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New Results on Neutrosophic Extended Triplet Groups Equipped with a Partial Order

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Received: 13 November 2019; Accepted: 10 December 2019; Published: 13 December 2019

Abstract: Neutrosophic extended triplet group (NETG) is a novel algebra structure and it is different from the classical group. The major concern of this paper is to present the concept of a partially ordered neutrosophic extended triplet group (po-NETG), which is a NETG equipped with a partial order that relates to its multiplicative operation, and consider properties and structure features of po-NETGs. Firstly, in a po-NETG, we propose the concepts of the positive cone and negative cone, and investigate the structure features of them. Secondly, we study the specificity of the positive cone in a partially ordered weak commutative neutrosophic extended triplet group (po-WCNETG). Finally, we introduce the concept of a po-NETG homomorphism between two po-NETGs, construct a po-NETG on a quotient set by providing a multiplication and a partial order, then we discuss some fundamental properties of them.

Keywords: partially ordered neutrosophic extended triplet group; positive cone; homomorphism; quotient set

1. Introduction

Groups play a very important role in algebraic structures [1–3], and have been applied in many other areas such as chemistry, physics, biology, etc. The concept of neutrosophic set theory is proposed by Smarandache in [4], which is the generalization of classical sets [5], fuzzy sets [6], and intuitionistic fuzzy sets [5,7]. Neutrosophic sets have received wide attention both on practical applications [8–10] and on theory as well [11,12]. The main idea of the concept of a neutrosophic triplet group (NTG), is defined in [13,14]. For an NTG $(G, *)$, every element a in G has its own neutral element (denoted by $neut(a)$) satisfying $a * neut(a) = neut(a) * a = a$, and there exists at least one opposite element (denoted by $anti(a)$) in G relative to $neut(a)$ satisfying $a * anti(a) = anti(a) * a = neut(a)$. Here, $neut(a)$ is not allowed to be equal to the classical identity element as a special case. By removing this restriction, the concept of neutrosophic extended triplet group (NETG), is presented in [13]. Many significant results and several studies on NTGs and NETGs can be found in [15–20]. On the other hand, some algebraic structures are equipped with a partial order that relates to the algebraic operations, such as ordered groups, ordered semigroups, ordered rings and so on [21–28].

Regarding these developments, as the motivation of this article, we will consider what it is like to endow a NETG with a partial order and introduce the concepts of partially ordered NETGs and positive cones. Then we consider a question: is a subset P of a NETG G the positive cone relative to some compatible order on G if P satisfies some conditions? To solve this problem, we investigate structure features of partially ordered NETGs and try to characterize the positive cones. Finally, we study properties of homomorphisms and quotient sets in partially ordered NETGs, and discuss the relationships between homomorphisms and congruences. In particular, the quotient set equipped with a special multiplication and a partial order provides a way to obtain a partially ordered NETG. All these results lay the groundwork for investigation of category properties of partially ordered NETGs.

The rest of this paper is organized as follows. In Section 2, we review some basic concepts, such as a neutrosophic extended triplet set, a neutrosophic extended triplet group, a weak commutative neutrosophic extended triplet group and a completely regular semigroup, and several results were published in [16,19]. In Section 3, we define a partially ordered neutrosophic extended triplet group and partially ordered weak commutative neutrosophic extended triplet group. Several of their interesting properties of partially ordered neutrosophic extended triplet group and partially weak commutative neutrosophic extended triplet group are explained. The homomorphisms and quotient sets of partially ordered neutrosophic extended triplet group are shown in Section 4. Finally, conclusions are given in Section 5.

2. Preliminaries

In this section, we recall some basic notions and results which will be used in this paper as indicated below.

Definition 1. *([13]) Let G be a non-empty set together with a binary operation $*$. Then G is called a neutrosophic extended triplet set if for any $a \in G$, there exist a neutral of "a" (denoted by $neut(a)$) and an opposite of "a" (denoted by $anti(a)$), such that $neut(a) \in G$, $anti(a) \in G$, and*

$$a * neut(a) = neut(a) * a = a;$$

$$a * anti(a) = anti(a) * a = neut(a).$$

The triplet $(a,\ neut(a),\ anti(a))$ is called a neutrosophic extended triplet.

