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Neutrosophic Modal Logic

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Abstract

We introduce now for the first time the neutrosophic modal logic. The Neutrosophic Modal Logic includes the neutrosophic operators that express the modalities. It is an extension of neutrosophic predicate logic and of neutrosophic propositional logic.

1. Introduction

The paper extends the fuzzy modal logic (Girle, 2010; Hájek & Harmancová, 1993; & Liau & Pen Lin, 1992), fuzzy environment (Hur et. al., 2006) and neutrosophic sets, numbers and operators (Liu et. al., 2014; Liu & Shi, 2015; Liu & Tang, 2016; Liu & Wang, 2016; Liu & Li, 2017; Liu & Tang, 2016; Liu et. al., 2016; Liu, 2016), together with the last developments of the neutrosophic environment {including *(t,i,f)*-neutrosophic algeb-raic structures, neutrosophic triplet structures, and neutrosophic overset / underset / offset} (Smarandache, 2016a; Smarandache & Ali, 2016; Smarandache, 2016b) passing through the symbolic neutrosophic logic (Smarandache, 2015), ultimately to neutrosophic modal logic.

This paper also answers Rivieccio's question on neutrosophic modalities.

All definitions, sections, and notions in-troduced in this paper were never done before, neither in my previous work nor in other researchers'.

Therefore, we introduce now the Neutrosophic Modal Logic and the Refined Neutrosophic Modal Logic. Then we can extend them to Symbolic Neutrosophic Modal Logic and Refined Symbolic Neutrosophic Modal Logic, using labels instead of numerical values.

There is a large variety of neutrosophic modal logics, as actually happens in classical modal logic too. Similarly, the neutrosophic accessibility relation and possible neutrosophic worlds have many interpretations, depending on each par-ticular application. Several neutrosophic modal applications are also listed.

Due to numerous applications of neutrosophic modal logic (see the examples throughout the paper), the introduction of the neutrosophic modal logic was needed.

Neutrosophic Modal Logic is a logic where some neutrosophic modalities have been included.

Let $\mathcal{P}$ be a neutrosophic proposition. We have the following types of *neutrosophic modalities*:

I. Neutrosophic Alethic Modalities (related to truth) has three neutros-ophic operators:
Neutrosophic Possibility: It is neutros-ophically possible that $\mathcal{P}$.
Neutrosophic Necessity: It is neutros-ophically necessary that $\mathcal{P}$.
Neutrosophic Impossibility: It is neutros-ophically impossible that $\mathcal{P}$.

II. Neutrosophic Temporal Modalities (related to time)
It was the neutrosophic case that $\mathcal{P}$.
It will neutrosophically be that $\mathcal{P}$.
And similarly:
It has always neutrosophically been that $\mathcal{P}$.
It will always neutrosophically be that $\mathcal{P}$.

III. Neutrosophic Epistemic Modalities (related to knowledge):
It is neutrosophically known that $\mathcal{P}$.

IV. Neutrosophic Doxastic Modalities (related to belief):
It is neutrosophically believed that $\mathcal{P}$.

V. Neutrosophic Deontic Modalities:
It is neutrosophically obligatory that $\mathcal{P}$.
It is neutrosophically permissible that $\mathcal{P}$.

2. Neutrosophic Alethic Modal Operators

The modalities used in classical (alethic) modal logic can be neutrosophicated by inserting the indeterminacy.

We insert the *degrees of possibility* and *degrees of necessity*, as refinement of classical modal operators.

2.1. Neutrosophic Possibility Operator

The classical Possibility Modal Operator «◊ *P*» meaning «It is possible that *P*» is extended to *Neutrosophic Possibility Operator*: $\lozenge_N \mathcal{P}$ meaning «It is (*t, i, f*)-possible that $\mathcal{P}$ », using Neutrosophic Probability, where «(t, i, f)-possible» means *t* % possible (chance that $\mathcal{P}$ occurs), *i* % indeterminate (indeterminate-chance that $\mathcal{P}$ occurs), and *f* % impossible (chance that $\mathcal{P}$ does not occur).

