n-Refined Neutrosophic Vector Spaces

Florentin Smarandache[1] and Mohammad Abobala[2]

[1] Department of Mathematics and Science, University of New Mexico, Gallup, NM 87301, USA
[2] Tishreen University, Faculty of Science, Department of Mathematics, Lattakia, Syria
[1]e-mail: smarand@unm.edu

[2]e-mail: mohammadabobala777@gmail.com

Abstract

This paper introduces the concept of n-refined neutrosophic vector spaces as a generalization of neutrosophic vector spaces, and it studies elementary properties of them. Also, this work discusses some corresponding concepts such as weak/strong n-refined neutrosophic vector spaces, and n-refined neutrosophic homomorphisms.

Keywords: n-Refined weak neutrosophic vector space, n-Refined strong neutrosophic vector space, n-Refined neutrosophic homomorphism.

1.Introduction

Neutrosophy as a part of philosophy founded by F. Smarandache to study origin, nature, and indeterminacies became a strong tool in studying algebraic concepts. Neutrosophic algebraic structures were defined and studied such as neutrosophic modules,and neutrosopohic vector spaces, etc.See [1,2,3,4,5,6,7,8,9]. In 2013 Smarandacheintroduceda perfect idea, when he extended the neutrosophic set to refined [n-valued] neutrosophic set, i.e. the truth value T is refined/split into types of sub-truths such as (T_1, T_2, …,) similarly indeterminacy I is refined/split into types of sub-indeterminacies (I_1, I_2, …,) and the falsehood F is refined/split into sub-falsehood (F_1, F_2,..,) [10,11]. Refined neutrosophic algebraic structures were studied such as refined neutrosophic rings, refined neutrosophic modules, and n-refined neutrosophic rings [4,12].

In this article authors try to define n-refined neutrosophic vector spaces, subspaces, and homomorphisms and to present some of their elementary properties.

For our purpose we use multiplication operation (defined in [12]) between indeterminacies $I_1, I_2, \dots, I_n$ as follows:

$$I_m I_s = I_{\min(m,s)}.$$

This work is a continuation of the study on the n-refined neutrosophic structures that began in [12].

2. Preliminaries

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Definition 2.1: [12]

Let (R,+,×) be a ring and $I_k; 1 \le k \le n$ be n indeterminacies. We define R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n$; $a_i \in R$} to be an n-refined neutrosophic ring.

Definition 4.3: [12]

(a) Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$ = {$a_0 + a_1 I + \cdots + a_n I_n$: $a_i \in P_i$}, where P_i is a subset of R, we define P to be an AH-subring if P_i is a subring of R for all . AHS-subring is defined by the condition $P_i = P_j$ for all i , j.

(b) P is an AH-ideal if$P_i$aretwo-side ideals of R for all i, the AHS-ideal is defined by the condition $P_i = P_j$ for all i , j.

(c) The AH-ideal P is said to be null if $P_i = R or P_i = \{0\}$ for all$i$.

Definition 2.3 :[5]

Let (V , + , ·) be a vector space over the field K; then (V(I) , + , ·) is called a weak neutrosophic vector space over the field K, and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field K(I).

Definition 2.4 : [5]

Let V(I) be a strong neutrosophic vector space over the neutrosophic field K(I) and W(I) be a non empty set of V(I) then W(I) is called a strong neutrosophic subspace if W(I) is itself a strong neutrosophic vector space.

Definition 2.6 :[5]

Let U(I) , W(I) be twostrong neutrosophic subspaces of V(I)and let $f: V(I) \to W(I)$, we say that f is a neutrosophic vector space homomorphism if

(a) f(I)=I,

(b) f is a vector space homomorphism.

We define the kernel of f by Ker(f) = { x∈ $V(I)$; f(x) = $0_{W(I)}$}.

Definition 2.7 :[5]

Let $v_1, v_2 .. v_s \in V(I)$and$x \in V(I)$; we say that x is a linear combination of { $v_i; i = 1, .., s$} if

x = $a_1 v_1 + \cdots + a_s v_s$ suchthat$a_i \in K(I)$.

The set{ $\boldsymbol{v_i; i = 1, .., s}$}is called linearly independent if $\boldsymbol{a_1 v_1 + \cdots + a_s v_s = 0}$ implies $\boldsymbol{a_i = 0}$for all i**.**

3. Main concepts and results

Definition 3.1:

Let (K,+,·) be a field, we say that $K_n(I) = K + KI_1 + \cdots + KI_n = \{a_0 + a_1 I_1 + \cdots + a_n I_n; a_i \in K\}$ is an n-refined neutrosophic field.

