Neutrosophic $\mathcal{N}$-ideals in Ternary Semigroups

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Abstract

The objective of this paper is to extend the concept of neutrosophic $\mathcal{N}$-ideals in semigroups to ternary semigroups and investigate some of its properties. Moreover, consider characterizations of neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideals by using the notion of neutrosophic $\mathcal{N}$-products. Furthermore, we show that the homomorphic preimage and the onto homomorphic image of neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideals are also neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideals in ternary semigroups.

1 Introduction

The notion of ternary algebraic systems was first introduced by Lehmer [9] in 1932 who investigated certain ternary algebraic systems, called triplexes, which turned out to be commutative ternary groups. The notion of ternary semigroups was known to Banach who, by an example, verified that a ternary semigroup does not necessarily reduce to an ordinary semigroup. The ideal theory in ternary semigroups was studied by Siosn [15]. In 2010, Santiago and Bala [14] developed the theory of ternary semigroups.

Key words and phrases: Neutrosophic $\mathcal{N}$-structure, neutrosophic $\mathcal{N}$-ideal, neutrosophic $\mathcal{N}$-product.

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Zadeh [18] introduced the degree of membership truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [2] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache [16] introduced the degree of indeterminacy/neutrality (i) and defined the neutrosophic set on three components $(t, i, f) = (\text{truth}, \text{indeterminacy}, \text{falsehood})$.

These three functions are completely independent. Later, Smarandache [17] considered a more general platform which extends the concepts of the classic sets and fuzzy sets, intuitionistic fuzzy sets and interval intuitionistic fuzzy sets. In 2009, Jun et al. [5] introduced a new function, called a negative-valued function, and constructed $\mathcal{N}$-structures. Khan et al. [7] discussed neutrosophic $\mathcal{N}$-structures and their applications in semigroups. This structure was studied by many mathematicians (e.g., [1, 11, 6, 8]). In 2019, Elavarasan et al. [4] introduced the notion of neutrosophic $\mathcal{N}$-ideals in semigroups and investigated some of their properties. Recently, Rattana and Chinram [12, 13] extended the concept of neutrosophic $\mathcal{N}$-structures in $n$-ary groupoids and ternary semigroups.

In this paper, we investigate the extension of neutrosophic $\mathcal{N}$-ideals from semigroups to ternary semigroups and study some of their properties. Moreover, we consider characterizations of neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideals by using the concept of neutrosophic $\mathcal{N}$-products. Furthermore, we show that the homomorphic preimage and the onto homomorphic image of neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideals are also a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal in ternary semigroups.

## 2 Preliminaries

A nonempty set $X$ with a ternary operation $[ ] : X \times X \times X \to X$, written as $(x_1, x_2, x_3) \mapsto [x_1x_2x_3]$, is called a ternary semigroup [9] if it satisfies the following associative law holds:

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$$

for all $x_1, x_2, x_3, x_4, x_5 \in X$.

Let $(S, \cdot)$ be a semigroup. Then, we define the ternary operation $[ ]$ on $S$ by $[abc] = (ab)c$ for all $a, b, c \in S$. So, $(S, [ ])$ is a ternary semigroup. This shows that every semigroup is a ternary semigroup. Conversely, Banach showed that a ternary semigroup does not necessarily reduce to a semigroup.
For example, \( S = \{ -i, 0, i \} \) is a ternary semigroup under the multiplication over complex numbers, while \( S = \{ -i, 0, i \} \) is not a semigroup under complex number multiplication.

For any nonempty subsets \( A, B \) and \( C \) of a ternary semigroup \( X \), let \( [ABC] = \{ abc \mid a \in A, b \in B, c \in C \} \).

A nonempty subset \( A \) of a ternary semigroup \( X \) is called a ternary sub-semigroup of \( X \) if \( [AAA] \subseteq A \); a left ideal of \( X \) if \( [XAX] \subseteq A \); a lateral ideal of \( X \) if \( [AXA] \subseteq A \); a right ideal of \( X \) if \( [AXA] \subseteq A \); an ideal of \( X \) if \( A \) is a left, right and lateral ideal of \( X \), see [3].

Let \( \{ a_i \mid i \in \Lambda \} \) be a family of real numbers. We have

\[
\bigvee \{ a_i \mid i \in \Lambda \} := \begin{cases} 
\max \{ a_i \mid i \in \Lambda \} & \text{if } \Lambda \text{ is finite;} \\
\sup \{ a_i \mid i \in \Lambda \} & \text{otherwise,}
\end{cases}
\]

\[
\bigwedge \{ a_i \mid i \in \Lambda \} := \begin{cases} 
\min \{ a_i \mid i \in \Lambda \} & \text{if } \Lambda \text{ is finite;} \\
\inf \{ a_i \mid i \in \Lambda \} & \text{otherwise.}
\end{cases}
\]

For any two real numbers \( a \) and \( b \), we write \( a \lor b \) and \( a \land b \) instead of \( \lor \{ a, b \} \) and \( \land \{ a, b \} \), respectively.

