Neutrosophic subsemigroups

ANNALS OF COMMUNICATIONS IN MATHEMATICS
Volume 1, Number 1 (2018), 1-10
ISSN: 2582-0818

NEUTROSOPHIC SUBSEMIGROUPS

G. MUHIUDDIN

ABSTRACT. In the present paper, we introduce the notion of (Φ, Ψ)-neutrosophic subsemigroups of a semigroup where $\Phi, \Psi \in \{\in, q, \in \vee q\}$, and related properties are investigated. We consider characterizations of an $(\in, \in)$-neutrosophic subsemigroup and an $(\in, \in \vee q)$-neutrosophic subsemigroup. Conditions for the neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in \vee q$-subsets to be subsemigroups are discussed. Finally, we discuss conditions for a neutrosophic set to be a $(q, \in \vee q)$-neutrosophic subsemigroup.

1. INTRODUCTION

The notion of neutrosophic set (NS) developed by Smarandache [8, 9, 10] is a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. Nowadays, the theory of neutrosophic sets became a very interesting and challenging topic of study as research point of view. The theory of neutrosophic sets is an important mathematical tool to deal with the indeterminate information and inconsistent information; and has vast applications in various directions (see e.g., [3], [4],[11], [11]). For more details we refer readers to the site http://fs.gallup.unm.edu/neutrosophy.htm. In the year 2015, Agboola and Davvaz introduced the concept of neutrosophic BCI/BCK-algebras and presented elementary properties of neutrosophic BCI/BCK-algebras. Further in the same year, they studied neutrosophic ideals of neutrosophic BCI-algebras (see [1] [2]). Recently, Muhiuddin et al. studied the notion of $(\in, \in)$-neutrosophic subalgebras and ideals in BCK/BCI-algebras [11].

Motivated by a lot of work on neutrosophic sets in various fields of research, in this paper, we introduce the notion of (Φ, Ψ)-neutrosophic subsemigroup of a semigroup S for $\Phi, \Psi \in \{\in, q, \in \vee q\}$, and investigate related properties. We provide characterizations of an $(\in, \in)$-neutrosophic subsubsemigroup and an $(\in, \in \vee q)$-neutrosophic subsubsemigroup. Given special sets, so called neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in \vee q$-subsets, we provide conditions for the neutrosophic $\in$-subsets, neutrosophic

2010 *Mathematics Subject Classification.* 06F35, 03G25, 03B52.
Key words and phrases. $(\in, \in)$-neutrosophic subsemigroup, $(\in, q)$-neutrosophic subsemigroup, $(q, \in)$-neutrosophic subsemigroup, (q, q)-neutrosophic subsemigroup, $(\in, \in \vee q)$-neutrosophic subsemigroup, $(q, \in \vee q)$-neutrosophic subsemigroup.

q-subsets and neutrosophic $\in\vee\, q$-subsets to be subsemigroups. We discuss conditions for a neutrosophic set to be a $(q, \in\vee\, q)$-neutrosophic subsubsemigroup.

2. Preliminaries

Let S be a semigroup. Let A and B be subsets of S. Then the multiplication of A and B is defined as follows:

$$AB = \{ab \in S \mid a \in A \text{ and } b \in B\}.$$

Let S be a semigroup. By a *subsemigroup* of S we mean a nonempty subset A of S such that $A^2 \subseteq A$.

A fuzzy set A in a set S of the form

$$A(y) := \begin{cases} t \in (0,1] & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases} \tag{2.1}$$

is said to be a *fuzzy point* with support x and value t and is denoted by (x,t).

For a fuzzy set A in a set S, a fuzzy point (x,t) is said to

- *contained* in A, denoted by $(x,t) \in A$ (see [7]), if $A(x) \geq t$.
- *be quasi-coincident with* A, denoted by $(x,t)\, q\, A$ (see [7]), if $A(x) + t > 1$.

For a fuzzy point (x,t) and a fuzzy set A in a set S, we say that

- $(x,t) \in\vee\, q\, A$ if $(x,t) \in A$ or $(x,t)\, q\, A$.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

If $\Lambda = \{1,2\}$, we will also use $a_1 \vee a_2$ and $a_1 \wedge a_2$ instead of $\bigvee\{a_i \mid i \in \Lambda\}$ and $\bigwedge\{a_i \mid i \in \Lambda\}$, respectively.

Let S be a non-empty set. A neutrosophic set (NS) in S (see [9]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in S\}$$

where $A_T : S \to [0,1]$ is a truth membership function, $A_I : S \to [0,1]$ is an indeterminate membership function, and $A_F : S \to [0,1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in S\}.$$

3. NEUTROSOPHIC SUBSEMIGROUPS OF SEVERAL TYPES

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set S, $\alpha, \beta \in (0, 1]$ and $\gamma \in [0, 1)$, we consider the following sets:

$$
\begin{aligned}
&T_\in(A; \alpha) := \{x \in S \mid A_T(x) \geq \alpha\},\\
&I_\in(A; \beta) := \{x \in S \mid A_I(x) \geq \beta\},\\
&F_\in(A; \gamma) := \{x \in S \mid A_F(x) \leq \gamma\},\\
&T_q(A; \alpha) := \{x \in S \mid A_T(x) + \alpha > 1\},\\
&I_q(A; \beta) := \{x \in S \mid A_I(x) + \beta > 1\},\\
&F_q(A; \gamma) := \{x \in S \mid A_F(x) + \gamma < 1\},\\
&T_{\in\vee\, q}(A; \alpha) := \{x \in S \mid A_T(x) \geq \alpha \text{ or } A_T(x) + \alpha > 1\},\\
&I_{\in\vee\, q}(A; \beta) := \{x \in S \mid A_I(x) \geq \beta \text{ or } A_I(x) + \beta > 1\},\\
&F_{\in\vee\, q}(A; \gamma) := \{x \in S \mid A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1\}.
\end{aligned}
$$

