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Research article

MBJ-neutrosophic hyper BCK-ideals in hyper BCK-algebras

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Abstract: In 2018, Takallo et al. introduced the concept of an MBJ-neutrosophic structure, which is a generalization of a neutrosophic structure, and applied it to a BCK/BCI-algebra. The aim of this study is to apply the notion of an MBJ-neutrosophic structure to a hyper BCK-algebra. The notions of the MBJ-neutrosophic hyper BCK-ideal, the MBJ-neutrosophic weak hyper BCK-ideal, the MBJ-neutrosophic s-weak hyper BCK-ideal and the MBJ-neutrosophic strong hyper BCK-ideal are introduced herein, and their relations and properties are investigated. These notions are discussed in connection with the MBJ-neutrosophic level cut sets.

Keywords: hyper BCK-algebra; MBJ-neutrosophic hyper BCK-ideal; MBJ-neutrosophic weak (s-weak) hyper BCK-ideal; MBJ-neutrosophic strong hyper BCK-ideal
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1. Introduction

BCK-algebras entered into pure mathematics in 1966 through the work of Imai and Iséki [1], and were applied to various mathematical fields, such as functional analysis, group theory, topology and probability theory, etc. The hyperstructure theory was born in 1934 when Marty introduced hypergroups. In this side, he explored and applied their properties to groups and other algebraic structures [2]. Algebraic hyperstructures reflect a natural generalization of classical algebraic structures. In an algebraic hyperstructure, the composition of two elements is a set, while in a classical algebraic structure, the composition of two elements is an element. As an extension of a BCK-algebra, Jun et al. [3] introduced an algebraic hyperstructure called a hyper BCK-algebra. They studied hyper BCK-ideals in hyper BCK-algebras. Saeid and Zahedi [4] studied quotient hyper BCK-algebras and in [5] Saeid et al. introduced weak implicative and implicative hyper K-ideals of hyper K-algebras. After that, many books and several articles have been published on hyper BCK-algebras and other hyper algebraic structures.

Zadeh [6] introduced fuzzy set theory in 1965 and in 1986 this concept has been generalized to intuitionistic fuzzy set theory by adding a non-membership function by Atanassov [7]. As a generalization of the classical set and (intuitionistic) fuzzy set theory, Smarandache [8, 9] launched a significant topic, that deals with indeterminacy, called neutrosophic set theory. In [10], Takallo et al. presented the notion of an MBJ-neutrosophic set as generalization of a neutrosophic set and they applied it to BCK/BCI-algebras. In an MBJ-neutrosophic set, the indeterminacy membership function is generalized to interval valued membership function. Next, Jun and Roh [11] introduced and studied the concept of an MBJ-neutrosophic ideal in BCK/BCI-algebras. In B-algebras, Manokaran and Prakasam [12] introduced the MBJ-neutrosophic subalgebra and Khalid et al. [13] defined and studied the MBJ-neutrosophic T-ideal. The notions of (intuitionistic) fuzzy sets, neutrosophic sets and other extensions of fuzzy sets have been applied to algebraic structures, decision making problems, etc. For algebraic structures, see [14–22] and for decision making problems, see [23, 24]. In an algebraic hyperstructure, Jun and Xin [25] discussed the topic of fuzzy set theory of hyper BCK-ideals in hyper BCK-algebras and in [26] Bakhshi et al. studied fuzzy (positive, weak) implicative hyper BCK-ideals. In 2004, Borzooei and Jun [27] studied the intuitionistic fuzzy set theory of hyper BCK-ideals in hyper BCK-algebras. In addition, Khademan et al. [28] studied neutrosophic set theory of hyper BCK-ideals in hyper BCK-algebras.

As no studies have been reported so far to generalize the above mentioned concepts, so the aim of this present article is:

(1) To apply the notion of an MBJ-neutrosophic structure to a hyper BCK-algebra.

(2) To define and study the notions of MBJ-neutrosophic (weak, s-weak, strong) hyper BCK-ideals of hyper BCK-algebras.

(3) To discuss MBJ-neutrosophic (weak, strong) hyper BCK-ideals in relation to MBJ-neutrosophic level cut sets.

To do so, the rest of the article is structured as follows: In Section 2, we review some elementary notions. In Section 3, we introduce the notions of the MBJ-neutrosophic hyper BCK-ideal, the MBJ-neutrosophic weak hyper BCK-ideal, the MBJ-neutrosophic s-weak hyper BCK-ideal and the MBJ-neutrosophic strong hyper BCK-ideal and investigate several properties. We discuss MBJ-neutrosophic (weak, strong) hyper BCK-ideal in relation to MBJ-neutrosophic level cut sets. Finally, in Section 4, we present the conclusion and future works of the study.

2. Preliminaries

In the current section, we remember some of the basic notions of hyper BCK-algebras which will be very helpful in further study of the paper. Let $\mathcal{H}$ be a hyper BCK-algebra in what follows, unless otherwise stated.

Let $\mathcal{H}$ be a non-empty set and let " $\diamond$ " be a mapping

$$\diamond : \mathcal{H} \times \mathcal{H} \to \mathcal{P}(\mathcal{H})\backslash\{\emptyset\}$$

which is said to be hyperoperation. For any two subsets $\mathcal{K}$ and $\mathcal{F}$, denote by $\mathcal{K} \diamond \mathcal{F}$, the set $\bigcup\{\varrho \diamond \tau \mid \varrho \in \mathcal{K}, \tau \in \mathcal{F}\}$. We shall use $\varrho \diamond \tau$ instead of $\{\varrho\} \diamond \tau$, $\varrho \diamond \{\tau\}$, or $\{\varrho\} \diamond \{\tau\}$.

By a hyper BCK-algebra $\mathcal{H}$ (see [3]), we mean a non-empty set $\mathcal{H}$ with a special element 0 and a hyperoperation $\diamond$, for all $\varrho, \tau, \eta \in \mathcal{H}$, that satisfies the following axioms:

(HI) $(\varrho \diamond \eta) \diamond (\tau \diamond \eta) = \varrho \diamond \tau$,
(HII) $(\varrho \diamond \tau) \diamond \eta = (\varrho \diamond \eta) \diamond \tau$,
(HIII) $\varrho \diamond \mathcal{H} \ll \{\varrho\}$,
(HIV) $\varrho \ll \tau$ and $\tau \ll \varrho$ imply $\varrho = \tau$,

for all $\varrho, \tau, \eta \in \mathcal{H}$, where $\varrho \ll \tau$ is defined by $0 \in \varrho \diamond \tau$ and for any $\mathcal{K}, \mathcal{F} \subseteq \mathcal{H}, \mathcal{K} \ll \mathcal{F}$ is defined by $\forall r \in \mathcal{K}, \exists t \in \mathcal{F}$ such that $r \ll t$.

In a hyper BCK-algebra $\mathcal{H}$ the axiom (HIII) is equivalent to the following axiom:

(HV) $\varrho \diamond \tau \ll \{\varrho\}$ for all $\varrho, \tau \in \mathcal{H}$.

Proposition 2.1. *[3] Every hyper BCK-algebra $\mathcal{H}$ satisfies the following conditions, for all $\varrho, \tau, \eta \in \mathcal{H}$ and for any non-empty subsets $\mathcal{K}, \mathcal{F}, \mathcal{G}$ of $\mathcal{H}$,*

(1) $\varrho \diamond 0 \ll \{\varrho\}, 0 \diamond \varrho = \{0\}, 0 \diamond 0 = \{0\}$,
(2) $0 \ll \varrho, \varrho \ll \varrho, \varrho \in \varrho \diamond 0$,
(3) $\varrho \diamond 0 \ll \{\tau\} \Rightarrow \varrho \ll \tau$,
(4) $\tau \ll \eta \Rightarrow \varrho \diamond \eta \ll \varrho \diamond \tau$,
(5) $\varrho \diamond \tau = \{0\} \Rightarrow \varrho \diamond \eta \ll \tau \diamond \eta, (\varrho \diamond \eta) \diamond (\tau \diamond \eta) = \{0\}$,
(6) $\mathcal{K} \subseteq \mathcal{F} \Rightarrow \mathcal{K} \ll \mathcal{F}$,
(7) $\mathcal{K} \ll \{0\} \Rightarrow \mathcal{K} = \{0\}$,
(8) $\mathcal{K} \ll \mathcal{K}, \mathcal{K} \diamond \mathcal{F} \ll \mathcal{K}, (\mathcal{K} \diamond \mathcal{F}) \diamond \mathcal{G} = (\mathcal{K} \diamond \mathcal{G}) \diamond \mathcal{F}$,
(9) $\mathcal{K} \diamond \{0\} = \{0\} \Rightarrow \mathcal{K} = \{0\}$.