If $\mathcal{P}(t_p, i_p, f_p)$ is a neutrosophic proposition, with t_p, i_p, f_p subsets of [0, 1], then the neutrosophic truth-value of the neutrosophic possibility operator is:

$$\lozenge_N \mathcal{P} = \left(\sup(t_p), \inf(i_p), \inf(f_p)\right), \qquad (1)$$

which means that if a proposition *P* is t_p true, i_p indeterminate, and f_p false, then the value of the neutrosophic possibility operator $\lozenge_N \mathcal{P}$ is: $\sup(t_p)$ possibility, $\inf(i_p)$ indeterminate-possibility, and $\inf(f_p)$ impossibility.

For example.

Let P = «It will be snowing tomorrow».

According to the meteorological center, the neutrosophic truth-value of $\mathcal{P}$ is:

$\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})$,

i.e. $[0.5, 0.6]$ true, $(0.2, 0.4)$ indeterminate, and $\{0.3, 0.5\}$ false.

Then the neutrosophic possibility operator is:

$\lozenge_N \mathcal{P} = (\sup[0.5, 0.6], \inf(0.2, 0.4), \inf\{0.3, 0.5\}) = (0.6, 0.2, 0.3)$,

i.e. 0.6 possible, 0.2 indeterminate-possibility, and 0.3 impossible.

2.2. Neutrosophic Necessity Operator

The classical Necessity Modal Operator «□*P*» meaning «It is necessary that *P*» is extended to *Neutrosophic Necessity Operator*: $\square_N \mathcal{P}$ meaning «It is (*t, i, f*)-necessary that $\mathcal{P}$ », using again the Neutrosophic Probability, where similarly «(t, i, f)-necessity» means *t* % necessary (surety that $\mathcal{P}$ occurs), *i* % indeterminate (indeterminate-surety that $\mathcal{P}$ occurs), and *f* % unnecessary (unsurely that $\mathcal{P}$ occurs).

If $\mathcal{P}(t_p, i_p, f_p)$ is a neutrosophic proposition, with t_p, i_p, f_p subsets of [0, 1], then the neutrosophic truth value of the neutrosophic necessity operator is:

$$\square_N \mathcal{P} = \left(\inf(t_p), \sup(i_p), \sup(f_p)\right), \qquad (2)$$

which means that if a proposition $\mathcal{P}$ is t_p true, i_p indeterminate, and f_p false, then the value of the neutrosophic necessity operator $\square_N \mathcal{P}$ is: $\inf(t_p)$ necessary, $\sup(i_p)$ indeterminate-necessity, and $\sup(f_p)$ unnecessary.

Taking the previous *example*:

$\mathcal{P}$ = «It will be snowing tomorrow», with $\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})$,

then the neutrosophic necessity operator is:

$\square_N \mathcal{P} = (\inf[0.5, 0.6], \sup(0.2, 0.4), \sup\{0.3, 0.5\}) = (0.5, 0.4, 0.5)$,
i.e. 0.5 necessary, 0.4 indeterminate-necessity, and 0.5 unnecessary.

2.3. Other Possibility and Necessity Operators

The previously defined neutrosophic pos-sibility and respectively neutrosophic necessity operators, for $\mathcal{P}(t_p, i_p, f_p)$ a neutrosophic propos-ition, with t_p, i_p, f_p subset-valued included in *[0, 1]*,

$\lozenge_N \mathcal{P}$ = (sup(t_p), inf(i_p), inf(f_p)),

$\square_N \mathcal{P}$ = (inf(t_p), sup(i_p), sup(f_p)),

work quite well for subset-valued (including interval-valued) neutrosophic components, but they fail for single-valued neutrosophic components because one gets $\lozenge_N \mathcal{P} = \square_N \mathcal{P}$.

Depending on the applications, more possibility and necessity operators may be defined.

Their definitions may work, mostly based on *max / min / min* for possibility operator and *min / max / max* for necessity operator (when dealing with single-valued neutrosophic components in *[0, 1]*), or based on *sup / inf / inf* for possibility operator and *inf / sup / sup* for necessity operator (when dealing with interval-valued or more general with subset-valued of neutrosophic components included in *[0, 1]*):

For examples.

