Neutrosophic deductive filters on BL-algebras

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Abstract. In this paper, we introduce the notions of neutrosophic deductive filter, Boolean neutrosophic deductive filter (BNDF) and implicative neutrosophic deductive filter (INDF) on BL-algebras as generalizations of the fuzzy corresponding versions. We also investigate some properties of these filters and drive some characterizations of them. The relation between BNDF and INDF is investigated and it is proved that every BNDF is an INDF, but the converse is true when certain condition is satisfied. Furthermore, we construct a quotient structure related to the neutrosophic deductive filter and prove certain isomorphism theorems.

Keywords: BL-algebra, neutrosophic deductive filter, quotient structure

1. Introduction

Fuzzy set theory was introduced by Zadeh in 1965 [11]. A fuzzy subset $A$ of a set $X$ is a function $\mu_A : X \rightarrow [0, 1]$, where for each $x \in X$, $\mu_A(x)$ represents the grade of membership of the element $x \in X$ to $A$. In [1], Atanassov introduced the intuitionistic fuzzy sets as a generalization of fuzzy sets. The intuitionistic fuzzy sets consider both membership degree and nonmembership degree.

In 1998, neutrosophy has been proposed by Smarandache [9] as a new branch of philosophy in order to formally represent neutralities. The fundamental thesis of neutrosophy is that every idea has not only a certain degree of truth and a certain degree of falsity but also an indeterminacy degree that have to be considered independently from each other. In neutrosophic set theory, indeterminacy is measured explicitly and independently. This assumption is very important in many applications such as information fusion in which we try to combine the data from different sensors. As an example, suppose there are 10 voters during a voting process. One possible situation is that there are three yes votes, two no votes and five undecided ones. We note that this example is beyond the scope of intuitionistic fuzzy set theory.

In 1960 Abraham Robinson introduced non-standard analysis as a formalization of analysis and a branch of mathematical logic. In non-standard analysis a nonzero number $\epsilon$ is said to be infinitely small, or infinitesimal if and only if for all positive integers $n$, $|\epsilon| \leq 1/n$. In this case the reciprocal $\delta = 1/\epsilon$ will be infinitely large, or simply infinite, meaning that for all positive integers $n$, we have $|\delta| > n$. The set of hyper-real numbers is an extension of the set of real numbers which includes the class of infinite numbers and the class of infinitesimal numbers. The non-standard unit interval is $[0^-, 1^+] = [0^-] \cup [0, 1] \cup [1^+]$. Here $0^-$ is the set of all hyper-real numbers $0 - \epsilon$, and $1^+$ is the set of all hyper-real numbers $1 + \lambda$, where $\epsilon$ and $\lambda$ are infinitesimal.

If $U$ is a set, a neutrosophic set defined on the universe $U$ assigns to each element $x \in U$, a triple $(T(x), I(x), F(x))$, where $T(x)$, $I(x)$ and $F(x)$ are standard or non-standard elements of $[0^-, 1^+]$. $T$ is the degree of membership in the set $U$, $I$ is the degree of indeterminacy-membership in the set $U$ and $F$ is the degree of nonmembership in the set $U$. In this paper we work with special neutrosophic sets that their neutrosophic elements are standard real numbers in $[0,1]$. Neutrosophy has laid the foundation for a whole family of new mathematical theories, such as neutrosophic
set theory, neutrosophic probability, neutrosophic statistics and neutrosophic logic. In recent years neutro-

BL-algebras provide an algebraic semantics for H👇ek's Basic Logic [2]. The main example of a BL-

implicative filter and fuzzy positive implicative filter on


tions of fuzzy prime filter, fuzzy Boolean filter, fuzzy
deductive filter and implicative neutrosophic deductive

set A provides an algebraic semantics for H👇ek's Basic Logic [2]. The main example of a BL-

properties and characterizations were investigated.

from the literature.

true. Furthermore, the condition under which an INDF

filter and investigate some of their properties. We drive

definition of BL-algebra is the unit interval [0,1] endowed with the

structure induced by a continuous t-norm. MV-algebras,

G"odel algebras and Product algebras are the most

important rule in studying these algebras. From the

Gödel algebras and Product algebras are the most

structure related to the neutrosophic deductive filter and

characterizations of these filters. Also, we inves-

tigate relation between BNDF and INDF and prove that

Let

Definition 2.2.

Note that there is no restrictions on the values of

Note that there is no restrictions on the values of

Definition 2.6.

The only inference rule is modus ponens (MP). We show

Proposition 2.5. [2] In BL the following statements hold:

Definition 2.2. [9] Let A and B be two neutrosophic

Definition 2.4. [2] The axioms of propositional H👇ek

Definition 2.1. [9] Let X be a set. A neutrosophic sub-

Nonmembership function. Here for each

Fuzzy neutrosophic-valued sets on X provide a triple

in the Hilbert-style

Definition 2.3. [9] Let A and B be two neutrosophic

Definition 2.5. [2] A BL-algebra is an algebra

(\mathcal{L}, \lor, \land, \oplus, \to, 0, 1) of type (2,2,2,0,0) such that

\begin{align*}
T_A(x) &\leq T_B(x), \quad I_A(x) \geq I_B(x), \quad F_A(x) \geq F_B(x), \\
&\quad \text{for all } x \in X.
\end{align*}

\begin{align*}
\mathcal{T}(x) &\leq \mathcal{T}(x), \quad \mathcal{I}(x) \geq \mathcal{I}(x), \quad \mathcal{F}(x) \geq \mathcal{F}(x).
\end{align*}

\begin{align*}
\mathcal{T}(x) &\leq \mathcal{T}(x), \quad \mathcal{I}(x) \geq \mathcal{I}(x), \quad \mathcal{F}(x) \geq \mathcal{F}(x).
\end{align*}

\begin{align*}
\mathcal{T}(x) &\leq \mathcal{T}(x), \quad \mathcal{I}(x) \geq \mathcal{I}(x), \quad \mathcal{F}(x) \geq \mathcal{F}(x).
\end{align*}

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(BL1) $(\mathcal{C}, \lor, \land, 0, 1)$ is a bounded lattice,
(BL2) $(\mathcal{C}, \circ, 1)$ is a commutative monoid,
(BL3) $x \circ y \leq z$ if and only if $x \leq y \to z$, for all $x, y, z \in \mathcal{C}$,
(BL4) $x \land y = x \circ (x \to y)$,
(BL5) $(x \to y) \lor (y \to x) = 1$.