Definition 2.2. Let $(\mathcal{H}, \diamond)$ be a hyper BCK-algebra. A subset $\mathcal{K}$ of $\mathcal{H}$ is called:

- A hyper BCK-ideal of $\mathcal{H}$ (see [3]) if
  (1) $0 \in \mathcal{K}$,
  (2) $\varrho \diamond \tau \ll \mathcal{K}, \tau \in \mathcal{K} \Rightarrow \varrho \in \mathcal{K}, \forall \varrho, \tau \in \mathcal{H}$.
- A weak hyper BCK-ideal of $\mathcal{H}$ (see [3]) if it satisfies (1) and
  (3) $\varrho \diamond \tau \subseteq \mathcal{K}, \tau \in \mathcal{K} \Rightarrow \varrho \in \mathcal{K}, \forall \varrho, \tau \in \mathcal{H}$,
- A strong hyper BCK-ideal of $\mathcal{H}$ (see [29]) if it satisfies (1) and
  (4) $(\varrho \diamond \tau) \cap \mathcal{K} \neq \emptyset, \tau \in \mathcal{K} \Rightarrow \varrho \in \mathcal{K}, \forall \varrho, \tau \in \mathcal{H}$,

By an interval $\tilde{u}$ we mean an interval $\tilde{u} = [u^-, u^+]$, where $0 \leq u^- \leq u^+ \leq 1$. The set of all closed intervals I is denoted by $[I]$. The interval $[u, u]$ is identified with the number u.

For two intervals $\tilde{u}_1 = [u_1^-, u_1^+]$ and $\tilde{u}_2 = [u_2^-, u_2^+]$, we define

$$r\max\{\tilde{u}_1, \tilde{u}_2\} = [\max\{u_1^-, u_2^-\}, \max\{u_1^+, u_2^+\}],$$

$$r\min\{\tilde{u}_1, \tilde{u}_2\} = [\min\{u_1^-, u_2^-\}, \min\{u_1^+, u_2^+\}],$$

Furthermore, we have

(1) $\tilde{u}_1 \geq \tilde{u}_2 \Leftrightarrow u_1^- \geq u_2^-, u_1^+ \geq u_2^+$,

(2) $\tilde{u}_1 \leq \tilde{u}_2 \Leftrightarrow u_1^- \leq u_2^-, u_1^+ \leq u_2^+$,
(3) $\tilde{u}_1 = \tilde{u}_2 \Leftrightarrow u_1^- = u_2^-, u_1^+ = u_2^+$.

Let $\mathcal{H}$ be a nonempty set. A function $\tilde{D} : \mathcal{H} \to [I]$ is said to be an interval-valued fuzzy set over a universe $\mathcal{H}$. Let $[I]^{\mathcal{H}}$ stands for the set of all interval-valued fuzzy sets $\mathcal{H}$. For any $\tilde{D} \in [I]^{\mathcal{H}}$ and $\varrho \in \mathcal{H}$, $\tilde{D} = [D^-(\varrho), D^+(\varrho)]$ is called the degree of membership of an element ϱ to $\tilde{D}$, where $D^-(\varrho) : \mathcal{H} \to I$ and $D^+(\varrho) : \mathcal{H} \to I$ are fuzzy sets over a universe $\mathcal{H}$ which are called a lower fuzzy set and an upper fuzzy set over $\mathcal{H}$, respectively. For simplicity, we denote $\tilde{D} = [D^-, D^+]$.

Let $\mathcal{H}$ be a nonempty set. A neutrosophic set over a universe $\mathcal{H}$ (see [9]) is a structure of the form:

$$\mathcal{D} = \{\langle \varrho; \mathcal{D}_T(\varrho), \mathcal{D}_I(\varrho), \mathcal{D}_F(\varrho)\rangle \mid \varrho \in \mathcal{H}\},$$

where $\mathcal{D}_T$, $\mathcal{D}_I$ and $\mathcal{D}_F$ are fuzzy sets over a universe $\mathcal{H}$, which are called a truth, an indeterminate and a false membership functions, respectively.

For the sake of simplicity, we shall use the symbol $\mathcal{D} = (\mathcal{D}_T, \mathcal{D}_I, \mathcal{D}_F)$ for the neutrosophic set

$$\mathcal{D} = \{\langle \varrho; \mathcal{D}_T(\varrho), \mathcal{D}_I(\varrho), \mathcal{D}_F(\varrho)\rangle \mid \varrho \in \mathcal{H}\}.$$

In [10], Takallo et el. introduced the idea of an MBJ-neutrosophic set as follows:

Definition 2.3. Let $\mathcal{H}$ be a nonempty set. By an MBJ-neutrosophic set over a universe $\mathcal{H}$, we mean a structure of the form:

$$\mathfrak{D} = \{\langle \varrho; \mathcal{M}_{\mathfrak{D}}(\varrho), \tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho), \mathcal{J}_{\mathfrak{D}}(\varrho)\rangle \mid \varrho \in \mathcal{H}\},$$

where $\mathcal{M}_{\mathfrak{D}}$ and $\mathcal{J}_{\mathfrak{D}}$ are fuzzy sets over a universe $\mathcal{H}$, which are called a truth and a false membership functions, respectively, and $\tilde{\mathcal{B}}_{\mathfrak{D}}$ is an interval-valued fuzzy set over a universe $\mathcal{H}$ which is called an indeterminate interval-valued membership function.

For the sake of simplicity, we shall use the symbol $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ for the MBJ-neutrosophic set

$$\mathfrak{D} = \{\langle \varrho; \mathcal{M}_{\mathfrak{D}}(\varrho), \tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho), \mathcal{J}_{\mathfrak{D}}(\varrho)\rangle \mid \varrho \in \mathcal{H}\}.$$

Given an MBJ-neutrosophic set $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ over a universe $\mathcal{H}$, we consider the following sets:

$$U(\mathcal{M}_{\mathfrak{D}}, \alpha) = \{\varrho \in H \mid \mathcal{M}_{\mathfrak{D}}(\varrho) \geq \alpha\},$$
$$U(\tilde{\mathcal{B}}_{\mathfrak{D}}, \tilde{\beta}) = \{\varrho \in H \mid \tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho) \geq \tilde{\beta}\},$$
$$L(\mathcal{J}_{\mathfrak{D}}, \gamma) = \{\varrho \in H \mid \mathcal{J}_{\mathfrak{D}}(\varrho) \leq \gamma\},$$

where $\alpha, \gamma \in [0, 1]$ and $\tilde{\beta} = [\beta^-, \beta^+] \in [I]$.

3. MBJ-neutrosophic hyper BCK-ideals

Definition 3.1. An MBJ-neutrosophic set $\mathfrak{D}$ on $\mathcal{H}$ is called an MBJ-neutrosophic hyper BCK-ideal of $\mathcal{H}$ if it satisfies:

(1) $(\forall \varrho, \tau \in \mathcal{H})\left(\varrho \ll \tau \Rightarrow \mathcal{M}_{\mathfrak{D}}(\varrho) \geq \mathcal{M}_{\mathfrak{D}}(\tau), \tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho) \geq \tilde{\mathcal{B}}_{\mathfrak{D}}(\tau), \mathcal{J}_{\mathfrak{D}}(\varrho) \leq \mathcal{J}_{\mathfrak{D}}(\tau) \right)$,

$$(2)\ (\forall \varrho,\tau\in\mathcal{H})\left(\begin{array}{l}\mathcal{M}_{\mathfrak{D}}(\varrho)\geq\min\left\{\inf\{\mathcal{M}_{\mathfrak{D}}(z)\mid z\in\varrho\diamond\tau\},\mathcal{M}_{\mathfrak{D}}(\tau)\right\}\\ \tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho)\geq r\min\left\{\inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(z)\mid z\in\varrho\diamond\tau\},\tilde{\mathcal{B}}_{\mathfrak{D}}(\tau)\right\}\\ \mathcal{J}_{\mathfrak{D}}(\varrho)\leq\max\left\{\sup\{\mathcal{J}_{\mathfrak{D}}(z)\mid z\in\varrho\diamond\tau\},\mathcal{J}_{\mathfrak{D}}(\tau)\right\}\end{array}\right).$$

Example 3.1. Let $\mathcal{H}=\{0,\varrho,\tau\}$ be a set with the hyperoperation " $\diamond$ ", which is given by Table 1.

Table 1. Tabular representation of the hyperoperation " $\diamond$ ".

$\diamond$	0	ϱ	τ
0	$\{0\}$	$\{0\}$	$\{0\}$
ϱ	$\{\varrho\}$	$\{0,\varrho\}$	$\{0,\varrho\}$
τ	$\{\tau\}$	$\{\varrho,\tau\}$	$\{0,\varrho,\tau\}$

Then, $\mathcal{H}$ is a hyper BCK-algebra (see [3]). Let $\mathfrak{D}=(\mathcal{M}_{\mathfrak{D}},\tilde{\mathcal{B}}_{\mathfrak{D}},\mathcal{J}_{\mathfrak{D}})$ be an MBJ-neutrosophic set over $\mathcal{H}$ given by Table 2.

Table 2. Tabular representation of $\mathfrak{D}=(\mathcal{M}_{\mathfrak{D}},\tilde{\mathcal{B}}_{\mathfrak{D}},\mathcal{J}_{\mathfrak{D}})$.

