Neutrosophic α-Continuous Multifunction In Neutrosophic Topological Spaces

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Abstract—Aim of this present paper is, we introduce and investigate a new class of continuous multivalued function is called Neutrosophic α-continuous multi valued function in Neutrosophic topological spaces and its properties and characterization are discussed details.

Keywords—Neutrosophic α-closed sets, Neutrosophic α-continuous, Neutrosophic α-continuous multi valued function, Neutrosophic topological spaces

I. INTRODUCTION

C.L. Chang [3] was introduced and developed fuzzy topological space by using L.A. Zadeh’s [18] fuzzy sets. Coker [4] introduced the notion of Intuitionistic fuzzy topological spaces by using Atanasssov’s [1] Intuitionistic fuzzy set. Neutrality the degree of indeterminacy, as an independent concept was introduced by Smarandache [7] in 1998. He also defined the Neutrosophic set on three component (t, f, i) = (Truth, Falsehood, Indeterminacy). The Neutrosophic crisp set concept was converted to Neutrosophic topological spaces by A.A. Salama [12]. I. Arokiarani [2] et al., introduced Neutrosophic α-closed sets. Wadei and saeid [17] are introduced Neutrosophic upper and lower pre continuous multivalued function and R.Dhavaseelan and et.al. [6] are investigated Neutrosophic semi continuous function. Aim of this present paper is, we introduce and investigate a new class of continuous multivalued function is called Neutrosophic α-continuous multivalued function in Neutrosophic topological spaces and its properties and characterization are discussed details.

II. PRELIMINARIES

In this section, we introduce the basic Definition for Neutrosophic sets and its operations. Throughout this paper, (X, τ) is called classical topological spaces on X (represent as CTSX), (Y, τN) is called Neutrosophic topological spaces on Y (represent as NUTSY). The family of all open set in X (α-Open in X, semi-open in X and pre-open in X respectively) is denoted by O(CTSX),( αO(CTSX), SO(CTSX) and PO(CTSX) respectively). The family of all Neutrosophic open set in Y (α-Open in Y, semi-open in Y and pre-open in Y respectively) is denoted by O(NUTSY), (αO(NUTSY), SO(NUTSY) and PO(NUTSY) respectively). The family of all closed set in X (α-closed in X, semi-closed in X and pre-closed in X respectively) is denoted by C(CTSX), (α-C(CTSX), SC(CTSX) and PS(CTSX) respectively). The family of all Neutrosophic Closed in Y (α-closed in Y, semi-closed in Y and pre-closed in Y respectively) is denoted by C(NUTSY),( αC(NUTSY), SC(NUTSY) and PC(NUTSY) respectively).

Definition 2.1[7]
Let X be a non-empty fixed set. A Neutrosophic set A is an object having the form A= {<x, μA(x), σA(x), γA(x)> : x ∈ X}. Where μA(x), σA(x) and γA(x) which represent Neutrosophic of the degree of membership function, the degree indeterminacy and the degree of non membership function respectively of each element x ∈ X to the set A with 0 ≤ μA(x) + σA(x) + γA(x) ≤ 1.

Remark 2.2[7]
we shall use the symbol
A =<x, μA, σA, γA> for the Neutrosophic set A = {<x, μA(x), σA(x), γA(x) : x ∈ X}.

Example 2.3[7]
Every Intuitionistic fuzzy set A is a non-empty set in X is obviously on Neutrosophic set having the form A = {<x, μA(x), 1-((μA(x) + γA(x)), γA(x) : x ∈ X}.

Definition 2.4[7]
we must introduce the Neutrosophic set 0N and 1N in X as follows:
0N be defined as:
0N = {<x, 0, 0, 1> : x ∈ X}
1N be defined as :
1N = {<x, 1, 0, 0> : x ∈ X}
Definition 2.5 [7]
Let \( A = \{ <x, \mu_A(x), \sigma_A(x), \gamma_A(x)> : x \in \mathbb{X} \} \) be a Neutrosophic set on \( X \), Then the complement of the set \( A (A^c) \) defined as
\[ A^c = \{ <x, \mu_A(x), \sigma_A(x), \mu_A(x)> : x \in \mathbb{X} \} \]

Definition 2.6 [7]
Let \( X \) be a non-empty set and Neutrosophic sets \( A \) and \( B \) in the form
\[ A = \{ <x, \mu_A(x), \sigma_A(x), \gamma_A(x)> : x \in \mathbb{X} \} \] and
\[ B = \{ <x, \mu_B(x), \sigma_B(x), \gamma_B(x)> : x \in \mathbb{X} \} \].
Then we consider \( A \) subsets of \( B (A \subseteq B) \),
defined as: \( A \subseteq B \iff \mu_A(x) \leq \mu_B(x), \sigma_A(x) \leq \sigma_B(x), \) and \( \gamma_A(x) \geq \gamma_B(x) \) for all \( x \in \mathbb{X} \)

Definition 2.7 [7]
Let \( X \) be a non-empty set, and Take \( A = \{ <x, \mu_A(x), \sigma_A(x), \gamma_A(x)> : x \in \mathbb{X} \} \) and \( B = \{ <x, \mu_B(x), \sigma_B(x), \gamma_B(x)> : x \in \mathbb{X} \} \) are Neutrosophic sets. Then
(i) \( A \cap B \) defined as: \( A \cap B = \{ <x, \mu_A(x), \mu_B(x), \sigma_A(x), \sigma_B(x), \gamma_A(x), \gamma_B(x)> : x \in \mathbb{X} \} \)
(ii) \( A \cup B \) defined as: \( A \cup B = \{ <x, \mu_A(x), \mu_B(x), \sigma_A(x), \sigma_B(x), \gamma_A(x), \gamma_B(x)> : x \in \mathbb{X} \} \)

