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Neutrosophic Soluble Groups, Neutrosophic Nilpotent Groups and Their Properties

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Abstract

The theory of soluble groups and nilpotent groups is old and hence a generalized on. In this paper, we introduced neutrosophic soluble groups and neutrosophic nilpotent groups which have some kind of indeterminacy. These notions are generalized to the classic notions of soluble groups and nilpotent groups. We also derive some new type of series which derived some new notions of soluble groups and nilpotent groups. They are mixed neutrosophic soluble groups and mixed neutrosophic nilpotent groups as well as strong neutrosophic soluble groups and strong neutrosophic nilpotent groups.

Key words: Soluble group, nilpotent group, neutrosophic group, neutrosophic soluble group, neutrosophic nilpotent group.

1. Introduction

Smarandache [15] in 1980 introduced neutrosophy which is a branch of philosophy that studies the origin and scope of neutralities and their interaction with ideational spectra. The concept of neutrosophic set and logic came into being due to neutrosophy, where each proposition is approximated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F. Neutrosophic sets are the generalization to all other traditional theories of logics. This mathematical framework is used to handle problems with uncertaint, imprecise, indeterminate, incomplete and inconsistent etc. Kandasamy and Smarandache apply the concept of indeterminacy factor in algebraic structures by inserting the indeterminate element I in the algebraic notions with respect to the opeartaion *. This phenomenon generates the corresponding neutrosophic algebraic notion. They called that indeterminacy element I, a neutrosophic element which is unknown in some sense. This approach a relatively large structure which contain the old classic alegebraic structure. In this way, they studied several neutrosophic algebraic structures in [9,10,11,12]. Some of them are neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic semigroups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, and neutrosophic bigroupoids and so on. Mumtaz et al.[1] introduced neutrosophic left almost semigroup in short neutrosophic LA-semigroup and their generalization [2]. Further, Mumtaz et al. studied neutrosophic LA-semigroup rings and their generalization.

Groups [5,7] are the most rich algebraic structures in the theory of algebra.They shared common features to all the algebraic structures. Soluble groups [13,14] are important notions in the theory of groups as they are studied on the basis of some kind of series structures of the subgroups of the group. A soluble group is constructed by using abelian groups through the extension. A nilpotent group [13] is one whose which has finite length of central series. Thus a nilplotent group is also a soluble group. It is a special type of soluble group because every soluble group has a abelian series. A huge amount of literature on soluble groups and nilpotent groups can be found in [6,8,16,17,18].

In this paper, we introduced neutrosophic soluble groups and neutrosophic nilpotent groups and investigate some of their propertied. The organization of this paper is as follows: In section 1, we give a brief introduction of neutrosophic algebraic structures in terms of I and soluble groups and nilpotent groups. In the next section 2, some basic concept have been studied which we have used in the rest of the paper. In section 3, we introduced neutrosophic soluble groups and investigate some of their basic properties. In section 4, the notions of neutrosophic nilpotent groups are introduced and studied their basic properties. Conclusion is placed in section 5.

2. Fundamental Concepts

Definition 2.1: Let $(G,*)$ be a group . Then the neutrosophic group is generated by G and I under $*$ denoted by $N(G)=\{\langle G\cup I\rangle,*\}$. The identity element is represented by e and $\{e\}$ represents the trivial subgroup of G. I is called the indeterminate element with the property $I^2=I$. For an integer n , $n+I$ and nI are neutrosophic elements and $0.I=0$. I^{-1} , the inverse of I is not defined and hence does not exist.

Definition 2.2: Let $N(G)$ be a neutrosophic group and H be a neutrosophic subgroup of $N(G)$. Then H is a neutrosophic normal subgroup of $N(G)$ if $xH=Hx$ for all $x\in N(G)$.

Definition 2.3: Let $N(G)$ be a neutrosophic group. Then center of $N(G)$ is denoted by $C(N(G))$ and defined as $C(N(G))=\{x\in N(G): ax=xa$ for all $a\in N(G)\}$.

