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Neutrosophic Quotient Submodules and Homomorphisms

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Abstract.: In this paper, we define two different kinds of neutrosophic submodules over a classical quotient R-module using single valued neutrosophic set. We also define neutrosophic submodule homomorphism and study the features of neutrosophic set under R-module homomorphism. Finally we conduct an investigation for the image and inverse image of neutrosophic submodule under classical homomorphim of R-module.

Key Words: Module, Neutrosophic set, Neutrosophic submodule, Quotient module, Homomorphism of neutrosophic submodule

1. Introduction

The Oxford English dictionary defines the term fuzzy as not clear or vague. In 1965, Lotfi A. Zadeh defined a fuzzy set which represent vague concepts and contexts expressed in natural language by means of graded membership of elements in $[0, 1]$ [19, 38]. In 1986 Attanassov put forward intuitionistic fuzzy set theory as a stereotype illustration of a set in which each component is concomitant with a membership grades and non membership grades [4]. In 1995, Smarandache outlined neutrosophic set in which each element of a set is represented by three differing types of membership values [34]. Neutrosophic set is a tool or a framework for sorting out vague, obscure and contrary data in the genuine world pragmatic problems([37, 7, 39, 36]). Neutrosophy is another part of theory and rationale that has focused nature's provenance and equability features. [17]. Each element of a neutrosophic set is defined by three contrasting types of registration estimates that talk to condensed, imprecise and absurd information ([13, 30, 29, 1, 17]).

The algebraic structure in pure mathematics cloning with uncertainty has been studied by some authors. In 1971, Azriel Rosenfield bestowed a seminal paper on fuzzy subgroup and W.J. Liu developed the idea of fuzzy normal subgroup and fuzzy subring. Consolidating neutrosophic set hypothesis with algebraic structures is a rising pattern in the region of mathematical research. In 2011, Isaac.P, P.P.John [16] recognized some algebraic nature of intuitionistic fuzzy submodule of a classical module. Neutrosophic algebraical structures and its properties provide us a solid mathematical foundation to clarify connected scientific ideas in designing, information mining and economic science([2],[26],[28]).

2. Preliminaries

Definition 2.1. *(*[3]*) A module M over a ring R, denoted as M_R, is an abelian group with a law of composition written '$+'$ and the map $R \times M \rightarrow M$, written $(\varrho, \vartheta) \rightsquigarrow \varrho\vartheta$, that satisfy these axioms*

(1) $1\vartheta = \vartheta$
(2) $(\varrho\tau)\vartheta = \varrho(\tau\vartheta)$
(3) $(\varrho + \tau)\vartheta = \varrho\vartheta + \tau\vartheta$
(4) $\varrho(\vartheta + \vartheta') = \varrho\vartheta + \varrho\vartheta' \ \forall \ \varrho, \tau \in R$ *and* $\vartheta, \vartheta` \in M$.

Definition 2.2. *(*[3]*) A submodule W of M_R is a nonempty subset that is closed under addition and scalar multiplication.*

Definition 2.3. [3] *A homomorphism $\Upsilon : V \rightarrow W$ of R-modules copies that of a linear transformation of vector spaces. It is a map compatible with the laws of composition:*

$$\Upsilon(\vartheta + \vartheta') = \Upsilon(\vartheta) + \Upsilon(\vartheta')\ and\ \Upsilon(\varrho\vartheta) = \varrho\Upsilon(\vartheta)$$

denoted as $Hom_R(M, N)$, $\forall\ \vartheta, \vartheta' \in V\ and\ \varrho \in R$. If Υ is bijective, then ϑ is isomorphic to W .

Definition 2.4. [6, 3] *The kernel of a homomorphism $\Upsilon : V \rightarrow W$, the collection of elements $\vartheta \in V$ in which $\Upsilon(\vartheta) = 0$, is a submodule of the domain V.*

The image of a homomorphism $\Upsilon : V \rightarrow W$, the collection of elements w in W such that $\Upsilon(\vartheta) = w,\ forall\ \vartheta \in\ V$, is a submodule of the range W.

Definition 2.5. [12, 3] *Let $N \subseteq M_R$. Then the quotient module M/N is the group of additive cosets $\eta + N,\ \eta \in M$.*

Remark 2.6. $[\eta]$ *represents the coset $\eta + N, \forall\ \eta \in M$*

Remark 2.7. $\varrho[\eta] = [\varrho\eta]\ \forall\ \varrho \in R$

Definition 2.8. [22, 15, 11, 5] *Let R be an integral domain. Then M_R is said to be divisible if $\forall\eta \in M$ can be divided by $\varrho \in R$, in the sense that,*

$$0 \neq \varrho \in R,\ \eta \in M \Rightarrow \eta = \varrho n\ for\ some\ n \in M$$

.

