Neutrosophic Topological Groups

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Abstract

In this work we continue the study of topological group structure of neutrosophic sets. Some basic properties of neutrosophic topological groups are investigated by giving an alternate definition for neutrosophic topological group. It is proved that automorphism of neutrosophic topological groups is neutrosophic homeomorphism.

1 Introduction

In 1965, Zadeh [10] introduced fuzzy logic. In 1983, Atanassov [2] introduced fuzzy intuitionistic set in which every element of a nonempty set has a degree of membership and a degree of nonmembership. In 1995, Smrandache [7] initiated the idea of a neutrosophic set of a nonempty set in which every element has a degree of membership, a degree of indeterminacy and a degree of nonmembership. In 2012, Salama and Alblowi [3] introduced neutrosophic topological spaces and in 2015, Sumathi and Arockiarani [8], defined the fuzzy neutrosophic groups. In 2016, Sumathi and Arockiarani [9] combined the neutrosophic algebraic structure of group and neutrosophic topological structure of topological space and linked by the requirement that the multiplication and inversion mappings are neutrosphic continuous.

In this paper we shall continue the study of neutrosphic topological groups by introducing an alternative definition in order to investigate and develope some basic theory.

2 Preliminaries

Definition 1 *[3] On a universe of discorse X a neutrosophic set (NS) A is defined as*

$$A = \{\langle t, T_A(t), I_A(t), F_A(t)\rangle : t \in X\},$$

where $T_A(t), I_A(t)$, and $F_A(t)$ represent the degree of membership function, the degree of indeterminacy and the degree of nonmembership respecively of each element $t \in X$ to the set A. Note that $0 \leq T_A(t) + I_A(t) + F_A(t) \leq 3$.

Definition 2 *[5, 3] The complement of NS A is denoted by A^c and is defined as*

$$A^c(t) = \{\langle t, T_{A^c}(t) = F_A(t), I_{A^c}(t) = 1 - I_A(t), F_{A^c}(t) = T_A(t)\rangle : t \in X\}$$

Definition 3 *[5, 3] Let $X \neq \phi$ and $A = \{\langle t, T_A(t), I_A(t), F_A(t)\rangle : t \in X\}$, $B = \{\langle t, T_B(t), I_B(t), F_B(t)\rangle : t \in X\}$ are neutrosophic sets. Then:*

1. $A \wedge B = \{\langle t, \min((T_A(t), T_B(t)), \min(I_A(t), I_B(t)), \max(F_A(t), F_B(t))\rangle : t \in X\}$

2. $A \vee B = \{\langle t, \max((T_A(t), T_B(t)), \max(I_A(t), I_B(t)), \min(F_A(t), F_B(t))\rangle : t \in X\}$

3. $A \leq B$ if for each $t \in X$, $T_A(t) \leq T_B(t)$, $I_A(t) \leq I_B(t)$, $F_A(t) \geq F_B(t)$

Definition 4 *[6] Let X and Y be two nonempty sets and let f be a function from a set X to a set Y. Let $A = \{\langle t, T_A(t), I_A(t), F_A(t)\rangle : t \in X\}$, $B = \{\langle u, T_B(u), I_B(u), F_B(u)\rangle : u \in Y\}$ be NS in X and Y respectively. Then*

1. $f^{\leftarrow}(B)$, the preimage of B under f is the NS in X defined by

$$f^{\leftarrow}(B) = \{\langle t, f^{\leftarrow}(T_B)(t), f^{\leftarrow}(I_B)(t), f^{\leftarrow}(F_B)(t)\rangle : t \in X\}$$

where, for all $t \in X$, $f^{\leftarrow}(T_B)(t) = T_B(f(t))$, $f^{\leftarrow}(I_B)(t) = I_B(f(t))$, $f^{\leftarrow}(F_B)(t) = F_B(f(t))$.

2. The image of A under f denoted by $f(A)$ is a NS in Y defined by

$$f(A) = (f(T_A), f(I_A), f(F_A)),$$

where for each $u \in Y$,

$$f(T_A)(u) = \left\{ \begin{array}{c} \vee_{t \in f^{\leftarrow}(u)} T_A(t) \text{ , if } f^{\leftarrow}(u) \neq \phi \\ 0 \text{ , otherwise} \end{array} \right\}$$

$$f(I_A)(u) = \left\{ \begin{array}{c} \vee_{t \in f^{\leftarrow}(u)} I_A(t) \text{ , if } f^{\leftarrow}(u) \neq \phi \\ 0 \text{ , otherwise} \end{array} \right\}$$

$$f(F_A)(u) = \left\{ \begin{array}{c} \vee_{t \in f^{\leftarrow}(u)} F_A(t) \text{ , if } f^{\leftarrow}(u) \neq \phi \\ 0 \text{ , otherwise} \end{array} \right\}$$

Definition 5 *[4] Let $\alpha, \beta, \gamma \in [0,1]$ and $\alpha+\beta+\gamma \leq 3$. A neutrosophic point $t_{(\alpha,\beta,\gamma)}$ (NP) of X is the neutrosophic set in X defined by*

$$t_{(\alpha,\beta,\gamma)}(u) = \left\{ \begin{array}{ll} (\alpha,\beta,\gamma), & if \ \ t = u \\ (0,0,1), & if \ \ t \neq u \end{array} \right\} \ ; \ for\ each\ u \in X.$$

A neutrosophic point is said to belong to a neutrosophic set $A = \{\langle t, T_A(t), I_A(t), F_A(t)\rangle : t \in X\}$ in X denoted by $t_{(\alpha,\beta,\gamma)} \in A$ if

$$\alpha \leq T_A(t), \beta \leq I_A(t), \text{ and } \gamma \geq F_A(t).$$

We denote the set of all neutrosophic points in X by NP(X).