If $\mathcal{L}$ is a BL-algebra $\mathcal{C}$ satisfies $\neg\neg x = x$, for each $x \in \mathcal{L}$, it is called an MV-algebra.

**Proposition 2.7.** [2, 6] In any BL-algebra $\mathcal{C}$, the following properties hold:

(R1) $x \leq y \Rightarrow x \to y = 1$,
(R2) $1 \to x = x, x \to 1 = 1, x \to x = 1 \Rightarrow x = 1$,
(R3) $x \leq y \Rightarrow z \to y \leq x \to z$,
(R4) $x \to (y \to z) = (x \land y) \to z = y \to (x \to z)$,
(R5) $x \leq z \Rightarrow (x \to z) \to y \leq z \to y$ and $y \to z \leq (y \to z) \to x$,
(R6) $z \to (x \to z) \to (y \to z), z \to y \leq (y \to z) \to (z \to x)$,
(R7) $(x \to y) \circ (y \to z) \leq x \to z$,
(R8) $x \to y \leq \neg \neg (x \to y)$, $x \leq \neg \neg x$, when $\neg \neg x = x \to 0$,
(R9) $\neg \neg (x \land y) = (\neg \neg x \land \neg \neg y)$,
(R10) $x \lor \neg x = 1 \Rightarrow x \land \neg x = 0$,
(M1) $x \circ y \leq x \land y$,
(M2) $x \leq y \Rightarrow \neg \neg x \leq \neg \neg y \equiv (y \to z) \leq (y \to z)$,
(M3) $y \to z \leq \neg \neg (y \to z), \neg \neg x \leq \neg \neg x$,
(M4) $x \land (y \lor z) = (x \land y) \lor (x \land z)$,
(M5) $x \lor (y \lor z) = (x \lor y) \lor (x \lor z)$,
(M6) $x \lor (y \lor z) = (x \lor y) \lor (x \lor y), (y \lor (x \lor z))$,
(M7) $x \land (y \land z) \leq (x \land y) \land (x \land z)$,
(M8) $x \land (y \lor z) = (x \land y) \lor (x \land z)$,
(M9) $x \lor (y \land z) = (x \lor y) \land (x \lor z)$,
(M10) $x \land (y \lor z) = (x \land y) \lor (x \land z)$,
(M11) $x \lor (y \land z) = (x \lor y) \land (x \lor z)$.

**Definition 2.8.** [2, 10] Let $F$ be a nonempty subset of a BL-algebra $\mathcal{C}$ such that $1 \in F$. $F$ is called:

(i) a filter on $\mathcal{C}$ if,

$(\forall x, y \in \mathcal{C}(x \in F \Rightarrow \neg \neg x \in F) \land \neg \neg (x \lor y) \in F)$

and $(\forall x, y \in \mathcal{C}(x \land y \in F \Rightarrow x \land \neg \neg y \in F)$,

(ii) a Boolean filter on $\mathcal{C}$, if it is a filter and moreover we have

$(\forall x \in \mathcal{C}(x \lor \neg x \in F)$.

(iii) an implicatitive filter on $\mathcal{C}$, if it is a filter and moreover for all $x, y, z \in \mathcal{C}$ we have

$[x \to y, x \to (y \to z) \in F \Rightarrow (x \to z) \in F].$

**Proposition 2.9.** [10] A nonempty subset $F$ of a BL-algebra $\mathcal{C}$ is a filter if and only if

(DS1) $1 \in F$,
(DS2) $(\forall x, y \in \mathcal{C}(x \in F, x \to y \in F \Rightarrow y \in F)$.

**Theorem 2.10.** [2] Let $F$ be a filter on a BL-algebra $\mathcal{C}$. Define the binary relation $\sim_F$ on $\mathcal{C}$ by

$x \sim_F y \Leftrightarrow (x = y \in F \land y = x \in F)$

Then $\sim_F$ is a congruence on $\mathcal{C}$, and the set of all congruence classes $\mathcal{C}/\sim_F = \{[x] : x \in \mathcal{C}\}$ with the following operations form a BL-algebra:

$[x] \cdot [y] = [x \land y], [x] \rightarrow [y] = [x \to y], [x] = [x \lor y], [x] \cap [y] = [x \land y]$

**Lemma 2.11.** [6] Let $F_1$ and $F_2$ be non filters on BL-algebra $\mathcal{C}$ which $F_1 \subseteq F_2$. Then $F_1$ is a filter on $F_2$ if $F_2 \setminus F_1$ is a filter on $\mathcal{C}/F_1$.

**Definition 2.12.** [6] The neutrosophic set $\mathcal{F}$ of a BL-algebra $\mathcal{C}$ has Sup-Inf Property if for any nonempty subset $S$ of $\mathcal{C}$, there exist $x_0, x_1, x_2 \in S$, such that

$\sup_{\mathcal{F}} F(x) = F(x_0), \inf_{\mathcal{F}} F(x) = F(x_1), \sup_{\mathcal{F}} F(x) = F(x_2)$

From now on, we use the same notations for corresponding logical and algebraic notions. Also, if there is no confusion, we use $\land$ and $\lor$ for minimum and maximum for real numbers.