Definition 2.4: Let G be a group and $H_1,H_2,...,H_n$ be the subgroups of . Then

$$\{e\}=H_0\le H_1\le H_2\le ...\le H_{n-1}\le H_n=G$$

is called subgroup series of G .

Definition 2.5: Let G be a group and 1 be the identity element. Then

$$\{e\}=H_0\lhd H_1\lhd H_2\lhd ...\lhd H_{n-1}\lhd H_n=G$$

is called subnormal series. That is H_j is normal subgroup of H_{j+1} for all j .

Definition 2.6: Let

$$\{e\}=H_0\lhd H_1\lhd H_2\lhd ...\lhd H_{n-1}\lhd H_n=G$$

be a subnormal series of G. If each H_j is normal in G for all j, then this subnormal series is called normal series.

Definition 2.7: A normal series

$$\{e\} = H_0 \lhd H_1 \lhd H_2 \lhd \ldots \lhd H_{n-1} \lhd H_n = G$$

is called an abelian series if the factor group H_{j+1}/H_j is an abelian group.

Definition 2.8: A group is called a soluble group if has an abelian series.

Definition 2.9: Let be a soluble group. Then length of the shortest abelian series of is called derived length.

Definition 2.10: Let be a group. The series

$$\{e\} = H_0 \lhd H_1 \lhd H_2 \lhd \ldots \lhd H_{n-1} \lhd H_n = G$$

is called central series if $H_{j+1}/H_j \subseteq Z\left(G/H_j\right)$ for all j.

Definition 2.11: A group is called a nilpotent group if has a central series.

3. Neutrosophic Soluble Groups

Definition 3.1: Let $N(G) = \langle G \cup I \rangle$ be a neutrosophic group and let $H_1, H_2, \ldots, \mathrm{H}_n$ be the neutrosophic subgroups of $N(G)$. Then a neutrosophic subgroup series is a chain of neutrosophic subgroups such that

$$\{e\} = H_0 \le H_1 \le H_2 \le \ldots \le H_{n-1} \le H_n = N(G).$$

Example 3.2: Let $N(G) = \langle \mathbb{Z} \cup I \rangle$ be a neutrosophic group of integers. Then the following are the neutrosophic subgroups series of the group $N(G)$. Where 1 is the identity element.

$$\{0\} \le 4\mathbb{Z} \le 2\mathbb{Z} \le \langle 2\mathbb{Z} \cup I \rangle \le \langle \mathbb{Z} \cup I \rangle,$$

$$\{0\} \le \langle 4\mathbb{Z} \cup I \rangle \le \langle 2\mathbb{Z} \cup I \rangle \le \langle \mathbb{Z} \cup I \rangle,$$

$$\{0\} \le 4\mathbb{Z} \le 2\mathbb{Z} \le \mathbb{Z} \le \langle \mathbb{Z} \cup I \rangle.$$

Definition 3.3: Let $\{e\} = H_0 \le H_1 \le H_2 \le \ldots \le H_{n-1} \le H_n = N(G)$ be a neutrosophic subgroup series of the neutrosophic group $N(G)$. Then this series of subgroups is called a strong neutrosophic subgroup series if each H_j is a neutrosophic subgroup of $N(G)$ for all j.

Example 3.4: Let $N(G)=\langle \mathbb{Z}\cup I\rangle$ be a neutrosophic group. Then the following neutrosophic subgroup series of $N(G)$ is a strong neutrosophic subgroup series:

$$\{0\}\leq\langle 4\mathbb{Z}\cup I\rangle\leq\langle 2\mathbb{Z}\cup I\rangle\leq\langle \mathbb{Z}\cup I\rangle .$$

Theorem 3.5: Every strong neutrosophic subgroup series is trivially a neutrosophic subgroup series but the converse is not true in general.

Definition 3.6: If some H_j's are neutrosophic subgroups and some H_k's are just subgroups of $N(G)$. Then that neutrosophic subgroups series is called mixed neutrosophic subgroup series.

