, 0.3, 0.9)^P(-0.5, -0.5, -0.4)^N\}$

Definition 3.8. The complement of IBNSs $A = \langle x, T_A^P(x), I_A^P(x), T_A^N(x), I_A^N(x), F_A^N(x) \rangle$ for all $x \in X$, is defined as 

$$(T^P(x))^C = F^P(x), (I^P(x))^C = 1 - I^P(x), (F^P(x))^C = T^P(x),$$

and

$$(T^N(x))^C = F^N(x), (I^N(x))^C = 1 - I^N(x), (F^N(x))^C = T^N(x),$$

for all $x \in X$.

4. INTUITIONISTIC BIPOLAR NEUTROSOPHIC GRAPHS

Definition 4.1. An Intuitionistic Bipolar Neutrosophic Graph (IBNG) is defined as a pair $G = (R, S), R = (A^P, A^N)$ and $S = (B^P, B^N)$ where

(i) $R = \{r_1, r_2, \ldots, r_n\}$ such that, $T_A^P : R \rightarrow [0, 1], I_A^P : R \rightarrow [0, 1], T_A^N : R \rightarrow [-1, 0], I_A^N : R \rightarrow [-1, 0], F_A^N : R \rightarrow [-1, 0]$ denote the degree of truth-membership, indeterminacy-membership and falsity-membership functions, respectively,

(ii) $S \subseteq R \times R$ where $T_B^P : R \times R \rightarrow [0, 1], I_B^P : R \times R \rightarrow [0, 1], T_B^N : R \times R \rightarrow [-1, 0], I_B^N : R \times R \rightarrow [-1, 0], F_B^N : R \times R \rightarrow [-1, 0]$,

(iii) $T_B^P (rs) \leq \min(T_A^P(r), T_A^P(s)), I_B^P (rs) \leq \min(I_A^P(r), I_A^P(s)), F_B^P (rs) \leq \max(F_A^P(r), F_A^P(s))$,

(iv) $T_B^P (rs) \wedge I_B^P (rs) \leq 0.5, T_B^P (rs) \wedge F_B^P (rs) \leq 0.5, I_B^P (rs) \wedge F_B^P (rs) \leq 0.5$,

(v) $0 \leq T_B^P (rs) + I_B^P (rs) + F_B^P (rs) \leq 2$,

(vi) $T_B^N (rs) \geq \max(T_A^N(r), T_A^N(s)), I_B^N (rs) \geq \max(I_A^N(r), I_A^N(s)),$

(vii) $T_B^N (rs) \vee I_B^N (rs) \geq -0.5, T_B^N (rs) \vee F_B^N (rs) \geq -0.5, I_B^N (rs) \vee F_B^N (rs) \geq -0.5$,

(viii) $0 \geq T_B^N (rs) + I_B^N (rs) + F_B^N (rs) \geq 2$. 
Example 4.2. Consider a IBNGs such that $A = \{a, b, c, d\}$, $B = \{ab, bc, cd\}$ by routine condition we have,

![Intuitionistic Bipolar Neutrosophic Graph](image1)

Figure 1: INTUITIONISTIC BIPOLAR NEUTROSOPHIC GRAPH

Definition 4.3. A graph $G' = (R', S')$ is said to be subgraph of $G = (R, S)$ if

$$(T'_A)^P(r) \leq T_A^P(r), (I'_A)^P(r) \leq I_A^P(r), (F'_A)^P(r) \geq F_A^P(r)$$

and

$$(T'_A)^N(r) \geq T_A^N(r), (I'_A)^N(r) \geq I_A^N(r), (F'_A)^N(r) \leq F_A^N(r)$$

for all $r \in R$ and

$$(T'_B)^P(rs) \leq T_B^P(rs), (I'_B)^P(rs) \leq I_B^P(rs), (F'_B)^P(rs) \geq F_B^P(rs)$$

and

$$(T'_B)^N(rs) \geq T_B^N(rs), (I'_B)^N(rs) \geq I_B^N(rs), (F'_B)^N(rs) \leq F_B^N(rs)$$

for all $rs \in S$.

Example 4.4. An IBNG subgraph is represented as Figure 2

![Intuitionistic Bipolar Neutrosophic Subgraph](image2)

Figure 2: INTUITIONISTIC BIPOLAR NEUTROSOPHIC SUBGRAPH

Definition 4.5. A graph $G' = (R', S')$ is said to be induced subgraph of $G = (R, S)$ if

$$(T'_A)^P(r) = T_A^P(r), (I'_A)^P(r) = I_A^P(r), (F'_A)^P(r) = F_A^P(r)$$

and

$$(T'_A)^N(r) = T_A^N(r), (I'_A)^N(r) = I_A^N(r), (F'_A)^N(r) = F_A^N(r)$$

for all $r \in R$.
for all \( r \in R \) and
\[
(T'_B)^P(rs) = T_B^P(rs), (I'_A)^P(rs) = I_B^P(rs), (F'_B)^P(rs) = F_B^P(rs),
\]
\[
(T'_B)^N(rs) = T_B^N(rs), (I'_B)^N(rs) = T_B^N(rs), (F'_B)^N(rs) = F_B^N(rs),
\]
for \( rs \in S \)

**Example 4.6.** An IBNG induced subgraph is represented as Figure 3.

![Figure 3: INTUIONISTIC BIPOLAR NEUTROSOPHIC INDUCED SUBGRAPH](image)

**Definition 4.7.** A graph \( G' = (R', S') \) is said to be spanning subgraph of \( G = (R, S) \) if
\[
(T'_B)^P(rs) \leq T_B^P(rs), (I'_A)^P(rs) \leq I_B^P(rs), (F'_B)^P(rs) \geq F_B^P(rs)
\]
\[
(T'_B)^N(rs) \geq T_B^N(rs), (I'_B)^N(rs) \geq T_B^N(rs), (F'_B)^N(rs) \leq F_B^N(rs),
\]
for all \( rs \in S \).

