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Article

Neutrosophic Quadruple BCI-Positive Implicative Ideals

Young Bae Jun [1], Seok-Zun Song [2,*] and Seon Jeong Kim [3]

1 Department of Mathematics Education, Gyeongsang National University, Jinju 52828, Korea; skywine@gmail.com
2 Department of Mathematics, Jeju National University, Jeju 63243, Korea
3 Department of Mathematics, Natural Science of College, Gyeongsang National University, Jinju 52828, Korea; skim@gnu.ac.kr
* Correspondence: szsong@jejunu.ac.kr

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Abstract: By considering an entry (i.e., a number, an idea, an object, etc.) which is represented by a known part (a) and an unknown part (bT, cI, dF) where T, I, F have their usual neutrosophic logic meanings and a, b, c, d are real or complex numbers, Smarandache introduced the concept of neutrosophic quadruple numbers. Using the concept of neutrosophic quadruple numbers based on a set, Jun et al. constructed neutrosophic quadruple BCK/BCI-algebras and implicative neutrosophic quadruple BCK-algebras. The notion of a neutrosophic quadruple BCI-positive implicative ideal is introduced, and several properties are dealt with in this article. We establish the relationship between neutrosophic quadruple ideal and neutrosophic quadruple BCI-positive implicative ideal. Given nonempty subsets I and J of a BCI-algebra, conditions for the neutrosophic quadruple (I, J)-set to be a neutrosophic quadruple BCI-positive implicative ideal are provided.

Keywords: neutrosophic quadruple BCK/BCI-number; neutrosophic quadruple BCK/BCI-algebra; neutrosophic quadruple (BCI-positive implicative) ideal

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1. Introduction

A BCK/BCI-algebra is a class of logical algebras introduced by K. Iséki (see [1,2]) and was extensively investigated by several researchers. Neutrosophic algebraic structures in BCK/BCI-algebras are discussed in the papers [3–10]. Smarandache introduced the notion of neutrosophic sets with wide applications in sciences (see [11–13]), which is a more general stage to extend the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. Smarandache [14] introduced the concept of neutrosophic quadruple numbers by considering an entry (i.e., a number, an idea, an object, etc.) which is represented by a known part (a) and an unknown part (bT, cI, dF), where T, I, F have their usual neutrosophic logic meanings and a, b, c, d are real or complex numbers. Using the notion of neutrosophic quadruple numbers based on a set, Jun et al. [15] constructed neutrosophic quadruple BCK/BCI-algebras and implicative neutrosophic quadruple BCK-algebras (see also [16]).

In this paper, we introduce the notion of a neutrosophic quadruple BCI-positive implicative ideal, and investigate several properties. We consider relations between neutrosophic quadruple ideal and neutrosophic quadruple BCI-positive implicative ideal. Given nonempty subsets I and J of a BCI-algebra U, we provide conditions for the neutrosophic quadruple (I, J)-set to be a neutrosophic quadruple BCI-positive implicative ideal.

2. Preliminaries

A *BCI-algebra* is a set U with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall p, q, r \in U)\ (((p * q) * (p * r)) * (r * q) = 0)$,
(II) $(\forall p, q \in U)\ ((p * (p * q)) * q = 0)$,
(III) $(\forall p \in U)\ (p * p = 0)$,
(IV) $(\forall p, q \in U)\ (p * q = 0,\ q * p = 0\ \Rightarrow\ p = q)$.

If a BCI-algebra U satisfies the following identity:

(V) $(\forall p \in U)\ (0 * p = 0)$,

then U is called a *BCK-algebra*. In a BCK/BCI-algebra U, the following conditions are valid.

$$(\forall p \in U)\ (p * 0 = p)\,, \tag{1}$$
$$(\forall p, q, r \in U)\ (p \leq q\ \Rightarrow\ p * r \leq q * r,\ r * q \leq r * p)\,, \tag{2}$$
$$(\forall p, q, r \in U)\ ((p * q) * r = (p * r) * q)\,, \tag{3}$$
$$(\forall p, q, r \in U)\ ((p * r) * (q * r) \leq p * q) \tag{4}$$

where $p \leq q$ if and only if $p * q = 0$.