$\mathfrak{D}$	$\mathcal{M}_{\mathfrak{D}}$	$\tilde{\mathcal{B}}_{\mathfrak{D}}$	$\mathcal{J}_{\mathfrak{D}}$
0	$\frac{1}{5}$	$[\frac{1}{3},0.71]$	$\frac{2}{9}$
ϱ	$\frac{1}{7}$	$[\frac{1}{6},0.51]$	$\frac{2}{7}$
τ	$\frac{1}{9}$	$[\frac{1}{9},0.21]$	$\frac{2}{5}$

It is routine to check that $\mathfrak{D}=(\mathcal{M}_{\mathfrak{D}},\tilde{\mathcal{B}}_{\mathfrak{D}},\mathcal{J}_{\mathfrak{D}})$ is an MBJ-neutrosophic hyper BCK-ideal of $\mathcal{H}$.

Proposition 3.2. *Let $\mathfrak{D}=(\mathcal{M}_{\mathfrak{D}},\tilde{\mathcal{B}}_{\mathfrak{D}},\mathcal{J}_{\mathfrak{D}})$ be an MBJ-neutrosophic hyper BCK-ideal of $\mathcal{H}$. Then,*

(i) $(\forall\varrho\in\mathcal{H})\left(\ \mathcal{M}_{\mathfrak{D}}(0)\geq\mathcal{M}_{\mathfrak{D}}(\varrho),\tilde{\mathcal{B}}_{\mathfrak{D}}(0)\geq\tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho),\mathcal{J}_{\mathfrak{D}}(0)\leq\mathcal{J}_{\mathfrak{D}}(\varrho)\ \right)$,

(ii) If $\mathfrak{D}=(\mathcal{M}_{\mathfrak{D}},\tilde{\mathcal{B}}_{\mathfrak{D}},\mathcal{J}_{\mathfrak{D}})$ satisfies

$$(\forall\mathcal{K}\subseteq\mathcal{H})(\exists\varrho_\circ,\tau_\circ,\eta_\circ\in\mathcal{K})\text{ such that }\left(\begin{array}{l}\mathcal{M}_{\mathfrak{D}}(\varrho_\circ)=\inf\{\mathcal{M}_{\mathfrak{D}}(\varrho)\mid\varrho\in\mathcal{K}\}\\ \tilde{\mathcal{B}}_{\mathfrak{D}}(\tau_\circ)=\inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(\tau)\mid\tau\in\mathcal{K}\}\\ \mathcal{J}_{\mathfrak{D}}(\eta_\circ)=\sup\{\mathcal{J}_{\mathfrak{D}}(\eta)\mid\eta\in\mathcal{K}\}\end{array}\right),\tag{3.1}$$

then

$$(\forall\varrho,\tau\in\mathcal{H})(\exists u,v,w\in\varrho\diamond\tau)\text{ such that }\left(\begin{array}{l}\mathcal{M}_{\mathfrak{D}}(\varrho)\geq\min\{\mathcal{M}_{\mathfrak{D}}(u),\mathcal{M}_{\mathfrak{D}}(\tau)\}\\ \tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho)\geq r\min\{\tilde{\mathcal{B}}_{\mathfrak{D}}(v),\tilde{\mathcal{B}}_{\mathfrak{D}}(\tau)\}\\ \mathcal{J}_{\mathfrak{D}}(\varrho)\leq\max\{\mathcal{J}_{\mathfrak{D}}(w),\mathcal{M}_{\mathfrak{D}}(\tau)\}\end{array}\right).\tag{3.2}$$

Proof. (i) Since $0\ll\varrho$ for all $\varrho\in\mathcal{H}$, it follows from Definition 3.1(1) that $\mathcal{M}_{\mathfrak{D}}(0)\geq\mathcal{M}_{\mathfrak{D}}(\varrho),\tilde{\mathcal{B}}_{\mathfrak{D}}(0)\geq\tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho)$ and $\mathcal{J}_{\mathfrak{D}}(0)\leq\mathcal{J}_{\mathfrak{D}}(\varrho)$ for all $\varrho\in\mathcal{H}$.
(ii) Suppose that $\mathfrak{D}=(\mathcal{M}_{\mathfrak{D}},\tilde{\mathcal{B}}_{\mathfrak{D}},\mathcal{J}_{\mathfrak{D}})$ satisfies the condition (3.1). For any $\varrho,\tau\in\mathcal{H}$, there exist $u_\circ,v_\circ,w_\circ\in\varrho\diamond\tau$ such that $\mathcal{M}_{\mathfrak{D}}(u_\circ)=\inf\{\mathcal{M}_{\mathfrak{D}}(u)\mid u\in\varrho\diamond\tau\}$, $\tilde{\mathcal{B}}_{\mathfrak{D}}(v_\circ)=\inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(v)\mid v\in\varrho\diamond\tau\}$ and $\mathcal{J}_{\mathfrak{D}}(w_\circ)=\sup\{\mathcal{J}_{\mathfrak{D}}(w)\mid w\in\varrho\diamond\tau\}$. It follows from Definition 3.1(2) that

$$\mathcal{M}_{\mathfrak{D}}(\varrho)\geq\min\left\{\inf\{\mathcal{M}_{\mathfrak{D}}(u)\mid u\in\varrho\diamond\tau\},\mathcal{M}_{\mathfrak{D}}(\tau)\right\}=\min\{\mathcal{M}_{\mathfrak{D}}(u_\circ),\mathcal{M}_{\mathfrak{D}}(\tau)\},$$

$$\tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho) \geq r\min\left\{\inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(v) \mid v \in \varrho \diamond \tau\}, \tilde{\mathcal{B}}_{\mathfrak{D}}(\tau)\right\} = r\min\{\tilde{\mathcal{B}}_{\mathfrak{D}}(v_\circ), \tilde{\mathcal{B}}_{\mathfrak{D}}(\tau)\}$$

and

$$\mathcal{J}_{\mathfrak{D}}(\varrho) \leq \max\left\{\sup\{\mathcal{J}_{\mathfrak{D}}(w) \mid w \in \varrho \diamond \tau\}, \mathcal{J}_{\mathfrak{D}}(\tau)\right\} = \max\{\mathcal{J}_{\mathfrak{D}}(w_\circ), \mathcal{J}_{\mathfrak{D}}(\tau)\}.$$

This completes the proof. □

Corollary 3.3. *In a finite hyper BCK-algebra, every MBJ-neutrosophic hyper BCK-ideal $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ over $\mathcal{H}$ satisfies the condition (3.2).*

Lemma 3.4 ([30]). *Let $\mathcal{K}$ be a subset of a hyper BCK-algebra $\mathcal{H}$. If $\mathcal{I}$ is a hyper BCK-ideal of $\mathcal{H}$ such that $\mathcal{K} \ll \mathcal{I}$, then $\mathcal{K}$ is contained in $\mathcal{I}$.*

Theorem 3.5. *An MBJ-neutrosophic set $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ over $\mathcal{H}$ is an MBJ-neutrosophic hyper BCK-ideal of $\mathcal{H}$ if and only if the nonempty sets $U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $L(\mathcal{J}_{\mathfrak{D}}, \gamma)$ are hyper BCK-ideals of $\mathcal{H}$ for all $(\alpha, \gamma) \in [0,1] \times [0,1]$ and $[\beta^-, \beta^+] \in [I]$.*

Proof. Assume that $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ is an MBJ-neutrosophic hyper BCK-ideal of $\mathcal{H}$. Let $(\alpha, \gamma) \in [0,1] \times [0,1]$ and $[\beta^-, \beta^+] \in [I]$ be such that $U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $L(\mathcal{J}_{\mathfrak{D}}, \gamma)$ are nonempty sets. It easy to see that $0 \in U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $0 \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $0 \in L(\mathcal{J}_{\mathfrak{D}}, \gamma)$ by proposition 3.2(i). Let $\varrho, \tau, u, v, a, b \in \mathcal{H}$ be such that $\varrho \diamond \tau \in U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $\tau \in U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $u \diamond v \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$, $v \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$, $a \diamond b \in L(\mathcal{J}_{\mathfrak{D}}, \gamma)$, and $b \in L(\mathcal{J}_{\mathfrak{D}}, \gamma)$. Then, $\varrho \diamond \tau \ll U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $\tau \in U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $u \diamond v \ll U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$, $v \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$, $a \diamond b \ll L(\mathcal{J}_{\mathfrak{D}}, \gamma)$, and $b \in L(\mathcal{J}_{\mathfrak{D}}, \gamma)$. It follows that

$$(\forall x \in \varrho \diamond \tau)(\exists x_\circ \in U(\mathcal{M}_{\mathfrak{D}}, \alpha) \text{ such that } x \ll x_\circ) \text{ and so } \mathcal{M}_{\mathfrak{D}}(x) \geq \mathcal{M}_{\mathfrak{D}}(x_\circ),$$
$$(\forall y \in u \diamond v)(\exists y_\circ \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+]) \text{ such that } y \ll y_\circ) \text{ and so } \tilde{\mathcal{B}}_{\mathfrak{D}}(y) \geq \tilde{\mathcal{B}}_{\mathfrak{D}}(y_\circ)$$

