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Neutrosophic Triangular Norms and Their Derived Residuated Lattices

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Abstract: Neutrosophic triangular norms (t-norms) and their residuated lattices are not only the main research object of neutrosophic set theory, but also the core content of neutrosophic logic. Neutrosophic implications are important operators of neutrosophic logic. Neutrosophic residual implications based on neutrosophic t-norms can be applied to the fields of neutrosophic inference and neutrosophic control. In this paper, neutrosophic t-norms, neutrosophic residual implications, and the residuated lattices derived from neutrosophic t-norms are investigated deeply. First of all, the lattice and its corresponding system are proved to be a complete lattice and a De Morgan algebra, respectively. Second, the notions of neutrosophic t-norms are introduced on the complete lattice discussed earlier. The basic concepts and typical examples of representable and non-representable neutrosophic t-norms are obtained. Naturally, De Morgan neutrosophic triples are defined for the duality of neutrosophic t-norms and neutrosophic t-conorms with respect to neutrosophic negators. Third, neutrosophic residual implications generated from neutrosophic t-norms and their basic properties are investigated. Furthermore, residual neutrosophic t-norms are proved to be infinitely $\vee$-distributive, and then some important properties possessed by neutrosophic residual implications are given. Finally, a method for producing neutrosophic t-norms from neutrosophic implications is presented, and the residuated lattices are constructed on the basis of neutrosophic t-norms and neutrosophic residual implications.

Keywords: neutrosophic sets; neutrosophic triangular norms; residuated lattices; representable neutrosophic t-norms; De Morgan neutrosophic triples; neutrosophic residual implications; infinitely $\vee$-distributive

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1. Introduction

Neutrosophic sets were firstly proposed by Smarandache [1] from a philosophical point of view in 1998, which is a generalization of fuzzy sets and intuitionistic fuzzy sets. However, it is difficult to apply neutrosophic sets to solve practical problems since the values of their three functions with respect to truth, indeterminacy and falsity lie in $]0-,1+[$. The definition of single-valued neutrosophic sets were introduced by Wang [2], whose values belong to [0,1]. With the development of neutrosophic set theory, single-valued neutrosophic sets and their applications have been investigated by more scholars. Single-valued neutrosophic sets were successfully applied to various decision making problems [3–8]. In addition, Zhang et al. studied the neutrosophic logic algebras and discussed neutrosophic filters and neutrosophic triplet groups, which are the important foundation of the development of neutrosophic logic theory [9–12]. To facilitate research, "single-valued neutrosophic sets" are abbreviated as "neutrosophic sets" in this paper. For neutrosophic sets, the truth-membership, indeterminacy-membership and falsity-membership are not restricted to each other, which is different from intuitionistic fuzzy sets. Picture fuzzy sets proposed by Cuong [13] in 2013 is a direct generalization of intuitionistic fuzzy sets, because their positive membership, neutral membership and negative membership are not independent completely. It is worth noting that picture fuzzy sets can be regarded as special neutrosophic sets [14,15], and also can be called standard neutrosophic sets [1,2,16–19].

Fuzzy logic plays a vital role in fuzzy set theory. T-norms, t-conorms, negators and implications are very important fuzzy logic operators. T-norms were originally defined by Menger [20], and then Schweizer and Sklar [21,22] redefined the t-norms which have been used to today. From the perspective of fuzzy logic, t-norms are the extension of intersection operation of fuzzy sets [23]. The algebraic properties of t-norms, for example, continuity, archimedean, strict, nilpotent and so on, are discussed in some papers [24–28]. Hu et al. studied t-norm extension operations [29]. Wang et al. discussed the lattice structure of algebra of fuzzy values [30]. The t-norms which satisfy the residual principle are an important class of t-norms, because they can produce fuzzy implications and constitute the residuated lattices [31–34]. Type-2 t-norms (t-conorms) and their residual operators on type-2 fuzzy sets were investigated by Li [25], which promote the development of fuzzy reference system. Intuitionistic fuzzy t-norms on intuitionistic fuzzy sets (L^*-fuzzy sets) were proposed by Deschrijver et al., they discussed t-representable intuitionistic fuzzy t-norms and their residual operators [35,36]. Picture fuzzy sets are particular L-fuzzy sets [37]. Picture fuzzy t-norms on picture fuzzy sets were introduced in [17,38,39], some basic picture fuzzy logic connectives and their properties for picture fuzzy sets are investigated in [40,41]. Some classes of representable picture fuzzy t-norms and representable picture fuzzy t-conorms on picture fuzzy sets and De Morgan picture operator triples in picture fuzzy logic are discussed [42]. Furthermore, a picture inference system is proposed by Son [43]. The residual operations, residual implications of uninorms were discussed by Baets [44]. Wang proposed the notions of residual implications (co-implications) of pseudo t-norms, left and right uninorms and studied some properties of infinitely $\vee$-distributive ($\wedge$-distributive) pseudo t-norms, left and right uninorms [45–47]. Then Liu introduced semi-uninorms and their residual implications [48].

Neutrosophic t-norms, neutrosophic t-conorms, neutrosophic negators and neutrosophic implications are important neutrosophic logic operators for neutrosophic sets. It is a very meaningful topic to discuss neutrosophic t-norms and their residual implications on neutrosophic sets. In the last few years, although Alkhazaleh discusses some neutrosophic t-norms and t-conorms in [49], Liu proposes aggregation operators based on Archimedean t-norms and t-conorms for neutrosophic numbers in [5], Smarandache discussed neutrosophic norms (n-norms), n-valued refined neutrosophic logic and its applications in physics [50,51], there are a few papers about basic neutrosophic logic connectives and their properties and neutrosophic logic inference systems and their applications in the field of control. Therefore, it is necessary to study neutrosophic logic operators and their properties, especially the application of neutrosophic residual implications in neutrosophic inference and neutrosophic control.

To achieve these goals, the definitions of neutrosophic t-norms should be given firstly. We can study neutrosophic logic and neutrosophic inference systems further only if neutrosophic t-norms and their residual implications are studied thoroughly. Thus, it is the main task of this paper to study neutrosophic t-norms and their residual implications. Section 2 presents some basic notions. In Section 3, the lattice structure of neutrosophic sets is analyzed and constructed systematically based on the first type inclusion relation on neutrosophic sets. In particular, we combine some basic algebraic operations: Union, intersection and complement and their related properties to prove that the system $(D^*; \vee_1, \wedge_1, ^c, 0_{D^*}, 1_{D^*})$ is a De Morgan algebra. In Section 4, we introduce neutrosophic t-norms (t-conorms), representable neutrosophic t-norms (t-conorms) and De Morgan neutrosophic triples. In addition, we present some important theorems and typical examples. In Section 5, the definitions of neutrosophic residual implications (co-implications) are obtained and their basic properties are discussed deeply. Moreover, residual neutrosophic t-norms (t-conorms) are proved to be infinitely $\vee$-distributive ($\wedge$-distributive), and then some important results related to residual neutrosophic t-norms and neutrosophic residual implications are given. Section 6 shows a method for obtaining neutrosophic t-norms from neutrosophic implications, and then proves that the system $(D^*; \vee_1, \wedge_1, \otimes, \rightarrow, 0_{D^*}, 1_{D^*})$ is a residuated lattice. In Section 7, we conclude the paper.

2. Preliminaries

Some basic concepts in fuzzy set theory will be reviewed in this section.

Definition 1. *([52]) Let U be a nonempty set. An intuitionistic fuzzy set M in U is characterized by a membership function $\mu_M(u)$ and a non-membership function $\nu_M(u)$. Then, an intuitionistic fuzzy set M can be denoted by*

$$M = \{(u, \mu_M(u), \nu_M(u)) \mid u \in U\},$$

where $\mu_M(u)$: $U \to [0,1]$ and $\nu_M(u)$: $U \to [0,1]$ with the condition $0 \leq \mu_M(u) + \nu_M(u) \leq 1$, for all $u \in U$. Here $\mu_M(u), \nu_M(u) \in [0,1]$ denote the membership and the non-membership functions of the intuitionistic fuzzy set M, respectively.

Definition 2. *([1]) Let U be a nonempty set. A neutrosophic set M in U is characterized by three functions: Truth-membership function $T_M(u)$, indeterminacy-membership function $I_M(u)$, and falsity-membership function $F_M(u)$. Here, $T_M(u)$: $U \to]^-0, 1^+[$, $I_M(u)$: $U \to]^-0, 1^+[$, and $F_M(u)$: $U \to]^-0, 1^+[$ with the condition $^-0 \leq \sup T_M(u) + \sup I_M(u) + \sup F_M(u) \leq 3^+$, for all $u \in U$.*

However, there are a lot of limitations in solving practical problems with neutrosophic sets, because their notions are given from a philosophical perspective. Thus, the concepts of single-valued neutrosophic sets are given by Wang et al. as follows.

Definition 3. *([2]) Let U be a nonempty set. A single-valued neutrosophic set M in U is characterized by three functions: Truth-membership function $T_M(u)$, indeterminacy-membership function $I_M(u)$, and falsity-membership function $F_M(u)$. Then, a single-valued neutrosophic set M can be denoted by*

$$M = \{\langle u, T_M(u), I_M(u), F_M(u)\rangle | u \in U\},$$

where $T_M(u), I_M(u), F_M(u) \in [0,1]$ with the condition $0 \leq T_M(u) + I_M(u) + F_M(u) \leq 3$, for all $u \in U$.

So far, scholars have described the inclusion relation of neutrosophic sets from three different angles. The first definition is proposed by Smarandache [1,18,53] and denoted as $\subseteq_1$; the second one is mentioned in [2,14,54] and denoted by $\subseteq_2$; the third one is presented in [14,38,39,54] and denoted by $\subseteq_3$. Furthermore, based on the correlation between union, intersection operations and inclusion relation, we can obtain three different types of union, intersection operations and their properties. In this paper, we consider the first type inclusion relation.

Definition 4. *([1,18,53]) Let U be a nonempty set. Suppose that $M = \{\langle u, T_M(u), I_M(u), F_M(u)\rangle | u \in U\}$ and $N = \{\langle u, T_N(u), I_N(u), F_N(u)\rangle | u \in U\}$ are two neutrosophic sets in U. The first type inclusion relation $\subseteq_1$ and its basic algebraic operations are defined as follows:*

(1) $M \subseteq_1 N$ *if and only if* $T_M(u) \leq T_M(u)$, $I_M(u) \geq I_N(u)$, $F_N(u) \geq F_N(u)$, *for all* $u \in U$;
(2) $M \cup_1 N = \{\langle u, \max(T_M(u), T_N(u)), \min(I_M(u), I_N(u)), \min(F_M(u), F_N(u))\rangle | u \in U\}$;
(3) $M \cap_1 N = \{\langle u, \min(T_M(u), T_N(u)), \max(I_M(u), I_N(u)), \max(F_M(u), F_N(u))\rangle | u \in U\}$;
(4) $M^c = \{\langle u, I_M(u), 1 - F_M(u), T_M(u)\rangle | u \in U\}$.

Definition 5. *([34]) Let $(L; \leq_L)$ be a complete lattice. A t-norm on $(L; \leq_L)$ is a commutative, associative, increasing mapping $\mathscr{T} : L^2 \to L$, which satisfies $\mathscr{T}(1_L, u) = u$, for all $u \in L$.*
A t-conorm on $(L; \leq_L)$ is a commutative, associative, increasing mapping $\mathscr{S} : L^2 \to L$, which satisfies $\mathscr{S}(0_L, u) = u$, for all $u \in L$.

Example 1. *([27]) Some basic t-norms and their residual implications on* $([0,1];\leq)$ *(Table 1) are defined as follows, for all* $u,v\in[0,1]$,

Table 1. Some basic t-norms and their residual implications on $([0,1];\leq)$.

t-Norms	Residual Implications
$T_M(u,v)=\min(u,v)$	$I_{GD}(u,v)=\begin{cases}1 & \text{if } u\leq v,\\ v & \text{otherwise}\end{cases}$
$T_P(u,v)=u\cdot v$	$I_{GG}(u,v)=\begin{cases}1 & \text{if } u\leq v,\\ \frac{v}{u} & \text{otherwise}\end{cases}$
$T_{LK}(u,v)=\max(u+v-1,0)$	$I_{LK}(u,v)=\min(1,1-u+v)$
$T_D(u,v)=\begin{cases}0 & \text{if } (u,v)\in[0,1[^2,\\ \min(u,v) & \text{otherwise}\end{cases}$	$I_{WB}(u,v)=\begin{cases}1 & \text{if } (u,v)\in[0,1[^2,\\ v & \text{otherwise}\end{cases}$
$T_{nM}(u,v)=\begin{cases}0 & \text{if } u+v\leq 1,\\ \min(u,v) & \text{otherwise}\end{cases}$	$I_{FD}(u,v)=\begin{cases}1 & \text{if } u\leq v,\\ \max(1-u,v) & \text{otherwise}\end{cases}$

Example 2. *([27]) Some basic t-conorms and their residual co-implications on* $([0,1];\leq)$ *(Table 2) are defined by, for all* $u,v\in[0,1]$,

Table 2. Some basic t-conorms and their residual co-implications on $([0,1];\leq)$.

t-Conorms	Residual Co-Implications
$S_M(u,v)=\max(u,v)$	$J_{GD}(u,v)=\begin{cases}0 & \text{if } u\geq v,\\ v & \text{otherwise}\end{cases}$
$S_P(u,v)=u+v-u\cdot v$	$J_{GG}(u,v)=\begin{cases}0 & \text{if } u\geq v,\\ \frac{v-u}{1-u} & \text{otherwise}\end{cases}$
$S_{LK}(u,v)=\min(u+v,1)$	$J_{LK}(u,v)=\max(0,v-u)$
$S_D(u,v)=\begin{cases}1 & \text{if } (u,v)\in]0,1]^2,\\ \max(u,v) & \text{otherwise}\end{cases}$	$J_{WB}(u,v)=\begin{cases}0 & \text{if } (u,v)\in]0,1]^2,\\ v & \text{otherwise}\end{cases}$
$S_{nM}(u,v)=\begin{cases}1 & \text{if } u+v\geq 1,\\ \max(u,v) & \text{otherwise}\end{cases}$	$J_{FD}(u,v)=\begin{cases}0 & \text{if } u\geq v,\\ \min(1-u,v) & \text{otherwise}\end{cases}$

3. The Lattice Structure of $(D^*;\leq_1)$

Now we consider the set D^* defined by,

$$D^*=\{u=(u_1,u_2,u_3)|u_1,u_2,u_3\in[0,1]\}.$$

As defined above, if $u\in D^*$, then u has three components: The first component u_1, the second component u_2 and the third component u_3.

