Neutrosophic Linear Space Theory

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Neutrosophic Linear Space Theory

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Abstract: In this Lecture, we give a review about neutrosophic linear spaces and their properties.

Main Concepts

Definition

Let \((V, +, .)\) be a vector space over the field \(K\) then \((V(I), +, .)\) is called a weak neutrosophic vector space over the field \(K\), and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field \(K(I)\).

Elements of \(V(I)\) have the form \(x + yI; x, y \in V\), i.e \(V(I)\) can be written as \(V(I) = V + VI\).

Definition

Let \(V(I)\) be a strong neutrosophic vector space over the neutrosophic field \(K(I)\) and \(W(I)\) be a non empty set of \(V(I)\) then \(W(I)\) is called a strong neutrosophic subspace if \(W(I)\) is itself a strong neutrosophic vector space.

Definition

Let \(U(I), W(I)\) be two strong neutrosophic subspaces of \(V(I)\) then we say that \(V(I)\) is a direct sum of \(U(I)\) and \(W(I)\) if and only if for each element \(x \in V(I)\) then \(x\) can be written uniquely as \(x = y + z\) such \(y \in U(I)\) and \(z \in W(I)\).

Definition

Let \(U(I), W(I)\) be two strong neutrosophic subspaces of \(V(I)\) and let \(f: V(I) \rightarrow W(I)\), we say that \(f\) is a neutrosophic vector space homomorphism if

(a) \(f(I) = I\).

(b) \(f\) is a vector space homomorphism.

We define the kernel of \(f\) by \(\text{Ker} f = \{ x \in V(I); f(x) = 0 \}\).

Definition

Let \(v_1, v_2, \ldots, v_s \in V(I)\) and \(x \in V(I)\) we say that \(x\) is a linear combination of \(\{ v_i; i = 1 \ldots s \}\) if

\[ X = a_1 v_1 + \cdots + a_s v_s \] such \(a_i \in K(I)\).

The set \(\{ v_i; i = 1 \ldots s \}\) is called linearly independent if \(a_1 v_1 + \cdots + a_s v_s = 0\) implies \(a_i = 0\) for all \(i\).

Theorem

If \(\{ v_1, \ldots, v_s \}\) is a basis of \(V(I)\) and \(f: V(I) \rightarrow W(I)\) is a neutrosophic vector space homomorphism then \(\{ f(v_1), \ldots, f(v_s) \}\) is a basis of \(W(I)\).

Definition
Let $V(I) = V+VI$ be a strong/weak neutrosophic vector space, the set

$$S = \{ x + yI; x \in P, y \in Q \},$$

where $P, Q$ are subspaces of $V$.

If $P = Q$ then $S$ is called an AHS-subspace of $V(I)$.

**Example**

We have $V = \mathbb{R}^2$ is a vector space, $P=<(0,1)>$, $Q=<(1,0)>$, are two subspaces of $V$. The set

$$S = \{ x + yI; x \in P, y \in Q \}$$

is an AH-subspace of $V(I)$.

The set $L = P + PI = \{ (0, a) + (0, b)I; b \in R \}$ is an AHS-subspace of $V(I)$.

**Theorem**

Let $V(I) = V+VI$ be a neutrosophic weak vector space, and let $S = P + QI$ be an AHS-subspace of $V(I)$, then $S$ is a subspace by the classical meaning.

**Theorem**

Let $V(I)$ be a neutrosophic strong vector space over a neutrosophic field $K(I)$, let $S=P+PI$ be an AHS-subspace. $S$ is a subspace of $V(I)$.

**Proof:**

Suppose that $x = a + bI, y = c + dI \in S; a, c, b, c \in P$, we have

Let $m = x + yI \in K(I)$ be a neutrosophic scalar, we find $m \cdot x = (a + c) + (b + d)I \in S$

since $y, a + y, b + x, b \in P$, thus we get the desired result.