Any BCI-algebra U satisfies the following conditions (see [17]):

$$(\forall p, q \in U)(p * (p * (p * q)) = p * q), \tag{5}$$
$$(\forall p, q \in U)(0 * (p * q) = (0 * p) * (0 * q)), \tag{6}$$
$$(\forall p, q \in U)(0 * (0 * (p * q)) = (0 * q) * (0 * p)). \tag{7}$$

By a *subalgebra* of a BCK/BCI-algebra U, we mean a nonempty subset S of U such that $p * q \in S$ for all $p, q \in S$. We say that a subset G of a BCK/BCI-algebra U is an *ideal* of U if it satisfies:

$$0 \in G, \tag{8}$$
$$(\forall p \in U)\ (\forall q \in G)\ (p * q \in G\ \Rightarrow\ p \in G)\,. \tag{9}$$

A subset G of a BCI-algebra U is called a *BCI-positive implicative ideal* of U (see [18,19]) if it satisfies (8) and

$$(\forall p, q, r \in U)\ (((p * r) * r) * (q * r) \in G,\ q \in G\ \Rightarrow\ p * r \in G)\,, \tag{10}$$

For further information regarding BCK/BCI-algebras and neutrosophic set theory, we refer the reader to the books [17,20] and the site [21] respectively. We will use neutrosophic quadruple numbers based on a set instead of real or complex numbers.

Let U be a set. A *neutrosophic quadruple U-number* is an ordered quadruple (a, pT, qI, rF), where $a, p, q, r \in U$ and T, I, F have their usual neutrosophic logic meanings (see [15]).

The set of all neutrosophic quadruple U-numbers which is denoted by $\mathcal{N}(U)$, that is,

$$\mathcal{N}(U) := \{(a, pT, qI, rF) \mid a, p, q, r \in U\},$$

is called the *neutrosophic quadruple set* based on U. In particular, if U is a BCK/BCI-algebra, then a neutrosophic quadruple U-number is called a *neutrosophic quadruple BCK/BCI-number* and $\mathcal{N}(U)$ is called the *neutrosophic quadruple BCK/BCI-set*.

We define a binary operation $\boxdot$ on the neutrosophic quadruple BCK/BCI-set $\mathcal{N}(U)$ by

$$(a, pT, qI, rF) \boxdot (b, uT, vI, wF) = (a * b, (p * u)T, (q * v)I, (z * w)F)$$

for all (a, pT, qI, rF), $(b, uT, vI, wF) \in \mathcal{N}(U)$. Given $a_1, a_2, a_3, a_4 \in U$, the neutrosophic quadruple BCK/BCI-number (a_1, a_2T, a_3I, a_4F) is denoted by $\tilde{a}$, that is,

$$\tilde{a} = (a_1, a_2T, a_3I, a_4F),$$

and the neutrosophic quadruple BCK/BCI-number $(0, 0T, 0I, 0F)$ is denoted by $\tilde{0}$, that is,

$$\tilde{0} = (0, 0T, 0I, 0F),$$

which is called the *zero neutrosophic quadruple BCK/BCI-number*. Then $(\mathcal{N}(U); \boxdot, \tilde{0})$ is a BCK/BCI-algebra (see [15]), which is called *neutrosophic quadruple BCK/BCI-algebra*, and it is simply denoted by $\mathcal{N}(U)$.

We define an order relation "$\ll$" and the equality "$=$" on the neutrosophic quadruple BCK/BCI-algebra $\mathcal{N}(U)$ as follows:

$$\tilde{p} \ll \tilde{q} \Leftrightarrow p_i \leq q_i \text{ for } i = 1, 2, 3, 4,$$
$$\tilde{p} = \tilde{q} \Leftrightarrow p_i = q_i \text{ for } i = 1, 2, 3, 4$$

for all $\tilde{p} = (p_1, p_2T, p_3I, p_4F)$, $\tilde{q} = (q_1, q_2T, q_3I, q_4F) \in \mathcal{N}(U)$. It is easy to verify that "$\ll$" is an equivalence relation on $\mathcal{N}(U)$.

Let U be a BCK/BCI-algebra. Given nonempty subsets I and J of U, consider the set

$$\mathcal{N}(I, J) := \{(a, pT, qI, rF) \in \mathcal{N}(U) \mid a, p \in I \;\&\; q, r \in J\},$$

which is called the *neutrosophic quadruple (I, J)-set*.

The neutrosophic quadruple (I, J)-set $\mathcal{N}(I, J)$ with $I = J$ is denoted by $\mathcal{N}(I)$, and it is called the *neutrosophic quadruple I-set*.