**Definition 4.8.** An IBNG \( G = (R, S) \) is called strong IBNG if
\[
(T'_B)^P(rs) = T_B^P(rs), (I'_A)^P(rs) = I_B^P(rs), (F'_B)^P(rs) = F_B^P(rs)
\]
\[
(T'_B)^N(rs) = T_B^N(rs), (I'_B)^N(rs) = T_B^N(rs), (F'_B)^N(rs) = F_B^N(rs),
\]
for all \( rs \in S \).

**Definition 4.9.** An IBNG \( G = (R, S) \) is called complete IBNG if
\[
(T'_B)^P(rs) = T_B^P(rs), (I'_A)^P(rs) = I_B^P(rs), (F'_B)^P(rs) = F_B^P(rs)
\]
\[
(T'_B)^N(rs) = T_B^N(rs), (I'_B)^N(rs) = T_B^N(rs), (F'_B)^N(rs) = F_B^N(rs),
\]
for all \( rs \in S \).
Definition 4.10. The Cartesian product of two IBNGs $G_1$ and $G_2$ is denoted by the pair $G_1 \times G_2 = (R_1 \times R_2, S_1 \times S_2)$ and defined as

$$T_{A_1 \times A_2}^P(kl) = T_{A_1}^P(k) \land T_{A_2}^P(l)$$
$$I_{A_1 \times A_2}^P(kl) = I_{A_1}^P(k) \land I_{A_2}^P(l)$$
$$F_{A_1 \times A_2}^P(kl) = F_{A_1}^P(k) \lor F_{A_2}^P(l)$$
$$T_{A_1 \times A_2}^N(kl) = T_{A_1}^N(k) \lor T_{A_2}^N(l)$$
$$I_{A_1 \times A_2}^N(kl) = I_{A_1}^N(k) \lor I_{A_2}^N(l)$$
$$F_{A_1 \times A_2}^N(kl) = F_{A_1}^N(k) \land F_{A_2}^N(l),$$

for all $kl \in R_1 \times R_2$. The membership value of the edges in $G_1 \times G_2$ can be calculated as,

1. $T_{B_1 \times B_2}^P(k_1 l_1)(k_2 l_2) = T_{A_1}^P(k_1) \land T_{A_2}^P(l_2)$
2. $T_{B_1 \times B_2}^N(k_1 l_1)(k_2 l_2) = T_{A_1}^N(k_1) \lor T_{A_2}^N(l_2)$

for all $k_1, k_2 \in S_1, l_1, l_2 \in S_2$.

Example 4.11. Consider $G_1 = (R_1, S_1)$ and $G_2 = (R_2, S_2)$ are two IBNG of $G = (R, S)$ respectively, as represented in Figure 4, now we get $G_1 \times G_2$ as follows Figure 5.

Theorem 4.1. The Cartesian product $G_1 \times G_2 = (R_1 \times R_2, S_1 \times S_2)$ of IBNG of IBNG $G_1$ and $G_2$ is an IBNG of $G_1 \times G_2$.

Proof. We consider:

Case 1: for $k \in R_1, l_1, l_2 \in S_2$

$$T_{(B_1 \times B_2)}^P((k l_1)(k l_2)) = T_{A_1}^P(k) \land T_{B_2}^P(l_1 l_2) \leq T_{A_1}^P(k) \land [T_{A_2}^P(l_1) \land T_{A_2}^P(l_2)] = [T_{A_1}^P(k) \land T_{A_2}^P(l_1)] \land [T_{A_1}^P(k) \land T_{A_2}^P(l_2)] = T_{(A_1 \times A_2)}^P(k, l_1) \land T_{(A_1 \times A_2)}^P(k, l_2)$$
$I_{(B_1 \times B_2)}^P((kl_1)(kl_2)) = I_{A_1}^P(k) \land I_{B_2}^P(l_1l_2)$
$\leq I_{A_1}^P(k) \land [I_{A_1}^P(l_1) \land I_{A_2}^P(l_2)]$
$= [I_{A_1}^P(k) \land I_{A_2}^P(l_1)] \land [I_{A_1}^P(k) \land I_{A_2}^P(l_2)]$
$= I_{(A_1 \times A_2)}^P(k, l_1) \land I_{(A_1 \times A_2)}^P(k, l_2)$

$F_{(B_1 \times B_2)}^P((kl_1)(kl_2)) = F_{A_1}^P(k) \lor F_{B_2}^P(l_1l_2)$
$\leq F_{A_1}^P(k) \lor [F_{A_2}^P(l_1) \lor F_{A_2}^P(l_2)]$
$= [F_{A_1}^P(k) \lor F_{A_2}^P(l_1)] \lor [F_{A_1}^P(k) \lor F_{A_2}^P(l_2)]$
$= F_{(A_1 \times A_2)}^P(k, l_1) \lor F_{(A_1 \times A_2)}^P(k, l_2)$
for all $kl_1, kl_2 \in G_1 \times G_2$.