3. Neutrosophic deductive filters on BL-algebras

In this section, we define the neutrosophic deductive filters and prove some properties of them. Furthermore, we characterize the neutrosophic decisive filter generated by a neutrosophic deductive set.

**Definition 3.1.** Suppose that $\Gamma$ and $\Delta$ are two subsets of $[0, 1]^\mathcal{C}$. We define the relation $\models$ as follows:

$\Gamma \models \Delta \Leftrightarrow \land \Gamma \leq \Delta$

If $\Gamma = 0$, then we define $\land \Gamma = (1, 0, 0)$, and if $\Delta = 0$, then we define $\land \Delta = (0, 1, 1)$.
From now on, if $\Gamma \vdash \Delta$ and $\Delta \vdash \Gamma$, we write $\Gamma \equiv \Delta$.

**Definition 3.2.** Let $\mathcal{L}$ be a BL-algebra and $\vdash$ be a consequence relation on the set of BL-formulas. A neutrosophic subset $\mathcal{F}$ of $\mathcal{L}$ is called a neutrosophic filter with respect to $\vdash$, if for each assignment $v$ into $\mathcal{L}$ and for every set $\Gamma \cup \{ \phi \}$ of BL-formulas, if $\Gamma \vdash \phi$, then $[\mathcal{F}(\gamma)](v) = 1$, where $\mathcal{F}(\gamma) = \{ \gamma(\phi) : \phi \in \Gamma \}$.

In particular, if $\vdash$ is presented by a Hilbert-style system, then we define $\vdash$ as follows: for all assignments $v$ and (MP) is the only inference rule.

**Lemma 3.4.** A neutrosophic subset $F$ of a BL-algebra $\mathcal{L}$ is a NDF iff for all formulas $\psi, \chi$ and each assignment $v$ into $\mathcal{L}$:

$(\text{NDF1}) \quad F(\psi)(v) = F(1)$,
$(\text{NDF2}) \quad F(\psi(|v\phi\chi|)) = F(F(\phi)(v))$.

**Proof.** This can be easily obtained from the fact that in a BL-algebra, all axioms of BL are evaluated to 1 under all assignments and (MP) is the only inference rule.

**Corollary 3.5.** A neutrosophic subset $F$ of a BL-algebra $\mathcal{L}$ is a NDF iff:

$(\text{NDF1}) \quad (\forall v \in \mathcal{L})(F(a) \land F(b) \land F(a \land b) = F(b))$.

**Theorem 3.8.** Let $F$ be a neutrosophic subset of $\mathcal{L}$. Then $F$ is a NDF if and only if for all formulas $\psi, \chi$ and all assignment $v$ into $\mathcal{L}$, if $\vdash_{\mathcal{L}} (\psi \rightarrow (\chi \rightarrow \psi))$ then $\mathcal{F}(\psi)(v)$.

**Proof.** Let $\vdash$ be a NDF on $\mathcal{L}$, $\mathcal{L}$ be an assignment into $\mathcal{L}$ and $\vdash_{\mathcal{L}} (\psi \rightarrow (\chi \rightarrow \psi))$, for some formulas $\psi, \chi$. By Lemma 3.4, we have $[\mathcal{F}(\psi)(v) \rightarrow (\chi \rightarrow \psi)]$, $[\mathcal{F}(\psi)(v) \rightarrow \chi)$, $[\mathcal{F}(\psi)(v)] \rightarrow [\mathcal{F}(\chi)(v)]$ and $[\mathcal{F}(\psi)(v) \rightarrow \chi)$. Thus, we obtain that $[\mathcal{F}(\chi)(v)] \rightarrow [\mathcal{F}(\chi)(v)]$, which completes the proof.

**Corollary 3.9.** Let $F$ be a neutrosophic subset of $\mathcal{L}$. Then $F$ is a NDF if and only if for all formulas $\psi, \chi$ and all assignment $v$ into $\mathcal{L}$, if $\vdash_{\mathcal{L}} (\psi \rightarrow (\chi \rightarrow \psi))$, then $\mathcal{F}(\psi)(v) \rightarrow \chi)$. Then $\mathcal{F}(\psi)(v) \rightarrow \chi)$.

**Theorem 3.10.** Let $F$ be a neutrosophic subset of $\mathcal{L}$. Then $F$ is a NDF if and only if for all formulas $\psi, \chi$ and all assignment $v$ into $\mathcal{L}$:

$(i) \quad \vdash_{\mathcal{L}} (\psi \rightarrow (\chi \rightarrow \psi))$ $\Rightarrow$ $[\mathcal{F}(\psi)(v)] \rightarrow [\mathcal{F}(\chi)(v)]$.
$(ii) \quad [\mathcal{F}(\psi)(v)] \Rightarrow [\mathcal{F}(\psi)(v)]$.

**Proof.** Suppose that $F$ is a NDF. Since $\vdash_{\mathcal{L}} (\psi \rightarrow (\chi \rightarrow \psi))$, we have $\vdash_{\mathcal{L}} (\psi \rightarrow (\chi \rightarrow \psi))$. So, by Corollary 3.8, it follows that for all assignments $v$, $[\mathcal{F}(\psi)(v)] \Rightarrow [\mathcal{F}(\chi)(v)]$.

**Example 3.7.** Let $\mathcal{L} = \{0, a, b, 1\}$. For all $x, y \in \mathcal{L}$, we define $x \lor y = \min(x, y), x \land y = \max(x, y)$ and $\Rightarrow$ as follows:

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Then $(\mathcal{L}, \lor, \land, \Rightarrow, \rightarrow, 0, 1)$ is a BL-algebra. The neutrosophic subset $\mathcal{F}$ of $\mathcal{L}$ defined by $\mathcal{F}(0) = \mathcal{F}(a) = (t_1, t_1, t_1), \mathcal{F}(b) = (t_2, t_2, t_2), \mathcal{F}(1) = (t_3, t_3, t_3)$, where $0 \leq t_1 < t_2 < t_3 \leq 1$ are three fixed real numbers in $[0, 1]$, is a NDF.
Proof.