Let $\mathcal{P}(t_p, i_p, f_p)$ be a neutrosophic proposition, with t_p, i_p, f_p single-valued of *[0, 1]*, then the neutrosophic truth-value of the neutrosophic possibility operator is:

$\lozenge_N \mathcal{P} = (max\{t_p,\ 1\text{-}f_p\},\ min\{i_p,\ 1\text{-}i_p\},\ min\{f_p,\ 1\text{-}\ t_p\})$

or

$\lozenge_N \mathcal{P} = (max\{t_p,\ 1\text{-}t_p\},\ min\{i_p,\ 1\text{-}i_p\},\ min\{f_p,\ 1\text{-}\ f_p\})$

or

$\lozenge_N \mathcal{P} = (1\text{-}\ f_p,\ i_p,\ f_p)$

{defined by Anas Al-Masarwah}.

Let $\mathcal{P}(t_p, i_p, f_p)$ be a neutrosophic proposition, with t_p, i_p, f_p single-valued of *[0, 1]*, then the neutrosophic truth-value of the neutrosophic necessity operator is:

$\square_N \mathcal{P} = (min\{t_p,\ 1\text{-}f_p\},\ max\{i_p,\ 1\text{-}i_p\},\ max\{f_p,\ 1\text{-}\ t_p\})$

or

$\square_N \mathcal{P} = (min\{t_p,\ 1\text{-}t_p\},\ max\{i_p,\ 1\text{-}i_p\},\ max\{f_p,\ 1\text{-}\ f_p\})$

or

$\square_N \mathcal{P} = (t_p,\ i_p,\ 1 - t_p)$

{defined by Anas Al-Masarwah}.

The above six defined operators may be extended from single-valued numbers of *[0, 1]* to interval and subsets of *[0, 1],* by simply replacing the subtractions of numbers by subtractions of intervals or subsets, and "*min*" by "*inf*", while "*max*" by "*sup*".

3. Connection between Neutrosophic Possibility Operator and Neutrosophic Necessity Operator

In classical modal logic, a modal operator is equivalent to the negation of the other:

$\lozenge P \leftrightarrow \neg\square\neg P$, (3)

$\square P \leftrightarrow \neg \lozenge \neg P$. (4)

In neutrosophic logic one has a class of neutrosophic negation operators. The most used one is:

$\underset{N}{\neg} P(t, i, f) = \bar{P}(f, 1 - i, t)$, (5)

where t, i, f are real subsets of the interval *[0, 1].*

Let's check what's happening in the neutros-ophic modal logic, using the previous *example*.

One had:

$\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})$,
then

$\underset{N}{\neg}\mathcal{P} = \bar{\mathcal{P}}(\{0.3, 0.5\}, 1 - (0.2, 0.4), [0.5, 0.6]) =$
$\bar{\mathcal{P}}(\{0.3, 0.5\}, 1 - (0.2, 0.4), [0.5, 0.6]) =$
$\bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6])$.

Therefore, denoting by $\underset{N}{\leftrightarrow}$ the neutrosophic equivalence, one has:

$\underset{N}{\neg}\underset{N}{\Box}\underset{N}{\neg}\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})\underset{N}{\leftrightarrow}$
$\underset{N}{\leftrightarrow}$ It is not neutrosophically necessary that «It will not be snowing tomorrow»
$\underset{N}{\leftrightarrow}$ It is not neutrosophically necessary that $\bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6])$
$\underset{N}{\leftrightarrow}$ It is neutrosophically possible that $\underset{N}{\neg}\bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6])$
$\underset{N}{\leftrightarrow}$ It is neutrosophically possible that $\mathcal{P}([0.5, 0.6], 1 - (0.6, 0.8), \{0.3, 0.5\})$
$\underset{N}{\leftrightarrow}$ It is neutrosophically possible that $\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})$
$\underset{N}{\leftrightarrow}\underset{N}{\Diamond}\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\}) = (0.6, 0.2, 0.3)$.

Let's check the second neutrosophic equivalence.

$\underset{N}{\neg}\underset{N}{\Diamond}\underset{N}{\neg}\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})\underset{N}{\leftrightarrow}$
$\underset{N}{\leftrightarrow}$ It is not neutrosophically possible that «It will not be snowing tomorrow»
$\underset{N}{\leftrightarrow}$ It is not neutrosophically possible that $\bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6])$
$\underset{N}{\leftrightarrow}$ It is neutrosophically necessary that $\underset{N}{\neg}\bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6])$
$\underset{N}{\leftrightarrow}$ It is neutrosophically necessary that $\mathcal{P}([0.5, 0.6], 1 - (0.6, 0.8), \{0.3, 0.5\})$
$\underset{N}{\leftrightarrow}$ It is neutrosophically necessary that $\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})$
$\underset{N}{\leftrightarrow}\underset{N}{\Box}\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\}) = (0.6, 0.2, 0.3)$.