**Definition**

(a) Let $V, W$ be two vector spaces, $L_V: V \to W$ be a linear transformation. The AHS-linear transformation can be defined as follows:

. Where $L_V$ is the restriction of $L$ on $V: L(V(I)) \to W(I); L(a + bI) = L_V(a) + L_V(b)I$

(b) If $S = P + QI$ is an AH-subspace of $V(I)$, $L(S) = L_V(P) + L_V(Q)I$.

(c) If $S = P + QI$ is an AH-subspace of $W(I)$, $L^{-1}(S) = L^{-1}_W(P) + L^{-1}_W(Q)I$.

(d) $AH - Ker L = Ker L_V + Ker L_Y I = \{ x + yI; x, y \in Ker L_V \}$.

**Theorem**

Let $W(I), V(I)$ be two neutrosophic strong/weak vector spaces, and $L: V(I) \to W(I)$ be an AHS-linear transformation, we have:

(a) $AH - Ker L$ is an AHS-subspace of $V(I)$.

(b) If $S = P + QI$ is an AH-subspace of $V(I)$, $L(S)$ is an AH-subspace of $W(I)$.

(c) If $S = P + QI$ is an AH-subspace of $W(I)$, $L^{-1}(S)$ is an AH-subspace of $V(I)$.
Let $W(I), V(I)$ be two neutrosophic strong vector spaces over a neutrosophic field $K(I)$, and $L: V(I) \to W(I)$ be an AHS-linear transformation, we have:

, for all $x, y \in V(I), m \in K(I), L(x + y) = L(x) + L(y), L(m \cdot x) = m \cdot L(x)$

Proof:

Suppose $x = a + bI, y = c + dI; a, b, c, d \in V, and m = s + tI \in K(I)$, we have

$L(x + y) = L([a + c] + [b + d]I) = L_v(a + c) + L_v(b + d)I = [L_v(a) + L_v(b)]I + [L_v(c) + L_v(d)]I = L(x) + L(y)$.

$m \cdot x = (s \cdot a) + (s \cdot b + t \cdot a + t \cdot b)I, L(m \cdot x) = L_v(s \cdot a) + L_v(s \cdot b + t \cdot a + t \cdot b)I$

$= s \cdot L_v(a) + [s \cdot L_v(b) + t \cdot L_v(a) + t \cdot L_v(b)]I = (s + tI) \cdot (L_v(a) + L_v(b))I = m \cdot L(x)$.

**Theorem**

Let $S = P + QI$ be an AH-subspace of a neutrosophic weak vector space $V(I)$ over a field $K$, suppose that $X$ is a basis of $P$, and $Y = \{y_j; 1 \leq j \leq m\}$ is a basis of $Q$ then $X \cup Y I$ is a basis of $S \times X = \{x_i; 1 \leq i \leq n\}$

**Definition**

Let $V(I)$ be a neutrosophic strong/weak vector space, $S = P + QI$ be an AH-subspace of $V(I)$, we define

AH-Quotient as:

$[x + P] + (y + Q)I; x, y \in V \cdot V(I)/S = V/P + (V/Q)I$

**Theorem**

Let $V(I)$ be a neutrosophic weak vector space over a field $K$, and $S = P + QI$ be an AH-subspace of $V(I)$. The AH-Quotient $V(I)/S$ is a vector space with respect to the following operations:

Addition: $[(x + P) + (y + Q)I] + [(a + P) + (b + Q)I] = (x + a + P) + (y + b + Q)I; x, y, a, b \in V$.

Multiplication by a neutrosophic scalar: $(m) \cdot [(x + P) + (y + Q)I] = (m \cdot x + P) + (m \cdot y + Q)I$

$x, y \in V and m \in K$

**Example**

We have $V = R^2$ is a vector space over the field $R$, $P = \langle(0, 1)\rangle$, $Q = \langle(1, 0)\rangle$, are two subspaces of $V$, $S = P + QI = \{(0, a) + (b, 0)I; a, b \in R\}$ is an AH-subspace of $V(I)$.

The AH-Quotient is $V(I)/S = \{(x, y) + P\} + [(a, b) + Q]I; x, y, a, b \in V\}.$

We clarify operations on $V(I)/S$ as follows:

are two elements in $V(I)/S, m = x = [(2, 1) + P] + [(1, 3) + Q]I, y = [(2, 5) + P] + [(1, 1) + Q]I$

$3$ is a scalar in $R$.