Let $G$ be a NDF, for each $\vdash \{ \psi, \phi \}$, then we get $\vdash \{ \psi, \phi \}$.

Thus $\vdash \{ \psi, \phi \}$.

Therefore, $\vdash \{ \psi, \phi \}$.

Example 3.16. Suppose that $\langle \omega, \chi \rangle \leq 0$ be the BL-algebra defined in Example 3.7. Define the...
neutrosophic subset \( F \) of \( L \) by \( F(0) = (t_1, t_1, t_1) \), \( F(a) = F(b) = (t_1, t_2, t_2) \), \( F(1) = (t_2, t_2, t_2) \) (0 ≤ \( t_1 < t_2 ≤ 1 \)) and the neutrosophic subset \( G \) of \( L \) by \( G(0) = G(a) = G(b) = (t_1, t_1, t_1), G(1) = (t_2, t_1, t_2) \). One can easily check that \( G = \{ F(x) \} \).

4. Boolean neutrosophic deductive filters

In this section we define and study the notion of Boolean neutrosophic deductive filters on BL-algebras.

Definition 4.1. Let \( F \) be a NDF on \( L \). \( F \) is called a Boolean neutrosophic deductive filter (briefly, BNDF) if \( F(1) \models F((\psi \vee \neg \psi)) \), for all formulas \( \psi \) and all assignments \( v \).

Example 4.2. Let \( L = \{0, a, b, 1\} \) be a chain with Cayley tables as follow:

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Define \( \wedge \) and \( \vee \) on \( L \) as min and max, respectively. Then \( (L, \leq) \) is a BL-algebra.

Proposition 4.3. Let \( F \) be a NDF on \( L \). \( F \) is a BNDF if and only if for all formulas \( \psi \) and all assignments \( v \) we have \( F((\psi \vee \neg \psi)) \models F((\psi \wedge \neg \psi)) \).

Proposition 4.4. Let \( F \) be a NDF on \( L \). Then, \( F \) is a BNDF if and only if for all formula \( \psi \) and all assignments \( v \) we have \( F((\psi \rightarrow \neg \psi)) \models F((\neg \psi \rightarrow \psi)) \).

Proposition 4.5. Let \( F \) be a NDF on \( L \). Then, \( F \) is a BNDF if and only if for all \( x \in L \) we have \( F(x \rightarrow x) = F((\neg x \rightarrow x)) = F((x \rightarrow x)) \).
On the other hand for each BL-provable formula \( \psi \) we have \( \mathcal{F}(\psi \rightarrow (\neg \chi \rightarrow \psi)) = \mathcal{F}(\psi \rightarrow (\neg \chi \rightarrow \psi)) \), \( \mathcal{F}(\psi \rightarrow (\neg \chi \rightarrow \psi)) \) proves (i).

Theorem 4.11. Let \( \mathcal{F} \) be a NDF on \( \mathcal{L} \). We say \( \mathcal{F} \) has Implicative Property, if for all formulas \( \psi, \chi, \) and all assignments \( \nu \), it satisfies:

\[
[\mathcal{F}(\psi \rightarrow (\neg \chi \rightarrow \psi)), \mathcal{F}(\psi \rightarrow \chi)]
\]

which proves that \( \mathcal{F} \) satisfies the Implicative Property, by Theorem 4.10 (i), (ii).

Conversely, suppose that \( \mathcal{F} \) satisfies the Implicative Property. By Theorem 4.10 (iii), replacing \( \psi \) by \( \neg \chi \rightarrow \psi \) and \( \chi \) by \( \chi \), we have \( \mathcal{F}(\psi \rightarrow \chi) = \mathcal{F}(\neg \chi \rightarrow \psi) \).

Theorem 4.12. A NDF \( \mathcal{F} \) on \( \mathcal{L} \) is a BNF if and only if it satisfies the Implicative Property.

Proof. Suppose that \( \mathcal{F} \) is a BNF on \( \mathcal{L} \). From \( \vdash_{BL} \chi \rightarrow (\neg \chi \rightarrow \chi) \), \( \vdash_{BL} \psi \rightarrow (\neg \chi \rightarrow \psi) \), \( \vdash_{BL} (\psi \rightarrow (\neg \chi \rightarrow \psi)) \), and \( \vdash_{BL} (\psi \rightarrow (\neg \chi \rightarrow \psi)) \), it follows that

\[
\mathcal{F}(\psi \rightarrow (\neg \chi \rightarrow \chi)) = \mathcal{F}(\psi \rightarrow (\neg \chi \rightarrow \psi))
\]

which proves that \( \mathcal{F} \) satisfies the Implicative Property, by Theorem 4.10 (i), (ii).

By Theorem 4.10 (iii), replacing \( \psi \) by \( \neg \chi \rightarrow \psi \) and \( \chi \) by \( \psi \), we get

\[
\mathcal{F}(\psi \rightarrow \chi) = \mathcal{F}(\neg \chi \rightarrow \psi)
\]

Theorem 4.13. Let \( \mathcal{F} \) be a NDF on \( \mathcal{L}, \psi, \chi, \) and \( \nu \) be an assignment on \( \mathcal{L} \). Then the following are equivalent:

(i) \( \mathcal{F} \) is a BNF,
(ii) \( \mathcal{F}(\psi \rightarrow (\neg \chi \rightarrow \psi)) \),
(iii) \( \mathcal{F}(\psi \rightarrow (\neg \chi \rightarrow \psi)) \),
(iv) \( \mathcal{F}(\psi \rightarrow \chi) \),
(v) \( \mathcal{F}(\psi \rightarrow (\neg \chi \rightarrow \psi)) \),
(vi) \( \mathcal{F}(\psi \rightarrow (\neg \chi \rightarrow \psi)) \).