Case 2: for $k \in R_2, l_1, l_2 \in S_1$

\[
T_{(B_1 \times B_2)}^p((l_1 k)(l_2 k)) = T_{B_1}^p(k) \land T_{B_2}^p(l_2)
\]
\[
\leq T_{A_2}^p(k) \land [T_{A_1}^p(l_1) \land T_{A_1}^p(l_2)]
\]
\[
= [T_{A_2}^p(k) \land T_{A_1}^p(l_1)] \land [T_{A_2}^p(k) \land T_{A_1}^p(l_2)]
\]
\[
= T_{(A_1 \times A_2)}^p(l_1, k) \land T_{(A_1 \times A_2)}^p(l_2, k)
\]
\[
I_{(B_1 \times B_2)}^p((l_1 k)(l_2 k)) = I_{B_1}^p(k) \land I_{B_2}^p(l_2)
\]
\[
\leq I_{A_2}^p(k) \land [I_{A_1}^p(l_1) \land I_{A_1}^p(l_2)]
\]
\[
= [I_{A_2}^p(k) \land I_{A_1}^p(l_1)] \land [I_{A_2}^p(k) \land I_{A_1}^p(l_2)]
\]
\[
= I_{(A_1 \times A_2)}^p(l_1, k) \land I_{(A_1 \times A_2)}^p(l_2, k)
\]
\[
F_{(B_1 \times B_2)}^p((l_1 k)(l_2 k)) = F_{B_1}^p(k) \lor F_{B_2}^p(l_2)
\]
\[
\leq F_{A_2}^p(k) \lor [F_{A_1}^p(l_1) \lor F_{A_1}^p(l_2)]
\]
\[
= [F_{A_2}^p(k) \lor F_{A_1}^p(l_1)] \lor [F_{A_2}^p(k) \lor F_{A_1}^p(l_2)]
\]
\[
= F_{(A_1 \times A_2)}^p(l_1, k) \lor F_{(A_1 \times A_2)}^p(l_2, k),
\]

for all $l_1 k, l_2 k \in G_1 \times G_2$.

Similarly, one can prove the result for negative part also. □

**Definition 4.12.** The Cross product of two IBNGs $G_1$ and $G_2$ is denoted by the pair $G_1 \times G_2 = (R_1 \times R_2, S_1 \times S_2)$ and defined as

(i) $T_{A_1 \times A_2}^p(kl) = T_{A_1}^p(k) \land T_{A_2}^p(l)$

(ii) $I_{A_1 \times A_2}^p(kl) = I_{A_1}^p(k) \land I_{A_2}^p(l)$

(iii) $F_{A_1 \times A_2}^p(kl) = F_{A_1}^p(k) \lor F_{A_2}^p(l)$

(iv) $T_{A_1 \times A_2}^N(kl) = T_{A_1}^N(k) \lor T_{A_2}^N(l)$

(v) $I_{A_1 \times A_2}^N(kl) = I_{A_1}^N(k) \land I_{A_2}^N(l)$

(vi) $F_{A_1 \times A_2}^N(kl) = F_{A_1}^N(k) \lor F_{A_2}^N(l)$

for all $k, l \in R_1 \times R_2$.

\[
T_{(B_1 \times B_2)}^p((k_1 l_1)(k_2 l_2)) = T_{B_1}^p(k_1 k_2) \land T_{B_2}^p(l_1 l_2)
\]
\[
I_{(B_1 \times B_2)}^p((k_1 l_1)(k_2 l_2)) = I_{B_1}^p(k_1 k_2) \land I_{B_2}^p(l_1 l_2)
\]
\[
F_{(B_1 \times B_2)}^p((k_1 l_1)(k_2 l_2)) = F_{B_1}^p(k_1 k_2) \lor F_{B_2}^p(l_1 l_2)
\]
\[
T_{(B_1 \times B_2)}^N((k_1 l_1)(k_2 l_2)) = T_{B_1}^N(k_1 k_2) \lor T_{B_2}^N(l_1 l_2)
\]
\[
I_{(B_1 \times B_2)}^N((k_1 l_1)(k_2 l_2)) = I_{B_1}^N(k_1 k_2) \land I_{B_2}^N(l_1 l_2)
\]
\[
F_{(B_1 \times B_2)}^N((k_1 l_1)(k_2 l_2)) = F_{B_1}^N(k_1 k_2) \lor F_{B_2}^N(l_1 l_2),
\]

for all $k_1 k_2 \in S_1, l_1 l_2 \in S_2$.

**Example 4.13.** Consider $G_1 = (R_1, S_1)$ and $G_2 = (R_2, S_2)$ are two IBNG of $G = (R, S)$ respectively, as represented in Figure 4. Now, we get cross product $G_1 \times G_2$ as follows Figure 6.
Theorem 4.2. Cross product $G_1 \times G_2 = (R_1 \times R_2, S_1 \times S_2)$ of two IBNG of $G_1$ and $G_2$ is an IBNG of $G_1 \times G_2$.