3. Neutrosophic Quadruple BCI-Positive Implicative Ideals

In what follows, let U and $\mathcal{N}(U)$ be a BCI-algebra and a neutrosophic quadruple BCI-algebra, respectively, unless otherwise specified.

Definition 1. *Given nonempty subsets I and J of U, if $\mathcal{N}(I, J)$ is a BCI-positive implicative ideal of $\mathcal{N}(U)$, we say $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.*

Example 1. *Consider a BCI-algebra $U = \{0, 1, a\}$ with the binary operation $*$, which is given in Table 1.*

Table 1. Cayley table for the binary operation "$*$".

*	0	1	*a*
0	0	0	*a*
1	1	0	*a*
a	*a*	*a*	0

Then the neutrosophic quadruple BCI-algebra $\mathcal{N}(U)$ has 81 elements. If we take $I = \{0, a\}$ and $J = \{0, a\}$, then

$$\mathcal{N}(I, J) = \{\tilde{0}, \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3, \tilde{\beta}_4, \tilde{\beta}_5, \tilde{\beta}_6, \tilde{\beta}_7, \tilde{\beta}_8, \tilde{\beta}_9, \tilde{\beta}_{10}, \tilde{\beta}_{11}, \tilde{\beta}_{12}, \tilde{\beta}_{13}, \tilde{\beta}_{14}, \tilde{\beta}_{15}\}$$

and it is routine to check that $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ where

$$\tilde{0} = (0, 0T, 0I, 0F),\ \tilde{\beta}_1 = (0, 0T, 0I, aF),\ \tilde{\beta}_2 = (0, 0T, aI, 0F),\ \tilde{\beta}_3 = (0, 0T, aI, aF),$$
$$\tilde{\beta}_4 = (0, aT, 0I, 0F),\ \tilde{\beta}_5 = (0, aT, 0I, aF),\ \tilde{\beta}_6 = (0, aT, aI, 0F),\ \tilde{\beta}_7 = (0, aT, aI, aF),$$

$$\tilde{\beta}_8 = (a, 0T, 0I, 0F),\ \tilde{\beta}_9 = (a, 0T, 0I, aF),\ \tilde{\beta}_{10} = (a, 0T, aI, 0F),\ \tilde{\beta}_{11} = (a, 0T, aI, aF),$$
$$\tilde{\beta}_{12} = (a, aT, 0I, 0F),\ \tilde{\beta}_{13} = (a, aT, 0I, aF),\ \tilde{\beta}_{14} = (a, aT, aI, 0F),\ \tilde{\beta}_{15} = (a, aT, aI, aF).$$

Proposition 1. *Given nonempty subsets I and J of U, the neutrosophic quadruple BCI-positive implicative ideal $\mathcal{N}(I,J)$ of $\mathcal{N}(U)$ satisfies the following assertions.*

$$(\forall \tilde{p}, \tilde{q}, \tilde{r} \in \mathcal{N}(U))(((\tilde{p} \boxdot \tilde{r}) \boxdot \tilde{r}) \boxdot (\tilde{q} \boxdot \tilde{r}) \in \mathcal{N}(I,J) \ \Rightarrow\ (\tilde{p} \boxdot \tilde{q}) \boxdot \tilde{r} \in \mathcal{N}(I,J)), \tag{11}$$
$$(\forall \tilde{p}, \tilde{q} \in \mathcal{N}(U))(((\tilde{p} \boxdot \tilde{q}) \boxdot \tilde{q}) \boxdot (\tilde{0} \boxdot \tilde{q}) \in \mathcal{N}(I,J) \ \Rightarrow\ \tilde{p} \boxdot \tilde{q} \in \mathcal{N}(I,J)). \tag{12}$$