$3 \cdot x = [(6, 3) + P] + [(3, 9) + Q]I, x + y = [(4, 6) + P] + [(2, 4) + Q]I$
Definition

Let (R, +, \cdot) be a ring and I_k; 1 \leq k \leq n be n indeterminacies. We define \( R_n(I) = \{ a_0 + a_1 I + \cdots + a_n I_n; a_i \in R \} \) to be n-refined neutrosophic ring.

Definition

(a) Let \( R_n(I) \) be an n-refined neutrosophic ring and \( P = \sum_{i=0}^{n} P_i I_i = \{ a_0 + a_1 I + \cdots + a_n I_n; a_i \in P_i \} \), where \( P_i \) is a subset of R, we define P to be an AH-subring if \( P_i \) is a subring of R for all \( i \). AHS-subring is defined by the condition \( P_i = P_j \) for all \( i, j \).

(b) P is an AH-ideal if \( P_i \) are two sided ideals of R for all \( i \), the AHS-ideal is defined by the condition \( P_i = P_j \) for all \( i, j \).

Definition

Let \((V, +, \cdot)\) be a vector space over the field K then \((V(I), +, \cdot)\) is called a weak neutrosophic vector space over the field K, and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field K(I).

Definition

Let \( V(I) \) be a strong neutrosophic vector space over the neutrosophic field K(I) and \( W(I) \) be a nonempty set of \( V(I) \), then \( W(I) \) is called a strong neutrosophic subspace if \( W(I) \) is itself a strong neutrosophic vector space.

Definition

Let \((K, +, \cdot)\) be a field, we say that \( K_n(I) = K + K I_1 + \cdots + K I_n = \{ a_0 + a_1 I_1 + \cdots + a_n I_n; a_i \in K \} \) is an n-refined neutrosophic field.

It is clear that \( K_n(I) \) is an n-refined neutrosophic ring but not a field in classical meaning.

Definition

Let \((V, +, \cdot)\) be a vector space over the field K. Then we say that \( V_n(I) = V + V I_1 + \cdots + V I_n = \{ x_0 + x_1 I_1 + \cdots + x_n I_n; x_i \in V \} \) is a weak n-refined neutrosophic vector space over the field K. Elements of \( V_n(I) \) are called n-refined neutrosophic vectors, elements of K are called scalars.

If we take scalars from the n-refined neutrosophic field \( K_n(I) \), we say that \( V_n(I) \) is a strong n-refined neutrosophic vector space over the n-refined neutrosophic field \( K_n(I) \). Elements of \( K_n(I) \) are called n-refined neutrosophic scalars.

Definition

Let \( V_n(I) \) be a weak n-refined neutrosophic vector space over the field K, a nonempty subset \( W_n(I) \) is called a weak n-refined neutrosophic subspace of \( V_n(I) \) if \( W_n(I) \) is a subspace of \( V_n(I) \) itself.

Definition

Let \( V_n(I) \) be a strong n-refined neutrosophic vector space over the n-refined neutrosophic field \( K_n(I) \), a nonempty subset \( W_n(I) \) is called a strong n-refined neutrosophic subspace of \( V_n(I) \) if \( W_n(I) \) is a submodule of \( V_n(I) \) itself.
Let $V(I) = V + VI$ be a strong/weak neutrosophic vector space, the set

is called an AH-subspace of $S = P + QI = \{x + yI; x \in P, y \in Q\}$, where $P$ and $Q$ are subspaces of $V$ V(I).

If $P = Q$ then $S$ is called an AHS-subspace of $V(I)$.

**Definition**

(a) Let $V$ and $W$ be two vector spaces, $L_V: V \rightarrow W$ be a linear transformation. The AHS-linear transformation can be defined as follows:

$L: V(I) \rightarrow W(I); L(a + bI) = L_V(a) + L_V(b)I$

(b) If $S = P + QI$ is an AH-subspace of $V(I)$, then $L(S) = L_V(P) + L_V(Q)I$.