and

$$(\forall z \in a \diamond b)(\exists z_\circ \in L(\mathcal{J}_{\mathfrak{D}}, \gamma) \text{ such that } z \ll z_\circ) \text{ and so } \mathcal{J}_{\mathfrak{D}}(z) \leq \mathcal{J}_{\mathfrak{D}}(z_\circ),$$

which imply that $\mathcal{M}_{\mathfrak{D}}(x) \geq \alpha$, $\tilde{\mathcal{B}}_{\mathfrak{D}}(y) \geq [\beta^-, \beta^+]$ and $\mathcal{J}_{\mathfrak{D}}(z) \leq \gamma$ for all $x \in \varrho \diamond \tau, y \in u \diamond v$ and $z \in a \diamond b$. Hence, $\inf\{\mathcal{M}_{\mathfrak{D}}(x) \mid x \in \varrho \diamond \tau\} \geq \alpha$, $\inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(y) \mid y \in u \diamond v\} \geq [\beta^-, \beta^+]$ and $\sup\{\mathcal{J}_{\mathfrak{D}}(z) \mid z \in a \diamond b\} \leq \gamma$, and so

$$\mathcal{M}_{\mathfrak{D}}(\varrho) \geq \min\left\{\inf\{\mathcal{M}_{\mathfrak{D}}(x) \mid x \in \varrho \diamond \tau\}, \mathcal{M}_{\mathfrak{D}}(\tau)\right\} \geq \min\{\alpha, \alpha\} = \alpha,$$
$$\tilde{\mathcal{B}}_{\mathfrak{D}}(u) \geq r\min\left\{\inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(y) \mid y \in u \diamond v\}, \tilde{\mathcal{B}}_{\mathfrak{D}}(v)\right\} \geq r\min\{[\beta^-, \beta^+], [\beta^-, \beta^+]\} = [\beta^-, \beta^+]$$

and

$$\mathcal{J}_{\mathfrak{D}}(a) \leq \max\left\{\sup\{\mathcal{J}_{\mathfrak{D}}(z) \mid z \in a \diamond b\}, \mathcal{J}_{\mathfrak{D}}(b)\right\} \leq \max\{\gamma, \gamma\} = \gamma.$$

Thus, $\varrho \in U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $u \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $a \in L(\mathcal{J}_{\mathfrak{D}}, \gamma)$ and therefore $U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $L(\mathcal{J}_{\mathfrak{D}}, \gamma)$ are hyper BCK-ideals of $\mathcal{H}$.

Conversely, assume that the nonempty sets $U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $L(\mathcal{J}_{\mathfrak{D}}, \gamma)$ are hyper BCK-ideals of $\mathcal{H}$ for all $(\alpha, \gamma) \in [0,1] \times [0,1]$ and $[\beta^-, \beta^+] \in [I]$. Let $\varrho, \tau, u, v, a, b \in \mathcal{H}$ be such that $\varrho \ll \tau$,

$u \ll v$, $a \ll b$, $\mathcal{M}_{\mathfrak{D}}(\tau) = \alpha$, $\tilde{\mathcal{B}}_{\mathfrak{D}}(v) = [\beta^-, \beta^+]$ and $\mathcal{J}_{\mathfrak{D}}(b) = \gamma$. Then, $\tau \in U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $v \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $b \in L(\mathcal{J}_{\mathfrak{D}}, \gamma)$, and so $\{\varrho\} \ll U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $\{u\} \ll U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $\{a\} \ll L(\mathcal{J}_{\mathfrak{D}}, \gamma)$. It follows that from Lemma 3.4 that $\{\varrho\} \subseteq U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $\{u\} \subseteq U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $\{a\} \subseteq L(\mathcal{J}_{\mathfrak{D}}, \gamma)$, i.e., $\varrho \in U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $u \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $a \in L(\mathcal{J}_{\mathfrak{D}}, \gamma)$. Hence, $\mathcal{M}_{\mathfrak{D}}(\varrho) \geq \alpha = \mathcal{M}_{\mathfrak{D}}(\tau)$, $\tilde{\mathcal{B}}_{\mathfrak{D}}(u) \succeq [\beta^-, \beta^+] = \tilde{\mathcal{B}}_{\mathfrak{D}}(v)$ and $\mathcal{J}_{\mathfrak{D}}(a) \leq \gamma = \mathcal{J}_{\mathfrak{D}}(b)$. For any $\varrho, \tau, u, v, a, b \in \mathcal{H}$, let

$$\alpha = \min\Big\{\inf\{\mathcal{M}_{\mathfrak{D}}(t_1) \mid t_1 \in \varrho \diamond \tau\}, \mathcal{M}_{\mathfrak{D}}(\tau)\Big\},$$
$$[\beta^-, \beta^+] = r\min\Big\{\inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(t_2) \mid t_2 \in u \diamond v\}, \tilde{\mathcal{B}}_{\mathfrak{D}}(v)\Big\}$$

and

$$\gamma = \max\Big\{\sup\{\mathcal{J}_{\mathfrak{D}}(t_3) \mid t_3 \in a \diamond b\}, \mathcal{J}_{\mathfrak{D}}(b)\Big\}.$$

Then, $\tau \in U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $v \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$, $b \in L(\mathcal{J}_{\mathfrak{D}}, \gamma)$ and

$$\mathcal{M}_{\mathfrak{D}}(t_4) \geq \inf\{\mathcal{M}_{\mathfrak{D}}(t_1) \mid t_1 \in \varrho \diamond \tau\} \geq \min\Big\{\inf\{\mathcal{M}_{\mathfrak{D}}(t_1) \mid t_1 \in \varrho \diamond \tau\}, \mathcal{M}_{\mathfrak{D}}(\tau)\Big\} = \alpha,$$
$$\tilde{\mathcal{B}}_{\mathfrak{D}}(t_5) \succeq \inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(t_2) \mid t_2 \in u \diamond v\} \succeq r\min\Big\{\inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(t_2) \mid t_2 \in u \diamond v\}, \tilde{\mathcal{B}}_{\mathfrak{D}}(v)\Big\} = [\beta^-, \beta^+]$$

and

$$\mathcal{J}_{\mathfrak{D}}(t_6) \leq \sup\{\mathcal{J}_{\mathfrak{D}}(t_3) \mid t_3 \in a \diamond b\} \leq \max\Big\{\sup\{\mathcal{J}_{\mathfrak{D}}(t_3) \mid t_3 \in a \diamond b\}, \mathcal{J}_{\mathfrak{D}}(b)\Big\} = \gamma.$$

for all $t_4 \in \varrho \diamond \tau$, $t_5 \in u \diamond v$ and $t_6 \in a \diamond b$, i.e, $t_4 \in U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $t_5 \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $t_6 \in L(\mathcal{J}_{\mathfrak{D}}, \gamma)$. Thus, $\varrho \diamond \tau \subseteq U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $u \diamond v \subseteq U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $a \diamond b \subseteq L(\mathcal{J}_{\mathfrak{D}}, \gamma)$, which imply from Proposition 2.1(6) that $\varrho \diamond \tau \ll U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $u \diamond v \ll U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $a \diamond b \ll L(\mathcal{J}_{\mathfrak{D}}, \gamma)$. Since $U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $L(\mathcal{J}_{\mathfrak{D}}, \gamma)$ are hyper BCK-ideals of $\mathcal{H}$, we have $\varrho \in U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $u \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $a \in L(\mathcal{J}_{\mathfrak{D}}, \gamma)$, which imply that

$$\mathcal{M}_{\mathfrak{D}}(\varrho) \geq \alpha = \min\Big\{\inf\{\mathcal{M}_{\mathfrak{D}}(t_1) \mid t_1 \in \varrho \diamond \tau\}, \mathcal{M}_{\mathfrak{D}}(\tau)\Big\},$$
$$\tilde{\mathcal{B}}_{\mathfrak{D}}(u) \succeq [\beta^-, \beta^+] = r\min\Big\{\inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(t_2) \mid t_2 \in u \diamond v\}, \tilde{\mathcal{B}}_{\mathfrak{D}}(v)\Big\}$$

and

$$\mathcal{J}_{\mathfrak{D}}(a) \leq \gamma = \max\Big\{\sup\{\mathcal{J}_{\mathfrak{D}}(t_3) \mid t_3 \in a \diamond b\}, \mathcal{J}_{\mathfrak{D}}(b)\Big\}.$$

Therefore, $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ is an MBJ-neutrosophic hyper BCK-ideal of $\mathcal{H}$. □

Now, we define and study the notions of an MBJ-neutrosophic weak (s-weak) hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$.

Definition 3.6. An MBJ-neutrosophic set $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ over $\mathcal{H}$ is called:

(1) An MBJ-neutrosophic weak hyper BCK-ideal of $\mathcal{H}$ if it satisfies Proposition 3.2(i) and Definition 3.1(2).
(2) An MBJ-neutrosophic s-weak hyper BCK-ideal of $\mathcal{H}$ if it satisfies Proposition 3.2(i) and (3.2).

Theorem 3.7. *Every MBJ-neutrosophic hyper BCK-ideal is an MBJ-neutrosophic weak hyper BCK-ideal.*

Proof. Straightforward. □

The converse of Theorem 3.7 is not true in general, as seen in the following example.