4. Neutrosophic Modal Equivalences

Neutrosophic Modal Equivalences hold within a certain accuracy, depending on the definitions of neutrosophic possibility operator and neutros-ophic necessity operator, as well as on the definition of the neutrosophic negation – employed by the experts depending on each application. Under these conditions, one may have the following neutrosophic modal equivalences:

$$\Diamond_N \mathcal{P}(t_p, i_p, f_p)\underset{N}{\leftrightarrow}\underset{N}{\neg}\underset{N}{\Box}\underset{N}{\neg}\mathcal{P}(t_p, i_p, f_p) \qquad (6)$$

$$\Box_N \mathcal{P}(t_p, i_p, f_p)\underset{N}{\leftrightarrow}\underset{N}{\neg}\underset{N}{\Diamond}\underset{N}{\neg}\mathcal{P}(t_p, i_p, f_p) \qquad (7)$$

For example, other definitions for the *neutros-ophic modal operators* may be:

$$\Diamond_N \mathcal{P}(t_p, i_p, f_p) = \left(\sup(t_p), \sup(i_p), \inf(f_p)\right), \qquad (8)$$

or

$$\Diamond_N \mathcal{P}(t_p, i_p, f_p) = \left(\sup(t_p), \frac{i_p}{2}, \inf(f_p)\right) \text{ etc.}, \qquad (9)$$

while

$$\Box_N \mathcal{P}(t_p, i_p, f_p) = \left(\inf(t_p), \inf(i_p), \sup(f_p)\right), \qquad (10)$$

or

$\square_N \mathcal{P}(t_p, i_p, f_p) = \left(\inf(t_p), 2i_p \cap [0,1], \sup(f_p)\right)$ (11)

etc.

5. Neutrosophic Truth Threshold

In neutrosophic logic, first we have to introduce a *neutrosophic truth threshold*, $TH = \langle T_{th}, I_{th}, F_{th} \rangle$, where T_{th}, I_{th}, F_{th} are subsets of [0, 1]. We use uppercase letters (T, I, F) in order to distinguish the neutrosophic components of the threshold from those of a proposition in general.

We can say that the proposition $\mathcal{P}(t_p, i_p, f_p)$ is *neutrosophically true* if:

$\inf(t_p) \geq \inf(T_{th})$ and $\sup(t_p) \geq \sup(T_{th})$; (12)

$\inf(i_p) \leq \inf(I_{th})$ and $\sup(t_p) \leq \sup(I_{th})$; (13)

$\inf(f_p) \leq \inf(F_{th})$ and $\sup(f_p) \leq \sup(F_{th})$. (14)

For the particular case when all T_{th}, I_{th}, F_{th} and t_p, i_p, f_p are single-valued numbers from the interval [0, 1], then one has:

The proposition $\mathcal{P}(t_p, i_p, f_p)$ is *neutrosophically true* if:

$t_p \geq T_{th}$;

$i_p \leq I_{th}$;

$f_p \leq F_{th}$.

The neutrosophic truth threshold is established by experts in accordance to each application.

6. Neutrosophic Semantics

Neutrosophic Semantics of the *Neutrosophic Modal Logic* is formed by a *neutrosophic frame* G_N, which is a *non-empty neutrosophic set*, whose elements are called *possible neutrosophic worlds*, and a *neutrosophic binary relation* $\mathcal{R}_N$, called *neutrosophic accesibility relation*, between the possible neutrosophic worlds. By notation, one has:

$\langle G_N, \mathcal{R}_N \rangle$.

A neutrosophic world w'_N that is neutrosophically accessible from the neutrosophic world w_N is symbolized as:

$w_N \mathcal{R}_N w'_N$.

In a *neutrosophic model* each neutrosophic proposition $\mathcal{P}$ has a *neutrosophic truth-value* $(t_{w_N}, i_{w_N}, f_{w_N})$ respectively to each *neutrosophic world* $w_N \in G_N$, where $t_{w_N}, i_{w_N}, f_{w_N}$ are subsets of [0, 1].

