Neutrosophic Linear Space Theory

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Abstract: In this Lecture, we give a review about neutrosophic linear spaces and their properties.

Main Concepts

Definition

Let (V , + , .) be a vector space over the field K then (V(I) , + ,.) is called a weak neutrosophic vector space over the field K, and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field K(I).

Elements of V(I) have the form $\boldsymbol{x + yI}$; $\boldsymbol{x, y} \in \boldsymbol{V}$, i.e V(I) can be written as $\boldsymbol{V(I) = V + VI}$.

Definition

Let V(I) be a strong neutrosophic vector space over the neutrosophic field K(I) and W(I) be a non empty set of V(I) then W(I) is called a strong neutrosophic subspace if W(I) is itself a strong neutrosophic vector space.

Definition

Let U(I) , W(I) be two strong neutrosophic subspaces of V(I) then we say that V(I) is a direct sum of U(I) and W(I) if and only if for each element x $\in \boldsymbol{V(I)}$ then x can be written uniquely as x = y + z such y$\in \boldsymbol{U(I)}$ and z $\in \boldsymbol{W(I)}$

Definition

Let U(I) , W(I) be two strong neutrosophic subspaces of V(I) and let $f: V(I) \to W(I)$, we say that f is a neutrosophic vector space homomorphism if

(a) f(I) = I.

(b) f is a vector space homomorphism.

We define the kernel of f by Ker f = { x$\in V(I)$; f(x) = 0 }.

Definition

Let $v_1, v_2 .. v_s \in V(I) and\ x \in V(I)$ we say that x is a linear combination of $\{ v_i ; i = 1..s\}$ if

X = $a_1 v_1 + \cdots + a_s v_s$ such $a_i \in K(I)$.

The set $\{ v_i ; i = 1..s\}$ is called linearly independent if $a_1 v_1 + \cdots + a_s v_s = 0$ implies $a_i = 0$ for all i.

Theorem

If { v_1 , ... v_s } is a basis of V(I) and $f: V(I) \to W(I)$ is a neutrosophic vector space homomrphism then { f(v_1) , ... $f(v_s)$} is a basis of W(I).

Definition

Let V(I) = V+VI be a strong/weak neutrosophic vector space, the set

is called an AH-subspace of $S = P + QI = \{x + yI; x \in P, y \in Q\}$, where P, Q are subspaces of V V(I).

If $P = Q$ then S is called an AHS-subspace of V(I).

Example

We have V = R^2 is a vector space, P=<(0,1)>, Q=<(1,0)>, are two subspaces of V. The set

$S = P + QI = \{(0, a) + (b, 0)I; a, b \in R\}$ is an AH-subspace of V(I).

The set $L = P + PI = \{(0, a) + (0, b)I\}; b \in R$ is an AHS-subspace of V(I).

Theorem

Let V(I) = V+VI be a neutrosophic weak vector space, and let $S = P + QI$ be an AH-subspace of V(I), then S is a subspace by the classical meaning.

Theorem

Let V(I) be a neutrosophic strong vector space over a neutrosophic field K(I), let S=P+PI be an AHS-subspace. S is a subspace of V(I).

Proof :

Suppose that $x = a + bI , y = c + dI \in S; a, c, b, c \in P$, we have

. Let $m = x + yI \in K(I)$ be a neutrosophic scalar, we find$x + y = (a + c) + (b + d)I \in S$

since $y.a + y.b + x.b \in P$, thus we get the desired result.$m.x = (x.a) + (y.a + y.b + x.b)I \in S$,

Definition

(a) Let V,W be two vector spaces, $L_V: V \rightarrow W$ be a linear transformation. The AHS-linear transformation cab be defined as follows:

. Where L_V is the restriction of L on V.$L: V(I) \rightarrow W(I); L(a + bI) = L_V(a) + L_V(b)I$

(b) If $S = P + QI$ is an AH-subspace of V(I), $L(S) = L_V(P) + L_V(Q)I$.