Definition 2.9. [23, 21, 14] *A submodule N of M_R is said to be a prime submodule of M if $\varrho\eta \in N, \varrho \in R, \eta \in N \Rightarrow either\ \varrho = 0\ or\ \eta \in N$.*

Definition 2.10. [32, 35] *A neutrosophic set P of the universal set X is defined as*

$$P = \{(\eta, t_P(\eta), i_P(\eta), f_P(\eta)) : \eta \in X\}$$

where $t_P, i_P, f_P : X \to (^-0, 1^+)$. *The three components* t_P, i_P *and* f_P *represent membership value (Percentage of truth), indeterminacy (Percentage of indeterminacy) and non membership value (Percentage of falsity) respectively. These components are functions of non standard unit interval* $(^-0, 1^+)$ [25].

Remark 2.11. [32, 13]

(1) *If* $t_P, i_P, f_P : X \to [0, 1]$, *then P is known as single valued neutrosophic set(SVNS).*
(2) *In this paper, we discuss about the algebraic structure* R*-module with underlying set as SVNS. For simplicity SVNS will be called neutrosophic set.*
(3) U^X *denotes the set of all neutrosophic subset of* X *or neutrosophic power set of* X.

Definition 2.12. [32, 24, 33] *Let* $P, Q \in U^X$. *Then P is contained in Q, denoted as* $P \subseteq Q$ *if and only if* $P(\eta) \leqslant Q(\eta)\ \forall \eta \in X$, *this means that*

$$t_P(\eta) \leq t_Q(\eta), i_P(\eta) \leq i_Q(\eta), f_P(\eta) \geq f_Q(\eta),\ \forall\ \eta \in X$$

Definition 2.13. [32, 27, 18] *The complement of* $P = \{(\eta, t_P(\eta), i_P(\eta), f_P(\eta)\}$ *is denoted by* P^C *and defined as* $P^C = \{\eta, f_P(\eta), 1 - i_P(\eta), t_P(\eta)\}$ *and* $(P^C)^C = P$

Definition 2.14. [9, 32, 18] *Let* $P, Q \in U^X$.

(1) *The union* $C = \{\eta, t_C(\eta), i_C(\eta), f_C(\eta) : \eta \in X\}$ *of P and Q* [24] *is denoted by* $C = P \cup Q$ *where*

$$t_C(\eta) = t_P(\eta) \vee t_Q(\eta)$$
$$i_C(\eta) = i_P(\eta) \vee i_Q(\eta)$$
$$f_C(\eta) = f_P(\eta) \wedge f_Q(\eta)$$

(2) *The intersection* $C = \{\eta, t_C(\eta), i_C(\eta), f_C(\eta) : \eta \in \eta\}$ *of P and Q* [24] *is denoted by* $C = P \cap Q$ *where*

$$t_C(\eta) = t_P(\eta) \wedge t_Q(\eta)$$
$$i_C(\eta) = i_P(\eta) \wedge i_Q(\eta)$$
$$f_C(\eta) = f_P(\eta) \vee f_Q(\eta)$$

Definition 2.15. [31] *The sum* $P + Q = \{\eta, t_{P+Q}(\eta), t_{P+Q}(\eta), t_{P+Q}(\eta) : \eta \in M_R\}$ *of two neutrosophic sets* P *and* Q *is a neutrosophic set of* M_R, *defined as follows*

$$t_{P+Q}(\eta) = \vee\{t_P(\theta) \wedge t_Q(\vartheta) | \eta = \theta + \vartheta, \theta, \vartheta \in M_R\}$$
$$i_{P+Q}(\eta) = \vee\{i_P(\theta) \wedge i_Q(\vartheta) | \eta = \theta + \vartheta, \theta, \vartheta \in M_R\}$$
$$f_{P+Q}(\eta) = \wedge\ f_P(\theta) \vee f_B(\vartheta) | \eta = \theta + \vartheta, \theta, \vartheta \in M_R\}$$

Definition 2.16. [33, 24]*For any neutrosophic subset* $P = \{(\eta, t_P(\eta), i_P(\eta), f_P(\eta)) : \eta \in X\}$, *the support* P^* *of the neutrosophic set P can be defined as*

$$P^* = \{\eta \in X, t_P(\eta) > 0, i_P(\eta) > 0, f_P(\eta) < 1\}$$

.

Definition 2.17. [8, 20, 35, 18] *The image of P, where $P \in U^X$, under the map $g : X \to Y$ is denoted by $g(P)$ and is defined as $g(P) = \{\theta, t_{g(P)}(\theta), i_{g(P)}(\theta), f_{g(P)}(\theta) : \theta \in Y\}$ where*

$$t_{g(P)}(\theta) = \begin{cases} \vee t_P(\eta) : \eta \in g^{-1}(\theta)) & g^{-1}(\theta) \neq \emptyset \\ 0 & \textit{otherwise} \end{cases}$$

$$i_{g(P)}(\theta) = \begin{cases} \vee i_P(\eta) : \eta \in g^{-1}(\theta)) & g^{-1}(\theta) \neq \emptyset \\ 0 & otherwise \end{cases}$$

$$f_{g(P)}(\theta) = \begin{cases} \wedge f_P(\eta) : \eta \in g^{-1}(\theta)) & g^{-1}(\theta) \neq \emptyset \\ 1 & otherwise \end{cases}$$