We denote the family of all functions from a nonempty set \( X \) to \([-1, 0]\) by \( \mathcal{F}(X, [-1, 0]) \). An element of \( \mathcal{F}(X, [-1, 0]) \) is called a negative-valued function from \( X \) to \([-1, 0]\) (briefly, \( \mathcal{N} \)-function on \( X \)). An ordered pair \((X, f)\) of \( X \) and an \( \mathcal{N} \)-function \( f \) on \( X \) is called an \( \mathcal{N} \)-structure. A neutrosophic \( \mathcal{N} \)-structure over \( X \) [7] is defined to be the structure

\[
X_N := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{(T_N(x), I_N(x), F_N(x))} \mid x \in X \right\}
\]

where \( T_N, I_N \) and \( F_N \) are \( \mathcal{N} \)-functions on \( X \) which are called the negative truth membership function, the negative indeterminacy membership function and the negative falsity membership function on \( X \), respectively.

Note that every neutrosophic \( \mathcal{N} \)-structure \( X_N \) over \( X \) satisfies the condition: \(-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0 \) for all \( x \in X \).

Let \( X_N := \frac{X}{(T_N, I_N, F_N)} \) and \( X_M := \frac{X}{(T_M, I_M, F_M)} \) be neutrosophic \( \mathcal{N} \)-structures over \( X \).

(i) \( X_N \) is called a neutrosophic \( \mathcal{N} \)-substructure of \( X_M \), denoted by \( X_N \subseteq X_M \), if it satisfies:
\[ T_N(x) \geq T_M(x), I_N(x) \leq I_M(x), F_N(x) \geq F_M(x) \]

for all \( x \in X \). If \( X_N \subseteq X_M \) and \( X_M \subseteq X_N \), we say that \( X_N = X_M \).

(ii) The union of \( X_N \) and \( X_M \) is defined to be a neutrosophic \( \mathcal{N} \)-structure

\[
X_{\cup M} := \frac{X}{(T_{\cup M}, I_{\cup M}, F_{\cup M})}
\]

where \( T_{\cup M}(x) = \min{T_N(x), T_M(x)} \), \( I_{\cup M}(x) = \max{I_N(x), I_M(x)} \) and \( F_{\cup M}(x) = \min{F_N(x), F_M(x)} \) for all \( x \in X \).

(iii) The intersection of \( X_N \) and \( X_M \) is defined to be a neutrosophic \( \mathcal{N} \)-structure

\[
X_{\cap M} := \frac{X}{(T_{\cap M}, I_{\cap M}, F_{\cap M})}
\]

where \( T_{\cap M}(x) = \max{T_N(x), T_M(x)} \), \( I_{\cap M}(x) = \min{I_N(x), I_M(x)} \) and \( F_{\cap M}(x) = \max{F_N(x), F_M(x)} \) for all \( x \in X \).

**Example 2.1.** Let \( X = \{x, y, z\} \) be a set and let \( X_N \) and \( X_M \) be the neutrosophic \( \mathcal{N} \)-structures over \( X \) which are given by

\[
X_N = \left\{ \begin{array}{ccc}
x & y & z \\
(-0.3, -0.5, -0.9) & (-0.8, -0.2, -0.1) & (-0.7, -0.4, -0.5) \\
\end{array} \right\},
\]

\[
X_M = \left\{ \begin{array}{ccc}
x & y & z \\
(-0.5, -0.3, -0.7) & (-0.1, -0.4, -0.8) & (-0.1, -0.5, -0.2) \\
\end{array} \right\}.
\]

Then, \( X_N \) and \( X_M \) are neutrosophic \( \mathcal{N} \)-structures over \( X \). Next, the union and intersection of \( X_N \) and \( X_M \) are defined as follows:

\[
X_{\cup M} = \left\{ \begin{array}{ccc}
x & y & z \\
(-0.5, -0.3, -0.9) & (-0.8, -0.2, -0.8) & (-0.7, -0.4, -0.5) \\
\end{array} \right\},
\]

\[
X_{\cap M} = \left\{ \begin{array}{ccc}
x & y & z \\
(-0.3, -0.5, -0.7) & (-0.1, -0.4, -0.1) & (-0.1, -0.5, -0.2) \\
\end{array} \right\}.
\]

For a subset \( A \) of a nonempty \( X \), consider the neutrosophic \( \mathcal{N} \)-structure over \( X \)

\[
\chi_A(X_N) = \frac{X}{(\chi_A(T)_N, \chi_A(I)_N, \chi_A(F)_N)},
\]

where
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\[ \chi_A(T)_N : X \to [-1,0], x \mapsto \begin{cases} -1 & \text{if } x \in A; \\ 0 & \text{otherwise}, \end{cases} \]

\[ \chi_A(I)_N : X \to [-1,0], x \mapsto \begin{cases} 0 & \text{if } x \in A; \\ -1 & \text{otherwise}, \end{cases} \]

\[ \chi_A(F)_N : X \to [-1,0], x \mapsto \begin{cases} -1 & \text{if } x \in A; \\ 0 & \text{otherwise}, \end{cases} \]

which is called the characteristic neutrosophic $\mathcal{N}$-structure of $A$.