For the definition of neutrosophic topology, we refer the reader to see [3].

Definition 6 *[4] Let $(X, \urcorner)$ be a neutrosophic topological space and A be a neutrosophic set in X. Then the induced neutrosophic topology (INT) on A is the collection of neutrosophic sets in A which are the intersection of neutrosophic open sets in X with A. Then the pair $(A, \urcorner_A)$ is called a neutrosophic subspace of $(X, \urcorner)$. The induced neutrosophic topology is denoted by $\urcorner_A$.*

Definition 7 *[3] Let $(X, \urcorner)$ be a neutrosophic topological space and $A = \{\langle t, T_A(t), I_A(t), F_A(t)\rangle : t \in X\}$ be a neutrosophic set in X. Then the neutrosophic closure of A is denoted by $\mathcal{N}Cl(A)$ and defined by $\mathcal{N}Cl(A) = \wedge\{K : K$ is a neutrosophic closed set in X and $A \leq K\}$.*

Definition 8 *[8] Let $(X, .)$ be a group and let A be a NS in X. Then A is said to be a neutrosophic group (NG) in X if it satisfies the following axioms for all $t, u \in X$,*

1) i) $T_A(tu) \geq T_A(t) \wedge T_A(u)$,
ii) $I_A(tu) \geq I_A(t) \wedge I_A(u)$,
iii) $F_A(tu) \leq F_A(t) \wedge F_A(u)$.
2) i) $T_A(t^{-1}) \geq T_A(t)$,
ii) $I_A(t^{-1}) \geq I_A(t)$,
iii) $F_A(t^{-1}) \leq F_A(t)$

Definition 9 *[9] Let X be a group and G be neutrosophic group in X. Let $a \in X$ be a fixed point. Then the neutrosophic left coset of neutrosophic group G by a is denoted by aG and is defined by*

$$aG(t) = G(a^{-1}t) = \left\{\langle t, T(a^{-1}t), I(a^{-1}t), F(a^{-1}t\rangle : t \in X)\right\}$$

Definition 10 *[9] Let X be a group and let G be neutrosophic group (NG) in X and e be the identity of X. We define the neutrosophic set G_e by:*

$$G_e = \{x \in X : T_G(x) = T_G(e), I_G(x) = I_G(e), F_G(x) = F_G(e)\}.$$

We note that for a neutrosophic group G in a group X, for every $x \in X$: $T_G(x^{-1}) = T_G(x)$, $I_G(x^{-1}) = I_G(x)$ and $F_G(x^{-1}) = F_G(x)$. Also for the identity e of the group X: $T_G(e) \geq T_G(x)$, $I_G(e) \geq I_G(x)$ and $F_G(e) \leq F_G(x)$.

Proposition 11 *Let G be a neutrosophic group in a group X . Then for all $x, y \in X$,*

1. $T_G\left(xy^{-1}\right) = T_G(e) \Rightarrow T_G(x) = T_G(y)$.
2. $I_G\left(xy^{-1}\right) = I_G(e) \Rightarrow I_G(x) = I_G(y)$.
3. $F_G\left(xy^{-1}\right) = F_G(e) \Rightarrow F_G(x) = F_G(y)$.

Proof. 1. $T_G(x) = T_G(\left(xy^{-1}\right)y) \geq T_G\left(xy^{-1}\right) \wedge T_G(y) = T_G(e) \wedge T_G(y) = T_G(y)$

$T_G(y) = T_G(\left(yx^{-1}\right)x) \geq T_G\left(yx^{-1}\right) \wedge T_G(x) = T_G(e) \wedge T_G(x) = T_G(x)$.

$\Rightarrow T_G(x) = T_G(y)$.

2 and 3 can be proved similarly. ■

Proposition 12 *Let X be a group. Then the following statements are equivalent;*

1. G is neutrosophic group in X.
2. For all $x, y \in X, T_G\left(xy^{-1}\right) \geq T_G(x) \wedge T_G(y), I_G\left(xy^{-1}\right) \geq I_G(x) \wedge I_G(y), F_G\left(xy^{-1}\right) \leq F_G(x) \vee F_G(y)$.