(c) If $S = P + QI$ is an AH-subspace of W(I), $L^{-1}(S) = L_W^{-1}(P) + L_W^{-1}(Q)I$.

(d) $AH - Ker\ L = KerL_V + KerL_V\ I = \{x + yI; x, y \in KerL_V\}$.

Theorem

Let W(I),V(I) be two neutrosophic strong/weak vector spaces, and $L: V(I) \rightarrow W(I)$ be an AHS-linear transformation, we have :

(a) $AH - Ker\ L$ is an AHS-subspace of V(I).

(b) If $S = P + QI$ is an AH-subspace of V(I), $L(S)$ is an AH-subspace of W(I).

(c) If $S = P + QI$ is an AH-subspace of W(I), $L^{-1}(S)$ is an AH-subspace of V(I).

Theorem

Let W(I),V(I) be two neutrosophic strong vector spaces over a neutrosophic field K(I), and $L: V(I) \to W(I)$ be an AHS-linear transformation, we have :

, for all $x, y \in V(I), m \in K(I)$.$L(x+y) = L(x) + L(y), L(m.x) = m.L(x)$

Proof :

Suppose $x = a + bI, y = c + dI; a, b, c, d \in V, and\ m = s + tI \in K(I)$, we have

$L(x+y) = L([a+c] + [b+d]I) = L_V(a+c) + L_V(b+d)I =$

$[L_V(a) + L_V(b)I] + [L_V(c) + L_V(d)I] = L(x) + L(y).$

$m.x = (s.a) + (s.b + t.a + t.b)I, L(m.x) = L_V(s.a) + L_V(s.b + t.a + t.b)I$

$= s.L_V(a) + [s.L_V(b) + t.L_V(a) + t.L_V(b)]I = (s + tI).(L_V(a) + L_V(b)I) = m.L(x).$

Theorem

Let S=P+QI be an AH-subspace of a neutrosophic weak vector space V(I) over a field K, suppose that

is a basis of P, and $Y = \{y_j; 1 \le j \le m\}$ is a basis of Q then $X \cup YI$ is a basis of S.$X = \{x_i; 1 \le i \le n\}$

Definition

Let V(I) be a neutrosophic strong/weak vector space, S=P+QI be an AH-subspace of V(I), we define

AH-Quotient as :

$= (x+P) + (y+Q)I; x, y \in V$.$V(I)/S = V/P + (V/Q)\ I$

Theorem

Let V(I) be a neutrosophic weak vector space over a field K, and S=P+QI be an AH-subspace of V(I). The AH-Quotient $V(I)/S$ is a vector space with respect to the following operations:

Addition: $[(x+P) + (y+Q)I] + [(a+P) + (b+Q)]I = (x+a+P) + (y+b+Q)I; x, y, a, b \in V.$

Multiplication by a neutrosophic scalar: $(m).[(x+P) + (y+Q)I] = (m.x+P) + (m.y+Q)I;$

.$x, y \in V\ and\ m \in K$

Example

We have V = R^2 is a vector space over the field R, P=<(0,1)>, Q=<(1,0)>, are two subspaces of V,

$S = P + QI = \{(0,a) + (b,0)I; a, b \in R\}$ is an AH-subspace of V(I).

The AH-Quotient is $V(I)/S = \{[(x,y)+P] + [(a,b)+Q]I; x, y, a, b \in V\}$.

We clarify operations on $V(I)/S$ as follows:

are two elements in $V(I)/S$, m =$x = [(2,1)+P] + [(1,3)+Q]I, y = [(2,5)+P] + [(1,1)+Q]I$ $3\ is\ a\ scalar\ in\ R.$

, 3.$x = [(6,3)+P] + [(3,9)+Q]I$.$x + y = [(4,6)+P] + [(2,4)+Q]I$

Definition

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n$; $a_i \in R$} to be n-refined neutrosophic ring.

