Neutrosophic quadruple ideals in neutrosophic quadruple BCI-algebras

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Abstract: In the present paper, we discuss the Neutrosophic quadruple q-ideals and (regular) neutrosophic quadruple ideals and investigate their related properties. Also, for any two nonempty subsets U and V of a BCI-algebra S, conditions for the set $NQ(U, V)$ to be a (regular) neutrosophic quadruple ideal and a neutrosophic quadruple q-ideal of a neutrosophic quadruple BCI-algebra $NQ(S)$ are discussed. Furthermore, we prove that let U, V, I and J be ideals of a BCI-algebra S such that $I \subseteq U$ and $J \subseteq V$. If I and J are q-ideals of S, then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Keywords: neutrosophic quadruple BCK/BCI-number, neutrosophic quadruple BCK/BCI-algebra, (regular) neutrosophic quadruple ideal, neutrosophic quadruple q-ideal.

1 Introduction

To deal with incomplete, inconsistent and indeterminate information, Smarandache introduced the notion of neutrosophic sets (see ([1], [2] and [3]). In fact, neutrosophic set is a useful mathematical tool which extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. Neutrosophic set theory has useful applications in several branches (see for e.g., [4], [5], [6] and [7]).

In [8], Smarandache considered an entry (i.e., a number, an idea, an object etc.) which is represented by a known part (a) and an unknown part (bT, cI, dF) where T, I, F have their usual neutrosophic logic meanings and a, b, c, d are real or complex numbers, and then he introduced the concept of neutrosophic quadruple numbers. Neutrosophic quadruple algebraic structures and hyperstructures are discussed in [9] and [10]. Recently, neutrosophic set theory has been applied to the BCK/BCI-algebras on various aspects (see for e.g., [11], [12] [13], [14], [15], [16], [17], [18], [19] and [20].) Using the notion of neutrosophic quadruple numbers based on a set, Jun et al. [21] constructed neutrosophic quadruple BCK/BCI-algebras. They investigated several properties, and considered ideal and positive implicative ideal in neutrosophic quadruple BCK-algebra, and closed ideal in neutrosophic quadruple BCI-algebra. Given subsets A and B of a neutrosophic quadruple BCK/BCI-algebra, they considered sets $NQ(U,V)$ which consists of neutrosophic quadruple BCK/BCI-numbers with a condition. They provided conditions for the set $NQ(U,V)$ to be a (positive implicative) ideal of a neutrosophic quadruple BCK-algebra, and the set $NQ(U,V)$ to be a (closed) ideal of a neutrosophic quadruple BCI-algebra. They gave an example to show that the set $\{\tilde{0}\}$ is not a positive implicative ideal in a neutrosophic quadruple BCK-algebra, and then they considered conditions for the set $\{\tilde{0}\}$ to be a positive implicative ideal in a neutrosophic quadruple BCK-algebra. Muhiuddin et al. [22] discussed several properties and (implicative) neutrosophic quadruple ideals in (implicative) neutrosophic quadruple BCK-algebras.

In this paper, we introduce the notions of (regular) neutrosophic quadruple ideal and neutrosophic quadruple q-ideal in neutrosophic quadruple BCI-algebras, and investigate related properties. Given nonempty subsets A and B of a BCI-algebra S, we consider conditions for the set $NQ(U,V)$ to be a (regular) neutrosophic quadruple ideal of $NQ(S)$ and a neutrosophic quadruple q-ideal of $NQ(S)$.

2 Preliminaries

We begin with the following definitions and properties that will be needed in the sequel.

A nonempty set S with a constant 0 and a binary operation $*$ is called a BCI-algebra if for all $x,y,z \in S$ the following conditions hold ([23] and [24]):

(I) $(((x*y)*(x*z))*(z*y)=0)$,

(II) $((x*(x*y))*y=0)$,

(III) $(x*x=0)$,

(IV) $(x*y=0,\ y*x=0 \ \Rightarrow\ x=y)$.

If a BCI-algebra S satisfies the following identity:

(V) $(\forall x \in S)\ (0*x=0)$,

then S is called a *BCK-algebra.* Define a binary relation $\leq$ on X by letting $x*y=0$ if and only if $x \leq y$. Then $(S,\leq)$ is a partially ordered set.

