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Neutrosophic deductive filters on BL-algebras

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Abstract. In this paper, we introduce the notions of neutrosophic deductive filter, Boolean neutrosophic deductive filter (BNDF) and implicative neutrosophic deductive filter (INDF) on BL-algebras as generalizations of the fuzzy corresponding versions. We also investigate some properties of these filters and drive some characterizations of them. The relation between BNDF and INDF is investigated and it is proved that every BNDF is an INDF, but the converse is true when certain condition is satisfied. Furthermore, we construct a quotient structure related to the neutrosophic deductive filter and prove certain isomorphism theorems.

Keywords: BL-algebra, neutrosophic deductive filter, quotient structure

1. Introduction

Fuzzy set theory was introduced by Zadeh in 1965 [11]. A fuzzy subset A of a set X is a function $\mu_A : X \to [0, 1]$, where for each $x \in X$, $\mu_A(x)$ represents the grade of membership of the element $x \in X$ to A. In [1], Atanassov introduced the intuitionistic fuzzy sets as a generalization of fuzzy sets. The intuitionistic fuzzy sets consider both membership degree and nonmembership degree.

In 1998, *neutrosophy* has been proposed by Smarandache [9] as a new branch of philosophy in order to formally represent neutralities. The fundamental thesis of neutrosophy is that every idea has not only a certain degree of truth and a certain degree of falsity but also an *indeterminacy* degree that have to be considered independently from each other. In neutrosophic set theory, indeterminacy is measured explicitly and independently. This assumption is very important in many applications such as information fusion in which we try to combine the data from different sensors. As an example, suppose there are 10 voters during a voting process. One possible situation is that there are three *yes* votes, two *no* votes and five *undecided* ones. We note that this example is beyond the scope of intuitionistic fuzzy set theory.

In 1960 Abraham Robinson introduced *non-standard analysis* as a formalization of analysis and a branch of mathematical logic. In non-standard analysis a nonzero number ε is said to be infinitely small, or infinitesimal if and only if for all positive integers n, $|\varepsilon| \leq 1/n$. In this case the reciprocal $\delta = 1/\varepsilon$ will be infinitely large, or simply infinite, meaning that for all positive integers n, we have $|\delta| > n$. The set of hyper-real numbers is an extension of the set of real numbers which includes the class of infinite numbers and the class of infinitesimal numbers. The *non-standard unit interval* is $]0^-, 1^+[= 0^- \cup [0, 1] \cup 1^+$. Here 0^- is the set of all hyper-real numbers $0 - \varepsilon$, and 1^+ is the set of all hyper-real numbers $1 + \lambda$, where ε and λ are infinitesimal.

If U is a set, a neutrosophic set defined on the universe U assigns to each element $x \in U$, a triple $(T(x), I(x), F(x))$, where $T(x)$, $I(x)$ and $F(x)$ are standard or non-standard elements of $]0^-, 1^+[$. T is the degree of membership in the set U, I is the degree of indeterminacy-membership in the set U and F is the degree of nonmembership in the set U. In this paper we work with special netrosophic sets that their neutrosophic elements are standard real numbers in [0,1].

Neutrosophy has laid the foundation for a whole family of new mathematical theories, such as neutrosophic

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set theory, neutrosophic probability, neutrosophic statistics and neutrosophic logic. In recent years neutrosophic algebraic structures have been investigated (see [3, 5]).

BL-algebras provide an algebraic semantics for Hájek's Basic Logic [2]. The main example of a BL-algebra is the unit interval [0,1] endowed with the structure induced by a continuous t-norm. MV-algebras, Gödel algebras and Product algebras are the most known classes of BL-algebras. Filter theory plays an important rule in studying these algebras. From the logical point of view, various filters correspond to various sets of provable formulas. In [4] and [7], the notions of fuzzy prime filter, fuzzy Boolean filter, fuzzy implicative filter and fuzzy positive implicative filter on BL-algebras were introduced and some of their properties and characterizations were investigated.

In this paper we generalize the concept of fuzzy filer on a BL-algebra and define the concept of *neutrosophic deductive filter*. We define Boolean neutrosophic deductive filter and implicative neutrosophic deductive filter and investigate some of their properties. We drive several characterizations of these filters. Also, we investigate relation between BNDF and INDF and prove that every BNDF is an INDF, but the converse may not be true. Furthermore, the condition under which an INDF is BNDF is established. Finally, we construct a quotient structure related to the neutrosophic deductive filter and prove some isomorphism theorems.

2. Preliminaries

In this section, we give some definitions and results from the literature.

Definition 2.1. [9] Let X be a set. A *neutrosophic subset* A of X is a triple (T_A, I_A, F_A) where $T_A : X \to [0, 1]$ is the membership function, $I_A : X \to [0, 1]$ is the indeterminacy function and $F_A : X \to [0, 1]$ is the nonmembership function. Here for each $x \in X$, $T_A(x)$, $I_A(x)$ and $F_A(x)$ are all standard real numbers in [0,1].

Note that there is no restrictions on the values of $T_A(x)$, $I_A(x)$ and $F_A(x)$ and we only have the obvious condition

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3.$$

Definition 2.2. [9] Let A and B be two neutrosophic sets on X. Define $A \leq B$ if and only if

$$T_A(x) \leq T_B(x), \; I_A(x) \geq I_B(x), \; F_A(x) \geq F_B(x).$$

for all $x \in X$.

Definition 2.3. [9] Let A and B be two neutrosophic sets on X. Define

$$A \wedge B = (T_A \wedge T_B, \; I_A \vee I_B, \; F_A \vee F_B)$$
$$A \vee B = (T_A \vee T_B, \; I_A \wedge I_B, \; F_A \wedge F_B)$$

where, $\wedge$ is the minimum and $\vee$ is the maximum between real numbers.

Definition 2.4. [2] The axioms of propositional Hajék Basic Logic in the Hilbert-style system are as the following:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,
(A2) $(\varphi \& \psi) \to \varphi$,
(A3) $(\varphi \& \psi) \to (\psi \& \varphi)$,
(A4) $(\varphi \& (\varphi \to \psi)) \to (\psi \& (\psi \to \varphi))$,
(A5) $(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi$
(A6) $((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$,
(A7) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A8) $0 \to \varphi$,

The only inference rule is modus pones (MP). We show the consequence relation of BL in the Hilbert-style axiomatization by $\vdash_{BL}$. If a formula φ is provable in BL, we write $\vdash_{BL} \varphi$.

Proposition 2.5. [2] *In BL the following statements hold:*

(1) $\vdash_{BL} (\varphi \to (\psi \to \varphi))$,
(2) $\vdash_{BL} (\varphi \wedge \psi \to \psi \wedge \varphi)$,
(3) $\vdash_{BL} (\varphi \wedge \psi \to \psi)$,
(4) $\vdash_{BL} \varphi \to \neg\neg\varphi$,
(5) $\vdash_{BL} (\varphi \to \psi) \to (\neg\psi \to \neg\varphi)$,
(6) $\vdash_{BL} \neg\varphi \to (\varphi \to \psi)$,
(7) $\vdash_{BL} (\varphi \to (\psi \to \chi)) \leftrightarrow (\psi \to (\varphi \to \chi))$,
(8) $\vdash_{BL} (\varphi \to (\psi \to \chi)) \leftrightarrow ((\varphi \& \psi) \to \chi))$,
(9) $\vdash_{BL} (\varphi \to \psi) \to ((\chi \to \varphi) \to (\chi \to \psi))$,
(10) $\vdash_{BL} (\varphi \to \psi) \to ((\varphi \to \chi) \to (\psi \to \chi))$,
(11) $\vdash_{BL} ((\varphi \to \neg\varphi) \to \neg\varphi) \leftrightarrow (\varphi \to \neg\neg\varphi)$,
(12) $\vdash_{BL} [(\varphi \vee \neg\varphi) \to (\psi \to \varphi)] \leftrightarrow [(\varphi \to (\psi \to \varphi)) \wedge (\neg\varphi \to (\psi \to \varphi))]$,
(13) $\vdash_{BL} \psi \Rightarrow \vdash_{BL} (\varphi \to (\varphi \to \psi))$,
(14) $\vdash_{BL} (\varphi \to \psi), \; \vdash_{BL} (\chi \to \theta) \Rightarrow \vdash_{BL} (\varphi \& \chi \to \psi \& \theta)$,

Definition 2.6. [2] A BL-algebra is an algebra $(\mathcal{L}, \vee, \wedge, \odot, \to, 0, 1)$ of type (2,2,2,2,0,0) such that

(BL1) $(\mathcal{L}, \vee, \wedge, 0, 1)$ is a bounded lattice,
(BL2) $(\mathcal{L}, \odot, 1)$ is a commutative monoid,
(BL3) $x \odot y \leq z$ if and only if $x \leq y \to z$, for all $x, y, z \in \mathcal{L}$,
(BL4) $x \wedge y = x \odot (x \to y)$,
(BL5) $(x \to y) \vee (y \to x) = 1$.

If a BL-algebra $\mathcal{L}$ satisfies $\neg\neg x = x$, for each $x \in \mathcal{L}$, it is called an *MV-algebra*.

Proposition 2.7. [2, 6] *In any* BL-*algebra* $\mathcal{L}$*, the following properties hold:*

(R1) $x \leq y \Leftrightarrow x \to y = 1$,
(R2) $1 \to x = x, x \to 1 = 1, x \to x = 1, 0 \to x = 1, x \to (y \to x) = 1$,
(R3) $x \leq y \to z \Leftrightarrow y \leq x \to z$,
(R4) $x \to (y \to z) = (x \odot y) \to z = y \to (x \to z)$,
(R5) $x \leq y \Rightarrow (z \to x \leq z \to y$ and $y \to z \leq x \to z)$,
(R6) $z \to y \leq (x \to z) \to (x \to y)$, $z \to y \leq (y \to x) \to (z \to x)$,
(R7) $(x \to y) \odot (y \to z) \leq x \to z$,
(R8) $\neg x = \neg\neg\neg x, x \leq \neg\neg x$, when $\neg x = x \to 0$,
(R9) $\neg x \wedge \neg y = \neg(x \vee y)$,
(R10) $x \vee \neg x = 1 \Rightarrow x \wedge \neg x = 0$,
(M1) $x \odot y \leq x \wedge y$,
(M2) $x \leq y \Rightarrow x \odot z \leq y \odot z$,
(M3) $y \to z \leq x \vee y \to x \vee z$,
(M4) $\neg x \vee \neg y = \neg(x \wedge y)$,
(M5) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$,
(M6) $x \vee y = ((x \to y) \to y) \wedge ((y \to x) \to x)$,
(M7) $x \to (y \vee z) = (x \to y) \vee (x \to z)$,
(M8) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
(B1) $\neg\neg(x \wedge y) = (\neg\neg x \wedge \neg\neg y)$, $\neg\neg(x \vee y) = (\neg\neg x \vee \neg\neg y)$, $\neg\neg(x \odot y) = (\neg\neg x \odot \neg\neg y)$,
(B2) $\neg(\neg\neg x \to x) = 0$, $\neg\neg(x \to y) = (\neg\neg x \to \neg\neg y)$.

Definition 2.8. [2, 10] Let F be a nonempty subset of a BL-algebra $\mathcal{L}$ such that $1 \in F$. F is called:

(i) a filter on $\mathcal{L}$, if

$$(\forall x, y \in \mathcal{L})(x, y \in F \Rightarrow x \odot y \in F) \text{ and}$$
$$(\forall x, y \in \mathcal{L})(x \in F,\ x \leq y \Rightarrow y \in F),$$

(ii) a *Boolean filter* on $\mathcal{L}$, if it is a filter and moreover we have

$$(\forall x \in \mathcal{L})(x \vee \neg x \in F),$$

(iii) an *implicative filter* on $\mathcal{L}$, if it is a filter and moreover for all $x, y, z \in \mathcal{L}$ we have

$$[x \to y,\ x \to (y \to z) \in F \Rightarrow (x \to z) \in F].$$

Proposition 2.9. [10] *A nonempty subset F of a BL-algebra $\mathcal{L}$ is a filter if and only if*

(DS1) $1 \in F$,
(DS2) $(\forall x,\ y \in \mathcal{L})(x \in F,\ x \to y \in F \Rightarrow y \in F)$.