Proof. For all $k_1 l_1, k_2 l_2 \in G_1 \times G_2$

$$T_{(B_1 \times B_2)}^p((k_1 l_1)(k_2 l_2)) = T_{B_1}^p(k_1 k_2) \land T_{B_2}^p(l_1 l_2)$$

$$\leq [T_{A_1}^p(k_1) \land T_{A_2}^p(k_2)] \land [T_{A_1}^n(l_1) \land T_{A_2}^n(l_2)]$$

$$= [T_{A_1}^p(k_1) \land T_{A_2}^p(l_1)] \land [T_{A_1}^p(k_2) \land T_{A_2}^p(l_2)]$$

$$= T_{(A_1 \times A_2)}^p(k_1 l_1) \land T_{(A_1 \times A_2)}^p(k_2 l_2),$$

$$I_{(B_1 \times B_2)}^p((k_1 l_1)(k_2 l_2)) = I_{B_1}^p(k_1 k_2) \land I_{B_2}^p(l_1 l_2)$$

$$\leq [I_{A_1}^p(k_1) \land I_{A_2}^p(k_2)] \land [I_{A_1}^n(l_1) \land I_{A_2}^n(l_2)]$$

$$= [I_{A_1}^p(k_1) \land I_{A_2}^p(l_1)] \land [I_{A_1}^p(k_2) \land I_{A_2}^p(l_2)]$$

$$= I_{(A_1 \times A_2)}^p(k_1 l_1) \land I_{(A_1 \times A_2)}^p(k_2 l_2),$$

$$F_{(B_1 \times B_2)}^p((k_1 l_1)(k_2 l_2)) = F_{B_1}^p(k_1 k_2) \lor F_{B_2}^p(l_1 l_2)$$

$$\leq [F_{A_1}^p(k_1) \lor F_{A_2}^p(k_2)] \lor [F_{A_1}^n(l_1) \lor F_{A_2}^n(l_2)]$$

$$= [F_{A_1}^p(k_1) \lor F_{A_2}^p(l_1)] \lor [F_{A_1}^p(k_2) \lor F_{A_2}^p(l_2)]$$

$$= F_{(A_1 \times A_2)}^p(k_1 l_1) \lor F_{(A_1 \times A_2)}^p(k_2 l_2)$$

Similarly, we can prove the result for negative part also. □
Definition 4.14. The lexicographic product of two IBNGs $G_1$ and $G_2$ is denoted by the pair $G_1 \bullet G_2 = (R_1 \bullet R_2, S_1 \bullet S_2)$ and defined as

\[
(i) T_{(A_1 \bullet A_2)}^{P}(k) = T_{A_{1}}^{P}(k) \land T_{A_{2}}^{P}(l)
\]
\[
I_{(A_1 \bullet A_2)}^{P}(k) = I_{A_{1}}^{P}(k) \land I_{A_{2}}^{P}(l)
\]
\[
F_{(A_1 \bullet A_2)}^{P}(k) = F_{A_{1}}^{P}(k) \lor F_{A_{2}}^{P}(l)
\]
\[
T_{(A_1 \bullet A_2)}^{N}(k) = T_{A_{1}}^{N}(k) \lor T_{A_{2}}^{N}(l)
\]
\[
I_{(A_1 \bullet A_2)}^{N}(k) = I_{A_{1}}^{N}(k) \lor I_{A_{2}}^{N}(l)
\]
\[
F_{(A_1 \bullet A_2)}^{N}(k) = F_{A_{1}}^{N}(k) \land F_{A_{2}}^{N}(l),
\]

for all $k, l \in R_1 \times R_2$

\[
(iii) T_{(B_1 \bullet B_2)}^{P}(k_1)(k_2) = T_{B_{1}}^{P}(k_1) \land T_{B_{2}}^{P}(l_1 l_2)
\]
\[
I_{(B_1 \bullet B_2)}^{P}(k_1)(k_2) = I_{B_{1}}^{P}(k_1) \land I_{B_{2}}^{P}(l_1 l_2)
\]
\[
F_{(B_1 \bullet B_2)}^{P}(k_1)(k_2) = F_{B_{1}}^{P}(k_1) \lor F_{B_{2}}^{P}(l_1 l_2)
\]
\[
T_{(B_1 \bullet B_2)}^{N}(k_1)(k_2) = T_{B_{1}}^{N}(k_1) \lor T_{B_{2}}^{N}(l_1 l_2)
\]
\[
I_{(B_1 \bullet B_2)}^{N}(k_1)(k_2) = I_{B_{1}}^{N}(k_1) \lor I_{B_{2}}^{N}(l_1 l_2)
\]
\[
F_{(B_1 \bullet B_2)}^{N}(k_1)(k_2) = F_{B_{1}}^{N}(k_1) \land F_{B_{2}}^{N}(l_1 l_2),
\]

for all $k \in R_1, l_1 l_2 \in S_2$.

Example 4.15. Lexicographic product of IBNG $G_1 = (R_1, S_1)$ and $G_2 = (R_2, S_2)$ shown in Figure 2 are defined as $G_1 \bullet G_2 = (R_1 \bullet R_2, S_1 \bullet S_2)$ and is represented in Figure 7.

Theorem 4.3. Lexicographic product $G_1 \bullet G_2 = (R_1 \bullet R_2, S_1 \bullet S_2)$ of two IBNG of $G_1$ and $G_2$ is an IBNG of $G_1 \bullet G_2$.