Furthermore, the inverse of g , denoted by $g^{-1} : Y \to X$ is defined by

$$g^{-1}(Q) = \{\eta, t_{g^{-1}(Q)}(\eta), i_{g^{-1}(Q)}(\eta), f_{g^{-1}(Q)}(\eta) : g(\eta) \in Q\}$$

where

$$t_{g^{-1}(Q)}(\eta) = t_Q(g(\eta)),\ i_{g^{-1}(Q)}(\eta) = i_Q(g(\eta)),\ f_{g^{-1}(Q)}(\eta) = f_Q(g(\eta)) \,\forall\, \eta \in X$$

Definition 2.18. [9, 10] *Let $P \in U^M$ where $M \in M_R$. Then the neutrosophic subset P of M is called a neutrosophic submodule of M if*

(1) $t_P(0) = 1, i_P(0) = 1, f_P(0) = 0$
(2) $t_P(\eta + \theta) \geq t_P(\eta) \wedge t_P(\theta)$
$i_P(\eta + \theta) \geq i_P(\eta) \wedge i_P(\theta)$
$f_P(\eta + \theta) \leq f_P(\eta) \vee f_P(\theta),\ for\ all\ \eta, \theta\ in\ M$
(3) $t_P(\gamma\eta) \geq t_P(\eta)$
$i_P(\gamma\eta) \geq i_P(\eta)$
$f_P(\gamma\eta) \leq f_P(\eta), for\ all\ \eta\ in\ M_R,\ for\ all\ \gamma\ in\ R$

Remark 2.19. *We denote neutrosophic submodules over a classical R-module using single valued neutrosophic set by $U(M)$.*

Remark 2.20. *If $P \in U(M)$, then the neutrosophic components of P can be denoted as $(t_P(\eta), i_P(\eta), f_P(\eta))$.*

Definition 2.21. [9] *Define the neutrosophic set $\gamma P = \{\eta, t_{\gamma P}(\eta), i_{\gamma P}(\eta), f_{\gamma P}(\eta) : \eta \in M, \gamma \in R\}$ of M_R where $P \in U^M$ as follows*

$$t_{\gamma P}(\eta) = \vee\{t_P(\theta) : \theta \in M_R, \eta = \gamma\theta\}$$

$$i_{\gamma P}(\eta) = \vee\{i_P(\theta) : \theta \in M_R, \eta = \gamma\theta\}$$

$$f_{\gamma P}(\eta) = \wedge\{f_P(\theta) : \theta \in M_R, \eta = \gamma\theta\}$$

3. Construction of Neutrosophic Quotient Submodules

In this precinct, we elucidate two different aspects or methods within the formation of neutrosophic quotient submodule of the classical $(M/N)_R$ where $N \subseteq M$.

Method 1:

Theorem 3.1. *If $P = \{\eta, t_P(\eta), i_P(\eta), f_P(\eta) : \eta \in M\} \in U(M)$ and $N \subseteq M$, then define ω, a neutrosophic set in M/N as follows.*

$$\omega = \{[\eta], t_\omega([\eta]), i_\omega([\eta]), f_\omega([\eta]) : \eta \in M\}$$

where

$$t_\omega([\eta]) = \vee\{t_P(u) : u \in [\eta]\}$$
$$i_\omega([\eta]) = \vee\{i_P(u) : u \in [\eta]\}$$
$$f_\omega([\eta]) = \wedge\{f_P(u) : u \in [\eta]\}$$

Then $\omega \in U(M/N)$

Proof. We have $t_\omega([0]) = \vee\{t_P(u) : u \in [0]\} = t_P(0) = 1$, similarly $i_\omega([0]) = 1$ and $f_\omega([0]) = 0$
Now for $\eta, \theta \in M$

$$\begin{aligned} t_\omega([\eta] + [\theta]) &= \vee\{t_P(u) : u \in [\eta] + [\theta]\} \\ &= \vee\{t_P(\zeta + \xi) : \zeta + \xi \in [\eta] + [\theta]\} \\ &\geq \vee\{t_P(\zeta + \xi) : \zeta \in [\eta], \xi \in [\theta]\} \\ &\geq \vee\{t_P(\zeta) \wedge t_P(\xi) : \zeta \in [\eta], \xi \in [\theta]\} \\ &= (\vee\{t_P(\zeta) : \zeta \in [\eta]\}) \wedge (\vee\{t_P(\xi) : \xi \in [\theta]\}) \\ &= t_\omega([\eta]) + t_\omega([\theta]) \end{aligned}$$

then correspondingly

$$i_\omega([\eta] + [\theta]) \geq i_\omega([\eta]) \wedge i_\omega([\theta])$$

and

$$f_Q([\eta] + [\theta]) \leq f_\omega([\eta]) \vee f_\omega([\theta])$$

Now for all $\varrho \; in \; R, \eta \; in \; M$,

$$\begin{aligned} t_\omega(\varrho[\eta]) &= t_\omega([\varrho\eta]) \\ &\geq \vee\{t_P(\varrho u) : \varrho u \in [\varrho\eta]\} \\ &\geq \vee\{t_P(u) : u \in [\eta]\} \\ &= t_\omega([\eta]) \end{aligned}$$