Proof. Let $\mathcal{N}(I,J)$ be a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ for any nonempty subsets I and J of U. Assume that $((\tilde{p} \boxdot \tilde{r}) \boxdot \tilde{r}) \boxdot (\tilde{q} \boxdot \tilde{r}) \in \mathcal{N}(I,J)$ for all $\tilde{p}, \tilde{q}, \tilde{r} \in \mathcal{N}(U)$. Since

$$\begin{aligned}(((\tilde{p} \boxdot \tilde{q}) \boxdot \tilde{r}) \boxdot \tilde{r}) \boxdot (\tilde{0} \boxdot \tilde{r}) &= (((\tilde{p} \boxdot \tilde{r}) \boxdot \tilde{r}) \boxdot \tilde{q}) \boxdot ((\tilde{q} \boxdot \tilde{q}) \boxdot \tilde{r}) \\ &= (((\tilde{p} \boxdot \tilde{r}) \boxdot \tilde{r}) \boxdot \tilde{q}) \boxdot ((\tilde{q} \boxdot \tilde{r}) \boxdot \tilde{q}) \\ &\leq ((\tilde{p} \boxdot \tilde{r}) \boxdot \tilde{r}) \boxdot (\tilde{q} \boxdot \tilde{r}),\end{aligned}$$

we have $(((\tilde{p} \boxdot \tilde{q}) \boxdot \tilde{r}) \boxdot \tilde{r}) \boxdot (\tilde{0} \boxdot \tilde{r}) \in \mathcal{N}(I,J)$. Since $\mathcal{N}(I,J)$ is a neutrosophic quadruple BCI-positive implicative ideal, it follows that $(\tilde{p} \boxdot \tilde{q}) \boxdot \tilde{r} \in \mathcal{N}(I,J))$. Hence (11) is valid. If we take $\tilde{q} = \tilde{0}$ and $\tilde{r} = \tilde{q}$ in (11), then we get (12). □

We consider relations between neutrosophic quadruple ideal and neutrosophic quadruple BCI-positive implicative ideal.

Theorem 1. *For any nonempty subsets I and J of U, if $\mathcal{N}(I,J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$, then it is a neutrosophic quadruple ideal of $\mathcal{N}(U)$.*

Proof. Assume that $\mathcal{N}(I,J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$. Let $\tilde{p} = (p_1, p_2T, p_3I, p_4F)$ and $\tilde{q} = (q_1, q_2T, q_3I, q_4F)$ be elements of $\mathcal{N}(U)$ such that $\tilde{q} \in \mathcal{N}(I,J)$ and $\tilde{p} \boxdot \tilde{q} \in \mathcal{N}(I,J)$. Then

$$((\tilde{p} \boxdot \tilde{0}) \boxdot \tilde{0}) \boxdot (\tilde{q} \boxdot \tilde{0}) = \tilde{p} \boxdot \tilde{q} \in \mathcal{N}(I,J),$$

which implies that $\tilde{p} = \tilde{p} \boxdot \tilde{0} \in \mathcal{N}(I,J)$. Therefore $\mathcal{N}(I,J)$ is a neutrosophic quadruple ideal of $\mathcal{N}(U)$. □

The converse of Theorem 1 is not true as seen in the following example.

Example 2. *Consider a BCI-algebra $U = \{0, 1, a\}$ with the binary operation $*$, which is given in Table 2.*

Table 2. Cayley table for the binary operation "$*$".

*	**0**	**1**	***a***
0	0	0	0
1	1	0	0
a	a	1	0

Then the neutrosophic quadruple BCI-algebra $\mathcal{N}(U)$ has 81 elements. If we take $I = \{0\}$ and $J = \{0\}$, then $\mathcal{N}(I,J) = \{\tilde{0}\}$ is a neutrosophic quadruple ideal of $\mathcal{N}(U)$. But it is not a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ since

$$(((a, aT, aI, aF) \boxdot (1, 1T, 1I, 1F)) \boxdot (1, 1T, 1I, 1F)) \boxdot (\tilde{0} \boxdot (1, 1T, 1I, 1F)) = \tilde{0} \in \mathcal{N}(I,J)$$

and $(a, aT, aI, aF) \boxdot (1, 1T, 1I, 1F) = (1, 1T, 1I, 1F) \notin \mathcal{N}(I, J)$.

Given nonempty subsets I and J of U, we provide conditions for the set $\mathcal{N}(I, J)$ to be a neutrosophic quadruple BCI-positive implicative ideal.