Example 3.2. Let $\mathcal{H} = \{0, \varrho, \tau\}$ be a hyper BCK-algebra as in Example 3.1. Let $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ be an MBJ-neutrosophic set over $\mathcal{H}$ given by Table 3.

Table 3. Tabular representation of $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$.

$\mathfrak{D}$	$\mathcal{M}_{\mathfrak{D}}$	$\tilde{\mathcal{B}}_{\mathfrak{D}}$	$\mathcal{J}_{\mathfrak{D}}$
0	0.8	$[\frac{1}{4}, 0.6]$	0.3
ϱ	0.2	$[\frac{1}{8}, 0.2]$	0.6
τ	0.6	$[\frac{1}{6}, 0.5]$	0.5

Then, $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ is an MBJ-neutrosophic weak hyper BCK-ideal of $\mathcal{H}$. Note that $\varrho \ll \tau$,

$$\mathcal{M}_{\mathfrak{D}}(\varrho) = 0.2 < 0.6 = \mathcal{M}_{\mathfrak{D}}(\tau),$$
$$\tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho) = [\frac{1}{8}, 0.2] \prec [\frac{1}{6}, 0.5] = \tilde{\mathcal{B}}_{\mathfrak{D}}(\tau)$$

and

$$\mathcal{J}_{\mathfrak{D}}(\varrho) = 0.6 > 0.5 = \mathcal{J}_{\mathfrak{D}}(\tau).$$

Hence, $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ is not an MBJ-neutrosophic hyper BCK-ideal of $\mathcal{H}$.

Theorem 3.8. *In a hyper BCK-algebra, every MBJ-neutrosophic s-weak hyper BCK-ideal is an MBJ-neutrosophic weak hyper BCK-ideal.*

Proof. Let $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ be an MBJ-neutrosophic s-weak hyper BCK-ideal of over $\mathcal{H}$ and let $\varrho, \tau, u, v, a, b \in \mathcal{H}$. Then, there exist $z_1 \in \varrho \diamond \tau$, $z_2 \in u \diamond v$ and $z_3 \in a \diamond b$ such that $\mathcal{M}_{\mathfrak{D}}(\varrho) \geq \min\left\{\mathcal{M}_{\mathfrak{D}}(z_1), \mathcal{M}_{\mathfrak{D}}(\tau)\right\}$, $\tilde{\mathcal{B}}_{\mathfrak{D}}(u) \succeq r\min\left\{\tilde{\mathcal{B}}_{\mathfrak{D}}(z_2), \tilde{\mathcal{B}}_{\mathfrak{D}}(v)\right\}$ and $\mathcal{J}_{\mathfrak{D}}(a) \leq \max\left\{\mathcal{J}_{\mathfrak{D}}(z_3), \mathcal{J}_{\mathfrak{D}}(b)\right\}$ by the condition (ii) of Proposition 3.2. Since $\mathcal{M}_{\mathfrak{D}}(z_1) \geq \inf\{\mathcal{M}_{\mathfrak{D}}(z_2) \mid z_2 \in \varrho \diamond \tau\}$, $\tilde{\mathcal{B}}_{\mathfrak{D}}(z_2) \succeq \inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(z_3) \mid z_3 \in u \diamond v\}$ and $\mathcal{J}_{\mathfrak{D}}(z_3) \leq \sup\{\mathcal{J}_{\mathfrak{D}}(z_4) \mid z_4 \in a \diamond b\}$, it follows that

$$\mathcal{M}_{\mathfrak{D}}(\varrho) \geq \min\left\{\inf\{\mathcal{M}_{\mathfrak{D}}(z_2) \mid z_2 \in \varrho \diamond \tau\}, \mathcal{M}_{\mathfrak{D}}(\tau)\right\},$$
$$\tilde{\mathcal{B}}_{\mathfrak{D}}(u) \succeq r\min\left\{\inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(z_3) \mid z_3 \in u \diamond v\}, \tilde{\mathcal{B}}_{\mathfrak{D}}(v)\right\}$$

and

$$\mathcal{J}_{\mathfrak{D}}(a) \leq \max\left\{\sup\{\mathcal{J}_{\mathfrak{D}}(z_4) \mid z_4 \in a \diamond b\}, \mathcal{J}_{\mathfrak{D}}(b)\right\}.$$

Therefore, $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ is an MBJ-neutrosophic weak hyper BCK-ideal of over $\mathcal{H}$. □

Question1. Is the converse of Theorem 3.8 true?

It is not easy to find an example of an MBJ-neutrosophic weak hyper BCK-ideal which is not an MBJ-neutrosophic s-weak hyper BCK-ideal. However, we give the following theorem.

Theorem 3.9. *Let $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ be an MBJ-neutrosophic weak hyper BCK-ideal of $\mathcal{H}$ which satisfies the condition (3.1) of Proposition 3.2. Then, $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ is an MBJ-neutrosophic s-weak hyper BCK-ideal of $\mathcal{H}$.*

Proof. For any $\varrho, \tau, u, v, a, b \in \mathcal{H}$, there exist $r_\circ \in \varrho \diamond \tau$, $s_\circ \in u \diamond v$ and $t_\circ \in a \diamond b$ such that $\mathcal{M}_{\mathfrak{D}}(r_\circ) = \inf\{\mathcal{M}_{\mathfrak{D}}(r) \mid r \in \varrho \diamond \tau\}$, $\tilde{\mathcal{B}}_{\mathfrak{D}}(s_\circ) = \inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(s) \mid s \in u \diamond v\}$ and $\mathcal{J}_{\mathfrak{D}}(t_\circ) = \sup\{\mathcal{J}_{\mathfrak{D}}(t) \mid t \in a \diamond b\}$. It follows that

$$\begin{aligned} \mathcal{M}_{\mathfrak{D}}(\varrho) \quad &\geq \min\Big\{\inf\{\mathcal{M}_{\mathfrak{D}}(r) \mid r \in \varrho \diamond \tau\}, \mathcal{M}_{\mathfrak{D}}(\tau)\Big\} \\ &= \min\Big\{\mathcal{M}_{\mathfrak{D}}(r_\circ), \mathcal{M}_{\mathfrak{D}}(\tau)\Big\}, \end{aligned}$$

$$\begin{aligned} \tilde{\mathcal{B}}_{\mathfrak{D}}(u) \quad &\geq r\min\Big\{\inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(s) \mid s \in u \diamond v\}, \tilde{\mathcal{B}}_{\mathfrak{D}}(v)\Big\} \\ &= r\min\Big\{\tilde{\mathcal{B}}_{\mathfrak{D}}(s_\circ), \tilde{\mathcal{B}}_{\mathfrak{D}}(v)\Big\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{J}_{\mathfrak{D}}(a) \quad &\leq \max\Big\{\sup\{\mathcal{J}_{\mathfrak{D}}(t) \mid t \in a \diamond b\}, \mathcal{J}_{\mathfrak{D}}(b)\Big\} \\ &= \max\Big\{\mathcal{J}_{\mathfrak{D}}(t_\circ), \mathcal{J}_{\mathfrak{D}}(b)\Big\}. \end{aligned}$$

Therefore, $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ is an MBJ-neutrosophic s-weak hyper BCK-ideal of over $\mathcal{H}$. □

Theorem 3.10. *An MBJ-neutrosophic set $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ over $\mathcal{H}$ is an MBJ-neutrosophic weak hyper BCK-ideal of $\mathcal{H}$ if and only if the nonempty sets $U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $L(\mathcal{J}_{\mathfrak{D}}, \gamma)$ are weak hyper BCK-ideals of $\mathcal{H}$ for all $(\alpha, \gamma) \in [0, 1] \times [0, 1]$ and $[\beta^-, \beta^+] \in [I]$.*

Proof. It is similar to the proof of Theorem 3.5. □

The following definition presents the concept of an MBJ-neutrosophic strong hyper BCK-ideal of a hyper BCK-algebra $\mathcal{H}$. Next, we study some properties of this concept.

Definition 3.11. An MBJ-neutrosophic set $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ over $\mathcal{H}$ is called an MBJ-neutrosophic strong hyper BCK-ideal of $\mathcal{H}$ if it satisfies:

(1) $(\forall \varrho, \tau \in \mathcal{H})\left(\begin{array}{c} \inf\{\mathcal{M}_{\mathfrak{D}}(z_1) \mid z_1 \in \varrho \diamond \varrho\} \geq \mathcal{M}_{\mathfrak{D}}(\varrho) \\ \mathcal{M}_{\mathfrak{D}}(\varrho) \geq \min\Big\{\sup\{\mathcal{M}_{\mathfrak{D}}(z_2) \mid z_2 \in \varrho \diamond \tau\}, \mathcal{M}_{\mathfrak{D}}(\tau)\Big\} \end{array}\right)$,

(2) $(\forall u, v \in \mathcal{H})\left(\begin{array}{c} \inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(z_2) \mid z_2 \in u \diamond u\} \geq \tilde{\mathcal{B}}_{\mathfrak{D}}(u) \\ \tilde{\mathcal{B}}_{\mathfrak{D}}(u) \geq r\min\Big\{\sup\{\tilde{\mathcal{B}}_{\mathfrak{D}}(z_3) \mid z_3 \in u \diamond v\}, \tilde{\mathcal{B}}_{\mathfrak{D}}(v)\Big\} \end{array}\right)$,

(3) $(\forall w, z \in \mathcal{H})\left(\begin{array}{c} \sup\{\mathcal{J}_{\mathfrak{D}}(z_3) \mid z_3 \in w \diamond w\} \leq \mathcal{J}_{\mathfrak{D}}(w) \\ \mathcal{J}_{\mathfrak{D}}(w) \leq \max\Big\{\inf\{\mathcal{J}_{\mathfrak{D}}(z_4) \mid z_4 \in w \diamond z\}, \mathcal{J}_{\mathfrak{D}}(z)\Big\} \end{array}\right)$.