Example 3.7: Let $N(G)=\langle \mathbb{Z}\cup I\rangle$ be a neutrosophic group. Then the following neutrosophic subgroup series of $N(G)$ is a mixed neutrosophic subgroup series:

$$\{0\}\leq 4\mathbb{Z}\leq 2\mathbb{Z}\leq\langle 2\mathbb{Z}\cup I\rangle\leq\langle \mathbb{Z}\cup I\rangle .$$

Theorem 3.8: Every mixed neutrosophic subgroup series is trivially a neutrosophic subgroup series but the converse is not true in general.

Definition 3.9: If H_j's in $\{e\}=H_0\leq H_1\leq H_2\leq ...\leq H_{n-1}\leq H_n=N(G)$ are only subgroups of the neutrosophic group $N(G)$, then that series is termed as subgroup series of the neutrosophic group $N(G)$.

Example 3.10: Let $N(G)=\langle \mathbb{Z}\cup I\rangle$ be a neutrosophic group. Then the following neutrosophic subgroup series of $N(G)$ is just a subgroup series:

$$\{0\}\leq 4\mathbb{Z}\leq 2\mathbb{Z}\leq \mathbb{Z}\leq\langle \mathbb{Z}\cup I\rangle .$$

Theorem 3.11: A neutrosophic group $N(G)$ has all three type of neutrosophic subgroups series.

Theorem 3.12: Every subgroup series of the group G is also a subgroup series of the neutrosophic group $N(G)$.

Proof: Since G is always contained in $N(G)$. This directly followed the proof.

Definition 3.13: Let $\{e\}=H_0\leq H_1\leq H_2\leq ...\leq H_{n-1}\leq H_n=N(G)$ be a neutrosophic subgroup series of the neutrosophic group $N(G)$. If

$$=H_0\lhd H_1\lhd H_2\lhd ...\lhd H_{n-1}\lhd H_n=N(G)\ldots\ldots\ldots\ldots\ldots(1)$$

That is each H_j is normal in H_{j+1}. Then (1) is called a neutrosophic subnormal series of the neutrosophic group $N(G)$.

Example 3.14: Let $N(G)=\langle A_4 \cup I\rangle$ be a neutrosophic group, where A_4 is the alternating subgroup of the permutation group S_4. Then the following is the neutrosophic subnormal series of the group $N(G)$.

$$1 \lhd C_2 \lhd V_4 \lhd \langle V_4 \cup I\rangle \lhd \langle A_4 \cup I\rangle .$$

Definition 3.15: A neutrosophic subnormal series is called strong neutrosophic subnormal series if all $H_j's$ are neutrosophic normal subgroups in (1) for all j.

Example 3.16: Let $N(G)=\langle \mathbb{Z} \cup I\rangle$ be a neutrosophic group of integers. Then the following is a strong neutrosophic subnormal series of $N(G)$.

$$1 \lhd \langle 4\mathbb{Z} \cup I\rangle \lhd \langle 2\mathbb{Z} \cup I\rangle \lhd \langle \mathbb{Z} \cup I\rangle .$$

Theorem 3.17: Every strong neutrosophic subnormal series is trivially a neutrosophic subnormal series but the converse is not true in general.

Definition 3.18: A neutrosophic subnormal series is called mixed neutrosophic subnormal series if some $H_j's$ are neutrosophic normal subgroups in (1) while some $H_k's$ are just normal subgroups in (1) for some j and k.

Example 3.19: Let $N(G)=\langle \mathbb{Z} \cup I\rangle$ be a neutrosophic group of integers. Then the following is a mixed neutrosophic subnormal series of $N(G)$.

$$1 \lhd 4\mathbb{Z} \lhd 2\mathbb{Z} \lhd \langle 2\mathbb{Z} \cup I\rangle \lhd \langle \mathbb{Z} \cup I\rangle .$$

Theorem 3.20: Every mixed neutrosophic subnormal series is trivially a neutrosophic subnormal series but the converse is not true in general.

Definition 3.21: A neutrosophic subnormal series is called subnormal series if all $H_j's$ are only normal subgroups in (1) for all j.

Theorem 3.22: Every subnormal series of the group   is also a subnormal series of the neutrosophic group $N(G)$.

Definition 3.23: If H_j are all normal neutrosophic subgroups in $N(G)$. Then the neutrosophic subnormal series (1) is called neutrosophic normal series.