Theorem 2.1. *Let S be a BCK/BCI-algebra. Then following conditions are hold:*

$$(\forall x \in S)\,(x*0=x)\,, \tag{2.1}$$
$$(\forall x,y,z \in S)\,(x \leq y \ \Rightarrow\ x*z \leq y*z,\ z*y \leq z*x)\,, \tag{2.2}$$
$$(\forall x,y,z \in S)\,((x*y)*z=(x*z)*y)\,, \tag{2.3}$$
$$(\forall x,y,z \in S)\,((x*z)*(y*z) \leq x*y) \tag{2.4}$$

where $x \leq y$ if and only if $x*y=0$.

Any BCI-algebra S satisfies the following conditions (see [25]):

$$(\forall x,y \in S)(x*(x*(x*y))=x*y), \tag{2.5}$$
$$(\forall x,y \in S)(0*(x*y)=(0*x)*(0*y)), \tag{2.6}$$
$$(\forall x,y \in S)(0*(0*(x*y))=(0*y)*(0*x)). \tag{2.7}$$

A nonempty subset A of a BCK/BCI-algebra S is called a *subalgebra* of S if $x * y \in A$ for all $x, y \in A$. A subset I of a BCK/BCI-algebra S is called an *ideal* of S if it satisfies:

$$0 \in I, \tag{2.8}$$

$$(\forall x \in S)\,(\forall y \in I)\,(x * y \in I \;\Rightarrow\; x \in I)\,. \tag{2.9}$$

An ideal I of a BCI-algebra S is said to be *regular* (see [26]) if it is also a subalgebra of S.

It is clear that every ideal of a BCK-algebra is regular (see [26]).

A subset I of a BCI-algebra S is called a *q-ideal* of S (see [27]) if it satisfies (2.8) and

$$(\forall x, y, z \in S)(x * (y * z) \in I,\, y \in I \;\Rightarrow\; x * z \in I). \tag{2.10}$$

We refer the reader to the books [25, 28] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

We consider neutrosophic quadruple numbers based on a set instead of real or complex numbers.

Definition 2.2 ([21])**.** Let S be a set. A *neutrosophic quadruple S-number* is an ordered quadruple (a, xT, yI, zF) where $a, x, y, z \in S$ and T, I, F have their usual neutrosophic logic meanings.

The set of all neutrosophic quadruple S-numbers is denoted by $NQ(S)$, that is,

$$NQ(S) := \{(a, xT, yI, zF) \mid a, x, y, z \in S\},$$

and it is called the *neutrosophic quadruple set* based on S. If S is a BCK/BCI-algebra, a neutrosophic quadruple S-number is called a *neutrosophic quadruple BCK/BCI-number* and we say that $NQ(S)$ is the *neutrosophic quadruple BCK/BCI-set*.

Let S be a BCK/BCI-algebra. We define a binary operation $\circledast$ on $NQ(S)$ by

$$(a, xT, yI, zF) \circledast (b, uT, vI, wF) = (a * b, (x * u)T, (y * v)I, (z * w)F)$$

for all (a, xT, yI, zF), $(b, uT, vI, wF) \in NQ(S)$. Given $a_1, a_2, a_3, a_4 \in S$, the neutrosophic quadruple BCK/BCI-number (a_1, a_2T, a_3I, a_4F) is denoted by $\tilde{a}$, that is,

$$\tilde{a} = (a_1, a_2T, a_3I, a_4F),$$

and the zero neutrosophic quadruple BCK/BCI-number $(0, 0T, 0I, 0F)$ is denoted by $\tilde{0}$, that is,

$$\tilde{0} = (0, 0T, 0I, 0F).$$

We define an order relation "$\ll$" and the equality "$=$" on $NQ(S)$ as follows:

$$\tilde{x} \ll \tilde{y} \Leftrightarrow x_i \leq y_i \text{ for } i = 1, 2, 3, 4,$$
$$\tilde{x} = \tilde{y} \Leftrightarrow x_i = y_i \text{ for } i = 1, 2, 3, 4$$

for all $\tilde{x}, \tilde{y} \in NQ(S)$. It is easy to verify that "$\ll$" is an equivalence relation on $NQ(S)$.