Let $X_N$ be a neutrosophic $\mathcal{N}$-structure over $X$ and let $\alpha, \beta, \gamma \in [-1,0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets:

\[ T^\alpha_N := \{ x \in X \mid T_N(x) \leq \alpha \}; \]

\[ I^\beta_N := \{ x \in X \mid I_N(x) \geq \beta \}; \]

\[ F^\gamma_N := \{ x \in X \mid F_N(x) \leq \gamma \}. \]

The set

\[ X_N(\alpha, \beta, \gamma) := \{ x \in X \mid T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma \} \]

is called a $(\alpha, \beta, \gamma)$-level set of $X_N$. Note that $X_N(\alpha, \beta, \gamma) = T^\alpha_N \cap I^\beta_N \cap F^\gamma_N$.

3 Main Results

In this section, we apply the concept of neutrosophic $\mathcal{N}$-ideals in semigroups to define the notion of neutrosophic $\mathcal{N}$-ideals in ternary semigroups and study some of its basic properties. Throughout this paper, we assume that $X$ is a ternary semigroup unless specified otherwise.

**Definition 3.1.** [13] Let $X_N$ be a neutrosophic $\mathcal{N}$-structure over $X$. Then, $X_N$ is said to be a neutrosophic $\mathcal{N}$-ternary subsemigroup of $X$ if it satisfies the following conditions:

(i) $T_N([xyz]) \leq \vee \{T_N(x), T_N(y), T_N(z)\}$;

(ii) $I_N([xyz]) \geq \wedge \{I_N(x), I_N(y), I_N(z)\}$;

(iii) $F_N([xyz]) \leq \vee \{F_N(x), F_N(y), F_N(z)\}$,

for all $x, y, z \in X$. 
Definition 3.2. A neutrosophic $\mathcal{N}$-structure $X_N$ over $X$ is called a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal of $X$ if it satisfies the following conditions:

(i) $T_N([xyz]) \leq T_N(z)$ (resp., $T_N([xyz]) \leq T_N(y)$, $T_N([xyz]) \leq T_N(x)$);

(ii) $I_N([xyz]) \geq I_N(z)$ (resp., $I_N([xyz]) \geq I_N(y)$, $I_N([xyz]) \geq I_N(x)$);

(iii) $F_N([xyz]) \leq F_N(z)$ (resp., $F_N([xyz]) \leq F_N(y)$, $F_N([xyz]) \leq F_N(x)$),

for all $x, y, z \in X$.

If $X_N$ is a neutrosophic $\mathcal{N}$-left, $\mathcal{N}$-lateral and $\mathcal{N}$-right ideal of $X$, then $X_N$ is called a neutrosophic $\mathcal{N}$-ideal of $X$.

Note that every neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal of a ternary semigroup is a neutrosophic $\mathcal{N}$-ternary subsemigroup, but the neutrosophic $\mathcal{N}$-ternary subsemigroup need not be a neutrosophic $\mathcal{N}$-left ideal or a neutrosophic $\mathcal{N}$-lateral ideal or a neutrosophic $\mathcal{N}$-right ideal as the following example shows.

Example 3.3. Let $X = \{a, b, c, d\}$ and define the ternary operation $[\ ]$ on $X$ as follows:

\[
\begin{array}{cccc|cccc|cccc}
[\ ] & a & b & c & d & a & b & c & d & a & b & c & d \\
\hline
\text{aa} & a & a & a & d & \text{ba} & b & b & b & d & \text{ca} & a & a & a & d \\
\text{ab} & a & a & a & d & \text{bb} & b & b & b & d & \text{cb} & a & a & a & d \\
\text{ac} & a & a & a & d & \text{bc} & b & b & b & d & \text{cc} & a & a & a & d \\
\text{ad} & d & d & d & d & \text{bd} & d & d & d & d & \text{cd} & d & d & d & d \\
\text{da} & d & d & d & d & \text{db} & d & d & d & d & \\
\text{dc} & d & d & d & d & \text{dd} & d & d & d & d \\
\end{array}
\]

Then, $(X, [\ ])$ is a ternary semigroup [10]. Define a neutrosophic $\mathcal{N}$-structure $X_N$ over $X$ as follows:

\[
T_N(a) = -0.6, \quad I_N(a) = -0.1, \quad F_N(a) = -0.9; \\
T_N(b) = -0.6, \quad I_N(b) = -0.1, \quad F_N(b) = -0.9; \\
T_N(c) = -0.4, \quad I_N(c) = -0.3, \quad F_N(c) = -0.8; \\
T_N(d) = -0.2, \quad I_N(d) = -0.7, \quad F_N(d) = -0.6.
\]
By routine calculations, \( X_N := \frac{X}{(T_N, I_N, F_N)} \) is a neutrosophic ternary \( \mathcal{N} \)-subsemigroup of \( X \), but it is not a neutrosophic \( \mathcal{N} \)-left ideal, because
\[
T_N([bda]) = -0.2 \nleq -0.6 = T_N(a),
I_N([bda]) = -0.7 \nleq -0.1 = I_N(a),
F_N([bda]) = -0.6 \nleq -0.9 = F_N(a).
\]

**Example 3.4.** Let \( X = \{a, b, c, d\} \) and define the ternary operation \([ \ ]\) on \( X \) as follows:

\[
\begin{array}{cccc}
[&] & a & b & c & d \\
aa & a & b & a & d \\
ab & a & b & a & d \\
ac & a & b & a & d \\
ad & d & d & d & d \\
ba & b & a & a & d \\
bb & b & a & a & d \\
bc & b & a & a & d \\
b & b & a & b & d \\
ca & c & a & a & d \\
cb & c & a & a & d \\
c & c & a & a & d \\
c & b & b & d & d \\
da & d & d & d & d \\
db & d & d & d & d \\
dc & d & d & d & d \\
dd & d & d & d & d
\end{array}
\]