Proof. $(1 \Rightarrow 2)$

Assume that G is fuzzy neutrosophic group, then

$$T_G\left(tu^{-1}\right) \geq T_G(t) \wedge T_G\left(u^{-1}\right) = T_G(t) \wedge T_G(u),$$

$$I_G\left(tu^{-1}\right) \geq I_G(t) \wedge I_G\left(u^{-1}\right) = I_G(t) \wedge I_G(u),$$

and

$$F_G\left(tu^{-1}\right) \leq F_G(t) \vee F_G\left(u^{-1}\right) = F_G(t) \vee F_G(u).$$

$(2 \Rightarrow 1)$

Since

$$T_G\left(tu^{-1}\right) \geq T_G(t) \wedge T_G(u),$$

Let $u = t$, then

$$\begin{aligned} T_G\left(tt^{-1}\right) &\geq T_G(t) \wedge T_G\left(t^{-1}\right) \\ &\Rightarrow T_G(e) \geq T_G(t) \end{aligned}$$

Hence,

$$\begin{aligned} T_G\left(u^{-1}\right) &= T_G\left(eu^{-1}\right) \geq T_G(e) \wedge T_G(u) = T_G(u) \\ &\Rightarrow T_G\left(u^{-1}\right) \geq T_G(u) \end{aligned}$$

and

$$\begin{aligned} T_G(tu) &= T_G\left(t(u^{-1})^{-1}\right) \geq T_G(t) \wedge T_G\left(u^{-1}\right) \geq T_G(t) \wedge T_G(u) \\ &\Rightarrow T_G(tu) \geq T_G(t) \wedge T_G(u) \end{aligned}$$

Similarly Since

$$I_G\left(tu^{-1}\right) \geq I_G(t) \wedge I_G(u),$$

Let $u = t$, then

$$\begin{aligned} I_G\left(tt^{-1}\right) &\geq I_G(t) \wedge I_G\left(t^{-1}\right) \\ &\Rightarrow I_G(e) \geq I_G(t) \end{aligned}$$

Hence,

$$\begin{aligned} I_G\left(u^{-1}\right) &= I_G\left(eu^{-1}\right) \geq I_G(e) \wedge I_G(u) = I_G(u) \\ &\Rightarrow I_G\left(u^{-1}\right) \geq I_G(u) \end{aligned}$$

and

$$\begin{aligned} I_G(tu) &= I_G\left(t(u^{-1})^{-1}\right) \geq I_G(t) \wedge I_G\left(u^{-1}\right) \geq I_G(t) \wedge I_G(u) \\ &\Rightarrow I_G(tu) \geq I_G(t) \wedge I_G(u) \end{aligned}$$

and since

$$F_G\left(tu^{-1}\right) \leq F_G(t) \vee F_G(u),$$

Let $u = t$, then

$$\begin{aligned} F_G\left(tt^{-1}\right) &\leq F_G(t) \vee F_G\left(t^{-1}\right) \\ &\Rightarrow F_G(e) \leq F_G(t) \end{aligned}$$

Hence,

$$\begin{aligned} F_G\left(u^{-1}\right) &= F_G\left(eu^{-1}\right) \leq F_G(e) \vee F_G(u) = F_G(u) \\ &\Rightarrow F_G\left(u^{-1}\right) \leq F_G(u) \end{aligned}$$

and

$$\begin{aligned} F_G(tu) &= F_G\left(t(u^{-1})^{-1}\right) \leq F_G(t) \vee F_G\left(u^{-1}\right) \leq F_G(t) \vee F_G(u) \\ &\Rightarrow F_G(tu) \leq F_G(t) \vee F_G(u) \end{aligned}$$

■

3 Neutrosophic Continuity

It is known by [4] that $f : (X, \daleth_X) \to (Y, \daleth_Y)$ is neutrosophic contionuous if the preimage of each open neutrosophic set in Y is open neutrosophic set in X. We extend this definition.

Definition 13 *Let $f : (X, \daleth_X) \to (Y, \daleth_Y)$ be a function on neutrosophic topological spaces. Then f is neutrosophic continuous at a neutrosophic point (NP) $x_{(\alpha,\beta,\gamma)}$, such that $x \in X$, if for each neutrosophic set V in $\daleth_Y$, containing neutrosophic point $f\left(x_{(\alpha,\beta,\gamma)}\right)$, there exists a neutrosophic set U in $\daleth_X$ such that $f(U) \leq V$.*

The relationship between neutrosophic continuity on a set X and neutrosophic continuity at FNP is stated in the following theorem.

Theorem 14 *Let $(X, \urcorner_X)$ and $(Y, \urcorner_Y)$ be two NTS and $f : (X, \urcorner_X) \to (Y, \urcorner_Y)$ be a mapping, then f is neutrosophic continuous if and only if f is neutrosophic continuous at neutrosophic point $x_{(\alpha,\beta,\gamma)}$, for each $x \in X$.*

Proof. Assume that f is neutrosophic continuous at each neutrosophic point $x_{(\alpha,\beta,\gamma)}$, for $x \in X$. Let A be a neutrosophic set in $\urcorner_Y$. Then for each neutrosophic point $x_{(\alpha,\beta,\gamma)} \in f^{\leftarrow}(A)$, $f\left(x_{(\alpha,\beta,\gamma)}\right) \in A$. There exists a neutrosophic set B_x in $\urcorner_X$ such that $x_{(\alpha,\beta,\gamma)} \in B_x$, and $f(B_x) \leq A$ or $B_x \leq f^{\leftarrow} f(B_x) \leq f^{\leftarrow}(A)$ or $x_\lambda \in B_x \leq f^{\leftarrow}(A)$ or

$$\vee_{x_{(\alpha,\beta,\gamma)} \in f^{\leftarrow}(A)} x_\lambda \in \vee_{x_{(\alpha,\beta,\gamma)} \in f^{\leftarrow}(A)} B_x = f^{\leftarrow}(A)$$

This proves that $f^{\leftarrow}(A)$ is neutrosophic set in $\urcorner_X$.