Proof. (i) \( \Rightarrow \) (ii) Since \( \vdash_{BL} \neg \chi \rightarrow (\neg \psi \rightarrow \psi) \), then by Corollary 3.11 we have \( \mathcal{F}(\psi \rightarrow (\neg \chi \rightarrow \psi)) \).

The other direction follows from Theorem 4.12, replacing \( \psi \) by a BL-provable formulas and \( \psi, \chi, \) by \( \psi \) in Definition 4.11.

5. Implicative neutrosophic deductive filters

In this section we define and study the notion of implicational deductive filter on BL-algebras. Also, we investigate some relations between BNDFs and INDFs.

Definition 5.1. A neutrosophic subset \( \mathcal{F} \) of \( \mathcal{L} \) is called an implicational neutrosophic deductive filter (briefly, INDF) if for all formulas \( \psi, \chi, \) and all assignments \( \nu \),

\[
\mathcal{F}(\psi \rightarrow (\neg \chi \rightarrow \psi)) \Rightarrow \mathcal{F}(\nu \rightarrow (\neg \chi \rightarrow \nu))
\]

As an immediate result we have:

Theorem 5.2. Every INDF is a NDF.

Proof. Let \( \mathcal{F} \) be an INDF on \( \mathcal{L} \). Then for each BL-provable formula \( \theta \), we have \( \mathcal{F}(\theta \rightarrow (\psi \rightarrow \chi)) \), \( \mathcal{F}(\theta \rightarrow (\nu \rightarrow \psi)) \), \( \mathcal{F}(\theta \rightarrow (\psi \rightarrow \chi)) \), then \( \mathcal{F}(\theta \rightarrow (\psi \rightarrow \chi)) \).

Thus \( \mathcal{F} \) is a NDF on \( \mathcal{L} \).

Proposition 5.3. A neutrosophic subset \( \mathcal{F} \) of \( \mathcal{L} \) is a INDF if and only if for all \( x, y, z \) in \( \mathcal{L} \).
Let $\mathcal{F}$ be a Neutrosophic deductive filter on a BL-algebra $\mathcal{L}$. Then the following statements are equivalent:

(i) $\mathcal{F}$ is an INDF on $\mathcal{L}$.
(ii) $\mathcal{F}(x \to (y \to z)) = \mathcal{F}(x \to (y \to z)) \land \mathcal{F}(y \to z) = \mathcal{F}(x \to (y \to z) \land \mathcal{F}(y \to z))$.
(iii) $\mathcal{F}(x \to (y \to z)) = \mathcal{F}(x \to (y \to z) \land \mathcal{F}(v \to z))$.
(iv) $\mathcal{F}(v \to z) = \mathcal{F}(v \to z)$.
(v) $\mathcal{F}(v \to z) = \mathcal{F}(v \to z)$.

Proof. Suppose that $\mathcal{F}$ is an INDF on $\mathcal{L}$. Then $\mathcal{F}(x \to (y \to z)) = \mathcal{F}(x \to (y \to z) \land \mathcal{F}(y \to z))$.

Since $\mathcal{F}$ is an INDF, $\mathcal{F}(x \to (y \to z)) \leq t$, $\mathcal{F}(x) \leq t$ and $\mathcal{F}(y) \leq t$. Hence, $\mathcal{F}$ is an INDF on $\mathcal{L}$.

Theorem 5.5. A NDF $\mathcal{F}$ on $\mathcal{L}$ is an INDF if and only if

\[ \mathcal{F}(x \to (y \to z)) = \mathcal{F}(x \to (y \to z) \land \mathcal{F}(y \to z)) \]

This implies that $\mathcal{F}$ and $\mathcal{F}$ are implicative filters on $\mathcal{L}$.

Conversely, suppose that for each $t \in [0, 1]$, if $\mathcal{F}$ is an INDF on $\mathcal{L}$, then $\mathcal{F}$ is an INDF on $\mathcal{L}$.

Thus $\mathcal{F}$ is an INDF on $\mathcal{L}$. Therefore, $\mathcal{F}$ is an INDF on $\mathcal{L}$.

Theorem 5.6. Every NDF is an INDF.

Proof. Let $\mathcal{F}$ be a NDF on $\mathcal{L}$. Then $\mathcal{F}(x \to (y \to z)) = \mathcal{F}(x \to (y \to z) \land \mathcal{F}(y \to z))$. Therefore, $\mathcal{F}$ is an INDF on $\mathcal{L}$, by Theorem 5.4 (ii).

Example 5.7. Let $\mathcal{L} = [0, a, b, 1]$ be a chain with

\[
\begin{array}{c|cccc}
  & 0 & a & b & 1 \\
0 & 0 & 0 & 0 & 0 \\
a & a & 0 & 1 & 1 \\
b & 0 & a & b & 0 \\
1 & 0 & a & b & 1 \\
\end{array}
\]

and $\mathcal{F}(x \to (y \to z)) = \mathcal{F}(x \to (y \to z) \land \mathcal{F}(y \to z))$. Hence, $\mathcal{F}$ is an INDF on $\mathcal{L}$.
Define $\land$ and $\lor$ on $\mathcal{L}$ as min and max, respectively. Then $(\mathcal{L}, \land, \lor, \neg, \top, \bot, 0, 1)$ is a BL-algebra. The neutrosophic subset $\mathcal{F}$ of $\mathcal{L}$ defined by $\mathcal{F}(0) = \mathcal{F}(a) = \mathcal{F}(b) = \mathcal{F}(t_1, t_2, t_1)$, for any $0 < t_1 < t_2 \leq 1$, is an INDF, if not it is not a BNDF.