**Definition**

Let $(V,+,.)$ be a vector space over a field $K$, $V_n(I)$ be the corresponding weak $n$-refined neutrosophic vector space over $K$. Consider the set $\{M_i; 0 \leq i \leq n\}$, where $M_i$ is a subspace of $V$. We say:

is a weak $n$-refined AH-$M_n(I) = M_0 + M_1I_1 + \cdots + M_nI_n = \{m_0 + m_1I_1 + \cdots + m_nI_n; m_i \in M_i\}$ subspace of the weak $n$-refined vector space $V_n(I)$.

We say that $M_n(I)$ is a weak $n$-refined AH-subspace if $M_i = M_j$ for all $i, j$.

**Definition**

Let $(V,+,.)$ be a vector space over a field $K$, $V_n(I)$ be the corresponding strong $n$-refined neutrosophic vector space over the $n$-refined neutrosophic field $K_n(I)$. Consider the set $\{M_i; 0 \leq i \leq n\}$, where $M_i$ is a subspace of $V$. We say:

is a strong $n$-refined AH-$M_n(I) = M_0 + M_1I_1 + \cdots + M_nI_n = \{m_0 + m_1I_1 + \cdots + m_nI_n; m_i \in M_i\}$ subspace of the strong $n$-refined vector space $V_n(I)$.

We say that $M_n(I)$ is a strong $n$-refined AH-subspace if $M_j = M_i$ for all $i, j$.

**Theorem**

Let $(V,+,.)$ be a vector space over a field $K$, $V_n(I)$ be the corresponding weak $n$-refined neutrosophic vector space over $K$, $M_n(I) = M_0 + M_1I_1 + \cdots + M_nI_n$ be a weak $n$-refined AH-subspace. Then

(a) $M_n(I)$ is a vector subspace of $V_n(I)$.

(b) If $X_i$ is a bases of $M_i$, $X = \bigcup_{i=0}^n X_iI_i$ is a bases of $M_n(I)$.

(c) $\dim(M_n(I)) = \sum_{i=0}^n \dim(M_i)$.

**Theorem**

Let $V$ be a vector space with $\dim(V) = n + 1$. Then $V$ is isomorphic to a weak AHS-subspace of the corresponding weak $n$-refined neutrosophic vector space.

**Proof:**
Let $M$ be any one dimensional subspace of $V$, $T = M + MI_1 + \cdots + MI_n$ is a weak AHS-subspace of the weak $n$-refined neutrosophic vector space $V_n(I)$. As a result of Theorem 3.3, we find $\dim(T) = n + 1 = \dim(V)$, thus $V$ is isomorphic to $T$.

**Example**

Let $V = R^3$ be a vector space over the field $R$, $V_2(I) = \{a + bl_1 + cl_2; a, b, c \in V\}$ is the corresponding weak 3-refined neutrosophic vector space, $M = \langle (1,0,0) \rangle$ is a subspace of $V$.

is a weak AHS-subspace of $T = M + MI_1 + MI_2 = \{(a,0,0) + (b,0,0)l_1 + (c,0,0)l_2; a, b, c \in R\}$ $V_3(I)$ with $\dim(T) = 3$, this implies $T \cong V$.

**Theorem**

Let $(V,+,.)$ be a vector space over a field $K$, $V_n(I)$ be the corresponding strong $n$-refined neutrosophic vector space over the $n$-refined neutrosophic field $K_n(I)$, $M_n(I) = M + MI_1 + \cdots + MI_n$ be a strong $n$-refined AHS-subspace. Then:

(a) $M_n(I)$ is a submodule of $V_n(I)$.

(b) If $Y$ is a bases of $M$, $X = \bigcup_{i=0}^n YI_i$ is a bases of $M_n(I)$.

(c) $\dim(M_n(I)) = \sum_{i=0}^n \dim(M) = n \cdot \dim(M)$.