Then, \((X,[ \ ]\)) is a ternary semigroup [10]. Now, define a neutrosophic \( \mathcal{N} \)-structure \( X_N \) over \( X \) as follows:
\[
T_N(a) = -0.9, \quad I_N(a) = -0.2, \quad F_N(a) = -0.8;
T_N(b) = -0.5, \quad I_N(b) = -0.4, \quad F_N(b) = -0.6;
T_N(c) = -0.3, \quad I_N(c) = -0.7, \quad F_N(c) = -0.2;
T_N(d) = -0.9, \quad I_N(d) = -0.2, \quad F_N(d) = -0.8.
\]

By routine computations, \( X_N := \frac{X}{(T_N, I_N, F_N)} \) is a neutrosophic \( \mathcal{N} \)-ideal of \( X \), but it is not a neutrosophic \( \mathcal{N} \)-lateral ideal, because
\[
T_N([bab]) = -0.5 \nleq -0.9 = T_N(a),
I_N([bab]) = -0.4 \nleq -0.2 = I_N(a),
F_N([bab]) = -0.6 \nleq -0.8 = F_N(a).
\]

In addition, \( X_N \) is also not a neutrosophic \( \mathcal{N} \)-right ideal of \( X \), because
\[
T_N([acb]) = -0.5 \nleq -0.9 = T_N(a),
I_N([acb]) = -0.4 \nleq -0.2 = I_N(a),
F_N([acb]) = -0.6 \nleq -0.8 = F_N(a).
\]
Throughout this section, we will prove the following theorems for neutrosophic $N$-left ideals. For neutrosophic $N$-lateral ideals and neutrosophic $N$-right ideals, one can prove similarly.

**Theorem 3.5.** Let $X_N$ be a neutrosophic $N$-structure over $X$ and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $X_N$ is a neutrosophic $N$-left (resp., $N$-lateral, $N$-right) ideal of $X$, then the $(\alpha, \beta, \gamma)$-level set of $X_N$ is a left (resp., lateral, right) ideal of $X$ whenever it is nonempty.

**Proof.** Assume that $X_N$ is a neutrosophic $N$-left ideal of $X$ and $X_N(\alpha, \beta, \gamma) \neq \emptyset$ for $\alpha, \beta, \gamma \in [-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Let $x, y, z \in X$ and $a \in X_N(\alpha, \beta, \gamma)$. Then, $T_N(a) \leq \alpha, I_N(a) \geq \beta$ and $F_N(a) \leq \gamma$. It follows that $T_N([xya]) \leq T_N(a) \leq \alpha, I_N([xya]) \geq I_N(a) \geq \beta$ and $F_N([xya]) \leq F_N(a) \leq \gamma$. Hence, $[xya] \in X_N(\alpha, \beta, \gamma)$. Therefore, $X_N(\alpha, \beta, \gamma)$ is a left ideal of $X$. \qed

**Theorem 3.6.** Let $X_N$ be a neutrosophic $N$-structure over $X$ and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $T_N^\alpha, I_N^\beta$ and $F_N^\gamma$ are left (resp., lateral, right) ideals of $X$, then $X_N$ is a neutrosophic $N$-left (resp., $N$-lateral, $N$-right) ideal of $X$.

**Proof.** Assume that $T_N^\alpha, I_N^\beta$ and $F_N^\gamma$ are left ideals of $X$. Suppose that $T_N([abc]) > T_N(c)$ for some $a, b, c \in X$. Then, $T_N([abc]) > t_\alpha \geq T_N(c)$ for some $t_\alpha \in [-1, 0]$. Hence, $c \in T_N^{t_\alpha}$, but $[abc] \not\in T_N^{t_\alpha}$, which is a contradiction. Thus, $T_N([xyz]) \leq T_N(z)$ for all $x, y, z \in X$. If $I_N([abc]) < I_N(c)$ for some $a, b, c \in X$, then $I_N([abc]) < t_\beta \leq I_N(c)$ for some $t_\beta \in (-1, 0]$. Thus, $c \in I_N^{t_\beta}$, but $[abc] \not\in I_N^{t_\beta}$. This is a contradiction. So $I_N([xyz]) \geq I_N(z)$ for some $x, y, z \in X$. Now, suppose that $F_N([abc]) > F_N(c)$ for some $a, b, c \in X$. Then, $F_N([abc]) > t_\gamma \geq F_N(c)$ for some $t_\gamma \in [-1, 0)$. This implies that $c \in F_N^{t_\gamma}$, but $[abc] \not\in F_N^{t_\gamma}$, which is a contradiction. Hence, $F_N([xyz]) \leq F_N(z)$ for all $x, y, z \in X$. Therefore, $X_N$ is a neutrosophic $N$-left ideal of $X$. \qed