Theorem 2.10. [2] *Let F be a filter on a BL-algebra $\mathcal{L}$. Define the binary relation $\sim_F$ on $\mathcal{L}$ by*

$$x \sim_F y \ \Leftrightarrow\ (x \to y \in F \text{ and } y \to x \in F)$$

Then $\sim_F$ is a congruence on $\mathcal{L}$, and the set of all congruence classes $\mathcal{L}/F = \{[x]_F : x \in \mathcal{L}\}$ with the following operations form a BL-algebra:

$$[x] \bullet [y] = [x \odot y],\ [x] \rightharpoonup [y] = [x \to y],$$
$$[x] \sqcup [y] = [x \vee y],\ [x] \sqcap [y] = [x \wedge y]$$

Lemma 2.11. [6] *Let F_1 and F_2 be two filters on* BL-*algebra $\mathcal{L}$ which $F_1 \subseteq F_2$. Then F_1 is a filter on F_2 and F_2/F_1 is a filter on $\mathcal{L}/F_1$.*

Definition 2.12. [6] The neutrosophic set $\mathcal{F}$ of a BL-algebra $\mathcal{L}$ has *Sup-Inf Property* if for any nonempty subset S of $\mathcal{L}$, there exist $x_0, x_1, x_2 \in S$, such that

$$\sup_{x\in S} T_{\mathcal{F}}(x) = T_{\mathcal{F}}(x_0),\ \inf_{x\in S} I_{\mathcal{F}}(x) = I_{\mathcal{F}}(x_1),$$
$$\inf_{x\in S} F_{\mathcal{F}}(x) = F_{\mathcal{F}}(x_2).$$

From now on, we use the same notations for corresponding logical and algebraic notions. Also, if there is no confusion, we use $\wedge$ and $\vee$ for minimum and maximum for real numbers.

3. Neutrosophic deductive filters on BL-algebras

In this section, we define the neutrosophic deductive filters and prove some properties of them. Furthermore, we characterize the neurosophic decuctive filter generated by a neutrosophic deductive set.

Definition 3.1. Suppose that Γ and Δ be two subsets of $[0, 1]^3$. We define the relation $\models$ as follows:

$$\Gamma \models \Delta \ \Leftrightarrow\ \wedge\Gamma \leq \wedge\Delta$$

If $\Gamma = \emptyset$, then we define $\wedge\Gamma = (1, 0, 0)$, and if $\Delta = \emptyset$, then we define $\wedge\Delta = (0, 1, 1)$.

From now on, if $\Gamma \models \Delta$ and $\Delta \models \Gamma$, we write $\Gamma = \Delta$.

Definition 3.2. Let $\mathcal{L}$ be a BL-algebra and $\vdash$ be a consequence relation on the set of BL-formulas. A neutrosophic subset $\mathcal{F}$ of $\mathcal{L}$ is called a neutrosophic filter with respect to $\vdash$, if for each assignment v into $\mathcal{L}$ and for every set $\Gamma \cup \{\varphi\}$ of BL-formulas, if $\Gamma \vdash \varphi$, then $\{\mathcal{F}(v(\Gamma))\} \models \mathcal{F}(v(\varphi))$, where $\mathcal{F}(v(\Gamma)) = \{\mathcal{F}(v(\gamma)) : \gamma \in \Gamma\}$.

In particular, if $\vdash$ is presented by a Hilbert-style system, for example if $\vdash$ is $\vdash_{BL}$, then it is enough to check the above condition for the inference rules (Γ, φ) and the axioms $(\emptyset, \varphi)$ of the proof system.

Definition 3.3. A neutrosophic subset F of a BL-algebra $\mathcal{L}$ is called a *neutrosophic deductive filter* (briefly, NDF), if $\mathcal{F}$ is a neutrosophic filter with respect to $\vdash_{BL}$.

Lemma 3.4. *A neutrosophic subset F of a BL-algebra $\mathcal{L}$ is a NDF iff for all formulas φ, ψ and each assignment v into $\mathcal{L}$:*

(NDF1) $\mathcal{F}(v(\varphi)) \models \mathcal{F}(1)$,
(NDF2) $\{\mathcal{F}(v(\varphi)),\ \mathcal{F}(v(\varphi \to \psi))\} \models \mathcal{F}(v(\psi))$.

Proof. This can be easily obtained from the fact that in a BL-algebra, all axioms of BL are evaluated to 1 under all assignments and (MP) is the only inference rule.

Corollary 3.5. *A neutrosophic subset F of a BL-algebra $\mathcal{L}$ is a NDF iff*

(NDF1) $(\forall a \in \mathcal{L})(\ \mathcal{F}(a) \models \mathcal{F}(1))$,
(NDF2) $(\forall a, b \in \mathcal{L})(\ \{\mathcal{F}(a),\ \mathcal{F}(a \to b)\} \models \mathcal{F}(b))$.

Corollary 3.6. *Let $\mathcal{F}$ be a NDF. Then we have*

(i) $(\forall a \in \mathcal{L})(\mathcal{F}(a) \leq \mathcal{F}(1))$,
(ii) $(\forall a, b \in \mathcal{L})(\mathcal{F}(a) \wedge \mathcal{F}(a \to b) \leq \mathcal{F}(b))$.

Example 3.7. Let $\mathcal{L} = \{0, a, b, 1\}$. For all $x, y \in \mathcal{L}$, we define $x \wedge y = min\{x, y\}$, $x \vee y = max\{x, y\}$ and $\odot$ and $\to$ as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(\mathcal{L}, \wedge, \vee, \odot, \to, 0, 1)$ is a BL-algebra. The neutrosophic subset $\mathcal{F}$ of $\mathcal{L}$ defined by $\mathcal{F}(0) = \mathcal{F}(a) = (t_1, t_3, t_3)$, $\mathcal{F}(b) = (t_2, t_2, t_2)$, $\mathcal{F}(1) = (t_3, t_1, t_1)$, where $0 \leq t_1 < t_2 < t_3 \leq 1$ are three fixed real numbers in [0,1], is a NDF.

Theorem 3.8. *Let $\mathcal{F}$ be a neutrosophic subset of $\mathcal{L}$. Then $\mathcal{F}$ is a NDF if and only if for all formulas φ, ψ and all assignment v into $\mathcal{L}$, if $\vdash_{BL} (\varphi \to (\psi \to \chi))$ then $\{\mathcal{F}(v(\varphi)),\ \mathcal{F}(v(\psi))\} \models \mathcal{F}(v(\chi))$.*

Proof. Let $\mathcal{F}$ be a NDF on $\mathcal{L}$, v be an assignment into $\mathcal{L}$ and $\vdash_{BL} (\varphi \to (\psi \to \chi))$, for some formulas φ, ψ. By Lemma 3.4, we have $\{\mathcal{F}(v(\varphi \to (\psi \to \chi)),\ \mathcal{F}(v(\varphi))\} \models \mathcal{F}(v(\psi \to \chi))$ and $\{\mathcal{F}(v(\psi \to \chi)),\ \mathcal{F}(v(\psi))\} \models \mathcal{F}(v(\chi))$. Then $\{\mathcal{F}(v(\varphi)),\ \mathcal{F}(v(\varphi \to (\psi \to \chi))),\ \mathcal{F}(v((\psi))\} \models \mathcal{F}(v(\chi))$. Now, since $\vdash_{BL} (\varphi \to (\psi \to \chi))$, so $\mathcal{F}(v(\varphi \to (\psi \to \chi))) = \mathcal{F}(1)$. Thus, we obtain that $\{\mathcal{F}(v(\varphi)),\ \mathcal{F}(v(\psi))\} \models \mathcal{F}(v(\chi))$.

Conversely, assume that the condition holds. We know if $\vdash_{BL} \psi$, then $\vdash_{BL} \varphi \to (\varphi \to \psi)$ and so $\{\mathcal{F}(v((\varphi)),\ \mathcal{F}(v((\varphi))\} \models \mathcal{F}(v(\psi)) = \mathcal{F}(1)$, therefore $\{\mathcal{F}(v((\varphi))\} \models \mathcal{F}(1)$. Now, since $\vdash_{BL} \varphi \to ((\varphi \to \psi) \to \psi)$, by the condition we get $\{\mathcal{F}(v(\varphi)),\ \mathcal{F}(v(\varphi \to \psi))\} \models \mathcal{F}(v(\psi))$, which completes the proof.

Corollary 3.9. *Let $\mathcal{F}$ be a neutrosophic subset of $\mathcal{L}$. Then $\mathcal{F}$ is a NDF if and only if for all formulas φ, ψ and all assignments v into $\mathcal{L}$, if $\vdash_{BL} (\varphi \& \psi) \to \chi$, then $\{\mathcal{F}(v(\varphi)),\ \mathcal{F}(v(\psi))\} \models \mathcal{F}(v(\chi))$.*

Theorem 3.10. *Let $\mathcal{F}$ be a neutrosophic subset of $\mathcal{L}$. Then $\mathcal{F}$ is a NDF if and only if for all formulas φ, ψ and all assignments v into $\mathcal{L}$*

(i) $\vdash_{BL} (\varphi \to \psi) \Rightarrow \{\mathcal{F}(v(\varphi))\} \models \mathcal{F}(v(\psi))$,
(ii) $\{\mathcal{F}(v(\varphi)),\ \mathcal{F}(v(\psi))\} \models \mathcal{F}(v(\varphi \& \psi))$.

Proof. Suppose that $\mathcal{F}$ is a NDF. Since $\vdash_{BL} (\varphi \& \varphi \to \varphi)$ and $\vdash_{BL} (\varphi \to \psi)$, we have $\vdash_{BL} (\varphi \& \varphi \to \psi)$. So, by Corollary 3.9, it follows that for all assignments v, $\{\mathcal{F}(v(\varphi)),\ \mathcal{F}(v(\varphi))\} \models \mathcal{F}(v(\psi))$, therefore $\{\mathcal{F}(v(\varphi))\} \models \mathcal{F}(v(\psi))$ which proves (i). Since $\vdash_{BL} (\varphi \& \psi \to \varphi \& \psi)$, by Corollary 3.9 we have $\{\mathcal{F}(v(\varphi)),\ \mathcal{F}(v(\psi))\} \models \mathcal{F}(v(\varphi \& \psi))$, proving (ii).

Conversely, assume that (i) and (ii) hold and $\vdash_{BL} ((\varphi \& \psi) \to \chi)$, for some formulas φ, ψ, χ. Then by (i) we have $\{\mathcal{F}(v(\varphi \& \psi))\} \models \mathcal{F}(v(\chi))$ and since by (ii) we have $\{\mathcal{F}(v(\varphi)),\ \mathcal{F}(v(\psi))\} \models \mathcal{F}(v(\varphi \& \psi))$, we get $\{\mathcal{F}(v(\varphi)),\ \mathcal{F}(v(\psi))\} \models \mathcal{F}(v(\chi))$. Therefore, the result is obtained by Corollary 3.9.

By (NDF2), Theorem 3.10 and (R1)-(R6), we get the following corollary.

Corollary 3.11. *For a NDF $\mathcal{F}$ on $\mathcal{L}$, we have:*

(1) $\vdash_{BL} (\varphi \to \psi) \Rightarrow \{\mathcal{F}(v(\varphi))\} \models \mathcal{F}(v(\varphi))$,

(2) $\models \mathcal{F}(v(\varphi \to \psi)) \Rightarrow \{\mathcal{F}(v(\varphi))\} \models \mathcal{F}(v(\psi))$,
(3) $\{\mathcal{F}(v(\varphi \& \psi))\} = \{\mathcal{F}(v(\varphi)),\ \mathcal{F}(v(\psi))\}$,
(4) $\{\mathcal{F}(v(\varphi \wedge \psi))\} = \{\mathcal{F}(v(\varphi)),\ \mathcal{F}(v(\psi))\}$,
(5) $\{\mathcal{F}(v(\varphi^n))\} = \mathcal{F}(v(\varphi))$, *where* $\varphi^n = \varphi \& ... \& \varphi$ (n *times*),
(6) $\{\mathcal{F}(v(\varphi)), \mathcal{F}(v(\neg\varphi))\} = (0, 1, 1)$,
(7) $\{\mathcal{F}(v(\varphi \to \psi))\} \models \mathcal{F}(v((\chi \to \varphi) \to (\chi \to \psi)))$,
(8) $\{\mathcal{F}(v(\varphi \to \psi))\} \models \mathcal{F}(v((\psi \to \chi) \to (\varphi \to \chi)))$,
(9) $\{\mathcal{F}(v(\varphi \to \psi))\} \models \mathcal{F}(v((\varphi \& \chi) \to (\psi \& \chi)))$,
(10) $\{\mathcal{F}(v(\varphi \to \psi)),\quad \mathcal{F}(v(\psi \to \chi))\} \models \mathcal{F}(v(\varphi \to \chi)))$,
(11) $\{\mathcal{F}(v(\varphi \to (\psi \to \chi))),\quad \mathcal{F}(v(\varphi \to \psi))\} \models \{\mathcal{F}(v(\varphi \to (\varphi \to \chi)))\}$.