In the same way, we can conclude

$$i_\omega(\varrho[\eta]) \geq i_\omega([\eta]) \; and \; f_\omega(\varrho[\eta]) \leq f_\omega([\eta])$$

Thus $\omega \in U(M/N)$. □

Method 2:

Theorem 3.2. *Let R be an integral domain and M be a divisible module over R. Consider a prime submodule N of M. If $P \in U(M)$, define a neutrosophic set ω in quotient module M/N defined as, for $\eta \in M$*

$$t_\omega([\eta]) = \begin{cases} 1 & [\eta] = N \\ \wedge\{t_P(u) : u \in [\eta]\} & otherwise \end{cases}, \ i_\omega(\eta) = \begin{cases} 1 & [\eta] = N \\ \wedge\{i_P(u) : u \in [\eta]\} & otherwise \end{cases}$$

and

$$f_\omega([\eta]) = \begin{cases} 0 & [\eta] = N \\ \vee\{f_P(u) : u \in [\eta]\} & otherwise \end{cases}$$

Then $\omega \in U(M/N)$.

Proof. Since $N \subseteq M$, the neutrosophic components of $[0] = (1, 1, 0)$
Now for $\eta, \theta \ in \ M$, consider

$$t_\omega([\eta] + [\theta]) = \begin{cases} 1 & [\eta] + [\theta] = N \\ \wedge\{t_P(u) : u \in [\eta] + [\theta]\} & otherwise \end{cases}$$

Case 1: If $[\eta] + [\theta] = N$, then clearly $t_\omega([\eta] + [\theta]) \geq t_\omega([\eta]) \wedge t_\omega([\theta])$, $i_\omega([\eta] + [\theta]) \geq i_\omega([\eta]) \wedge i_\omega([\theta])$ and $f_\omega([\eta] + [\theta]) \leq f_\omega([\eta]) \vee f_\omega([\theta])$

Case 2: If $[m] + [n] \neq N$, then

$$\begin{aligned} t_\omega([\eta] + [\theta]) &= \wedge\{t_P(u) : u \in [\eta] + [\theta]\} \\ &= \wedge\{t_P((\eta + \zeta) + (\theta + \xi) : \zeta, \xi \in N\} \\ &\geq \wedge\{t_P(\eta + \zeta) \wedge t_P(\theta + \xi) : \zeta, \xi \in N\} \\ &\geq (\wedge\{t_P(\eta + \zeta) : \zeta \in N\}) \wedge (\wedge\{t_P(\theta + \xi) : \xi \in N\}) \\ &= (\wedge\{t_P(v_1) : v_1 \in [\eta]\}) \wedge (\wedge\{t_P(v_2) : v_2 \in [\theta]\}) \\ &= t_\omega([\eta]) \wedge t_\omega([\theta]) \end{aligned}$$

In the same manner,

$$i_\omega([\eta] + [\theta]) \geq i_\omega([\eta]) \wedge i_\omega([\theta])$$

and

$$f_\omega([\eta] + [\theta]) \leq f_\omega([\eta]) \vee f_\omega([\theta]).$$

For $\varrho \ in \ R, \eta \ in \ M$, consider

$$t_\omega(\varrho[\eta]) = \begin{cases} 1 & \varrho[\eta] = N \\ \wedge\{t_A(u) : u \in \varrho[\eta]\} & otherwise \end{cases}$$

Case 3: If $\varrho[\eta] = N$, $t_\omega(\varrho[\eta]) = 1 \geq t_\omega([\eta])$, similarly $i_\omega(\varrho[\eta]) \geq i_\omega([\eta]) \ and$ $f_\omega(\varrho[\eta]) \leq f_\omega([\eta])$

Case 4: If $\varrho[\eta] \neq N \Rightarrow \varrho\eta \notin N \Rightarrow \varrho \neq 0, \eta \notin N$ and

$$\begin{aligned} t_\omega(\varrho[\eta]) &= t_\omega([\varrho\eta]) \\ &= \wedge\{t_P(u) : u \in [\varrho\eta]\} \\ &= \wedge\{t_P(\varrho\eta + v) : v \in N\} \\ &= \wedge\{t_P(\varrho\eta + \varrho\theta) : \theta \in N\} \\ &= \wedge\{t_P(\varrho(\eta + \theta)) : \theta \in N\} \\ &= \wedge\{t_P(\eta + \theta) : \theta \in N\} \\ &= \wedge\{t_P(w) : w \in [\eta]\} \\ &= t_\omega([\eta]) \end{aligned}$$

Correspondingly $i_\omega(\varrho[\eta]) \geq i_\omega([\eta])\ and\ f_\omega(\varrho[\eta]) \leq f_\omega([\eta])$
Hence $\omega \in U(M/N)$