A *neutrosophic actual world* can be similarly noted as in classical modal logic as $w_N *$.

Formalization

Let S_N be a set of neutrosophic propositional variables.

7. Neutrosophic Formulas

1. Every neutrosophic propositional variable $\mathcal{P} \in S_N$ is a neutrosophic formula.

2. If *A, B* are neutrosophic formulas, then $\underset{N}{\neg} A$, $A \underset{N}{\wedge} B$, $A \underset{N}{\vee} B$, $A \underset{N}{\rightarrow} B$, $A \underset{N}{\leftrightarrow} B$, and $\underset{N}{\lozenge} A$, $\underset{N}{\square} A$, are also neutrosophic formulas, where $\underset{N}{\neg}$, $\underset{N}{\wedge}$, $\underset{N}{\vee}$, $\underset{N}{\rightarrow}$, $\underset{N}{\leftrightarrow}$, and $\underset{N}{\lozenge}$, $\underset{N}{\square}$ represent the neutrosophic negation, neutrosophic intersection, neutrosophic union, neutros-ophic implication, neutrosophic equivalence, and neutrosophic possibility operator, neutrosophic necessity operator respectively.

8. Accesibility Relation in a Neutrosophic Theory

Let G_N be a set of neutrosophic worlds w_N such that each w_N chracterizes the propositions (formulas) of a given neutrosophic theory τ.

We say that the neutrosophic world w'_N is accesible from the neutrosophic world w_N, and we write: $w_N \mathcal{R}_N w'_N$ or $\mathcal{R}_N(w_N, w'_N)$, if for any proposition (formula) $\mathcal{P} \in w_N$, meaning the neutrosophic truth-value of $\mathcal{P}$ with respect to w_N is

$\mathcal{P}\left(t_p^{w_N}, i_p^{w_N}, f_p^{w_N}\right)$,

one has the neutrophic truth-value of $\mathcal{P}$ with respect to w'_N

$\mathcal{P}\left(t_p^{w'_N}, i_p^{w'_N}, f_p^{w'_N}\right)$,

where

$\inf\left(t_p^{w'_N}\right) \geq \inf\left(t_p^{w_N}\right)$ and $\sup\left(t_p^{w'_N}\right) \geq \sup\left(t_p^{w_N}\right)$; (15)

$\inf\left(i_p^{w'_N}\right) \leq \inf\left(i_p^{w_N}\right)$ and $\sup\left(i_p^{w'_N}\right) \leq \sup\left(i_p^{w_N}\right)$; (16)

$\inf\left(f_p^{w'_N}\right) \leq \inf\left(f_p^{w_N}\right)$ and $\sup\left(f_p^{w'_N}\right) \leq \sup\left(f_p^{w_N}\right)$ (17)

(in the general case when $t_p^{w_N}, i_p^{w_N}, f_p^{w_N}$ and $t_p^{w'_N}, i_p^{w'_N}, f_p^{w'_N}$ are subsets of the interval [0, 1]).

But in the instant of $t_p^{w_N}, i_p^{w_N}, f_p^{w_N}$ and $t_p^{w'_N}, i_p^{w'_N}, f_p^{w'_N}$ as single-values in [0, 1], the above inequalities become:

$t_p^{w'_N} \geq t_p^{w_N}$, (18)

$i_p^{w'_N} \leq i_p^{w_N}$, (19)

$f_p^{w'_N} \leq f_p^{w_N}$. (20)

9. Applications

If the neutrosophic theory τ is the Neutros-ophic Mereology, or Neutrosophic Gnosisology, or Neutrosophic Epistemology etc., the neutrosophic accesibility relation is defined as above.

9.1. Neutrosophic n-ary Accesibility Relation

We can also extend the classical *binary* accesibility relation $\mathcal{R}$ to a *neutrosophic n-ary accesibility relation*

$\mathcal{R}_N^{(n)}$, for *n* integer ≥ 2.

Instead of the classical $R(w, w')$, which means that the world w' is accesible from the world w, we generalize it to:

$\mathcal{R}_N^{(n)}\left(w_{1_N}, w_{2_N}, \ldots, w_{n_N}; w'_N\right)$,

which means that the neutrosophic world w'_N is accesible from the neutrosophic worlds $w_{1_N}, w_{2_N}, \ldots, w_{n_N}$ all together.