Corollary 3.12. *For a NDF $\mathcal{F}$ on $\mathcal{L}$, the following hold:*

(1) $x \le y \Rightarrow \mathcal{F}(x) \le \mathcal{F}(y)$,
(2) $\mathcal{F}(x \to y) = \mathcal{F}(1) \ \Rightarrow\ \mathcal{F}(x) \le \mathcal{F}(y)$,
(3) $\mathcal{F}(x \odot y) = \mathcal{F}(x) \wedge \mathcal{F}(y)$,
(4) $\mathcal{F}(x \wedge y) = \mathcal{F}(x) \wedge \mathcal{F}(y)$,
(5) $\mathcal{F}(x^n) = \mathcal{F}(x)$, *where* $x^n = x \odot x ... \odot x$ (n *times*),
(6) $\mathcal{F}(x) \wedge \mathcal{F}(\neg x) = \mathcal{F}(0) = (0, 1, 1)$,
(7) $\mathcal{F}(x \to y) \le \mathcal{F}((z \to x) \to (z \to y))$,
(8) $\mathcal{F}(x \to y) \le \mathcal{F}((y \to z) \to (x \to z))$,
(9) $\mathcal{F}(x \to y) \le \mathcal{F}(x \odot z \to y \odot z)$,
(10) $\mathcal{F}(x \to z) \le \mathcal{F}(x \to y) \wedge \mathcal{F}(y \to z)$,
(11) $\mathcal{F}(x \to (y \to z)) \wedge \mathcal{F}(x \to y) \le \mathcal{F}(x \to (x \to z))$.

Proposition 3.13. *Let Λ be a nonempty set and $\mathcal{F}_i$ be a NDF, for each $i \in \Lambda$. Then $\bigwedge_{i\in\Lambda} \mathcal{F}_i$ is a NDF.*

Definition 3.14. Let $\mathcal{F}$ be a neutrosophic subset of $\mathcal{L}$ and $\mathcal{G}$ be a NDF. $\mathcal{G}$ is said to be generated by $\mathcal{F}$, if $\mathcal{F} \le \mathcal{G}$ and for any NDF $\mathcal{H}$, $\mathcal{F} \le \mathcal{H}$ implies that $\mathcal{G} \le \mathcal{H}$. The NDF generated by $\mathcal{F}$ will be denoted by $\langle\mathcal{F}\rangle$.

Theorem 3.15. *Let $\mathcal{F}$ be a neutrosophic subset of $\mathcal{L}$. Then for each formula ψ and assignment v we have $\langle\mathcal{F}\rangle(v(\psi)) = \bigvee\{\mathcal{F}(v(\varphi_1)) \wedge ... \wedge \mathcal{F}(v(\varphi_n)) \mid\ \vdash_{BL} ((\varphi_1 \& ... \& \varphi_n) \to \psi)$, for some $n \in N, \varphi_1, ..., \varphi_n \in Fm\}$.*

Proof. We first prove that $\langle\mathcal{F}\rangle$ is a NDF. Obviously, $\langle\mathcal{F}\rangle(v(\psi)) \models \mathcal{F}(1)$, for each formula ψ. Now, considering the formulas φ, ψ, we observe that if there exist $n, m \in \mathbf{N}$ and formulas $\varphi_1, ..., \varphi_n, \psi_1, ..., \psi_m$ such that $\vdash_{BL} (\varphi_1 \& ... \& \varphi_n \to \varphi)$, $\vdash_{BL} (\psi_1 \& ... \& \psi_m \to (\varphi \to \psi))$, then we get $\vdash_{BL} (\varphi_1 \& ... \& \varphi_n \& \psi_1 \& ... \& \psi_m \to (\varphi \& (\varphi \to \psi)))$. Hence, since by Proposition 2, $\vdash_{BL} ((\varphi \& (\varphi \to \psi)) \to \psi)$, we get

$$\vdash_{BL} (\varphi_1 \& ... \& \varphi_n \& \psi_1 \& ... \& \psi_m \to \psi)$$

Thus
$\mathcal{F}(v(\varphi_1)) \wedge ... \wedge \mathcal{F}(v(\varphi_n)) \wedge \mathcal{F}(v(\psi_1)) \wedge ... \wedge \mathcal{F}(v(\psi_m)) \le \langle\mathcal{F}\rangle(v(\psi))$,
and then

$$\begin{aligned}
&\langle\mathcal{F}\rangle(v(\varphi)) \wedge \langle\mathcal{F}\rangle(v(\varphi \to \psi)) \\
&= (\vee\{\mathcal{F}(v(\varphi_1)) \wedge ... \wedge \mathcal{F}(v(\varphi_n)) \mid\ \vdash_{BL} \\
&\quad ((\varphi_1 \& ... \& \varphi_n) \to \varphi), \text{ for some } n \in \mathbf{N}, \\
&\quad \varphi_1, ..., \varphi_n \in Fm\}) \\
&\wedge(\vee\{\mathcal{F}(v(\psi_1)) \wedge ... \wedge \mathcal{F}(v(\psi_m)) \mid\ \vdash_{BL} \\
&\quad ((\psi_1 \& ... \& \psi_m) \to (\varphi \to \psi)), \text{ for some } m \in \mathbf{N}, \\
&\psi_1, ..., \psi_m \in Fm\}) \\
&= (\vee\{\mathcal{F}(v(\varphi_1)) \wedge ... \wedge \mathcal{F}(v(\varphi_n)) \\
&\wedge\mathcal{F}(v(\psi_1)) \wedge ... \wedge \mathcal{F}(v(\psi_m))\} \mid \\
&\quad \vdash_{BL} ((\varphi_1 \& ... \& \varphi_n) \to \varphi), \\
&\quad \vdash_{BL} ((\psi_1 \& ... \& \psi_m) \to (\varphi \to \psi)), \\
&\quad \text{for some } n, m \in \mathbf{N}, \varphi_1, ..., \varphi_n, \psi_1, ..., \psi_m \in Fm\}) \\
&\le \langle\mathcal{F}\rangle(v(\psi))
\end{aligned}$$

Therefore, $\langle\mathcal{F}\rangle$ is a NDF.

Now, Since $\vdash_{BL} (\varphi \& \varphi) \to \varphi$, then by definition of $\langle\mathcal{F}\rangle$, it follows that $\langle\mathcal{F}\rangle(v(\varphi)) \ge \mathcal{F}(v(\varphi \& \varphi))$. Also, $\mathcal{F}(v(\varphi \& \varphi)) = \{\mathcal{F}(v(\varphi)), \mathcal{F}(v(\varphi))\} = \mathcal{F}(v(\varphi))$, by Corollary 3.11. So, $\langle\mathcal{F}\rangle(v(\varphi)) \ge \mathcal{F}(v(\varphi))$. Therefore, $\mathcal{F} \le \langle\mathcal{F}\rangle$. Finally, suppose that $\mathcal{H}$ is a NDF such that $\mathcal{F} \le \mathcal{H}$ and φ is a formula. Then,

$$\begin{aligned}
\langle\mathcal{F}\rangle(v(\varphi)) &= \bigvee\{\mathcal{F}(v(\varphi_1)) \wedge ... \wedge \mathcal{F}(v(\varphi_n)) \mid \\
&\quad \vdash_{BL} ((\varphi_1 \& ... \& \varphi_n) \to \varphi), n \in \mathbf{N}, \\
&\quad \varphi_1, ..., \varphi_n \in Fm\} \\
&\le \bigvee\{\mathcal{H}(v(\varphi_1)) \wedge ... \wedge \mathcal{H}(v(\varphi_n)) \mid \\
&\quad \vdash_{BL} ((\varphi_1 \& ... \& \varphi_n) \to \varphi), n \in \mathbf{N}, \\
&\quad \varphi_1, ..., \varphi_n \in Fm\} \\
&\le \bigvee\{\mathcal{H}(v(\varphi))\} = \mathcal{H}(v(\varphi)),
\end{aligned}$$

Therefore, $\langle\mathcal{F}\rangle \le \mathcal{H}$, which completes the proof.

Example 3.16. Suppose that $(\mathcal{L}, \wedge, \vee, \odot, \to, 0, 1)$ be the BL-algebra defined in Example 3.7. Define the neutrosophic subset $\mathcal{F}$ of $\mathcal{L}$ by $\mathcal{F}(0) = (t_1, t_1, t_1)$, $\mathcal{F}(a) = \mathcal{F}(b) = (t_1, t_2, t_2)$, $\mathcal{F}(1) = (t_2, t_2, t_2)$ $(0 \leq t_1 < t_2 \leq 1)$ and the neutrosophic subset $\mathcal{G}$ of $\mathcal{L}$ by $\mathcal{G}(0) = \mathcal{G}(a) = \mathcal{G}(b) = (t_1, t_1, t_1)$, $\mathcal{G}(1) = (t_2, t_1, t_1)$. One can easily check that $\mathcal{G} = \langle \mathcal{F} \rangle$.

4. Boolean neutrosophic deductive filters

In this section we define and study the notion of Boolean neutrosophic deductive filter on BL-algebras.

Definition 4.1. Let $\mathcal{F}$ be a NDF on $\mathcal{L}$. $\mathcal{F}$ is called a *Boolean neutrosophic deductive filter* (briefly, BNDF) if $\mathcal{F}(1) \models \mathcal{F}(v(\varphi \vee \neg\varphi))$, for all formula φ and all assignments v.

Example 4.2. Let $\mathcal{L} = \{0, a, b, 1\}$ be a chain with cayley tables as follow:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	a	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	b	1	1
1	0	a	b	1

Define $\wedge$ and $\vee$ on $\mathcal{L}$ as min and max, respectively. Then $(\mathcal{L}, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra. The neutrosophic subset $\mathcal{F}$ of $\mathcal{L}$ defined by $\mathcal{F}(0) = (t_1, t_2, t_2)$, $\mathcal{F}(a) = \mathcal{F}(b) = \mathcal{F}(1) = (t_2, t_1, t_1)$, where $0 \leq t_1 < t_2 \leq 1$ are two fixed real numbers in [0,1], is a BNDF.

Proposition 4.3. *Let $\mathcal{F}$ be a NDF on $\mathcal{L}$. $\mathcal{F}$ is a BNDF if and only if $\mathcal{F}(x \vee \neg x) = \mathcal{F}(1)$, for all $x \in \mathcal{L}$.*

Since $\varphi \vee \psi = ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$, by Corollary 3.11, we have the following proposition.

Proposition 4.4. *Let $\mathcal{F}$ be a NDF on $\mathcal{L}$. Then, $\mathcal{F}$ is a BNDF if and only if for all formula φ and all assignments v:*

$$\mathcal{F}(v((\varphi \rightarrow \neg\varphi) \rightarrow \neg\varphi))) = \mathcal{F}(v((\neg\varphi \rightarrow \varphi) \rightarrow \varphi))) = \mathcal{F}(1).$$

Proposition 4.5. *Let $\mathcal{F}$ be a NDF on $\mathcal{L}$. Then, $\mathcal{F}$ is a BNDF if and only if for all $x \in \mathcal{L}$,*

$$\mathcal{F}((x \rightarrow \neg x) \rightarrow \neg x) = \mathcal{F}((\neg x \rightarrow x) \rightarrow x) = \mathcal{F}(1).$$

Definition 4.6. Let $\mathcal{F}$ be a NDF on $\mathcal{L}$. Then, for each $t \in [0, 1]$ we define $\mathcal{F}_t = (T_{\mathcal{F}_t}, I_{\mathcal{F}_t}, F_{\mathcal{F}_t})$, where $T_{\mathcal{F}t} = \{x \in \mathcal{L} \ : \ T_{\mathcal{F}} \geq t\}$, $I_{\mathcal{F}t} = \{x \in \mathcal{L} \ : \ I_{\mathcal{F}} \leq t\}$, $F_{\mathcal{F}t} = \{x \in \mathcal{L} \ : \ F_{\mathcal{F}} \leq t\}$.