The order relation $\leq_1$ on D^* can also be defined by, for all $u,v\in D^*$,

$$u\leq_1 v \text{ if and only if } u_1\leq v_1,\ u_2\geq v_2,\ u_3\geq v_3.$$

Proposition 1. $(D^*;\leq_1)$ *is a partially ordered set.*

Proof.

(1) Reflexivity: $u\leq_1 u$, for all $u\in D^*$.

(2) Anti-symmetry: If $u\leq_1 v$ and $v\leq_1 u$, then it is obvious that $u=v$, for all $u,v\in D^*$.

(3) Transitivity: If $u\leq_1 v$ and $v\leq_1 w$, then $u_1\leq w_1, u_2\geq w_2, u_3\geq w_3$, that is, $u\leq_1 w$, for all $u,v,w\in D^*$.

□

Proposition 2. *The operations $\wedge_1$ and $\vee_1$ are defined by, for all $u, v \in D^*$,*

$$u \wedge_1 v = \begin{cases} u & if\ u \leq_1 v, \\ v & if\ v \leq_1 u, \\ (\min(u_1, v_1), \max(u_2, v_2), \max(u_3, v_3)) & otherwise. \end{cases}$$

$$u \vee_1 v = \begin{cases} u & if\ v \leq_1 u, \\ v & if\ u \leq_1 v, \\ (\max(u_1, v_1), \min(u_2, v_2), \min(u_3, v_3)) & otherwise. \end{cases}$$

Then $u \wedge_1 v$ is called the greatest lower bound of u, v, denoted by $\inf(u, v)$; $u \vee_1 v$ is called the least upper bound of u, v, denoted by $\sup(u, v)$. That is, $(D^*; \leq_1)$ is a lattice.

Proof. According to the definitions above, if either $u \leq_1 v$ or $v \leq_1 u$, $u \wedge_1 v$ is the greatest lower bound of u and v. Thus, $u \wedge_1 v = \inf(u, v)$. Similarly, $u \vee_1 v = \sup(u, v)$ can be obtained.

Now, we assume that neither $u \leq_1 v$ nor $v \leq_1 u$. According to the definitions above, $u \wedge_1 v = (\min(u_1, v_1), \max(u_2, v_2), \max(u_3, v_3))$, $u \vee_1 v = (\max(u_1, v_1), \min(u_2, v_2), \min(u_3, v_3))$.

(i) To prove $u \wedge_1 v = \inf(u, v)$, we denote

$$\kappa = (\kappa_1, \kappa_2, \kappa_3) = (\min(u_1, v_1), \max(u_2, v_2), \max(u_3, v_3)).$$

Since $u_1 \geq \min(u_1, v_1) = \kappa_1$, $u_2 \leq \max(u_2, v_2) = \kappa_2$, $u_3 \leq \max(u_3, v_3) = \kappa_3$, $\kappa \leq_1 u$. Similarly, we have $\kappa \leq_1 v$. Thus, κ is the lower bound of u and v. Furthermore, κ is the greatest lower bound of u and v. In fact, assume $a = (a_1, a_2, a_3) \in D^*$ with the condition $a \leq_1 u$ and $a \leq_1 v$. Then, $a_1 \leq u_1, a_2 \geq u_2, a_3 \geq u_3$ and $a_1 \leq v_1, a_2 \geq v_2, a_3 \geq v_3$. Therefore, $a_1 \leq \min(u_1, v_1) = \kappa_1$, $a_2 \geq \max(u_2, v_2) = \kappa_2$, $a_3 \geq \max(u_3, v_3) = \kappa_3$. Hence, $a \leq_1 \kappa$. To sum up, $\kappa = (\min(u_1, v_1), \max(u_2, v_2), \max(u_3, v_3))$ is the greatest lower bound of u and v.

(ii) To prove $u \vee_1 v = \sup(u, v)$, we denote

$$\omega = (\omega_1, \omega_2, \omega_3) = (\max(u_1, v_1), \min(u_2, v_2), \min(u_3, v_3)).$$

Since $u_1 \leq \max(u_1, v_1) = \omega_1$, $u_2 \geq \min(u_2, v_2) = \omega_2$, $u_3 \geq \min(u_3, v_3) = \omega_3$, $u \leq_1 \omega$. Similarly, we have $v \leq_1 \omega$. Thus, ω is the upper bound of u and v. Furthermore, ω is the least upper bound of u and v. In fact, assume $b = (b_1, b_2, b_3) \in D^*$ with the condition $b \geq_1 u$ and $b \geq_1 v$. Then, $b_1 \geq u_1, b_2 \leq u_2, b_3 \leq u_3$ and $b_1 \geq v_1, b_2 \leq v_2, b_3 \leq v_3$. Therefore, $b_1 \geq \max(u_1, v_1) = \omega_1$, $b_2 \leq \min(u_2, v_2) = \omega_2$, $b_3 \leq \min(u_3, v_3) = \omega_3$. Hence, $b \geq_1 \omega$. To sum up, $\omega = (\max(u_1, v_1), \min(u_2, v_2), \min(u_3, v_3))$ is the least upper bound of u and v.

(i) and (ii) show that $u \wedge_1 v = \inf(u, v)$, $u \vee_1 v = \sup(u, v)$, for all $u, v \in D^*$. Then $(D^*; \leq_1)$ is a lattice. □

The first, second and third projection mapping pr_1, pr_2 and pr_3 on D^* are defined as follows, $pr_1(u) = u_1$, $pr_2(u) = u_2$ and $pr_3(u) = u_3$, for all $u \in D^*$.

Proposition 3. $(D^*; \leq_1)$ *is a complete lattice.*

Proof. Let B be a nonempty subset of D^*, we have

$$\inf B = (\inf pr_1 B, \inf pr_2 B, \inf pr_3 B),$$

where $\inf pr_1 B = \inf\{u_1 \mid u_1 \in [0,1], \exists u = (u_1, u_2, u_3) \in B\}$, $\inf pr_2 B = \sup\{u_2 \mid u_2 \in [0,1], \exists u = (u_1, u_2, u_3) \in B\}$, $\inf pr_3 B = \sup\{u_3 \mid u_3 \in [0,1], \exists u = (u_1, u_2, u_3) \in B\}$.

And

$$\sup B = (\sup pr_1 B, \sup pr_2 B, \sup pr_3 B),$$

where $\sup pr_1 B = \sup\{u_1 \mid u_1 \in [0,1], \exists u = (u_1, u_2, u_3) \in B\}$, $\sup pr_2 B = \inf\{u_2 \mid u_2 \in [0,1], \exists u = (u_1, u_2, u_3) \in B\}$, $\sup pr_3 B = \inf\{u_3 \mid u_3 \in [0,1], \exists u = (u_1, u_2, u_3) \in B\}$. □

The maximum and minimum of D^* are denoted by $1_{D^*} = (1,0,0)$ and $0_{D^*} = (0,1,1)$, respectively.

Note that, if u and v are incomparable with respect to $\leq_1$, for all $u, v \in D^*$, then the relationship between u and v can be denoted as $u \parallel_{\leq_1} v$.

Obviously, each neutrosophic set $M = \{\langle u, T_M(u), I_M(u), F_M(u)\rangle \mid u \in U\}$ corresponds to a D^*-fuzzy set. That is, there exits a mapping

$$M\colon U \to D^*; u \longmapsto (T_M(u), I_M(u), F_M(u)).$$

Based on the relationship between neutrosophic sets and D^*-fuzzy sets, the triple formed by the three membership degrees of neutrosophic sets is an element of D^*. Therefore, we can obtain more compact formulas for neutrosophic sets, analyze and extend some operators defined in the fuzzy case for neutrosophic sets by using the lattice $(D^*; \leq_1)$.

For example, the intersection of two neutrosophic sets M and N in a universe U is defined as

$$M \cap_1 N = \{\langle u, \min(T_M(u), T_N(u)), \max(I_M(u), I_N(u)), \max(F_M(u), F_N(u))\rangle | u \in U\}.$$

Using the lattice $(D^*; \leq_1)$, we can get, for all $u \in U$,

$$(M \cap_1 N)(u) = (\min(T_M(u), T_N(u)), \max(I_M(u), I_N(u)), \max(F_M(u), F_N(u))) = M(u) \wedge_1 N(u).$$

Definition 6. *The complement of u is defined by, for all $u \in D^*$,*

$$u^c = (u_3, 1 - u_2, u_1).$$

Proposition 4. *Let $u, v, w \in D^*$. Then*

(1) $u \wedge_1 u = u$, $u \vee_1 u = u$;
(2) $u \wedge_1 v = v \wedge_1 u$, $u \vee_1 v = v \vee_1 u$;
(3) $(u \wedge_1 v) \wedge_1 w = u \wedge_1 (v \wedge_1 w)$, $(u \vee_1 v) \vee_1 w = u \vee_1 (v \vee_1 w)$;
(4) $u \wedge_1 (v \vee_1 u) = u$, $u \vee_1 (v \wedge_1 u) = u$;
(5) $u \leq_1 v$ *if and only if* $u \vee_1 v = v$, $u \wedge_1 v = u$;
(6) $(u^c)^c = u$.

Proposition 5. *Let $u, v \in D^*$. Then*

(1) $(u \wedge_1 v)^c = u^c \vee_1 v^c$;
(2) $(u \vee_1 v)^c = u^c \wedge_1 v^c$.

Proof. (1) Suppose $u, v \in D^*$. If $u \leq_1 v$, that is, $u_1 \leq v_1, u_2 \geq v_2, u_3 \geq v_3$. By Definition 6, we have $u^c \geq_1 v^c$. Thus, $(u \wedge_1 v)^c = u^c \vee_1 v^c$. Similarly, if $u \geq_1 v$, then $u^c \leq_1 v^c$ and $(u \wedge_1 v)^c = u^c \vee_1 v^c$. If $u \parallel_{\leq_1} v$, then $u \wedge_1 v = (\min(u_1, v_1), \max(u_2, v_2), \max(u_3, v_3))$. Thus, $(u \wedge_1 v)^c = (\max(u_3, v_3), 1 - \max(u_2, v_2), \min(u_1, v_1))$. Since $u^c \vee_1 v^c = (u_3, 1 - u_2, u_1) \vee_1 (v_3, 1 - v_2, v_1) = (\max(u_3, v_3), \min(1 - u_2, 1 - v_2), \min(u_1, v_1)) = (\max(u_3, v_3), 1 - \max(u_2, v_2), \min(u_1, v_1))$. Hence, $(u \wedge_1 v)^c = u^c \vee_1 v^c$.
(2) Similarly, we can get $(u \vee_1 v)^c = u^c \wedge_1 v^c$. □

Proposition 6. *The system $(D^*; \wedge_1, \vee_1, ^c, 0_{D^*}, 1_{D^*})$ is a De Morgan algebra.*

Proof. By Propositions 1–5 and the definition of the generalized De Morgan algebra [14,55], we can get that $(D^*; \wedge_1, \vee_1, ^c, 0_{D^*}, 1_{D^*})$ is a generalized De Morgan algebra. Furthermore, we can prove that $(D^*; \wedge_1, \vee_1, ^c, 0_{D^*}, 1_{D^*})$ is a distributive lattice, that is, for all $u, v, w \in D^*$ such that $u \wedge_1 (v \vee_1 w) = (u \wedge_1 v) \vee_1 (u \wedge_1 w)$.

(1) For all $u, v, w \in D^*$, if any two of them are comparable, then there are six situations as follows:

Case 1: If $u \leq_1 v \leq_1 w$, then $u \wedge_1 (v \vee_1 w) = u \wedge_1 w = u$, $(u \wedge_1 v) \vee_1 (u \wedge_1 w) = u \vee_1 u = u$.
Case 2: If $u \leq_1 w \leq_1 v$, then $u \wedge_1 (v \vee_1 w) = u \wedge_1 v = u$, $(u \wedge_1 v) \vee_1 (u \wedge_1 w) = u \vee_1 u = u$.
Case 3: If $w \leq_1 u \leq_1 v$, then $u \wedge_1 (v \vee_1 w) = u \wedge_1 v = u$, $(u \wedge_1 v) \vee_1 (u \wedge_1 w) = u \vee_1 w = u$.
Case 4: If $v \leq_1 u \leq_1 w$, then $u \wedge_1 (v \vee_1 w) = u \wedge_1 w = u$, $(u \wedge_1 v) \vee_1 (u \wedge_1 w) = v \vee_1 u = u$.
Case 5: If $w \leq_1 v \leq_1 u$, then $u \wedge_1 (v \vee_1 w) = u \wedge_1 v = v$, $(u \wedge_1 v) \vee_1 (u \wedge_1 w) = v \vee_1 w = v$.
Case 6: If $v \leq_1 w \leq_1 u$, then $u \wedge_1 (v \vee_1 w) = u \wedge_1 w = w$, $(u \wedge_1 v) \vee_1 (u \wedge_1 w) = v \vee_1 w = w$.