Theorem 2.3 ([21])**.** *If S is a BCK/BCI-algebra, then $(NQ(S); \circledast, \tilde{0})$ is a BCK/BCI-algebra.*

We say that $(NQ(S); \circledast, \tilde{0})$ is a *neutrosophic quadruple BCK/BCI-algebra,* and it is simply denoted by $NQ(S)$.

Let S be a BCK/BCI-algebra. Given nonempty subsets A and B of S, consider the set

$$NQ(U,V) := \{(a, xT, yI, zF) \in NQ(S) \mid a, x \in U \;\&\; y, z \in V\},$$

which is called the *neutrosophic quadruple* (U,V)*-set*.

The set $NQ(U,U)$ is denoted by $NQ(U)$, and it is called the *neutrosophic quadruple* U*-set*.

3 (Regular) neutrosophic quadruple ideals

Definition 3.1. Given nonempty subsets U and V of a BCI-algebra S, if the neutrosophic quadruple (U,V)-set $NQ(U,V)$ is a (regular) ideal of a neutrosophic quadruple BCI-algebra $NQ(S)$, we say $NQ(U,V)$ is a *(regular) neutrosophic quadruple ideal* of $NQ(S)$.

Question 1. *If U and V are subalgebras of a BCI-algebra S, then is the neutrosophic quadruple (U,V)-set $NQ(U,V)$ a neutrosophic quadruple ideal of $NQ(S)$?*

The answer to Question 1 is negative as seen in the following example.

Example 3.2. Consider a BCI-algebra $S = \{0, 1, a, b, c\}$ with the binary operation $*$, which is given in Table 1. Then the neutrosophic quadruple BCI-algebra $NQ(S)$ has 625 elements. Note that $U = \{0, a\}$ and $V = \{0, b\}$

Table 1: Cayley table for the binary operation "$*$"

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0

are subalgebras of S. The neutrosophic quadruple (U,V)-set $NQ(U,V)$ consists of the following elements:

$$NQ(U,V) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}, \tilde{8}, \tilde{9}, \tilde{10}, \tilde{11}, \tilde{12}, \tilde{13}, \tilde{14}, \tilde{15}\}$$

where
$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{1} = (0, 0T, 0I, bF)$, $\tilde{2} = (0, 0T, bI, 0F)$,
$\tilde{3} = (0, 0T, bI, bF)$, $\tilde{4} = (0, aT, 0I, 0F)$, $\tilde{5} = (0, aT, 0I, bF)$,
$\tilde{6} = (0, aT, bI, 0F)$, $\tilde{7} = (0, aT, bI, bF)$, $\tilde{8} = (a, 0T, 0I, 0F)$,
$\tilde{9} = (a, 0T, 0I, bF)$, $\tilde{10} = (a, 0T, bI, 0F)$, $\tilde{11} = (a, 0T, bI, bF)$,
$\tilde{12} = (a, aT, 0I, 0F)$, $\tilde{13} = (a, aT, 0I, bF)$,
$\tilde{14} = (a, aT, bI, 0F)$, $\tilde{15} = (a, aT, bI, bF)$.

If we take $(1, aT, bI, 0F) \in NQ(S)$, then $(1, aT, bI, 0F) \notin NQ(U, V)$ and

$$(1, aT, bI, 0F) \circledast \tilde{9} = \tilde{15} \in NQ(U, V).$$

Hence the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is not a neutrosophic quadruple ideal of $NQ(S)$.

We consider conditions for the neutrosophic quadruple (U, V)-set $NQ(U, V)$ to be a regular neutrosophic quadruple ideal of $NQ(S)$.

Lemma 3.3 ([21]). *If U and V are subalgebras (resp., ideals) of a BCI-algebra S, then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple subalgebra (resp., ideal) of $NQ(S)$.*

Theorem 3.4. *Let U and V be subalgebras of a BCI-algebra S such that*

$$(\forall x, y \in S)(x \in U \text{ (resp., } V),\ y \notin U \text{ (resp., } V) \ \Rightarrow\ y * x \notin U \text{ (resp., } V)). \tag{3.1}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.