Definition 2.8 [7]
We can easily generalize the operation of intersection and union in Definition 2.7 to arbitrary family of Neutrosophic sets as follows:
Let \( \{ A_j : j \in J \} \) be an arbitrary family of Neutrosophic sets in \( X \), then
(i) \( \bigcap_{j \in J} A_j \) defined as: \( \bigcap_{j \in J} A_j = \{ <x, \mu_{A_j}(x), \sigma_{A_j}(x), \gamma_{A_j}(x)> : x \in \mathbb{X} \} \)
(ii) \( \bigcup_{j \in J} A_j \) defined as: \( \bigcup_{j \in J} A_j = \{ <x, \mu_{A_j}(x), \sigma_{A_j}(x), \gamma_{A_j}(x)> : x \in \mathbb{X} \} \)

Proposition 2.9 [9]
For all \( A \) and \( B \) are two Neutrosophic sets then the following condition are true:
(1) \( A \cap B = A \cap B \)
(2) \( A \cup B = A \cup B \)

Definition 2.10 [10]
A Neutrosophic topology is a non-empty set \( X \) is a family \( \tau_X \) of Neutrosophic subsets in \( X \) satisfying the following axioms:
(i) \( \emptyset, X \in \tau_X \).
(ii) \( \forall G_1, G_2 \in \tau_X \), \( G_1 \cap G_2 \in \tau_X \).
(iii) \( \forall G_i \in \tau_X \) for every \( G_i \in \tau_X, i \in J \)
the pair \( (X, \tau_X) \) is called a Neutrosophic topological space.
The element Neutrosophic topological spaces of \( \tau_X \) are called Neutrosophic open sets.
A Neutrosophic set \( A \) is closed if and only if \( A^c \) is Neutrosophic open.

Definition 2.11[10]
Let \( \{ A_G : G \in \mathcal{G} \} \) be Neutrosophic topological spaces and \( A = \{ <x, \mu_A(x), \sigma_A(x), \gamma_A(x)> : x \in \mathbb{X} \} \) be a Neutrosophic set in \( X \). Then the Neutrosophic closure and Neutrosophic interior of \( A \) are defined by
\( \text{Neu-cl}(A) = \bigcap_{K \subseteq \mathcal{G}} K : K \) is a Neutrosophic closed set in \( X \) and \( A \subseteq K \).
\( \text{Neu-int}(A) = \bigcup_{G \in \mathcal{G}} G : G \) is a Neutrosophic open set in \( X \) and \( A \subseteq G \).

Definition 2.12[8]
Let \( \{ X, \tau_X \} \) be Neutrosophic topological spaces and \( \mathcal{A} = \{ <x, \mu_A(x), \sigma_A(x), \gamma_A(x)> : x \in \mathbb{X} \} \) be a Neutrosophic set in \( X \). Then \( \mathcal{A} \) is called if Neutrosophic semi-open if \( \mathcal{A} \subseteq \text{Neu-cl}(\text{Neu-int}(A)) \).
The complement of Neutrosophic semi-open set is called Neutrosophic semi-closed.

Definition 2.13[10]
Let \( \{ X, \tau_X \} \) be Neutrosophic topological spaces and \( \mathcal{A} = \{ <x, \mu_A(x), \sigma_A(x), \gamma_A(x)> : x \in \mathbb{X} \} \) be a Neutrosophic set in \( X \). Then \( \mathcal{A} \) is called if Neutrosophic \( a \)-open set if \( \mathcal{A} \subseteq \text{Neu-cl}(\text{Neu-int}(A)) \).
The complement of Neutrosophic \( a \)-open set is called Neutrosophic \( a \)-closed.

Definition 2.14[10]
Let \( \{ X, \tau_X \} \) be Neutrosophic topological spaces and \( \mathcal{A} = \{ <x, \mu_A(x), \sigma_A(x), \gamma_A(x)> : x \in \mathbb{X} \} \) be a Neutrosophic set in \( X \). Then \( \mathcal{A} \) is called if Neutrosophic pre-open set if \( \mathcal{A} \subseteq \text{Neu-int}(\text{Neu-cl}(A)) \).
The complement of Neutrosophic pre-open set is called Neutrosophic pre-closed.

Remark: 2.15[11]
Let \( A \) be a Neutrosophic topological space \( (X, \tau_X) \). Then
(i) \( \text{Neu-a-cl}(A) = A \cup \text{Neu-cl}(\text{Neu-int}(\text{Neu-cl}(A))) \).
(ii) \( \text{Neu-a-int}(A) = A \cup \text{Neu-int}(\text{Neu-cl}(\text{Neu-int}(A))) \).

Definition 2.16[9]
Take \( r,s,t \) are belongs to real numbers 0 to 1 such that \( 0 \leq r + s + t \leq 1 \). An Neutrosophic point \( \beta_{(r,s,t)} \) is Neutrosophic set defined by
\[ \beta_{(r,s,t)} = \begin{cases} (r,s,t) \text{ if } x = p \\ (0,0,1) \text{ if } x \neq p \end{cases} \]
Take \( p(r,s,t) = \langle p_r, p_s, p_t \rangle \) where \( p_r, p_s, p_t \) are represent Neutrosophic topological spaces the degree of membership function, the degree indeterminacy and the degree of non-membership function respectively of each element \( x \in X \) to the set \( A \)

**Definition 2.17**

A Neutrosophic set \( A \) in \( Y \) is said to be quasi-coincident (q-coincident) with a Neutrosophic set \( B \) denoted by \( A \cap B \), if and only if there exists \( \exists Y \) such that \( A(y) + B(y) > 1 \).