Proof. We consider two cases:

Case 1: for $k \in R_1, l_1 l_2 \in S_2$

\[
T_{(B_1 \bullet B_2)}^{P}((k_1)(k_2)) = T_{B_{1}}^{P}(k_1) \land T_{B_{2}}^{P}(l_1 l_2)
\]
\[
\leq T_{B_{1}}^{P}(k_1) \land [T_{B_{2}}^{P}(l_1) \land T_{B_{2}}^{P}(l_2)]
\]
\[
= [T_{A_{1}}^{P}(k) \land T_{A_{2}}^{P}(l_1)] \land [T_{A_{1}}^{P}(k) \land T_{A_{2}}^{P}(l_2)]
\]
\[
= T_{(A_1 \bullet A_2)}^{P}(k_1, l_1) \land T_{(A_1 \bullet A_2)}^{P}(k_2, l_2)\]
for all \( kl \in \) $S_1 \times S_2$. 

Case 2: For all \( k_1k_2 \in S_1 \), \( l_1l_2 \in S_2 \)

Similarly, we can prove the result for negative part also.
Definition 4.16. The strong product of two IBNGs $G_1$ and $G_2$ is denoted by the pair $G_1 \boxtimes G_2 = (R_1 \boxtimes R_2, S_1 \boxtimes S_2)$ and defined as

$$
(i) T^P_{(A_1 \boxtimes A_2)}(kl) = T^P_{A_1}(k) \wedge T^P_{A_2}(l),
I^P_{(A_1 \boxtimes A_2)}(kl) = I^P_{A_1}(k) \wedge I^P_{A_2}(l),
F^P_{(A_1 \boxtimes A_2)}(kl) = F^P_{A_1}(k) \lor F^P_{A_2}(l),
T^N_{(A_1 \boxtimes A_2)}(kl) = T^N_{A_1}(k) \lor T^N_{A_2}(l),
I^N_{(A_1 \boxtimes A_2)}(kl) = I^N_{A_1}(k) \lor I^N_{A_2}(l),
F^N_{(A_1 \boxtimes A_2)}(kl) = F^N_{A_1}(k) \land F^N_{A_2}(l),
$$

for all $k, l \in R_1 \boxtimes R_2$

$$
(ii) T^P_{(B_1 \boxtimes B_2)}(k_1)(k_2) = T^P_{A_1}(k) \wedge T^P_{B_2}(l_1l_2),
I^P_{(B_1 \boxtimes B_2)}(k_1)(k_2) = I^P_{A_1}(k) \wedge I^P_{B_2}(l_1l_2),
F^P_{(B_1 \boxtimes B_2)}(k_1)(k_2) = F^P_{A_1}(k) \lor F^P_{B_2}(l_1l_2),
T^N_{(B_1 \boxtimes B_2)}(k_1)(k_2) = T^N_{A_1}(k) \lor T^N_{B_2}(l_1l_2),
I^N_{(B_1 \boxtimes B_2)}(k_1)(k_2) = I^N_{A_1}(k) \lor I^N_{B_2}(l_1l_2),
F^N_{(B_1 \boxtimes B_2)}(k_1)(k_2) = F^N_{A_1}(k) \land F^N_{B_2}(l_1l_2),
$$

for all $k \in R_1, l_1l_2 \in S_2$

$$
(iii) T^P_{(B_1 \boxtimes B_2)}(k_1)(l_2) = T^P_{A_2}(l) \wedge T^P_{B_2}(k_1k_2),
I^P_{(B_1 \boxtimes B_2)}(k_1)(l_2) = I^P_{A_2}(l) \wedge I^P_{B_2}(k_1k_2),
F^P_{(B_1 \boxtimes B_2)}(k_1)(l_2) = F^P_{A_2}(l) \lor F^P_{B_2}(k_1k_2),
T^N_{(B_1 \boxtimes B_2)}(k_1)(l_2) = T^N_{A_2}(l) \lor T^N_{B_2}(k_1k_2),
I^N_{(B_1 \boxtimes B_2)}(k_1)(l_2) = I^N_{A_2}(l) \lor I^N_{B_2}(k_1k_2),
F^N_{(B_1 \boxtimes B_2)}(k_1)(l_2) = F^N_{A_2}(l) \land F^N_{B_2}(k_1k_2),
$$

for all $k_1, k_2 \in S_1, l \in R_2$

$$
(iv) T^P_{(B_1 \boxtimes B_2)}(k_1)(k_2l_2) = T^P_{B_1}(k_1k_2) \wedge T^P_{B_2}(l_1l_2),
I^P_{(B_1 \boxtimes B_2)}(k_1)(k_2l_2) = I^P_{B_1}(k_1k_2) \wedge I^P_{B_2}(l_1l_2),
F^P_{(B_1 \boxtimes B_2)}(k_1)(k_2l_2) = F^P_{B_1}(k_1k_2) \lor F^P_{B_2}(l_1l_2),
T^N_{(B_1 \boxtimes B_2)}(k_1)(k_2l_2) = T^N_{B_1}(k_1k_2) \lor T^N_{B_2}(l_1l_2),
I^N_{(B_1 \boxtimes B_2)}(k_1)(k_2l_2) = I^N_{B_1}(k_1k_2) \lor I^N_{B_2}(l_1l_2),
F^N_{(B_1 \boxtimes B_2)}(k_1)(k_2l_2) = F^N_{B_1}(k_1k_2) \land F^N_{B_2}(l_1l_2),
$$

for all $k_1, k_2 \in S_1, l_1l_2 \in S_2$.