**Theorem 5.8.** Let $\mathcal{F}$ be an INDF on $\mathcal{L}$. $\mathcal{F}$ is a BNDF if and only if for all formulas $\psi$, $\chi$ and all assignments $v$:

$$F((\psi \rightarrow \chi)) = F((\psi \rightarrow \chi))$$

(5.1)

**Proof.** Suppose that $\mathcal{F}$ is a BNDF. From $\vdash_{\mathcal{L}} \neg \psi \rightarrow (\psi \rightarrow \phi)$, it follows that $\vdash_{\mathcal{L}} \neg \psi \rightarrow (\psi \rightarrow \phi)$ and since by Proposition 2, $\vdash_{\mathcal{L}} \neg \psi \rightarrow (\psi \rightarrow \phi)$, then we have $\vdash_{\mathcal{L}} \neg \neg \psi \rightarrow (\neg \psi \rightarrow \phi) = (\neg \neg \psi \rightarrow \neg \psi \rightarrow \phi)$ in addition, since $\vdash_{\mathcal{L}} \neg \psi \rightarrow (\neg \psi \rightarrow \phi)$ and $\vdash_{\mathcal{L}} \neg \psi \rightarrow (\neg \psi \rightarrow \phi)$ we have $\vdash_{\mathcal{L}} \neg \neg \psi \rightarrow (\neg \neg \psi \rightarrow \phi)$.

Therefore, $\vdash_{\mathcal{L}} \neg \neg \psi \rightarrow (\neg \neg \psi \rightarrow \phi)$ and hence $F((\neg \neg \psi \rightarrow \phi)) = F((\neg \neg \psi \rightarrow \phi))$, by Corollary 3.11.

Since by Theorem 4.13 (ii), $F((\neg \neg \psi \rightarrow \phi)) = F((\neg \neg \psi \rightarrow \phi))$, then $F((\neg \neg \psi \rightarrow \phi)) = F((\neg \neg \psi \rightarrow \phi))$.

Conversely, let $\mathcal{F}$ be an INDF on $\mathcal{L}$ and (5.1) holds.

Then, in order to prove that $\mathcal{F}$ is a BNDF, it is enough to show that $F((\neg \neg \psi \rightarrow \phi)) = F((\neg \neg \psi \rightarrow \phi))$. By (5.1), by Proposition 4.4, $\vdash_{\mathcal{L}} \neg \psi \rightarrow \neg \neg \neg \psi$ and $\vdash_{\mathcal{L}} \neg \neg \neg \psi \rightarrow \neg \neg \psi$, then it follows that $F((\neg \neg \psi \rightarrow \phi)) = F((\neg \neg \psi \rightarrow \phi))$.

Hence, $F((\neg \neg \psi \rightarrow \phi)) = F((\neg \neg \psi \rightarrow \phi))$, Obviously, $F((\neg \neg \psi \rightarrow \phi)) = F((\neg \neg \psi \rightarrow \phi))$, and $\vdash_{\mathcal{L}} \neg \neg \neg \neg \psi \rightarrow \neg \neg \psi \rightarrow \phi$.

**Corollary 5.10.** Let $\mathcal{L}$ be a MV-Algebra and $\mathcal{F}$ be a NDF on $\mathcal{L}$. Then, the following statements are equivalent:

(i) $\mathcal{F}$ is a BNDF on $\mathcal{L}$.

(ii) $\mathcal{F}$ is an INDF.

(iii) $F((\neg \neg \psi \rightarrow \phi)) = F((\neg \neg \psi \rightarrow \phi))$.

(iv) $F((\neg \neg \psi \rightarrow \phi)) = F((\neg \neg \psi \rightarrow \phi))$.

(v) $F((\neg \neg \psi \rightarrow \phi)) = F((\neg \neg \psi \rightarrow \phi))$.

(vi) $F((\neg \neg \psi \rightarrow \phi)) = F((\neg \neg \psi \rightarrow \phi))$.

(vii) $F((\neg \neg \psi \rightarrow \phi)) = F((\neg \neg \psi \rightarrow \phi))$.

6. Quotient structures

In this section we define the quotient structure for neutrosophic deductive filters and study some of its properties.

Let $\mathcal{F}$ be a NDF on $\mathcal{L}$ and $x \in \mathcal{L}$. The neutrosophic set $\mathcal{F}^*(x) = \{T(x, y) \mid y \in \mathcal{L}\}$ which is defined by $\mathcal{F}^*(x) = \{T(x, y) \mid y \in \mathcal{L}\}$, where $T(x, y) = T(x \rightarrow y)$ and $F(x, y) = F(x \rightarrow y)$ and $F(x, y) = F(x \rightarrow y)$.

Let $\mathcal{F}$ be a MV-Algebra and $\mathcal{F}$ be a NDF on $\mathcal{L}$. Then, the following statements are equivalent:

(i) $\mathcal{F}$ is a BNDF on $\mathcal{L}$.

(ii) $\mathcal{F}$ is an INDF.

(iii) $F((\neg \neg \psi \rightarrow \phi)) = F((\neg \neg \psi \rightarrow \phi))$.

(iv) $F((\neg \neg \psi \rightarrow \phi)) = F((\neg \neg \psi \rightarrow \phi))$.

(v) $F((\neg \neg \psi \rightarrow \phi)) = F((\neg \neg \psi \rightarrow \phi))$.

(vi) $F((\neg \neg \psi \rightarrow \phi)) = F((\neg \neg \psi \rightarrow \phi))$.

(vii) $F((\neg \neg \psi \rightarrow \phi)) = F((\neg \neg \psi \rightarrow \phi))$.
\textbf{Theorem 6.5.} Let $\mathcal{F}$ be a NDF on $\mathcal{L}$. Then $\mathcal{L}/\mathcal{F}$ is an algebra.