Thus, $u \wedge_1 (v \vee_1 w) = (u \wedge_1 v) \vee_1 (u \wedge_1 w)$.

(2) For all $u, v, w \in D^*$, if at least two of them are not comparable, then $u \wedge_1 (v \vee_1 w) = u \wedge_1 (\max(v_1, w_1), \min(v_2, w_2), \min(v_3, w_3)) = (\min(u_1, \max(v_1, w_1)), \max(u_2, \min(v_2, w_2)), \max(u_3, \min(v_3, w_3))) = (\max(\min(u_1, v_1), \min(u_1, w_1)), \min(\max(u_2, v_2), \max(u_2, w_2)), \min(\max(u_3, v_3), \max(u_3, w_3))) = (\min(u_1, v_1), \max(u_2, v_2), \max(u_3, v_3)) \vee (\min(u_1, w_1), \max(u_2, w_2), \max(u_3, w_3)) = (u \wedge_1 v) \vee_1 (u \wedge_1 w)$.

Therefore, $(D^*; \wedge_1, \vee_1, ^c, 0_{D^*}, 1_{D^*})$ is a De Morgan algebra. □

Considering the second type inclusion relation on neutrosophic sets which is dual of the first type inclusion relation, we get that $(D^*; \wedge_2, \vee_2, ^c, (0,0,1), (1,1,0))$ is also a De Morgan algebra.

From this, Proposition 2.2 (see [14]) can be easily proved by using Proposition 6. That is, Proposition 2.2 (see [14]) is a corollary of Proposition 6.

In short, in combination with the conclusions given in [14], we find that neutrosophic net is different from intuitionistic fuzzy set.

4. Neutrosophic t-Norms and De Morgan Neutrosophic Triples

Section 3 proposes that $(D^*; \leq_1)$ is a complete lattice, Section 4 will introduce the notions of neutrosophic t-norms (t-conorms) on $(D^*; \leq_1)$.

Definition 7. *A neutrosophic t-norm is a function $\mathscr{T}\colon (D^*)^2 \to D^*$ that satisfies the following conditions, for all $u, v, w \in D^*$:*

(NT1) $\mathscr{T}(u, v) = \mathscr{T}(v, u)$;
(NT2) $\mathscr{T}(u, \mathscr{T}(v, w)) = \mathscr{T}(v, \mathscr{T}(u, w))$;
(NT3) $\mathscr{T}(u, v) \leq_1 \mathscr{T}(u', v')$, *where* $u \leq_1 u'$, $v \leq_1 v'$;
(NT4) $\mathscr{T}(u, 1_{D^*}) = u$.

Definition 8. *A neutrosophic t-conorm is a function $\mathscr{S}\colon (D^*)^2 \to D^*$ that satisfies the following conditions, for all $u, v, w \in D^*$:*

(NS1) $\mathscr{S}(u, v) = \mathscr{S}(v, u)$;
(NS2) $\mathscr{S}(u, \mathscr{S}(v, w)) = \mathscr{S}(v, \mathscr{S}(u, w))$;
(NS3) $\mathscr{S}(u, v) \leq_1 \mathscr{S}(u', v')$, *where* $u \leq_1 u'$, $v \leq_1 v'$;
(NS4) $\mathscr{S}(u, 0_{D^*}) = u$.

Some basic neutrosophic t-norms (t-conorms) on $(D^*; \leq_1)$ are presented as follows:

Example 3. *Some neutrosophic t-norms are defined by, for all $u, v \in D^*$:*

(1) $\mathscr{T}_M(u, v) = (T_M(u_1, v_1), S_M(u_2, v_2), S_M(u_3, v_3))$;
(2) $\mathscr{T}_P(u, v) = (T_P(u_1, v_1), S_P(u_2, v_2), S_P(u_3, v_3))$;
(3) $\mathscr{T}_{LK}(u, v) = (T_{LK}(u_1, v_1), S_{LK}(u_2, v_2), S_{LK}(u_3, v_3))$;

(4) $\mathscr{T}_D(u,v) = (T_D(u_1,v_1), S_D(u_2,v_2), S_D(u_3,v_3))$;
(5) $\mathscr{T}_{nM}(u,v) = (T_{nM}(u_1,v_1), S_{nM}(u_2,v_2), S_{nM}(u_3,v_3))$;
(6) $\mathscr{T}(u,v) = (T_P(u_1,v_1), S_{LK}(u_2,v_2), S_D(u_3,v_3))$;
(7) $\mathscr{T}(u,v) = (T_D(u_1,v_1), S_P(u_2,v_2), S_P(u_3,v_3))$;
(8) $\mathscr{T}(u,v) = (T_{LK}(u_1,v_1), S_{LK}(u_2,v_2), S_M(u_3,v_3))$;
(9) $\mathscr{T}(u,v) = (T_M(u_1,v_1), S_P(u_2,v_2), S_M(u_3,v_3))$.

Example 4. *Some neutrosophic t-conorms are defined by, for all $u,v \in D^*$:*

(1) $\mathscr{S}_M(u,v) = (S_M(u_1,v_1), T_M(u_2,v_2), T_M(u_3,v_3))$;
(2) $\mathscr{S}_P(u,v) = (S_P(u_1,v_1), T_P(u_2,v_2), T_P(u_3,v_3))$;
(3) $\mathscr{S}_{LK}(u,v) = (S_{LK}(u_1,v_1), T_{LK}(u_2,v_2), T_{LK}(u_3,v_3))$;
(4) $\mathscr{S}_D(u,v) = (S_D(u_1,v_1), T_D(u_2,v_2), T_D(u_3,v_3))$;
(5) $\mathscr{S}_{nM}(u,v) = (S_{nM}(u_1,v_1), T_{nM}(u_2,v_2), T_{nM}(u_3,v_3))$;
(6) $\mathscr{S}(u,v) = (S_M(u_1,v_1), T_P(u_2,v_2), T_{LK}(u_3,v_3))$;
(7) $\mathscr{S}(u,v) = (S_P(u_1,v_1), T_{LK}(u_2,v_2), T_{LK}(u_3,v_3))$;
(8) $\mathscr{S}(u,v) = (S_{LK}(u_1,v_1), T_{LK}(u_2,v_2), T_M(u_3,v_3))$;
(9) $\mathscr{S}(u,v) = (S_D(u_1,v_1), T_P(u_2,v_2), T_{nM}(u_3,v_3))$.

Furthermore, the representation theorems of neutrosophic t-norms (t-conorms) are proposed as follows:

Theorem 1. *Let $\mathscr{T}$ be a binary operation on D^*. Then, for all $u,v \in D^*$,*

$$\mathscr{T}(u,v) = (T(u_1,v_1), S_1(u_2,v_2), S_2(u_3,v_3))$$

is a neutrosophic t-norm, where S_1, S_2 are t-conorms, T is a t-norm on $[0,1]$.

Proof. (NT1) Let S_1, S_2 be two t-conorms, T is a t-norm on $[0,1]$. Since $T(u_1,v_1) = T(v_1,u_1)$, $S_1(u_2,v_2) = S_1(v_2,u_2)$, $S_2(u_3,v_3) = S_2(v_3,u_3)$, $\mathscr{T}(u,v) = \mathscr{T}(v,u)$, for all $u,v \in D^*$.

(NT2) $\mathscr{T}(1_{D^*},u) = (T(1,u_1), S_1(0,u_2), S_2(0,u_3)) = (u_1,u_2,u_3) = u$, for all $u \in D^*$.

(NT3) For all $u,u',v,v' \in D^*$ with the condition $u \leq_1 u'$, $v \leq_1 v'$, we have $T(u_1,v_1) \leq T(u'_1,v'_1)$, $S_1(u_2,v_2) \geq S_1(u'_2,v'_2)$, $S_2(u_3,v_3) \geq S_2(u'_3,v'_3)$. Therefore, $\mathscr{T}(u,v) \leq_1 \mathscr{T}(u',v')$.

(NT4) $\mathscr{T}(u,\mathscr{T}(v,w)) = \mathscr{T}(u,(T(v_1,w_1), S_1(v_2,w_2), S_2(v_3,w_3))) = (T(u_1,T(v_1,w_1)), S_1(u_2, S_1(v_2,w_2)), S_2(u_3,S_2(v_3,w_3))) = (T(v_1,T(u_1,w_1)), S_1(v_2,S_1(u_2,w_2)), S_2(v_3, S_2(u_3,w_3))) = \mathscr{T}(v, \mathscr{T}(u,w))$, for all $u,v,w \in D^*$.

Hence, $\mathscr{T}(u,v)$ is a neutrosophic t-norm. □

Theorem 2. *Let $\mathscr{S}$: $(D^*)^2 \to D^*$ be a mapping. Then, for all $u,v \in D^*$,*

$$\mathscr{S}(u,v) = (S(u_1,v_1), T_1(u_2,v_2), T_2(u_3,v_3))$$

is a neutrosophic t-conorm, where S is a t-conorm, T_1, T_2 are t-norms on $[0,1]$.

Proof. The proof is similar to that of Theorem 1. □

Theorem 1 proposes a way to construct neutrosophic t-norms on D^* with t-norms and t-conorms which are defined on $[0,1]$. Unfortunately, the converse is not always true. It is not always possible to find two t-conorms S_1, S_2, a t-norm T on $[0,1]$ such that $\mathscr{T} = (T, S_1, S_2)$.

To distinguish these two kinds of neutrosophic t-norms, we introduce the notions of representable neutrosophic t-norms.

Definition 9. *A neutrosophic t-norm $\mathscr{T}$ is called representable, if and only if, there exist two t-conorms S_1, S_2 and a t-norm T on $[0,1]$ satisfying, for any $u,v \in D^*$,*

$$\mathscr{T}(u,v) = (T(u_1,v_1), S_1(u_2,v_2), S_2(u_3,v_3)).$$

Definition 10. *A neutrosophic t-norm $\mathscr{T}$ is called standard representable, if and only if, there exists a t-norm T and a t-conorm S on $[0,1]$ satisfying, for any $u, v \in D^*$,*

$$\mathscr{T}(u,v) = (T(u_1,v_1), S(u_2,v_2), S(u_3,v_3)).$$

Definition 11. *A N-dual representable neutrosophic t-norm $\mathscr{T}$ defined by, for any $u, v \in D^*$,*

$$\mathscr{T}(u,v) = (T(u_1,v_1), S(u_2,v_2), S(u_3,v_3)).$$

where T is a t-norm on $[0,1]$ and S is the N-dual t-conorm of T, that is, $T(u,v) = 1 - S(1-u, 1-v)$.

Definition 12. *A first N-dual representable neutrosophic t-norm $\mathscr{T}$ defined by, for any $u, v \in D^*$,*

$$\mathscr{T}(u,v) = (T(u_1,v_1), S_1(u_2,v_2), S_2(u_3,v_3)).$$

where T is a t-norm on $[0,1]$ and S_1 is the N-dual t-conorm of T, S_2 is a t-conorm on $[0,1]$.

Definition 13. *A second N-dual representable neutrosophic t-norm $\mathscr{T}$ defined by, for any $u, v \in D^*$,*

$$\mathscr{T}(u,v) = (T(u_1,v_1), S_1(u_2,v_2), S_2(u_3,v_3))$$

where T is a t-norm on $[0,1]$ and S_2 is the N-dual t-conorm of T, S_1 is a t-conorm on $[0,1]$.

Notice that the N-dual representable neutrosophic t-norms are not only the standard representable neutrosophic t-norms, but also the first N-dual representable neutrosophic t-norms and the second N-dual representable neutrosophic t-norms. Those neutrosophic t-norms presented in Example 3 are all representable neutrosophic t-norms, and (1)–(5) are N-dual representable neutrosophic t-norms, (8) is a first N-dual representable neutrosophic t-norm, (9) is a second N-dual representable neutrosophic t-norm.

Definition 14. *A neutrosophic t-conorm $\mathscr{S}$ is called representable, if and only if, there exists a t-conorm S and two t-norms T_1, T_2 on $[0,1]$ satisfying, for any $u, v \in D^*$,*

$$\mathscr{S}(u,v) = (S(u_1,v_1), T_1(u_2,v_2), T_2(u_3,v_3)).$$

For neutrosophic t-conorms, the rest of the related concepts can be obtained by contrasting with Definitions 10–13 of neutrosophic t-norms above.

The following propositions present a method for constructing new representable neutrosophic t-norms (t-conorms) with intuitionistic fuzzy t-norms (t-conorms).