We say $T_\in(A; \alpha)$, $I_\in(A; \beta)$ and $F_\in(A; \gamma)$ are *neutrosophic $\in$-subsets*; $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are *neutrosophic q-subsets*; and $T_{\in\vee\, q}(A; \alpha)$, $I_{\in\vee\, q}(A; \beta)$ and $F_{\in\vee\, q}(A; \gamma)$ are *neutrosophic $\in \vee\, q$-subsets*. For $\Phi \in \{\in, q, \in\vee\, q\}$, the element of $T_\Phi(A; \alpha)$ (resp., $I_\Phi(A; \beta)$ and $F_\Phi(A; \gamma)$) is called a *neutrosophic T_Φ-point* (resp., *neutrosophic I_Φ-point* and *neutrosophic F_Φ-point*) with value α (resp., β and γ). It is clear that

$$T_{\in\vee\, q}(A; \alpha) = T_\in(A; \alpha) \cup T_q(A; \alpha), \tag{3.1}$$
$$I_{\in\vee\, q}(A; \beta) = I_\in(A; \beta) \cup I_q(A; \beta), \tag{3.2}$$
$$F_{\in\vee\, q}(A; \gamma) = F_\in(A; \gamma) \cup F_q(A; \gamma). \tag{3.3}$$

Definition 3.1. Given $\Phi, \Psi \in \{\in, q, \in\vee\, q\}$, a neutrosophic set $A = (A_T, A_I, A_F)$ in a semigroup S is called a (Φ, Ψ)-*neutrosophic subsemigroup* of S if the following assertions are valid.

$$
\begin{aligned}
&x \in T_\Phi(A; \alpha_x),\ y \in T_\Phi(A; \alpha_y) \ \Rightarrow\ xy \in T_\Psi(A; \alpha_x \wedge \alpha_y),\\
&x \in I_\Phi(A; \beta_x),\ y \in I_\Phi(A; \beta_y) \ \Rightarrow\ xy \in I_\Psi(A; \beta_x \wedge \beta_y),\\
&x \in F_\Phi(A; \gamma_x),\ y \in F_\Phi(A; \gamma_y) \ \Rightarrow\ xy \in F_\Psi(A; \gamma_x \vee \gamma_y)
\end{aligned}
\tag{3.4}
$$

for all $x, y \in S$, $\alpha_x, \alpha_y, \beta_x, \beta_y, \in (0, 1]$ and $\gamma_x, \gamma_y \in [0, 1)$.

Theorem 3.1. *A neutrosophic set $A = (A_T, A_I, A_F)$ in a semigroup S is an $(\in, \in)$-neutrosophic subsemigroup of S if and only if it satisfies:*

$$(\forall x, y \in S) \begin{pmatrix} A_T(xy) \geq A_T(x) \wedge A_T(y) \\ A_I(xy) \geq A_I(x) \wedge A_I(y) \\ A_F(xy) \leq A_F(x) \vee A_F(y) \end{pmatrix}. \tag{3.5}$$

Proof. Assume that $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subsemigroup of S. If there exist $x, y \in S$ such that $A_T(xy) < A_T(x) \wedge A_T(y)$, then

$$A_T(xy) < \alpha_t \leq A_T(x) \wedge A_T(y)$$

for some $\alpha_t \in (0, 1]$. It follows that $x, y \in T_\in(A; \alpha_t)$ but $xy \notin T_\in(A; \alpha_t)$. Hence $A_T(xy) \geq A_T(x) \wedge A_T(y)$ for all $x, y \in S$. Similarly, we show that

$$A_I(xy) \geq A_I(x) \wedge A_I(y)$$

for all $x, y \in S$. Suppose that there exist $a, b \in S$ and $\gamma_f \in [0,1]$ be such that $A_F(ab) > \gamma_f \geq A_F(a) \vee A_F(b)$. Then $a, b \in F_\in(A;\gamma_f)$ and $ab \notin F_\in(A;\gamma_f)$, which is a contradiction. Therefore $A_F(xy) \leq A_F(x) \vee A_F(y)$ for all $x, y \in S$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in S which satisfies the condition (3.5). Let $x, y \in S$ be such that $x \in T_\in(A;\alpha_x)$ and $y \in T_\in(A;\alpha_y)$. Then $A_T(x) \geq \alpha_x$ and $A_T(y) \geq \alpha_y$, which imply that $A_T(xy) \geq A_T(x) \wedge A_T(y) \geq \alpha_x \wedge \alpha_y$, that is, $xy \in T_\in(A;\alpha_x \wedge \alpha_y)$. Similarly, if $x \in I_\in(A;\beta_x)$ and $y \in I_\in(A;\beta_y)$ then $xy \in I_\in(A;\beta_x \wedge \beta_y)$. Now, let $x \in F_\in(A;\gamma_x)$ and $y \in F_\in(A;\gamma_y)$ for $x, y \in S$. Then $A_F(x) \leq \gamma_x$ and $A_F(y) \leq \gamma_y$, and so $A_F(xy) \leq A_F(x) \vee A_F(y) \leq \gamma_x \vee \gamma_y$. Hence $xy \in F_\in(A;\gamma_x \vee \gamma_y)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subsemigroup of S. □