Example 4.17. Strong product of IBNG $G_1 = (R_1, S_1)$ and $G_2 = (R_2, S_2)$ shown in Figure 2 is defined as $G_1 \boxtimes G_2 = (R_1 \boxtimes R_2, S_1 \boxtimes S_2)$ and is represented in Figure 8.

Theorem 4.4. Strong product $G_1 \boxtimes G_2 = (R_1 \boxtimes R_2, S_1 \boxtimes S_2)$ of two IBNG of $G_1$ and $G_2$ is an IBNG of $G_1 \boxtimes G_2$. 

Case 1: for $k \in R_1, l_1 l_2 \in S_2$

$$T\left((k_1)(k_2)\right) = T_{A_1}(k) \land T_{A_2}(l_1 l_2)$$

$$\leq T_{A_1}(k) \land [T_{A_2}(l_1) \land T_{A_2}(l_2)]$$

$$= [T_{A_1}(k) \land T_{A_2}(l_1)] \land [T_{A_1}(k) \land T_{A_2}(l_2)]$$

$$= T_{(A_1, A_2)}(k, l_1) \land T_{(A_1, A_2)}(k, l_2)$$

$$I\left((k_1)(k_2)\right) = I_{A_1}(k) \land I_{A_2}(l_1 l_2)$$

$$\leq I_{A_1}(k) \land [I_{A_2}(l_1) \land I_{A_2}(l_2)]$$

$$= [I_{A_1}(k) \land I_{A_2}(l_1)] \land [I_{A_1}(k) \land I_{A_2}(l_2)]$$

$$= I_{(A_1, A_2)}(k, l_1) \land I_{(A_1, A_2)}(k, l_2)$$

$$F\left((k_1)(k_2)\right) = F_{A_1}(k) \lor F_{A_2}(l_1 l_2)$$

$$\leq F_{A_1}(k) \lor [F_{A_2}(l_1) \lor F_{A_2}(l_2)]$$

$$= [F_{A_1}(k) \lor F_{A_2}(l_1)] \lor [F_{A_1}(k) \lor F_{A_2}(l_2)]$$

$$= F_{(A_1, A_2)}(k, l_1) \lor F_{(A_1, A_2)}(k, l_2)$$

for all $k_1, k_2 \in R_1 \land R_2$.

Case 2: for $k \in R_2, l_1 l_2 \in S_1$

$$T\left((l_1)(l_2)\right) = T_{A_2}(k) \land T_{A_1}(l_1 l_2)$$

$$\leq T_{A_2}(k) \land [T_{A_1}(l_1) \land T_{A_1}(l_2)]$$

$$= [T_{A_2}(k) \land T_{A_1}(l_1)] \land [T_{A_2}(k) \land T_{A_1}(l_2)]$$

$$= T_{(A_1, A_2)}(l_1, k) \land T_{(A_1, A_2)}(l_2, k)$$
The composition of two IBNGs

Definition 4.18. The composition of two IBNGs $G_1$ and $G_2$ is denoted by the pair $G_1 \circ G_2 = (R_1 \circ R_2, S_1 \circ S_2)$ and defined as

$$(i) T_{(A_1 \circ A_2)}^P (kl) = T_{A_1}^P (k) \land T_{A_2}^P (l)$$

$$(ii) I_{(A_1 \circ A_2)}^P (kl) = I_{A_1}^P (k) \land I_{A_2}^P (l)$$

$$(iii) F_{(A_1 \circ A_2)}^P (kl) = F_{A_1}^P (k) \lor F_{A_2}^P (l)$$

$$(iv) T_{(A_1 \circ A_2)}^N (kl) = T_{A_1}^N (k) \lor T_{A_2}^N (l)$$

$$(v) I_{(A_1 \circ A_2)}^N (kl) = I_{A_1}^N (k) \lor I_{A_2}^N (l)$$

$$(vi) F_{(A_1 \circ A_2)}^N (kl) = F_{A_1}^N (k) \lor F_{A_2}^N (l)$$

for all $l_1, k_1 \in \mathbb{R}_1 \otimes \mathbb{R}_2$. $\square$

Case 3: For all $k_1 k_2 \in S_1, l_1 l_2 \in S_2$,

$T_{(B_1 \otimes B_2)}^P ((k_1 l_1)(k_2 l_2)) = T_{B_1}^P (k_1 k_2) \land T_{B_2}^P (l_1 l_2)$

$F_{(B_1 \otimes B_2)}^P ((k_1 l_1)(k_2 l_2)) = F_{B_1}^P (k_1 k_2) \lor F_{B_2}^P (l_1 l_2)$