Proof. By Lemma 3.3, $NQ(U, V)$ is a neutrosophic quadruple subalgebra of $NQ(S)$. Hence it is clear that $\tilde{0} \in NQ(U, V)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F) \in NQ(S)$ and $\tilde{y} = (y_1, y_2T, y_3I, y_4F) \in NQ(S)$ be such that $\tilde{y} \circledast \tilde{x} \in NQ(U, V)$ and $\tilde{x} \in NQ(U, V)$. Then $x_i \in U$ and $x_j \in V$ for $i = 1, 2$ and $j = 3, 4$. Also,

$$\begin{aligned}\tilde{y} \circledast \tilde{x} &= (y_1, y_2T, y_3I, y_4F) \circledast (x_1, x_2T, x_3I, x_4F)\\ &= (y_1 * x_1, (y_2 * x_2)T, (y_3 * x_3)I, (y_4 * x_4)F) \in NQ(U, V),\end{aligned}$$

and so $y_1 * x_1 \in U$, $y_2 * x_2 \in U$, $y_3 * x_3 \in V$ and $y_4 * x_4 \in V$. If $\tilde{y} \notin NQ(U, V)$, then $y_i \notin A$ or $y_j \notin B$ for some $i = 1, 2$ and $j = 3, 4$. It follows from (3.1) that $y_i * x_i \notin U$ or $y_j * x_j \notin V$ for some $i = 1, 2$ and $j = 3, 4$. This is a contradiction, and so $\tilde{y} \in NQ(U, V)$. Thus $NQ(U, V)$ is a neutrosophic quadruple ideal of $NQ(S)$, and therefore $NQ(U, V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$. □

Corollary 3.5. *Let U be a subalgebra of a BCI-algebra S such that*

$$(\forall x, y \in S)(x \in U,\ y \notin U \ \Rightarrow\ y * x \notin U). \tag{3.2}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.

Theorem 3.6. *Let U and V be subsets of a BCI-algebra S. If any neutrosophic quadruple ideal $NQ(U, V)$ of $NQ(S)$ satisfies $\tilde{0} \circledast \tilde{x} \in NQ(U, V)$ for all $\tilde{x} \in NQ(U, V)$, then $NQ(U, V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.*

Proof. For any $\tilde{x}, \tilde{y} \in NQ(U, V)$, we have

$$(\tilde{x} \circledast \tilde{y}) \circledast \tilde{x} = (\tilde{x} \circledast \tilde{x}) \circledast \tilde{y} = \tilde{0} \circledast \tilde{y} \in NQ(U, V).$$

Since $NQ(U, V)$ is an ideal of $NQ(S)$, it follows that $\tilde{x} \circledast \tilde{y} \in NQ(U, V)$. Hence $NQ(U, V)$ is a neutrosophic quadruple subalgebra of $NQ(S)$, and therefore $NQ(U, V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$. □

Corollary 3.7. *Let U be a subset of a BCI-algebra S. If any neutrosophic quadruple ideal $NQ(U)$ of $NQ(S)$ satisfies $\tilde{0} \circledast \tilde{x} \in NQ(U)$ for all $\tilde{x} \in NQ(U)$, then $NQ(U)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.*

Theorem 3.8. *If U and V are ideals of a finite BCI-algebra S, then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.*

Proof. By Lemma 3.3, $NQ(U, V)$ is a neutrosophic quadruple ideal of $NQ(S)$. Since S is finite, $NQ(S)$ is also finite. Assume that $|NQ(S)| = n$. For any element $\tilde{x} \in NQ(U, V)$, consider the following $n+1$ elements:

$$\tilde{0}, \tilde{0} \circledast \tilde{x}, (\tilde{0} \circledast \tilde{x}) \circledast \tilde{x}, \cdots, (\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{n\text{-times}}.$$

Then there exist natural numbers p and q with $p > q$ such that

$$(\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{p\text{-times}} = (\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{q\text{-times}}.$$