□

Corollary 3.3. *If N is contained in M where $M \in M_R$ and R is a field and $P \in U(M)$. Then by the theorem 3.2, $\omega \in U(M/N)$.*

Definition 3.4. *If $P \in U(M)\ and\ N \subseteq M$, then the restriction of a neutrosophic submodule P to N is represented by $P|_N$ and it is a neutrosophic set of N defined as $P|_N = \{\theta, t_{P|_N}(\theta), i_{P|_N}(\theta), f_{P|_N}(\theta)\}$ where $\forall\ \theta \in N$ and*

$$t_{P|_N}(\theta) = t_P(\theta)$$

$$i_{P|_N}(\theta) = i_P(\theta)$$

$$f_{P|_N}(\theta) = f_P(\theta)$$

Proposition 3.5. *If $P \in U(M)\ and\ N \subseteq M$, then $P|_N \in U(N)$.*

Proof. If $P = \{\eta, t_P(\eta), i_P(\eta), f_P(\eta) : \eta \in M\} \in U(M)$ and $N \subseteq M$ then the

$$t_{P|_N}(0) = t_P(0) = 1,\ i_{P|_N}(0) = i_P(0) = 1\ and\ f_{P|_N}(0) = f_P(0) = 0.$$

Now $\varrho\ in\ R,\ \eta\ in\ M$

$$\begin{aligned} t_{P|_N}(\varrho\eta) &= t_P(\varrho\eta) \\ &\geq t_P(\eta) \\ &= t_{P|_N}(\eta) \end{aligned}$$

Similarly $i_{P|_N}(\varrho\eta) \geq i_{P|_N}(\eta)\ and\ f_{P|_N}(\varrho\eta) \leq f_{P|_N}(\eta)$
Now $\theta, \vartheta \in N$

$$\begin{aligned} t_{P|_N}(\theta + \vartheta) &= t_P(\theta + \vartheta) \\ &\geq t_P(\theta) \wedge t_P(\vartheta) \\ &= t_{P|_N}(\theta) \wedge t_{P|_N}(\vartheta) \end{aligned}$$

Similarly $i_{P|_N}(\theta + \vartheta) \geq i_{P|_N}(\theta) \wedge i_{P|_N}(\vartheta),\ f_{P|_N}(\theta + \vartheta) \leq f_{P|_N}(\theta) \vee f_{P|_N}(\vartheta)$
Thus $P|_N \in U(N)$ □

Remark 3.6. *Let $P, Q \in U(M)$ and $P \subseteq Q$. Then $P^* \subseteq Q^*$ and $Q|_{Q^*} \in U(Q^*)$. Now define Q, a neutrosophic set of Q^*/P^* where for $\eta \in Q^*$*

$$t_\omega([\eta]) = \vee\{t_P(\zeta) : \zeta \in [\eta]\}$$

$$i_\omega([\eta]) = \vee\{i_P(\zeta) : \zeta \in [\eta]\}$$

$$f_\omega([\eta]) = \wedge\{f_P(\zeta) : u \in [\eta]\}$$

Then by the theorem 3.2, $\omega \in U(Q^*/P^*)$ and it is denoted as Q/P.

Remark 3.7. *We write P_N for the neutrosophic quotient submodule ω of M/N, i.e. $P_N \in U(M/N)$*

4. Homomorphisms of Neutrosophic Submodules

In this section we study about the inherent attributes of the image and inverse image of a neutrosophic set and a neutrosophic submodule under classical module homomorphism and the homomorphism properties of neutrosophic submodules.

Let M and N be the R modules and $\Upsilon \in Hom_R(M, N)$. Also $P \in U(M)$ and $Q \in U(N)$

Definition 4.1. *$\Upsilon \in Hom_R(M, N)$ is called a weak neutrosophic homomorphism of P onto Q if $\Upsilon(P) \subseteq Q$ and we denote it as $P \sim Q$.*

*$\Upsilon \in Hom_R(M, N)$ is called a **neutrosophic homomorphism** of P onto Q if $\Upsilon(P) = Q$ and we represent it as $P \approx Q$.*

Theorem 4.2. *Let $P,\ Q \in U^M$ and $\Upsilon \in Hom_R(M, N)$. Then*

(1) $\Upsilon(P + Q) = \Upsilon(P) + \Upsilon(Q)$
(2) $\Upsilon(rP) = r\Upsilon(P)\ \forall\ r \in R$
(3) $\Upsilon(r_1 P + r_2 Q) = r_1\Upsilon(P) + r_2\Upsilon(Q)\ \forall\ r_1, r_2 \in R$

Proof. **(1):** We have

$$\Upsilon(P + Q)(\theta) = \{\theta, t_{\Upsilon(P+Q)}(\theta), i_{\Upsilon(P+Q)}(\theta), f_{\Upsilon(P+Q)}(\theta) : \theta \in N\}$$

and

$$(\Upsilon(P) + \Upsilon(Q))(\theta) = \{\theta, t_{\Upsilon(P)+\Upsilon(Q)}(\theta), i_{\Upsilon(P)+\Upsilon(Q)}(\theta), f_{\Upsilon(P)+\Upsilon(Q)}(\theta) : \theta \in N\}$$