A Neutrosophic set \( \Gamma \) of \( Y \) is called a Neutrosophic neighborhood of a fuzzy point \( y_\alpha \) in \( Y \) if there exists a Neutrosophic open set \( \mu \) in \( Y \) such that \( y_\alpha \in \mu \leq \Gamma \)

**Remark 2.18**

\[ A \cap B \Rightarrow A \subseteq B \]

**Definition 2.19**[9]

Let \( X \) and \( Y \) be two finite sets. Define \( f : X \rightarrow Y \).

1. Neutrosophic continuous (Neu continuous in short) if the pre image (inverse image) of Neutrosophic closed sets (Neu - closed in short) in \( Y \) are Neutrosophic closed sets (Neu - closed in short) in \( X \).
2. Neutrosophic \( \alpha \)-continuous (Neu \( \alpha \) - continuous in short) if the pre image (inverse image) of Neutrosophic closed sets (Neu - closed in short) in \( Y \) are Neutrosophic \( \alpha \)-closed sets (Neu \( \alpha \) - closed in short) in \( X \).
3. Neutrosophic semi-continuous (Neu semi - continuous in short) if the pre image (inverse image) of Neutrosophic closed sets (Neu - closed in short) in \( Y \) are Neutrosophic semi - closed sets (Neu semi - closed in short) in \( X \).

**Definition 2.21.**

Let \( (X, \tau) \) be a topological space in the classical sense and \( (Y, \tau_{NY}) \) be an Neutrosophic topological space. \( F : (X, \tau) \rightarrow (Y, \tau_{NY}) \) is called a Neutrosophic multifunction if and only if for each \( x \in X \), \( F(x) \) is a Neutrosophic set in \( Y \)

**Definition 2.22.**

For a Neutrosophic multifunction \( F : (X, \tau) \rightarrow (Y, \tau_{NY}) \), the upper inverse \( F^+ (\Gamma) \) and the lower inverse \( F^- (\Gamma) \) of a Neutrosophic set \( \Gamma \) in \( Y \) are defined as follows:

\[ F^+ (\Gamma) = \{ x \in X \mid F(x) \leq \Gamma \} \]

\[ F^- (\Gamma) = \{ x \in X \mid F(x) \geq \Gamma \} \]

**Lemma 2.23.**

For a Neutrosophic multifunction \( F : (X, \tau) \rightarrow (Y, \tau_{NY}) \), we have \( F^- (1- \Gamma) = X - F^+ (\Gamma) \), for any Neutrosophic set \( \Gamma \) in \( Y \).

**Definition 2.24**[6]

A Neutrosophic multifunction \( F : (X, \tau) \rightarrow (Y, \tau_{NY}) \) is said to be

1. Neutrosophic upper semi continuous at a point \( x \in X \) if for any \( \Gamma \in O(CTSY) \), \( \Gamma \) containing \( F(x) \) (that is \( F_\Gamma (x) \leq \Gamma \)), there exist \( \exists x \in U \in O(CTSX) \) such that \( F(U) \leq \Gamma \).
2. Neutrosophic lower semi continuous at a point \( x \in X \) if for any \( \Gamma \in O(CTSY) \), with \( F(x) \) \( \Gamma \), there exist \( \exists x \in U \in O(CTSX) \) such that \( U \subseteq F^- (\Gamma) \).
3. Neutrosophic upper semi continuous (Neutrosophic lower semi continuous) if it is Neutrosophic upper semi continuous (Neutrosophic lower semi continuous) at each point \( x \in X \).

**Definition 2.25**[17]

A Neutrosophic multifunction \( F : (X, \tau) \rightarrow (Y, \tau_{NY}) \) is said to be

1. Neutrosophic upper pre -continuous at a point \( x \in X \) if for any \( \Gamma \in O(CTSY) \), \( \Gamma \) containing \( F(x) \) (that is \( F(x) \leq \Gamma \)), there exist \( \exists x \in U \in O(CTSX) \) such that \( F(U) \leq \Gamma \).
2. Neutrosophic lower pre - continuous at a point \( x \in X \) if for any \( \Gamma \in O(CTSY) \), with \( F(x) \) \( \Gamma \), there exist \( \exists x \in U \in O(CTSX) \) such that \( U \subseteq F^- (\Gamma) \).
3. Neutrosophic upper pre-continuous (Neutrosophic lower pre-continuous) if it is Neutrosophic upper pre-continuous (Neutrosophic lower pre-continuous) at each point \( x \in X \).
Definition 2.26.
A Neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_NY)$ is said to be
(i).Neutrosophic upper quasi -continuous at a point $x \in x$ for any $\Gamma \in O(NUTSY)$, $\Gamma$ containing $F(x)$ (that is , $F(x) \leq \Gamma$), there exist $x \in U \in SO(CTSX)$ such that $F(U) \subseteq \Gamma$. (that is $U \subseteq F'(\Gamma)$)
(ii).Neutrosophic lower quasi -continuous at a point $x \in X$ if for any $\Gamma \in O(NUTSY)$, with $F(x)\Gamma$, there exist $x \in U \in SO(CTSX)$ such that $U \subseteq F^- (\Gamma)$.
(iii).Neutrosophic upper quasi-continuous (Neutrosophic lower quasi-continuous) if it is Neutrosophic upper quasi-continuous (Neutrosophic lower quasi-continuous) at each point $x \in X$.

Definition 2.27
Let $A$ be a Neutrosophic set in Neutrosophic fuzzy topology space$(Y, \tau_NY)$,Then $V$ is said to be a neighborhood of $A$ in $Y$ if there exist an Neutrosophic open set $U$ of $Y$ such that $A \subseteq U \subseteq V$.