Definition

(a) Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$ = { $a_0 + a_1 I + \cdots + a_n I_n$: $a_i \in P_i$}, where P_i is a subset of R, we define P to be an AH-subring if P_i is a subring of R for all i . AHS-subring is defined by the condition $P_i = P_j$ for all i , j.

(b) P is an AH-ideal if P_i are two sided ideals of R for all i, the AHS-ideal is defined by the condition $P_i = P_j$ for all i , j.

Definition

Let (V,+,.) be a vector space over the field K then (V(I),+,.) is called a weak neutrosophic vector space over the field K, and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field K(I).

Definition

Let V(I) be a strong neutrosophic vector space over the neutrosophic field K(I) and W(I) be a non empty set of V(I), then W(I) is called a strong neutrosophic subspace if W(I) is itself a strong neutrosophic vector space.

Definition

Let (K,+,.) be a field, we say that $K_n(I) = K + KI_1 + \cdots + KI_n = \{a_0 + a_1 I_1 + \cdots + a_n I_n; a_i \in K\}$ is an n-refined neutrosophic field.

It is clear that $K_n(I)$ is an n-refined neutrosophic ring but not a field in classical meaning**.**

Definition

Let (V,+,.) be a vector space over the field K. Then we say that $V_n(I) = V + VI_1 + \cdots + VI_n = \{x_0 + x_1 I_1 + \cdots + x_n I_n;\ x_i \in V\}$ is a weak n-refined neutrosophic vector space over the field K. Elements of $V_n(I)$ are called n-refined neutrosophic vectors, elements of K are called scalars.

If we take scalars from the n-refined neutrosophic field $K_n(I)$, we say that $V_n(I)$ is a strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$. Elements of $K_n(I)$ are called n-refined neutrosophic scalars.

Definition

Let $V_n(I)$ be a weak n-refined neutrosophic vector space over the field K, a nonempty subset $W_n(I)$ is called a weak n-refined neutrosophic subspace of $V_n(I)$ if $W_n(I)$ is a subspace of $V_n(I)$ itself.

Definition

Let $V_n(I)$ be a strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$, a nonempty subset $W_n(I)$ is called a strong n-refined neutrosophic subspace of $V_n(I)$ if $W_n(I)$ is a submodule of $V_n(I)$ itself.

Definition

Let V(I) = V+VI be a strong/weak neutrosophic vector space, the set

is called an AH-subspace of $S = P + QI = \{x + yI; x \in P, y \in Q\}$, where P and Q are subspaces of V V(I).

If $P = Q$ then S is called an AHS-subspace of V(I).

Definition

(a) Let V and W be two vector spaces, $L_V: V \to W$ be a linear transformation. The AHS-linear transformation can be defined as follows:

. $L: V(I) \to W(I); L(a + bI) = L_V(a) + L_V(b)I$

(b) If $S = P + QI$ is an AH-subspace of V(I), $L(S) = L_V(P) + L_V(Q)I$.

Definition

Let (V,+,.) be a vector space over a field K, $V_n(I)$ be the corresponding weak n-refined neutrosophic vector space over K. Consider the set $\{M_i; 0 \leq i \leq n\}$, where M_i is a subspace of V. We say:

is a weak n-refined AH-$M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n = \{m_0 + m_1 I_1 + \cdots + m_n I_n;\ m_i \in M_i\}$ subspace of the weak n-refined vector space $V_n(I)$.

We say that $M_n(I)$ is a weak n-refined AH-subspace if $M_j = M_i\ for\ all\ i, j$.

Definition

Let (V,+,.) be a vector space over a field K, $V_n(I)$ be the corresponding strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$. Consider the set $\{M_i; 0 \leq i \leq n\}$, where M_i is a subspace of V. We say:

is a strong n-refined AH-$M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n = \{m_0 + m_1 I_1 + \cdots + m_n I_n;\ m_i \in M_i\}$ subspace of the strong n-refined vector space $V_n(I)$.