**Remark**

If $V_n(I)$ is a strong $n$-refined neutrosophic vector space over the $n$-refined neutrosophic field $K_n(I)$, and

is a strong $n$-refined AH-subspace, then it is not supposed to be a $M_n(I) = M_0 + M_1I_1 + \cdots + M_nI_n$ submodule.

We clarify it by the following example.

**Example**

Let $V = R^2$ be a vector space over $R$, $V_2(I) = R_2^2(I) = \{(a,b) + (c,d)I_1 + (e,f)I_2; a, b, c, d, e, f \in R\}$ be the corresponding strong 2-refined neutrosophic vector space over the neutrosophic field $R_2(I)$.

are two subspaces of $V$, $T = M + NI_1 + NI_2$ is a strong $n$-refined subpace of $M = \langle 0,1 \rangle$, $N = \langle (1,0) \rangle > V_2(I)$.

$x = (0,1) + (2,0)I_1 + (1,0)I_2 \in T, r = 1 + 1.I_1 + 1.I_2 \in R_2(I)$

$+1.(0,1)I_2 + r.x = 1.(0,1) + 1.(0,1)I_1 + 1.(0,1)I_2 + 1.(2,0)I_1I_1 + 1.(2,0)I_1I_1 + 1.(1,0)I_1I_2$

$1.(2,0)I_1I_2 + 1.(2,0)I_2I_2 = 0; \text{ } 1.\in \bigg[0,1\bigg); \text{ } 1.\in \bigg[0,1\bigg); \text{ } 1.\in \bigg[0,1\bigg); \text{ } 1.\in \bigg[0,1\bigg)$

$r.x$ does not belong to $T$, thus $T$ is not a submodule. $M = \langle 0,1 \rangle + (5,1)I_1 + (2,2)I_2$

**Definition**

Let $V_n(I)$ be a weak/strong $n$-refined neutrosophic vector space, $M_n(I) = M_0 + M_1I_1 + \cdots + M_nI_n$ be two weak/strong AH-subspaces of $V_n(I)$, we define: $W_n(I) = W_0 + W_1I_1 + \cdots + W_nI_n$

(a) $M_n(I) \cap W_n(I) = (M_0 \cap W_0) + (M_1 \cap W_1)I_1 + \cdots + (M_n \cap W_n)I_n$.

(b) $M_n(I) + W_n(I) = (M_0 + W_0) + (M_1 + W_1)I_1 + \cdots + (M_n + W_n)I_n$.
Theorem

Let $V_n(I)$ be a weak n-refined neutrosophic vector space, $M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$, be two weak AH-subspaces of $V_n(I)$. Then $W_n(I) = W_0 + W_1 I_1 + \cdots + W_n I_n$ are two weak AH-subspaces of $V_n(I), M_n(I) \cap W_n(I), M_n(I) + W_n(I)$.

Theorem

Let $V_n(I)$ be a strong n-refined neutrosophic vector space, $M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$, be two strong AH-subspaces of $V_n(I)$. Then $W_n(I) = W_0 + W_1 I_1 + \cdots + W_n I_n$.

(a) $M_n(I) \cap W_n(I)$ is a strong AH-subspaces of $V_n(I)$.

(b) $M_n(I) + W_n(I)$ is not supposed to be a strong AH-subspace of $V_n(I)$.

Definition

Let $V,W$ be two vector spaces over the field $K$, $f_i: V \to W$; $0 \leq i \leq n + 1$ be $n + 1$ linear transformations, $V_n(I), W_n(I)$ be the corresponding weak n-refined neutrosophic vector spaces over the field $K$ respectively. We say:

(a) $f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n$ is a weak AH-linear transformation.

(b) If $f_i = f_j$ for all $i,j$, we call $f$ a weak AHS-linear transformation.

Definition

Let $V,W$ be two vector spaces over the field $K$, $f_i: V \to W$; $0 \leq i \leq n + 1$ be $n + 1$ linear transformations, $V_n(I), W_n(I)$ be the corresponding strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field $K_n(I)$ respectively. We say:

(a) $f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n$ is a strong AH-linear transformation.

(b) If $f_i = f_j$ for all $i,j$, we call $f$ a strong AHS-linear transformation.