Proposition 7. *Let $T(x,y)$ be a representable intuitionistic fuzzy t-norm: $T(x,y) = (t(x_1,y_1), s_2(x_3,y_3))$, for all $x = (x_1,x_3)$, $y = (y_1,y_3) \in L^*$, where t is a t-norm, s_2 is a t-conorm on $[0,1]$. Assume that s_1 is a t-conorm on $[0,1]$, satisfying, $0 \le t(u_1,v_1) + s_1(u_2,v_2) + s_2(u_3,v_3) \le 3$. Then $\mathscr{T}(u,v) = (t(u_1,v_1), s_1(u_2,v_2), s_2(u_3,v_3))$ is a representable neutrosophic t-norm, for any $u, v \in D^*$.*

Proposition 8. *Let $S(x,y)$ be a representable intuitionistic fuzzy t-conorm: $S(x,y) = (s(x_1,y_1), t_2(x_3,y_3))$, for all $u = (x_1,x_3)$, $v = (y_1,y_3) \in L^*$, where s is a t-conorm, t_2 is a t-norm on $[0,1]$. Suppose that t_1 is a t-norm on $[0,1]$, satisfying, $0 \le s(u_1,v_1) + t_1(u_2,v_2) + t_2(u_3,v_3) \le 3$. Then $\mathscr{S}(u,v) = (s(u_1,v_1), t_1(u_2,v_2), t_2(u_3,v_3))$ is a representable neutrosophic t-conorm, for all $u, v \in D^*$.*

De Morgan triple is the perfect combination of a fuzzy t-norm, a fuzzy t-conorm and a fuzzy negator because it describes the duality of a fuzzy t-norm and a fuzzy t-conorm with respect to a fuzzy negator. Thus, it is necessary to discuss De Morgan neutrosophic triples. First of all, neutrosophic negators as the extension of fuzzy negators, as well as intuitionistic negators can be defined as follows:

Definition 15. *A neutrosophic negator is a function $\mathscr{N}: D^* \to D^*$ that satisfies the following conditions:*

(NN1) $\mathscr{N}(u) \geq_1 \mathscr{N}(v)$, *for all $u, v \in D^*$ such that $u \leq_1 v$;*
(NN2) $\mathscr{N}(0_{D^*}) = 1_{D^*}$;
(NN3) $\mathscr{N}(1_{D^*}) = 0_{D^*}$.

If $\mathscr{N}(\mathscr{N}(u)) = u$, for all $u \in D^*$, then $\mathscr{N}$ is called an involutive neutrosophic negator.

The mapping $\mathscr{N}_s$: $D^* \to D^*$ defined by, for all $(u_1, u_2, u_3) \in D^*$,

$$\mathscr{N}_s(u_1, u_2, u_3) = (u_3, 1 - u_2, u_1)$$

is an involutive neutrosophic negator. Then we call it the standard neutrosophic negator. Of course, $\mathscr{N}(u) = (u_3, 1 - u_3, u_1)$, $\mathscr{N}(u) = (u_3, u_1, u_1)$ are neutrosophic negators.

Definition 16. *Let $\mathscr{T}$ be a neutrosophic t-norm, $\mathscr{S}$ be a neutrosophic t-conorm, $\mathscr{N}$ be a neutrosophic negator. The triple $(\mathscr{T}, \mathscr{N}, \mathscr{S})$ satisfied the following conditions, for all $u, v \in D^*$,*

$$\mathscr{N}(\mathscr{S}(u,v)) = \mathscr{T}(\mathscr{N}(u), \mathscr{N}(v));$$
$$\mathscr{N}(\mathscr{T}(u,v)) = \mathscr{S}(\mathscr{N}(u), \mathscr{N}(v))$$

is called a De Morgan neutrosophic triple. Moreover, $\mathscr{T}$ and $\mathscr{S}$ are dual with respect to $\mathscr{N}$.

Theorem 3. *Let $\mathscr{N}$ be an involutive neutrosophic negator.*

(1) If $\mathscr{S}$ is a neutrosophic t-conorm, then the operator $\mathscr{T}$ defined by

$$\mathscr{T}(u,v) = \mathscr{N}(\mathscr{S}(\mathscr{N}(u), \mathscr{N}(v))),$$

is a neutrosophic t-norm. Furthermore, $(\mathscr{T}, \mathscr{N}, \mathscr{S})$ is a De Morgan neutrosophic triple.

(2) If $\mathscr{T}$ is a neutrosophic t-norm, then the operator $\mathscr{S}$ defined by

$$\mathscr{S}(u,v) = \mathscr{N}(\mathscr{T}(\mathscr{N}(u), \mathscr{N}(v))),$$

is a neutrosophic t-conorm. Furthermore, $(\mathscr{T}, \mathscr{N}, \mathscr{S})$ is a De Morgan neutrosophic triple.

Proof. (1) Let $\mathscr{N}$ be an involutive neutrosophic negator, $\mathscr{S}$ be a neutrosophic t-conorm.

(NT1) For any $u, v \in D^*$, $\mathscr{T}(u,v) = \mathscr{N}(\mathscr{S}(\mathscr{N}(u), \mathscr{N}(v))) = \mathscr{N}(\mathscr{S}(\mathscr{N}(v), \mathscr{N}(u))) = \mathscr{T}(v,u)$, because $\mathscr{S}$ is commutative. Thus, $\mathscr{T}$ is commutative.

(NT2) For any $u, v, w \in D^*$, $\mathscr{T}(u, \mathscr{T}(v,w)) = \mathscr{T}(u, \mathscr{N}(\mathscr{S}(\mathscr{N}(v), \mathscr{N}(w)))) = \mathscr{N}(\mathscr{S}(\mathscr{N}(u), \mathscr{N}(\mathscr{N}(\mathscr{S}(\mathscr{N}(v), \mathscr{N}(w)))))) = \mathscr{N}(\mathscr{S}(\mathscr{N}(u), \mathscr{S}(\mathscr{N}(v), \mathscr{N}(w)))) = \mathscr{N}(\mathscr{S}(\mathscr{N}(v), \mathscr{S}(\mathscr{N}(u), \mathscr{N}(w)))) = \mathscr{T}(v, \mathscr{T}(u,w))$, because $\mathscr{S}$ is associative and $\mathscr{N}$ is involutive. Thus, $\mathscr{T}$ is associative.

(NT3) Let $u, u', v, v' \in D^*$ with the condition $u \leq_1 u'$, $v \leq_1 v'$. Then $\mathscr{N}(u) \geq_1 \mathscr{N}(u')$, $\mathscr{N}(v) \geq_1 \mathscr{N}(v')$, because $\mathscr{N}$ is non-increasing. Since $\mathscr{S}$ is non-decreasing in its every variable, $\mathscr{S}(\mathscr{N}(u), \mathscr{N}(v)) \geq_1 \mathscr{S}(\mathscr{N}(u'), \mathscr{N}(v'))$. Thus, $\mathscr{N}(\mathscr{S}(\mathscr{N}(u), \mathscr{N}(v))) \leq_1 \mathscr{N}(\mathscr{S}(\mathscr{N}(u'), \mathscr{N}(v')))$, that is, $\mathscr{T}(u,v) \leq_1 \mathscr{T}(u',v')$. Hence, $\mathscr{T}$ is non-decreasing.

(NT4) For any $u \in D^*$, $\mathscr{T}(u, 1_{D^*}) = \mathscr{N}(\mathscr{S}(\mathscr{N}(u), 0_{D^*})) = \mathscr{N}(\mathscr{N}(u)) = u$.

Therefore, $\mathscr{T}$ is a neutrosophic t-norm.

Furthermore, $(\mathscr{T}, \mathscr{N}, \mathscr{S})$ is a De Morgan neutrosophic triple.

(2) Similarly, assume that $\mathscr{T}$ is a neutrosophic t-norm, $\mathscr{S}$ can be proved to be a neutrosophic t-conorm and $(\mathscr{T}, \mathscr{N}, \mathscr{S})$ will be a De Morgan neutrosophic triple. □

Proposition 9. *Suppose that $(\mathscr{T}, \mathscr{N}, \mathscr{S})$ is a De Morgan neutrosophic triple, $\mathscr{N}$ is a standard neutrosophic negator. Then, for all $u \in D^*$,*

(1) $\mathscr{T}(u, 1_{D^*}) = u$ *if and only if* $\mathscr{S}(u, 0_{D^*}) = u$.
(2) $\mathscr{T}(u, 1_{D^*}) = (u_1, 0, u_3)$ *if and only if* $\mathscr{S}(u, 0_{D^*}) = (u_1, 1, u_3)$.
(3) $\mathscr{T}(u, 1_{D^*}) = (u_1, 1, u_3)$ *if and only if* $\mathscr{S}(u, 0_{D^*}) = (u_1, 0, u_3)$.

Example 5. *Some neutrosophic t-norms and neutrosophic t-conorms are dual with respect to* $\mathscr{N}_s$.

(1) $\mathscr{T}_M(u,v) = (T_M(u_1,v_1), S_M(u_2,v_2), S_M(u_3,v_3))$, $\mathscr{S}_M(u,v) = (S_M(u_1,v_1), T_M(u_2,v_2), T_M(u_3,v_3))$.

Indeed, $\mathscr{T}_M(\mathscr{N}(u), \mathscr{N}(v)) = \mathscr{T}_M((u_3, 1-u_2, u_1), (v_3, 1-v_2, v_1)) = (T_M(u_3,v_3), S_M(1-u_2, 1-v_2), S_M(u_1,v_1))$, *then* $\mathscr{N}(\mathscr{T}_M(\mathscr{N}(u), \mathscr{N}(v))) = (S_M(u_1,v_1), 1 - S_M(1-u_2, 1-v_2), T_M(u_3,v_3)) = (S_M(u_1,v_1), T_M(u_2,v_2), T_M(u_3,v_3)) = \mathscr{S}_M(u,v)$. *Thus,* $\mathscr{T}_M$ *and* $\mathscr{S}_M$ *are dual with respect to* $\mathscr{N}_s$.

(2) $\mathscr{T}_P(u,v) = (T_P(u_1,v_1), S_P(u_2,v_2), S_P(u_3,v_3))$, $\mathscr{S}_P(u,v) = (S_P(u_1,v_1), T_P(u_2,v_2), T_P(u_3,v_3))$.
(3) $\mathscr{T}(u,v) = (T_{LK}(u_1,v_1), S_P(u_2,v_2), S_P(u_3,v_3))$, $\mathscr{S}(u,v) = (S_P(u_1,v_1), T_P(u_2,v_2), T_{LK}(u_3,v_3))$.
(4) $\mathscr{T}(u,v) = (\frac{1}{2}(u_1 + v_1 - 1 + u_1 \cdot v_1) \vee 0, S_M(u_2,v_2), S_M(u_3,v_3))$, $\mathscr{S}(u,v) = (S_M(u_1,v_1), T_M(u_2,v_2), \frac{1}{2}(u_3 + v_3 - 1 + u_3 \cdot v_3) \vee 0)$.
(5) $\mathscr{T}(u,v) = (T_P(u_1,v_1), S_M(u_2,v_2), S_M(u_3,v_3))$, $\mathscr{S}(u,v) = (S_M(u_1,v_1), T_M(u_2,v_2), T_P(u_3,v_3))$.
(6) $\mathscr{T}(u,v) = (T_P(u_1,v_1), S_{LK}(u_2,v_2), S_{LK}(u_3,v_3))$, $\mathscr{S}(u,v) = (S_{LK}(u_1,v_1), T_{LK}(u_2,v_2), T_P(u_3,v_3))$.

Representable neutrosophic t-norms are mainly analyzed and discussed above. As for non-representable neutrosophic t-norms, we give the following theorem:

Theorem 4. *Let* $\mathscr{T}\colon (D^*)^2 \to D^*$ *be a mapping. Then, for all* $u, v \in D^*$,

$$\mathscr{T}(u,v) = \begin{cases} u & if\ v = 1_{D^*}, \\ v & if\ u = 1_{D^*}, \\ (\min(u_1,v_1), \max(1-u_1, 1-v_1), \max(u_3,v_3)) & otherwise. \end{cases}$$

is a non-representable neutrosophic t-norm.

Proof. Firstly, $\mathscr{T}$ is a neutrosophic t-norm. In fact,

(NT1) Obviously, $\mathscr{T}(u,v) = \mathscr{T}(v,u)$, for all $u, v \in D^*$.

(NT2) If $u = 1_{D^*}$ or $v = 1_{D^*}$, we can easily prove that $\mathscr{T}(u, \mathscr{T}(v,w)) = \mathscr{T}(\mathscr{T}(u,v), w)$. If $u \neq 1_{D^*}$ and $v \neq 1_{D^*}$, $\mathscr{T}(u, \mathscr{T}(v,w)) = (\min(u_1, \min(v_1,w_1)), \max(1-u_1, 1-\min(v_1,w_1)), \max(u_3, \max(v_3,w_3))) = (\min(u_1,v_1,w_1), \max(1-u_1, 1-v_1, 1-w_1), \max(u_3,v_3,w_3)) = (\min(\min(u_1,v_1),w_1), \max(1-\min(u_1,v_1), 1-w_1), \max(\max(u_3,v_3),w_3)) = \mathscr{T}(\mathscr{T}(u,v),w)$.

(NT3) $\mathscr{T}(1_{D^*}, u) = u$.

(NT4) If $u = 1_{D^*}$ or $v = 1_{D^*}$, we can easily prove that $\mathscr{T}$ is non-decreasing in every variable. If $u \neq 1_{D^*}$ and $v \neq 1_{D^*}$, let $u, u', v, v' \in D^*$ with the condition $u \leq_1 u'$, $v \leq_1 v'$. Then $u_1 \leq u'_1$, $v_1 \leq v'_1$, $u_3 \geq u'_3$, $v_3 \geq v'_3$. Thus, $\min(u_1,v_1) \leq \min(u'_1,v'_1)$, $\max(1-u_1, 1-v_1) \geq \max(1-u'_1, 1-v'_1)$, $\max(u_3,v_3) \geq \max(u'_3,v'_3)$. That is, $\mathscr{T}(u,v) \leq_1 \mathscr{T}(u',v')$. Therefore, $\mathscr{T}$ is a neutrosophic t-norm.