Conversly, let V be an open neutrosophic set in Y containing the neutrosophic point $f\left(x_{(\alpha,\beta,\gamma)}\right)$.This gives $x_{(\alpha,\beta,\gamma)} \in f^{\leftarrow}(V)$. By hypothesis $U = f^{\leftarrow}(V)$ is neutrosophic set in $\urcorner_X$, such that $x_{(\alpha,\beta,\gamma)} \in U = f^{\leftarrow}(V)$ or $f(U) = ff^{\leftarrow}(V) \leq V$. This proves that f is neutrosophic continuous at neutrosophic point $x_{(\alpha,\beta,\gamma)}$. ■

Corollary 15 *A mapping $f : (X, \urcorner_X) \to (Y, \urcorner_Y)$ is neutrosophic contionuous if and only if the preimage of each closed neutrosophic set in Y is closed neutrosophic set in X.*

4 Neutrosophic topological groups

Let X be a group and U, V two neutrosophic sets in X. We define the product UV of neutrosophic sets U, V, and the inverse V^{-1} of V as follows:

$$UV(x) = \{\langle x, T_{UV}(x), I_{UV}(x), F_{UV}(x)\rangle : x \in X\}$$

Where

$$T_{UV}(x) = \sup\{\min\{T_U(x_1), T_V(x_2)\}\}$$
$$I_{UV}(x) = \sup\{\min\{I_U(x_1), I_V(x_2)\}\}$$
$$F_{UV}(x) = \inf\{\max\{F_U(x_1), F_V(x_2)\}\}$$

where $x = x_1.x_2$ and for $V = \{\langle x, T_V(x), I_V(x), F_V(x)\rangle : x \in X\}$, we have $V^{-1} = \{\langle x, T_V(x^{-1}), I_V(x^{-1}), F_V(x^{-1})\rangle : x \in X\}$.

Neutrosophic topological groups were first defined by Sumathi [9] with the help of mappings. We give an alternative definition by using containment relation between sets.

Definition 16 *[9] Let X be a group and $\urcorner$ a neutrosophic topology on X. Let G be a neutrosophic group in X and let $(G, \urcorner_G)$ be a neutrosophic subspace of $(X, \urcorner)$. Then G is called a neutrosophic topological group (NTG) if it satisfies the following conditions:*

1) The mapping $\alpha : (G, \urcorner_G) \times (G, \urcorner_G) \to (G, \urcorner_G)$ defined by $\alpha(x, y) = xy$, for all $x, y \in X$, is relatively neutrosophic continuous.

2) The mapping $\beta : (G, \urcorner_G) \to (G, \urcorner_G)$ defined by $\beta(x) = x^{-1}$, for all $x \in X$, is relatively neutrosophic continuous.

We give an equivalent definition as follows:

Let X be a group and $\urcorner$ a neutrosophic topology on X. Let G be a neutrosophic group in X and let $(G, \urcorner_G)$ be a neutrosophic subspace of $(X, \urcorner)$. Then G is called a neutrosophic topological group (NTG) if it satisfies the following conditions:

1) for all $x, y \in X$ and every neutrosophic set W in $\urcorner_G$ containing the neutrosophic point $xy_{(s,t,u)}$, there exist neutrosophic sets U and V in $\urcorner_G$ containing the neutrosophic ponits $x_{(p,q,r)}$ and $y_{(l,m,n)}$, respectively such that $UV \leq W$.

2) for all $x \in X$ and open neutrosophic set W containig neutrosophic point $x^{-1}_{(p_1,q_1,r_1)}$, there exists an open neutrosophic sets U containing $x_{(p,q,r)}$ such that $U^{-1} \leq W$.

or equivalently

Let X be a group and $\urcorner$ a neutrosophic topology on X. Let G be a neutrosophic group in X and let $(G, \urcorner_G)$ be a neutrosophic subspace of $(X, \urcorner)$. Then G is called a neutrosophic topological group if it satisfies the following condition:

for all $x, y \in X$ and every neutrosophic set W in $\urcorner_G$ containing the neutrosophic point p_{xy} there exist neutrosophic sets U and V in $\urcorner_G$ containing the neutrosophic points $x_{(p,q,r)}$ and $y_{(l,m,n)}$, respectively such that $UV^{-1} \leq W$.

Definition 17 *Let G be a neutrosophic group of a group X. Then for fixed $a \in X$, the left translation $l_a : (G, \urcorner_G) \to (G, \urcorner_G)$ is defined by $l_a(x) = ax$, for all $x \in X$,where*

$$ax = \{\langle x, T_G(ax), I_G(ax), F_G(ax)\rangle : x \in X\}$$

Similarly, right translation $r_a : (G, \urcorner_G) \to (G, \urcorner_G)$ is defined as: $r_a(y) = ya$, for all $y \in X$ where

$$ya = \{\langle y, T_G(ya), I_G(ya), F_G(ya)\rangle : y \in X\}.$$

Lemma 18 *[9] Let X be a group with neutrosophic topology $\urcorner$ and G be a NTG in X. Then for each $a \in G_e$, the translation l_a and r_a are relatively neutrosophic homeomorphisms of $(G, \urcorner_G)$ into itself.*

Remark 19 *If G is a NTG on X carrying neutrosophic topology $\urcorner$, then the translations r_a and l_a, $a \in X$, in general are not relatively neutrosophic continuous mappings of $(G, \urcorner_G)$ into itself. However Lemma 18 is the special case.*

Theorem 20 *Let G be a NTG in neutrosophic topological space $(X, \urcorner)$. Let A be an open neutrosophic set of $(G, \urcorner_G)$ and $x \in G_e$, then xA and Ax are open neutrosophic sets.*

Proof. Since A is open neutrosophic set of G and $x \in G_e$, $l_x : G \to G$ is neutrosophic homeomorphism. This implies that $l_x(A) = xA$ is open neutrosophic set in G. Similarly, Ax is open neutrosophic set in G.