Definition

Let $V_n(I), W_n(I)$ be two weak/strong n-refined neutrosophic vector spaces,

be a weak/strong $f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n$ AH-linear transformation. We define:

(a) $AH - \text{Ker}(f) = \text{Ker}(f_0) + \text{Ker}(f_1)I_1 + \cdots + \text{Ker}(f_n)I_n$.

(b) $AH - \text{Im}(f) = \text{Im}(f_0) + \text{Im}(f_1)I_1 + \cdots + \text{Im}(f_n)I_n$.

Theorem

Let $V_n(I), W_n(I)$ be two weak n-refined neutrosophic vector spaces, be a weak AH-$f: V_n(I) \to W_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1)I_1 + \cdots + f_n(a_n)I_n = \sum_{i=0}^{n} f_i(a_i)I_i$ linear transformation. Then:
(a) $AH - Ker(f)$ is a weak AH-subspace of $V_n(I)$.

(b) $AH - Im(f)$ is a weak AH-subspace of $W_n(I)$.

(c) If $M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$ is a weak AH-subspace of $V_n(I)$, $f(M_n(I))$ is a weak AH-subspace of $W_n(I)$.

**Theorem**

Let $V_n(I), W_n(I)$ be two strong $n$-refined neutrosophic vector spaces over the $n$-refined neutrosophic field $K_n(I)$,

be a strong AH-$f$: $V_n(I) \rightarrow W_n(I); f(\sum_{i=0}^n a_i I_i) = f_0(a_0) + f_1(a_1) I_1 + \cdots + f_n(a_n) I_n = \sum_{i=0}^n f_i(a_i) I_i$

linear transformation. Then:

(a) $AH - Ker(f)$ is a strong AH-subspace of $V_n(I)$.

(b) $AH - Im(f)$ is a strong AH-subspace of $W_n(I)$.

(c) If $M_n(I) = M_0 + M_1 I_1 + \cdots + M_n I_n$ is a strong AH-subspace of $V_n(I)$, $f(M_n(I))$ is a strong AH-subspace of $W_n(I)$.

**Theorem**

Let $V_n(I), W_n(I)$ be two weak $n$-refined neutrosophic vector spaces over the field $K$,

be a weak AH-$f$: $V_n(I) \rightarrow W_n(I); f(\sum_{i=0}^n a_i I_i) = f_0(a_0) + f_1(a_1) I_1 + \cdots + f_n(a_n) I_n = \sum_{i=0}^n f_i(a_i) I_i$

linear transformation. Then:

for all $x, y \in V_n(I), r \in K, f(x + y) = f(x) + f(y), f(r . x) = r . f(x)$

**Theorem**

Let $V_n(I), W_n(I)$ be two strong $n$-refined neutrosophic vector spaces over the $n$-refined neutrosophic field $K_n(I)$,

be a strong AH-$f$: $V_n(I) \rightarrow W_n(I); f(\sum_{i=0}^n a_i I_i) = f_0(a_0) + f_1(a_1) I_1 + \cdots + f_n(a_n) I_n = \sum_{i=0}^n f_i(a_i) I_i$

linear transformation. Then:

for all $x, y \in V_n(I), r \in K, f(x + y) = f(x) + f(y), f(r . x) = r . f(x)$

**Theorem**

Let $V_n(I), W_n(I), U_n(I)$ be three weak $n$-refined neutrosophic vector spaces over the field $K$,

$f: W_n(I) \rightarrow U_n(I); f(\sum_{i=0}^n a_i I_i) = f_0(a_0) + f_1(a_1) I_1 + \cdots + f_n(a_n) I_n = \sum_{i=0}^n f_i(a_i) I_i$

$g: V_n(I) \rightarrow W_n(I); g(\sum_{i=0}^n a_i I_i) = g_0(a_0) + g_1(a_1) I_1 + \cdots + g_n(a_n) I_n = \sum_{i=0}^n g_i(a_i) I_i$

be two weak AH-linear transformations. Then:

(a) $fog = \sum_{i=0}^n (f_iog_i)$.