**Theorem 3.7.** The intersection of two neutrosophic $N$-left (resp., $N$-lateral, $N$-right) ideals of $X$ is also a neutrosophic $N$-left (resp., $N$-lateral, $N$-right) ideal.
Proof. Let $X_N := \frac{X}{(T_N, I_N, F_N)}$ and $X_M := \frac{X}{(T_M, I_M, F_M)}$ be two neutrosophic $\mathcal{N}$-left ideals of $X$. For every $x, y, z \in X$, we have

$$T_{N\cap M}([xyz]) = T_N([xyz]) \vee T_M([xyz]) \leq T_N(z) \vee T_N(z) = T_{N\cap M}(z),$$

$$I_{N\cap M}([xyz]) = I_N([xyz]) \wedge I_M([xyz]) \geq I_N(z) \wedge I_N(z) = I_{N\cap M}(z),$$

$$F_{N\cap M}([xyz]) = F_N([xyz]) \vee F_M([xyz]) \leq F_N(z) \vee F_N(z) = F_{N\cap M}(z).$$

Consequently, $X_{N\cap M}$ is a neutrosophic $\mathcal{N}$-left ideal of $X$. 

\[ \square \]

Corollary 3.8. If \{\(X_i\mid i \in \Lambda\)\} be a family of neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideals of $X$, then $X_{\bigcap i \in \Lambda} \cap N_i$ is also a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal of $X$.

Let $X_N := \frac{X}{(T_N, I_N, F_N)}$, $X_M := \frac{X}{(T_M, I_M, F_M)}$ and $X_L := \frac{X}{(T_L, I_L, F_L)}$ be neutrosophic $\mathcal{N}$-structures over $X$. The neutrosophic $\mathcal{N}$-product [13] of $X_N, X_M$ and $X_L$ is defined by

$$X_N \circ X_M \circ X_L := \frac{X}{(T_{N\circ M\circ L}, I_{N\circ M\circ L}, F_{N\circ M\circ L})}$$

$$= \left\{ \frac{x}{(T_{N\circ M\circ L}(x), I_{N\circ M\circ L}(x), F_{N\circ M\circ L}(x))} \mid x \in X \right\}$$

where

$$T_{N\circ M\circ L}(x) = \begin{cases} \bigwedge_{x=[pqr]} \{T_N(p) \vee T_M(q) \vee T_L(r)\} & \text{if } \exists p, q, r \in X \text{ such that } x = [pqr] \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{N\circ M\circ L}(x) = \begin{cases} \bigvee_{x=[pqr]} \{I_N(p) \wedge I_M(q) \wedge I_L(r)\} & \text{if } \exists p, q, r \in X \text{ such that } x = [pqr] \\ -1 & \text{otherwise,} \end{cases}$$

$$F_{N\circ M\circ L}(x) = \begin{cases} \bigwedge_{x=[pqr]} \{F_N(p) \vee F_M(q) \vee F_L(r)\} & \text{if } \exists p, q, r \in X \text{ such that } x = [pqr] \\ 0 & \text{otherwise.} \end{cases}$$

For any $x \in X$, the element $\frac{X}{(T_{N\circ M\circ L}, I_{N\circ M\circ L}, F_{N\circ M\circ L})}$ is simply denoted by

$$(X_N \circ X_M \circ X_L)(x) := (T_{N\circ M\circ L}(x), I_{N\circ M\circ L}(x), F_{N\circ M\circ L}(x)).$$
Theorem 3.9. Let $A$ be a nonempty subset of $X$. Then the following statements are equivalent:

(i) $A$ is a left (resp., lateral, right) ideal of $X$;

(ii) the characteristic neutrosophic $\mathcal{N}$-structure $\chi_A(X_N)$ over $X$ is a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal of $X$.

Proof. (i) $\Rightarrow$ (ii) Assume that $A$ is a left ideal of $X$. Let $x, y, z \in X$. If $z \notin A$, then $\chi_A(T)_N([xyz]) \leq 0 = \chi_A(T)_N(z)$; $\chi_A(I)_N([xyz]) \geq -1 = \chi_A(I)_N(z)$ and $\chi_A(F)_N([xyz]) \leq 0 = \chi_A(F)_N(z)$. On the other hand, suppose that $z \in A$. Then, $[xyz] \in A$. It follows that $\chi_A(T)_N([xyz]) = -1 = \chi_A(T)_N(z)$, $\chi_A(I)_N([xyz]) = 0 = \chi_A(T)_N(z)$ and $\chi_A(F)_N([xyz]) = -1 = \chi_A(F)_N(z)$. Therefore, $\chi_A(X_N)$ is a neutrosophic $\mathcal{N}$-left ideal of $X$.

(ii) $\Rightarrow$ (i) Assume that $\chi_A(X_N)$ is a neutrosophic $\mathcal{N}$-left ideal of $X$. Let $x, y \in X$ and $a \in A$. Then, $\chi_A(T)_N([xya]) \leq \chi_A(T)_N(a) = -1$, $\chi_A(I)_N([xya]) \geq \chi_A(I)_N(a) = 0$, $\chi_A(F)_N([xya]) \leq \chi_A(F)_N(a) = -1$. Hence, $\chi_A(T)_N([xya]) = -1$, $\chi_A(I)_N([xya]) = 0$ and $\chi_A(F)_N([xya]) = -1$. This implies that $[xya] \in A$. Consequently, $A$ is a left ideal of $X$. □

Theorem 3.10. Let $\chi_A(X_N), \chi_B(X_N)$ and $\chi_C(X_N)$ be characteristic neutrosophic $\mathcal{N}$-structures over $X$ for any subsets $A, B$ and $C$ of $X$. Then the following statements hold:

(i) $\chi_A(X_N) \cup \chi_B(X_N) = \chi_{A \cup B}(X_N)$;

(ii) $\chi_A(X_N) \cap \chi_B(X_N) = \chi_{A \cap B}(X_N)$;

(iii) $\chi_A(X_N) \odot \chi_B(X_N) \odot \chi_C(X_N) = \chi_{[ABC]}(X_N)$.