Example 3.3. Let $\mathcal{H} = \{0, \varrho, \tau\}$ be a set with the hyperoperation " $\diamond$ ", which is given by Table 4.

Table 4. Tabular representation of the hyperoperation " $\diamond$ ".

$\diamond$	0	ϱ	τ
0	$\{0\}$	$\{0\}$	$\{0\}$
ϱ	$\{\varrho\}$	$\{0\}$	$\{\varrho\}$
τ	$\{\tau\}$	$\{\tau\}$	$\{0, \tau\}$

Then, $\mathcal{H}$ is a hyper BCK-algebra (see [3]). Let $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ be an MBJ-neutrosophic set over $\mathcal{H}$ given by Table 5.

Table 5. Tabular representation of $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$.

$\mathfrak{D}$	$\mathcal{M}_{\mathfrak{D}}$	$\tilde{\mathcal{B}}_{\mathfrak{D}}$	$\mathcal{J}_{\mathfrak{D}}$
0	0.63	$[\frac{1}{4}, 0.71]$	0.30
ϱ	0.43	$[\frac{1}{6}, 0.51]$	0.50
τ	0.32	$[\frac{1}{8}, 0.31]$	0.70

It is routine to check that $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ an MBJ-neutrosophic strong hyper BCK-ideal of $\mathcal{H}$.

Proposition 3.12. *Every MBJ-neutrosophic strong hyper BCK-ideal $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ over $\mathcal{H}$ satisfying the following conditions:*

(1) $(\forall \varrho \in \mathcal{H}) \left(\mathcal{M}_{\mathfrak{D}}(0) \geq \mathcal{M}_{\mathfrak{D}}(\varrho), \tilde{\mathcal{B}}_{\mathfrak{D}}(0) \geq \tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho), \mathcal{J}_{\mathfrak{D}}(0) \leq \mathcal{J}_{\mathfrak{D}}(\varrho) \right)$,

(2) $(\forall \varrho, \tau \in \mathcal{H}) \left(\varrho \ll \tau \Rightarrow \mathcal{M}_{\mathfrak{D}}(\varrho) \geq \mathcal{M}_{\mathfrak{D}}(\tau), \tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho) \geq \tilde{\mathcal{B}}_{\mathfrak{D}}(\tau), \mathcal{J}_{\mathfrak{D}}(\varrho) \leq \mathcal{J}_{\mathfrak{D}}(\tau) \right)$,

(3) $(\forall z, \varrho, \tau \in \mathcal{H})\left(z \in \varrho \diamond \tau \Rightarrow \begin{pmatrix} \mathcal{M}_{\mathfrak{D}}(\varrho) \geq \min\left\{\mathcal{M}_{\mathfrak{D}}(z), \mathcal{M}_{\mathfrak{D}}(\tau)\right\} \\ \tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho) \geq r\min\left\{\tilde{\mathcal{B}}_{\mathfrak{D}}(z), \tilde{\mathcal{B}}_{\mathfrak{D}}(\tau)\right\} \\ \mathcal{J}_{\mathfrak{D}}(\varrho) \leq \max\left\{\mathcal{J}_{\mathfrak{D}}(z), \mathcal{J}_{\mathfrak{D}}(\tau)\right\} \end{pmatrix}\right)$.

Proof. (1) Since $0 \in \varrho \diamond \varrho \; \forall \varrho \in \mathcal{H}$, we have

$$\mathcal{M}_{\mathfrak{D}}(0) \geq \inf\{\mathcal{M}_{\mathfrak{D}}(z_1) \mid z_1 \in \varrho \diamond \varrho\} \geq \mathcal{M}_{\mathfrak{D}}(\varrho),$$
$$\tilde{\mathcal{B}}_{\mathfrak{D}}(0) \geq \inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(z_2) \mid z_2 \in \varrho \diamond \varrho\} \geq \tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho)$$

and

$$\mathcal{J}_{\mathfrak{D}}(0) \leq \sup\{\mathcal{J}_{\mathfrak{D}}(z_3) \mid z_3 \in \varrho \diamond \varrho\} \leq \mathcal{J}_{\mathfrak{D}}(\varrho)$$

for all $\varrho \in \mathcal{H}$.

(2) Let $\varrho, \tau \in \mathcal{H}$ be such that $\varrho \ll \tau$. Then, $0 \in \varrho \diamond \tau$ and thus $\sup\{\mathcal{M}_{\mathfrak{D}}(z_1) \mid z_1 \in \varrho \diamond \tau\} \geq \mathcal{M}_{\mathfrak{D}}(0)$, $\sup\{\tilde{\mathcal{B}}_{\mathfrak{D}}(z_2) \mid z_2 \in \varrho \diamond \tau\} \geq \tilde{\mathcal{B}}_{\mathfrak{D}}(0)$ and $\inf\{\mathcal{J}_{\mathfrak{D}}(z_3) \mid z_3 \in \varrho \diamond \tau\} \leq \mathcal{J}_{\mathfrak{D}}(0)$. It follows from Definition 3.11 and (1) that

$$\begin{aligned} \mathcal{M}_{\mathfrak{D}}(\varrho) \quad & \geq \min\{\sup\{\mathcal{M}_{\mathfrak{D}}(z_1) \mid z_1 \in \varrho \diamond \tau\}, \mathcal{M}_{\mathfrak{D}}(\tau)\} \\ & \geq \min\{\mathcal{M}_{\mathfrak{D}}(0), \mathcal{M}_{\mathfrak{D}}(\tau)\} \\ & = \mathcal{M}_{\mathfrak{D}}(\tau), \end{aligned}$$

$$\begin{aligned} \tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho) \quad & \geq r\min\{\sup\{\tilde{\mathcal{B}}_{\mathfrak{D}}(z_2) \mid z_2 \in \varrho \diamond \tau\}, \tilde{\mathcal{B}}_{\mathfrak{D}}(\tau)\} \\ & \geq \min\{\tilde{\mathcal{B}}_{\mathfrak{D}}(0), \tilde{\mathcal{B}}_{\mathfrak{D}}(\tau)\} \\ & = \tilde{\mathcal{B}}_{\mathfrak{D}}(\tau), \end{aligned}$$

and

$$\begin{aligned}\mathcal{J}_{\mathfrak{D}}(\varrho) &\leq \max\{\inf\{\mathcal{J}_{\mathfrak{D}}(z_3) \mid z_3 \in \varrho \diamond \tau\}, \mathcal{J}_{\mathfrak{D}}(\tau)\}\\ &\leq \min\{\mathcal{J}_{\mathfrak{D}}(0), \mathcal{J}_{\mathfrak{D}}(\tau)\}\\ &= \mathcal{J}_{\mathfrak{D}}(\tau),\end{aligned}$$

i.e., $\mathcal{M}_{\mathfrak{D}}(\varrho) \geq \mathcal{M}_{\mathfrak{D}}(\tau)$, $\tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho) \geq \tilde{\mathcal{B}}_{\mathfrak{D}}(\tau)$ and $\mathcal{J}_{\mathfrak{D}}(\varrho) \leq \mathcal{J}_{\mathfrak{D}}(\tau)$ for all $\varrho, \tau \in \mathcal{H}$ with $\varrho \ll \tau$.
(3) Let $z, \varrho, \tau \in \mathcal{H}$ be such that $z \in \varrho \diamond \tau$. Then,

$$\begin{aligned}\mathcal{M}_{\mathfrak{D}}(\varrho) &\geq \min\{\sup\{\mathcal{M}_{\mathfrak{D}}(z_1) \mid z_1 \in \varrho \diamond \tau\}, \mathcal{M}_{\mathfrak{D}}(\tau)\}\\ &\geq \min\{\mathcal{M}_{\mathfrak{D}}(z), \mathcal{M}_{\mathfrak{D}}(\tau)\},\end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho) &\geq r\min\{\sup\{\tilde{\mathcal{B}}_{\mathfrak{D}}(z_2) \mid z_2 \in \varrho \diamond \tau\}, \tilde{\mathcal{B}}_{\mathfrak{D}}(\tau)\}\\ &\geq r\min\{\tilde{\mathcal{B}}_{\mathfrak{D}}(z), \tilde{\mathcal{B}}_{\mathfrak{D}}(\tau)\},\end{aligned}$$

and

$$\begin{aligned}\mathcal{J}_{\mathfrak{D}}(\varrho) &\leq \max\{\inf\{\mathcal{J}_{\mathfrak{D}}(z_3) \mid z_3 \in \varrho \diamond \tau\}, \mathcal{J}_{\mathfrak{D}}(\tau)\}\\ &\leq \max\{\mathcal{J}_{\mathfrak{D}}(z), \mathcal{J}_{\mathfrak{D}}(\tau)\}\end{aligned}$$

for all $z, \varrho, \tau \in \mathcal{H}$ with $z \in \varrho \diamond \tau$. □

Corollary 3.13. *If $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ is an MBJ-neutrosophic strong hyper BCK-ideal over $\mathcal{H}$, then the condition (2) of Definition 3.1 is valid.*