Hence

$$\begin{aligned}
\tilde{0} &= ((\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{p \text{ times}}) \circledast ((\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{q \text{ times}}) \\
&= ((\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{q \text{ times}})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{p-q \text{ times}}) \circledast ((\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{q \text{ times}}) \\
&= (\cdots((\tilde{0} \circledast \tilde{x})\underbrace{\circledast \tilde{x}) \circledast \cdots) \circledast \tilde{x}}_{p-q \text{ times}} \in NQ(U, V).
\end{aligned}$$

Since $NQ(U, V)$ is an ideal of $NQ(S)$, it follows that $\tilde{0} \circledast \tilde{x} \in NQ(U, V)$. Therefore $NQ(U, V)$ is a regular neutrosophic quadruple ideal of $NQ(S)$ by Theorem 3.6. □

Corollary 3.9. *If U is an ideal of a finite BCI-algebra S, then the neutrosophic quadruple U-set $NQ(U)$ is a regular neutrosophic quadruple ideal of $NQ(S)$.*

4 Neutrosophic quadruple q-ideals

Definition 4.1. Given nonempty subsets U and V of S, if the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a q-ideal of a neutrosophic quadruple BCI-algebra $NQ(S)$, we say $NQ(U, V)$ is a *neutrosophic quadruple q-ideal* of $NQ(S)$.

Example 4.2. Consider a BCI-algebra $S = \{0, 1, a\}$ with the binary operation $*$, which is given in Table 2. Then the neutrosophic quadruple BCI-algebra $NQ(S)$ has 81 elements. If we take $U = \{0, 1\}$ and $V = \{0, 1\}$, then

$$NQ(U, V) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}, \tilde{4}, \tilde{5}, \tilde{6}, \tilde{7}, \tilde{8}, \tilde{9}, \tilde{10}, \tilde{11}, \tilde{12}, \tilde{13}, \tilde{14}, \tilde{15}\}$$

is a neutrosophic quadruple q-ideal of $NQ(S)$ where
$\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{1} = (0, 0T, 0I, 1F)$, $\tilde{2} = (0, 0T, 1I, 0F)$,
$\tilde{3} = (0, 0T, 1I, 1F)$, $\tilde{4} = (0, 1T, 0I, 0F)$, $\tilde{5} = (0, 1T, 0I, 1F)$,
$\tilde{6} = (0, 1T, 1I, 0F)$, $\tilde{7} = (0, 1T, 1I, 1F)$, $\tilde{8} = (1, 0T, 0I, 0F)$,
$\tilde{9} = (1, 0T, 0I, 1F)$, $\tilde{10} = (1, 0T, 1I, 0F)$, $\tilde{11} = (1, 0T, 1I, 1F)$,

Table 2: Cayley table for the binary operation "$*$"

$*$	0	1	a
0	0	0	a
1	1	0	a
a	a	a	0

$\tilde{12} = (1, 1T, 0I, 0F)$, $\tilde{13} = (1, 1T, 0I, 1F)$,
$\tilde{14} = (1, 1T, 1I, 0F)$, $\tilde{15} = (1, 1T, 1I, 1F)$.

Theorem 4.3. *For any nonempty subsets U and V of a BCI-algebra S, if the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$, then it is both a neutrosophic quadruple subalgebra and a neutrosophic quadruple ideal of $NQ(S)$.*

Proof. Assume that $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$. Since $\tilde{0} \in NQ(U, V)$, we have $0 \in U$ and $0 \in V$. Let $x, y, z \in S$ be such that $x * (y * z) \in U \cap V$ and $y \in U \cap V$. Then $(y, yT, yI, yF) \in NQ(U, V)$ and

$$\begin{aligned}&(x, xT, xI, xF) \circledast ((y, yT, yI, yF) \circledast (z, zT, zI, zF))\\&= (x, xT, xI, xF) \circledast (y * z, (y * z)T, (y * z)I, (y * z)F)\\&= (x * (y * z), (x * (y * z))T, (x * (y * z))I, (x * (y * z))F) \in NQ(U, V).\end{aligned}$$

Since $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$, it follows that

$$(x * z, (x * z)T, (x * z)I, (x * z)F) = (x, xT, xI, xF) \circledast (z, zT, zI, zF) \in NQ(U, V).$$

Hence $x * z \in U \cap V$, and therefore U and V are q-ideals of S. Since every q-ideal is both a subalgebra and an ideal, it follows from Lemma 3.3 that $NQ(U, V)$ is both a neutrosophic quadruple subalgebra and a neutrosophic quadruple ideal of $NQ(S)$. □

The converse of Theorem 4.3 is not true as seen in the following example.