It is clear that $K_n(I)$ is an n-refined neutrosophic field, but not a field in the classical meaning.

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Example 3.2 :

Let $K = Q$be the field of rationals. The corresponding 3-refined neutrosophic field is

$$Q_3(I) = \{a + bI_1 + cI_2 + dI_3; a, b, c, d \in Q\}.$$

Definition 3.3 :

Let (V,+,·) be a vector space over the field K. Then we say that $V_n(I) = V + VI_1 + \cdots + VI_n = \{x_0 + x_1I_1 + \cdots + x_nI_n;\ x_i \in V\}$ is a weak n-refined neutrosophic vector space over the field K. Elements of $V_n(I)$are called n-refined neutrosophic vectors, elements of K are called scalars.

If we take scalars from the n-refined neutrosophicfield $K_n(I)$, we say that $V_n(I)$ is a strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$. Elements of $K_n(I)$ are called n-refined neutrosophic scalars.

Remark 3.4:

If we take n=1 we get the classical neutrosophic vector space.

Addition on $V_n(I)$is defined as:

$$\sum_{i=0}^{n} a_iI_i + \sum_{i=0}^{n} b_iI_i = \sum_{i=0}^{n} (a_i + b_i)I_i.$$

Multiplication by a scalar $m \in K$is defined as:

$$m \cdot \sum_{i=0}^{n} a_iI_i = \sum_{i=0}^{n} (m.a_i)I_i.$$

Multiplication by an n-refined neutrosophic scalar $m = \sum_{i=0}^{n} m_iI_i \in K_n(I)$is defined as:

$$(\sum_{i=0}^{n} m_iI_i) \cdot (\sum_{i=0}^{n} a_iI_i) = \sum_{i,j=0}^{n} (m_i.a_j)I_iI_j,$$

where $a_i \in V, m_i \in K, I_iI_j = I_{\min(i,j)}$.

Theorem 3.5 :

Let (V,+,·) be a vector space over the field K. Then a weak n-refined neutrosophic vector space $V_n(I)$ is a vector space over the field K. A strong n-refined neutrosophic vector space is not a vector space but a module over the n-refinedneutrosophic field K_n(I).

Proof:

It is similar to that of Theorem 2.3 in [5].

Example 3.6:

Let $V = Z_2$ be the finite vector space of integers modulo 2 over itself:

(a) The corresponding weak 2-refined neutrosophic vector space over the field Z_2 is

$V_n(I) = \{0, 1, I_1, I_2, I_1 + I_2, 1 + I_1 + I_2, 1 + I_1, 1 + I_2\}$.

Definition 3.7:

Let $V_n(I)$ be a weak n-refined neutrosophic vector space over the field K; a nonempty subset $W_n(I)$ is called a weak n-refined neutrosophic subspace of $V_n(I)$ if $W_n(I)$ is a subspace of $V_n(I)$ itself.

Definition 3.8:

Let $V_n(I)$ be a strong n-refined neutrosophic vector space over the n-refined neutrosophic field$K_n(I)$; a nonempty subset $W_n(I)$ is called a strong n-refined neutrosophic subspace of $V_n(I)$ if $W_n(I)$ is a submodule of$V_n(I)$ itself.

Theorem 3.9:

Let $V_n(I)$ be a weak n-refined neutrosophic vector space over the field K, $W_n(I)$ be a nonempty subset of $V_n(I)$. Then $W_n(I)$ is a weak n-refined neutrosophic subspace if and only if:

$x + y \in W_n(I), m \cdot x \in W_n(I)$for all $x, y \in W_n(I), m \in K$.

Proof:

It holds directly from the condition of subspace.

Theorem 3.10:

Let $V_n(I)$ be a strong n-refined neutrosophic vector space over an n-refined neutrosophic field $K_n(I)$, $W_n(I)$ be a nonempty subset of $V_n(I)$. Then $W_n(I)$ is a strong n-refined neutrosophic subspace if and only if:

$x + y \in W_n(I), m \cdot x \in W_n(I)$for all $x, y \in W_n(I), m \in K_n(I)$.

Proof:

It holds directly from the condition of submodule.

Example 3.11:

Let $V = R^2$ be a vector space over the field R, $W = < (0,1) >$ is a subspace of V, $R_2^2(I) = \{(a,b) + (m,s)I_1 + (k,t)I_2; a,b,m,s,k,t \in R\}$ is the corresponding weak/strong 2-refined neutrosophic vector space.