Corollary 21 *Let G be a NTG in neutrosophic topological space $(X, \urcorner)$. Let A be an open neutrosophic set of $(G, \urcorner_G)$ and $B \leq G_e$. Then AB and BA are open neutrosophic sets .*

■

Proof. $AB = \vee \{Ab : b \in B\}$ is open neutrosophic subsets of G being the union of open neutrosophic subsets of G. ■

Lemma 22 *Let G be a NTG in neutrosophic topology $(X, \urcorner)$. Then*

1) The inverse function $f : G \to G$ defined by $f(x) = x^{-1}$, for all $x \in X$ is relatively neutrosophic homeomorphism.

2) The inner automorphism $h : G \to G$ defined by

$$h(g) = aga^{-1} = \left\{\left\langle g, T_G\left(aga^{-1}\right), I_G\left(aga^{-1}\right), F_G\left(aga^{-1}\right)\right\rangle\right\}$$

where $g \in X$ and $a \in G_e$ is relatively neutrosophic homeomorphism.

Proof. 1) Clearly f is one-to-one. Since $f(G) = \{\langle x, f \;\; T_G(x), \; f \; I_G(x), f \; F_G(x)\rangle : x \in G\}$ where

$$\begin{aligned} f \; T_G(x) &= \left\{ \begin{array}{cc} \vee_{y \in f^{\leftarrow}(x)} T_G(y), & \text{if } f^{\leftarrow}(x) \neq 0 \\ 0 \quad , & \text{otherwise.} \end{array} \right\} \\ &= \left\{ \begin{array}{cc} T_G\left(x^{-1}\right), & \text{if } f^{\leftarrow}(x) \neq 0 \\ 0 \quad , & \text{otherwise.} \end{array} \right\} \\ &= \left\{ \begin{array}{cc} T_G(x), & \text{if } f^{\leftarrow}(x) \neq 0 \\ 0 \quad , & \text{otherwise.} \end{array} \right\} \end{aligned}$$

Also

$$f \; I_G(x) = I_G(x) \text{ and } f \; F_G(x) = F_G(x).$$

Thus $f(G) = \{\langle x, T_G(x), I_G(x), F_G(x)\rangle : x \in G\} = G$.

Also f is neutrosophic continuous by definition because $(G, \urcorner_G)$ is neutrosophic topological group. The only thing left to prove is that f is relatively neutrosophic open mapping. Since $f^{\leftarrow}(x) = x^{-1}$ is relatively neutrosophic continuous, Hence for every $x \in X$, f is relatively neutrosophic open. Thus f is relatively neutrosophic homeomorphism.

2) Since r_a and l_a are relatively neutrosophic homeomorphism and $r_a^{\leftarrow} = r_{a^{-1}}$. The inner automorphism h is a composition of $r_{a^{-1}}$ and l_a. Hence h is a relative neutrosophic homeomorphism. ■

Corollary 23 *Let A be an open neutrosophic set in a NTG G. Then A^{-1} is relatively open neutrosophic set.*

Proof. Let $f : G \to G$ be a mapping defined by $f(x) = x^{-1}$, for each $x \in X$. Since f is relatively neutrosophic homeomorphism, therefore $f(A) = A^{-1}$ is relatively open neutrosophic set. ■

Corollary 24 *Let F be a closed neutrosophic set in a NTG G. Then aF, Fa, F^{-1} are relatively closed neutrosophic sets, for $a \in G_e$.*

Proof. Let $l_a : G \to G$ be a mapping defined by $l_a(g) = ag$, for each $g \in X$, and fixed $a \in G_e$. Then l_a is relatively neutrosophic homeomorphism, also since preimage of a closed neutrosophic set is closed neutrosophic set, therefore, $l_a(F) = aF$ is relatively closed neutrosophic set, where $a \in G_e$. Similarly, Fa is relatively closed neutrosophic set, where $a \in G_e$. Now let $\gamma : G \to G$ be a mapping defined by $\gamma(x) = x^{-1}$, for each $x \in X$. Then γ is relatively neutrosophic homeomorphism, also since preimage of a closed neutrosophic set is closed neutrosophic set, therefore $\gamma(U) = U^{-1}$ is relatively closed neutrosophic set. ■

Theorem 25 *Let G be a NTG in a group X, and e be the identity of X. If $a \in G_e$ and W is a neighbourhood of e such that*

$$T_W(e) = 1, I_W(e) = 1, F_W(e) = 0,$$

then aW is a neighbourhood of a such that $aW(a) = 1_N$.