Theorem 3.24: Every neutrosophic normal series is a neutrosophic subnormal series but the converse is not true.

For the converse, see the following Example.

Example 3.25: Let $N(G)=\langle A_4 \cup I\rangle$ be a neutrosophic group, where A_4 is the alternating subgroup of the permutation group S_4. Then the following are the neutrosophic subnormal series of the group $N(G)$.

$$1 \lhd C_2 \lhd V_4 \lhd \langle V_4 \cup I \rangle \lhd \langle A_4 \cup I \rangle .$$

This series is not neutrosophic normal series as C_2 (cyclic group of order 2) is not normal in V_4 (Klein four group).

Similarly we can define strong neutrosophic normal series, mixed neutrosophic normal series and normal series respectively on the same lines of the neutrosophic group $N(G)$.

Definition 3.26: The neutrosophic normal series

$$1 = H_0 \lhd H_1 \lhd H_2 \lhd ... \lhd H_{n-1} \lhd H_n = N(G) \ldots\ldots\ldots (2)$$

is called neutrosophic abelian series if the factor group H_{j+1}/H_j are all abelian for all j.

Example 3.27: Let $N(G) = \langle S_3 \cup I \rangle$ be a neutrosophic group, where S_3 is the permutation group. Then the following is the neutrosophic abelian series of the group $N(G)$.

$$1 \lhd A_3 \lhd \langle A_3 \cup I \rangle \lhd \langle S_3 \cup I \rangle .$$

We explain it as following:

Since $\langle S_3 \cup I \rangle / \langle A_3 \cup I \rangle \simeq \mathbb{Z}_2$ and $\mathbb{Z}_2$ is cyclic which is abelian. Thus $\langle S_3 \cup I \rangle / \langle A_3 \cup I \rangle$ is an abelian neutrosophic group.

Also,

$\langle A_3 \cup I \rangle / A_3 \simeq \mathbb{Z}_2$ and this is factor group is also cyclic and every cyclic group is abelian. Hence $\langle A_3 \cup I \rangle / A_3$ is also ablian group. Finally,

$A_3 / I \simeq \mathbb{Z}_3$ which is again abelian group. Therefore the series is a neutrosophic abelian series of the group $N(G)$.

Thus on the same lines, we can define strong neutrosophic abelian series, mixed neutrosophic abelian series and abelian series of the neutrosophic group $N(G)$.

Definition 3.28: A neutrosophic group $N(G)$ is called neutrosophic soluble group if $N(G)$ has a neutrosophic abelian series.

Example 3.29: Let $N(G) = \langle S_3 \cup I \rangle$ be a neutrosophic group, where S_3 is the permutation group. Then the following is the neutrosophic abelian series of the group $N(G)$,

$$1 \lhd A_3 \lhd \langle A_3 \cup I \rangle \lhd \langle S_3 \cup I \rangle .$$

Then clearly $N(G)$ is a neutrosophic soluble group.

Theorem 3.30: Every abelian series of a group is also an abelian series of the neutrosophic group $N(G)$.

Theorem 3.31: If a group is a soluble group, then the neutrosophic group $N(G)$ is also soluble neutrosophic group.

Theorem 3.32: If the neutrosophic group $N(G)$ is an abelian neutrosophic group, then $N(G)$ is a neutrosophic soluble group.

Theorem 3.33: If $N(G) = \mathrm{C}(N(G))$, then $N(G)$ is a neutrosophic soluble group.

Proof: Suppose the $N(G) = \mathrm{C}(N(G))$. Then it follows that $N(G)$ is a neutrosophic abelian group. Hence by above Theorem 3.35, $N(G)$ is a neutrosophic soluble group.

Theorem 3.34: If the neutrosophic group $N(G)$ is a cyclic neutrosophic group, then $N(G)$ is a neutrosophic soluble group.

Definition 3.35: A neutrosophic group $N(G)$ is called strong neutrosophic soluble group if $N(G)$ has a strong neutrosophic abelian series.