Theorem 3.2. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in)$-neutrosophic subsemigroup of a semigroup S, then neutrosophic q-subsets $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subsemigroups of S for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$ whenever they are nonempty.*

Proof. Let $x, y \in T_q(A;\alpha)$. Then $A_T(x) + \alpha > 1$ and $A_T(y) + \alpha > 1$. It follows that

$$\begin{aligned} A_T(xy) + \alpha &\geq (A_T(x) \wedge A_T(y)) + \alpha \\ &= (A_T(x) + \alpha) \wedge (A_T(y) + \alpha) > 1 \end{aligned}$$

and so that $xy \in T_q(A;\alpha)$. Hence $T_q(A;\alpha)$ is a subsemigroup of S. Similarly, we can prove that $I_q(A;\beta)$ is a subsemigroup of S. Now let $x, y \in F_q(A;\gamma)$. Then $A_F(x)+\gamma < 1$ and $A_F(y) + \gamma < 1$, which imply that

$$\begin{aligned} A_F(xy) + \gamma &\leq (A_F(x) \vee A_F(y)) + \gamma \\ &= (A_F(x) + \alpha) \vee (A_F(y) + \alpha) < 1. \end{aligned}$$

Hence $xy \in F_q(A;\gamma)$ and $F_q(A;\gamma)$ is a subsemigroup of S. □

Theorem 3.3. *If $A = (A_T, A_I, A_F)$ is a $(q, \in\vee q)$-neutrosophic subsemigroup of a semigroup S, then neutrosophic q-subsets $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subsemigroups of S for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0,0,5)$ whenever they are nonempty.*

Proof. Let $x, y \in T_q(A;\alpha)$. Then $xy \in T_{\in\vee q}(A;\alpha)$, and so $xy \in T_\in(A;\alpha)$ or $xy \in T_q(A;\alpha)$. If $xy \in T_\in(A;\alpha)$, then $A_T(xy) \geq \alpha > 1 - \alpha$ since $\alpha > 0.5$. Hence $xy \in T_q(A;\alpha)$. Therefore $T_q(A;\alpha)$ is a subsemigroup of S. Similarly, we prove that $I_q(A;\beta)$ is a subsemigroup of S. Let $x, y \in F_q(A;\gamma)$. Then $xy \in F_{\in\vee q}(A;\gamma)$, and so $xy \in F_\in(A;\gamma)$ or $xy \in F_q(A;\gamma)$. If $xy \in F_\in(A;\gamma)$, then $A_F(xy) \leq \gamma < 1-\gamma$ since $\gamma \in [0,0,5)$. Hence $xy \in F_q(A;\gamma)$, and therefore $F_q(A;\gamma)$ is a subsemigroup of S. □

We provide characterizations of an $(\in, \in\vee q)$-neutrosophic subsemigroup.

Theorem 3.4. *A neutrosophic set $A = (A_T, A_I, A_F)$ in a semigroup S is an $(\in, \in\vee q)$-neutrosophic subsemigroup of S if and only if it satisfies:*

$$(\forall x, y \in S) \left(\begin{array}{l} A_T(xy) \geq \bigwedge\{A_T(x), A_T(y), 0.5\} \\ A_I(xy) \geq \bigwedge\{A_I(x), A_I(y).0.5\} \\ A_F(xy) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \end{array} \right). \tag{3.6}$$

Proof. Suppose that $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subsemigroup of S and let $x, y \in S$. If $A_T(x) \wedge A_T(y) < 0.5$, then $A_T(xy) \geq A_T(x) \wedge A_T(y)$. For, assume that $A_T(xy) < A_T(x) \wedge A_T(y)$ and choose α_t such that

$$A_T(xy) < \alpha_t < A_T(x) \wedge A_T(y).$$

Then $x \in T_\in(A;\alpha_t)$ and $y \in T_\in(A;\alpha_t)$ but $xy \notin T_\in(A;\alpha_t)$. Also $A_T(xy)+\alpha_t < 1$, i.e., $xy \notin T_q(A;\alpha_t)$. Thus $xy \notin T_{\in\vee q}(A;\alpha_t)$, a contradiction. Therefore $A_T(xy) \geq \bigwedge\{A_T(x), A_T(y), 0.5\}$ whenever $A_T(x) \wedge A_T(y) < 0.5$. Now suppose that $A_T(x) \wedge A_T(y) \geq 0.5$. Then $x \in T_\in(A;0.5)$ and $y \in T_\in(A;0.5)$, which imply that $xy \in T_{\in\vee q}(A;0.5)$. Hence $A_T(xy) \geq 0.5$. Otherwise, $A_T(xy)+0.5 < 0.5+0.5 = 1$, a contradiction. Consequently, $A_T(xy) \geq \bigwedge\{A_T(x), A_T(y), 0.5\}$ for all $x, y \in S$. Similarly, we know that $A_I(xy) \geq \bigwedge\{A_I(x), A_I(y), 0.5\}$ for all $x, y \in S$. Suppose that $A_F(x) \vee A_F(y) > 0.5$. If $A_F(xy) > A_F(x) \vee A_F(y) := \gamma_f$, then $x, y \in F_\in(A;\gamma_f)$, $xy \notin F_\in(A;\gamma_f)$ and $A_F(xy)+\gamma_f > 2\gamma_f > 1$, i.e., $xy \notin F_q(A;\gamma_f)$. This is a contradiction. Hence $A_F(xy) \leq \bigvee\{A_F(x), A_F(y), 0.5\}$ whenever $A_F(x)\vee A_F(y) > 0.5$. Now, assume that $A_F(x)\vee A_F(y) \leq 0.5$. Then $x, y \in F_\in(A;0.5)$ and so $xy \in F_{\in\vee q}(A;0.5)$. Thus $A_F(xy) \leq 0.5$ or $A_F(xy)+0.5 < 1$. If $A_F(xy) > 0.5$, then $A_F(xy)+0.5 > 0.5+0.5 = 1$, a contradiction. Thus $A_F(xy) \leq 0.5$, and so $A_F(xy) \leq \bigvee\{A_F(x), A_F(y), 0.5\}$ whenever $A_F(x)\vee A_F(y) \leq 0.5$. Therefore $A_F(xy) \leq \bigvee\{A_F(x), A_F(y), 0.5\}$ for all $x, y \in S$.