\textbf{Proof.} By Lemma 6.3, the operations $\wedge$, $\vee$, $\odot$ and $\rightarrow$ on $\mathcal{L} \mathcal{F}$ are well-defined. We only need to prove $\mathcal{L}/\mathcal{F}$ satisfies the axioms of $\mathcal{L}$-algebras. The axioms (BL1), (BL2), (BL4) and (BL5) can be easily proved.

Let $\mathcal{F}$, $\mathcal{F}'$, $\mathcal{F}'' \in \mathcal{L} \mathcal{F}$, then by Corollary 6.4

\[(\mathcal{F}' \cap \mathcal{F}'' \leq \mathcal{F}') \iff (\mathcal{F}'' \leq \mathcal{F}') \]

\[\iff (x \rightarrow y = \mathcal{F}(1)) \]

\[\iff (x \rightarrow (y \rightarrow z) = \mathcal{F}(1)) \]

\[\iff \mathcal{F} \leq \mathcal{F}' \]

\[\iff \mathcal{F} \leq \mathcal{F}'' \]

\[\iff \mathcal{F} \leq \mathcal{F}' \rightarrow \mathcal{F}'' \]

\section{7. Isomorphism theorems}

In this section we prove three isomorphism theorems concerning quotients of neutrosophic deductive filters.

We note that, since $\mathcal{F}$ is a filter on $\mathcal{L}$, then by Theorem 2.10, $\mathcal{L} / \mathcal{F}$ is a BL-algebra.

\textbf{Theorem 7.1.} Let $\mathcal{F}$ be a NDF on $\mathcal{L}$. Then $\mathcal{L} / \mathcal{F} \cong \mathcal{L} / \mathcal{F}$. 

\textbf{Proof.} Define a map $\Phi: \mathcal{L} / \mathcal{F} \rightarrow \mathcal{L} / \mathcal{F}$ by $\Phi(\mathcal{F}(x)) = \mathcal{F}$. We prove that $\Phi$ is an isomorphism. Suppose that $\mathcal{F}' \cap \mathcal{F}'' = \mathcal{F}' \wedge \mathcal{F}''$ and $\mathcal{F}' \cap \mathcal{F}'' = \mathcal{F}' \cdot \mathcal{F}''$ and $\mathcal{F}' \cap \mathcal{F}'' = \mathcal{F}' \div \mathcal{F}''$ and $\mathcal{F}' \cap \mathcal{F}'' = \mathcal{F}' - \mathcal{F}''$.

We note that the lattice order $\leq$ on $\mathcal{L} / \mathcal{F}$ is defined by $\mathcal{F}' \leq \mathcal{F}''$ if and only if $\mathcal{F}' \wedge \mathcal{F}'' = \mathcal{F}'$. 

\textbf{Definition 7.3.} Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be BL-algebras and $\mathcal{F}$ be a NDF on $\mathcal{L}$ which ker$\mathcal{F} = \mathcal{F}$. Then $\mathcal{L}_1 / \mathcal{F}$ is isomorphic to $\mathcal{L}_2$.

\textbf{Corollary 7.2.} Let $f: \mathcal{L} \rightarrow \mathcal{L}'$ be a homomorphism of $\mathcal{L}$-algebras and $\mathcal{F}$ be a NDF on $\mathcal{L}$ which ker$\mathcal{F} = \mathcal{F}$. Then, $\mathcal{L}_1 / \mathcal{F} \cong f(\mathcal{L}_1)$.

\textbf{Definition 7.4.} Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be BL-algebras and $\mathcal{F}$ be a NDF on $\mathcal{L}$ which ker$\mathcal{F} = \mathcal{F}$. Then $\mathcal{L}_1 / \mathcal{F}$ is defined by

\[f(\mathcal{F}(l)) = \begin{cases} (f(\mathcal{F}(l)), 1, 1) & \text{if } f(\mathcal{F}(l)) \neq \emptyset \\ (0, 1, 1) & \text{otherwise} \end{cases} \]
for any $t_2 \in \mathcal{L}_2$, where
\[ T_{f_2}(l_1) = \lor \{ T_f(l_1) : l_1 \in \mathcal{L}_1, f(l_1) = t_1 \}, \]
\[ I_{f_2}(l_1) = \land \{ I_f(l_1) : l_1 \in \mathcal{L}_1, f(l_1) = t_1 \}, \]
\[ F_{f_2}(l_1) = \land \{ F_f(l_1) : l_1 \in \mathcal{L}_1, f(l_1) = t_1 \}. \]

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be BL-algebras and $F_1$ and $F_2$ be neurosophic sets of $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. A homomorphism $f$ of $\mathcal{L}_1$ onto $\mathcal{L}_2$ is called weak homomorphism of $F_1$ into $F_2$, if $f(F_1) \subseteq F_2$. In this case, we say that $F_1$ is weakly homomorphic to $F_2$ and we write $F_1 \sim F_2$ or simply $F_1 \sim F_2$. A homomorphism $f$ of $\mathcal{L}_1$ onto $\mathcal{L}_2$ is called a homomorphism of $F_1$ into $F_2$, if $f(F_1) = F_2$. In this case, we say that $F_1$ is homomorphic to $F_2$ and we write $F_1 \cong F_2$ or simply $F_1 \cong F_2$. If $f$ is bijective, we say that $F_1$ is isomorphic to $F_2$ and we write $F_1 \approx F_2$ or simply $F_1 \approx F_2$.

For a neurosophic subset $F$ of $\mathcal{L}$, define $F^* = \{ x \in \mathcal{L} : T_F(x) > 0, I_F(x) \leq \top, F_F(x) < \top \}$.