(b) $fog$ is a weak AH-linear transformation between $V_n(I), U_n(I)$.

**Theorem**
Let $V_n(I), W_n(I), U_n(I)$ be three strong n-refined neutrosophic vector spaces over the n-refined neutrosophic field $K$,

$$f: W_n(I) \to U_n(I); f(\sum_{i=0}^{n} a_i I_i) = f_0(a_0) + f_1(a_1) I_1 + \cdots + f_n(a_n) I_n = \sum_{i=0}^{n} f_i(a_i) I_i$$

$$g: V_n(I) \to W_n(I); g(\sum_{i=0}^{n} a_i I_i) = g_0(a_0) + g_1(a_1) I_1 + \cdots + g_n(a_n) I_n = \sum_{i=0}^{n} g_i(a_i) I_i$$

be two strong AH-linear transformations. Then:

(a) $fog = \sum_{i=0}^{n}(f_iog_i)$.

(b) $fog$ is a strong AH-linear transformation between $V_n(I), U_n(I)$.

**Definition**

Let $(R, +, \times)$ be a ring and $I_k$; $1 \leq k \leq n$ be $n$ indeterminacies. We define $R_n(I) = \{a_0 + a_1 I_1 + a_2 I_2 + \cdots + a_n I_n; a_i \in R\}$ to be n-cyclic refined neutrosophic ring.

**Operations on $R_n(I)$** are defined as:

$$\sum_{i=0}^{n} x_i I_i + \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i + y_i) I_i; \sum_{i=0}^{n} x_i I_i \times \sum_{i=0}^{n} y_i I_i = \sum_{i=0}^{n} (x_i \times y_i) I_i I_j = \sum_{i,j=0}^{n} (x_i \times y_i) I_{i+j \text{mod} n}$$

Where $\times$ is the multiplication on the ring $R$, and $x I_0 = x$, for all $x \in R$.

**Definition**

Let $(K, +, \times)$ be a field, we say that $K_n(I) = K + K I_1 + \cdots + K I_n = \{a_0 + a_1 I_1 + a_2 I_2 + \cdots + a_n I_n; a_i \in K\}$ is a n-cyclic refined neutrosophic field.

**Definition**

Let $(V, +, \times)$ be any vector space over a field $K$. Then we say that $V_n(I) = V + V I_1 + \cdots + V I_n = \{x_0 + x_1 I_1 + \cdots + x_n I_n; x_i \in V\}$ is a weak n-cyclic refined neutrosophic vector space over the field $K$. Elements of $V_n(I)$ are called n-cyclic refined neutrosophic vectors, elements of $K$ are called scalars.

If we take scalars from the n-cyclic refined neutrosophic field $K_n(I)$, we say that $V_n(I)$ is a strong n-cyclic refined neutrosophic vector space over the n-cyclic refined neutrosophic field $K_n(I)$. Elements of $K_n(I)$ n-cyclic refined neutrosophic scalars.

**Remark**

Multiplication by an n-cyclic refined neutrosophic scalar $m = \sum_{i=0}^{n} m_i I_i \in k_n(I)$ is defined as:

$$\left(\sum_{i=0}^{n} m_i I_i\right) \times \left(\sum_{i=0}^{n} a_i I_i\right) = \sum_{i,j=0}^{n} (m_i a_j) I_{i+j \text{mod} n}$$

Where $a_i \in V$, $m_i \in K$, $I_i I_j = I_{i+j \text{mod} n}$.

**Definition**

Let $V_n(I)$ be a weak n-cyclic refined neutrosophic vector space over the n-cyclic refined neutrosophic field $K$; a nonempty $W_n(I)$ is called a weak n-cyclic refined neutrosophic vector subspace of $V_n(I)$ if $W_n(I)$ is a subspace of $V_n(I)$ itself.

**Definition**
Let $V_n(I)$ be a strong $n$-cyclic refined neutrosophic vector space over then-cyclic refined neutrosophic field $K_n(I)$. A nonempty subset $W_n(I)$ is called a strong $n$-cyclic refined neutrosophic vector submodule of $V_n(I)$ if $W_n(I)$ is a submodule of $V_n(I)$ itself.