Conversely, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in S which satisfies the condition (3.6). Let $x, y \in S$ and $\alpha_x, \alpha_y, \beta_x, \beta_y, \gamma_x, \gamma_y \in [0,1]$. If $x \in T_\in(A;\alpha_x)$ and $y \in T_\in(A;\alpha_y)$, then $A_T(x) \geq \alpha_x$ and $A_T(y) \geq \alpha_y$. If $A_T(xy) < \alpha_x \wedge \alpha_y$, then $A_T(x) \wedge A_T(y) \geq 0.5$. Otherwise, we have

$$A_T(xy) \geq \bigwedge\{A_T(x), A_T(y), 0.5\} = A_T(x) \wedge A_T(y) \geq \alpha_x \wedge \alpha_y,$$

a contradiction. It follows that

$$A_T(xy) + \alpha_x \wedge \alpha_y > 2A_T(xy) \geq 2\bigwedge\{A_T(x), A_T(y), 0.5\} = 1$$

and so that $xy \in T_q(A;\alpha_x \wedge \alpha_y) \subseteq T_{\in\vee q}(A;\alpha_x \wedge \alpha_y)$. Similarly, if $x \in I_\in(A;\beta_x)$ and $y \in I_\in(A;\beta_y)$, then $xy \in I_{\in\vee q}(A;\beta_x \wedge \beta_y)$. Now, let $x \in F_\in(A;\gamma_x)$ and $y \in F_\in(A;\gamma_y)$. Then $A_F(x) \leq \gamma_x$ and $A_F(y) \leq \gamma_y$. If $A_F(xy) > \gamma_x \vee \gamma_y$, then $A_F(x) \vee A_F(y) \leq 0.5$ because if not, then

$$A_F(xy) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \leq A_F(x) \vee A_F(y) \leq \gamma_x \vee \gamma_y,$$

which is a contradiction. Hence

$$A_F(xy) + \gamma_x \vee \gamma_y < 2A_F(xy) \leq 2\bigvee\{A_F(x), A_F(y), 0.5\} = 1,$$

and so $xy \in F_q(A;\gamma_x \vee \gamma_y) \subseteq F_{\in\vee q}(A;\gamma_x \vee \gamma_y)$. Therefore $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subsemigroup of S. □

Theorem 3.5. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee q)$-neutrosophic subsemigroup of a semigroup S, then neutrosophic q-subsets $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are subsemigroups of S for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$ whenever they are nonempty.*

Proof. Assume that $T_q(A;\alpha)$, $I_q(A;\beta)$ and $F_q(A;\gamma)$ are nonempty for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$. Let $x, y \in T_q(A;\alpha)$. Then $A_T(x)+\alpha > 1$ and $A_T(y)+\alpha > 1$. It follows from Theorem 3.4 that

$$\begin{aligned} A_T(xy) + \alpha &\geq \bigwedge\{A_T(x), A_T(y), 0.5\} + \alpha \\ &= \bigwedge\{A_T(x)+\alpha, A_T(y)+\alpha, 0.5+\alpha\} \\ &> 1, \end{aligned}$$

that is, $xy \in T_q(A;\alpha)$. Hence $T_q(A;\alpha)$ is a subsemigroup of S. By the similar way, we can induce that $I_q(A;\beta)$ is a subsemigroup of S. Now, let $x, y \in F_q(A;\gamma)$. Then

$A_F(x) + \gamma < 1$ and $A_F(y) + \gamma < 1$. Using Theorem 3.4, we have

$$\begin{aligned} A_F(xy) + \gamma &\leq \bigvee\{A_F(x), A_F(y), 0.5\} + \gamma \\ &= \bigvee\{A_F(x) + \gamma, A_F(y) + \gamma, 0.5 + \gamma\} \\ &< 1, \end{aligned}$$

and so $xy \in F_q(A;\gamma)$. Therefore $F_q(A;\gamma)$ is a subsemigroup of S. □

Theorem 3.6. *For a neutrosophic set $A = (A_T, A_I, A_F)$ in a semigroup S, if the nonempty neutrosophic $\in \vee\, q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are subsemigroups of S for all $\alpha, \beta \in (0,1]$ and $\gamma \in [0,1)$, then $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q)$-neutrosophic subsemigroup of S.*