We say that $M_n(I)$ is a strong n-refined AH-subspace if $M_j = M_i\ for\ all\ i, j$.

Theorem

Let (V,+,.) be a vector space over a field K, $V_n(I)$ be the corresponding weak n-refined neutrosophic vector space over K, $M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$ be a weak n-refined AH-subspace. Then

(a) $M_n(I)$ is a vector subspace of $V_n(I)$.

(b) If X_i is a bases of M_i, $X = \bigcup_{i=0}^{n} X_i I_i$ is a bases of $M_n(I)$.

(c) $\dim\big(M_n(I)\big) = \sum_{i=0}^{n} \dim(M_i)$.

Theorem

Let V be a vector space with $\dim(V) = n + 1$. Then V is isomorphic to a weak AHS-subspace of the corresponding weak n-refined neutrosophic vector space.

Proof:

Let M be any one dimensional subspace of V, $T = M + MI_1 + \dots + MI_n$ is a weak AHS-subspace of the weak n-refined neutrosophic vector space $V_n(I)$. As a result of Theorem 3.3 , we find $\dim(T) = n + 1 = \dim(V)$, thus V is isomorphic to T.

Example

Let $V = R^3$ be a vector space over the field R, $V_3(I) = \{a + bI_1 + cI_2 + dI_3; a, b, c, d \in V\}$ is the corresponding weak 3-refined neutrosophic vector space, $M =< (1,0,0) >$ is a subspace of V.

is a weak AHS-subspace of $T = M + MI_1 + MI_2 = \{(a, 0,0) + (b, 0,0)I_1 + (c, 0,0)I_2; a, b, c \in R\}$ $V_3(I)$ with $\dim(T) = 3$, this implies $T \cong V$.

Theorem

Let (V,+,.) be a vector space over a field K, $V_n(I)$ be the corresponding strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$, $M_n(I) = M + MI_1 + \dots + MI_n$ be a strong n-refined AHS-subspace. Then:

(a) $M_n(I)$ is a submodule of $V_n(I)$.

(b) If Y is a bases of M, $X = \bigcup_{i=0}^{n} YI_i$ is a bases of $M_n(I)$.

(c) $\dim\big(M_n(I)\big) = \sum_{i=0}^{n} \dim(M) = n. \dim(M)$.

Remark

If $V_n(I)$ is a strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$, and

is a strong n-refined AH-subspace, then it is not supposed to be a $M_n(I) = M_0 + M_1I_1 + \dots + M_nI_n$ submodule.

We clarify it by the following example.

Example

Let $V = R^2$ be a vector space over R, $V_2(I) = R_2^2(I) = \{(a, b) + (c, d)I_1 + (e, f)I_2; a, b, c, d, e, f \in R\}$ be the corresponding strong 2-refined neutrosophic vector space over the neutrosophic field $R_2(I)$.

are two subspaces of V, $T = M + NI_1 + NI_2$ is a strong AH-subspace of $M =< 0,1 >, N =< (1,0) >$ $V_2(I)$.

,$x = (0,1) + (2,0)I_1 + (1,0)I_2 \in T, r = 1 + 1. I_1 + 1. I_2 \in R_2(I)$

$+1. (0,1)I_2$ $+r. x = 1. (0,1) + 1. (0,1)I_1 + 1. (0,1)I_2 + 1. (2,0)I_1I_1 + 1. (2,0)I_1 + 1. (1,0)I_1I_2$ $1. (2,0)I_1I_2 + 1. (2,0)I_2I_2 = (0,1) + [(0,1) + (2,0) + (1,0) + (2,0)]I_1 + [(0,1) + (0,1) + (2,0)]I_2 =$

, $r. x$ does not belong to T, thus T is not a submodule.$(0,1) + (5,1)I_1 + (2,2)I_2$

Definition

Let $V_n(I)$ be a weak/strong n-refined neutrosophic vector space, $M_n(I) = M_0 + M_1I_1 + \dots + M_nI_n$,

be two weak/strong AH-subspaces of $V_n(I)$, we define:$W_n(I) = W_0 + W_1I_1 + \dots + W_nI_n$

(a) $M_n(I) \cap W_n(I) = (M_0 \cap W_0) + (M_1 \cap W_1)I_1 + \dots + (M_n \cap W_n)I_n$.