III.NEUTROSOPHIC LOWER $\alpha$ - CONTINUOUS MULTIFUNCTION

In this section, we introduce the Definition for Neutrosophic Lower $\alpha$ - continuous multifunction and its properties

Definition 3.1.
A Neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_NY)$ is said to be
(i).Neutrosophic lower $\alpha$- continuous at a point $x \in X$ if for any $\Gamma \in O(NUTSY)$, with $F(x)\Gamma$, there exist $x \in U \in \alpha O(CTSX)$ such that $U \subseteq F^- (\Gamma)$.
(ii).Neutrosophic lower $\alpha$-continuous if it is Neutrosophic lower $\alpha$-continuous at each point $x \in X$.

Theorem 3.2
Every Neutrosophic lower semi continuous multifunction is Neutrosophic lower $\alpha$ continuous multifunction.

Proof:
Take for any $F \in O(NUTSY)$, with $F(x)\Gamma$. By our assumption, there exist $x \in U \in O(CTSX)$ such that $U \subseteq F^- (\Gamma)$.This implies, there exist $x \in U \in \alpha O(CTSX)$ such that $U \subseteq F^- (\Gamma)$.since open sets are $\alpha$-open set in $X$.

Remark 3.3
Converse of the above theorem need not be true.

Example 3.4
Consider $X=\{a, b, c\}$, $Y=[0,1]$ and take $\tau =\{\emptyset, \{a\}, X\}$ and $\tau_NY =\{0, 1_N, \beta(0.25, 0.25, 0.5), \beta(0.3, 0.3, 0.4)\}$ are topology and Neutrosophic topology on $X$ and $Y$ respectively. We using the notion Neutrosophic point (constant) $\beta(r,s,t)$ $=\beta(y,\beta,\beta,\beta), \forall Y$.Define the Neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_NY)$ by $F(a)=\beta(0.7, 0.2, 0.1)$, $F(b)=\beta(0.4, 0.3, 0.3)$, $F(c)=\beta(0.75, 0.1, 0.15)$. $F$ is Neutrosophic lower $\alpha$ continuous multifunction but not Neutrosophic lower semi continuous multifunction. Since $F^-(\beta(0.25, 0.25, 0.5)) =\{a, c\}$ and $F^-(\beta(0.3, 0.3, 0.4)) =\{a, c\}$ are $\alpha$-open set in $X$ but not open set in $X$.

Theorem 3.5
Every Neutrosophic lower $\alpha$continuous multifunction is Neutrosophic lower quasi semi continuous multifunction.

Proof:
For any $F \in O(NUTSY)$, with $F(x)\Gamma$. By our assumption, there exist $x \in U \in \alpha O(CTSX)$ such that $U \subseteq F^- (\Gamma)$.This implies, there exist $x \in U \in SO(CTSX)$ such that $U \subseteq F^- (\Gamma)$.since $\alpha$-open sets are semi open set in $X$.

Remark 3.6
Converse of the above theorem need not be true.

Example 3.7
Consider $X=\{a, b, c\}$, $Y=[0,1]$ and take $\tau =\{\emptyset, \{a\}, \{a, c\}, X\}$ and $\tau_NY =\{0, 1_N, \beta(0.25, 0.25, 0.5), \beta(0.3, 0.3, 0.4)\}$ are topology and Neutrosophic topology on $X$ and $Y$ respectively. We using the notion Neutrosophic point (constant) $\beta(r,s,t)$ $=\beta(y,\beta,\beta,\beta,\beta), \forall Y$.Define the Neutrosophic multifunction $F : (X, \tau) \to (Y, \tau_NY)$ by $F(a)=\beta(0.7, 0.2, 0.1)$, $F(b)=\beta(0.4, 0.3, 0.3)$, $F(c)=\beta(0.75, 0.1, 0.15)$. $F$ is Neutrosophic lower quasi semi continuous multifunction but not Neutrosophic lower $\alpha$ semi continuous multifunction. Since $F^-\beta((0.25, 0.25, 0.5)) =\{a, c\}$ and $F^-\beta((0.3, 0.3, 0.4)) =\{a, c\}$ are semi-open set in $X$ but not $\alpha$- open set in $X$.

Theorem 3.8
Every Neutrosophic lower $\alpha$ continuous multifunction is Neutrosophic lower pre continuous multifunction.

Proof:
For any $F \in O(NUTSY)$, with $F(x)\Gamma$. By our assumption, there exist $x \in U \in \alpha O(CTSX)$ such that $U \subseteq F^- (\Gamma)$.This implies, there exist $x \in U \in PO(CTSX)$ such that $U \subseteq F^- (\Gamma)$.since $\alpha$-open sets are an open pre set in $X$.

Remark 3.9
Converse of the above theorem need not be true.
Example:3.10
Consider $X=\{a, b, c, d\}$, $Y=[0,1]$ and take $\tau=\{\emptyset, \{a\}, \{b, c\}, X\}$ and $\tau_{N_Y}=\{0_N, 1_N, \hat{\beta}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \hat{\beta}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)\}$ are topology and Neutrosophic topology on $X$ and $Y$ respectively. We using the notion Neutrosophic point (constant) $\beta(r,s,t)=<y, \beta_Y, \beta_Y, \forall y>$. Define the Neutrosophic multifunction $F: (X, \tau) \rightarrow (Y, \tau_{N_Y})$ by $F(a)=\hat{\beta}\left(\frac{2}{3}, \frac{1}{2}, \frac{1}{2}\right), F(b)=\hat{\beta}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), F(c)=\hat{\beta}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $F(d)=\hat{\beta}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. $F$ is Neutrosophic lower pre continuous multifunction but not Neutrosophic lower $\alpha$-continuous multifunction. Since $F^{-1}(\hat{\beta}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)) = \{a, b, c\}$ and $F^{-1}(\hat{\beta}\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)) = \{a, b, c\}$ are pre-open set in $X$ but not $\alpha$-open set in $X$.