Proof. Let $T_{\in\vee\, q}(A;\alpha)$ be a subsemigroup of S and assume that

$$A_T(xy) < \bigwedge\{A_T(x), A_T(y), 0.5\}$$

for some $x, y \in S$. Then there exists $\alpha \in (0, 0.5]$ such that

$$A_T(xy) < \alpha \leq \bigwedge\{A_T(x), A_T(y), 0.5\}.$$

It follows that $x, y \in T_{\in}(A;\alpha) \subseteq T_{\in\vee\, q}(A;\alpha)$, and so that $xy \in T_{\in\vee\, q}(A;\alpha)$. Hence $A_T(xy) \geq \alpha$ or $A_T(xy) + \alpha > 1$. This is a contradiction, and so

$$A_T(xy) \geq \bigwedge\{A_T(x), A_T(y), 0.5\}$$

for all $x, y \in S$. Similarly, we show that

$$A_I(xy) \geq \bigwedge\{A_I(x), A_I(y), 0.5\}$$

for all $x, y \in S$. Now let $F_{\in\vee\, q}(A;\gamma)$ be a subsemigroup of S and assume that

$$A_F(xy) > \bigvee\{A_F(x), A_F(y), 0.5\}$$

for some $x, y \in S$. Then

$$A_F(xy) > \gamma \geq \bigvee\{A_F(x), A_F(y), 0.5\}, \tag{3.7}$$

for some $\gamma \in [0.5, 1)$, which implies that $x, y \in F_{\in}(A;\gamma) \subseteq F_{\in\vee\, q}(A;\gamma)$. Thus $xy \in F_{\in\vee\, q}(A;\gamma)$. From (3.7), we have $xy \notin F_{\in}(A;\gamma)$ and $A_F(xy) + \gamma > 2\gamma \geq 1$, i.e., $xy \notin F_q(A;\gamma)$. This is a contradiction, and hence

$$A_F(xy) \leq \bigvee\{A_F(x), A_F(y), 0.5\}$$

for all $x, y \in S$. Using Theorem 3.4, we know that $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q)$-neutrosophic subsemigroup of S. □

Theorem 3.7. *If $A = (A_T, A_I, A_F)$ is an $(\in, \in\vee\, q)$-neutrosophic subsemigroup of a semigroup S, then nonempty neutrosophic $\in\vee\, q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are subsemigroups of S for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$.*

Proof. Assume that $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are nonempty for all $\alpha, \beta \in (0, 0.5]$ and $\gamma \in [0.5, 1)$. Let $x, y \in I_{\in\vee\, q}(A;\beta)$. Then

$$x \in I_{\in}(A;\beta) \text{ or } x \in I_q(A;\beta),$$

and

$$y \in I_{\in}(A;\beta) \text{ or } y \in I_q(A;\beta).$$

Hence we have the following four cases:

(i) $x \in I_\in(A;\beta)$ and $y \in I_\in(A;\beta)$,
(ii) $x \in I_\in(A;\beta)$ and $y \in I_q(A;\beta)$,
(iii) $x \in I_q(A;\beta)$ and $y \in I_\in(A;\beta)$,
(iv) $x \in I_q(A;\beta)$ and $y \in I_q(A;\beta)$.

The first case implies that $xy \in I_{\in\vee\, q}(A;\beta)$. For the second case, $y \in I_q(A;\beta)$ induces $A_I(y) > 1-\beta \geq \beta$, that is, $y \in I_\in(A;\beta)$. Thus $xy \in I_{\in\vee\, q}(A;\beta)$. Similarly, the third case implies $xy \in I_{\in\vee\, q}(A;\beta)$. The last case induces $A_I(x) > 1-\beta \geq \beta$ and $A_I(y) > 1-\beta \geq \beta$, that is, $x \in I_\in(A;\beta)$ and $y \in I_\in(A;\beta)$. Hence $xy \in I_{\in\vee\, q}(A;\beta)$. Therefore $I_{\in\vee\, q}(A;\beta)$ is a subsemigroup of S for all $\beta \in (0, 0.5]$. By the similar way, we show that $T_{\in\vee\, q}(A;\alpha)$ is a subsemigroup of S for all $\alpha \in (0, 0.5]$. Let $x, y \in F_{\in\vee\, q}(A;\gamma)$. Then

$$A_F(x) \leq \gamma \text{ or } A_F(x) + \gamma < 1,$$

and

$$A_F(y) \leq \gamma \text{ or } A_F(y) + \gamma < 1.$$

If $A_F(x) \leq \gamma$ and $A_F(y) \leq \gamma$, then

$$A_F(xy) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \leq \bigvee\{\gamma, 0.5\} = \gamma$$

by Theorem 3.4, and so $xy \in F_\in(A;\gamma) \subseteq F_{\in\vee\, q}(A;\gamma)$. If $A_F(x) \leq \gamma$ and $A_F(y) + \gamma < 1$, then

$$A_F(xy) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \leq \bigvee\{\gamma, 1-\gamma, 0.5\} = \gamma$$

by Theorem 3.4. Thus $xy \in F_\in(A;\gamma) \subseteq F_{\in\vee\, q}(A;\gamma)$. Similarly, if $A_F(x) + \gamma < 1$ and $A_F(y) \leq \gamma$, then $xy \in F_{\in\vee\, q}(A;\gamma)$. Finally, assume that $A_F(x) + \gamma < 1$ and $A_F(y) + \gamma < 1$. Then

$$A_F(xy) \leq \bigvee\{A_F(x), A_F(y), 0.5\} \leq \bigvee\{1-\gamma, 0.5\} = 0.5 < \gamma$$

by Theorem 3.4. Hence $xy \in F_\in(A;\gamma) \subseteq F_{\in\vee\, q}(A;\gamma)$. Consequently, $F_{\in\vee\, q}(A;\gamma)$ is a subsemigroup of S for all $\gamma \in [0.5, 1)$. □