Proof. Since W is a neighbourhood of e such that $T_W(e) = 1, I_W(e) = 1, F_W(e) = 0$, there exists an open neutrosophic set U such that $U \subseteq W$, and $T_U(e) = T_W(e) = 1$, $I_U(e) = I_W(e) = 1$, $F_U(e) = F_W(e) = 0$. Let $l_a : G \to G$ be a left translation defined by $l_a(g) = ag$, for each $g \in X$. Then l_a is a neutrosophic homeomorphism. Thus aU is an open neutrosophic set. Now,

$$\begin{aligned} aU(a) &= \{\langle a, T_{aU}(a), I_{aU}(a), F_{aU}(a)\rangle\} \\ &= \{\langle a, T_U(aa^{-1}), I_U(aa^{-1}), F_U(aa^{-1})\rangle\} \\ &= \{\langle a, T_U(e), I_U(e), F_U(e)\rangle\} \\ &= \{\langle a, 1, 1, 0\rangle\} \end{aligned}$$

Also,

$$\begin{aligned} aW(x) &= \{\langle x, T_{aW}(x), I_{aW}(x), F_{aW}(x)\rangle : x \in X\} \\ &= \{\langle x, T_W(a^{-1}x), I_W(a^{-1}x), F_W(a^{-1}x)\rangle : x \in X\} \\ &\geq \{\langle x, T_U(a^{-1}x), I_U(a^{-1}x), F_U(a^{-1}x)\rangle : x \in X\} \\ &= \{\langle x, T_{aU}(x), I_{aU}(x), F_{aU}(x)\rangle\} \\ &= aU(x) \end{aligned}$$

$$aW(x) \geq aU(x), \text{ for each } x \in X$$

and

$$\begin{aligned} aW(a) &= \{\langle a, T_{aW}(a), I_{aW}(a), F_{aW}(a)\rangle\} \\ &= \{\langle a, T_W(aa^{-1}), I_W(aa^{-1}), F_W(aa^{-1})\rangle\} \\ &= \{\langle a, T_W(e), I_W(e), F_W(e)\rangle\} \\ &= \{\langle a, 1, 1, 0\rangle\} \end{aligned}$$

or

$$aW(a) = \{\langle a, 1, 1, 0\rangle\}$$

Thus, there exist an open neutrosophic set aU such that $aU \subseteq aW$ and $aU(a) = aW(a) = \{\langle a, 1, 1, 0\rangle\}$ ■

The definitions of neutrosophic cover and neutrosophic compactness are given below.

Definition 26 *Let B be a neutrosophic subset of a nonempty set X. A collection $\mathcal{A}$ of neutrosophic sets is a neutrosophic cover of B if and only if for each $x \in X$, $\sup\limits_{A\in\mathcal{A}} T_A(x) = 1$, $\sup\limits_{A\in\mathcal{A}} I_A(x) = 1$ and $\inf\limits_{A\in\mathcal{A}} F_A(x) = 0$.*

Definition 27 *Let $(X, \urcorner)$ be neutrosophic topological space. A collection $\mathcal{A}$ of neutrosophic sets is an neutrosophic cover of a neutrosophic set N if and only if $N \leq \vee_{A\in\mathcal{A}} A$. It is an neutrosophic open cover if and only if each member of $\mathcal{A}$ is an open neutrosophic set.*

Definition 28 *A neutrosophic set V is neutrosophic compact if and only if every neutrosophic open cover has a finite neutrosophic subcover.*

Definition 29 *Let $\mathcal{A}$ be a neutrosophic open cover of a neutrosophic topological space $(X, \urcorner)$. Then $(X, \urcorner)$ is compact neutrosophic topological space (CNTS) if and only if $\mathcal{A}$ has a finite subcover.*

Theorem 30 *Let X be a CNTS and Y be a NTS. Let $f : (X, \urcorner_X) \to (Y, \urcorner_Y)$ be a neutrosophic continuous function from X onto the Y. Then Y is compact neutrosophic topological space.*

Proof. Let $\mathcal{A}$ be a neutrosophic open cover of Y. Then $Y = \{A : A \in \mathcal{A}\}$. $Y = \vee_{A\in\mathcal{A}} A$. This implies

$$\begin{aligned}
f^{\leftarrow}(Y) &= f^{\leftarrow}(\vee_{A\in\mathcal{A}} A) \\
&= \vee_{A\in\mathcal{A}} f^{\leftarrow}(A) \\
&= \vee_{A\in\mathcal{A}} \{\langle x, f^{\leftarrow}(T_A(x)), f^{\leftarrow}(I_A(x)), f^{\leftarrow}(F_A(x))\rangle : x \in X\} \\
&= \vee_{A\in\mathcal{A}} \{\langle x, T_A(f(x)), I_A(f(x)), F_A(f(x))\rangle : x \in X\} \\
&= \left\{\left\langle x, \sup_{A\in\mathcal{A}} T_L(f(x)), \sup_{A\in\mathcal{A}} I_L(f(x)), \inf_{A\in\mathcal{A}} F_L(f(x))\right\rangle : x \in X\right\} \\
&= \{\langle x, 1, 1, 0\rangle : x \in X\}
\end{aligned}$$