Theorem 2. *If I and J are BCI-positive implicative ideal of U, then $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.*

Proof. Assume that I and J are BCI-positive implicative ideal of U. Obviously $\tilde{0} \in \mathcal{N}(I, J)$. Let $\tilde{p} = (p_1, p_2T, p_3I, p_4F)$, $\tilde{q} = (q_1, q_2T, q_3I, q_4F)$ and $\tilde{r} = (r_1, r_2T, r_3I, r_4F)$ be elements of $\mathcal{N}(U)$ such that $\tilde{q} \in \mathcal{N}(I, J)$ and $((\tilde{p} \boxdot \tilde{r}) \boxdot \tilde{r}) \boxdot (\tilde{q} \boxdot \tilde{r}) \in \mathcal{N}(I, J)$. Then $q_i \in I$ and $q_j \in J$ for $i = 1, 2$ and $j = 3, 4$. Also

$$\begin{aligned}
((\tilde{p} \boxdot \tilde{r}) \boxdot \tilde{r}) \boxdot (\tilde{q} \boxdot \tilde{r}) &= (((p_1, p_2T, p_3I, p_4F) \boxdot (r_1, r_2T, r_3I, r_4F)) \boxdot (r_1, r_2T, r_3I, r_4F)) \\
&\qquad \boxdot ((q_1, q_2T, q_3I, q_4F) \boxdot (r_1, r_2T, r_3I, r_4F)) \\
&= ((p_1 * r_1, (p_2 * r_2)T, (p_3 * r_3)I, (p_4 * r_4)F) \boxdot (r_1, r_2T, r_3I, r_4F)) \\
&\qquad \boxdot (q_1 * r_1, (q_2 * r_2)T, (q_3 * r_3)I, (q_4 * r_4)F) \\
&= (((p_1 * r_1) * r_1, ((p_2 * r_2) * r_2)T, ((p_3 * r_3) * r_3)I, ((p_4 * r_4) * r_4)F)) \\
&\qquad \boxdot (q_1 * r_1, (q_2 * r_2)T, (q_3 * r_3)I, (q_4 * r_4)F) \\
&= (((p_1 * r_1) * r_1) * (q_1 * r_1), (((p_2 * r_2) * r_2) * (q_2 * r_2))T, \\
&\qquad (((p_3 * r_3) * r_3) * (q_3 * r_3))I, (((p_4 * r_4) * r_4) * (q_4 * r_4))F) \\
&\in \mathcal{N}(I, J),
\end{aligned}$$

and so $((p_i * r_i) * r_i) * (q_i * r_i) \in I$ and $((p_j * r_j) * r_j) * (q_j * r_j) \in J$ for $i = 1, 2$ and $j = 3, 4$. it follows from (10) that $p_i * r_i \in I$ and $p_j * r_j \in J$ for $i = 1, 2$ and $j = 3, 4$. Hence

$$\begin{aligned}
\tilde{p} \boxdot \tilde{r} &= (p_1, p_2T, p_3I, p_4F) \boxdot (r_1, r_2T, r_3I, r_4F) \\
&= (p_1 * r_1, (p_2 * r_2)T, (p_3 * r_3)I, (p_4 * r_4)F) \in \mathcal{N}(I, J).
\end{aligned}$$

Therefore $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$. □

Corollary 1. *If I is a BCI-positive implicative ideal of U, then $\mathcal{N}(I)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.*

Theorem 3. *Let I and J be ideals of U which satisfies the following condition.*

$$(\forall p, q \in U)(((p * q) * q) * (0 * q) \in I \cap J \;\Rightarrow\; p * q \in I \cap J). \tag{13}$$

Then $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Proof. Obviously $\tilde{0} \in \mathcal{N}(I, J)$. Let $\tilde{p} = (p_1, p_2T, p_3I, p_4F)$, $\tilde{q} = (q_1, q_2T, q_3I, q_4F)$ and $\tilde{r} = (r_1, r_2T, r_3I, r_4F)$ be elements of $\mathcal{N}(U)$ such that $\tilde{r} \in \mathcal{N}(I, J)$ and $((\tilde{p} \boxdot \tilde{q}) \boxdot \tilde{q}) \boxdot (\tilde{r} \boxdot \tilde{q}) \in \mathcal{N}(I, J)$. Then $r_1, r_2 \in I$, $r_3, r_4 \in J$ and

$$\begin{aligned}
((\tilde{p} \boxdot \tilde{q}) \boxdot \tilde{q}) \boxdot (\tilde{r} \boxdot \tilde{q}) &= (((p_1 * q_1) * q_1) * (r_1 * q_1), (((p_2 * q_2) * q_2) * (r_2 * q_2))T, \\
&\qquad (((p_3 * q_3) * q_3) * (r_3 * q_3))I, (((p_4 * q_4) * q_4) * (r_4 * q_4))F) \\
&\in \mathcal{N}(I, J),
\end{aligned}$$