Example 4.4. Consider a BCI-algebra $S = \{0, a, b, c\}$ with the binary operation $*$, which is given in Table 3.

Table 3: Cayley table for the binary operation "$*$"

$*$	0	a	b	c
0	0	c	b	a
a	a	0	c	b
b	b	a	0	c
c	c	b	a	0

Then the neutrosophic quadruple BCI-algebra $NQ(S)$ has 256 elements. If we take $A = \{0\}$ and $B = \{0\}$, then $NQ(U, V) = \{\tilde{0}\}$ is both a neutrosophic quadruple subalgebra and a neutrosophic quadruple ideal of $NQ(S)$. If we take $\tilde{x} := (c, bT, 0I, aF)$, $\tilde{z} := (a, bT, 0I, cF) \in NQ(S)$, then

$$\begin{aligned}\tilde{x} \circledast (\tilde{0} \circledast \tilde{z}) &= (c, bT, 0I, aF) \circledast (\tilde{0} \circledast (a, bT, 0I, cF)) \\ &= (c, bT, 0I, aF) \circledast (c, bT, 0I, aF) = \tilde{0} \in NQ(U, V).\end{aligned}$$

But

$$\begin{aligned}\tilde{x} \circledast \tilde{z} &= (c, bT, 0I, aF) \circledast (a, bT, 0I, cF) \\ &= (c * a, (b * b)T, (0 * 0)I, (a * c)F) \\ &= (b, 0T, 0I, bF) \notin NQ(U, V).\end{aligned}$$

Therefore $NQ(U, V)$ is not a neutrosophic quadruple q-ideal of $NQ(S)$.

We provide conditions for the neutrosophic quadruple (U, V)-set $NQ(U, V)$ to be a neutrosophic quadruple q-ideal.

Theorem 4.5. *If U and V are q-ideals of a BCI-algebra S, then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.*

Proof. Suppose that U and V are q-ideals of a BCI-algebra S. Obviously, $\tilde{0} \in NQ(U, V)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(S)$ be such that $\tilde{x} \circledast (\tilde{y} \circledast \tilde{z}) \in NQ(U, V)$ and $\tilde{y} \in NQ(U, V)$. Then $y_i \in A$, $y_j \in B$ for $i = 1, 2$ and $j = 3, 4$, and

$$\begin{aligned}\tilde{x} \circledast (\tilde{y} \circledast \tilde{z}) &= (x_1, x_2T, x_3I, x_4F) \circledast ((y_1, y_2T, y_3I, y_4F) \circledast (z_1, z_2T, z_3I, z_4F)) \\ &= (x_1, x_2T, x_3I, x_4F) \circledast (y_1 * z_1, (y_2 * z_2)T, (y_3 * z_3)I, (y_4 * z_4)F) \\ &= (x_1 * (y_1 * z_1), (x_2 * (y_2 * z_2))T, (x_3 * (y_3 * z_3))I, (x_4 * (y_4 * z_4))F) \\ &\in NQ(U, V),\end{aligned}$$

that is, $x_i * (y_i * z_i) \in U$ and $x_j * (y_j * z_j) \in B$ for $i = 1, 2$ and $j = 3, 4$. It follows from (2.10) that $x_i * z_i \in U$ and $x_j * z_j \in V$ for $i = 1, 2$ and $j = 3, 4$. Thus

$$\tilde{x} \circledast \tilde{z} = (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \in NQ(U, V), \tag{4.1}$$

and therefore $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$. □

Corollary 4.6. *If A is a q-ideal of a BCI-algebra S, then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.*