Theorem 3.8. *If $A = (A_T, A_I, A_F)$ is a $(q, \in\vee q)$-neutrosophic subsemigroup of a semigroup S, then nonempty neutrosophic $\in\vee q$-subsets $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are subsemigroups of S for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$.*

Proof. Assume that $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are nonempty for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$. Let $x, y \in T_{\in\vee\, q}(A;\alpha)$. Then

$$x \in T_\in(A;\alpha) \text{ or } x \in T_q(A;\alpha).$$

and

$$y \in T_\in(A;\alpha) \text{ or } y \in T_q(A;\alpha).$$

If $x \in T_q(A;\alpha)$ and $y \in T_q(A;\alpha)$, then obviously $xy \in T_{\in\vee\, q}(A;\alpha)$. Suppose that $x \in T_\in(A;\alpha)$ and $y \in T_q(A;\alpha)$. Then $A_T(x) + \alpha \geq 2\alpha > 1$, i.e., $x \in T_q(A;\alpha)$. It follows that $xy \in T_{\in\vee\, q}(A;\alpha)$. Similarly, if $x \in T_q(A;\alpha)$ and $y \in T_\in(A;\alpha)$, then $xy \in T_{\in\vee\, q}(A;\alpha)$. Now, let $x, y \in F_{\in\vee\, q}(A;\gamma)$. Then

$$x \in F_\in(A;\gamma) \text{ or } x \in F_q(A;\gamma),$$

and

$$y \in F_{\in}(A;\gamma) \text{ or } y \in F_q(A;\gamma).$$

If $x \in F_q(A;\gamma)$ and $y \in F_q(A;\gamma)$, then clearly $xy \in F_{\in\vee\, q}(A;\gamma)$. If $x \in F_{\in}(A;\gamma)$ and $y \in F_q(A;\gamma)$, then $A_F(x) + \gamma \leq 2\gamma < 1$, i.e., $x \in F_q(A;\gamma)$. It follows that $xy \in F_{\in\vee\, q}(A;\gamma)$. Similarly, if $x \in F_q(A;\gamma)$ and $y \in F_{\in}(A;\gamma)$, then $xy \in F_{\in\vee\, q}(A;\gamma)$. Finally, assume that $x \in F_{\in}(A;\gamma)$ and $y \in F_{\in}(A;\gamma)$. Then $A_F(x) + \gamma \leq 2\gamma < 1$ and $A_F(y) + \gamma \leq 2\gamma < 1$, that is, $x \in F_q(A;\gamma)$ and $y \in F_q(A;\gamma)$. Therefore $xy \in F_{\in\vee\, q}(A;\gamma)$. Consequently, $T_{\in\vee\, q}(A;\alpha)$, $I_{\in\vee\, q}(A;\beta)$ and $F_{\in\vee\, q}(A;\gamma)$ are subsemigroups of S for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0.5)$. □

Given a neutrosophic set $A = (A_T, A_I, A_F)$ in a set S, we consider:

$$S_0^1 := \{x \in S \mid A_T(x) > 0,\ A_I(x) > 0,\ A_F(x) < 1\}.$$

Theorem 3.9. *If a neutrosophic set $A = (A_T, A_I, A_F)$ in a semigroup S is an $(\in, \in)$-neutrosophic subsemigroup of S, then the set S_0^1 is a subsemigroup of S.*

Proof. Let $x, y \in S_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$ and $A_F(y) < 1$. Suppose that $A_T(xy) = 0$. Note that $x \in T_{\in}(A; A_T(x))$ and $y \in T_{\in}(A; A_T(y))$. But $xy \notin T_{\in}(A; A_T(x) \wedge A_T(y))$ because $A_T(xy) = 0 < A_T(x) \wedge A_T(y)$. This is a contradiction, and thus $A_T(xy) > 0$. By the similar way, we show that $A_I(xy) > 0$. Note that $x \in F_{\in}(A; A_F(x))$ and $y \in F_{\in}(A; A_F(y))$. If $A_F(xy) = 1$, then $A_F(xy) = 1 > A_F(x) \vee A_F(y)$, and so $xy \notin F_{\in}(A; A_F(x) \vee A_F(y))$. This is impossible. Hence $xy \in S_0^1$, and therefore S_0^1 is a subsemigroup of S. □

Theorem 3.10. *If a neutrosophic set $A = (A_T, A_I, A_F)$ in a semigroup S is an $(\in, q)$-neutrosophic subsemigroup of S, then the set S_0^1 is a subsemigroup of S.*