Theorem 4.7. *Let $\mathcal{F}$ be a NDF on $\mathcal{L}$. Then, $\mathcal{F}$ is a BNDF if and only if for each $t \in [0, 1]$, $\emptyset \neq T_{\mathcal{F}t}$, $\emptyset \neq I_{\mathcal{F}t}$, $\emptyset \neq F_{\mathcal{F}t}$ and all of them be Boolean filters on $\mathcal{L}$.*

Proof. Suppose that $\mathcal{F}$ is a BNDF, and $\emptyset \neq T_{\mathcal{F}t}$, $\emptyset \neq I_{\mathcal{F}t}$ and $\emptyset \neq F_{\mathcal{F}t}$, for some $t \in [0, 1]$. Then, there exist $x_0 \in T_{\mathcal{F}t}$, $x_1 \in I_{\mathcal{F}t}$ and $x_2 \in F_{\mathcal{F}t}$. So, for each $x \in \mathcal{L}$, $T_{\mathcal{F}}(x \vee \neg x) = T_{\mathcal{F}}(1) \geq T_{\mathcal{F}}(x_0) \geq t$ and hence $x \vee \neg x \in T_{\mathcal{F}t}$. Similarly, $x \vee \neg x \in I_{\mathcal{F}t}$ and $x \vee \neg x \in F_{\mathcal{F}t}$. Thus, $T_{\mathcal{F}t}$, $I_{\mathcal{F}t}$ and $F_{\mathcal{F}t}$ are Boolean filters on $\mathcal{L}$.

Conversely, suppose that $\emptyset \neq T_{\mathcal{F}t}$, $\emptyset \neq I_{\mathcal{F}t}$ and $\emptyset \neq F_{\mathcal{F}t}$ are Boolean filters on $\mathcal{L}$, for each $t \in [0, 1]$. Then, $T_{T_{\mathcal{F}}(1)}, I_{I_{\mathcal{F}}(1)}$ and $F_{F_{\mathcal{F}}(1)}$ are Boolean filters and so $x \vee \neg x \in T_{T_{\mathcal{F}}(1)}, x \vee \neg x \in I_{I_{\mathcal{F}}(1)}$ and $x \vee \neg x \in F_{F_{\mathcal{F}}(1)}$. This implies that $\mathcal{F}(x \vee \neg x) = \mathcal{F}(1)$, for all $x \in \mathcal{L}$, and then $\mathcal{F}$ is a BNDF, by Proposition 4.3.

Corollary 4.8. *Let $\mathcal{F}$ be a NDF on $\mathcal{L}$. Then $\mathcal{F}$ is a BNDF if and only if $T_{T_{\mathcal{F}}(1)}$, $I_{I_{\mathcal{F}}(1)}$ and $F_{F_{\mathcal{F}}(1)}$ are Boolean filters.*

Theorem 4.9. *Let $\mathcal{F}$ and $\mathcal{G}$ be two NDFs on $\mathcal{L}$, which $\mathcal{F} \leq \mathcal{G}$, $\mathcal{F}(1) = \mathcal{G}(1)$. If $\mathcal{F}$ is a BNDF, then $\mathcal{G}$ is a BNDF too.*

Proof. Use Definition 4.1.

Theorem 4.10. *Let $\mathcal{F}$ be a NDF on $\mathcal{L}$, φ, ψ, χ be formulas and v be an assignment on $\mathcal{L}$. Then the following are equivalent:*

(i) $\{\mathcal{F}(v(\varphi \rightarrow (\neg\chi \rightarrow \psi))), \quad \mathcal{F}(v(\psi \rightarrow \chi)\} \models \mathcal{F}(v(\varphi \rightarrow \chi)$
(ii) $\{\mathcal{F}(v(\varphi \rightarrow (\neg\chi \rightarrow \chi)))\} \models \mathcal{F}(v(\varphi \rightarrow \chi))$
(iii) $\mathcal{F}(v(\varphi \rightarrow (\neg\chi \rightarrow \chi))) = \mathcal{F}(v(\varphi \rightarrow \chi))$
(iv) $\{\mathcal{F}(v(\psi \rightarrow (\varphi \rightarrow (\neg\chi \rightarrow \chi)))), \quad \mathcal{F}(v(\psi))\} \models \mathcal{F}(v(\varphi \rightarrow \chi))$

Proof. (i) $\Rightarrow$ (ii) It is enough to let $\psi = \chi$ in (i).

(ii) $\Rightarrow$ (iii) It follows from $\vdash_{BL} (\varphi \rightarrow \chi) \rightarrow (\neg\chi \rightarrow (\varphi \rightarrow \chi))$, $\vdash_{BL} (\neg\chi \rightarrow (\varphi \rightarrow \chi)) \leftrightarrow (\varphi \rightarrow (\neg\chi \rightarrow \chi))$ and Theorem 3.10.

(iii) $\Rightarrow$ (iv) Since $\mathcal{F}$ is a NDF, $\{\mathcal{F}(v(\psi \rightarrow (\varphi \rightarrow (\neg\chi \rightarrow \chi)))), \quad \mathcal{F}(v(\psi))\} \models \mathcal{F}(v(\varphi \rightarrow (\neg\chi \rightarrow \chi)))$. Then the result is obtained by $\mathcal{F}(\varphi \rightarrow (\neg\chi \rightarrow \chi)) = \mathcal{F}(\varphi \rightarrow \chi)$.

(iv) $\Rightarrow$ (i) By Corollary 3.11, we have $\{\mathcal{F}(v(\varphi \rightarrow (\neg\chi \rightarrow \psi))), \quad \mathcal{F}(v(\psi \rightarrow \chi))\} = \{\mathcal{F}(v((\varphi \odot \neg\chi) \rightarrow \psi)), \quad \mathcal{F}(v(\psi \rightarrow \chi))\} \models \mathcal{F}(v((\varphi \odot \neg\chi) \rightarrow \chi))$ and since $\mathcal{F}(v((\varphi \odot \neg\chi) \rightarrow \chi)) = \mathcal{F}(v(\varphi \rightarrow (\neg\chi \rightarrow \chi)))$, we have $\{\mathcal{F}(v(\varphi \rightarrow (\neg\chi \rightarrow \psi))), \ \mathcal{F}(v(\psi \rightarrow \chi))\} \models \mathcal{F}(v(\varphi \rightarrow (\neg\chi \rightarrow \chi)))$.

On the other hand for each BL-provable formula ψ we have $\mathcal{F}(v(\varphi \to (\neg\chi \to \chi))) = \{\mathcal{F}(v(\psi \to (\varphi \to (\neg\chi \to \chi)))),\ \mathcal{F}(v(\psi))\} \models \mathcal{F}(v(\varphi \to \chi))$ by (iv). Thus we get $\{\mathcal{F}(v(\varphi \to (\neg\chi \to \psi))),\ \mathcal{F}(v(\psi \to \chi))\} \models \mathcal{F}(v(\varphi \to \chi))$, which proves (i).

Definition 4.11. Let $\mathcal{F}$ be a NDF on $\mathcal{L}$. We say $\mathcal{F}$ has *Implicative Property*, if for all formulas φ, ψ, χ and all assignments v, it satisfies:

$$\{\mathcal{F}(v(\varphi \to (\neg\chi \to \psi))),\ \mathcal{F}(v(\psi \to \chi))\} \models \mathcal{F}(v(\varphi \to \chi))$$

Theorem 4.12. *A NDF $\mathcal{F}$ on $\mathcal{L}$ is a BNDF if and only if it satisfies the Implicative Property.*

Proof. Suppose that $\mathcal{F}$ is a BNDF on $\mathcal{L}$. From $\vdash_{BL} (\chi \vee \neg\chi) \to (\varphi \to \chi) \leftrightarrow (\chi \to (\varphi \to \chi)) \wedge (\neg\chi \to (\varphi \to \chi)) \leftrightarrow \neg\chi \to (\varphi \to \chi) \leftrightarrow \varphi \to (\neg\chi \to \chi)$, it follows that

$$\mathcal{F}(v(\varphi \to (\neg\chi \to \chi))) = \mathcal{F}(v((\chi \vee \neg\chi) \to (\varphi \to \chi)))$$
$$= \{\mathcal{F}(v((\chi \vee \neg\chi) \to (\varphi \to \chi))),\ \mathcal{F}(v(\chi \vee \neg\chi))\}$$
$$\models \mathcal{F}(v(\varphi \to \chi))$$

which proves that $\mathcal{F}$ satisfies the Implicative Property, by Theorem 4.10 (i), (ii).

Conversely, suppose that $\mathcal{F}$ satisfies the Implicative Property. By Theorem 4.10 (iii), replacing φ by $\neg\varphi \to \varphi$ and χ by φ, we have $\mathcal{F}(v((\neg\varphi \to \varphi) \to \varphi)) = \mathcal{F}(v((\neg\varphi \to \varphi) \to (\neg\varphi \to \varphi))) = \mathcal{F}(1)$

and replacing φ by $\varphi \to \neg\varphi$ and χ by $\neg\varphi$, we get $\mathcal{F}(v((\varphi \to \neg\varphi) \to \neg\varphi)) = \mathcal{F}(v((\varphi \to \neg\varphi) \to (\neg\neg\varphi \to \neg\varphi))) = \mathcal{F}(v((\varphi \to \neg\varphi) \to (\varphi \to \neg\varphi))) = \mathcal{F}(1)$. Then $\mathcal{F}(v(\varphi \vee \neg\varphi)) = \{\mathcal{F}(v((\varphi \to \neg\varphi) \to \neg\varphi)),\ \mathcal{F}(v((\neg\varphi \to \varphi) \to \varphi))\} = \mathcal{F}(1)$. Thus $\mathcal{F}$ is a BNDF on $\mathcal{L}$.

Theorem 4.13. *Let $\mathcal{F}$ be a NDF on $\mathcal{L}$, φ, ψ, χ be formulas and v be an assignment on $\mathcal{L}$. Then the following are equivalent:*

(i) $\mathcal{F}$ is a BNDF,
(ii) $\mathcal{F}(v(\varphi)) = \mathcal{F}(v(\neg\varphi \to \varphi))$,
(iii) $\mathcal{F}(v((\varphi \to \psi) \to \varphi)) \models \mathcal{F}(v(\varphi))$,
(iv) $\mathcal{F}(v((\varphi \to \psi) \to \varphi)) = \mathcal{F}(v(\varphi))$,
(v) $\{\mathcal{F}(v(\chi \to ((\varphi \to \psi) \to \varphi))), \mathcal{F}(v(\chi))\} \models \mathcal{F}(v(\varphi))$.

Proof. (i) $\Rightarrow$ (ii) Since $\vdash_{BL} \varphi \to (\neg\varphi \to \varphi)$, then by Corollary 3.11 we have $\mathcal{F}(v(\varphi)) \models \mathcal{F}(v(\neg\varphi \to \varphi))$. The other direction follows from Theorem 4.12, replacing φ by a BL-provable formulas and ψ, χ by φ in Definition 4.11.

(ii) $\Rightarrow$ (iii) From $\vdash_{BL} \neg\varphi \to (\varphi \to \psi)$, we get $\vdash_{BL} ((\varphi \to \psi) \to \varphi) \to (\neg\varphi \to \varphi)$. Thus, $\mathcal{F}(v((\varphi \to \psi) \to \varphi)) \models \mathcal{F}(v(\neg\varphi \to \varphi)) = \mathcal{F}(v(\varphi))$.

(iii) $\Rightarrow$ (iv) It is enough to prove that $\mathcal{F}(v(\varphi)) \models \mathcal{F}(v((\varphi \to \psi) \to \varphi))$ and this follows from $\vdash_{BL} \varphi \to ((\varphi \to \psi) \to \varphi)$.

(iv) $\Rightarrow$ (v) Since $\mathcal{F}$ is a NDF, then $\{\mathcal{F}(v(\chi \to ((\varphi \to \psi) \to \varphi))), \mathcal{F}(v(\chi))\} \models \mathcal{F}(v((\varphi \to \psi) \to \varphi)) = \mathcal{F}(v(\varphi))$.

(v) $\Rightarrow$ (i) Let $\mathcal{F}$ be a NDF. In order to verify that $\mathcal{F}$ is a BNDF, by Theorems 4.10 and 4.12, it is enough to prove that $\mathcal{F}(v(\varphi \to (\neg\chi \to \chi))) \models \mathcal{F}(v(\varphi \to \chi))$, for all formulas φ, χ and all assignments v. Since, $\vdash_{BL} \chi \to (\varphi \to \chi)$ we have $\vdash_{BL} (\neg(\varphi \to \chi)) \to \neg\chi$ and then $\vdash_{BL} (\neg\chi \to (\varphi \to \chi)) \to (\neg(\varphi \to \chi) \to (\varphi \to \chi))$. From (v), replacing φ by $\varphi \to \chi$, χ by a BL-provable formula θ and ψ by a contradiction (In this case the formula $\varphi \to \psi$ is equivalent to $\neg(\varphi \to \chi)$), we get $\mathcal{F}(v(\varphi \to (\neg\chi \to \chi))) = \mathcal{F}(v(\neg\chi \to (\varphi \to \chi))) \models \mathcal{F}(v(\neg(\varphi \to \chi) \to (\varphi \to \chi))) = \{\mathcal{F}(v(\theta \to (\neg(\varphi \to \chi) \to (\varphi \to \chi)))), \mathcal{F}(v(\theta))\} \models \mathcal{F}(v(\varphi \to \chi))$.