**Lemma 7.4.** Let $F$ be a NDF on $\mathcal{L}$, $\mathcal{T}_F(1) > 0$, $I_F(1) < \top$ and $F_F(1) < \top$. Then $F^{*}$ is a filter on $\mathcal{L}$.

**Proof.** The proof is easy.

Recall that $F_1 \subseteq F_2$ means that $T_{F_1}(x) \leq T_{F_2}(x)$, $I_{F_1}(x) \leq I_{F_2}(x)$ and $F_{F_1}(x) \leq F_{F_2}(x)$, for any $x \in \mathcal{L}$. Now, if $F_1 \subseteq F_2$ and $x \in F_2^*$, then $\mathcal{L} \leq T_{F_1}(x) \leq T_{F_2}(x)$, $\top \geq I_{F_1}(x) \geq I_{F_2}(x)$ and $\top \geq F_{F_1}(x) \geq F_{F_2}(x)$, which implies that $F_1^* \subseteq F_2^*$. Hence $F_2^*$ is a filter on $F_2^{*}$, and $F_2^{*}$ is a BL-algebra, by Lemma 2.11.

**Theorem 7.5.** Let $F_1$ and $F_2$ be two NDFs on $\mathcal{L}$, $F_1 \subseteq F_2$ and $F_2$ has Sup-inf property. Define $\xi : F_2^{*} \rightarrow [0, 1]$ by
\[ \xi([x]_{F_2}) = (T_{F_2}(x), I_{F_2}(x), F_{F_2}(x)) \]
where $T_{F_2}(x) = \lor \{ T_{F_2}(x) : x \in [x]_{F_2} \}$, $I_{F_2}(x) = \land \{ I_{F_2}(x) : x \in [x]_{F_2} \}$ and $F_{F_2}(x) = \land \{ F_{F_2}(x) : x \in [x]_{F_2} \}$, for all $[x]_{F_2} \in F_2^{*}$. Then $\xi$ is a neurosophic set.

**Proof.** Since $F_2$ has Sup-inf property, there exist $y_1, y_2 \in [x]_{F_2}$ such that $T_{F_2}([x]_{F_2}) = T_{F_2}(y_1)$, $I_{F_2}([x]_{F_2}) = I_{F_2}(y_1)$ and $F_{F_2}([x]_{F_2}) = F_{F_2}(y_1)$. Obviously, $\xi([x]_{F_2}) = (T_{F_2}(y_1), I_{F_2}(y_1), F_{F_2}(y_1))$ is a neurosophic set, which completes the proof. We call the neurosophic set $\xi$ defined in Theorem 7.5, the quotient neurosophic set on $F_2$ relative to $F_1$ and denote it by $F_2/F_1$.

**Theorem 7.6.** Let $F_1$ and $F_2$ be two NDFs on $\mathcal{L}$, $F_1 \subseteq F_2$ and $F_2$ has Sup-inf property. Then $F_2/F_1 \approx F_2/F_2$.

**Proof.** Let $f : F_2 \rightarrow F_2/F_1$ be the natural epimorphism and $[x]_{F_2} \subseteq F_2/F_1$. Then $f(F_2/F_1) = (\lor \{ f(F_2)(x) : x \in [x]_{F_2} \}) = (\lor \{ f(F_2)(y) : y \in [x]_{F_2} \}) = (\lor \{ f(F_2)(z) : z \in [x]_{F_2} \}) = (\lor \{ f(F_2)(y) : y \in [x]_{F_2} \}) = (\lor \{ f(F_2)(y) : y \in [x]_{F_2} \}) = (\lor \{ f(F_2)(y) : y \in [x]_{F_2} \})$. Therefore, $F_2/F_1 \approx F_2/F_2$.

**Theorem 7.7.** Let $F_1$ and $F_2$ be two NDFs on BL-algebras $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, $F_1 \approx F_2$ and $F_1$ has Sup-inf property. Then there exists a NDF $F_3$ such that $F_3 \subseteq F_1$ and $F_3/F_1 \approx F_2/F_2$.

**Proof.** Since $F_1 \approx F_2$, there is an homomorphism $f$ from $\mathcal{L}_1$ onto $\mathcal{L}_2$ such that $f(F_1) = F_2$. Define the neurosophic set $F_3$ as follows:
\[ F_3(x) = \begin{cases} F_1(x), & x \in \ker(f) \\ \top, & otherwise \end{cases} \]
for any $x \in \mathcal{L}_1$. It is easy to show that $F_3$ is a NDF on $\mathcal{L}_1$. Since $F_1 \approx F_2$, then $F_3/F_1 \approx F_2/F_2$. Therefore, $F_3/F_1 \approx F_2/F_2$.

**Theorem 7.8.** Let $F_1$ and $F_2$ be two NDFs on $\mathcal{L}$ such that $F_1 \subseteq F_2$. Then $F_2/F_1 \approx F_2/F_2$.

**Proof.** The proof is easy.
Lemma 7.9. Let $F_1$, $F_2$ and $F_3$ be NDFs on $L$, $F_1 \leq F_2 \leq F_3$ and $F_2$, $F_3$ have Sup-Inf property. Then $(F_2/F_1)$ and $(F_3/F_1)$ are neutrosophic subsets of $L$ such that $(F_2/F_1) \leq (F_3/F_1)$.

Proof. Use Theorem 7.5.

Theorem 7.10. Let $F_1$, $F_2$ and $F_3$ be NDFs on $L$, $F_1 \leq F_2 \leq F_3$ and $F_2$, $F_3$ have Sup-Inf property, such that $F_3/F_1$, $F_2/F_1$ are NDFs. Then

$$(F_3/F_1)/(F_2/F_1) \approx (F_3/F_2).$$

Proof. It can be proved by using Theorem 2 and Lemmas 7.8, 7.9.

References