9.2. Neutrosophic Kripke Frame

$k_N = \langle G_N, R_N \rangle$ is a neutrosophic Kripke frame, since:

$i.$ G_N is an arbitrary non-empty neutrosophic set of *neutrosophic worlds*, or *neutrosophic states*, or *neutrosophic situations*.

$ii.$ $R_N \subseteq G_N \times G_N$ is a *neutrosophic accesibility relation* of the neutrosophic Kripke frame. Actually, one has a degree of accessibility, degree of indeterminacy, and a degree of non-accessibility.

9.3. Neutrosophic (t, i, f)-Assignement

The Neutrosophic *(t, i, f)*-Assignement is a neutrosophic mapping

$v_N: S_N \times G_N \to [0,1] \times [0,1] \times [0,1]$ (21)

where, for any neutrosophic proposition $\mathcal{P} \in S_N$ and for any neutrosophic world w_N, one defines:

$v_N\left(P, w_N\right) = \left(t_p^{w_N}, i_p^{w_N}, f_p^{w_N}\right) \in [0,1] \times [0,1] \times [0,1]$ (22)

which is the neutrosophical logical truth value of the neutrosophic proposition $\mathcal{P}$ in the neutros-ophic world w_N.

9.4. Neutrosophic Deducibility

We say that the neutrosophic formula $\mathcal{P}$ is neutrosophically deducible from the neutrosophic Kripke frame k_N, the neutrosophic (*t, i, f*) – assignment v_N, and the neutrosophic world w_N, and we write as:

$$k_N, v_N, w_N \underset{N}{\vDash} \mathcal{P}. \qquad (23)$$

Let's make the notation:

$\alpha_N(\mathcal{P};\ k_N, v_N, w_N)$

that denotes the neutrosophic logical value that the formula $\mathcal{P}$ takes with respect to the neutrosophic Kripke frame k_N, the neutrosophic (*t, i, f*)-assignment v_N, and the neutrosphic world w_N.

We define α_N by neutrosophic induction:

1. $\alpha_N(\mathcal{P};\ k_N, v_N, w_N) \overset{def}{=} v_N(\mathcal{P}, w_N)$ if $\mathcal{P} \in S_N$ and $w_N \in G_N$.
2. $\alpha_N\left(\underset{N}{\neg}\mathcal{P};\ k_N, v_N, w_N\right) \overset{def}{=} \underset{N}{\neg}[\alpha_N(\mathcal{P};\ k_N, v_N, w_N)]$.
3. $\alpha_N\left(\mathcal{P}\underset{N}{\wedge}Q;\ k_N, v_N, w_N\right) \overset{def}{=} [\alpha_N(\mathcal{P};\ k_N, v_N, w_N)]\underset{N}{\wedge}[\alpha_N(Q;\ k_N, v_N, w_N)]$
4. $\alpha_N\left(\mathcal{P}\underset{N}{\vee}Q;\ k_N, v_N, w_N\right) \overset{def}{=} [\alpha_N(\mathcal{P};\ k_N, v_N, w_N)]\underset{N}{\vee}[\alpha_N(Q;\ k_N, v_N, w_N)]$
5. $\alpha_N\left(\mathcal{P}\underset{N}{\rightarrow}Q;\ k_N, v_N, w_N\right) \overset{def}{=} [\alpha_N(\mathcal{P};\ k_N, v_N, w_N)]\underset{N}{\rightarrow}[\alpha_N(Q;\ k_N, v_N, w_N)]$
6. $\alpha_N\left(\underset{N}{\Diamond}\mathcal{P};\ k_N, v_N, w_N\right) \overset{def}{=} \langle\sup, \inf, \inf\rangle\{\alpha_N(\mathcal{P};\ k_N, v_N, w'_N),\ w' \in G_N \text{ and } w_N R_N w'_N\}$.
7. $\alpha_N\left(\underset{N}{\Box}\mathcal{P};\ k_N, v_N, w_N\right) \overset{def}{=} \langle\inf, \sup, \sup\rangle\{\alpha_N(\mathcal{P};\ k_N, v_N, w'_N),\ w'_N \in G_N \text{ and } w_N R_N w'_N\}$.
8. $\underset{N}{\vDash}\mathcal{P}$ if and only if $w_N * \vDash \mathcal{P}$ (a formula $\mathcal{P}$ is neutrosophically deducible if and only if $\mathcal{P}$ is neutrosophically deducible in the actual neutrosophic world).