$for all k_1 l_1, k_2 l_2 \in R_1 \otimes R_2$. Similarly, we can prove the result for negative part also.
for all $k, l \in R_1 \circ R_2$

\[(ii) T_{(B_1 \circ B_2)}^P(kl_1)(kl_2) = T_{A_1}^P(k) \land T_{B_2}^P(l_1l_2)\]

\[I_{(B_1 \circ B_2)}^P(kl_1)(kl_2) = I_{A_1}^P(k) \land I_{B_2}^P(l_1l_2)\]

\[F_{(B_1 \circ B_2)}^P(kl_1)(kl_2) = F_{A_1}^P(k) \lor F_{B_2}^P(l_1l_2)\]

\[T_{(B_1 \circ B_2)}^N(kl_1)(kl_2) = T_{A_1}^N(k) \lor T_{B_2}^N(l_1l_2)\]

\[I_{(B_1 \circ B_2)}^N(kl_1)(kl_2) = I_{A_1}^N(k) \lor I_{B_2}^N(l_1l_2)\]

\[F_{(B_1 \circ B_2)}^N(kl_1)(kl_2) = F_{A_1}^N(k) \land F_{B_2}^N(l_1l_2),\]

for all $k \in R_1, l_1l_2 \in S_2$.

\[(iii) T_{(B_1 \circ B_2)}^P(k_1l)(k_2l) = T_{A_2}^P(l) \land T_{B_2}^P(k_1k_2)\]

\[I_{(B_1 \circ B_2)}^P(k_1l)(k_2l) = I_{A_2}^P(l) \land I_{B_2}^P(k_1k_2)\]

\[F_{(B_1 \circ B_2)}^P(k_1l)(k_2l) = F_{A_2}^P(l) \lor F_{B_2}^P(k_1k_2)\]

\[T_{(B_1 \circ B_2)}^N(k_1l)(k_2l) = T_{A_2}^N(l) \lor T_{B_2}^N(k_1k_2)\]

\[I_{(B_1 \circ B_2)}^N(k_1l)(k_2l) = I_{A_2}^N(l) \lor I_{B_2}^N(k_1k_2)\]

\[F_{(B_1 \circ B_2)}^N(k_1l)(k_2l) = F_{A_2}^N(l) \land F_{B_2}^N(k_1k_2),\]

for all $k_1k_2 \in S_1, l \in R_2$.

\[(iv) T_{(B_1 \circ B_2)}^P(k_1l_1)(k_2l_2) = T_{B_1}(k_1k_2) \land T_{A_2}^P(l_1) \land T_{A_2}^P(l_2)\]

\[I_{(B_1 \circ B_2)}^P(k_1l_1)(k_2l_2) = I_{B_1}(k_1k_2) \land I_{A_2}^P(l_1) \land I_{A_2}^P(l_2)\]

\[F_{(B_1 \circ B_2)}^P(k_1l_1)(k_2l_2) = F_{B_1}(k_1k_2) \lor F_{A_2}^P(l_1) \lor F_{A_2}^P(l_2)\]

\[T_{(B_1 \circ B_2)}^N(k_1l_1)(k_2l_2) = T_{B_1}(k_1k_2) \lor T_{A_2}^N(l_1) \lor T_{A_2}^N(l_2)\]

\[I_{(B_1 \circ B_2)}^N(k_1l_1)(k_2l_2) = I_{B_1}(k_1k_2) \lor I_{A_2}^N(l_1) \lor I_{A_2}^N(l_2)\]

\[F_{(B_1 \circ B_2)}^N(k_1l_1)(k_2l_2) = F_{B_1}(k_1k_2) \land F_{A_2}^N(l_1) \land F_{A_2}^N(l_2),\]

for all $k_1k_2 \in S_1, l_1l_2 \in S_2$ such that $l_1 \neq l_2$.

**Example 4.19.** Composition of IBNG $G_1 = (R_1, S_1)$ and $G_2 = (R_2, S_2)$ shown in Figure 2 is defined as $G_1 \circ G_2 = (R_1 \circ R_2, S_1 \circ S_2)$ and is represented in Figure 9.

**Theorem 4.5.** Composition $G_1 \circ G_2 = (R_1 \circ R_2, S_1 \circ S_2)$ of two IBNG of $G_1$ and $G_2$ is an IBNG of $G_1 \circ G_2$.

**Proof.** There are three cases:

Case 1: for $k \in R_1, l_1l_2 \in S_2$

\[T_{(B_1 \circ B_2)}^P((kl_1)(kl_2)) = T_{A_1}^P(k) \land T_{B_2}^P(l_1l_2)\]

\[\leq T_{A_1}^P(k) \land [T_{A_2}^P(l_1) \land T_{A_2}^P(l_2)]\]

\[= [T_{A_1}^P(k) \land T_{A_2}^P(l_1)] \land [T_{A_1}^P(k) \land T_{A_2}^P(l_2)]\]

\[= T_{(A_1 \circ A_2)}^P(k, l_1) \land T_{(A_1 \circ A_2)}^P(k, l_2)\]
\[ I_{(B_1 \circ B_2)}((k_1)(k_2)) = I_{A_1}^P(k) \land I_{B_2}^P(l_1 l_2) \]
\[ \leq I_{A_1}^P(k) \land [I_{A_1}^P(l_1) \land I_{A_2}^P(l_2)] \]
\[ = [I_{A_1}^P(k) \land I_{A_2}^P(l_1)] \land [I_{A_1}^P(k) \land I_{A_2}^P(l_2)] \]
\[ = I_{(A_1 \circ A_2)}(k, l_1) \land I_{(A_1 \circ A_2)}(k, l_2) \]
\[ F_{(B_1 \circ B_2)}((k_1)(k_2)) = F_{A_1}^P(k) \lor F_{B_2}^P(l_1 l_2) \]
\[ \leq F_{A_1}^P(k) \lor [F_{A_2}^P(l_1) \lor F_{A_2}^P(l_2)] \]
\[ = [F_{A_1}^P(k) \lor F_{A_2}^P(l_1)] \lor [F_{A_1}^P(k) \lor F_{A_2}^P(l_2)] \]
\[ = F_{(A_1 \circ A_2)}(k, l_1) \lor F_{(A_1 \circ A_2)}(k, l_2) \]

for all \( k_1, k_2 \in R_1 \circ R_2 \).