Proof. (i) Let $x \in X$. If $x \in A \cup B$, then $x \in A$ or $x \in B$. Thus,

$(\chi_A(T)_N \cup \chi_B(T)_N)(x) = \chi_A(T)_N(x) \land \chi_B(T)_N(x) = -1 = \chi_{A \cup B}(T)_N(x),$

$(\chi_A(I)_N \cup \chi_B(I)_N)(x) = \chi_A(I)_N(x) \lor \chi_B(I)_N(x) = 0 = \chi_{A \cup B}(I)_N(x),$

$(\chi_A(F)_N \cup \chi_B(F)_N)(x) = \chi_A(F)_N(x) \land \chi_B(F)_N(x) = -1 = \chi_{A \cup B}(F)_N(x).$

So, $\chi_A(X_N) \cup \chi_B(X_N) = \chi_{A \cup B}(X_N)$. If $x \notin A \cup B$, then

$(\chi_A(T)_N \cup \chi_B(T)_N)(x) = \chi_A(T)_N(x) \land \chi_B(T)_N(x) = 0 = \chi_{A \cup B}(T)_N(x),$

$(\chi_A(I)_N \cup \chi_B(I)_N)(x) = \chi_A(I)_N(x) \lor \chi_B(I)_N(x) = -1 = \chi_{A \cup B}(I)_N(x),$

$(\chi_A(F)_N \cup \chi_B(F)_N)(x) = \chi_A(F)_N(x) \land \chi_B(F)_N(x) = 0 = \chi_{A \cup B}(F)_N(x).$
Hence, $\chi_A(X_N) \cup \chi_B(X_N) = \chi_{A \cup B}(X_N)$.

(ii) The proof is similar to (i).

(iii) Let $x \in X$. If $x \not\in [ABC]$, then

$$
(\chi_A(T)_N \odot \chi_B(T)_N \odot \chi_C(T)_N)(x) = 0 = \chi_{[ABC]}(T)_N(x),
$$

$$
(\chi_A(I)_N \odot \chi_B(I)_N \odot \chi_C(I)_N)(x) = -1 = \chi_{[ABC]}(I)_N(x),
$$

$$
(\chi_A(F)_N \odot \chi_B(F)_N \odot \chi_C(F)_N)(x) = 0 = \chi_{[ABC]}(F)_N(x).
$$

Thus, $\chi_A(X_N) \odot \chi_B(X_N) \odot \chi_C(X_N) = \chi_{[ABC]}(X_N)$. If $x \in [ABC]$, then $x = [abc]$ for some $a \in A, b \in B$ and $c \in C$. It follows that

$$
(\chi_A(T)_N \odot \chi_B(T)_N \odot \chi_C(T)_N)(x) = \bigwedge_{x=[pqr]} \{ \chi_A(T)_N(p) \lor \chi_B(T)_N(q) \lor \chi_C(T)_N(r) \}
$$

$$
\leq \chi_A(T)_N(a) \lor \chi_B(T)_N(b) \lor \chi_C(T)_N(c)
$$

$$
= -1 = \chi_{[ABC]}(T)_N(x),
$$

$$
(\chi_A(I)_N \odot \chi_B(I)_N \odot \chi_C(I)_N)(x) = \bigvee_{x=[pqr]} \{ \chi_A(I)_N(p) \land \chi_B(I)_N(q) \land \chi_C(I)_N(r) \}
$$

$$
\geq \chi_A(I)_N(a) \land \chi_B(I)_N(b) \land \chi_C(I)_N(c)
$$

$$
= 0 = \chi_{[ABC]}(I)_N(x),
$$

$$
(\chi_A(F)_N \odot \chi_B(F)_N \odot \chi_C(F)_N)(x) = \bigwedge_{x=[pqr]} \{ \chi_A(F)_N(p) \lor \chi_B(F)_N(q) \lor \chi_C(F)_N(r) \}
$$

$$
\leq \chi_A(F)_N(a) \lor \chi_B(F)_N(b) \lor \chi_C(F)_N(c)
$$

$$
= -1 = \chi_{[ABC]}(F)_N(x).
$$

Therefore, $\chi_A(X_N) \odot \chi_B(X_N) \odot \chi_C(X_N) = \chi_{[ABC]}(X_N)$.

\[ \square \]

**Theorem 3.11.** Let $X_L$ be a neutrosophic $N$-structure over $X$. Then $X_L$ is a neutrosophic $N$-left ideal of $X$ if and only if $X_N \odot X_M \odot X_L \subseteq X_L$ for every neutrosophic $N$-structures over $X$.