Proof. Note that $\mathcal{M}_{\mathfrak{D}}(z) \geq \inf\{\mathcal{M}_{\mathfrak{D}}(z) \mid z \in \varrho \diamond \tau\}$, $\tilde{\mathcal{B}}_{\mathfrak{D}}(z) \geq \inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(z) \mid z \in \varrho \diamond \tau\}$ and $\mathcal{J}_{\mathfrak{D}}(z) \leq \sup\{\mathcal{M}_{\mathfrak{D}}(z) \mid z \in \varrho \diamond \tau\}$ for all $z, \varrho, \tau \in \mathcal{H}$ with $z \in \varrho \diamond \tau$. Hence, the condition (2) of Definition 3.1 follows from Proposition 3.12(2). □

Theorem 3.14. *Every MBJ-neutrosophic strong hyper BCK-ideal is an MBJ-neutrosophic hyper BCK-ideal.*

Proof. Straightforward. □

The converse of Theorem 3.14 is not true in general. That is, an MBJ-neutrosophic hyper *BCK*-ideal may not be an MBJ-neutrosophic strong hyper *BCK*-ideal.

Example 3.4. Let $\mathcal{H} = \{0, \varrho, \tau\}$ be a hyper *BCK*-algebra as in Example 3.1. Let $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ be an MBJ-neutrosophic set over $\mathcal{H}$ given by Table 6.

Table 6. Tabular representation of $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$.

$\mathfrak{D}$	$\mathcal{M}_{\mathfrak{D}}$	$\tilde{\mathcal{B}}_{\mathfrak{D}}$	$\mathcal{J}_{\mathfrak{D}}$
0	0.63	$[\frac{1}{6}, \frac{1}{4}]$	0.21
ϱ	0.63	$[\frac{1}{9}, \frac{1}{5}]$	0.32
τ	0.32	$[\frac{1}{12}, \frac{1}{6}]$	0.39

Then, $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ is an MBJ-neutrosophic hyper BCK-ideal of $\mathcal{H}$, but it is not an MBJ-neutrosophic strong hyper BCK-ideal of $\mathcal{H}$, since

$$\mathcal{M}_{\mathfrak{D}}(\tau) = 0.32 < 0.63 = \mathcal{M}_{\mathfrak{D}}(\varrho) = \min\left\{\sup\{\mathcal{M}_{\mathfrak{D}}(z) \mid z \in \tau \diamond \varrho\}, \mathcal{M}_{\mathfrak{D}}(\varrho)\right\},$$
$$\tilde{\mathcal{B}}_{\mathfrak{D}}(\tau) = [\frac{1}{12}, \frac{1}{6}] \prec [\frac{1}{9}, \frac{1}{5}] = \tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho) = r\min\left\{\sup\{\tilde{\mathcal{B}}_{\mathfrak{D}}(z) \mid z \in \tau \diamond \varrho\}, \tilde{\mathcal{B}}_{\mathfrak{D}}(\varrho)\right\}$$

and

$$\mathcal{J}_{\mathfrak{D}}(\tau) = 0.39 > 0.32 = \mathcal{J}_{\mathfrak{D}}(\varrho) = \max\left\{\inf\{\mathcal{J}_{\mathfrak{D}}(z) \mid z \in \tau \diamond \varrho\}, \mathcal{J}_{\mathfrak{D}}(\varrho)\right\}.$$

Theorem 3.15. *Let $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ be an MBJ-neutrosophic set over $\mathcal{H}$. If $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ is an MBJ-neutrosophic strong hyper BCK-ideal of $\mathcal{H}$, then the nonempty sets $U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $L(\mathcal{J}_{\mathfrak{D}}, \gamma)$ are strong hyper BCK-ideals of $\mathcal{H}$ for all $(\alpha, \gamma) \in [0, 1] \times [0, 1]$ and $[\beta^-, \beta^+] \in [I]$.*

Proof. Assume that $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ is an MBJ-neutrosophic set over $\mathcal{H}$. Let $(\alpha, \gamma) \in [0, 1] \times [0, 1]$ and $[\beta^-, \beta^+] \in [I]$ be such that $U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $L(\mathcal{J}_{\mathfrak{D}}, \gamma)$ are nonempty sets. Then there exist $r \in U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $t \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $s \in L(\mathcal{J}_{\mathfrak{D}}, \gamma)$, and so $\mathcal{M}_{\mathfrak{D}}(r) \geq \alpha$, $\tilde{\mathcal{B}}_{\mathfrak{D}}(t) \succeq [\beta^-, \beta^+]$ and $\mathcal{J}_{\mathfrak{D}}(s) \leq \gamma$. Clearly, $0 \in U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $0 \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $0 \in L(\mathcal{J}_{\mathfrak{D}}, \gamma)$ by Proposition 3.12 (1). Let $\varrho, \tau, u, v, a, b \in \mathcal{H}$ be such that $\tau \in U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $(\varrho \diamond \tau) \cap U(\mathcal{M}_{\mathfrak{D}}, \alpha) \neq \phi$, $v \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$, $(u \diamond v) \cap U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+]) \neq \phi$, $b \in L(\mathcal{J}_{\mathfrak{D}}, \gamma)$ and $(a \diamond b) \cap L(\mathcal{J}_{\mathfrak{D}}, \gamma) \neq \phi$. Then, there exist $r_\circ \in (\varrho \diamond \tau) \cap U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $t_\circ \in (u \diamond v) \cap U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $s_\circ \in (a \diamond b) \cap L(\mathcal{J}_{\mathfrak{D}}, \gamma)$. Hence, $\mathcal{M}_{\mathfrak{D}}(r_\circ) \geq \alpha$, $\tilde{\mathcal{B}}_{\mathfrak{D}}(t_\circ) \succeq [\beta^-, \beta^+]$ and $\mathcal{J}_{\mathfrak{D}}(s_\circ) \leq \gamma$. It follows that

$$\begin{aligned}
\mathcal{M}_{\mathfrak{D}}(\varrho) &\geq \min\left\{\sup\{\mathcal{M}_{\mathfrak{D}}(r) \mid r \in \varrho \diamond \tau\}, \mathcal{M}_{\mathfrak{D}}(\tau)\right\} \geq \min\{\mathcal{M}_{\mathfrak{D}}(r_\circ), \mathcal{M}_{\mathfrak{D}}(\tau)\} \geq \alpha, \\
\tilde{\mathcal{B}}_{\mathfrak{D}}(u) &\succeq r\min\left\{\sup\{\tilde{\mathcal{B}}_{\mathfrak{D}}(t) \mid t \in u \diamond v\}, \tilde{\mathcal{B}}_{\mathfrak{D}}(v)\right\} \succeq r\min\{\tilde{\mathcal{B}}_{\mathfrak{D}}(t_\circ), \tilde{\mathcal{B}}_{\mathfrak{D}}(v)\} \succeq [\beta^-, \beta^+], \\
\mathcal{J}_{\mathfrak{D}}(a) &\leq \max\left\{\inf\{\mathcal{J}_{\mathfrak{D}}(s) \mid s \in a \diamond b\}, \mathcal{J}_{\mathfrak{D}}(b)\right\} \leq \max\{\mathcal{J}_{\mathfrak{D}}(s_\circ), \mathcal{J}_{\mathfrak{D}}(b)\} \leq \gamma.
\end{aligned}$$

Hence, $\varrho \in U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $u \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $a \in L(\mathcal{J}_{\mathfrak{D}}, \gamma)$. Therefore, $U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $L(\mathcal{J}_{\mathfrak{D}}, \gamma)$ are strong hyper BCK-ideals of $\mathcal{H}$. □

Theorem 3.16. *Let $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ be an MBJ-neutrosophic set over $\mathcal{H}$ which satisfies the condition:*

$$(\forall \mathcal{K} \subseteq \mathcal{H})(\exists \varrho_\circ, \tau_\circ, \eta_\circ \in \mathcal{K}) \text{ such that } \begin{pmatrix} \mathcal{M}_{\mathfrak{D}}(\varrho_\circ) = \sup\{\mathcal{M}_{\mathfrak{D}}(\varrho) \mid \varrho \in \mathcal{K}\} \\ \tilde{\mathcal{B}}_{\mathfrak{D}}(\tau_\circ) = \sup\{\tilde{\mathcal{B}}_{\mathfrak{D}}(\tau) \mid \tau \in \mathcal{K}\} \\ \mathcal{J}_{\mathfrak{D}}(\eta_\circ) = \inf\{\mathcal{J}_{\mathfrak{D}}(\eta) \mid \eta \in \mathcal{K}\} \end{pmatrix}. \tag{3.3}$$

If the nonempty sets $U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $L(\mathcal{J}_{\mathfrak{D}}, \gamma)$ are strong hyper BCK-ideals of $\mathcal{H}$ for all $(\alpha, \gamma) \in [0, 1] \times [0, 1]$ and $[\beta^-, \beta^+] \in [I]$, then $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ is an MBJ-neutrosophic strong hyper BCK-ideal of $\mathcal{H}$.