Secondly, for a representable neutrosophic t-norm $\mathscr{T}$, there exists a t-norm T and two t-conorms S_1, S_2 on $[0,1]$ such that, for all $u = (u_1,u_2,u_3), v = (v_1,v_2,v_3) \in D^*$, $\mathscr{T}(u,v) = (T(u_1,v_1), S_1(u_2,v_2), S_2(u_3,v_3))$. Let $u = (0.2, 0.5, 0.7)$, $u' = (0.3, 0.5, 0.7)$, $v = (0.4, 0.5, 0.9)$. From $\mathscr{T}(u,v) = (0.2, 0.8, 0.9)$ and $\mathscr{T}(u',v) = (0.3, 0.7, 0.9)$, we get $S_1(u_2,v_2) = 0.8$ and $S_1(u'_2,v_2) = 0.7$, so $S_1(u_2,v_2) \neq S_1(u'_2,v_2)$. Hence $S_1(u,v)$ is not independent from u_1, thus $\mathscr{T}$ is not representable. □

Furthermore, the dual neutrosophic t-conorm of $\mathscr{T}$ with respect to the standard neutrosophic negator $\mathscr{N}_s$ is $\mathscr{S}$ defined by, for all $u, v \in D^*$,

$$\mathscr{S}(u,v)=\begin{cases} u & if\ v=0_{D^*},\\ v & if\ u=0_{D^*},\\ (\max(u_1,v_1),\min(u_3,v_3),\min(u_3,v_3)) & otherwise.\end{cases}$$

Then, $\mathscr{S}$ is not representable.

Remark 1. *Let $\mathscr{T}$ be a non-representable neutrosophic t-norm on D^*, $\mathscr{S}$ be a neutrosophic t-conorm which is dual to $\mathscr{T}$ with respect to the standard neutrosophic negator $\mathscr{N}_s$. Then, $\mathscr{S}$ is not representable. Conversely, the dual neutrosophic t-norm with respect to an involutive neutrosophic negator $\mathscr{N}$ on D^* of a non-representable neutrosophic t-conorm is not representable.*

Example 6. *Let $\mathscr{T}\colon (D^*)^2 \to D^*$ be a mapping. Then, for all $u,v \in D^*$,*

$$\mathscr{T}(u,v)=\begin{cases} u & if\ v=1_{D^*},\\ v & if\ u=1_{D^*},\\ (\min(u_1,v_1),\max(1-u_1,1-v_1),\max(1-u_1,1-v_1)) & otherwise.\end{cases}$$

is a non-representable neutrosophic t-norm.

Meanwhile, the dual neutrosophic t-conorm $\mathscr{S}$ of $\mathscr{T}$ with respect to $\mathscr{N}_s$ is presented by, for all $u,v \in D^*$,

$$\mathscr{S}(u,v)=\begin{cases} u & if\ v=0_{D^*},\\ v & if\ u=0_{D^*},\\ (\max(1-u_3,1-v_3),\min(u_3,v_3),\min(u_3,v_3)) & otherwise.\end{cases}$$

Then, $\mathscr{S}$ is not representable, too.

Example 7. *Let $\mathscr{S}$ be a mapping: $(D^*)^2 \to D^*$. Then, for all $u,v \in D^*$,*

$$\mathscr{S}(u,v)=\begin{cases} u & if\ v=0_{D^*},\\ v & if\ u=0_{D^*},\\ (\max(1-u_3,1-v_3),\min(u_2,v_2),\min(u_3,v_3)) & otherwise.\end{cases}$$

is a non-representable neutrosophic t-conorm.

Meanwhile, the dual neutrosophic t-norm $\mathscr{T}$ of $\mathscr{S}$ with respect to $\mathscr{N}_s$ is presented by, for all $u,v \in D^*$,

$$\mathscr{T}(u,v)=\begin{cases} u & if\ v=1_{D^*},\\ v & if\ u=1_{D^*},\\ (\min(u_1,v_1),\max(u_2,v_2),\max(1-u_1,1-v_1)) & otherwise.\end{cases}$$

Then, $\mathscr{T}$ is not representable, too.

5. Neutrosophic Residual Implications of Neutrosophic t-Norms

This section will introduce the notions of neutrosophic residual implications on the complete lattice D^*, investigate basic properties of neutrosophic residual implications, and give some important conclusions between neutrosophic t-norms and neutrosophic residual implications after proving that residual neutrosophic t-norms are $\vee$-distributive. Firstly, we give the notions of neutrosophic implications on D^*.

Definition 17. *A neutrosophic implication is a function $\mathscr{I}\colon (D^*)^2 \to D^*$ that satisfies the following conditions,*

- *(NI1) $\mathscr{I}$ is non-increasing with respect to $\leq_1$ in its first variable, that is, $\mathscr{I}(u,v) \geq_1 \mathscr{I}(u',v)$, where $u,u',v \in D^*$ and $u \leq_1 u'$;*
- *(NI2) $\mathscr{I}$ is non-decreasing with respect to $\leq_1$ in its second variable, that is, $\mathscr{I}(u,v) \leq_1 \mathscr{I}(u,v')$, where $u,v,v' \in D^*$ and $v \leq_1 v'$;*
- *(NI3) $\mathscr{I}(0_{D^*},0_{D^*}) = 1_{D^*}$;*

(NI4) $\mathscr{I}(1_{D^*}, 1_{D^*}) = 1_{D^*}$;
(NI5) $\mathscr{I}(1_{D^*}, 0_{D^*}) = 0_{D^*}$.

Definition 18. *A function $\mathscr{I}\colon (D^*)^2 \to D^*$ is called a neutrosophic residual implication, if there exits a neutrosophic t-norm $\mathscr{T}$ such that*

$$\mathscr{I}(u,v) = \sup\{w | w \in D^*, \mathscr{T}(u,w) \leq_1 v\}.$$

If $\mathscr{I}$ is a neutrosophic residual implication generated from a neutrosophic t-norm $\mathscr{T}$, then it will be denoted by $\mathscr{I}_{\mathscr{T}}$.

Furthermore, a neutrosophic t-norm $\mathscr{T}$ satisfies the residual principle if and only if, for all $u, v, w \in D^*$,

$$\mathscr{T}(u,w) \leq_1 v \text{ if and only if } w \leq_1 \mathscr{I}_{\mathscr{T}}(u,v).$$

Similarly, we can get the definitions of neutrosophic co-implications:

Definition 19. *A neutrosophic co-implication is a function $\mathscr{J}\colon (D^*)^2 \to D^*$ that satisfies the following conditions:*

(NJ1) $\mathscr{J}$ is non-increasing with respect to $\leq_1$ in its first variable, that is, $\mathscr{J}(u,v) \geq_1 \mathscr{J}(u',v)$, where $u, u', v \in D^*$ and $u \leq_1 u'$;
(NJ2) $\mathscr{J}$ is non-decreasing with respect to $\leq_1$ in its second variable, that is, $\mathscr{J}(u,v) \leq_1 \mathscr{J}(u,v')$, where $u, v, v' \in D^*$ and $v \leq_1 v'$;
(NJ3) $\mathscr{J}(0_{D^*}, 0_{D^*}) = 0_{D^*}$;
(NJ4) $\mathscr{J}(1_{D^*}, 1_{D^*}) = 0_{D^*}$;
(NJ5) $\mathscr{J}(0_{D^*}, 1_{D^*}) = 1_{D^*}$.

Definition 20. *A function $\mathscr{J}\colon (D^*)^2 \to D^*$ is called a neutrosophic residual co-implication, if there exits a neutrosophic t-conorm $\mathscr{S}$ such that*

$$\mathscr{J}(u,v) = \inf\{w | w \in D^*, \mathscr{S}(u,w) \geq_1 v\}.$$

If $\mathscr{J}$ is a neutrosophic residual co-implication generated from a neutrosophic t-conorm $\mathscr{S}$, then it will be denoted by $\mathscr{J}_{\mathscr{S}}$.

Furthermore, a neutrosophic t-conorm $\mathscr{S}$ satisfies the residual principle if and only if, for all $u, v, w \in D^*$,

$$\mathscr{S}(u,w) \geq_1 v \text{ if and only if } w \geq_1 \mathscr{I}_{\mathscr{S}}(u,v).$$

Using the description of the above definitions, we can easily obtain the neutrosophic residual implications of neutrosophic t-norms discussed in Section 4.

Example 8. *The neutrosophic residual implications of the representable neutrosophic t-norms of Example 3 are given by, for all $u, v \in D^*$,*

(1) $\mathscr{I}_{\mathscr{T}_M}(u,v) = (I_{GD}(u_1,v_1), J_{GD}(u_2,v_2), J_{GD}(u_3,v_3))$;
(2) $\mathscr{I}_{\mathscr{T}_P}(u,v) = (I_{GG}(u_1,v_1), J_{GG}(u_2,v_2), J_{GG}(u_3,v_3))$;
(3) $\mathscr{I}_{\mathscr{T}_{LK}}(u,v) = (I_{LK}(u_1,v_1), J_{LK}(u_2,v_2), J_{LK}(u_3,v_3))$;
(4) $\mathscr{I}_{\mathscr{T}_D}(u,v) = (I_{WB}(u_1,v_1), J_{WB}(u_2,v_2), J_{WB}(u_3,v_3))$;
(5) $\mathscr{I}_{\mathscr{T}_{nM}}(u,v) = (I_{FD}(u_1,v_1), J_{FD}(u_2,v_2), J_{FD}(u_3,v_3))$;
(6) $\mathscr{I}_{\mathscr{T}}(u,v) = (I_{GG}(u_1,v_1), J_{LK}(u_2,v_2), J_{WB}(u_3,v_3))$;
(7) $\mathscr{I}_{\mathscr{T}}(u,v) = (I_{WB}(u_1,v_1), J_{GG}(u_2,v_2), J_{GG}(u_3,v_3))$;
(8) $\mathscr{I}_{\mathscr{T}}(u,v) = (I_{LK}(u_1,v_1), J_{LK}(u_2,v_2), J_{GD}(u_3,v_3))$;

(9) $\mathscr{I}_{\mathscr{T}}(u,v) = (I_{GD}(u_1,v_1), J_{GG}(u_2,v_2), J_{GD}(u_3,v_3))$.

Example 9. *The neutrosophic residual co-implications of the representable neutrosophic t-conorms of Example 4 are given by, for all $u,v \in D^*$,*

(1) $\mathscr{J}_{\mathscr{S}_M}(u,v) = (J_{GD}(u_1,v_1), I_{GD}(u_2,v_2), I_{GD}(u_3,v_3))$;
(2) $\mathscr{J}_{\mathscr{S}_P}(u,v) = (J_{GG}(u_1,v_1), I_{GG}(u_2,v_2), I_{GG}(u_3,v_3))$;
(3) $\mathscr{J}_{\mathscr{S}_{LK}}(u,v) = (J_{LK}(u_1,v_1), I_{LK}(u_2,v_2), I_{LK}(u_3,v_3))$;
(4) $\mathscr{J}_{\mathscr{S}_D}(u,v) = (J_{WB}(u_1,v_1), I_{WB}(u_2,v_2), I_{WB}(u_3,v_3))$;
(5) $\mathscr{I}_{\mathscr{S}_{nM}}(u,v) = (J_{FD}(u_1,v_1), I_{FD}(u_2,v_2), I_{FD}(u_3,v_3))$;
(6) $\mathscr{J}_{\mathscr{S}}(u,v) = (J_{GD}(u_1,v_1), I_{GG}(u_2,v_2), I_{LK}(u_3,v_3))$;
(7) $\mathscr{J}_{\mathscr{S}}(u,v) = (J_{GG}(u_1,v_1), I_{LK}(u_2,v_2), I_{LK}(u_3,v_3))$;
(8) $\mathscr{J}_{\mathscr{S}}(u,v) = (J_{LK}(u_1,v_1), I_{LK}(u_2,v_2), I_{GD}(u_3,v_3))$;
(9) $\mathscr{J}_{\mathscr{S}}(u,v) = (J_{WB}(u_1,v_1), I_{GG}(u_2,v_2), I_{FD}(u_3,v_3))$.

As we all know, t-conorms are dual operators of t-norms on $[0,1]$, in the same way, residual co-implications are dual operators of residual implications on $[0,1]$, with respect to $N(u) = 1-u$. Neutrosophic residual co-implications of neutrosophic t-conorms are dual operators of neutrosophic residual implications of neutrosophic t-norms, just as that neutrosophic t-conorms are dual operators of neutrosophic t-norms with respect to $\mathscr{N}_s$. As Examples 8 and 9 above show, if $\mathscr{S}$ is the dual neutrosophic t-conorm of a neutrosophic t-norm $\mathscr{T}$, then the neutrosophic residual co-implication $\mathscr{J}_{\mathscr{S}}$ is the dual operator of the neutrosophic residual implication of $\mathscr{I}_{\mathscr{T}}$.

Next, we will introduce the most important theorem in this section, which gives the sufficient condition that the residual operators induced by neutrosophic t-norms must be neutrosophic implications.

Theorem 5. *Let $\mathscr{T}$ be a neutrosophic t-norm on D^* with the neutral element 1_{D^*}. Then, for all $u,v \in D^*$,*

$$\mathscr{I}_{\mathscr{T}}(u,v) = \sup\{w|w \in D^*, \mathscr{T}(u,w) \leq_1 v\}$$

is a neutrosophic implication.