that is,

$$((p_i * q_i) * q_i) * (r_i * q_i) \in I \text{ and } ((p_j * q_j) * q_j) * (r_j * q_j) \in J \tag{14}$$

for $i = 1, 2$ and $j = 3, 4$. Note that

$$\begin{aligned}&(((p_k * q_k) * q_k) * (0 * q_k)) * (((p_k * q_k) * q_k) * (r_k * q_k)) \\ &\leq (r_k * q_k) * (0 * q_k) \leq r_k * 0 = r_k\end{aligned}$$

for $k = 1, 2, 3, 4$ by (I), (1) and (4). Since I and J are ideals of U, it follows that

$$\begin{aligned}&(((p_i * q_i) * q_i) * (0 * q_i)) * (((p_i * q_i) * q_i) * (r_i * q_i)) \in I, \\ &(((p_j * q_j) * q_j) * (0 * q_j)) * (((p_j * q_j) * q_j) * (r_j * q_j)) \in J\end{aligned} \tag{15}$$

for $i = 1, 2$ and $j = 3, 4$. Combining (14) and (15), we get

$$((p_i * q_i) * q_i) * (0 * q_i) \in I \text{ and } ((p_j * q_j) * q_j) * (0 * q_j) \in J$$

for $i = 1, 2$ and $j = 3, 4$. Using (13) implies that $p_i * q_i \in I$ and $p_j * q_j \in J$ for $i = 1, 2$ and $j = 3, 4$. Thus

$$\begin{aligned}\tilde{p} \boxdot \tilde{q} &= (p_1, p_2T, p_3I, p_4F) \boxdot (q_1, q_2T, q_3I, q_4F) \\ &= (p_1 * q_1, (p_2 * q_2)T, (p_3 * q_3)I, (p_4 * q_4)F) \in \mathcal{N}(I, J).\end{aligned}$$

Therefore $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$. □

Corollary 2. *Let I be an ideal of U which satisfies the following condition.*

$$(\forall p, q \in U)(((p * q) * q) * (0 * q) \in I \ \Rightarrow \ p * q \in I). \tag{16}$$

Then $\mathcal{N}(I)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Theorem 4. *Let I and J be ideals of U which satisfies the following condition.*

$$(\forall p, q, r \in U)(((p * r) * (q * r)) * r \in I \cap J \ \Rightarrow \ (p * q) * r \in I \cap J). \tag{17}$$

Then $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Proof. Suppose that $((p * q) * q) * (0 * q) \in I \cap J$ for all $p, q \in U$. Then $((p * q) * (0 * q)) * q = ((p * q) * q) * (0 * q) \in I \cap J$, which implies from (17) and (1) that $p * q = (p * 0) * q \in I \cap J$. Therefore $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ by Theorem 3. □

Corollary 3. *Let I be an ideal of U which satisfies the following condition.*

$$(\forall p, q, r \in U)(((p * r) * (q * r)) * r \in I \ \Rightarrow \ (p * q) * r \in I). \tag{18}$$

Then $\mathcal{N}(I)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Theorem 5. *Let I and J be subsets of U such that*

$$0 \in I \cap J, \tag{19}$$

$$(\forall p, q, r \in U)((((p * q) * q) * (0 * q)) * r \in I \cap J, r \in I \cap J \ \Rightarrow p * q \in I \cap J). \tag{20}$$

Then $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Proof. If we take $q = 0$ in (20) and use (1) and (III), then

$$(\forall p, r \in U)(p * r \in I \cap J,\ r \in I \cap J \ \Rightarrow p \in I \cap J).$$

Hence I and J are ideals of U. Assume that $((p * q) * q) * (0 * q) \in I \cap J$ for all $p, q \in U$. Then

$$(((p * q) * q) * (0 * q)) * 0 = ((p * q) * q) * (0 * q) \in I \cap J,$$

It follows from (19) and (20) that $p * q \in I \cap J$. Consequently, $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ by Theorem 3. □

Corollary 4. *Let I be a subset of U such that*

$$0 \in I, \tag{21}$$
$$(\forall p, q, r \in U)((((p * q) * q) * (0 * q)) * r \in I,\ r \in I \ \Rightarrow p * q \in I). \tag{22}$$