Remark 3.11. We obtain the following diagram from the results we discussed above.

Diagram-1

IV. NEUTROSOPHIC UPPER $\alpha$-CONTINUOUS MULTIFUNCTION.

In this section, we introduce the definition for Neutrosophic upper $\alpha$-continuous multifunction and its properties.

Definition 4.1.
A Neutrosophic multifunction $F : (X, \tau) \rightarrow (Y, \tau_{N_Y})$ is said to be
(i). Neutrosophic upper $\alpha$-continuous at a point $x \in X$ if for any $\Gamma \in O(NUTSY)$, $\Gamma$ containing $F(x)$ (that is, $F(x) \leq \Gamma$), there exists $x \in U \in \alpha O(CTSX)$ such that $F(U) \leq \Gamma$ (that is, $U \cap F'(\Gamma)$).
(ii). Neutrosophic upper $\alpha$-continuous if it is Neutrosophic upper $\alpha$-continuous at each point $x \in X$.

Theorem:4.2
Every Neutrosophic upper semi continuous multifunction is Neutrosophic upper $\alpha$-continuous multifunction.

Proof:
For any $\Gamma \in O(NUTSY)$, $\Gamma$ containing $F(x)$. By our assumption, there exists $x \in U \in O(CTSX)$ such that $F(U) \leq \Gamma$. This implies there exists $x \in U \in \alpha O(CTSX)$ such that $F(U) \leq \Gamma$. Since open sets are $\alpha$-open sets in $X$.

Remark:4.3:
Converse of the above theorem need not be true.

Example:4.4
Consider $X=\{a, b, c\}$, $Y=[0,1]$ and take $\tau=\{\emptyset, \{b\}, X\}$ and $\tau_{N_Y}=\{0_N, 1_N, \hat{\beta}(0.7, 0.1, 0.2), \hat{\beta}(0.3, 0.4, 0.3)\}$ are topology and Neutrosophic topology on $X$ and $Y$ respectively. We using the notion Neutrosophic point (constant) $\beta(r,s,t)=<y, \beta_Y, \beta_Y, \forall y>$. Define the Neutrosophic multifunction $F: (X, \tau) \rightarrow (Y, \tau_{N_Y})$ by $F(a)=\hat{\beta}(0.3, 0.1, 0.6), F(b)=\hat{\beta}(0.5, 0.2, 0.3), F(c)=\hat{\beta}(0.8, 0.1, 0.1)$. $F$ is Neutrosophic upper $\alpha$-continuous Neutrosophic multifunction but not Neutrosophic upper semi continuous Neutrosophic multifunction. Since $F^{-1}(\hat{\beta}(0.7, 0.1, 0.2)) = \{a, b\}$ and $F^{-1}(\hat{\beta}(0.3, 0.4, 0.3)) = \{a, b\}$ are $\alpha$-open set in $X$ but not open set in $X$.

Theorem:4.5
Every Neutrosophic upper $\alpha$-continuous multifunction is Neutrosophic upper quasi semi continuous multifunction.

Proof:
For any $\Gamma \in O(NUTSY)$, $\Gamma$ containing $F(x)$. By our assumption, there exists $x \in U \in \alpha O(CTSX)$ such that $F(U) \leq \Gamma$. This implies there exists $x \in U \in S O(CTSX)$ such that $F(U) \leq \Gamma$. Since $\alpha$-open sets are semi-open set in $X$. 

Where $A \rightarrow B$ represents $A$ implies $B$.
Remark: 4.6
Converse of the above theorem need not be true.

Example: 4.7

Let $X = \{a, b, c\}$, $Y = [0, 1]$ and take $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\tau_{NY} = \{0_N, 1_N, \widetilde{\beta}(\frac{4}{6}, \frac{1}{2}, \frac{1}{2}), \widetilde{\beta}(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})\}$ be topology and Neutrosophic topology on X and Y respectively. We using the notion Neutrosophic point (constant) $f(r,s,t) = (y, \beta, \delta, \kappa)$, $\forall y$. Define the Neutrosophic multifunction $F: (X, \tau) \rightarrow (Y, \tau_{NY})$ by $F(a) = \widetilde{\beta}(\frac{4}{6}, \frac{1}{2}, \frac{1}{2})$, $F(b) = \widetilde{\beta}(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$, $F(c) = \widetilde{\beta}(\frac{6}{7}, \frac{1}{4}, \frac{1}{14})$ is Neutrosophic upper $\alpha$ continuous Neutrosophic multifunction but not Neutrosophic upper semi continuous Neutrosophic multifunction. Since $F^*(\widetilde{\beta}(\frac{1}{3}, \frac{2}{3}, \frac{2}{3})) = \{a, b\}$ are semi-open in X but not $\alpha$-open set in X.

Theorem: 4.8
Every Neutrosophic upper $\alpha$ continuous multifunction is Neutrosophic upper pre continuous multifunction.

Proof:
For any $\Gamma \in O(\text{NUTSY})$, $\Gamma$ containing $F(x)$. By our assumption, there exists $x \in U \in \alpha O(\text{CTS}X)$ such that $F(U) \leq \Gamma$. This implies there exists $x \in U \in PO(\text{CTS}X)$ such that $F(U) \leq \Gamma$ since $\alpha$-open sets are pre-open set in X.

Remark: 4.9
Converse of the above theorem need not be true.