**Proof.** Assume that $X_L$ is a neutrosophic $N$-left ideal of $X$. Let $X_N$ and $X_M$ be neutrosophic $N$-structures over $X$. Let $x \in X$. Obviously, $X_N \odot X_M \odot X_L \subseteq X_L$ for all $a, b, c \in X$ such that $x \neq [abc]$. Suppose that there exist $a, b, c \in X$ such that $x = [abc]$. We obtain

$$
T_L(x) = T_L([abc]) \leq T_L(c) \leq T_N(a) \lor T_M(b) \lor T_L(c),
$$

$$
I_L(x) = I_L([abc]) \geq I_L(c) \geq I_N(a) \land I_M(b) \land I_L(c),
$$

$$
F_L(x) = F_L([abc]) \leq F_L(c) \leq F_N(a) \lor F_M(b) \lor F_L(c).
$$
Theorem 3.13. Let \( N \) be a neutrosophic \( N \)-structure over \( X \). Then, \( N \) is a neutrosophic \( N \)-right ideal of \( X \) if and only if \( X_L \otimes X_M \otimes X_N \subseteq X_M \) for every neutrosophic \( N \)-structures \( X_L \) and \( X_M \) over \( X \).
Theorem 3.14. Let $X_A, X_N$ and $X_M$ be neutrosophic $\mathcal{N}$-structures over $X$. If $X_A$ is a neutrosophic $\mathcal{N}$-left ideal of $X$, then $X_A \odot X_N \odot X_M$ is also a neutrosophic $\mathcal{N}$-left ideal of $X$.

Proof. Assume that $X_A$ is a neutrosophic $\mathcal{N}$-left ideal of $X$. Let $x, y, z \in X$. If there exist $a, b, c \in X$ such that $z = [abc]$, then $[xyz] = [xy[abc]] = [[xya]bc]$. Then,

$$T_{A\odot N\odot M}(z) = \bigwedge_{z=[abc]} \{T_A(a) \lor T_N(b) \lor T_M(c)\} \geq \bigwedge_{[xyz]=[[xya]bc]} \{T_A([xya]) \lor T_N(b) \lor T_M(c)\}$$

$$= \bigwedge_{[xyz]=[[xya]bc]} \{T_A(t) \lor T_N(b) \lor T_M(c)\} = T_{A\odot N\odot M}([xyz]),$$

$$I_{A\odot N\odot M}(z) = \bigvee_{z=[abc]} \{I_A(a) \land I_N(b) \land I_M(c)\} \leq \bigvee_{[xyz]=[[xya]bc]} \{I_A([xya]) \land I_N(b) \land I_M(c)\}$$

$$= \bigvee_{[xyz]=[[xya]bc]} \{I_A(t) \land I_N(b) \land I_M(c)\} = I_{A\odot N\odot M}([xyz]),$$

$$F_{A\odot N\odot M}(z) = \bigwedge_{z=[abc]} \{F_A(a) \lor F_N(b) \lor F_M(c)\} \geq \bigwedge_{[xyz]=[[xya]bc]} \{F_A([xya]) \lor F_N(b) \lor F_M(c)\}$$

$$= \bigwedge_{[xyz]=[[xya]bc]} \{F_A(t) \lor F_N(b) \lor F_M(c)\} = F_{A\odot N\odot M}([xyz]),$$

Therefore, $X_A \odot X_N \odot X_M$ is a neutrosophic $\mathcal{N}$-left ideal of $X$. $\square$

Similarly, we have the following theorem:

Theorem 3.15. Let $X_A, X_N$ and $X_M$ be neutrosophic $\mathcal{N}$-structures over $X$. If $X_A$ is a neutrosophic $\mathcal{N}$-right ideal of $X$, then $X_N \odot X_M \odot X_A$ is also a neutrosophic $\mathcal{N}$-right ideal of $X$.

Let $f : X \rightarrow Y$ be a function of sets. If $Y_M := \frac{Y}{(T_M, I_M, F_M)}$ is a neutrosophic $\mathcal{N}$-structure over $Y$, the preimage [13] of $Y_M$ under $f$ is defined to be a neutrosophic $\mathcal{N}$-structure

$$f^{-1}(Y_M) := \frac{X}{(f^{-1}(T_M), f^{-1}(I_M), f^{-1}(F_M))}$$

over $X$ where $f^{-1}(T_M)(x) = T_M(f(x))$, $f^{-1}(I_M)(x) = I_M(f(x))$ and $f^{-1}(F_M)(x) = F_M(f(x))$ for all $x \in X$. 

**Theorem 3.16.** Let \( f : X \to Y \) be a homomorphism of ternary semigroups. If \( Y_M := (T_M, I_M, F_M) \) is a neutrosophic \( N \)-left (resp., \( N \)-lateral, \( N \)-right) ideal of \( Y \), then the preimage of \( Y_M \) under \( f \) is a neutrosophic \( N \)-left (resp., \( N \)-lateral, \( N \)-right) ideal of \( X \).