The collection of all neutrosophic sets of the form $f^{\leftarrow}(A)$, for A in $\mathcal{A}$ is a neutrosophic open cover of X, which has a finite subcover. However, if f is onto then $f\ f^{\leftarrow}(A) = A$, for any neutrosophic set A in Y. Hence, the collection of images of elements of the subcover which covers Y is a finite subcollection of $\mathcal{A}$ and consequently Y is neutrosophic compact. ■

Theorem 31 *Let $(X_1, \urcorner_1)$ and $(X_2, \urcorner_2)$ be two compact neutrosophic topological spaces. Then the product neutrosophic topological space $(X, \urcorner)$ is also neutrosophic compact.*

Proof. The neutrosophic product topology $\urcorner$ on X has the set of neutrosophic product spaces of the form

$$A_1 \times A_2 \ ; \ A_1 \in \urcorner_1 \text{ and } A_2 \in \urcorner_2.$$

The neutrosophic set $A_1 \times A_2$ has membership function, indeterminacy function and nonmembership function are given by

$$A_1 \times A_2 = \left\{ \begin{array}{c} \langle (x_1, x_2), T_{A_1 \times A_2}(x_1, x_2), I_{A_1 \times A_2}(x_1, x_2), F_{A_1 \times A_2}(x_1, x_2) \rangle : \\ x_1 \in A_1 \text{ and } x_2 \in A_2 \end{array} \right\}$$

where

$$\begin{aligned} T_{A_1 \times A_2}(x_1, x_2) &= \min\{T_{A_1}(x_1), T_{A_2}(x_2)\} \\ I_{A_1 \times A_2}(x_1, x_2) &= \min\{I_{A_1}(x_1), I_{A_2}(x_2)\} \\ F_{A_1 \times A_2}(x_1, x_2) &= \max\{F_{A_1}(x_1), F_{A_2}(x_2)\} \end{aligned}$$

Let the product neutrosophic topological space $(X, \urcorner)$ has a neutrosophic open cover $\mathcal{A} = \{B_i : i \in \nabla\}$. Since each B_i can be expressed as

$$B_i = A_1^i \times A_2^i; A_1^i \in \urcorner_1, A_2^i \in \urcorner_2, \forall i \in \nabla$$

Let $y \in X_2$. Then $X_1 \times \{y\}$ is a subset of X say $S_y = X_1 \times \{y\}$ .For any $\delta > 0$, $V_{y,\delta}$ be the subcollection of $\mathcal{A}$ such that $A_1^i \times A_2^i \in V_{y,\delta}$ if and only if for atleast one point

$$x_1 \in X_1, T_{A_1}^i(x_1) > 1 - \delta, I_{A_1}^i(x_1) > 1 - \delta, F_{A_1}^i(x_1) < \delta$$

and

$$T_{A_2}^i(y) > 1 - \delta, I_{A_2}^i(y) > 1 - \delta, F_{A_2}^i(y) < \delta$$

Then $V_{y,\delta}$ forms an open neutrosophic cover of subset of δ_y. To check this note that if $(x, y) \in \delta_y$, by definition of $\mathcal{A}$, there exists a subfamily $\{A_1^k \times A_2^k; k \in \nabla\}$ of $\mathcal{A}$ such that

$$\lim_{k\to\infty} T_{A_1^k \times A_2^k}(x, y) = 1, \lim_{k\to\infty} I_{A_1^k \times A_2^k}(x, y) = 1, \lim_{k\to\infty} F_{A_1^k \times A_2^k}(x, y) = 0$$

Hence for large k , $A_1^k \times A_2^k \in V_{y,\delta}$. Assume that $\{A_1^k \times A_2^k; k \in \nabla\}$ is a subcollection of $V_{y,\delta}$. If we take all $(t, y) \in S_y$, $V_{y,\delta}$ makes open neutrosophic cover of S_y. Let

$$W_{y,\delta} = \{A_1^k : A_1^k \times A_2^k \in V_{y,\delta}\}$$

Then $W_{y,\delta}$ is an open neutrosophic cover of $(X_1, \urcorner_1)$. Since for any $x \in X_1$, there exists a subcollection $\{A_1^k \times A_2^k; k \in \nabla\}$ of $V_{y,\delta}$ such that

$$\lim_{k\to\infty} T_{A_1^k \times A_2^k}(x, y) = 1, \lim_{k\to\infty} I_{A_1^k \times A_2^k}(x, y) = 1, \lim_{k\to\infty} F_{A_1^k \times A_2^k}(x, y) = 0$$

Consequently,

$$\lim_{k\to\infty} T_{A_1^k}(x) = 1, \lim_{k\to\infty} I_{A_1^k}(x) = 1, \lim_{k\to\infty} F_{A_1^k}(x) = 0$$

Since $(X_1, \urcorner_1)$ is neutrosophic compact, therefore, there exists a finite neutrosophic subcover of $W_{y,\delta}$, say $Z_{y,\delta}$. For each $A_1^i \in Z_{y,\delta}$, select one A_2^i such that $A_1^i \times A_2^i \in V_{y,\delta}$. In this way a finite collection $\left\{A_1^i \times A_2^i\right\}$ say $H_{y,\delta}$, is constructed and the finite collection corresponding A_2^i's will be say $G_{y,\delta}$. Elements of $G_{y,\delta}$ are $\urcorner_2$-open neutrosophic sets. Let the collection of intersection of members of $G_{y,\delta}$ is $D_{y,\delta}$. Then $D_{y,\delta}$ is open in $(X_2, \urcorner_2)$. Do this for all $y \in X_2$ and for all $\delta > 0$. Clearly, the collection $\{D_{y,\delta} : y \in X_2, \delta > 0\}$ forms an open neutrosophic cover say $\{D_{y_i,\delta_i}; i = 1, 2, 3, ..., m\}$. Finally $\{H_{y_i,\delta_i}; i = 1, 2, 3, ..., m\}$ makes a finite neutrosophic subcover of $\mathcal{A}$. Since it is a finite collection of finite families, therefore it is finite. ■