Therefore, $\mathcal{F}$ is a BNDF on $\mathcal{L}$.

5. Implicative neutrosophic deductive filters

In this section we define and study the notion of implicative deductive filter on BL-algebras. Also, we investigate some relations between BNDFs and INDFs.

Definition 5.1. A neutrosophic subset $\mathcal{F}$ of $\mathcal{L}$ is called an *implicative neutrosophic deductive filter* (briefly, INDF) if for all formulas φ, ψ, χ and all asignments v

$$\{\mathcal{F}(v(\varphi \to (\psi \to \chi))),\ \mathcal{F}(v(\varphi \to \psi))\} \models \mathcal{F}(v(\varphi \to \chi)).$$

As an immediate result we have:

Theorem 5.2. *Every INDF is a NDF.*

Proof. Let $\mathcal{F}$ be an INDF on $\mathcal{L}$. Then for each BL-provable formula θ, we have $\{\mathcal{F}(v(\theta \to (\psi \to \chi))),\ \mathcal{F}(v(\theta \to \psi))\} \models \mathcal{F}(v(\theta \to \chi))$, so $\{\mathcal{F}(v(\theta) \to v(\psi \to \chi))),\ \mathcal{F}(v(\theta) \to v(\psi))\} \models \mathcal{F}(v(\theta) \to v(\chi))$, then $\{\mathcal{F}(1 \to v(\psi \to \chi))),\ \mathcal{F}(1 \to v(\psi))\} \models \mathcal{F}(1 \to v(\chi))$, which implies that

$$\{\mathcal{F}(v(\psi \to \chi))),\ \mathcal{F}(v(\psi))\} \models \mathcal{F}(v(\chi))$$

Thus $\mathcal{F}$ is a NDF on $\mathcal{L}$.

Proposition 5.3. *A neutrosophic subset $\mathcal{F}$ of $\mathcal{L}$ is a INDF if and only if for all $x, y, z \in \mathcal{L}$*

$$\mathcal{F}(x \to z) \geq \mathcal{F}(x \to (y \to z)) \wedge \mathcal{F}(x \to y).$$

Proof. It can be easily verified by Definition 5.1.

Theorem 5.4. *Let $\mathcal{F}$ be a NDF on $\mathcal{L}$, then the following statements are equivalent:*

(i) $\mathcal{F}$ is an INDF,
(ii) $\mathcal{F}(v(\varphi \to (\varphi \to \psi))) \models \mathcal{F}(v(\varphi \to \psi))$,
(iii) $\mathcal{F}(v(\varphi \to \psi)) = \mathcal{F}(v(\varphi \to (\varphi \to \psi)))$,
(iv) $\mathcal{F}(v(\varphi \to (\psi \to \chi))) \models \mathcal{F}(v((\varphi \to \psi) \to (\varphi \to \chi)))$,
(v) $\mathcal{F}(v(\varphi \to (\psi \to \chi))) = \mathcal{F}(v((\varphi \to \psi) \to (\varphi \to \chi)))$,
(vi) $\mathcal{F}(v((\varphi \odot \psi) \to \chi)) = \mathcal{F}(v((\varphi \wedge \psi) \to \chi))$.

Proof. (i) $\Rightarrow$ (ii) It is enough to put $\psi = \varphi$ and $\chi = \psi$, in the definition.

(ii) $\Rightarrow$ (iii) It follows from $\vdash_{BL} (\varphi \to \psi) \to (\varphi \to (\varphi \to \psi))$.

(iii) $\Rightarrow$ (i) Using Corollary 3.11, we have $\{\mathcal{F}(v(\varphi \to (\psi \to \chi))),\ \mathcal{F}(v(\varphi \to \psi))\} \models \{\mathcal{F}(v(\varphi \to (\varphi \to \chi)))\} = \mathcal{F}(v(\varphi \to \chi))$

(i) $\Rightarrow$ (iv) Suppose that $\mathcal{F}$ is an INDF on $\mathcal{L}$. Then, $\{\mathcal{F}(v(\varphi \to (\psi \to \chi))),\ \mathcal{F}(v(\varphi \to ((\psi \to \chi) \to ((\varphi \to \psi) \to \chi))))\} \models \{\mathcal{F}(v(\varphi \to ((\varphi \to \psi) \to \chi)))\} = \{\mathcal{F}(v((\varphi \to \psi) \to (\varphi \to \chi)))\}$. Since $\vdash_{BL} (\psi \to \chi) \to ((\varphi \to \psi) \to (\varphi \to \chi))$, then $\{\mathcal{F}(v(\varphi \to (\psi \to \chi))),\ \mathcal{F}(1)\} \models \{\mathcal{F}(v((\varphi \to \psi) \to (\varphi \to \chi)))\}$. Therefore, $\mathcal{F}(v(\varphi \to (\psi \to \chi))) \models \{\mathcal{F}(v((\varphi \to \psi) \to (\varphi \to \chi)))\}$.

(iv) $\Rightarrow$ (v) It is easy.

(v) $\Rightarrow$ (vi) We have $\mathcal{F}(v(\varphi \& \psi \to \chi)) = \mathcal{F}(v(\varphi \to (\psi \to \chi))) = \mathcal{F}(v((\varphi \to \psi) \to (\varphi \to \chi))) = \mathcal{F}(v(((\varphi \to \psi) \& \varphi) \to \chi)) = \mathcal{F}(v((\varphi \& (\varphi \to \psi)) \to \chi)) = \mathcal{F}(v(\varphi \wedge \psi \to \chi))$.

(vi) $\Rightarrow$ (i) By Corollary 3.11, we have $\mathcal{F}(v(\varphi \to \chi)) = \mathcal{F}(v((\varphi \wedge \varphi) \to \chi)) = \mathcal{F}(v((\varphi \& \varphi) \to \chi)) = \mathcal{F}(v(\varphi \to (\varphi \to \chi)))$, also by Corollary 3.11 $\{\mathcal{F}(v(\varphi \to \psi)),\ \mathcal{F}(v(\varphi \to (\psi \to \chi)))\} = \{\mathcal{F}(v(\varphi \to \psi)),\ \mathcal{F}(v(\psi \to (\varphi \to \chi)))\} \models \mathcal{F}(v(\varphi \to (\varphi \to \chi)))$, therefore $\{\mathcal{F}(v(\varphi \to \psi)),\ \mathcal{F}(v(\varphi \to (\psi \to \chi)))\} \models \mathcal{F}(v(\varphi \to (\varphi \to \chi)))$. Hence, $\mathcal{F}$ is an INDF on $\mathcal{L}$.

Theorem 5.5. *A NDF $\mathcal{F}$ on $\mathcal{L}$ is an INDF if and only if for each $t \in [0, 1]$, $\emptyset \neq T_{\mathcal{F}t}$, $\emptyset \neq I_{\mathcal{F}t}$, $\emptyset \neq F_{\mathcal{F}t}$ and all of them are implicative filters on $\mathcal{L}$.*

Proof. Suppose that $\mathcal{F}$ is an INDF, $\emptyset \neq T_{\mathcal{F}t}$, $\emptyset \neq I_{\mathcal{F}t}$ and $\emptyset \neq F_{\mathcal{F}t}$, for some $t \in [0, 1]$. Let $x_0 \in T_{\mathcal{F}t}$, $x_1 \in I_{\mathcal{F}t}$ and $x_2 \in F_{\mathcal{F}t}$. Then, $T_{\mathcal{F}}(x_0) \geq t$, $I_{\mathcal{F}}(x_1) \leq t$ and $F_{\mathcal{F}}(x_1) \leq t$.

Since $\mathcal{F}$ is an INDF, $T_{\mathcal{F}}(1) \geq T_{\mathcal{F}}(x_0) \geq t$, $I_{\mathcal{F}}(1) \leq I_{\mathcal{F}}(x_1) \leq t$ and $F_{\mathcal{F}}(1) \leq F_{\mathcal{F}}(x_2) \leq t$, i.e. $1 \in T_{\mathcal{F}t}$, $1 \in I_{\mathcal{F}t}$ and $1 \in F_{\mathcal{F}t}$. Now, suppose that $x \to y,\ x \to (y \to z) \in T_{\mathcal{F}t}$, $x \to y,\ x \to (y \to z) \in I_{\mathcal{F}t}$ and $x \to y,\ x \to (y \to z) \in F_{\mathcal{F}t}$, for some $x, y, z \in \mathcal{L}$. Then,

$$\mathcal{F}(x \to z) \geq \mathcal{F}(x \to y) \wedge \mathcal{F}(x \to (y \to z)) \geq t$$

Thus $x \to z \in T_{\mathcal{F}t}$, $x \to z \in I_{\mathcal{F}t}$ and $x \to z \in F_{\mathcal{F}t}$. This implies that $T_{\mathcal{F}t}$, $I_{\mathcal{F}t}$ and $F_{\mathcal{F}t}$ are implicative filters on $\mathcal{L}$.

Conversely, suppose that for each $t \in [0, 1]$, $\emptyset \neq T_{\mathcal{F}t}$, $\emptyset \neq I_{\mathcal{F}t}$ and $\emptyset \neq F_{\mathcal{F}t}$ are implicative filters on $\mathcal{L}$. Obviously, $\emptyset \neq T_{T_{\mathcal{F}}(x)}$, $\emptyset \neq I_{I_{\mathcal{F}}(x)}$ and $\emptyset \neq F_{F_{\mathcal{F}}(x)}$, for any $x \in \mathcal{L}$. Then, $T_{T_{\mathcal{F}}(x)}$, $I_{I_{\mathcal{F}}(x)}$ and $F_{F_{\mathcal{F}}(x)}$ are implicative filters on $\mathcal{L}$ and so $1 \in T_{T_{\mathcal{F}}(x)}$, $1 \in I_{I_{\mathcal{F}}(x)}$ and $1 \in F_{F_{\mathcal{F}}(x)}$, i.e. $\mathcal{F}(1) \geq \mathcal{F}(x)$. Let $t_0 = T_{\mathcal{F}}(x \to y) \wedge T_{\mathcal{F}}(x \to (y \to z))$, $t_1 = I_{\mathcal{F}}(x \to y) \vee I_{\mathcal{F}}(x \to (y \to z))$ and $t_2 = F_{\mathcal{F}}(x \to y) \vee F_{\mathcal{F}}(x \to (y \to z))$, for some $x, y, z \in \mathcal{L}$. Then, $x \to y,\ x \to (y \to z) \in T_{\mathcal{F}t_0}$, $x \to y,\ x \to (y \to z) \in I_{\mathcal{F}t_1}$ and $x \to y,\ x \to (y \to z) \in F_{\mathcal{F}t_2}$ and so $x \to z \in T_{\mathcal{F}t_0}$, $x \to z \in I_{\mathcal{F}t_1}$ and $x \to z \in F_{\mathcal{F}t_2}$. Hence,

$$\begin{aligned} T_{\mathcal{F}}(x \to z) &\geq t_0 \\ &= T_{\mathcal{F}}(x \to (y \to z)) \wedge T_{\mathcal{F}}(x \to y), \\ I_{\mathcal{F}}(x \to z) &\leq t_1 \\ &= I_{\mathcal{F}}(x \to (y \to z)) \vee I_{\mathcal{F}}(x \to y), \\ F_{\mathcal{F}}(x \to z) &\leq t_2 \\ &= F_{\mathcal{F}}(x \to (y \to z)) \vee F_{\mathcal{F}}(x \to y). \end{aligned}$$

Thus, $\mathcal{F}$ is an INDF on $\mathcal{L}$.

Theorem 5.6. *Every BNDF is an INDF.*

Proof. Let $\mathcal{F}$ be a BNDF on $\mathcal{L}$. From $\vdash_{BL} ((\varphi \vee \neg\varphi) \to (\varphi \to \chi)) \leftrightarrow [((\varphi \to (\varphi \to \chi)) \wedge (\neg\varphi \to (\varphi \to \chi)))] \leftrightarrow (\varphi \to (\varphi \to \chi))$ it follows that $\{\mathcal{F}(v(\varphi \to (\varphi \to \chi)))\} = \{\mathcal{F}(v((\varphi \vee \neg\varphi) \to (\varphi \to \chi)))\} = \{\mathcal{F}(v((\varphi \vee \neg\varphi) \to (\varphi \to \chi))), \mathcal{F}(v(\varphi \vee \neg\varphi))\} \models \mathcal{F}(v(\varphi \to \chi))$. Therefore, $\mathcal{F}$ is an INDF on $\mathcal{L}$, by Theorem 5.4 (ii).