(b) $M_n(I) + W_n(I) = (M_0 + W_0) + (M_1 + W_1)I_1 + \dots + (M_n + W_n)I_n$.

Theorem

Let $V_n(I)$ be a weak n-refined neutrosophic vector space, $M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$,

be two weak AH-subspaces of $V_n(I)$. Then:$W_n(I) = W_0 + W_1 I_1 + \cdots + W_n I_n$

are two weak AH-subspaces of $V_n(I)$.$M_n(I) \cap W_n(I), M_n(I) + W_n(I)$

Theorem

Let $V_n(I)$ be a strong n-refined neutrosophic vector space, $M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$,

be two strong AH-subspaces of $V_n(I)$. Then:$W_n(I) = W_0 + W_1 I_1 + \cdots + W_n I_n$

(a) $M_n(I) \cap W_n(I)$ is a strong AH-subspaces of $V_n(I)$.

(b) $M_n(I) + W_n(I)$ is not supposed to be a strong AH-subspace of $V_n(I)$.

Definition

Let V,W be two vector spaces over the field K, $f_i: V \to W; 0 \leq i \leq n+1$ be $n+1$ linear transformations, $V_n(I), W_n(I)$ be the corresponding weak n-refined neutrosophic vector spaces over the field K respectively. We say:

(a) $f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ is a weak AH-linear transformation.

(b) If $f_i = f_j \; for \; all \; i, j$, we call f a weak AHS-linear transformation.

Definition

Let V,W be two vector spaces over the field K, $f_i: V \to W; 0 \leq i \leq n+1$ be $n+1$ linear transformations, $V_n(I), W_n(I)$ be the corresponding strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field $K_n(I)$ respectively. We say:

(a) $f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ is a strong AH-linear transformation.

(b) If $f_i = f_j \; for \; all \; i, j$, we call f a strong AHS-linear transformation.

Definition

Let $V_n(I), W_n(I)$ be two weak/strong n-refined neutrosophic vector spaces,

be a weak/strong $f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ AH-linear transformation. We define:

(a) $AH - Ker(f) = Ker(f_0) + Ker(f_1)I_1 + \cdots + Ker(f_n)I_n$.

(b) $AH - Im(f) = Im(f_0) + Im(f_1)I_1 + \cdots + Im(f_n)I_n$.

Theorem

Let $V_n(I), W_n(I)$ be two weak n-refined neutrosophic vector spaces,

be a weak AH-$f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ linear transformation. Then:

(a) $AH-Ker(f)$ is a weak AH-subspace of $V_n(I)$.

(b) $AH-Im(f)$ is a weak AH-subspace of $W_n(I)$.

(c) If $M_n(I)=M_0+M_1I_1+\cdots+M_nI_n$ is a weak AH-subspace of $V_n(I)$, $f(M_n(I))$ is a weak AH-subspace of $W_n(I)$.

Theorem

Let $V_n(I), W_n(I)$ be two strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field $K_n(I)$,

be a strong AH-$f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_iI_i) = f_0(a_0)+f_1(a_1)I_1+\cdots+f_n(a_n)I_n=\sum_{i=0}^{n} f_i(a_i)I_i$ linear transformation. Then:

(a) $AH-Ker(f)$ is a strong AH-subspace of $V_n(I)$.

(b) $AH-Im(f)$ is a strong AH-subspace of $W_n(I)$.