If $\Upsilon^{-1}(\theta) = \phi$, then $t_{\Upsilon(P+Q)}(\theta) = 0,\ i_{\Upsilon(P+Q)}(\theta) = 0$ and $f_{\Upsilon(P+Q)}(\theta) = 1$ also,

$$t_{\Upsilon(P)+\Upsilon(Q)}(\theta) = \vee\{t_{\Upsilon(P)}(\kappa) \wedge t_{\Upsilon(Q)}(\nu); y = \kappa + \nu, \kappa, \nu \in N\} = 0$$

since $\Upsilon^{-1}(\kappa) = \phi$ or $\Upsilon^{-1}(\nu) = \phi$ as $\Upsilon^{-1}(\theta) = \phi$

Thus $t_{\Upsilon(P+Q)}(\theta) = t_{\Upsilon(P)+\Upsilon(Q)}(\theta)$

If $\Upsilon^{-1}(\theta) \neq \phi$, then

$$\begin{aligned} t_{\Upsilon(P+Q)}(\theta) &= \vee\{t_{P+Q}(\eta) : \eta \in M, \theta = g(\eta)\} \\ &= \vee\{\vee\{t_P(\rho) \wedge t_Q(\varsigma) : \rho, \varsigma \in M, \eta = \rho + \varsigma\}; \theta = \Upsilon(\eta)\} \\ &= \vee\{\vee\{t_P(\rho) \wedge t_Q(\varsigma) : \rho, \varsigma \in M\} : \theta = \Upsilon(\rho) + \Upsilon(\varsigma)\} \\ &= \vee(\{\vee\{t_P(\rho) : \kappa = \Upsilon(\rho)\}\} \wedge \{\vee\{t_Q(\varsigma) : \nu = \Upsilon(\varsigma)\}\} : \theta = \kappa + \nu\}) \\ &= \vee\{t_{\Upsilon(P)}(\rho) \wedge t_{\Upsilon(Q)}(\varsigma) : \theta = \kappa + \nu\} \\ &= t_{\Upsilon(P)+t_{\Upsilon}(Q)}(\theta) \end{aligned}$$

Thus in both cases $t_{\Upsilon(P+Q)}(\theta) = t_{\Upsilon(P)+\Upsilon(Q)}(\theta)$. In the same way, we can prove that
$i_{\Upsilon(P+Q)}(\theta) = i_{\Upsilon(P)+\Upsilon(Q)}(\theta)\ and\ f_{\Upsilon(P+Q)}(\theta) = f_{\Upsilon(P)+\Upsilon(Q)}(\theta)$

(2) We have

$$\Upsilon(rP) = \{\theta, t_{\Upsilon(rP)}(\theta), i_{\Upsilon(rP)}(\theta), f_{\Upsilon(rP)}(\theta) : \theta \in N\}$$

and

$$r\Upsilon(P) = \{\theta, t_{r\Upsilon(P)}(\theta), i_{r\Upsilon(P)}(\theta), f_{r\Upsilon(P)}(\theta) : \theta \in N\}$$

If $\Upsilon^{-1}(\theta) = \phi,\ then\ t_{\Upsilon(rP)}(\theta) = 0$. Also

$$t_{r\Upsilon(P)}(\theta) = \vee\{t_{\Upsilon(P)}(\vartheta) : \vartheta \in N, \theta = r\vartheta\} = 0$$

Thus $t_{\Upsilon(rP)}(\theta) = t_{r\Upsilon(P)}(\theta)$
If $\Upsilon^{-1}(\theta) \neq \phi$, then

$$\begin{aligned} t_{\Upsilon(rP)}(\theta) &= \vee\{t_{rP}(\eta) : \eta \in M, \theta = \Upsilon(\eta)\} \\ &= \vee\{\vee\{t_P(u) : u \in M, \eta = ru\}\} \\ &= \vee\{\vee\{t_P(u) : u \in M, \theta = \Upsilon(ru)\}\} \\ &= \vee\{\vee\{t_P(u) : u \in M, \theta = r\Upsilon(u)\}\} \\ &= \vee\{t_{r\Upsilon(P)}(u) : \theta = r\Upsilon(u)\} \\ &= \vee t_{r\Upsilon(P)}(\theta) \end{aligned}$$

Thus we get $t_{\Upsilon(rP)}(\theta) = t_{r\Upsilon(P)}(\theta)\ \forall \theta \in N$. Similarly we get

$$i_{\Upsilon(rP)}(\theta) = i_{r\Upsilon(P)}(\theta)\ f_{\Upsilon(rP)}(\theta) = f_{r\Upsilon(P)}(\theta)$$

(3) This follows from $(1)\ and\ (2)$. □

Theorem 4.3. *If $P \in U(M)\ and\ \Upsilon \in Hom_R(M, N)$, then $\Upsilon(P) \in U(N)$.*