Proof. From Definition 18, $\mathscr{I}_{\mathscr{T}}(u,1_{D^*}) = \sup\{w|w \in D^*, \mathscr{T}(u,w) \leq_1 1_{D^*}\} = 1_{D^*}$, for all $u \in D^*$. Therefore, $\mathscr{I}_{\mathscr{T}}(1_{D^*},1_{D^*}) = 1_{D^*}$. Since $\mathscr{T}$ is non-decreasing, $\mathscr{I}_{\mathscr{T}}(1_{D^*},0_{D^*}) = \sup\{w|w \in D^*, \mathscr{T}(w,1_{D^*}) \leq_1 0_{D^*}\} = \sup\{w|w \in D^*, w \leq_1 0_{D^*}\} = 0_{D^*}$. $\mathscr{I}_{\mathscr{T}}(0_{D^*},0_{D^*}) = \sup\{w|w \in D^*, \mathscr{T}(w,0_{D^*}) \leq_1 0_{D^*}\} = 1_{D^*}$. Let $u,u' \in D^*$ with the condition $u \leq_1 u'$. Since the non-decreasingness of $\mathscr{T}$, $\{w|w \in D^*, \mathscr{T}(u,w) \leq_1 v\} \supseteq_1 \{w|w \in D^*, \mathscr{T}(u',w) \leq_1 v\}$, then $\sup\{w|w \in D^*, \mathscr{T}(u,w) \leq_1 v\} \geq_1 \sup\{w|w \in D^*, \mathscr{T}(u',w) \leq_1 v\}$. Thus, $\mathscr{I}_{\mathscr{T}}(u,v) \geq_1 \mathscr{I}_{\mathscr{T}}(u',v)$. That is $\mathscr{I}_{\mathscr{T}}$ is non-increasing with respect to $\leq_1$ in its first variable. Let $v,v' \in D^*$ with the condition $v \leq_1 v'$. Since the non-decreasingness of $\mathscr{T}$, $\{w|w \in D^*, \mathscr{T}(u,w) \leq_1 v\} \subseteq_1 \{w|w \in D^*, \mathscr{T}(u,w) \leq_1 v'\}$, then $\sup\{w|w \in D^*, \mathscr{T}(u,w) \leq_1 v\} \leq_1 \sup\{w|w \in D^*, \mathscr{T}(u,w) \leq_1 v'\}$. Thus, $\mathscr{I}_{\mathscr{T}}(u,v) \leq_1 \mathscr{I}_{\mathscr{T}}(u,v')$. That is $\mathscr{I}_{\mathscr{T}}$ is non-decreasing with respect to $\leq_1$ in its second variable. □

For neutrosophic residual implications, there are several important properties as follows:

Theorem 6. *Suppose that $\mathscr{T}$ is a neutrosophic t-norm on D^* with the neutral element 1_{D^*}, $\mathscr{I}_{\mathscr{T}}$ is a neutrosophic residual implication. Then, for all $u,v,w \in D^*$,*

(1) $\mathscr{I}_{\mathscr{T}}(0_{D^*},v) = 1_{D^*}$;
(2) $\mathscr{I}_{\mathscr{T}}(u,1_{D^*}) = 1_{D^*}$;
(3) $\mathscr{I}_{\mathscr{T}}(u,u) = 1_{D^*}$;
(4) $\mathscr{I}_{\mathscr{T}}(1_{D^*},v) = v$;
(5) $\mathscr{I}_{\mathscr{T}}(u,v) \geq_1 v$;

(6) $\mathscr{I}_{\mathscr{T}}(u,v) = 1_{D^*}$ *if and only if* $u \leq_1 v$;
(7) $u \leq_1 \mathscr{I}_{\mathscr{T}}(v,w)$ *if and only if* $v \leq_1 \mathscr{I}_{\mathscr{T}}(u,w)$;
(8) $u \leq_1 \mathscr{I}_{\mathscr{T}}(v, \mathscr{T}(u,v))$;
(9) $\mathscr{I}_{\mathscr{T}}(\mathscr{T}(u,v), \mathscr{T}(u,w)) \geq_1 \mathscr{I}_{\mathscr{T}}(v,w)$.

Proof. For all $u, v \in D^*$,

The proofs of (1)–(4) can be directly obtained by Definition 18.

(5) Since $\mathscr{I}$ is non-increasing with respect to $\leq_1$ in its first variable, $\mathscr{I}_{\mathscr{T}}(u,v) \geq_1 \mathscr{I}_{\mathscr{T}}(1_{D^*}, v) = v$.

(6) On the one hand, since $u \leq_1 v$, $\mathscr{T}(1_{D^*}, u) \leq_1 v$. Thus, $\mathscr{I}_{\mathscr{T}}(u,v) \geq_1 1_{D^*}$, that is $\mathscr{I}_{\mathscr{T}}(u,v) = 1_{D^*}$. On the other hand, if $\mathscr{I}_{\mathscr{T}}(u,v) = 1_{D^*}$, then $\mathscr{T}(1_{D^*}, u) \leq_1 v$. Thus, $u \leq_1 v$.

(7) Since $u \leq_1 \mathscr{I}_{\mathscr{T}}(v,w)$, $\mathscr{T}(v,u) \leq_1 w$. Thus, $v \leq_1 \mathscr{I}_{\mathscr{T}}(u,w)$. Similarly, it follows from $v \leq_1 \mathscr{I}_{\mathscr{T}}(u,w)$ that $u \leq_1 \mathscr{I}_{\mathscr{T}}(v,w)$.

(8) Since $\mathscr{T}(u,v) \leq_1 \mathscr{T}(u,v)$, $u \leq_1 \mathscr{I}_{\mathscr{T}}(v, \mathscr{T}(u,v))$.

(9) $\mathscr{I}_{\mathscr{T}}(\mathscr{T}(u,v), \mathscr{T}(u,w)) = \sup\{t \mid t \in D^*, \mathscr{T}(\mathscr{T}(u,v),t) \leq_1 \mathscr{T}(u,w)\} = \sup\{t \mid t \in D^*, \mathscr{T}(u, \mathscr{T}(v,t)) \leq_1 \mathscr{T}(u,w)\} \geq_1 \sup\{t \mid t \in D^*, \mathscr{T}(v,t) \leq_1 w\} = \mathscr{I}_{\mathscr{T}}(v,w)$. □

Example 10. *Example 8 shows some neutrosophic residual implications of representable neutrosophic t-norms, furthermore, it is easy to verify that neutrosophic residual implications of representable neutrosophic t-norms satisfy the properties described in Theorem 6.*

For non-representable neutrosophic t-norms, take the neutrosophic t-norm $\mathscr{T}$ presented in Theorem 4 for example, then, for all $u, v \in D^*$,

$$\mathscr{I}_{\mathscr{T}}(u,v) = \begin{cases} 1_{D^*} & \text{if } v = 1_{D^*}, \\ v & \text{if } u = 1_{D^*}, \\ \left(I_{GD}(u_1,v_1), \begin{cases} u_1 & \text{if } u_1 \leq 1 - v_2, \\ 0 & \text{otherwise} \end{cases}, J_{GD}(u_3,v_3)\right) & \text{otherwise} \end{cases}.$$

is a neutrosophic implication and satisfies the properties given in Theorem 6.

Similarly, we have the following two important theorems of neutrosophic t-conorm on D^*:

Theorem 7. *Assume that $\mathscr{S}$ is a neutrosophic t-conorm on D^* with the neutral element 0_{D^*}. Then, for all $u, v \in D^*$,*

$$\mathscr{J}_{\mathscr{S}}(u,v) = \inf\{w \mid w \in D^*, \mathscr{S}(u,w) \geq_1 v\}.$$

is a neutrosophic co-implication.

Proof. From Definition 20, we can prove it using the proven ways of Theorem 5. □

Theorem 8. *Assume that $\mathscr{S}$ is a neutrosophic t-conorm on D^* with the neutral element 0_{D^*}, $\mathscr{J}_{\mathscr{S}}$ is a neutrosophic residual co-implication. Then, for all $u, v, w \in D^*$,*

(1) $\mathscr{J}_{\mathscr{S}}(1_{D^*}, v) = 0_{D^*}$;
(2) $\mathscr{J}_{\mathscr{S}}(u, 0_{D^*}) = 0_{D^*}$;
(3) $\mathscr{J}_{\mathscr{S}}(u,u) = 0_{D^*}$;
(4) $\mathscr{J}_{\mathscr{S}}(0_{D^*}, v) = v$;
(5) $\mathscr{J}_{\mathscr{S}}(u,v) \leq_1 v$;
(6) $\mathscr{J}_{\mathscr{S}}(u,v) = 0_{D^*}$ *if and only if* $u \geq_1 v$;
(7) $u \geq_1 \mathscr{J}_{\mathscr{S}}(v,w)$ *if and only if* $v \geq_1 \mathscr{J}_{\mathscr{S}}(u,w)$;
(8) $u \geq_1 \mathscr{J}_{\mathscr{S}}(v, \mathscr{S}(u,v))$;
(9) $\mathscr{J}_{\mathscr{S}}(\mathscr{S}(u,v), \mathscr{S}(u,w)) \leq_1 \mathscr{J}_{\mathscr{S}}(v,w)$.

Example 11. *Example 9 shows some neutrosophic residual co-implications of representable neutrosophic t-conorms; furthermore, it is easy to verify that neutrosophic residual co-implications of representable neutrosophic t-conorms satisfy the properties described in Theorem 8.*

For non-representable neutrosophic t-conorms, take the neutrosophic t-conorm $\mathscr{S}$ presented in Theorem 4 for example, then, for all $u, v \in D^*$,

$$\mathscr{I}_{\mathscr{T}}(u,v) = \begin{cases} 0_{D^*} & \text{if } v = 0_{D^*}, \\ v & \text{if } u = 0_{D^*}, \\ (J_{GD}(u_1,v_1), \begin{cases} 1 & \text{if } u_3 \leq v_2, \\ v_2 & \text{otherwise} \end{cases}, I_{GD}(u_3,v_3)) & \text{otherwise} \end{cases}.$$

is a neutrosophic co-implication and it satisfies the properties given in Theorem 8.

In Definition 18, $\mathscr{I}_{\mathscr{T}}$ is called the neutrosophic residual implication. At the same time, $\mathscr{T}$ is called the residual neutrosophic t-norm. Then, some important properties of the residual neutrosophic t-norm will be discussed below.

Definition 21. *[45] A binary operation H on a complete lattice L is called left (right) infinitely $\vee$-distributive, if for all $u \in L$,*

$$H(\sup_{w \in W} w, v) = \sup_{w \in W} H(w,v) \; (H(u, \sup_{w \in W} w) = \sup_{w \in W} H(u,w));$$

H is called left (right) infinitely $\wedge$-distributive, if for all $u \in L$,

$$H(\inf_{w \in W} w, v) = \inf_{w \in W} H(w,v) \; (H(u, \inf_{w \in W} w) = \inf_{w \in W} H(u,w)),$$

where $W \subseteq L$. H is called infinitely $\vee$-distributive ($\wedge$-distributive) on L, if H is both left and right infinitely $\vee$-distributive ($\wedge$-distributive).

Theorem 9. *Assume that $\mathscr{T}$ is a residual neutrosophic t-norm on D^* with the neutral element 1_{D^*}. Then $\mathscr{T}$ is infinitely $\vee$-distributive on D^*.*

Proof. Let $W \subseteq D^*$. If $W = \emptyset$, then $\mathscr{T}(\sup_{w \in W} w, v) = \mathscr{T}(0_{D^*}, v) = 0_{D^*} = \sup_{w \in W} \mathscr{T}(w,v)$, for all $v \in D^*$. If $W \neq \emptyset$, since $\mathscr{T}$ is non-decreasing, $\mathscr{T}(\sup_{w \in W} w, v) \geq_1 \sup_{w \in W} \mathscr{T}(w,v)$, for any $v \in D^*$. Suppose $m = \sup_{w \in W} \mathscr{T}(w,v)$, then $\mathscr{T}(w,v) \leq_1 m$, now we have $w \in \{t \mid t \in D^*, \mathscr{T}(t,v) \leq_1 m\}$. By Definition 18, $w \leq_1 \mathscr{I}_{\mathscr{T}}(v,m)$, for all $w \in W$. Thus, $\sup_{w \in W} w \leq_1 \mathscr{I}_{\mathscr{T}}(v,m)$. Since $\mathscr{T}$ is non-decreasing, $\mathscr{T}(\sup_{w \in W} w, v) \leq_1 \mathscr{T}(\mathscr{I}_{\mathscr{T}}(v,m), v)$. Since $\mathscr{T}(u,w) \leq_1 v$ if and only if $w \leq_1 \mathscr{I}_{\mathscr{T}}(u,v)$, and $\mathscr{I}_{\mathscr{T}}(v,m) \leq_1 \mathscr{I}_{\mathscr{T}}(v,m)$, $\mathscr{T}(\mathscr{I}_{\mathscr{T}}(v,m), v) \leq_1 m = \sup_{w \in W} \mathscr{T}(w,v)$. Therefore, $\mathscr{T}(\sup_{w \in W} w, v) \leq_1 \sup_{w \in W} \mathscr{T}(w,v)$. □

Theorem 10. *Assume that $\mathscr{T}$ is a residual neutrosophic t-norm on D^* with the neutral element 1_{D^*}. Then, for all $u, v \in D^*$,*

(1) $\mathscr{T}(\mathscr{I}_{\mathscr{T}}(u,v), u) \leq_1 v$. *In particularly,* $\mathscr{T}(\mathscr{I}_{\mathscr{T}}(u,u), u) = u$, $\mathscr{T}(\mathscr{I}_{\mathscr{T}}(u, 0_{D^*}), u) = 0_{D^*}$;
(2) $\mathscr{I}_{\mathscr{T}}(\mathscr{T}(u,v), w) = \mathscr{I}_{\mathscr{T}}(v, \mathscr{I}_{\mathscr{T}}(u,w))$;
(3) $\mathscr{I}_{\mathscr{T}}(\sup_{u \in U} u, v) = \inf_{u \in U} \mathscr{I}_{\mathscr{T}}(u,v)$;
(4) $\mathscr{T}(\mathscr{I}_{\mathscr{T}}(u,v), \mathscr{I}_{\mathscr{T}}(v,w)) \leq_1 \mathscr{I}_{\mathscr{T}}(u,w)$;
(5) $\mathscr{T}(\mathscr{I}_{\mathscr{T}}(w, 1_{D^*}), \mathscr{I}_{\mathscr{T}}(u,v)) \leq_1 \mathscr{I}_{\mathscr{T}}(\mathscr{T}(u,w), v)$;
(6) $\mathscr{I}_{\mathscr{T}}(u, \mathscr{I}_{\mathscr{T}}(v,w)) = \mathscr{I}_{\mathscr{T}}(v, \mathscr{I}_{\mathscr{T}}(u,w))$;
(7) $\mathscr{T}(\mathscr{I}_{\mathscr{T}}(u, \mathscr{T}(v,u)), u) = \mathscr{T}(u,v)$;

(8) $\mathscr{I}_{\mathscr{T}}(u, \mathscr{T}(\mathscr{I}_{\mathscr{T}}(u,v),u)) = \mathscr{I}_{\mathscr{T}}(u,v)$.