(c) If $M_n(I)=M_0+M_1I_1+\cdots+M_nI_n$ is a strong AH-subspace of $V_n(I)$, $f(M_n(I))$ is a strong AH-subspace of $W_n(I)$.

Theorem

Let $V_n(I), W_n(I)$ be two weak n-refined neutrosophic vector spaces over the field K,

be a weak AH-$f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_iI_i) = f_0(a_0)+f_1(a_1)I_1+\cdots+f_n(a_n)I_n=\sum_{i=0}^{n} f_i(a_i)I_i$ linear transformation. Then:

for all $x,y \in V_n(I), r \in K. f(x+y)=f(x)+f(y), f(r.x)=r.f(x)$

Theorem

Let $V_n(I), W_n(I)$ be two strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field $K_n(I)$,

be a strong AH-$f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_iI_i) = f_0(a_0)+f_1(a_1)I_1+\cdots+f_n(a_n)I_n=\sum_{i=0}^{n} f_i(a_i)I_i$ linear transformation. Then:

for all $x,y \in V_n(I), r \in K_n(I). f(x+y)=f(x)+f(y), f(r.x)=r.f(x)$

Theorem

Let $V_n(I), W_n(I), U_n(I)$ be three weak n-refined neutrosophic vector spaces over the field K,

,$f: W_n(I) \to U_n(I); f(\sum_{i=0}^{n} a_iI_i) = f_0(a_0)+f_1(a_1)I_1+\cdots+f_n(a_n)I_n=\sum_{i=0}^{n} f_i(a_i)I_i$

,$g: V_n(I) \to W_n(I); g(\sum_{i=0}^{n} a_iI_i) = g_0(a_0)+g_1(a_1)I_1+\cdots+g_n(a_n)I_n=\sum_{i=0}^{n} g_i(a_i)I_i$

be two weak AH-linear transformations. Then:

(a) $fog=\sum_{i=0}^{n}(f_iog_i)$.

(b) fog is a weak AH-linear transformation between $V_n(I), U_n(I)$.

Theorem

Let $V_n(I), W_n(I), U_n(I)$ be three strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field K,

$,f: W_n(I) \to U_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$

$,g: V_n(I) \to W_n(I); g(\sum_{i=0}^{n} a_i I_i) = g_0(a_0) + g_1(a_1)I_1 + \cdots + g_n(a_n)I_n = \sum_{i=0}^{n} g_i(a_i)I_i$

be two strong AH-linear transformations. Then:

(a) $fog = \sum_{i=0}^{n}(f_i o g_i)$.

(b) fog is a strong AH-linear transformation between $V_n(I), U_n(I)$.

Definition

Let $(\boldsymbol{R}, +, \times)$ be a ring and$\boldsymbol{I_k}; \boldsymbol{1} \leq \boldsymbol{k} \leq \boldsymbol{n}$ be $\boldsymbol{n}$ indeterminacies. We define $\boldsymbol{R_n}(\boldsymbol{I}) = \{\boldsymbol{a_0} + \boldsymbol{a_1 I_1} + \boldsymbol{a_2 I_2} + \cdots + \boldsymbol{a_n I_n}; \boldsymbol{a_i} \in \boldsymbol{R}\}$ to be n-cyclic refined neutrosophic ring.

Operations on $\boldsymbol{R_n}(\boldsymbol{I})$ are defined as:

$\sum_{i=0}^{n} \boldsymbol{x_i I_i} + \sum_{i=0}^{n} \boldsymbol{y_i I_i} = \sum_{i=0}^{n}(\boldsymbol{x_i} + \boldsymbol{y_i})\boldsymbol{I_i}, \sum_{i=0}^{n} \boldsymbol{x_i I_i} \times \sum_{i=0}^{n} \boldsymbol{y_i I_i} = \sum_{i,j=0}^{n}(\boldsymbol{x_i} \times \boldsymbol{y_i})\boldsymbol{I_i I_j} = \sum_{i,j=0}^{n}(\boldsymbol{x_i} \times \boldsymbol{y_i})\boldsymbol{I}_{(\boldsymbol{i+jmodn})}$

Where $\times$ is the multiplication on the ring $\boldsymbol{R}$, and $\boldsymbol{x I_0} = \boldsymbol{x}$, for all $\boldsymbol{x} \in \boldsymbol{R}$.