Example 5.7. Let $\mathcal{L} = \{0, a, b, 1\}$ be a chain with cayley tables as follows:

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Define $\wedge$ and $\vee$ on $\mathcal{L}$ as min and max, respectively. Then $(\mathcal{L}, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra. The neutrosophic subset $\mathcal{F}$ of $\mathcal{L}$ defined by $\mathcal{F}(0) = \mathcal{F}(a) = \mathcal{F}(b) = (t_1, t_2, t_2)$, $\mathcal{F}(1) = (t_2, t_1, t_1)$, where $0 \le t_1 < t_2 \le 1$ are two fixed real numbers in [0,1], is an INDF, but it is not a BNDF.

Theorem 5.8. *Let $\mathcal{F}$ be an INDF on $\mathcal{L}$. $\mathcal{F}$ is a BNDF if and only if for all formulas φ, ψ and all assignments v:*

$$\mathcal{F}(v((\varphi \rightarrow \psi) \rightarrow \psi)) = \mathcal{F}(v((\psi \rightarrow \varphi) \rightarrow \varphi)) \quad (5.1)$$

Proof. Suppose that $\mathcal{F}$ is a BNDF. From $\vdash_{BL} \varphi \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)$, it follows that $\vdash_{BL} (\neg((\psi \rightarrow \varphi) \rightarrow \varphi)) \rightarrow \neg\varphi$ and since by Proposition 2, $\vdash_{BL} \neg\varphi \rightarrow (\varphi \rightarrow \psi))$, then we have $\vdash_{BL} (\neg((\psi \rightarrow \varphi) \rightarrow \varphi)) \rightarrow (\varphi \rightarrow \psi))$, so $\vdash_{BL} ((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow (\neg((\psi \rightarrow \varphi) \rightarrow \varphi) \rightarrow \psi)$. In addition, since $\vdash_{BL} \psi \rightarrow (\varphi \vee \psi)$ and $\vdash_{BL} (\varphi \vee \psi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)$ we have $\vdash_{BL} \psi \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)$. Thus $\vdash_{BL} (\neg((\psi \rightarrow \varphi) \rightarrow \varphi) \rightarrow \psi) \rightarrow (\neg((\psi \rightarrow \varphi) \rightarrow \varphi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi))$. Therefore, $\vdash_{BL} ((\varphi \rightarrow \psi) \rightarrow \psi) \rightarrow (\neg((\psi \rightarrow \varphi) \rightarrow \varphi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi))$ and hence $\mathcal{F}(v((\varphi \rightarrow \psi) \rightarrow \psi)) \models \mathcal{F}(v(\neg((\psi \rightarrow \varphi) \rightarrow \varphi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)))$, by Corollary 3.11.

Since by Theorem 4.13 (ii), $\mathcal{F}(v((\psi \rightarrow \varphi) \rightarrow \varphi)) = \mathcal{F}(v(\neg((\psi \rightarrow \varphi) \rightarrow \varphi) \rightarrow ((\psi \rightarrow \varphi) \rightarrow \varphi)))$, then $\mathcal{F}(v((\varphi \rightarrow \psi) \rightarrow \psi)) \models \mathcal{F}(v((\psi \rightarrow \varphi) \rightarrow \varphi))$. Similarly, we can prove that $\mathcal{F}(v((\psi \rightarrow \varphi) \rightarrow \varphi)) \models \mathcal{F}(v((\varphi \rightarrow \psi) \rightarrow \psi))$. Then, $\mathcal{F}(v((\varphi \rightarrow \psi) \rightarrow \psi)) = \mathcal{F}(v((\psi \rightarrow \varphi) \rightarrow \varphi))$.

Conversely, Let $\mathcal{F}$ be an INDF on $\mathcal{L}$ and (5.1) holds. Then, in order to prove that $\mathcal{F}$ is a BNDF, it is enough to show that $\mathcal{F}(v((\varphi \rightarrow \neg\varphi) \rightarrow \neg\varphi)) = \mathcal{F}(1)$, by Proposition 2. Since by Proposition 4.4, $\vdash_{BL} ((\varphi \rightarrow \neg\varphi) \rightarrow \neg\varphi) \leftrightarrow (\varphi \rightarrow \neg\neg\varphi)$ and $\vdash_{BL} \varphi \rightarrow \neg\neg\varphi$, then it follows that $\mathcal{F}(v((\varphi \rightarrow \neg\varphi) \rightarrow \neg\varphi)) = \mathcal{F}(v(\varphi \rightarrow \neg\neg\varphi)) = \mathcal{F}(1)$, which completes the proof.

Theorem 5.9. *Let $\mathcal{F}$ be an INDF on $\mathcal{L}$. $\mathcal{F}$ is a BNDF if and only if $\mathcal{F}(v(\neg\neg\varphi)) = \mathcal{F}(v(\varphi))$, for each formula φ and each assignment v.*

Proof. Assume that $\mathcal{F}(v(\neg\neg\varphi)) = \mathcal{F}(v(\varphi))$. Then, $\mathcal{F}(v(\neg\varphi \rightarrow \neg\neg\varphi)) = \mathcal{F}(v(\neg\neg\varphi))$, by Theorem 5.4 (iii). Also we have $\mathcal{F}(v(\neg\varphi \rightarrow \varphi)) \models \mathcal{F}(v(\neg\varphi \rightarrow \neg\neg\varphi))$. Hence $\mathcal{F}(v(\neg\varphi \rightarrow \varphi)) \models \mathcal{F}(v(\varphi))$. Obviously, $\mathcal{F}(v(\varphi)) \models \mathcal{F}(v(\neg\varphi \rightarrow \varphi))$. Thus $\mathcal{F}$ is a BNDF on $\mathcal{L}$, , by Theorem 4.13. The converse is obtained by using Theorem 5.8.

Corollary 5.10. *Let $\mathcal{L}$ be a MV-Algebra and $\mathcal{F}$ be a NDF on $\mathcal{L}$. Then, the following statements are equivalent:*

(i) $\mathcal{F}$ is a BNDF on $\mathcal{L}$,
(ii) $\mathcal{F}$ is an INDF,
(iii) $\mathcal{F}(v(\varphi \rightarrow \chi)) = \mathcal{F}(v(\varphi \rightarrow (\varphi \rightarrow \chi)))$,
(iv) $\mathcal{F}(v(\varphi \rightarrow \chi)) = \mathcal{F}(v(\varphi \rightarrow (\neg\chi \rightarrow \chi)))$,
(v) $\mathcal{F}(v((\varphi \rightarrow \psi) \rightarrow \varphi)) = \mathcal{F}(v(\varphi))$,
(vi) $\mathcal{F}(v(\varphi \rightarrow (\psi \rightarrow \chi))) = \mathcal{F}(v((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)))$,
(vii) $\mathcal{F}(v((\varphi \odot \psi) \rightarrow \chi)) = \mathcal{F}(v((\varphi \wedge \psi) \rightarrow \chi))$.

6. Quotient structures

In this section we define the quotient structure for neutrosophic deductive filters and study some of its properties.

Let $\mathcal{F}$ be a NDF on $\mathcal{L}$ and $x \in \mathcal{L}$. The neutrosophic set $\mathcal{F}^x : \mathcal{L} \rightarrow [0, 1]^3$ which is defined by $\mathcal{F}^x(y) = (T_{\mathcal{F}^x}(y), I_{\mathcal{F}^x}(y), F_{\mathcal{F}^x}(y))$, for any $y \in \mathcal{L}$, where $T_{\mathcal{F}^x}(y) = T_{\mathcal{F}}(x \rightarrow y) \wedge T_{\mathcal{F}}(y \rightarrow x)$, $I_{\mathcal{F}^x}(y) = I_{\mathcal{F}}(x \rightarrow y) \vee I_{\mathcal{F}}(y \rightarrow x)$ and $F_{\mathcal{F}^x}(y) = F_{\mathcal{F}}(x \rightarrow y) \vee F_{\mathcal{F}}(y \rightarrow x)$, is called the *neutrosophic coset* (briefly, NC) of $\mathcal{F}$. Denote the set of all NCs of $\mathcal{F}$ by $\mathcal{L}/\mathcal{F}$.

Now, we have the following lemma.

Lemma 6.1. *Let $\mathcal{F}$ be a NDF on $\mathcal{L}$. Then, $\mathcal{F}^x = \mathcal{F}^y$ if and only if $\mathcal{F}(x \rightarrow y) = \mathcal{F}(y \rightarrow x) = \mathcal{F}(1)$.*

Proof. Assume that $\mathcal{F}^x = \mathcal{F}^y$, for some $x, y \in \mathcal{L}$. Then $(T_{\mathcal{F}^x}(x), I_{\mathcal{F}^x}(x), F_{\mathcal{F}^x}(x)) = \mathcal{F}^x(x) = \mathcal{F}^y(x) = (T_{\mathcal{F}^y}(x), I_{\mathcal{F}^y}(x), F_{\mathcal{F}^y}(x))$, so, $T_{\mathcal{F}^x}(x) = T_{\mathcal{F}^y}(x)$, $I_{\mathcal{F}^x}(x) = I_{\mathcal{F}^y}(x)$ and $F_{\mathcal{F}^x}(x) = F_{\mathcal{F}^y}(x)$. Hence, $T_{\mathcal{F}}(1) = T_{\mathcal{F}}(x \rightarrow x) = T_{\mathcal{F}}(y \rightarrow x) \wedge T_{\mathcal{F}}(x \rightarrow y)$, $I_{\mathcal{F}}(1) = I_{\mathcal{F}}(x \rightarrow x) = I_{\mathcal{F}}(y \rightarrow x) \vee I_{\mathcal{F}}(x \rightarrow y)$ and $F_{\mathcal{F}}(1) = F_{\mathcal{F}}(x \rightarrow x) = F_{\mathcal{F}}(y \rightarrow x) \vee F_{\mathcal{F}}(x \rightarrow y)$ which implies that $T_{\mathcal{F}}(y \rightarrow x) = T_{\mathcal{F}}(x \rightarrow y) = T_{\mathcal{F}}(1)$, $I_{\mathcal{F}}(y \rightarrow x) = I_{\mathcal{F}}(x \rightarrow y) = I_{\mathcal{F}}(1)$ and $F_{\mathcal{F}}(y \rightarrow x) = F_{\mathcal{F}}(x \rightarrow y) = F_{\mathcal{F}}(1)$. Then, $\mathcal{F}(x \rightarrow y) = (T_{\mathcal{F}}(x \rightarrow y), I_{\mathcal{F}}(x \rightarrow y), F_{\mathcal{F}}(x \rightarrow y)) = (T_{\mathcal{F}}(1), I_{\mathcal{F}}(1), F_{\mathcal{F}}(1)) = \mathcal{F}(1)$ and similarly, $\mathcal{F}(y \rightarrow x) = \mathcal{F}(1)$.

Conversely, assume that $\mathcal{F}(x \rightarrow y) = \mathcal{F}(y \rightarrow x) = \mathcal{F}(1)$. Then, $T_{\mathcal{F}}(x \rightarrow y) = T_{\mathcal{F}}(y \rightarrow x) = T_{\mathcal{F}}(1)$, $I_{\mathcal{F}}(x \rightarrow y) = I_{\mathcal{F}}(y \rightarrow x) = I_{\mathcal{F}}(1)$ and $F_{\mathcal{F}}(x \rightarrow y) = F_{\mathcal{F}}(y \rightarrow x) = F_{\mathcal{F}}(1)$. Thus, by Corollary 3.12, for each $x, y, z \in \mathcal{L}$, we obtains that $\mathcal{F}(z \rightarrow x) \ge \mathcal{F}(z \rightarrow y) \wedge \mathcal{F}(y \rightarrow x) = \mathcal{F}(z \rightarrow y)$ and $\mathcal{F}(x \rightarrow z) \ge \mathcal{F}(x \rightarrow y) \wedge \mathcal{F}(y \rightarrow z) = \mathcal{F}(y \rightarrow z)$, hence $\mathcal{F}^x(z)$

$= \mathcal{F}(z \to x) \wedge \mathcal{F}(x \to z) \geq \mathcal{F}(z \to y) \wedge \mathcal{F}(y \to z) = \mathcal{F}^y(z)$. Similarly, $\mathcal{F}^y(z) \geq \mathcal{F}^x(z)$, therefore, $\mathcal{F}^x = \mathcal{F}^y$.

Suppose that $\mathcal{F}$ be a NDF on $\mathcal{L}$. Let $\mathcal{F}_{\mathcal{F}(1)} = \{x \in \mathcal{L} : \mathcal{F}(x) = \mathcal{F}(1)\}$, then it is easy to verify that $\mathcal{F}_{\mathcal{F}(1)} = T_{T(1)} \cap I_{I(1)} \cap F_{F(1)}$. Hence $\mathcal{F}_{\mathcal{F}(1)}$ is a filter on $\mathcal{L}$.