Corollary 4.7. *If $\{0\}$ is a q-ideal of a BCI-algebra S, then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ for any ideals U and V of S.*

Corollary 4.8. *If $\{0\}$ is a q-ideal of a BCI-algebra S, then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ for any ideal U of S.*

Theorem 4.9. *Let U and V be ideals of a BCI-algebra S such that*

$$(\forall x, y, z \in S)(x * (y * z) \in U \cap V \ \Rightarrow \ (x * y) * z \in U \cap V). \tag{4.2}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. It is clear that $\tilde{0} \in NQ(U, V)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $NQ(S)$ be such that $\tilde{x} \circledast (\tilde{y} \circledast \tilde{z}) \in NQ(U, V)$ and $\tilde{y} \in NQ(U, V)$. Then $y_1, y_2 \in U$, $y_3, y_4 \in V$ and

$$\begin{aligned}
\tilde{x} \circledast (\tilde{y} \circledast \tilde{z}) &= (x_1, x_2T, x_3I, x_4F) \circledast ((y_1, y_2T, y_3I, y_4F) \circledast (z_1, z_2T, z_3I, z_4F)) \\
&= (x_1, x_2T, x_3I, x_4F) \circledast (y_1 * z_1, (y_2 * z_2)T, (y_3 * z_3)I, (y_4 * z_4)F) \\
&= (x_1 * (y_1 * z_1), (x_2 * (y_2 * z_2))T, (x_3 * (y_3 * z_3))I, (x_4 * (y_4 * z_4))F) \\
&\in NQ(U, V),
\end{aligned}$$

that is, $x_i * (y_i * z_i) \in U$ and $x_j * (y_j * z_j) \in V$ for $i = 1, 2$ and $j = 3, 4$. It follows from (2.3) and (4.2) that $(x_i * z_i) * y_i = (x_i * y_i) * z_i \in U$ and $(x_j * z_j) * y_j = (x_j * y_j) * z_j \in V$ for $i = 1, 2$ and $j = 3, 4$. Since U and V are ideals of S, we have $x_i * z_i \in U$ and $x_j * z_j \in V$ for $i = 1, 2$ and $j = 3, 4$. Thus

$$\tilde{x} \circledast \tilde{z} = (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \in NQ(U, V), \tag{4.3}$$

and therefore $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$. □

Corollary 4.10. *Let U be an ideal of a BCI-algebra S such that*

$$(\forall x, y, z \in S)(x * (y * z) \in U \ \Rightarrow \ (x * y) * z \in U). \tag{4.4}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Theorem 4.11. *Let U and V be ideals of a BCI-algebra S such that*

$$(\forall x, y \in S)(x * (0 * y) \in U \cap V \ \Rightarrow \ x * y \in U \cap V). \tag{4.5}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. Assume that $x * (y * z) \in U \cap V$ for all $x, y, z \in S$. Note that

$$\begin{aligned}
((x * y)) * (0 * z)) * (x * (y * z)) &= ((x * y) * (x * (y * z))) * (0 * z) \\
&\leq ((y * z) * y) * (0 * z) \\
&= (0 * z) * (0 * z) = 0 \in U \cap V
\end{aligned}$$

Thus $(x * y) * (0 * z) \in U \cap V$ since U and V are ideals of S. It follows from (4.9) that $(x * y) * z \in U \cap V$. Using Theorem 4.9, $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$. □

Corollary 4.12. *Let U be an ideal of a BCI-algebra S such that*

$$(\forall x, y \in S)(x * (0 * y) \in U \ \Rightarrow \ x * y \in U). \tag{4.6}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Theorem 4.13. *Let U and V be ideals of a BCI-algebra S such that*

$$(\forall x, y \in S)(x \in U \cap U \ \Rightarrow \ x * y \in U \cap V). \tag{4.7}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. Assume that $x * (y * z) \in U \cap V$ and $y \in U \cap V$ for all $x, y, z \in S$. Using (2.3) and (4.7), we get $(x * z) * (y * z) = (x * (y * z)) * z \in U \cap V$ and $y * z \in U \cap V$. Since U and V are ideals of S, it follows that $x * z \in U \cap V$. Hence U and V are q-ideals of S, and therefore $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ by Theorem 4.5. □