Proof. Let $x, y \in S_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$ and $A_F(y) < 1$. If $A_T(xy) = 0$, then

$$A_T(xy) + A_T(x) \wedge A_T(y) = A_T(x) \wedge A_T(y) \leq 1.$$

Hence $xy \notin T_q(A; A_T(x) \wedge A_T(y))$, which is a contradiction since $x \in T_{\in}(A; A_T(x))$ and $y \in T_{\in}(A; A_T(y))$. Thus $A_T(xy) > 0$. Similarly, we get $A_I(xy) > 0$. Assume that $A_F(xy) = 1$. Then

$$A_F(xy) + A_F(x) \vee A_F(y) = 1 + A_F(x) \vee A_F(y) \geq 1,$$

that is, $xy \notin F_q(A; A_F(x) \vee A_F(y))$. This is a contradiction because of $x \in F_{\in}(A; A_F(x))$ and $y \in F_{\in}(A; A_F(y))$. Hence $A_F(xy) < 1$. Consequently, $xy \in S_0^1$ and S_0^1 is a subsemigroup of S. □

Theorem 3.11. *If a neutrosophic set $A = (A_T, A_I, A_F)$ in a semigroup S is a $(q, \in)$-neutrosophic subsemigroup of S, then the set S_0^1 is a subsemigroup of S.*

Proof. Let $x, y \in S_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$ and $A_F(y) < 1$. It follows that $A_T(x) + 1 > 1$, $A_T(y) + 1 > 1$, $A_I(x) + 1 > 1$, $A_I(y) + 1 > 1$, $A_F(x) + 0 < 1$ and $A_F(y) + 0 < 1$. Hence $x, y \in T_q(A; 1) \cap I_q(A; 1) \cap F_q(A; 0)$. If $A_T(xy) = 0$ or $A_I(xy) = 0$, then $A_T(xy) < 1 = 1 \wedge 1$ or $A_I(xy) < 1 = 1 \wedge 1$. Thus $xy \notin T_q(A; 1 \wedge 1)$ or $xy \notin I_q(A; 1 \wedge 1)$, a contradiction. Hence $A_T(xy) > 0$ and $A_I(xy) > 0$. If $A_F(xy) = 1$, then $xy \notin F_q(A; 0 \vee 0)$ which is a contradiction. Thus $A_F(xy) < 1$. Therefore $xy \in S_0^1$ and the proof is complete. □

Theorem 3.12. *If a neutrosophic set $A = (A_T, A_I, A_F)$ in a semigroup S is a (q, q)-neutrosophic subsemigroup of S, then the set S_0^1 is a subsemigroup of S.*

Proof. Let $x, y \in S_0^1$. Then $A_T(x) > 0$, $A_I(x) > 0$, $A_F(x) < 1$, $A_T(y) > 0$, $A_I(y) > 0$ and $A_F(y) < 1$. Hence $A_T(x) + 1 > 1$, $A_T(y) + 1 > 1$, $A_I(x) + 1 > 1$, $A_I(y) + 1 > 1$, $A_F(x) + 0 < 1$ and $A_F(y) + 0 < 1$. Hence $x, y \in T_q(A; 1) \cap I_q(A; 1) \cap F_q(A; 0)$. If $A_T(xy) = 0$ or $A_I(xy) = 0$, then

$$A_T(xy) + 1 \wedge 1 = 0 + 1 = 1$$

or

$$A_I(xy) + 1 \wedge 1 = 0 + 1 = 1,$$

and so $xy \notin T_q(A; 1 \wedge 1)$ or $xy \notin I_q(A; 1 \wedge 1)$. This is impossible, and thus $A_T(xy) > 0$ and $A_I(xy) > 0$. If $A_F(xy) = 1$, then $A_F(xy) + 0 \vee 0 = 1$, that is, $xy \notin F_q(A; 0 \vee 0)$. This is a contradiction, and so $A_F(xy) < 1$. Therefore $xy \in S_0^1$ and the proof is complete. $\square$

We provide conditions for a neutrosophic set to be a $(q, \in \vee\, q)$-neutrosophic subsemigroup.

Theorem 3.13. *For a subsemigroup Q of a semigroup S, let $A = (A_T, A_I, A_F)$ be a neutrosophic set in S such that*

$$(\forall x \in Q)\,(A_T(x) \geq 0.5, A_I(x) \geq 0.5, A_F(x) \leq 0.5)\,, \tag{3.8}$$

$$(\forall x \in S \setminus Q)\,(A_T(x) = 0, A_I(x) = 0, A_F(x) = 1)\,. \tag{3.9}$$

Then $A = (A_T, A_I, A_F)$ is a $(q, \in \vee\, q)$-neutrosophic subsemigroup of S.