**Theorem**

Let $V_n(I)$ be a weak $n$-cyclic refined neutrosophic vector space over the $n$-cyclic refined neutrosophic field $K_n$, $W_n(I)$ be a nonempty subset of $V_n(I)$. Then $W_n(I)$ is a weak $n$-cyclic refined neutrosophic subspace if only if:

for all $x, y \in W_n(I), m \in K, x + y \in W_n(I), m \times x \in W_n(I)$

**proof:**

it holds directly from the condition of subspace.

**Definition**

Let $V_n(I)$ be a weak $n$-cyclic refined neutrosophic vector space over the field $K$, $x$ be an arbitrary element of $V_n(I)$, we say that $x$ is a linear combination of $\{ x_1, x_2, ..., x_m \} \subseteq V_n(I)$ if $x = (a_1 \times x_1) + (a_2 \times x_2) + \cdots + (a_m \times x_m); a_i \in K(I), x_i \in V_n(I)$.

**Definition**

Let $V_n(I)$ be a strong $n$-cyclic refined neutrosophic vector space over the $n$-cyclic refined neutrosophic field $K_n(I)$, $x$ be an arbitrary element of $V_n(I)$, we say that $x$ is a linear combination of $\{ x_1, x_2, ..., x_m \} \subseteq V_n(I)$ if $x = (a_1 \times x_1) + (a_2 \times x_2) + \cdots + (a_m \times x_m); a_i \in K_n(I), x_i \in V_n(I)$.

**Definition**

Let $X = \{ x_1, x_2, ..., x_m \}$ be a subset of a weak $n$-cyclic refined neutrosophic vector space $V_n(I)$ over the field $K$, $X$ is a weak linearly independent set if $\sum_{i=0}^{n} a_i \times x_i = 0$ implies $a_i = 0; a_i \in K$.

**Definition**

Let $X = \{ x_1, x_2, ..., x_m \}$ be a subset of a strong $n$-cyclic refined neutrosophic vector space $V_n(I)$ over the $n$-cyclic refined neutrosophic field $K_n(I)$, $X$ is a weak linearly independent set if $\sum_{i=0}^{n} a_i \times x_i = 0$ implies $a_i = 0; a_i \in K_n(I)$.

**Definition**

Let $V_n(I), W_n(I)$ be two strong $n$-cyclic refined neutrosophic vector space over the $n$-cyclic refined neutrosophic field $K_n(I)$, let $f: V_n(I) \rightarrow U_n(I)$ be a well defined map. It is called a strong $n$-cyclic refined neutrosophic homomorphism if:

$f((a \times x) + (b \times y)) = a \times f(x) + b \times f(y)$ for all $x, y \in V_n(I), a, b \in K_n(I)$.

A weak $n$-cyclic refined neutrosophic homomorphism can be defined as the same.

**Definition**

Let $f: V_n(I) \rightarrow U_n(I)$ be a weak/strong $n$-cyclic refined neutrosophic homomorphism, we define:

(a) $\text{Ker}(f) = \{ x \in V_n(I); f(x) = 0 \}$. 
(b) \( \text{Im}(f) = \{ y \in U_n(I); \exists x \in V_n(I) \text{and } y = f(x) \} \).

**Theorem**

Let \( f: V_n(I) \rightarrow U_n(I) \) be a weak n-cyclic refined neutrosophic homomorphism. Then

(a) \( \text{Ker}(f) \) is a weak n-cyclic refined neutrosophic subspace of \( V_n(I) \).

(b) \( \text{Im}(f) \) is a weak n-cyclic refined neutrosophic subspace of \( U_n(I) \).

**Theorem**

Let \( f: V_n(I) \rightarrow U_n(I) \) be a strong n-cyclic refined neutrosophic homomorphism. Then

(a) \( \text{Ker}(f) \) is a strong n-cyclic refined neutrosophic subspace of \( V_n(I) \).

(b) \( \text{Im}(f) \) is a strong n-cyclic refined neutrosophic subspace of \( U_n(I) \).

**References**