Corollary 4.14. *Let U be an ideal of a BCI-algebra S such that*

$$(\forall x, y \in S)(x \in U \ \Rightarrow \ x * y \in U). \tag{4.8}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Theorem 4.15. *Let U, V, I and J be ideals of a BCI-algebra S such that $I \subseteq U$ and $J \subseteq V$. If I and J are q-ideals of S, then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.*

Proof. Let $x, y, z \in S$ be such that $x * (0 * y) \in U \cap V$. Then

$$(x * (x * (0 * y))) * (0 * y) = (x * (0 * y)) * (x * (0 * y)) = 0 \in I \cap J$$

by (2.3) and (III). Since I and J are q-ideals of S, it follows from (2.3) and (2.10) that

$$(x * y) * (x * (0 * y)) = (x * (x * (0 * y))) * y \in I \cap J \subseteq U \cap V$$

Since U and V are ideals of S, we have $x * y \in U \cap V$. Therefore $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ by Theorem 4.11. □

Corollary 4.16. *Let U and I be ideals of a BCI-algebra S such that $I \subseteq U$. If I is a q-ideal of S, then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.*

Theorem 4.17. *Let U, V, I and J be ideals of a BCI-algebra S such that $I \subseteq U$, $J \subseteq V$ and*

$$(\forall x, y, z \in S)(x * (y * z) \in I \cap J \ \Rightarrow \ (x * y) * z \in I \cap J). \tag{4.9}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. Let $x, y, z \in S$ be such that $x * (y * z) \in I \cap J$ and $y \in I \cap J$. Then

$$(x * z) * y = (x * y) * z \in I \cap J$$

by (2.3) and (4.9). Since I and J are ideals of S, it follows that $x * z \in I \cap J$. This shows that I and J are q-ideals of S. Therefore $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ by Theorem 4.15. □

Corollary 4.18. *Let U and I be ideals of a BCI-algebra S such that $I \subseteq U$ and*

$$(\forall x, y, z \in S)(x * (y * z) \in I \ \Rightarrow \ (x * y) * z \in I). \tag{4.10}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Theorem 4.19. *Let U, V, I and J be ideals of a BCI-algebra S such that $I \subseteq U$, $J \subseteq V$ and*

$$(\forall x, y \in S)(x \in I \cap J \;\Rightarrow\; x * y \in I \cap J). \tag{4.11}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. By the proof of Theorem 4.13, we know that I and J are q-ideals of S. Hence $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$ by Theorem 4.15. □

Corollary 4.20. *Let U and I be ideals of a BCI-algebra S such that $I \subseteq U$ and*

$$(\forall x, y \in S)(x \in I \;\Rightarrow\; x * y \in I). \tag{4.12}$$

Then the neutrosophic quadruple A-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Theorem 4.21. *Let U, V, I and J be ideals of a BCI-algebra S such that $I \subseteq U$, $J \subseteq V$ and*

$$(\forall x, y \in S)(x * (0 * y) \in I \cap J \;\Rightarrow\; x * y \in I \cap J). \tag{4.13}$$

Then the neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Proof. Assume that $x*(y*z) \in I \cap J$ For all $x, y, z \in S$. Then $(x*y)*z \in I \cap J$ by the proof of Theorem 4.11. It follows from Theorem 4.17 that neutrosophic quadruple (U, V)-set $NQ(U, V)$ is a neutrosophic quadruple q-ideal of $NQ(S)$. □

Corollary 4.22. *Let U and I be ideals of a BCI-algebra S such that $I \subseteq U$ and*

$$(\forall x, y \in S)(x * (0 * y) \in I \;\Rightarrow\; x * y \in I). \tag{4.14}$$

Then the neutrosophic quadruple U-set $NQ(U)$ is a neutrosophic quadruple q-ideal of $NQ(S)$.

Future Work: Using the results of this paper, we will aply it to another algebraic structures, for example, MV-algebras, BL-algebras, MTL-algebras, R_0-algebras, hoops, (ordered) semigroups and (semi, near) rings etc.

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