Case 2: for \( k \in R_2, l_1 l_2 \in S_1 \)
\[ T_{(B_1 \circ B_2)}((l_1 k)(l_2 k)) = T_{A_2}^P(k) \land T_{B_1}^P(l_1 l_2) \]
\[ \leq T_{A_2}^P(k) \land [T_{A_1}^P(l_1) \land T_{A_1}^P(l_2)] \]
\[ = [T_{A_2}^P(k) \land T_{A_1}^P(l_1)] \land [T_{A_2}^P(k) \land T_{A_1}^P(l_2)] \]
\[ = T_{(A_1 \circ A_2)}(l_1, k) \land T_{(A_1 \circ A_2)}(l_2, k) \]
\[ I_{(B_1 \circ B_2)}((l_1 k)(l_2 k)) = I_{A_2}^P(k) \land I_{B_1}^P(l_1 l_2) \]
\[ \leq I_{A_2}^P(k) \land [I_{A_1}^P(l_1) \land I_{A_1}^P(l_2)] \]
\[ = [I_{A_2}^P(k) \land I_{A_1}^P(l_1)] \land [I_{A_2}^P(k) \land I_{A_1}^P(l_2)] \]
\[ = I_{(A_1 \circ A_2)}(l_1, k) \land I_{(A_1 \circ A_2)}(l_2, k) \]
\[ F_{(B_1 \circ B_2)}((l_1 k)(l_2 k)) = F_{A_2}^P(k) \lor F_{B_1}^P(l_1 l_2) \]
\[ \leq F_{A_2}^P(k) \lor [F_{A_1}^P(l_1) \lor F_{A_1}^P(l_2)] \]
\[ = [F_{A_2}^P(k) \lor F_{A_1}^P(l_1)] \lor [F_{A_2}^P(k) \lor F_{A_1}^P(l_2)] \]
\[ = F_{(A_1 \circ A_2)}(l_1, k) \lor F_{(A_1 \circ A_2)}(l_2, k) \]
Case 3: For $k_1, k_2 \in S_1, l_1, l_2 \in R_2$ such that $l_1 \neq l_2$

$$T_{(B_1 \circ B_2)}^P((k_1 l_1)(k_2 l_2)) = T_{B_1}^P(k_1, k_2) \wedge T_{A_2}^P(l_1) \wedge T_{A_2}^P(l_2) \leq [T_{A_1}^P(k_1) \wedge T_{A_1}^P(k_2)] \wedge [T_{A_2}^P(l_1) \wedge T_{A_2}^P(l_2)] = [I_{A_1}^P(k_1) \wedge T_{A_2}^P(l_1)] \wedge [T_{A_1}^P(k_2) \wedge T_{A_2}^P(l_2)] = T_{(A_1 \circ A_2)}^P(k_1 l_1) \wedge T_{(A_1 \circ A_2)}^P(k_2 l_2)$$

$$I_{(B_1 \circ B_2)}^P((k_1 l_1)(k_2 l_2)) = I_{B_1}^P(k_1, k_2) \wedge I_{A_2}^P(l_1) \wedge I_{A_2}^P(l_2) \leq [I_{A_1}^P(k_1) \wedge I_{A_1}^P(k_2)] \wedge [I_{A_2}^P(l_1) \wedge I_{A_2}^P(l_2)] = [I_{A_1}^P(k_1) \wedge I_{A_2}^P(l_1)] \wedge [I_{A_1}^P(k_2) \wedge I_{A_2}^P(l_2)] = I_{(A_1 \circ A_2)}^P(k_1 l_1) \wedge I_{(A_1 \circ A_2)}^P(k_2 l_2)$$

$$F_{(B_1 \circ B_2)}^P((k_1 l_1)(k_2 l_2)) = F_{B_1}^P(k_1, k_2) \vee F_{A_2}^P(l_1) \vee F_{A_2}^P(l_2) \leq [F_{A_1}^P(k_1) \vee F_{A_1}^P(k_2)] \vee [F_{A_2}^P(l_1) \vee F_{A_2}^P(l_2)] = [F_{A_1}^P(k_1) \vee F_{A_2}^P(l_1)] \vee [F_{A_1}^P(k_2) \vee F_{A_2}^P(l_2)] = F_{(A_1 \circ A_2)}^P(k_1 l_1) \vee F_{(A_1 \circ A_2)}^P(k_2 l_2)$$

for all $k_1 l_1, k_2 l_2 \in R_1 \circ R_2$. Similarly, we can prove the result for negative part also. \(\square\)

5. Conclusions

In this work, a new concept of intuitionistic bipolar neutrosophic set with the operations like union, intersection and complement have been developed. Also, an application to intuitionistic bipolar neutrosophic graph with examples have established. Further, we presented the Cartesian product, cross product, lexicographic product and strong product with suitable examples. In future, isomorphic properties will be investigated.

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