Corollary 6.2. *Let $\mathcal{F}$ be a NDF on $\mathcal{L}$. Then, $\mathcal{F}^x = \mathcal{F}^y$ if and only if $x \sim_{\mathcal{F}_{\mathcal{F}(1)}} y$, where*

$$x \sim_{\mathcal{F}_{\mathcal{F}(1)}} y \;\Leftrightarrow\; x \to y \in \mathcal{F}_{\mathcal{F}(1)},\; y \to x \in \mathcal{F}_{\mathcal{F}(1)}.$$

Let $\mathcal{F}$ be a NDF on $\mathcal{L}$. For any $\mathcal{F}^x, \mathcal{F}^y \in \mathcal{L}/\mathcal{F}$, define $\mathcal{F}^x \wedge \mathcal{F}^y = \mathcal{F}^{x\wedge y}$, $\mathcal{F}^x \vee \mathcal{F}^y = \mathcal{F}^{x\vee y}$, $\mathcal{F}^x \odot \mathcal{F}^y = \mathcal{F}^{x\odot y}$, and $\mathcal{F}^x \to \mathcal{F}^y = \mathcal{F}^{x\to y}$.

Now, we get the following lemma.

Lemma 6.3. *Let $\mathcal{F}$ be a NDF on $\mathcal{L}$, $\mathcal{F}^x = \mathcal{F}^z$ and $\mathcal{F}^y = \mathcal{F}^w$, for some $x, y, z, w \in \mathcal{L}$. Then, $\mathcal{F}^x \vee \mathcal{F}^y = \mathcal{F}^z \vee \mathcal{F}^w$, $\mathcal{F}^x \wedge \mathcal{F}^y = \mathcal{F}^z \wedge \mathcal{F}^w$, $\mathcal{F}^x \odot \mathcal{F}^y = \mathcal{F}^z \odot \mathcal{F}^w$, and $\mathcal{F}^x \to \mathcal{F}^y = \mathcal{F}^z \to \mathcal{F}^w$.*

Proof. Assume that $\mathcal{F}^x = \mathcal{F}^z$ and $\mathcal{F}^y = \mathcal{F}^w$, for some $x, y, z, w \in \mathcal{L}$. By Corollary 6.2, $x \sim_{\mathcal{F}_{\mathcal{F}(1)}} z$ and $y \sim_{\mathcal{F}_{\mathcal{F}(1)}} w$. Since, by Theorem 2, $\sim_{\mathcal{F}_{\mathcal{F}(1)}}$ is a congruence on $\mathcal{L}$, then $x \wedge y \sim_{\mathcal{F}_{\mathcal{F}(1)}} z \wedge w$, $x \vee y \sim_{\mathcal{F}_{\mathcal{F}(1)}} z \vee w$, $x \odot y \sim_{\mathcal{F}_{\mathcal{F}(1)}} z \odot w$ and $x \to y \sim_{\mathcal{F}_{\mathcal{F}(1)}} z \to w$. This implies that $\mathcal{F}^x \wedge \mathcal{F}^y = \mathcal{F}^{x\wedge y} = \mathcal{F}^{z\wedge w} = \mathcal{F}^z \wedge \mathcal{F}^w$ and similarly, $\mathcal{F}^x \vee \mathcal{F}^y = \mathcal{F}^z \vee \mathcal{F}^w$, $\mathcal{F}^x \odot \mathcal{F}^y = \mathcal{F}^z \odot \mathcal{F}^w$ and $\mathcal{F}^x \to \mathcal{F}^y = \mathcal{F}^z \to \mathcal{F}^w$.

We note that the lattice order $\leq$ on $\mathcal{L}/\mathcal{F}$ is defined by $\mathcal{F}^x \leq \mathcal{F}^y$ if and only if $\mathcal{F}^x \vee \mathcal{F}^y = \mathcal{F}^y$.

Lemma 6.4. *Let $\mathcal{F}$ be a NDF on $\mathcal{L}$. Then, $\mathcal{F}^x \leq \mathcal{F}^y$ if and only if $\mathcal{F}(x \to y) = \mathcal{F}(1)$.*

Proof. Let $x, y \in \mathcal{L}$. Then,

$$\begin{aligned}
&\mathcal{F}^x \leq \mathcal{F}^y \Leftrightarrow \mathcal{F}^{x\vee y} = \mathcal{F}^x \vee \mathcal{F}^y = \mathcal{F}^y \Leftrightarrow \\
&T_{\mathcal{F}^{x\vee y}} = T_{\mathcal{F}^y},\ I_{\mathcal{F}^{x\vee y}} = I_{\mathcal{F}^y},\ F_{\mathcal{F}^{x\vee y}} = F_{\mathcal{F}^y} \\
\Leftrightarrow\ & T_{\mathcal{F}}(1) = T_{\mathcal{F}}(y \to (x \vee y)) = T_{\mathcal{F}}(x \vee y \to y), \\
& I_{\mathcal{F}}(1) = I_{\mathcal{F}}(y \to (x \vee y)) = I_{\mathcal{F}}(x \vee y \to y), \\
& F_{\mathcal{F}}(1) = F_{\mathcal{F}}(y \to (x \vee y)) = F_{\mathcal{F}}(x \vee y \to y) \\
\Leftrightarrow\ & T_{\mathcal{F}}(x \to y) = T_{\mathcal{F}}(1),\ I_{\mathcal{F}}(x \to y) = I_{\mathcal{F}}(1), \\
& F_{\mathcal{F}}(x \to y) = F_{\mathcal{F}}(1) \\
\Leftrightarrow\ & \mathcal{F}(x \to y) \\
=\ & (T_{\mathcal{F}}(x \to y), I_{\mathcal{F}}(x \to y), F_{\mathcal{F}}(x \to y)) \\
=\ & (T_{\mathcal{F}}(1), I_{\mathcal{F}}(1), F_{\mathcal{F}}(1)) = \mathcal{F}(1).
\end{aligned}$$

Theorem 6.5. *Let $\mathcal{F}$ be a NDF on $\mathcal{L}$. Then $(\mathcal{L}/\mathcal{F}, \wedge, \vee, \odot, \to, \mathcal{F}^1, \mathcal{F}^0)$ is a BL-algebra.*

Proof. By Lemma 6.3, the operations $\wedge$, $\vee$, $\odot$ and $\to$ on $\mathcal{L}/\mathcal{F}$ are well-defined. We only need to prove $\mathcal{L}/\mathcal{F}$ satisfies the axioms of BL-algebras. The axioms (BL1), (BL2), (BL4) and (BL5) can be easily proved. Let $\mathcal{F}^x, \mathcal{F}^y, \mathcal{F}^z \in \mathcal{L}/\mathcal{F}$, then by Corollary 6.4

$$\begin{aligned}
(\mathcal{F}^x \odot \mathcal{F}^y \leq \mathcal{F}^z) &\Leftrightarrow (\mathcal{F}^{x\odot y} \leq \mathcal{F}^z) \\
&\Leftrightarrow \mathcal{F}((x \odot y) \to z) = \mathcal{F}(1) \\
&\Leftrightarrow \mathcal{F}(x \to (y \to z)) = \mathcal{F}(1) \\
&\Leftrightarrow \mathcal{F}^x \leq \mathcal{F}^{y\to z} \\
&\Leftrightarrow \mathcal{F}^x \leq \mathcal{F}^y \to \mathcal{F}^z.
\end{aligned}$$

7. Isomorphism theorems

In this section we prove three isomorphism theorems concerning quotients of neutrosophic deductive filters.

We note that, since $\mathcal{F}_{\mathcal{F}(1)}$ is a filter on $\mathcal{L}$, then by Theorem 2.10, $\mathcal{L}/\mathcal{F}_{\mathcal{F}(1)}$ is a BL-algebra.

Theorem 7.1. *Let $\mathcal{F}$ be a NDF on $\mathcal{L}$. Then $\mathcal{L}/\mathcal{F} \simeq \mathcal{L}/\mathcal{F}_{\mathcal{F}(1)}$.*

Proof. Define a map $\Phi : \mathcal{L}/\mathcal{F} \longrightarrow \mathcal{L}/\mathcal{F}_{\mathcal{F}(1)}$ by $\Phi(\mathcal{F}^x) = [x]_{\mathcal{F}_{\mathcal{F}(1)}}$. We prove that Φ is an isomorphism. Suppose that $\mathcal{F}^x, \mathcal{F}^y \in \mathcal{L}/\mathcal{F}$. Then $\mathcal{F}^x = \mathcal{F}^y$ if and only if $x \sim_{\mathcal{F}_{\mathcal{F}(1)}} y$ if and only if $[x]_{\mathcal{F}_{\mathcal{F}(1)}} = [y]_{\mathcal{F}_{\mathcal{F}(1)}}$, which implies that Φ is an one-to-one function. Obviously, Φ is surjective. Now, $\Phi(\mathcal{F}^x \odot \mathcal{F}^y) = \Phi(\mathcal{F}^{x\odot y}) = [x \odot y]_{\mathcal{F}_{\mathcal{F}(1)}} = [x]_{\mathcal{F}_{\mathcal{F}(1)}} \bullet [y]_{\mathcal{F}_{\mathcal{F}(1)}} = \Phi(\mathcal{F}^x) \bullet \Phi(\mathcal{F}^y)$. Similarly, it can be obtained that $\Phi(\mathcal{F}^x \to \mathcal{F}^y) = \Phi(\mathcal{F}^x) \rightharpoonup \Phi(\mathcal{F}^y)$. Thus, Φ is an isomorphism and hence $\mathcal{L}/\mathcal{F} \simeq \mathcal{L}/\mathcal{F}_{\mathcal{F}(1)}$.

Corollary 7.2. *Let $f : \mathcal{L} \longrightarrow \mathcal{L}'$ be a homomorphism of BL-algebras and $\mathcal{F}$ be a NDF on $\mathcal{L}$ which $kerf = \mathcal{F}_{\mathcal{F}(1)}$. Then, $\mathcal{L}/\mathcal{F} \simeq f(\mathcal{L})$.*

Definition 7.3. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be BL-algebras and f be a map from $\mathcal{L}_1$ into $\mathcal{L}_2$. Also let $\mathcal{F}$ be a neutrosophic subset of $\mathcal{L}_1$. The neutrosophic subset $f(\mathcal{F})$ of $\mathcal{L}_2$ is defined by

$$f(\mathcal{F})(l_2) = \begin{cases} (T_{f(\mathcal{F})}(l_2),\ I_{f(\mathcal{F})}(l_2),\ F_{f(\mathcal{F})}(l_2)), & f^{-1}(l_2) \neq \emptyset \\ (0, 1, 1), & \text{otherwise} \end{cases}$$

for any $l_2 \in \mathcal{L}_2$, where

$$T_{f(\mathcal{F})}(l_2) = \vee\{T_{\mathcal{F}}(l_1) : l_1 \in \mathcal{L}_1,\ f(l_1) = l_2\},$$
$$I_{f(\mathcal{F})}(l_2) = \wedge\{I_{\mathcal{F}}(l_1) : l_1 \in \mathcal{L}_1,\ f(l_1) = l_2\},$$
$$F_{f(\mathcal{F})}(l_2) = \wedge\{F_{\mathcal{F}}(l_1) : l_1 \in \mathcal{L}_1,\ f(l_1) = l_2\}.$$

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be BL-algebras and $\mathcal{F}_1$ and $\mathcal{F}_2$ be neutrosophic sets of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. A homomorphism f of $\mathcal{L}_1$ onto $\mathcal{L}_2$ is called *weak homomorphism* of $\mathcal{F}_1$ into $\mathcal{F}_2$, if $f(\mathcal{F}_1) \leq \mathcal{F}_2$. In this case, we say that $\mathcal{F}_1$ is weakly homomorphic to $\mathcal{F}_2$ and we write $\mathcal{F}_1 \sim^f \mathcal{F}_2$ or simply $\mathcal{F}_1 \sim \mathcal{F}_2$. If f is bijective, we say that $\mathcal{F}_1$ is weakly isomorphic to $\mathcal{F}_2$ and we write $\mathcal{F}_1 \simeq^f \mathcal{F}_2$ or simply $\mathcal{F}_1 \simeq \mathcal{F}_2$. A homomorphism f of $\mathcal{L}_1$ onto $\mathcal{L}_2$ is called a *homomorphism* of $\mathcal{F}_1$ into $\mathcal{F}_2$, if $f(\mathcal{F}_1) = \mathcal{F}_2$. In this case, we say that $\mathcal{F}_1$ is homomorphic to $\mathcal{F}_2$ and we write $\mathcal{F}_1 \approx^f \mathcal{F}_2$ or simply $\mathcal{F}_1 \approx \mathcal{F}_2$. If f is bijective, we say that $\mathcal{F}_1$ is isomorphic to $\mathcal{F}_2$ and we write $\mathcal{F}_1 \cong^f \mathcal{F}_2$ or simply $\mathcal{F}_1 \cong \mathcal{F}_2$(See [8]).