$W_2(I) = \{a_0 + a_1I_1 + a_2I_2\} = \{(0,x) + (0,y)I_1 + (0,z)I_2; x,y,z \in R\}$is a weak 2-refined neutrosophic subspace of the weak 2-refined neutrosophic vector space $R_2^2(I)$ over the field R.

$W_2(I) = \{a_0 + a_1I_1 + a_2I_2\} = \{(0,x) + (0,y)I_1 + (0,z)I_2; x,y,z \in R\}$is a strong 2-refined neutrosophic subspace of the strong 2-refined neutrosophic vector space $R_2^2(I)$ over the n-refined neutrosophic field $R_2(I)$.

Definition 3.12:

Let $V_n(I)$ be a weak n-refined neutrosophic vector space over the field K, x be an arbitrary element of $V_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \subseteq V_n(I)$, or$x = a_1x_1 + a_2x_2 + \cdots + a_mx_m$: $a_i \in K, x_i \in V_n(I)$.

Example 3.13:

Consider the weak 2-refined neutrosophic vector space in Example 3.11,

$x = (0,2) + (1,3)I \in R_2^2(I)$, $x = 2(0,1) + 1(1,0)I_1 + 3(0,1)I_2$, i.e x is a linear combination of the set $\{(0,1), (1,0)I_1, (0,1)I_2\}$ over the field R.

Definition 3.14:

Let $V_n(I)$ be a strong n-refined neutrosophic vector space over an n-refined neutrosophic field $K_n(I)$, x be an arbitrary element of $V_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \subseteq V_n(I)$ is $x = a_1x_1 + a_2x_2 + \dots + a_mx_m$: $a_i \in K_n(I), x_i \in V_n(I)$.

Example 3.15:

Consider the strong 2-refined neutrosophic vector space $R_2^2(I) = \{(a,b) + (m,s)I_1 + (k,t)I_2; a,b,m,s,k,t \in R\}$ over the 2-refined neutrosophic field $R_2(I)$,

$x = (0,2) + (3,3)I_1 + (-1,0)I_2 = (2 + I_1) \cdot (0,1) + (1 + I_2) \cdot (1,1)I_1 + (I_1 - I_2) \cdot (1,0)I_2$, hence x is a linear combination of the set $\{(0,1), (1,1)I_1, (1,0)I_2\}$ over the 2-refined neutrosophic field $R_2(I)$.

Definition 3.16:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a weak n-refined neutrosophic vector space V_n(I) over the field K, X is a weak linearly independent set if $\sum_{i=0}^{m} a_ix_i = 0 \ implies a_i = 0;\ a_i \in K$.

Definition 3.17:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a strong n-refined neutrosophic vector space V_n(I) over the n-refined neutrosophic field $K_n(I)$, X is a weak linearly independent set if $\sum_{i=0}^{m} a_ix_i = 0 \ implies a_i = 0;\ a_i \in K_n(I)$.

Definition 3.18:

Let $V_n(I), W_n(I)$ be two strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$, let $f: V_n(I) \to U_n(I)$ be a well defined map. It is called a strong n-refined neutrosophic homomorphism if:

$f(a.x + b.y) = a.f(x) + b.f(y)$for all $x, y \in V_n(I), a, b \in K_n(I)$.

A weak n-refined neutrosophic homomorphism can be defined as the same.

We can understand the strong n-refined homomorphism as a module homomorphism, weak n-refined neutrosophic homomorphism can be understood as a vector space homomorphism.

Remark:

The previous definition of n-refined homomorphism between two strong/weak n-refined vector spaces is a classical homomorphism between two modules/spaces. We can not add a similar condition to the concept of neutrosophic homomorphism ($f(I_i) = I_i$), since I_i is not supposed to be an element of $V_n(I)$if V has more than one dimension for example. According to our definition,*Ker(f)* will be a subspace (which is different from classical neutrosophic vector space case) sicne$f$ was defined as a classical homomorphism without any additional condition.

Definition 3.19:

Let $f: V_n(I) \to U_n(I)$ be a weak/strong n-refined neutrosophic homomorphism, we define:

(a) $Ker(f) = \{x \in V_n(I); f(x) = 0\}$.

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(b) $Im(f) = \{y \in U_n(I);\ \exists x \in V_n(I) \text{and} y = f(x)\}$.