Example: 4.91
Consider $X = \{a, b, c, d\}$, $Y = [0, 1]$ and take $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\tau_{NY} = \{0_N, 1_N, \widetilde{\beta}(0.5, 0.25, 0.25), \widetilde{\beta}(0.3, 0.3, 0.4)\}$ be topology and Neutrosophic topology on X and Y respectively. We using the notion Neutrosophic point (constant) $f(r,s,t) = (y, \beta, \delta, \kappa)$, $\forall y$. Define the Neutrosophic multifunction $F: (X, \tau) \rightarrow (Y, \tau_{NY})$ by $F(a) = \widetilde{\beta}(0.4, 0.2, 0.4)$, $F(b) = \widetilde{\beta}(0.4, 0.35, 0.15)$, $F(c) = \widetilde{\beta}(0.5, 0.25, 0.25)$ and $F(d) = \widetilde{\beta}(0.4, 0.3, 0.3)$, $F$ is Neutrosophic upper $\alpha$ continuous Neutrosophic multifunction but not Neutrosophic upper $\alpha$ continuous Neutrosophic multifunction. Since $F^*(\widetilde{\beta}(0.5, 0.25, 0.25)) = \{a, b, c\}$ and $F^*(\widetilde{\beta}(0.3, 0.3, 0.4)) = \{a, b, c\}$ are pre-open in X but not $\alpha$-open set in X.

Remark 4.92
We obtain the following diagram from the results we discussed above.

Diagram-II

V. PROPERTIES ABOUT LOWER AND UPPER $\alpha$ CONTINUOUS MULTIFUNCTION

In this section, we derive some application about Lower and Upper $\alpha$ Continuous Multifunction.

Theorem 5.1.
Let $F: (X, \tau) \rightarrow (Y, \tau_{NY})$ be an Neutrosophic multifunction and let $x \in \chi$. Then the following statements are equivalent:

(a) $F$ is Neutrosophic lower $\alpha$-continuous at $x$.

(b) Every $\tilde{\Omega} \in O(\text{NUTSY})$ with $F(x)q\tilde{\Omega}$, implies $x \in s\text{Cl} \left(\text{Int} \ F^{-1}(\tilde{\Omega})\right)$.

(c) For any $x \in U \in SO(\text{CTS}X)$ and for any $\tilde{\Omega} \in O(\text{NUTSY})$ with $F(x)q\tilde{\Omega}$, there exists a non empty open set $B \in \text{W}$ such that $F(v)q\tilde{\Omega}$, for all $v \in B$. 

Where $A \implies B$ represents $A$ implies $B$.
Proof.
(a) \(\Rightarrow\) (b). Assume that \(F\) is Neutrosophic lower \(\alpha\)-continuous at \(x\) and \(x \in X\) and \(\overline{\Omega} \in O(\text{NUTSY})\) with \(F(x)q\overline{\Omega}\). By our assumption, there exist \(W \in aO(\text{CTSX})\) such that \(x \in W\) and \(F(W)q\overline{\Omega}\). Thus, \(x \in W \subset F^{-}(\overline{\Omega})\). Then, \(W \in aO(\text{CTSX})\) implies \(W \subset sCl(\text{Int}(W))\). Let \(x \in W\). Then \(x \in X\). Let \(x \in W\). Then \(x \in X\). Let \(x \in W\). Then \(x \in X\). Let \(x \in W\). Then \(x \in X\).

Theorem 5.2.
For an Neutrosophic multifunction \(F : (X, \tau) \rightarrow (Y, \tau_{NY})\) and let \(x \in X\), the following statements are equivalent:
(1) \(F\) is Neutrosophic lower \(\alpha\)-continuous.
(2) \(F^{-}(G) \in aO(\text{CTSX})\) for every \(G \in O(\text{NUTSY})\).
(3) \(s\text{Int}(Cl(F^{+}(\overline{B})) \subset F^{+}(\text{Int} \text{Cl}(\overline{B}))\) for any Neutrosophic set \(\overline{B}\) of \(Y\).
(4) \(s\text{Int}(\text{Cl}(F^{+}(\overline{B})) \subset F^{+}(\text{Int} \text{Cl}(\overline{B}))\) for each subset \(A\) of \(X\).

Proof.
(1) \(\Rightarrow\) (2).
Let \(\overline{B}\) be any Neutrosophic open set of \(Y\) and \(x \in F^{-}(\overline{G})\). So \(F(x)q\overline{G}\), since \(F\) is Neutrosophic lower \(\alpha\) - continuous multifunction, it follows that \(x \in s\text{Cl}(\text{Int}(F^{-}(\overline{G}))\). Hence \(F^{-}(\overline{G}) \in aO(X)\). Let \(x\) be arbitrarily chosen in \(X\) and \(\overline{G}\) be any Neutrosophic open set of \(Y\) such that \(F(x)q\overline{G}\), so \(x \in F^{-}(\overline{G})\). By hypothesis \(F^{-}(\overline{G}) \in aO(\text{CTSX})\), we have \(x \in F^{-}(\overline{G}) \subset s\text{Cl}(\text{Int}(F^{-}(\overline{G}))\) and thus \(F\) is Neutrosophic lower \(\alpha\)-continuous at \(x\). As \(x\) is arbitrarily chosen, \(F\) Neutrosophic lower \(\alpha\)-continuous. (2) \(\Rightarrow\) (3). It follows from the fact that \([F^{-}(\overline{A})]^{c} = F^{+}(\overline{A}^{c})\) for every Neutrosophic set \(\overline{A}\) of \(Y\) and compliment of every open set is always closed. (3) \(\Rightarrow\) (4). Let \(B\) be any Neutrosophic open set of \(Y\) since \(\text{Cl}(\overline{B})\) is Neutrosophic closed set in \(Y\). Then by (3), \(F^{+}(\text{Cl}(\overline{B})\) is an \(\alpha\) - closed set in \(X\). Thus we have \(F^{+}(\text{Cl}(\overline{B})) \supset s\text{Int}(\text{Cl}(F^{+}(\overline{B}))\) .