Definition

Let $(K, +, \times)$ be a field, we say that $K_n(I) = K + KI_1 + \cdots + KI_n = \{a_0 + a_1 I_1 + a_2 I_2 + \cdots + a_n I_n; a_i \in K\}$ is a n-cyclic refined neutrosophic field.

Definition

Let $(V, +, \times)$ be any vector space over a field K. Then we say that $V_n(I) = V + VI_1 + \cdots + VI_n = \{x_0 + x_1 I_1 + \cdots + x_n I_n; x_i \in V\}$is a weak n-cyclic refined neutrosophic vector space over the field K.Elements of $V_n(I)$are called n-cyclic refined neutrosophic vectors, elements of $K$are called scalars.

If we take scalars from the n-cyclic refined neutrosophic field $K_n(I)$, we say that $V_n(I)$ is a strong n-cyclic refined neutrosophic vector space over thea n-cyclic refined neutrosophic field $K_n(I)$. Elements of $K_n(I)$n-cyclic refined neutrosophic scalars.

Remark.

Multiplication by an n-cyclic refined neutrosophic scalar $m = \sum_{i=0}^{n} m_i I_i \in k_n(I)$is defined as:

$$\left(\sum_{i=0}^{n} m_i I_i\right) \times \left(\sum_{i=0}^{n} a_i I_i\right) = \sum_{i,j=0}^{n} (m_i a_j) I_i I_j$$

Where $a_i \in V$, $m_i \in K$, $I_i I_j = \boldsymbol{I}_{(\boldsymbol{i+jmodn})}$.

Definition

Let $V_n(I)$be a weak n-cyclic refined neutrosophic vector space over the n-cyclic refined neutrosophic field K; a nonempty $W_n(I)$ is called a weak n-cyclic refined neutrosophic vector subspaceof $V_n(I)$if $W_n(I)$ is a subspace of $V_n(I)$ itself.

Definition

Let $V_n(I)$be a strong n-cyclic refined neutrosophic vector space over then-cyclic refined neutrosophic field $K_n(I)$. A nonempty subset $W_n(I)$ is called a strong n-cyclic refined neutrosophic vector submodule of $V_n(I)$ if $W_n(I)$ is a submodule of $V_n(I)$ itself.

Theorem

Let $V_n(I)$ be a weak n-cyclic refined neutrosophic vector space over the n-cyclic refined neutrosophic field$K$, $W_n(I)$ be a nonempty subset of $V_n(I)$. Then $W_n(I)$ is a weak n-cyclic refined neutrosophic subspace if only if:

for all $x, y \in W_n(I), m \in K.x + y \in W_n(I), m \times x \in W_n(I)$

proof:

it holds directly from the condition of subspace.

Definition

Let $\boldsymbol{V_n(I)}$ be a weak n-cyclic refined neutrosophic vector space over the field $\boldsymbol{K}$, $\boldsymbol{x}$ be an arbitrary element of $\boldsymbol{V_n(I)}$, we say that $\boldsymbol{x}$ is a linear combination of $\{\boldsymbol{x_1, x_2, \dots, x_m}\} \subseteq \boldsymbol{V_n(I)}$ if $\boldsymbol{x = (a_1 \times x_1) + (a_2 \times x_2) + \dots + (a_m \times x_m)}$: $\boldsymbol{a_i \in K(I), x_i \in V_n(I)}$.