Proof. We have $\Upsilon(P) = \{(\theta, t_{\Upsilon(P)}(\theta), i_{\Upsilon(P)}(\theta), f_{\Upsilon(P)}(\theta)) : y \in N\}$. Then

$$t_{\Upsilon(P)}(0) = \vee\{t_P(\eta) : \eta \in M,\ \Upsilon(\eta) = 0\} = t_P(0) = 1$$

Similarly $i_{\Upsilon(P)}(0) = 1$ and $f_{\Upsilon(P)}(0) = 0$
Now, let $\kappa, \nu \in N$

If $\Upsilon^{-1}(\kappa) = \phi\ or\ \Upsilon^{-1}(\nu) = \phi$**,:** then correspondingly $t_{\Upsilon(P)}(\kappa) = 0\ or\ t_{\Upsilon(P)}(\nu) = 0$
$\Rightarrow t_{\Upsilon(P)}(\kappa) \wedge t_{\Upsilon(P)}(\nu) = 0$ and so $t_{\Upsilon(P)}(\kappa + \nu) \geq t_{\Upsilon(P)}(\kappa) \wedge t_{\Upsilon(P)}(\nu)$.

If $\Upsilon^{-1}(\kappa) \neq \phi \neq \Upsilon^{-1}(\nu)$**,:** then

$$\begin{aligned} t_{\Upsilon(P)}(\kappa+\nu) &= \vee\{t_P(\eta) : \eta \in M, \kappa+\nu = \Upsilon(\eta)\} \\ &= \vee\{t_P(\rho+\varsigma) : \rho, \varsigma \in M, \kappa+\nu = \Upsilon(\rho+\varsigma)\} \\ &\geq \vee\{t_P(\rho+\varsigma) : \rho \in \Upsilon^{-1}(\kappa), \varsigma \in \Upsilon^{-1}(\nu)\} \\ &= \vee\{t_P(\rho+\varsigma) : \rho, \varsigma \in M,\ \Upsilon(\rho) = \kappa, \Upsilon(\varsigma) = \nu\} \\ &\geq \vee\{t_P(\rho) \wedge t_P(\varsigma) : \rho, \varsigma \in M,\ \Upsilon(\rho) = \kappa, \Upsilon(\varsigma) = \nu\} \\ &\geq (\vee\{t_P(\rho), \rho \in M,\ \Upsilon(\rho) = \kappa\}) \wedge (\vee\{t_P(\varsigma) : \varsigma \in M, \Upsilon(\varsigma) = \nu\}) \\ &= t_{\Upsilon(P)}(\kappa) \wedge t_{\Upsilon(P)}(\nu) \end{aligned}$$

Similarly we can prove that

$$i_{\Upsilon(P)}(\kappa+\nu) \geq i_{\Upsilon(P)}(\kappa) \wedge i_{\Upsilon(P)}(\nu)$$

and

$$f_{\Upsilon(P)}(\kappa+\nu) \leq f_{\Upsilon(P)}(\kappa) \vee f_{\Upsilon(P)}(\nu)$$

If $\Upsilon^{-1}(\theta) = \phi,\ \theta \in N,$**:** then $t_{\Upsilon(P)}(\theta) = 0\ \Rightarrow t_{\Upsilon(P)}(\varrho\theta) \geq t_{\Upsilon(P)}(\theta), \forall \varrho \in R$
If $\Upsilon^{-1}(\theta) \neq \phi,\ \theta \in N,$**:** then

$$\begin{aligned} t_{\Upsilon(P)}(r\theta) &= \vee\{t_P(\eta) : \eta \in M, \varrho\theta = \Upsilon(\eta)\} \\ &\geq \vee\{t_P(\varrho\rho) : \varrho\rho \in M, \varrho\theta = \Upsilon(\varrho\rho)\} \\ &\geq \vee\{t_P(\varrho\rho) : r\rho \in M, \rho \in \Upsilon^{-1}(\theta)\} \\ &= \vee\{t_P(r\rho) : r\rho \in M, \theta = \Upsilon(\rho)\} \\ &\geq \vee\{t_P(\rho) : \rho \in M, \theta = \Upsilon(\rho)\} \\ &= t_{\Upsilon(P)}(\theta) \end{aligned}$$

Similarly we can prove that $i_{\Upsilon(P)}(\varrho\theta) \geq i_{\Upsilon(P)}(\theta)\ and\ f_{\Upsilon(P)}(\varrho\theta) \leq f_{\Upsilon(P)}(\theta), \forall\theta \in N$.

Thus $\Upsilon(P) \in U(N)$ □

Theorem 4.4. *If* $Q \in U(N)\ and\ \Upsilon \in Hom_R(M,N)$*, then* $\Upsilon^{-1}(Q) \in U(M)$.