For a neutrosophic subset $\mathcal{F}$ of $\mathcal{L}$, define $\mathcal{F}^* = \{x \in \mathcal{L},\ T_{\mathcal{F}}(x) > \overline{0},\ I_{\mathcal{F}}(x) < \overline{1},\ F_{\mathcal{F}}(x) < \overline{1}\}$.

Lemma 7.4. *Let $\mathcal{F}$ be a NDF on $\mathcal{L}$, $T_{\mathcal{F}}(1) > \overline{0}$, $I_{\mathcal{F}}(1) < \overline{1}$ and $F_{\mathcal{F}}(1) < \overline{1}$. Then $\mathcal{F}^*$ is a filter on $\mathcal{L}$.*

Proof. The proof is easy.

Recall that $\mathcal{F}_1 \leq \mathcal{F}_2$ means that $T_{\mathcal{F}_1}(x) \leq T_{\mathcal{F}_2}(x)$, $I_{\mathcal{F}_1}(x) \geq I_{\mathcal{F}_2}(x)$ and $F_{\mathcal{F}_1}(x) \geq F_{\mathcal{F}_2}(x)$, for any $x \in \mathcal{L}$. Now, if $\mathcal{F}_1 \leq \mathcal{F}_2$ and $x \in \mathcal{F}_1{}^*$, then $\overline{0} < T_{\mathcal{F}_1}(x) \leq T_{\mathcal{F}_2}(x)$, $\overline{1} > I_{\mathcal{F}_1}(x) \geq I_{\mathcal{F}_2}(x)$ and $\overline{1} > F_{\mathcal{F}_1}(x) \geq F_{\mathcal{F}_2}(x)$ and so $x \in \mathcal{F}_2{}^*$, which implies that $\mathcal{F}_1^* \subseteq \mathcal{F}_2^*$. Hence $\mathcal{F}_1^*$ is a filter on $\mathcal{F}_2^*$ and $\mathcal{F}_2^*/\mathcal{F}_1^*$ is a BL-algebra, by Lemma 2.11.

Theorem 7.5. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two NDFs on $\mathcal{L}$, $\mathcal{F}_1 \leq \mathcal{F}_2$ and $\mathcal{F}_2$ has Sup-Inf property. Define $\xi : \mathcal{F}_2{}^*/\mathcal{F}_1{}^* \longmapsto [0, 1]$ by*

$$\xi([x]_{\mathcal{F}_1^*}) = (T_\xi([x]_{\mathcal{F}_1^*}),\ I_\xi([x]_{\mathcal{F}_1^*}),\ F_\xi([x]_{\mathcal{F}_1^*}))$$

where $T_\xi([x]_{\mathcal{F}_1^*}) = \vee\{T_{\mathcal{F}_2}(y)|\ y \in [x]_{\mathcal{F}_1^*}\}$, $I_\xi([x]_{\mathcal{F}_1^*}) = \wedge\{I_{\mathcal{F}_2}(y)|\ y \in [x]_{\mathcal{F}_1^*}\}$ and $F_\xi([x]_{\mathcal{F}_1^*}) = \wedge\{F_{\mathcal{F}_2}(y)|\ y \in [x]_{\mathcal{F}_1^*}\}$, for all $[x]_{\mathcal{F}_1^*} \in \mathcal{F}_2{}^*/\mathcal{F}_1{}^*$. Then ξ is a neutrosophic set.

Proof. Since, $\mathcal{F}_2$ has Sup-Inf property, there exist $y_0, y_1, y_2 \in [x]_{\mathcal{F}_1^*}$ such that $T_\xi([x]_{\mathcal{F}_1^*}) = T_{\mathcal{F}_2}(y_0)$, $I_\xi([x]_{\mathcal{F}_1^*}) = I_{\mathcal{F}_2}(y_1)$ and $F_\xi([x]_{\mathcal{F}_1^*}) = F_{\mathcal{F}_2}(y_2)$. Obviously, $\xi([x]_{\mathcal{F}_1^*}) = (T_{\mathcal{F}_2}(y_0), I_{\mathcal{F}_2}(y_1), F_{\mathcal{F}_2}(y_2))$ is a neutrosophic set, which completes the proof. We call the neutrosophic set ξ defined in Theorem 7.5, the *quotient neutrosophic set* on $\mathcal{F}_2$ relative to $\mathcal{F}_1$ and denote it by $\mathcal{F}_2/\mathcal{F}_1$.

Theorem 7.6. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two NDFs on $\mathcal{L}$, $\mathcal{F}_1 \leq \mathcal{F}_2$ and $\mathcal{F}_2$ has Sup-Inf property. Then*

$$\mathcal{F}_2 |_{\mathcal{F}_2^*} \approx \mathcal{F}_2/\mathcal{F}_1.$$

Proof. Let $f : \mathcal{F}_2^* \longrightarrow \mathcal{F}_2^*/\mathcal{F}_1^*$ be the natural epimorphism and $[x]_{\mathcal{F}_1^*} \in \mathcal{F}_2^*/\mathcal{F}_1^*$. Then $f(\mathcal{F}_2 |_{\mathcal{F}_2^*})([x]_{\mathcal{F}_1^*}) = (\vee\{T_{\mathcal{F}_2}(z)|z \in \mathcal{F}_2^*, f(z) = [x]_{\mathcal{F}_1^*}\}, \wedge\{I_{\mathcal{F}_2}(z)|z \in \mathcal{F}_2^*, f(z) = [x]_{\mathcal{F}_1^*}\}, \wedge\{F_{\mathcal{F}_2}(z)|z \in \mathcal{F}_2^*, f(z) = [x]_{\mathcal{F}_1^*}\}) = (\vee \{T_{\mathcal{F}_2}(y)|y \in [x]_{\mathcal{F}_1^*}\}, \wedge\{I_{\mathcal{F}_2}(y)|y \in [x]_{\mathcal{F}_1^*}\}, \wedge\{F_{\mathcal{F}_2}(y)|y \in [x]_{\mathcal{F}_1^*}\}) = (\mathcal{F}_2/\mathcal{F}_1)([x]_{\mathcal{F}_1^*})$. Therefore, $\mathcal{F}_2 |_{\mathcal{F}_2^*} \approx \mathcal{F}_2/\mathcal{F}_1$.

Theorem 7.7. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two NDFs on BL-algebras $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, $\mathcal{F}_1 \approx \mathcal{F}_2$ and $\mathcal{F}_1$ has Sup-Inf property. Then there exists a NDF $\mathcal{F}_3$ such that $\mathcal{F}_3 \leq \mathcal{F}_1$ and $\mathcal{F}_1/\mathcal{F}_3 \cong \mathcal{F}_2 |_{\mathcal{F}_2^*}$.*

Proof. Since $\mathcal{F}_1 \approx \mathcal{F}_2$, there is an homomorphism f from $\mathcal{L}_1$ onto $\mathcal{L}_2$ such that $f(\mathcal{F}_1) = \mathcal{F}_2$. Define the neutrosophic set $\mathcal{F}_3$ as follows:

$$\mathcal{F}_3(x) = \begin{cases} (T_{\mathcal{F}_1}(x),\ I_{\mathcal{F}_1}(x),\ F_{\mathcal{F}_1}(x)), & x \in ker(f) \\ (\overline{0}, \overline{1}, \overline{1}), & \text{otherwise} \end{cases}$$

for any $x \in \mathcal{L}_1$. It is easy to show that $\mathcal{F}_3$ is a NDF on $\mathcal{L}_1$. Since $\mathcal{F}_1 \approx \mathcal{F}_2$, then $f(\mathcal{F}_1^*) = \mathcal{F}_2^*$. Let $g = f |_{\mathcal{F}_1^*}$, then g is a homomorphism from $\mathcal{F}_1^*$ onto $\mathcal{F}_2^*$ and $ker(g) = \mathcal{F}_3^*$. Thus, by the first isomorphism theorem, there exists an isomorphism $h : \mathcal{F}_1{}^*/\mathcal{F}_3{}^* \longrightarrow \mathcal{F}_2^*$ such that $h([x]_{\mathcal{F}_3^*}) = g(x) = f(x)$, for any $x \in \mathcal{F}_1{}^*$. Now, $h(\mathcal{F}_1/\mathcal{F}_3)(z) = (\vee\{T_{\mathcal{F}_1/\mathcal{F}_3}([x]_{\mathcal{F}_3^*}) \mid x \in \mathcal{F}_1{}^*, h([x]_{\mathcal{F}_3^*}) = z\}, \wedge\{I_{\mathcal{F}_1/\mathcal{F}_3}([x]_{\mathcal{F}_3^*}) \mid x \in \mathcal{F}_1{}^*, h([x]_{\mathcal{F}_3^*}) = z\}, \wedge\{F_{\mathcal{F}_1/\mathcal{F}_3}([x]_{\mathcal{F}_3^*}) \mid x \in \mathcal{F}_1{}^*, h([x]_{\mathcal{F}_3^*}) = z\}) = (\vee\{\vee \{T_{\mathcal{F}_1}(y) : y \in [x]_{\mathcal{F}_3^*}\} | x \in \mathcal{F}_1{}^*, g(x) = z\}, \wedge\{\wedge\{I_{\mathcal{F}_1}(y) : y \in [x]_{\mathcal{F}_3^*}\} \mid x \in \mathcal{F}_1{}^*, g(x) = z\}, \wedge\{\wedge\{F_{\mathcal{F}_1}(y) : y \in [x]_{\mathcal{F}_3^*}\} \mid x \in \mathcal{F}_1{}^*, g(x) = z\}) = (\vee\{T_{\mathcal{F}_1}(y)|y \in \mathcal{F}_1{}^*, g(y) = z\}, \wedge\{I_{\mathcal{F}_1}(y)|y \in \mathcal{F}_1{}^*, g(y) = z\}, \wedge\{F_{\mathcal{F}_1}(y)|y \in \mathcal{F}_1{}^*, g(y) = z\}) = (\vee\{T_{\mathcal{F}_1}(y)|y \in \mathcal{L}_1, f(y) = z\}, \wedge \{I_{\mathcal{F}_1}(y)|y \in \mathcal{L}_1, f(y) = z\}, \wedge\{F_{\mathcal{F}_1}(y)|y \in \mathcal{L}_1, f(y) = z\}) = \mathcal{F}_2(z)$ for any $z \in \mathcal{F}_2{}^*$. Therefore $\mathcal{F}_1/\mathcal{F}_3 \cong \mathcal{F}_2 |_{\mathcal{F}_2^*}$.

Lemma 7.8. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two NDFs on $\mathcal{L}$ such that $\mathcal{F}_1 \leq \mathcal{F}_2$. Then $\mathcal{F}_2^*/\mathcal{F}_1^* = (\mathcal{F}_2/\mathcal{F}_1)^*$.*

Proof. The proof is easy.

Lemma 7.9. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ be NDFs on $\mathcal{L}$, $\mathcal{F}_1 \leq \mathcal{F}_2 \leq \mathcal{F}_3$ and $\mathcal{F}_2, \mathcal{F}_3$ have Sup-Inf property. Then $(\mathcal{F}_2/\mathcal{F}_1)$ and $(\mathcal{F}_3/\mathcal{F}_1)$ are neurosophic subsets of $\mathcal{L}$ such that $(\mathcal{F}_2/\mathcal{F}_1) \leq (\mathcal{F}_3/\mathcal{F}_1)$.*

Proof. Use Theorem 7.5.

Theorem 7.10. *Let $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ be NDFs on $\mathcal{L}$, $\mathcal{F}_1 \leq \mathcal{F}_2 \leq \mathcal{F}_3$ and $\mathcal{F}_2, \mathcal{F}_3$ have Sup-Inf property, such that $\mathcal{F}_3/\mathcal{F}_1,\ \mathcal{F}_2/\mathcal{F}_1$ are NDFs. Then*

$$(\mathcal{F}_3/\mathcal{F}_1)/(\mathcal{F}_2/\mathcal{F}_1) \approx \mathcal{F}_3/\mathcal{F}_2.$$

Proof. It can be proved by using Theorem 2 and Lemmas 7.8, 7.9.

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