Theorem 3.36: Every strong neutrosophic soluble group $N(G)$ is trivially a neutrosophic soluble group but the converse is not true.

Definition 3.37: A neutrosophic group $N(G)$ is called mixed neutrosophic soluble group if $N(G)$ has a mixed neutrosophic abelian series.

Theorem 3.38: Every mixed neutrosophic soluble group $N(G)$ is trivially a neutrosophic soluble group but the converse is not true.

Definition 3.39: A neutrosophic group $N(G)$ is called soluble group if $N(G)$ has an abelian series.

Definition 3.40: Let $N(G)$ be a neutrosophic soluble group. Then length of the shortest neutrosophic abelian series of $N(G)$ is called derived length.

Example 3.41: Let $N(G) = \langle \mathbb{Z} \cup I \rangle$ be a neutrosophic soluble group. The following is a neutrosophic abelian series of the group $N(G)$.

$$1 \triangleleft 4\mathbb{Z} \triangleleft 2\mathbb{Z} \triangleleft \langle 2\mathbb{Z} \cup I \rangle \triangleleft \langle \mathbb{Z} \cup I \rangle .$$

Then $N(G)$ has derived length 4.

Remark 3.42: Neutrosophic group of derive length zero is trivial neutrosophic group.

Proposition 3.43: Every neutrosophic subgroup of a neutrosophic soluble group is soluble.

Proposition 3.44: Quotient neutrosophic group of a neutrosophic soluble group is soluble.

4. Neutrosophic Nilpotent Groups

Definition 4.1: Let $N(G)$ be a neutrosophic group. The series

$$1 = H_0 \lhd H_1 \lhd H_2 \lhd \ldots \lhd H_{n-1} \lhd H_n = N(G) \ldots\ldots\ldots\ldots\ldots (3)$$

is called neutrosophic central series if $H_{j+1}/H_j \subseteq C\left(N(G)/H_j \right)$ for all j.

Definition 4.2: A neutrosophic group $N(G)$ is called a neutrosophic nilpotent group if $N(G)$ has a neutrosophic central series.

Theorem 4.3: Every neutrosophic central series is a neutrosophic abelian series.

Theorem 4.4: If $N(G) = \mathrm{C}(N(G))$, then $N(G)$ is a neutrosophic nilpotent group.

Theorem 4.5: Every neutrosophic nilpotent group $N(G)$ is a neutrosophic soluble group.

Theorem 4.6: All neutrosophic abelian groups are neutrosophic nilpotent groups.

Theorem 4.7: All neutrosophic cyclic groups are neutrosophic nilpotent groups.

Theorem 4.8: The direct product of two neutrosophic nilpotent groups is nilpotent.

Definition 4.9: Let $N(G)$ be a neutrosophic group. Then the neutrosophic central series (3) is called strong neutrosophic central series if all H_j ' s are neutrosophic normal subgroups for all j.

Theorem 4.10: Every strong neutrosophic central series is trivially a neutrosophic central series but the converse is not true in general.

Theorem 4.11: Every strong neutrosophic central series is a strong neutrosophic abelian series.

Definition 4.12: A neutrosophic group $N(G)$ is called strong neutrosophic nilpotent group if $N(G)$ has a strong neutrosophic central series.

Theorem 4.13: Every strong neutrosophic nilpotent group is trivially a neutrosophic nilpotent group.

Theorem 4.14: Every strong neutrosophic nilpotent group is also a strong neutrosophic soluble group.

Definition 4.15: Let $N(G)$ be a neutrosophic group. Then the neutrosophic central series (3) is called mixed neutrosophic central series if some $H_j's$ are neutrosophic normal subgroups while some $H_k's$ are just normal subgroups for j,k.

Theorem 4.16: Every mixed neutrosophic central series is trivially a neutrosophic central series but the converse is not true in general.

Theorem 4.17: Every mixed neutrosophic central series is a mixed neutrosophic abelian series.

Definition 4.18: A neutrosophic group $N(G)$ is called mixed neutrosophic nilpotent group if $N(G)$ has a mixed neutrosophic central series.

Theorem 4.19: Every mixed neutrosophic nilpotent group is trivially a neutrosophic nilpotent group.