Proof. We can directly prove (1)–(6) directly by the method of Theorems 4.3 and 4.6–4.9 in [46]; The proofs of (7) and (8) can be obtained directly from Theorem 3.5 in [48]. □

Naturally, we can prove that a residual neutrosophic t-conorm is infinitely $\wedge$-distributive, and then we can get some important properties of a residual neutrosophic t-conorm on D^*.

Theorem 11. *Assume that $\mathscr{S}$ is a residual neutrosophic t-conorm on D^* with the neutral element 0_{D^*}. Then $\mathscr{S}$ is infinitely $\wedge$-distributive on D^*.*

Theorem 12. *Assume that $\mathscr{S}$ is a residual neutrosophic t-conorm on D^* with the neutral element 0_{D^*}. Then, for all $u, v \in D^*$,*

(1) $\mathscr{S}(\mathscr{J}_{\mathscr{S}}(u,v),u) \geq_1 v$. *In particularly,* $\mathscr{S}(\mathscr{J}_{\mathscr{S}}(u,u),u) = u$, $\mathscr{S}(\mathscr{J}_{\mathscr{S}}(u,1_{D^*}),u) = 1_{D^*}$;
(2) $\mathscr{J}_{\mathscr{S}}(\mathscr{S}(u,v),w) = \mathscr{J}_{\mathscr{S}}(v, \mathscr{J}_{\mathscr{S}}(u,w))$;
(3) $\mathscr{J}_{\mathscr{S}}(\inf_{u\in U} u, v) = \sup_{u\in U} \mathscr{J}_{\mathscr{S}}(u,v)$;
(4) $\mathscr{S}(\mathscr{J}_{\mathscr{S}}(u,v), \mathscr{J}_{\mathscr{S}}(v,w)) \geq_1 \mathscr{J}_{\mathscr{S}}(u,w)$;
(5) $\mathscr{S}(\mathscr{J}_{\mathscr{S}}(w,1_{D^*}), \mathscr{J}_{\mathscr{S}}(u,v)) \geq_1 \mathscr{J}_{\mathscr{S}}(\mathscr{S}(u,w),v)$;
(6) $\mathscr{J}_{\mathscr{S}}(u, \mathscr{J}_{\mathscr{S}}(v,w)) = \mathscr{J}_{\mathscr{S}}(v, \mathscr{J}_{\mathscr{S}}(u,w))$;
(7) $\mathscr{S}(\mathscr{J}_{\mathscr{S}}(u, \mathscr{S}(v,u)),u) = \mathscr{S}(u,v)$;
(8) $\mathscr{J}_{\mathscr{S}}(u, \mathscr{S}(\mathscr{J}_{\mathscr{S}}(u,v),u)) = \mathscr{J}_{\mathscr{S}}(u,v)$.

Proof. The proofs of (1)–(6) can be obtained directly from Theorems 3.2 and 3.5–3.8 in [47]; the proofs of (7) and (8) can be obtained directly from Theorem 3.5 in [48]. □

6. Neutrosophic t-Norms Induced by Neutrosophic Implications on D^*

From Theorem 5, we know that neutrosophic implications can be induced by neutrosophic t-norms. In this section, the dual situation will be considered. Then, residuated lattices can be constructed on the basis of neutrosophic t-norms and their corresponding neutrosophic residual implications.

Definition 22. *Let $\mathscr{I}\colon (D^*)^2 \to D^*$ be a neutrosophic implication. The induced operator $\mathscr{T}_{\mathscr{I}}$ by $\mathscr{I}$ is defined as follows:*

$$\mathscr{T}_{\mathscr{I}}(u,v) = \inf\{w \mid w \in D^*, v \leq_1 \mathscr{I}(u,w)\}, \text{for all } u, v \in D^*.$$

Remark 2. *(1) $\mathscr{T}_{\mathscr{I}}(u,v)$ is a non-empty set, since $\mathscr{I}(u,1_{D^*}) = 1_{D^*}$, for all $u \in D^*$.*
(2) $\mathscr{T}_{\mathscr{I}}$ defined above is not always a neutrosophic t-norm. For example, for all $u, v \in D^*$,

$$\mathscr{I}(u,v) = (1 - u_1 + u_1 v_1, \begin{cases} 0 & \text{if } u_2 \geq v_2, \\ v_2 & \text{otherwise} \end{cases}, \begin{cases} 0 & \text{if } u_3 \geq v_3, \\ v_3 & \text{otherwise} \end{cases})$$

is a neutrosophic implication. However, $\mathscr{T}_{\mathscr{I}}$ is not a neutrosophic t-norm, because $\mathscr{T}_{\mathscr{I}}(1_{D^*}, v) = (1, v_2, v_3) \neq v$.

Theorem 13. *Let $\mathscr{I}$ be a neutrosophic implication on D^*. The induced operator $\mathscr{T}_{\mathscr{I}}$ by $\mathscr{I}$:*

$$\mathscr{T}_{\mathscr{I}}(u,v) = \inf\{w \mid w \in D^*, v \leq_1 \mathscr{I}(u,w)\}$$

is a neutrosophic t-norm if $\mathscr{I}$ satisfies the following conditions, for all $u, v, w \in D^*$:

(1) $u \leq_1 \mathscr{I}(v,w)$ *if and only if* $v \leq_1 \mathscr{I}(u,w)$;
(2) $\mathscr{I}(\mathscr{I}(u,v),w) = \mathscr{I}(u, \mathscr{I}(v,w))$;
(3) $\mathscr{I}(u,v) = 1_{D^*}$ *if and only if* $u \leq_1 v$;

(4) $\mathscr{I}(1_{D^*}, u) = u.$

Proof. Firstly, we prove that $\mathscr{T}_{\mathscr{I}}$ is a neutrosophic t-norm.
(NT1) From (1), we can directly get $\mathscr{T}_{\mathscr{I}}(u, v) = \mathscr{T}_{\mathscr{I}}(v, u)$, for all $u, v \in D^*$.

(NT2) From (1) and (2), $\mathscr{T}_{\mathscr{I}}(\mathscr{T}_{\mathscr{I}}(u, v), w) = \mathscr{T}_{\mathscr{I}}(w, \mathscr{T}_{\mathscr{I}}(u, v)) = \inf\{t \mid t \in D^*, \mathscr{T}_{\mathscr{I}}(u, v) \leq_1 \mathscr{I}(w, t)\} = \inf\{t \mid t \in D^*, u \leq_1 \mathscr{I}(v, \mathscr{I}(w, t))\} = \inf\{t \mid t \in D^*, v \leq_1 \mathscr{I}(u, \mathscr{I}(w, t))\} = \inf\{t \mid t \in D^*, v \leq_1 \mathscr{I}(w, \mathscr{I}(u, t))\} = \inf\{t \mid t \in D^*, \mathscr{T}_{\mathscr{I}}(v, w) \leq_1 \mathscr{I}(u, t)\} = \mathscr{T}_{\mathscr{I}}(\mathscr{T}_{\mathscr{I}}(v, w), u) = \mathscr{T}_{\mathscr{I}}(u, \mathscr{T}_{\mathscr{I}}(v, w))$.

(NT3) Since $\mathscr{T}_{\mathscr{I}}(u, 1_{D^*}) = \mathscr{T}_{\mathscr{I}}(1_{D^*}, u) = \inf\{t \mid t \in D^*, u \leq_1 \mathscr{I}(1_{D^*}, t)\} = \inf\{t \mid t \in D^*, u \leq_1 t\} = u$, $\mathscr{T}_{\mathscr{I}}(u, 1_{D^*}) = \mathscr{T}_{\mathscr{I}}(1_{D^*}, u) = u$.

(NT4) Assume $u, u', v, v' \in D^*$ with the condition $u \leq_1 u'$, $v \leq_1 v'$. Since $\mathscr{I}$ is a neutrosophic implication, $\mathscr{I}(u', t) \leq_1 \mathscr{I}(u, t)$, for all $t \in D^*$. For any $t_0 \in \{t \mid t \in D^*, v' \leq_1 \mathscr{I}(u', t)\}$, it follows that $v' \leq_1 \mathscr{I}(u', t_0)$. Since $v \leq_1 v'$, and $\mathscr{I}(u', t_0) \leq_1 \mathscr{I}(u, t_0)$, $v \leq_1 \mathscr{I}(u, t_0)$, that is, $t_0 \in \{t \mid t \in D^*, v \leq_1 \mathscr{I}(u, t)\}$. Thus, $\{t \mid t \in D^*, v' \leq_1 \mathscr{I}(u', t)\} \subseteq_1 \{t \mid t \in D^*, v \leq_1 \mathscr{I}(u, t)\}$. Hence, $\inf\{t \mid t \in D^*, v \leq_1 \mathscr{I}(u, t)\} \leq_1 \inf\{t \mid t \in D^*, v' \leq_1 \mathscr{I}(u', t)\}$, that is, $\mathscr{T}_{\mathscr{I}}(u, v) \leq_1 \mathscr{T}_{\mathscr{I}}(u', v')$.

Therefore, $\mathscr{T}_{\mathscr{I}}$ is a neutrosophic t-norm. □

Theorem 13 describes the conditions that an induced operator $\mathscr{T}_{\mathscr{I}}$ by $\mathscr{I}$ is a neutrosophic t-norm. Moreover, we can construct neutrosophic t-norms with neutrosophic implications according to these conditions.

Next, some important properties of the residual neutrosophic implication on D^* will be discussed.

Theorem 14. *Let $\mathscr{I}$ be a residual neutrosophic implication on D^*. Then $\mathscr{I}(u, \inf_{w \in W} w) = \inf_{w \in W} \mathscr{I}(u, w)$, for all $u \in D^*$, $W \subseteq_1 D^*$.*

Proof. Let $W \subseteq_1 D^*$. If $W = \emptyset$, then $\mathscr{I}(u, \inf_{w \in W} w) = \mathscr{I}(u, 1_{D^*}) = 1_{D^*} = \inf_{w \in W} \mathscr{I}(u, w)$, for any $u \in D^*$. If $W \neq \emptyset$, since $\mathscr{I}$ is non-decreasingness in its second variable, $\mathscr{I}(u, \inf_{w \in W} w) \leq_1 \inf_{w \in W} \mathscr{I}(u, w)$, for all $u \in D^*$. Suppose $n = \inf_{w \in W} \mathscr{I}(u, w)$, then $n \leq_1 \mathscr{I}(u, w)$, now we have $w \in \{t \mid t \in D^*, n \leq_1 \mathscr{I}(u, t)\}$. By Definition 22, $\mathscr{T}_{\mathscr{I}}(n, u) \leq_1 w$, for all $w \in W$. Thus, $\mathscr{T}_{\mathscr{I}}(n, u) \leq_1 \inf_{w \in W} w$. Since $\mathscr{I}$ is non-decreasingness in its second variable, $\mathscr{I}(u, \inf_{w \in W} w) \geq_1 \mathscr{I}(u, \mathscr{T}_{\mathscr{I}}(n, u))$. Since $u \leq_1 \mathscr{I}(v, w)$ if and only if $w \geq_1 \mathscr{T}_{\mathscr{I}}(u, v)$, and $\mathscr{T}_{\mathscr{I}}(n, u) \geq_1 \mathscr{T}_{\mathscr{I}}(n, u)$, $\mathscr{I}(u, \mathscr{T}_{\mathscr{I}}(n, u)) \geq_1 n = \inf_{w \in W} \mathscr{I}(u, w)$. Therefore, $\mathscr{I}(u, \inf_{w \in W} w) \geq_1 \inf_{w \in W} \mathscr{I}(u, w)$. □

From Theorem 14, we know that a residual neutrosophic implication satisfies infinitively $\wedge$-distributive in its second variable.

Theorem 15. *Assume that $\mathscr{I}$ is a residual neutrosophic implication on D^*. Then, for all $u, v \in D^*$,*

(1) $\mathscr{T}_{\mathscr{I}}(u, \mathscr{I}(u, \mathscr{T}_{\mathscr{I}}(u, v))) = \mathscr{T}_{\mathscr{I}}(u, v)$;
(2) $\mathscr{I}(u, \mathscr{T}_{\mathscr{I}}(u, \mathscr{I}(u, v))) = \mathscr{I}(u, v)$.

Proof. Let $u, v \in D^*$.

(1) $\mathscr{T}_{\mathscr{I}}(u, \mathscr{I}(u, \mathscr{T}_{\mathscr{I}}(u, v))) = \inf\{t \mid t \in D^*, \mathscr{I}(u, \inf\{t \mid t \in D^*, v \leq_1 \mathscr{I}(u, t)\}) \leq_1 \mathscr{I}(u, t)\} = \inf\{t \mid t \in D^*, \inf\{\mathscr{I}(u, t) \mid t \in D^*, v \leq_1 \mathscr{I}(u, t)\} \leq_1 \mathscr{I}(u, t)\} = \inf\{t \mid t \in D^*, v \leq_1 \mathscr{I}(u, t)\} = \mathscr{T}_{\mathscr{I}}(u, v)$.
(2) $\mathscr{I}(u, \mathscr{T}_{u, \mathscr{I}}(\mathscr{I}(u, v))) = \mathscr{I}(u, \inf\{t \mid t \in D^*, \mathscr{I}(u, v) \leq_1 \mathscr{I}(u, t)\}) = \inf\{\mathscr{I}(u, t) \mid t \in D^*, \mathscr{I}(u, t) \geq_1 \mathscr{I}(u, v)\} = \mathscr{I}(u, v)$.