(4) \(\Rightarrow\) (5).
Let \(A\) be an arbitrary subset of \(X\). Let us put \(F(A) = \overline{B}\). Then \(A \subset F^{+}(\overline{B})\).
Therefore, \(s\text{Int}(\text{Cl}(A)) \subset s\text{Int}(F^{+}(\overline{B}))\subset F^{+}(\text{Cl}(\overline{B}))\).
Therefore, \(F\text{Int}(\text{Cl}(A)) \subset F\text{Int}(F^{+}(\overline{B}))\subset F^{+}(\text{Cl}(\overline{B}))\subset \text{Cl}(F(A)).\)

Put \(\overline{A} = F^{+}(\overline{B})\). Then \(\text{Cl}(A) \subset \overline{B}\).
Therefore, \(\text{Cl}(\text{Cl}(A)) \subset \text{Cl}(\overline{B})\).
Then \(\text{Cl}(\overline{B}) = \overline{B}\).
Thus \(\text{Cl}(\text{Cl}(A)) = \text{Cl}(\overline{B})\).
Hence \(\text{Cl}(\text{Cl}(A)) \subset F^{+}(\text{Cl}(\overline{B}))\subset F^{+}(\text{Cl}(F(A)))\subset \text{Cl}(F(A)).\)

Consequently \(F^{+}(\text{Cl}(\overline{B})) \subset F^{+}(\overline{B})\) but \(F^{+}(\text{Cl}(\overline{B})) \supset F^{+}(\overline{B})\).
Thus \(F^{+}(\overline{B})\) is \(\alpha\) - closed set in \(X\). (6) \(\Rightarrow\) (7).
Let \(\overline{B}\) be any Neutrosophic closed set of \(Y\). Put \(\overline{A} = F^{+}(\overline{B})\).
Then \(\text{Cl}(\text{Cl}(\overline{B})) \subset \text{Cl}(\overline{B})\).
Thus \(\text{Cl}(\overline{B}) = \overline{B}\).
Consequently \(\text{Cl}(\overline{B}) \subset \text{Cl}(\overline{B})\).
Thus \(\text{Cl}(\overline{B}) = \overline{B}\).
Replacing \(\overline{B}\) by \(F(A)\).
This implies \(\text{Cl}(\text{Cl}(\overline{B})) \subset \text{Cl}(\overline{B})\).
This implies \(\text{Cl}(\text{Cl}(\overline{B})) \subset \text{Cl}(\overline{B})\).
This implies \(\text{Cl}(\text{Cl}(\overline{B})) = \text{Cl}(\overline{B})\).
(7) \(\Rightarrow\) (8).
It is clearly true. (8) \(\Rightarrow\) (1).
Let \(x \in X\) and \(\overline{Y}\) be a Neutrosophic set in such that \(F(x)q\overline{Y}\).
Thus \(x \in F^{-}(\overline{Y})\).
We have to prove that \(F^{-}(\overline{Y})\) is \(\alpha\) - open set in \(X\). We have \(F(\text{Cl}(\text{Cl}(\overline{B})) \subset \text{Cl}(\overline{B}) \subset \text{Cl}(\overline{B})\).
Hence \(F^{-}(\overline{Y})\) is \(\alpha\) - open set in \(X\).

Theorem 5.3.
For an Neutrosophic multifunction $F \colon (X, τ) \to (Y, τ_NY)$ and let $x \in X$, the following statements are equivalent:
(a) $F$ is Neutrosophic upper $α$-continuous at $x$.
(b) For each Neutrosophic open set $\tilde{G}$ of $Y$ with $F(x) \subset \tilde{G}$, there results the relation $x \in sCl(\text{Int}(F^+(\tilde{G})))$.
(c) For any semi-open set $U \subset X$ containing $x$ and for any Neutrosophic open set $\tilde{G}$ of $Y$, $F(x) \subset \tilde{G}$, there exists a non empty open set $V \subset U$ such that $F(V) \subset \tilde{G}$.

Proof.
(a)⇒(b). Let $x \in X$ and $\tilde{G}$ be any Neutrosophic open set of $Y$ such that $F(x) \subset \tilde{G}$, there is a $U \in aO(CTXS)$ such that $x \in U$ and $F(u) \subset \tilde{G}$, for all $u \in U$. Thus $x \in U \subset F^+(\tilde{G})$. Since $U \in aO(X), U \subset sCl(\text{Int}(U)) \subset sCl(\text{Int}(F^+(\tilde{G})))$. Hence, $x \in sCl(\text{Int}(F^+(\tilde{G})))$. (b)⇒(c). Let $\tilde{B}$ be any Neutrosophic open set of $Y$ such that $F(x) \subset \tilde{B}$, then $x \in sCl(\text{Int}(F^+(\tilde{B})))$. Let $U \subset X$ be any semi-open set such that $x \in U$. Then $U \cap \text{Int}(F^+(\tilde{B})) \neq \emptyset$. Put $V = U \cap \text{Int}(F^+(\tilde{B}))$. Then $V$ is an semi-open set in $X$, $V \subset U$, $V \neq \emptyset$ and $F(V) \subset \tilde{G}$. (c)⇒(a). Let $\{U_\delta\}$ be the system of the semi-open sets in $X$ containing $x$. For any semi-open set $U \subset X$ such that $x \in U$ and $\tilde{G}$ be any Neutrosophic open set of $Y$ such that $F(x) \subset \tilde{G}$, there exists a non empty open set $\tilde{G}_U \subset U$ such that $F(\tilde{G}_U) \subset \tilde{G}$.