Definition

Let $\boldsymbol{V_n(I)}$ be a strong n-cyclic refined neutrosophic vector space over the n-cyclic refined neutrosophic field $\boldsymbol{K_n(I)}$, $\boldsymbol{x}$ be an arbitrary element of $\boldsymbol{V_n(I)}$, we say that $\boldsymbol{x}$ is a linear combination of $\{\boldsymbol{x_1, x_2, \dots, x_m}\} \subseteq \boldsymbol{V_n(I)}$if $\boldsymbol{x = (a_1 \times x_1) + (a_2 \times x_2) + \dots + (a_m \times x_m)}$: $\boldsymbol{a_i \in K_n(I), x_i \in V_n(I)}$.

Definition

Let $\boldsymbol{X} = \{\boldsymbol{x_1, x_2, \dots, x_m}\}$ be a subset of a weak n-cyclic refined neutrosophic vector space $\boldsymbol{V_n(I)}$ over the field $\boldsymbol{K}$, $\boldsymbol{X}$ is a weak linearly independent set if $\sum_{i=0}^{n} \boldsymbol{a_i \times x_i = 0}$ implies $\boldsymbol{a_i = 0}$; $\boldsymbol{a_i \in K}$.

Definition

Let $\boldsymbol{X} = \{\boldsymbol{x_1, x_2, \dots, x_m}\}$ be a subset of a strong n-cyclic refined neutrosophic vector space $\boldsymbol{V_n(I)}$ over the n-cyclic refined neutrosophicfield $\boldsymbol{K_n(I)}$, $\boldsymbol{X}$ is a weak linearly independent set if $\sum_{i=0}^{n} \boldsymbol{a_i \times x_i = 0}$ implies $\boldsymbol{a_i = 0}$; $\boldsymbol{a_i \in K_n(I)}$.

Definition

Let $\boldsymbol{V_n(I), W_n(I)}$ be two strong n-cyclic refined neutrosophic vector space over the n-cyclic refined neutrosophic field $\boldsymbol{K_n(I)}$, let $\boldsymbol{f: V_n(I) \to U_n(I)}$ be a well defined map. It is called a strong n-cyclic refined neutrosophic homomorphism if:

$\boldsymbol{f((a \times x) + (b \times y)) = a \times f(x) + b \times f(y)}$ for all $\boldsymbol{x, y \in V_n(I), a, b \in K_n(I)}$.

A weak n-cyclic refined neutrosophic homomorphism can be defined as the same.

Definition

Let $\boldsymbol{f: V_n(I) \to U_n(I)}$ be a weak/strong n-cyclic refined neutrosophic homomorphism, we define:

(a) $\boldsymbol{Ker(f) = \{x \in V_n(I); f(x) = 0\}}$.

(b) $\boldsymbol{Im(f)} = \{\boldsymbol{y} \in \boldsymbol{U_n(I)};\ \exists \boldsymbol{x} \in \boldsymbol{V_n(I)} \textbf{and } \boldsymbol{y} = \boldsymbol{f(x)}\}$.

Theorem

Let $\boldsymbol{f}: \boldsymbol{V_n(I)} \rightarrow \boldsymbol{U_n(I)}$ be a weak n-cyclic refined neutrosophic homomorphism. Then

(a) $\boldsymbol{Ker(f)}$ is a weak n-cyclic refined neutrosophic subspace of $\boldsymbol{V_n}$(I).

(b) $\boldsymbol{Im(f)}$ is a weak nn-cyclic refined neutrosophic subspace of $\boldsymbol{U_n(I)}$.

Theorem

Let $\boldsymbol{f}: \boldsymbol{V_n(I)} \rightarrow \boldsymbol{U_n(I)}$ be a strong n-cyclic refined neutrosophic homomorphism. Then

(a) $\boldsymbol{Ker(f)}$ is a strong n- cyclic refined neutrosophic subspace of $\boldsymbol{V_n}$(I).

(b) $\boldsymbol{Im(f)}$ is a strong n- cyclic refined neutrosophic subspace of $\boldsymbol{U_n(I)}$.

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