Theorem 3.20:

Let $f: V_n(I) \to U_n(I)$ be a weak n-refined neutrosophic homomorphism. Then

(a) $Ker(f)$is a weak n-refined neutrosophic subspace of V_n(I).

(b) $Im(f)$is a weak n-refined neutrosophic subspace of $U_n(I)$.

Proof:

(a) $f$is a vector space homomorphism since $V_n(I), U_n(I)$ are vector spaces, hence $Ker(f)$ is a subspace of the vector space $V_n(I)$, thus $Ker(f)$ is a weak n-refined neutrosophic subspace of $V_n(I)$.

(b) It holds by similar argument.

Theorem 3.21:

Let $f: V_n(I) \to U_n(I)$ be a strong n-refined neutrosophic homomorphism. Then

(a) $Ker(f)$is a strong n-refined neutrosophic subspace of V_n(I).

(b) $Im(f)$is a strong n-refined neutrosophic subspace of $U_n(I)$.

Proof:

(a) $f$is a module homomorphism since $V_n(I), U_n(I)$ are modules over the n-refined neutrosophic field $K_n(I)$, hence $Ker(f)$ is a submodule of the vector space $V_n(I)$, thus $Ker(f)$ is a strong n-refined neutrosophic subspace of $V_n(I)$.

(b) Holds by similar argument.

Example 3.22:

Let $R_2^2(I) = \{x_0 + x_1 I_1 + x_2 I_2;\ x_0, x_1, x_2 \in R^2\}$, $R_2^3(I) = \{y_0 + y_1 I_1 + y_2 I_2;\ y_0, y_1, y_2 \in R^3\}$ be two weak 2-refined neutrosophic vector space over the field R. Consider $f: R_2^2(I) \to R_2^3(I)$, where

$f[(a,b) + (m,n)I_1 + (k,s)I_2] = (a,0,0) + (m,0,0)I_1 + (k,0,0)I_2$, f is a weak 2-refined neutrosophic homomorphism over the field R.

$Ker(f) = \{(0,b) + (0,n)I_1 + (0,s)I_2; b,n,s \in R\}$.

$Im(f) = \{(a,0,0) + (m,0,0)I_1 + (k,0,0)I_2; a,m,k \in R\}$.

Example 3.23:

Let $W_2(I) = <(0,0,1)I_1> = \{q.(0,0,a)I_1; a \in R, q \in R_2(I)\}$, $U_2(I) = <(0,1,0)I_1> = \{q.(0,a,0)I_1; a \in R; q \in R_2(I)\}$ be two strong 2-refined neutrosophic subspaces of the strong 2-refined neutrosophic vector space $R_2^3(I)$ over 2-refined neutrosophic field R_2(I). Define $f: W_2(I) \to U_2(I); f[q(0,0,a)I_1] = q(0,a,0)I_1; q \in R_2(I)$.

$f$is a strong 2-refinedneutrosophic homomorphism:

Let $A = q_1(0,0,a)I_1, B = q_2(0,0,b)I_1 \in W_2(I);\ q_1, q_2 \in R_2(I)$, we have

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$$A + B = (q_1 + q_2)(0,0,a+b)I_1, f(A+B) = (q_1 + q_2).(0,a+b,0)I_1 = f(A) + f(B).$$

Let $m = c + dI_1 + eI_2 \in R_2(I)$ be a 2-refined neutrosophic scalar, we have

$m \cdot A = c \cdot q_1(0,0,a)I_1 + d \cdot q_1(0,0,a)I_1I_1 + e \cdot q_1(0,0,a)I_2I_1 = q_1(0,0,c.a + d.a + e.a)I_1$,

$f(m.A) = q_1(0, c.a + d.a + e.a, 0)I_1 = m \cdot f(A)$, hence f is a strong 2-refined neutrosophic homomorphism.

$Ker(f) = (0,0,0) + (0,0,0)I_1 + (0,0,0)I_2$.

$Im(f) = U_2(I)$.

Remark 3.24:

A union of two n-refined neutrosophic vector spaces $V_n(I)$and$W_n(I)$ is not supposed to be an n-refined neutrosophic vector space, since the addition operation can not be defined. For example consider $V = R^3, W = R^2, n = 2$.

5. Conclusion

In this paper we have introducedthe concept of weak/strong n-refined neutrosophic vector space. Also, some related concepts such as weak/strong n-refined neutrosophic subspace, weak/strong n-refined neutrosophic homomorphism have been presented and studied.

Future research

Authors hope that some corresponding notions will be studied in future such as weak/strong n-refined neutrosophic basis, and AH-subspaces.

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