Let $W = U_{\delta \in U_\delta} G_\delta$. Then $W$ is open, $x \in sCl(W)$ and $F(w) \subset \tilde{G}$. for all $w \in W$ . Put $S = W \cup \{x\}$, then $W \subset S \subset sCl(W)$. Thus $S \in aO(CTXS), x \in S$ and $F(w) \subset \tilde{G}$, for all $w \in S$. Hence $F$ is Neutrosophic upper $α$-continuous at $x$.

Theorem 5.4.
For an Neutrosophic multifunction $F \colon (X, τ) \to (Y, τ_Y)$ and let $x \in X$, the following statements are equivalent:
(a) $F$ is Neutrosophic upper $α$-continuous.
(b) $F^+(\tilde{G}) \in aO(CTXS)$, for every Neutrosophic open set $\tilde{G}$ of $Y$.
(c) $F^- (\tilde{B}) \in aC(CTXS)$ for each Neutrosophic closed set $\tilde{B}$ of $Y$.
(d) For each point $x \in X$ and for each neighborhood $\tilde{V}$ of $F(x)$ in $Y$, $F^+(\tilde{V})$ is a $α$- neighborhood of $x$.
(e) For each point $x \in X$ and for each neighborhood $\tilde{V}$ of $F(x)$ in $Y$, there is an $α$- neighborhood $U$ of $x$ such that $F(U) \subset \tilde{V}$.
(f) $aC(F^-(\tilde{B})) \subset F^- (\text{Cl}(\tilde{B}))$ for each Neutrosophic closed set $\tilde{B}$ of $Y$.

Proof.
(a)⇒(b). Let $\tilde{B}$ be any Neutrosophic open set of $Y$ and $x \in F^+(\tilde{B})$. We get $x \in sCl(\text{Int}(F^+(\tilde{B})))$. Hence, $F^+(\tilde{B}) \in aO(CTXS)$ (b)⇒(a). Let $x \in X$ and $\tilde{G}$ be any Neutrosophic open set of $Y$ such that $F(x) \subset \tilde{G}$, so $x \in F^+(\tilde{G})$. By assumption $F^+(\tilde{G}) \in aO(CTXS)$, we have $x \in F^+(\tilde{G}) \in sCl(\text{Int}(F^+(\tilde{G})))$ and Thus $F$ is Neutrosophic upper $α$ - continuous at $x$. Hence $F$ is Neutrosophic upper $α$-continuous.

Thus we have $F^- (\text{Cl}(\tilde{B})) \supset sInt(\text{Cl}(F^- (\text{Cl}(\tilde{B})))) \supset sInt(\text{Cl}(F^-(\tilde{B}))) \supset F^-(\tilde{B})) (sInt(\text{Cl}(F^-(\tilde{B}))) \supset aC(F^-(\tilde{B})))$. (f)⇒(g). Let $\tilde{B}$ be any Neutrosophic open set of $Y$, we have $aC(F^-(\tilde{B})) = F^+(\tilde{B}) (sInt(\text{Cl}(F^-(\tilde{B}))) \supset F^-(\text{Cl}(\tilde{B})))$. (g)⇒(c). Let $\tilde{B}$ be any Neutrosophic closed set of $Y$. Then we have $sCl(\text{Cl}(F^-(\tilde{B}))) \subset F^-(\tilde{B}) \subset sInt(\text{Cl}(F^-(\tilde{B}))) \subset F^-(\text{Cl}(\tilde{B})$. Hence $F^-(\tilde{B}) \in aC(CTXS)$ (b)⇒(d). Let $x \in X$ and $\tilde{V}$ be a neighborhood of $F(x)$ in $Y$. Then there is an Neutrosophic open set $\tilde{G}$ of $Y$, such that $(x) \subset \tilde{G} \subset \tilde{V}$. Hence, $x \in F^+(\tilde{G}) \subset F^+(\tilde{V})$. Now by hypothesis $F^+(\tilde{G}) \in aO(CTXS)$, and Thus $F^+(\tilde{V})$ is an $α$- neighborhood of $x$. (c)⇒(e). Let $x \in X$ and $\tilde{V}$ be a neighborhood of $F(x)$ in $Y$. Put $U = F^+(\tilde{V})$. Then $U$ is an $α$-neighborhood of $x$ and $F(U) \subset \tilde{V}$. (e)⇒(a). Let $x \in X$ and $\tilde{V}$ be a Neutrosophic set in $Y$. such that $F(x) \subset \tilde{V}$. Being an Neutrosophic open set in $Y$, it is a neighborhood of $x$ and according to the hypothesis there is an $α$-neighborhood $U$ of $x$ such that $F(U) \subset \tilde{V}$. Therefore there is $A \in aO(CTXS)$ such that $x \in A \subset U$ and hence $F(A) \subset F(U) \subset \tilde{V}$.

Corollary 3.10.
For a multifunction $F : X \to Y$ and point $x \in X$ the following statements are equivalent:
(a) $F$ is lower $α$-continuous at $x$.
(b) For each non-empty open set $B$ of $Y$ with $F(x) \cap B \neq \emptyset$, implies $x \in sCl(\text{Int}(F^-(B)))$.
(c) For any semi-open set $U$ of $X$ containing $x$ and for any non-empty open set $B$ of $Y$ with $F(x) \cap B \neq \emptyset$, there exists a non empty open set $V \subset U$ such that $F(x) \cap B \neq \emptyset$, for all $v \in V$. 

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