Then $\mathcal{N}(I)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Theorem 6. *Let I, J, G and H be ideals of U such that $G \subseteq I$ and $H \subseteq J$. If G and H are BCI-positive implicative ideals of U, then $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.*

Proof. Let $p, q, r \in U$ be such that $((p * q) * q) * (0 * q) \in I \cap J$. Then

$$\begin{aligned}&(((p * (((p * q) * q) * (0 * q))) * q) * q) * (0 * q)\\ &= (((p * q) * q) * (0 * q)) * (((p * q) * q) * (0 * q))\\ &= 0 \in G \cap H,\end{aligned}$$

and so $(p * q) * (((p * q) * q) * (0 * q)) = (p * (((p * q) * q) * (0 * q))) * q \in G \cap H \subseteq I \cap J$ since G and H are BCI-positive implicative ideals of U. Thus $p * q \in I \cap J$, and therefore $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ by Theorem 3. □

Corollary 5. *Let I and G be ideals of U such that $G \subseteq I$. If G is a BCI-positive implicative ideal of U, then $\mathcal{N}(I)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.*

Theorem 7. *Let I, J, G and H be ideals of U such that $G \subseteq I$, $H \subseteq J$ and*

$$(\forall p, q \in U)(((p * q) * q) * (0 * q) \in G \cap H \ \Rightarrow \ p * q \in G \cap H). \tag{23}$$

Then $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Proof. Let $p, q, r \in U$ be such that $r \in G \cap H$ and $((p * q) * q) * (r * q) \in G \cap H$. Since

$$(((p * q) * q) * (0 * q)) * (((p * q) * q) * (r * q)) \leq (r * q) * (0 * q) \leq r * 0 = r \in G \cap H,$$

we have $((p * q) * q) * (0 * q) \in G \cap H$. It follows from (23) that $p * q \in G \cap H$. Hence G and H are BCI-positive implicative ideals of U, and therefore $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ by Theorem 6. □

Corollary 6. *Let I and G be ideals of U such that $G \subseteq I$ and*

$$(\forall p, q \in U)(((p * q) * q) * (0 * q) \in G \ \Rightarrow \ p * q \in G). \tag{24}$$

Then $\mathcal{N}(I)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Theorem 8. *Let I, J, G and H be ideals of U such that $G \subseteq I$, $H \subseteq J$ and*

$$(\forall p, q \in U)(((p * r) * (q * r)) * r \in G \cap H \;\Rightarrow\; (p * q) * r \in G \cap H). \tag{25}$$

Then $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

Proof. Let $p, q \in U$ be such that $((p * q) * q) * (0 * q) \in G \cap H$. Then

$$((p * q) * (0 * q)) * q = ((p * q) * q) * (0 * q) \in G \cap H.$$

It follows from (25) and (1) that $p * q = (p * 0) * q \in G \cap H$. Hence $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ by Theorem 7. □

Proof. If we put $q = 0$ and $r = q$ in (25), then we have the condition (23). Hence $\mathcal{N}(I, J)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$ by Theorem 7. □

Corollary 7. *Let I and G be ideals of U such that $G \subseteq I$ and*

$$(\forall p, q \in U)(((p * r) * (q * r)) * r \in G \;\Rightarrow\; (p * q) * r \in G). \tag{26}$$

Then $\mathcal{N}(I)$ is a neutrosophic quadruple BCI-positive implicative ideal of $\mathcal{N}(U)$.

4. Conclusions

By considering an entry (i.e., a number, an idea, an object, etc.) which is represented by a known part (a) and an unknown part (bT, cI, dF) where T, I, F have their usual neutrosophic logic meanings and a, b, c, d are real or complex numbers, Smarandache have introduced the concept of neutrosophic quadruple numbers. Using the notion of neutrosophic quadruple numbers based on a set (instead of real or complex numbers), Jun et al. have constructed neutrosophic quadruple BCK/BCI-algebras and implicative neutrosophic quadruple BCK-algebras. In this manuscript, we have introduced the concept of a neutrosophic quadruple BCI-positive implicative ideal, and investigated several properties. We have discussed relations between neutrosophic quadruple ideal and neutrosophic quadruple BCI-positive implicative ideal. Given nonempty subsets I and J of a BCI-algebra U, we have provided conditions for the neutrosophic quadruple (I, J)-set to be a neutrosophic quadruple BCI-positive implicative ideal. In the forthcoming research and papers, we will continue these ideas and will define new notions in several algebraic structures.

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