Proof. We have $\Upsilon^{-1}(Q)(\eta) = \{\eta, t_{\Upsilon^{-1}(Q)}(\eta), i_{\Upsilon^{-1}(Q)}(\eta), f_{\Upsilon^{-1}(Q)}(\eta) : \eta \in M\}$ where

$$t_{\Upsilon^{-1}(Q)}(\eta) = t_Q(\Upsilon(\eta)),\ i_{\Upsilon^{-1}(Q)}(\eta) = i_Q(\Upsilon(\eta))\ and\ f_{\Upsilon^{-1}(Q)}(x) = f_Q(\Upsilon(\eta)).$$

Now
$t_{\Upsilon^{-1}(Q)}(0) = t_Q(\Upsilon(0)) = t_Q(0) = 1$. Similarly we can write $i_{\Upsilon^{-1}(Q)}(0) = 1\ and\ f_{\Upsilon^{-1}(Q)}(0) = 0$
Now $\forall\ \eta, \theta \in M$

$$\begin{aligned} t_{\Upsilon^{-1}(Q)}(\eta+\theta) &= t_Q(\Upsilon(\eta+\theta)) \\ &= t_Q(\Upsilon(\eta)+\Upsilon(\theta)) \\ &\geq t_Q(\Upsilon(\eta)) \wedge t_Q(\Upsilon(\theta)) \\ &= t_{\Upsilon^{-1}Q}(\eta) \wedge t_{\Upsilon^{-1}Q}(\theta) \end{aligned}$$

Similarly we can prove that

$$i_{\Upsilon^{-1}(Q)}(\eta+\theta) \geq i_{\Upsilon^{-1}Q}(\eta) \wedge i_{\Upsilon^{-1}Q}(\theta)$$

and

$$f_{\Upsilon^{-1}(Q)}(\eta+\theta) \geq f_{\Upsilon^{-1}Q}(\eta) \wedge f_{\Upsilon^{-1}Q}(\theta)$$

Now $\forall \eta \in M, \varrho \in R$

$$\begin{aligned} t_{\Upsilon^{-1}(Q)}(\varrho\eta) &= t_Q(\Upsilon(\varrho\eta)) \\ &= t_Q(\varrho\Upsilon(\eta)) \\ &\geq t_Q(\Upsilon(\eta)) \\ &= t_{\Upsilon^{-1}(Q)}(\eta) \end{aligned}$$

Similarly $i_{\Upsilon^{-1}(Q)}(\varrho\eta) \geq i_{\Upsilon^{-1}(Q)}(\eta)$ *and* $f_{\Upsilon^{-1}(Q)}(\varrho\eta) \leq f_{\Upsilon^{-1}(Q)}(\eta)$.
Thus $\Upsilon^{-1}(Q) \in U(M)$ □

Theorem 4.5. *Let $\Upsilon \in Hom_R(M, M^{\circledast})$ be a neutrosophic module homomorphism of P onto Q, where $P \in U(M)$ and $Q \in U(\Upsilon(M))$. Then the map $\Pi : M/N \to M^{\circledast}$, defined by $\Pi([\eta]) = \Upsilon(\eta),\ \eta \in M$ is a neutrosophic quotient module homomorphism of P_N on to Q, where $P_N \in U(M/N)$ and $N \subseteq M$.*

Proof. Given that $\Upsilon : M \to M^{\circledast}$ be neutrosophic module homomorphism of P onto Q, $\Rightarrow \Upsilon(P) = Q$. Then to prove that $\Pi : M/N \to M^{\circledast}$ *where* $\Pi([\eta]) = \Upsilon(\eta)$ is neutrosophic module homomorphism of P_N onto Q.
First we prove that $\Pi \in Hom_R(M/N, M^{\circledast})$. Let $\varrho_1, \varrho_2 \in R, \rho, \varsigma \in M$, then

$$\begin{aligned} \Pi([\varrho_1[\rho] + \varrho_2[\varsigma]) &= \Pi(\varrho_1(\rho+N) + \varrho_2(\varsigma+N)) \\ &= \Pi(\varrho_1\rho + \varrho_2\varsigma + N) \\ &= \Pi([\varrho_1\rho + \varrho_2\varsigma]) \\ &= \Upsilon(\varrho_1\rho + \varrho_2\varsigma) \\ &= \varrho_1\Upsilon(\rho) + \varrho_2\Upsilon(\varsigma) \\ &= \varrho_1\Pi([\rho]) + \varrho_2\Pi([\varsigma]) \end{aligned}$$

For any $r \in R,\ [\eta] \in M/N$, then

$$\begin{aligned} \Pi(r[m]) &= \Pi(r(\eta+N)) \\ &= \Pi(r\eta+N) \\ &= \Pi([r\eta]) \\ &= \Upsilon(r\eta) \\ &= r\Upsilon(\eta) \\ &= r\Pi([\eta]) \end{aligned}$$

$\Rightarrow\ \Pi \in Hom_R(M/N, M^{\circledast})$. Then to prove that $\Pi(P_N) = Q$, Now

$$\Pi(P_N)(\vartheta) = \{\vartheta, t_{\psi_{P_N}}(\vartheta), i_{\psi_{P_N}}(\vartheta), f_{\psi_{P_N}}(\vartheta) : \vartheta \in \Pi(M/N)\}$$