Corollary 32 *Let $\{X_i, \urcorner_i; i = 1, 2, .., m\}$ be a finite collection of compact neutrosophic topological spaces. Then the product neutrosophic topological space $(X, \urcorner)$ is also neutrosophic compact.*

Corollary 33 *Let $\{X_i, \urcorner_i; i = 1, 2, .., m\}$ be a finite collection of countabily compact neutrosophic topological spaces. Then the product neutrosophic topological space $(X, \urcorner)$ is also countably neutrosophic compact.*

Remark 34 *The product of countable family of compact (countabily compact) neutrosophic topological spaces may not be neutrosophic compact (countabily compact).*

Example 35 *Let $X = \mathbb{R}$. Let $A_n; n > 0$, and $n \in \mathbb{Z}$, be a neutrosophic set in X such that*

$$A_n = \{\langle x, T_{A_n}(x) = 1 - 1/n, \quad I_{A_n}(x) = 1 - 1/n, \; F_{A_n}(x) = 1/n\rangle : x \in X\}$$

Let $X_n = X$; $n = 1, 2, ...$ with $\urcorner_n = \{0_N, 1_N, A_n\}$. Since X_n is the only open neutrosophic cover of X_n, therefore, $(X_n, \urcorner_n); n = 1, 2, ...$ is CNTS. Now construct the product topology $(Y, \urcorner)$ of $(X_n, \urcorner_n)$ that is $Y = \Pi_{n=1}^{\infty} X_n$. Consider the projection mapping $\wp$ from Y onto X. Then the neutrosophic set $\wp_n^{\leftarrow}[A_n]$ has the membership function, indeterminacy function and nonmembership function as follow:

$$T_{\wp_n^{\leftarrow}[A_n]}(y) = 1 - 1/n; \forall y \in Y = \Pi_{n=1}^{\infty} X_n,$$
$$I_{\wp_n^{\leftarrow}[A_n]}(y) = 1 - 1/n; \forall y \in Y = \Pi_{n=1}^{\infty} X_n,$$
$$F_{\wp_n^{\leftarrow}[A_n]}(y) = 1/n; \forall y \in Y = \Pi_{n=1}^{\infty} X_n.$$

Thus $\mathcal{S} = \{0_N, 1_N, \wp_n^{\leftarrow}[A_n]; n = 1, 2, ...\}$ is a subbase for the product topology $\urcorner$. The product topology $\urcorner$ generated by $\mathcal{S}$ is $\urcorner$ itself, that is $\urcorner = \mathcal{S}$. Thus, the family $\{\wp_n^{\leftarrow}[A_n]; n = 1, 2, ...\}$ is an open neutrosophic cover of $(Y, \urcorner)$ which has no finite neutrosophic subcover.

Definition 36 *Let X be a group and S be a neotrusophic set in X. Then S is called the neutrosophic semigroup in X if and only if for all $t, u \in X$,*

$$T_S(xy) \geq \min\{T_S(x), T_S(y)\},$$

$$I_S(xy) \geq \min\{I_S(x), I_S(y)\},$$

$$F_S(xy) \leq \max\{F_S(x), F_S(y)\}.$$

Definition 37 *Let S be a neutrosophic semigroup in a group X with induced fuzzy neutrosophic topology $\urcorner_S$. Then S is called a neutrosophic topological semigroup (NTSG) in X if and only if the mapping*

$$\phi : (S, \urcorner_S) \times (S, \urcorner_S) \rightarrow (S, \urcorner_S)$$

defined by

$$\phi(x, y) = xy; \ \forall \ x, y \in X,$$

is relatively neutrosophic continuous.

Proposition 38 *Let X be a group. Let S be a neutrosophic topological semigroup in X and G be a NTG in X. Let A and B be neutrosophic sets in S and let C be FNS of G. Then*

1) If A and B are neutrosophic compact then AB is neutrosophic compact.
2) If C is neutrosophic compact then C^{-1} is neutrosophic compact.

Proof. 1) Let $\phi : S \times S \rightarrow S$ be a mapping defined by $\phi(x, y) = xy$; for each $x, y \in X$, then ϕ is neutrosophic continuous. Since $A \times B$ is neutrosophic compact and the image of a neutrosophic compact set under a neutrosophic continuous mapping is neutrosophic compact, therefore $\phi(A, B) = \phi(A \times B) = AB$ is neutrosophic compact.

2) Let $\phi : S \rightarrow S$ be a mapping defined by $\phi(x) = x^{-1}$; for each $x \in X$. Then ϕ is neutrosophic continuous and the neutrosophic continuous image of a neutrosophic compact set is neutrosophic compact, therefore, $\phi(C) = C^{-1}$, is neutrosophic compact. ■

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