We should remark that α_N has a degree of truth (t_{α_N}), a degree of indeterminacy (i_{α_N}), and a degree of falsehood (f_{α_N}), which are in the general case subsets of the interval [0, 1].

Applying $\langle\sup, \inf, \inf\rangle$ to α_N is equivalent to calculating:

$\langle\sup(t_{\alpha_N}), \inf(i_{\alpha_N}), \inf(f_{\alpha_N})\rangle$,

and similarly

$\langle\inf, \sup, \sup\rangle\alpha_N = \langle\inf(t_{\alpha_N}), \sup(i_{\alpha_N}), \sup(f_{\alpha_N})\rangle$.

10. Refined Neutrosophic Modal Single-Valued Logic

Using neutrosophic (*t, i, f*) - thresholds, we refine for the first time the neutrosophic modal logic as:

10.1. Refined Neutrosophic Possibility Operator

$\underset{N}{\Diamond_1}\mathcal{P}_{(t,i,f)}$ = «It is very little possible (degree of possibility t_1) that $\mathcal{P}$», corresponding to the threshold (t_1, i_1, f_1), i.e. $0 \le t \le t_1$, $i \ge i_1$, $f \ge f_1$, for t_1 a very little number in [0, 1];

$\underset{N}{\Diamond_2}\mathcal{P}_{(t,i,f)}$ = «It is little possible (degree of pos-sibility t_2) that $\mathcal{P}$», corresponding to the threshold (t_2, i_2, f_2), i.e. $t_1 < t \le t_2$, $i \ge i_2 > i_1$, $f \ge f_2 > f_1$;

… … …

and so on;

$\underset{N}{\Diamond_m}\mathcal{P}_{(t,i,f)}$ = «It is possible (with a degree of possibility t_m) that $\mathcal{P}$», corresponding to the threshold (t_m, i_m, f_m), i.e. $t_{m-1} < t \le t_m$, $i \ge i_m > i_{m-1}$, $f \ge f_m > f_{m-1}$.

10.2. Refined Neutrosophic Necessity Operator

$\underset{N}{\square_1}\mathcal{P}_{(t,i,f)}$ = «It is a small necessity (degree of necessity t_{m+1}) that $\mathcal{P}$», i.e. $t_m < t \le t_{m+1}$, $i \ge i_{m+1} \ge i_m$, $f \ge f_{m+1} > f_m$;

$\underset{N}{\square_2}\mathcal{P}_{(t,i,f)}$ = «It is a little bigger necessity (degree of necessity t_{m+2}) that $\mathcal{P}$», i.e. $t_{m+1} < t \le t_{m+2}$, $i \ge i_{m+2} > i_{m+1}$, $f \ge f_{m+2} > f_{m+1}$;

… … …

and so on;

$\underset{N}{\square_k}\mathcal{P}_{(t,i,f)}$ = «It is a very high necessity (degree of necessity t_{m+k}) that $\mathcal{P}$», i.e. $t_{m+k-1} < t \le t_{m+k} = 1$, $i \ge i_{m+k} > i_{m+k-1}$, $f \ge f_{m+k} > f_{m+k-1}$.

11. Application of the Neutrosophic Threshold

We have introduced the term of (*t, i, f*)-physical law, meaning that a physical law has a degree of truth (*t*), a degree of indeterminacy (*i*), and a degree of falsehood (*f*). A physical law is 100% true, 0% indeterminate, and 0% false in perfect (ideal) conditions only, maybe in laboratory.

But our actual world ($w_N *$) is not perfect and not steady, but continously changing, varying, fluctuating.

For example, there are physicists that have proved a universal constant (*c*) is not quite universal (i.e. there are special conditions where it does not apply, or its value varies between $(c-\varepsilon, c+\varepsilon)$, for $\varepsilon > 0$ that can be a tiny or even a bigger number).

Thus, we can say that a proposition $\mathcal{P}$ is *neutrosophically nomological necessary*, if $\mathcal{P}$ is neutrosophically true at all possible neutrosophic worlds that obey the (*t, i, f*)-physical laws of the actual neutrosophic world $w_N *$.