Theorem 4.20: Every mixed neutrosophic nilpotent group is also a mixed neutrosophic soluble group.

Definition 4.21: Let $N(G)$ be a neutrosophic group. Then the neutrosophic central series (3) is called central series if all $H_j's$ are only normal subgroups for all j.

Theorem 4.22: Every central series is an abelian series.

Definition 4.23: A neutrosophic group $N(G)$ is called nilpotent group if $N(G)$ has a central series.

Theorem 4.24: Every nilpotent group is also a soluble group.

Theorem 4.25: If G is nilpotent group, then $N(G)$ is also a neutrosophic nilpotent group.

5. Conclusion

In this paper, we initiated the study of neutrosophic soluble groups and neutrosophic nilpotent groups which are the generalization of soluble groups and nilpotent groups. We also investigate their properties. Strong neutrosophic soluble and strong neutrosophic nilpotent groups are introduced which are completely new in their nature and properties. We also study the notions of mixed neutrosophic soluble groups and mixed neutrosophic nilpotent groups. These notions are studied on the basis of their serieses. In future, a lot of study can be carried out on neutrosophic nilpotent groups and neutrosophic soluble groups and their related properties.

References

1. M. Ali, M. Shabir, M. Naz and F. Smarandache, Neutrosophic Left Almost Semigroup, Neutrosophic Sets and Systems, 3 (2014), 18-28.
2. M. Ali, F. Smarandache, M. Shabir, and M. Naz, Neutrosophic Bi-LA-semigroup, and Neutrosophic N-LA-semigroup, Neutrosophic Sets and Systems, 4 (2014), 19-24.
3. M. Ali, F. Smarandache, M. Shabir and L. Vladareanu. Generalization of Neutrosophic Rings and Neutrosophic Fields. Neutrosophic Sets and Systems, 5, (2014), 9-14.
4. M. Ali, F. Smarandache, and M. Shabir, New Research on Neutrosophic Algebraic Structures, Europa Nova. ASBL 3E clos du Paranasse Brussels, 1000, Belgium.
5. Bechtell, and Homer, The theory of groups. Addison-Wesley, (1971).
6. F. V. Haeseler, Automatic Sequences (De Gruyter Expositions in Mathematics, 36). (2002), Berlin: Walter de Gruyter.

7. Isaacs, I. Martin, Finite group theory, (2008). American Mathematical Society.
8. J. Joseph, Rotman, An introduction to the theory of groups. Graduate texts in mathematics 148, (4 ed.),
9. (1995).
10. W. B. V. Kandasamy, and F. Smarandache, Basic Neutrosophic Algebraic Structures and their Applications to Fuzzy and Neutrosophic Models, Hexis, 149 pp., 2004.
11. W. B. V. Kandasamy, and F. Smarandache, Neutrosophic Rings, Hexis, Phoenix, 2006.
12. W. B. V. Kandasamy, and F. Smarandache, N-Algebraic Structures and S-N-Algebraic Structures, 209 pp., Hexis, Phoenix, 2006.
13. W. B. V. Kandasamy, and F. Smarandache, Some Neutrosophic Algebraic Structures and Neutrosophic N-Algebraic Structures, 219 p., Hexis, 2006.
14. A. I. Malcev, "Generalized nilpotent algebras and their associated groups", Mat. Sbornik N.S. **25** (67), (1949), 347–366.
15. Palmer, W. Theodore, Banach algebras and the general theory of *-algebras. (1994), Cambridge, UK: Cambridge University Press.
16. F. Smarandache, A Unifying Field in Logics. Neutrosophy, Neutrosophic Probability, Set and Logic. Rehoboth: American Research Press, (1999).
17. U. Stammbach, Homology in group theory, Lecture Notes in Mathematics, Volume 359, Springer-Verlag, New York, 1973.
18. Tabachnikova, Olga, Smith, and Geoff, Topics in Group Theory (Springer Undergraduate Mathematics Series). Berlin: Springer, (2000).
19. Zassenhaus, and Hans, The theory of groups. (1999). New York: Dover Publications.