□

Summarizing the results in Theorem 5 and 13, we get the following theorem.

Theorem 16. *(1) Assume that $\mathscr{T}$ is a neutrosophic t-norm on D^*. Then $\mathscr{I}_{\mathscr{T}}(u, \inf_{v \in V} v) = \inf_{v \in V} \mathscr{I}_{\mathscr{T}}(u, v)$ and $\mathscr{T} = \mathscr{T}_{\mathscr{I}_{\mathscr{T}}}$;*
(2) Let $\mathscr{I}$ be a neutrosophic implication on D^*. Then $\mathscr{T}_{\mathscr{I}}$ which satisfies the conditions presented in Theorem 13 is a infinitely $\vee$-distributive neutrosophic t-norm, and $\mathscr{I} = \mathscr{I}_{\mathscr{T}_{\mathscr{I}}}$.

Proof. (1) From Theorem 5, $\mathscr{I}_{\mathscr{T}}$ is a neutrosophic implication. Next, we prove $\mathscr{I}_{\mathscr{T}}(u, \inf_{v \in V} v) = \inf_{v \in V} \mathscr{I}_{\mathscr{T}}(u, v)$, for all $u \in D^*$, $V \subseteq_1 D^*$. Suppose $V \subseteq_1 D^*$. If $V = \emptyset$, then $\mathscr{I}_{\mathscr{T}}(u, \inf_{v \in V} v) = \mathscr{I}_{\mathscr{T}}(u, 1_{D^*}) = 1_{D^*} = \inf_{v \in V} \mathscr{I}_{\mathscr{T}}(u, v)$, for all $u \in D^*$. If $V \neq \emptyset$, then $\mathscr{I}_{\mathscr{T}}(u, \inf_{v \in V} v) = \sup\{t \mid t \in D^*, \mathscr{T}(t, u) \leq_1 \inf_{v \in V} v\} = \sup\{t \mid t \in D^*, \forall v \in V, \mathscr{T}(t, u) \leq_1 v\} = \sup\{t \in D^* \mid \forall v \in V, t \leq_1 \mathscr{I}_{\mathscr{T}}(u, v)\} = \sup\{t \in D^* \mid t \leq_1 \inf_{v \in V} \mathscr{I}_{\mathscr{T}}(u, v)\} = \inf_{v \in V} \mathscr{I}_{\mathscr{T}}(u, v)$. Finally, from Definitions 18 and 22, we get $\mathscr{T}_{\mathscr{I}_{\mathscr{T}}}(u, v) = \inf\{t \mid t \in D^*, v \leq_1 \mathscr{I}_{\mathscr{T}}(u, t)\} = \inf\{t \mid t \in D^*, \mathscr{T}(u, v) \leq_1 t\} = \mathscr{T}(u, v)$, for all $u, v \in D^*$. Thus, $\mathscr{T} = \mathscr{T}_{\mathscr{I}_{\mathscr{T}}}$.
(2) From Definition 22 and Theorem 13, $\mathscr{T}_{\mathscr{I}}$ is a neutrosophic t-norm. Next, we prove $\mathscr{T}_{\mathscr{I}}(\sup_{u \in U} u, v) = \sup_{u \in U} \mathscr{T}_{\mathscr{I}}(u, v)$, for all $v \in D^*$, $U \subseteq_1 D^*$. Suppose $U \subseteq_1 D^*$. If $U = \emptyset$, then $\mathscr{T}_{\mathscr{I}}(\sup_{u \in U} u, v) = \mathscr{T}_{\mathscr{I}}(0_{D^*}, v) = 0_{D^*} = \sup_{u \in U} \mathscr{T}_{\mathscr{I}}(u, v)$, for all $v \in D^*$. If $U \neq \emptyset$, then $\mathscr{T}_{\mathscr{I}}(\sup_{u \in U} u, v) = \inf\{t \mid t \in D^*, v \leq_1 \mathscr{I}(\sup_{u \in U} u, t)\} = \inf\{t \mid t \in D^*, \forall u \in U, v \leq_1 \mathscr{I}(u, t)\} = \inf\{t \mid t \in D^*, \forall u \in U, t \geq_1 \mathscr{T}_{\mathscr{I}}(u, v)\} = \inf\{t \mid t \in D^*, t \geq_1 \sup_{u \in U} \mathscr{T}_{\mathscr{I}}(u, v)\} = \sup_{u \in U} \mathscr{T}_{\mathscr{I}}(u, v)$. Since $\mathscr{T}_{\mathscr{I}}$ satisfies the commutative law, $\mathscr{T}_{\mathscr{I}}(u, \sup_{v \in V} v) = \sup_{v \in V} \mathscr{T}_{\mathscr{I}}(u, v)$. Hence, $\mathscr{T}_{\mathscr{I}}$ is infinitely $\vee$-distributive. At last, from Definitions 18 and 22, we get $\mathscr{I}_{\mathscr{T}_{\mathscr{I}}}(u, v) = \sup\{t \mid t \in D^*, \mathscr{T}_{\mathscr{I}}(t, u) \leq_1 v\} = \sup\{t \mid t \in D^*, t \leq_1 \mathscr{I}(u, v)\} = \mathscr{I}(u, v)$, for all $u, v \in D^*$. Thus, $\mathscr{I} = \mathscr{I}_{\mathscr{T}_{\mathscr{I}}}$. □

Sections 4 and 5 mainly discuss neutrosophic t-norms and their residual implications, then, we can get a residuated lattice by using these two neutrosophic logic operators as follows:

Theorem 17. *Let $\mathscr{T}$ be a neutrosophic t-norm on D^*. Suppose $(D^*; \vee_1, \wedge_1, ^c, 0_{D^*}, 1_{D^*})$ is a system on D^*. For all $u, v \in D^*$, we define:*

$$u \otimes v = \mathscr{T}_{\mathscr{I}_{\mathscr{T}}}(u, v); u \to v = \mathscr{I}_{\mathscr{T}}(u, v).$$

Then, $(D^*; \vee_1, \wedge_1, \otimes, \to, 0_{D^*}, 1_{D^*})$ is a residuated lattice.

Proof. Firstly, from Proposition 3, we know that $(D^*; \vee_1, \wedge_1, 0_{D^*}, 1_{D^*})$ is a bounded lattice.
Then, we prove that $(D^*; \otimes, 1_{D^*})$ is a commutative monoid. (1) For any $u \in D^*$, $1_{D^*} \otimes u = \inf\{t \mid t \in D^*, 1_{D^*} \leq_1 \mathscr{I}_{\mathscr{T}}(u, t)\} = \inf\{t \mid t \in D^*, \mathscr{I}_{\mathscr{T}}(u, t) = 1_{D^*}\} = \inf\{t \mid t \in D^*, u \leq_1 t\} = u$, $u \otimes 1_{D^*} = \inf\{t \mid t \in D^*, u \leq_1 \mathscr{I}_{\mathscr{T}}(1_{D^*}, t)\} = \inf\{t \mid t \in D^*, u \leq_1 t\} = u$. Thus, $1_{D^*} \otimes u = u \otimes 1_{D^*} = u$. (2) Theorem 16 proves that $\mathscr{T}_{\mathscr{I}_{\mathscr{T}}} = \mathscr{T}$ is a neutrosophic t-norm. Thus, $\mathscr{T}$ satisfies the commutative law, that is, $u \otimes v = v \otimes u$. (3) Similarly, $\mathscr{T}$ satisfies the associative law, that is, $u \otimes (v \otimes w) = (u \otimes v) \otimes w$.

Finally, we prove that $\otimes$ is a binary operation for which the equivalence

$$u \otimes v \leq_1 w \text{ if and only if } v \leq_1 u \to w$$

holds for all $u, v, w \in D^*$. On the one hand, by the definition of $\otimes$, we have $u \otimes v = \inf\{w \mid w \in D^*, v \leq_1 u \to w\}$, then $u \otimes v \leq_1 w$. Thus, $v \leq_1 u \to w$. On the other hand, from the definition of $\to$, we have $u \to w = \sup\{v \mid v \in D^*, u \otimes v \leq_1 w\}$. Thus, $u \otimes v \leq_1 w$.
Therefore, $(D^*; \vee_1, \wedge_1, \otimes, \to, 0_{D^*}, 1_{D^*})$ is a residuated lattice. □

Example 12. *Suppose $(D^*; \vee_1, \wedge_1, ^c, 0_{D^*}, 1_{D^*})$ is a system on D^*. For all $u, v \in D^*$, we define:*

$$u \otimes v = \mathscr{T}_{\mathscr{I}_{\mathscr{T}}}(u,v);$$
$$u \to v = \mathscr{I}_{\mathscr{T}}(u,v).$$

where $\mathscr{T}_{\mathscr{I}_{\mathscr{T}}}(u,v)$ *is that presented in Theorem 4,* $\mathscr{I}_{\mathscr{T}}(u,v)$ *is that presented in Example 10.*

Then, $(D^*; \vee_1, \wedge_1, \otimes, \to, 0_{D^*}, 1_{D^*})$ *is a residuated lattice.*

Proof. Firstly, from Proposition 3, we know that $(D^*; \vee_1, \wedge_1, 0_{D^*}, 1_{D^*})$ is a bounded lattice. Then, we prove that $(D^*; \otimes, 1_{D^*})$ is a commutative monoid. (1) For any $u \in D^*$, by the definition of $\otimes$, we get $1_{D^*} \otimes u = u \otimes 1_{D^*} = u$. (2) Obviously, $u \otimes v = v \otimes u$. (3) Suppose $u, v, w \in D^*$. If at least one of them is equal to 1_{D^*}, then $u \otimes (v \otimes w) = (u \otimes v) \otimes w$. Otherwise, $u \otimes (v \otimes w) = (\min(u_1, \min(v_1, w_1)), \max(1-u_1, \max(1-v_1, 1-w_1)), \max(u_3, \max(v_3, w_3))) = (\min(u_1, v_1, w_1), \max(1-u_1, 1-v_1, 1-w_1), \max(u_3, v_3, w_3)) = (\min(v_1, \min(u_1, w_1)), \max(1-v_1, \max(1-u_1, 1-w_1)), \max(v_3, \max(u_3, w_3))) = (u \otimes v) \otimes w$.

Finally, we will prove

$$u \otimes v \leq_1 w \Leftrightarrow v \leq_1 u \to w, \text{ for all } u, v, w \in D^*.$$

If $u = 1_{D^*}$, $u \otimes v = v \leq_1 w$ if and only if $v \leq_1 u \to w \Rightarrow v \leq_1 1_{D^*} \to w \Rightarrow v \leq_1 w$; If $v = 1_{D^*}$, $u \otimes v = u \otimes 1_{D^*} = u \leq_1 w \Rightarrow u \leq_1 1_{D^*}$ if and only if $v \leq_1 u \to w \Rightarrow 1_{D^*} \leq 1_{D^*}$; If $u \neq 1_{D^*}$ and $v \neq 1_{D^*}$, $u \otimes v = (\min(u_1, \min(v_1, w_1)), \max(1-u_1, \max(1-v_1, 1-w_1)), \max(u_3, \max(v_3, w_3))) \leq_1 w$ if and only if

$$v \leq_1 \begin{cases} 1_{D^*} & \text{if } w = 1_{D^*}, \\ w & \text{if } u = 1_{D^*}, \\ (I_{GD}(u_1, w_1), \begin{cases} u_1 & \text{if } u_1 \leq 1 - w_2, \\ 0 & \text{otherwise} \end{cases}, J_{GD}(u_3, w_3)) & \text{otherwise} \end{cases}$$

that is, $v \leq_1 u \to w$.

Therefore, $(D^*; \vee_1, \wedge_1, \otimes, \to, 0_{D^*}, 1_{D^*})$ is a residuated lattice. □

7. Conclusions

Neutrosophic logic plays a vital role in neutrosophic set theory. Neutrosophic t-norms, t-conorms, negators and implications are very important neutrosophic logic operators. In this paper, under the first type inclusion relation, the lattice structure of neutrosophic sets is discussed, $(D^*; \leq_1)$ and $(D^*; \vee_1, \wedge_1, ^c, 0_{D^*}, 1_{D^*})$ are proved to be a complete lattice and De Morgan algebra, respectively. On the complete lattice $(D^*; \leq_1)$, we introduce the definitions of neutrosophic t-norms, t-conorms, negators and their operations. Furthermore, De Morgan neutrosophic triples are defined, which describe that neutrosophic t-norms and t-conorms are dual with respect to the standard neutrosophic negator. Then, we introduce neutrosophic residual implications (co-implications) on the complete lattice $(D^*; \leq_1)$, propose a theorem which shows that residual operations induced by neutrosophic t-norms are neutrosophic implications, investigate basic properties for neutrosophic residual implications (co-implications), prove that residual neutrosophic t-norms are infinitely $\vee$-distributive, and give some important results for residual neutrosophic t-norms and neutrosophic residual implications. Finally, we introduce neutrosophic operations produced by neutrosophic implications, discuss the conditions that the neutrosophic operations are neutrosophic t-norms, and then construct residuated lattices. Based on these results, we will consider their applications in neutrosophic inference systems in the future.

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