In other words, at each possible neutrosophic world w_N, neutrosophically accesible from $w_N *$, one has:

$$\mathcal{P}\left(t_p^{w_N}, i_p^{w_N}, f_p^{w_N}\right) \ge TH(T_{th}, I_{th}, F_{th}), \quad (24)$$

i.e. $t_p^{w_N} \ge T_{th}$, $i_p^{w_N} \le I_{th}$, and $f_p^{w_N} \ge F_{th}$. (25)

12. Neutrosophic Mereology

Neutrosophic Mereology means the theory of the neutrosophic relations among the parts of a whole, and the neutrosophic relations between the parts and the whole.

A neutrosophic relation between two parts, and similarly a neutrosophic relation between a part and the whole, has a degree of connectibility (*t*), a degree of indeterminacy (*i*), and a degree of disconnectibility (*f*).

12.1. Neutrosophic Mereological Threshold

Neutrosophic Mereological Threshold is def-ined as:

$$TH_M = (\min(t_M), \max(i_M), \max(f_M)) \qquad (26)$$

where t_M is the set of all degrees of con-nectibility between the parts, and between the parts and the whole;

i_M is the set of all degrees of indeterminacy between the parts, and between the parts and the whole;

f_M is the set of all degrees of disconnectibility between the parts, and between the parts and the whole.

We have considered all degrees as single-valued numbers.

13. Neutrosophic Gnosisology

Neutrosophic Gnosisology is the theory of (*t, i, f*)-knowledge, because in many cases we are not able to completely (100%) find whole knowledge, but only a part of it (*t* %), another part remaining unknown (*f* %), and a third part indeterminate (unclear, vague, contradictory) (*i* %), where *t, i, f* are subsets of the interval *[0, 1]*.

13.1. Neutrosophic Gnosisological Threshold

Neutrosophic Gnosisological Threshold is defined, similarly, as:

$$TH_G = (\min(t_G), \max(i_G), \max(f_G)) \qquad (27)$$

where t_G is the set of all degrees of knowledge of all theories, ideas, propositions etc.,

i_G is the set of all degrees of indeterminate-knowledge of all theories, ideas, propositions etc.,

f_G is the set of all degrees of non-knowledge of all theories, ideas, propositions etc.

We have considered all degrees as single-valued numbers.

14. Neutrosophic Epistemology

And *Neutrosophic Epistemology*, as part of the Neutrosophic Gnosisology, is the theory of (*t, i, f*)-scientific knowledge. Science is infinite. We know only a small part of it (*t*%), another big part is yet to be discovered (*f*%), and a third part indeterminate (unclear, vague, contradictory) (*i*%). Of course, *t, i, f* are subsets of [0, 1].

14.1. Neutrosophic Epistemological Threshold

Neutrosophic Epistemological Threshold is defined as:

$$TH_E = (\min(t_E), \max(i_E), \max(f_E)) \qquad (28)$$

where t_E is the set of all degrees of scientific knowledge of all scientific theories, ideas, propositions etc.,

i_E is the set of all degrees of indeterminate scientific knowledge of all scientific theories, ideas, propositions etc.,

f_E is the set of all degrees of non-scientific knowledge of all scientific theories, ideas, propositions etc.

We have considered all degrees as single-valued numbers.

15. Conclusions

We have introduced for the first time the Neutrosophic Modal Logic and the Refined Neutrosophic Modal Logic.

Symbolic Neutrosophic Logic can be connected to the neutrosophic modal logic too, where instead of numbers we may use labels, or instead of quantitative neutrosophic logic we may have a quantitative neutrosophic logic. As an extension, we may introduce *Symbolic Neutrosophic Modal Logic* and *Refined Symbolic Neutrosophic Modal Logic*, where the symbolic neutrosophic modal operators (and the symbolic neutrosophic accessibility relation) have qualitative values (labels) instead on numerical values (subsets of the interval [0, 1]).

Applications of neutrosophic modal logic are to neutrosophic modal metaphysics. Similarly to classical modal logic, there is a plethora of neutrosophic modal logics. Neutrosophic modal logics is governed by a set of neutrosophic axioms and neutrosophic rules. The neutrosophic accessibility relation has various interpretations, depending on the applications. Similarly, the notion of possible neutrosophic worlds has many interpretations, as part of possible neutrosophic semantics.

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