**Proof.** Assume that \( f^{-1}(Y_M) := (f^{-1}(T_M), f^{-1}(I_M), f^{-1}(F_M)) \) is the preimage of \( Y_M \) under \( f \). Let \( x, y, z \in X \). Then,

\[
\begin{align*}
  f^{-1}(T_M)([xyz]) &= T_M(f([xyz])) = T_M([f(x)f(y)f(z)]) \leq T_M(f(z)) = f^{-1}(T_M)(z), \\
  f^{-1}(I_M)([xyz]) &= I_M(f([xyz])) = I_M([f(x)f(y)f(z)]) \geq I_M(f(z)) = f^{-1}(I_M)(z), \\
  f^{-1}(F_M)([xyz]) &= F_M(f([xyz])) = F_M([f(x)f(y)f(z)]) \leq F_M(f(z)) = f^{-1}(F_M)(z).
\end{align*}
\]

Hence, \( f^{-1}(Y_M) \) is a neutrosophic \( N \)-left ideal of \( X \). \( \square \)

Let \( f : X \to Y \) be an onto function of sets. If \( X_N := (T_N, I_N, F_N) \) is a neutrosophic \( N \)-structure over \( X \), then the image \([13]\) of \( X_N \) under \( f \) is defined to be a neutrosophic \( N \)-structure

\[
f(X_N) := \frac{Y}{(f(T_N), f(I_N), f(F_N))}
\]

over \( Y \) where

\[
\begin{align*}
  f(T_N)(y) &= \bigwedge_{x \in f^{-1}(y)} T_N(x), \quad f(I_N)(y) = \bigvee_{x \in f^{-1}(y)} I_N(x), \\
  f(F_N)(y) &= \bigwedge_{x \in f^{-1}(y)} F_N(x).
\end{align*}
\]

**Theorem 3.17.** For an onto homomorphism \( f : X \to Y \) of ternary semigroups, let \( X_N := (T_N, I_N, F_N) \) be a neutrosophic \( N \)-structure over \( X \) such that for any nonempty subset \( A \) of \( X \) there exists \( x_0 \in A \) such that \( T_N(x_0) = \bigwedge_{z \in A} T_N(z), I_N(x_0) = \bigvee_{z \in A} I_N(z) \) and \( F_N(x_0) = \bigwedge_{z \in A} F_N(z) \). If \( X_N \) is a neutrosophic \( N \)-left (resp., \( N \)-lateral, \( N \)-right) ideal of \( X \), then the image of \( X_N \) under \( f \) is a neutrosophic \( N \)-left (resp., \( N \)-lateral, \( N \)-right) ideal of \( Y \).

**Proof.** Assume that \( f(X_N) := \frac{X}{(f(T_N), f(I_N), f(F_N))} \) is the image of \( X_N \) under \( f \). Let \( a, b, c \in Y \). Then, \( f^{-1}(a) \neq \emptyset, f^{-1}(b) \neq \emptyset \) and \( f^{-1}(c) \neq \emptyset \) in \( X \).
Thus, there exist $x_a \in f^{-1}(a), x_b \in f^{-1}(b)$ and $x_c \in f^{-1}(c)$ such that

\[
\begin{align*}
T_N(x_a) &= \bigwedge_{z \in f^{-1}(a)} T_N(z), \quad I_N(x_a) = \bigvee_{z \in f^{-1}(a)} I_N(z), \quad F_N(x_a) = \bigwedge_{z \in f^{-1}(a)} F_N(z), \\
T_N(x_b) &= \bigwedge_{z \in f^{-1}(b)} T_N(z), \quad I_N(x_b) = \bigvee_{z \in f^{-1}(b)} I_N(z), \quad F_N(x_b) = \bigwedge_{z \in f^{-1}(b)} F_N(z), \\
T_N(x_c) &= \bigwedge_{z \in f^{-1}(c)} T_N(z), \quad I_N(x_c) = \bigvee_{z \in f^{-1}(c)} I_N(z), \quad F_N(x_c) = \bigwedge_{z \in f^{-1}(c)} F_N(z).
\end{align*}
\]

It turns out that

\[
\begin{align*}
f(T_N)([abc]) &= \bigwedge_{x \in f^{-1}([abc])} T_N(x) \leq T_N([x_ax_bx_c]) \leq T_N(x_c) = \bigwedge_{z \in f^{-1}(c)} T_N(z) = f(T_N)(c), \\
f(I_N)([abc]) &= \bigvee_{x \in f^{-1}([abc])} I_N(x) \geq I_N([x_ax_bx_c]) \geq I_N(x_c) = \bigvee_{z \in f^{-1}(c)} I_N(z) = f(I_N)(c), \\
f(F_N)([abc]) &= \bigwedge_{x \in f^{-1}([abc])} F_N(x) \leq F_N([x_ax_bx_c]) \leq F_N(x_c) = \bigwedge_{z \in f^{-1}(c)} F_N(z) = f(F_N)(c).
\end{align*}
\]

Therefore, $f(X_N)$ is a neutrosophic $\mathcal{N}$-left ideal of $Y$.

4 Conclusion

In this paper, we have introduced the concept of neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideals in ternary semigroups and investigated several their properties. We have also discussed characterizations of neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideals by using the notion of neutrosophic $\mathcal{N}$-products. Finally, we have shown that the homomorphic preimage and the onto homomorphic image of a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal are also a neutrosophic $\mathcal{N}$-left (resp., $\mathcal{N}$-lateral, $\mathcal{N}$-right) ideal in ternary semigroups. In our future study, we will define the concept of neutrosophic $\mathcal{N}$-bi-ideals in ternary semigroups and investigate their properties.

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References