Proof. Assume that the nonempty sets $U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $L(\mathcal{J}_{\mathfrak{D}}, \gamma)$ are strong hyper BCK-ideals of $\mathcal{H}$ for all $(\alpha, \gamma) \in [0, 1] \times [0, 1]$ and $[\beta^-, \beta^+] \in [I]$. Then, $\varrho \in U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $\tau \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $\eta \in L(\mathcal{J}_{\mathfrak{D}}, \gamma)$ for some $\varrho, \tau, \eta \in \mathcal{H}$, and so $\varrho \diamond \varrho \ll \{\varrho\} \subseteq U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $\tau \diamond \tau \ll \{\tau\} \subseteq U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$

and $\eta \diamond \eta \ll \{\eta\} \subseteq L(\mathcal{J}_{\mathfrak{D}}, \gamma)$. By Lemma 3.4, we have $\varrho \diamond \varrho \subseteq U(\mathcal{M}_{\mathfrak{D}}, \alpha)$, $\tau \diamond \tau \subseteq U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$ and $\eta \diamond \eta \subseteq L(\mathcal{J}_{\mathfrak{D}}, \gamma)$. Thus, for any $a \in \varrho \diamond \varrho, b \in \tau \diamond \tau$ and $c \in \eta \diamond \eta$, we get $a \in U(\mathcal{M}_{\mathfrak{D}}, \alpha), b \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [\beta^-, \beta^+])$, and $c \in L(\mathcal{J}_{\mathfrak{D}}, \gamma)$. Hence, $\mathcal{M}_{\mathfrak{D}}(a) \geq \alpha, \tilde{\mathcal{B}}_{\mathfrak{D}}(b) \succeq [\beta^-, \beta^+]$ and $\mathcal{J}_{\mathfrak{D}}(c) \leq \gamma$. It follows that

$$\begin{aligned} \inf\{\mathcal{M}_{\mathfrak{D}}(a) \mid a \in \varrho \diamond \varrho\} &\geq \alpha = \mathcal{M}_{\mathfrak{D}}(\varrho), \\ \inf\{\tilde{\mathcal{B}}_{\mathfrak{D}}(b) \mid b \in \tau \diamond \tau\} &\succeq [\beta^-, \beta^+] = \tilde{\mathcal{B}}_{\mathfrak{D}}(\tau) \\ \sup\{\mathcal{J}_{\mathfrak{D}}(c) \mid c \in \eta \diamond \eta\} &\leq \gamma = \mathcal{J}_{\mathfrak{D}}(\eta). \end{aligned}$$

For any $\varrho, \tau, u, v, w, z \in \mathcal{H}$. Taking

$$\begin{aligned} r &= \min\left\{\sup\{\mathcal{M}_{\mathfrak{D}}(a) \mid a \in \varrho \diamond \tau\}, \mathcal{M}_{\mathfrak{D}}(\tau)\right\}, \\ \tilde{t} &= [t^-, t^+] = r\min\left\{\sup\{\tilde{\mathcal{B}}_{\mathfrak{D}}(b) \mid b \in u \diamond v\}, \tilde{\mathcal{B}}_{\mathfrak{D}}(v)\right\}, \\ s &= \max\left\{\inf\{\mathcal{J}_{\mathfrak{D}}(c) \mid c \in w \diamond z\}, \mathcal{J}_{\mathfrak{D}}(z)\right\}. \end{aligned}$$

Then by assumption, $U(\mathcal{M}_{\mathfrak{D}}, r)$, $U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [t^-, t^+])$ and $L(\mathcal{J}_{\mathfrak{D}}, s)$ are strong hyper BCK-ideals of $\mathcal{H}$. The condition (3.3) implies that there exist $a_\circ \in \varrho \diamond \tau$, $b_\circ \in u \diamond v$ and $c_\circ \in w \diamond z$ such that $\mathcal{M}_{\mathfrak{D}}(a_\circ) = \sup\{\mathcal{M}_{\mathfrak{D}}(a) \mid a \in \varrho \diamond \tau\}$, $\tilde{\mathcal{B}}_{\mathfrak{D}}(b_\circ) = \sup\{\tilde{\mathcal{B}}_{\mathfrak{D}}(b) \mid b \in u \diamond v\}$ and $\mathcal{J}_{\mathfrak{D}}(c_\circ) = \inf\{\mathcal{J}_{\mathfrak{D}}(c) \mid c \in w \diamond z\}$. Hence,

$$\begin{aligned} \mathcal{M}_{\mathfrak{D}}(a_\circ) &= \sup\{\mathcal{M}_{\mathfrak{D}}(a) \mid a \in \varrho \diamond \tau\} \geq \min\left\{\sup\{\mathcal{M}_{\mathfrak{D}}(a) \mid a \in \varrho \diamond \tau\}, \mathcal{M}_{\mathfrak{D}}(\tau)\right\} = r, \\ \tilde{\mathcal{B}}_{\mathfrak{D}}(b_\circ) &= \sup\{\tilde{\mathcal{B}}_{\mathfrak{D}}(b) \mid b \in u \diamond v\} \succeq r\min\left\{\sup\{\tilde{\mathcal{B}}_{\mathfrak{D}}(b) \mid b \in u \diamond v\}, \tilde{\mathcal{B}}_{\mathfrak{D}}(v)\right\} = [t^-, t^+], \\ \mathcal{J}_{\mathfrak{D}}(c_\circ) &= \inf\{\mathcal{J}_{\mathfrak{D}}(c) \mid c \in w \diamond z\} \leq \max\left\{\inf\{\mathcal{J}_{\mathfrak{D}}(c) \mid c \in w \diamond z\}, \mathcal{M}_{\mathfrak{D}}(z)\right\} = s. \end{aligned}$$

This imply that $a_\circ \in U(\mathcal{M}_{\mathfrak{D}}, r)$, $b_\circ \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [t^-, t^+])$ and $c_\circ \in L(\mathcal{J}_{\mathfrak{D}}, s)$. Hence, $(\varrho \diamond \tau) \cap U(\mathcal{M}_{\mathfrak{D}}, r) \neq \phi$, $(u \diamond v) \cap U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [t^-, t^+]) \neq \phi$, $(w \diamond z) \cap L(\mathcal{J}_{\mathfrak{D}}, s) \neq \phi$ and thus $\varrho \in U(\mathcal{M}_{\mathfrak{D}}, r)$, $u \in U(\tilde{\mathcal{B}}_{\mathfrak{D}}, [t^-, t^+])$ and $w \in L(\mathcal{J}_{\mathfrak{D}}, s)$. It follows that

$$\begin{aligned} \mathcal{M}_{\mathfrak{D}}(\varrho) &\geq r = \min\left\{\sup\{\mathcal{M}_{\mathfrak{D}}(a) \mid a \in \varrho \diamond \tau\}, \mathcal{M}_{\mathfrak{D}}(\tau)\right\}, \\ \tilde{\mathcal{B}}_{\mathfrak{D}}(u) &\succeq [t^-, t^+] = r\min\left\{\sup\{\tilde{\mathcal{B}}_{\mathfrak{D}}(b) \mid b \in u \diamond v\}, \tilde{\mathcal{B}}_{\mathfrak{D}}(v)\right\}, \\ \mathcal{J}_{\mathfrak{D}}(c) &\leq s = \max\left\{\inf\{\mathcal{J}_{\mathfrak{D}}(c) \mid c \in w \diamond z\}, \mathcal{J}_{\mathfrak{D}}(z)\right\}. \end{aligned}$$

Therefore, $\mathfrak{D} = (\mathcal{M}_{\mathfrak{D}}, \tilde{\mathcal{B}}_{\mathfrak{D}}, \mathcal{J}_{\mathfrak{D}})$ is an MBJ-neutrosophic strong hyper BCK-ideal of $\mathcal{H}$. □

4. Conclusions

In this paper, we have applied the MBJ-neutrosophic set to hyper BCK-algebra. We have presented the concepts of the MBJ-neutrosophic hyper BCK-ideal, the MBJ-neutrosophic weak hyper BCK-ideal, the MBJ-neutrosophic s-weak hyper BCK-ideal and the MBJ-neutrosophic strong hyper BCK-ideal, and have discussed related properties and their relations. We have investigated MBJ-neutrosophic (weak, s-weak, strong) hyper BCK-ideals in relation to level cut sets. In the future work, we will use the concept and results in this paper to study other hyper algebraic structures, for instance, hyper BCI-algebra, hyper hoop, hyper MV-algebra and hyper B-algebra.

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Conflict of interest

We declare that we have no conflict of interest.

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