Proof. Assume that $x \in I_q(A; \beta_x)$ and $y \in I_q(A; \beta_y)$ for all $x, y \in S$ and $\beta_x, \beta_y \in [0, 1]$. Then $A_I(x) + \beta_x > 1$ and $A_I(y) + \beta_y > 1$. If $xy \notin Q$, then $x \in S \setminus Q$ or $y \in S \setminus Q$ since Q is a subsemigroup of S. Hence $A_I(x) = 0$ or $A_I(y) = 0$, which imply that $\beta_x > 1$ or $\beta_y > 1$. This is a contradiction, and so $xy \in Q$. If $\beta_x \wedge \beta_y > 0.5$, then $A_I(xy) + \beta_x \wedge \beta_y > 1$, i.e., $xy \in I_q(A; \beta_x \wedge \beta_y)$. If $\beta_x \wedge \beta_y \leq 0.5$, then $A_I(xy) \geq 0.5 \geq \beta_x \wedge \beta_y$, i.e., $xy \in I_\in(A; \beta_x \wedge \beta_y)$. Hence $xy \in I_{\in\vee\, q}(A; \beta_x \wedge \beta_y)$. Similarly, if $x \in T_q(A; \alpha_x)$ and $y \in T_q(A; \alpha_y)$ for all $x, y \in S$ and $\alpha_x, \alpha_y \in [0, 1]$, then $xy \in T_{\in\vee\, q}(A; \alpha_x \wedge \alpha_y)$. Now let $x, y \in S$ and $\gamma_x, \gamma_y \in [0, 1]$ be such that $x \in F_q(A; \gamma_x)$ and $y \in F_q(A; \gamma_y)$. Then $A_F(x) + \gamma_x < 1$ and $A_F(y) + \gamma_y < 1$. It follows that $xy \in Q$. In fact, if not then $x \in S \setminus Q$ or $y \in S \setminus Q$ since Q is a subsemigroup of S. Hence $A_F(x) = 1$ or $A_F(y) = 1$, which imply that $\gamma_x < 0$ or $\gamma_y < 0$. This is a contradiction, and so $xy \in Q$. If $\gamma_x \vee \gamma_y \geq 0.5$, then $A_F(xy) \leq 0.5 \leq \gamma_x \vee \gamma_y$, that is, $xy \in F_\in(A; \gamma_x \vee \gamma_y)$. If $\gamma_x \vee \gamma_y < 0.5$, then $A_F(xy) + \gamma_x \vee \gamma_y < 1$, that is, $xy \in F_q(A; \gamma_x \vee \gamma_y)$. Hence $xy \in F_{\in\vee\, q}(A; \gamma_x \vee \gamma_y)$, and consequently $A = (A_T, A_I, A_F)$ is a $(q, \in \vee\, q)$-neutrosophic subsemigroup of S. $\square$

Combining Theorems 3.3 and 3.13, we have the following corollary.

Corollary 3.14. *For a subsemigroup Q of S, if $A = (A_T, A_I, A_F)$ is a neutrosophic set in S satisfying conditions (3.8) and (3.9), then $T_q(A; \alpha)$, $I_q(A; \beta)$ and $F_q(A; \gamma)$ are subsemigroups of S for all $\alpha, \beta \in (0.5, 1]$ and $\gamma \in [0, 0, 5)$ whenever they are nonempty.*

4. CONCLUSION

In this paper, we introduce the notion of (Φ, Ψ)-neutrosophic subsemigroup of a semigroup S for $\Phi, \Psi \in \{\in, q, \in \vee\, q\}$, and investigate related properties. We provide characterizations of an $(\in, \in)$-neutrosophic subsubsemigroup and an $(\in, \in \vee\, q)$-neutrosophic subsubsemigroup. Given special sets, so called neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in \vee q$-subsets, we provide conditions for the neutrosophic $\in$-subsets, neutrosophic q-subsets and neutrosophic $\in \vee q$-subsets to be subsemigroups. We discuss conditions for a neutrosophic set to be a $(q, \in \vee q)$-neutrosophic subsubsemigroup. We hope that this work will provide a deep impact on the upcoming research in this field and other related algebraic structures studies to open up new horizons of interest and innovations. As future directions, one can further study the neutrosophic set theory in different algebras such as BCK/BCI-algebras, BL-algebras, EQ-algebras, B-algebras, MV-algebras, Q-algebras, etc.

5. Acknowledgements

The author is grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

[1] A.A.A. Agboola and B. Davvaz, On neutrosophic ideals of neutrosophic BCI-algebras, Critical Review. Volume X, (2015), 93–103.

[2] A.A.A. Agboola and B. Davvaz, Introduction to neutrosophic BCI/BCK-algebras, Inter. J. Math. Math. Sci. Volume 2015, Article ID 370267, 6 pages.

[3] S. Broumi and F. Smarandache, Correlation coefficient of interval neutrosophic sets. Appl. Mech. Mater. 436 (2013), 511517.

[4] H.D. Cheng and Y. Guo, A new neutrosophic approach to image thresholding. New Math. Nat. Comput. 42(008), 291–308.

[5] Y. Guo and H.D. Cheng, New neutrosophic approach to image segmentation. Pat. Recognit. 42 (2009), 587–595.

[6] G. Muhiuddin, H. Bordbar, F. Smarandache, Y. B. Jun, Further results on $(\in, \in)$-neutrosophic subalgebras and ideals in BCK/BCI-algebras, Neutrosophic Sets and Systems, Vol. 20 (2018), 36–43.

[7] P. M. Pu and Y. M. Liu, Fuzzy topology I, Neighborhood structure of a fuzzy point and Moore-Smith convergence, J. Math. Anal. Appl. **76** (1980), 571–599.

[8] F. Smarandache, Neutrosophy, Neutrosophic Probability, Set, and Logic, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998. http://fs.gallup.unm.edu/eBook-neutrosophics6.pdf (last edition online).

[9] F. Smarandache, A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability, American Reserch Press, Rehoboth, NM, 1999.

[10] F. Smarandache, Neutrosophic set-a generalization of the intuitionistic fuzzy set, Int. J. Pure Appl. Math. 24(3) (2005), 287–297.

[11] J. Ye, Similarity measures between interval neutrosophic sets and their multicriteria decision-making method. J. Intell. Fuzzy Syst. 26 (2014), 165–172.

G. Muhiuddin
Department of Mathematics, University of Tabuk, Tabuk 71491, Saudi Arabia
Email address: chishtygm@gmail.com