Definition 2. *([13]) Let $(G, *)$ be a neutrosophic extended triplet set. If $(G, *)$ is a semigroup, then G is called a neutrosophic extended triplet group (for short, NETG).*

Proposition 1. *([[16] Theorems 1 and 2]) Let $(G, *)$ be a NETG. The following properties hold: $\forall a \in G$*

(1) $neut(a)$ *is unique;*
(2) $neut(a) * neut(a) = neut(a)$;
(3) $neut(neut(a)) = neut(a)$.

Notice that $anti(a)$ may be not unique for every element a in a NETG $(G, *)$. To avoid confusion, we use the following notations:

$anti(a)$ denotes any certain one opposite of a and $\{anti(a)\}$ denotes the set of all opposites of a.

Proposition 2. *([[19], Theorem 1]) Let $(G, *)$ be a NETG. The following properties hold: $\forall\ a \in G$, $\forall\ p,\ q \in \{anti(a)\}$*

(1) $p * neut(a) \in \{anti(a)\}$;
(2) $p * neut(a) = q * neut(a) = neut(a) * q$;
(3) $neut(p * neut(a)) = neut(a)$;
(4) $a \in \{anti(p * neut(a))\}$;
(5) $anti(p * neut(a)) * neut(p * neut(a)) = a$.

Definition 3. *([16]) Let $(G, *)$ be a NETG. If $a * neut(b) = neut(b) * a$ $(\forall a \in G,\ \forall b \in G)$, then G is called a weak commutative neutrosophic extended triplet group (WCNETG).*

Proposition 3. *([[16], Theorem 2]) Let $(G, *)$ be a NETG. Then G is a WCNETG iff G satisfies the following conditions: $\forall a \in G,\ \forall b \in G$*

(1) $neut(a) * neut(b) = neut(b) * neut(a)$;
(2) $neut(a) * neut(b) * a = a * neut(b)$.

Proposition 4. *([[16], Theorem 3]) Let $(G, *)$ be a WCNETG. The following properties hold: $\forall a \in G,\ \forall b \in G$*

(1) $neut(a) * neut(b) = neut(b * a)$;
(2) $anti(a) * anti(b) \in \{anti(b * a)\}$.

Definition 4. *([29]) A semigroup $(S, *)$ will be called completely regular if there exists a unary operation $a \mapsto a^{-1}$ on S with the properties:*

$$(a^{-1})^{-1} = a,\ a * a^{-1} * a = a,\ a * a^{-1} = a^{-1} * a.$$

Proposition 5. *([[19], Theorem 2]) Let $(G, *)$ be a groupoid. Then G is a NETG iff it is a completely regular semigroup.*

Note 1. In semigroup theory, a^{-1} is called the inverse element of a and it is unique. However, in a NETG, $anti(a)$ is called an opposite element of a and it may not be unique. From Proposition 5, we get that for arbitrary element a of a NETG $(G, *)$, if we define a unary operation $a \mapsto a^{-1}$ by $a^{-1} = anti(a) * neut(a)$, then $(G, *)$ is a completely regular semigroup.

In the following, we will regard all NETGs as completely regular semigroups, in which $a^{-1} = anti(a) * neut(a)$ for arbitrary element a. Then by Proposition 2, we have in a NETG $(G, *)$, for each $a \in G$, $a^{-1} \in \{anti(a)\}$ and $a^{-1} * a = a * a^{-1} = neut(a)$.

3. Partially Ordered NETGs

An NETG is a special set endowed with a multiplicative operation. Assuming that we introduce a partial order which is compatible with multiplication in a NETG, we will get the definition of partially ordered NETGs as indicated below.

Definition 5. *Let $(G, *)$ be a NETG. If there exists a partial order relation $\leq$ on G such that $a \leq b$ implying $c * a \leq c * b$ and $a * c \leq b * c$ for all $a \in G,\ b \in G,\ c \in G$, then $\leq$ is called a compatible partial order on G, and $(G, *, \leq)$ is called a partially ordered NETG (for short, po-NETG).*

Similarly, if $(G, *)$ is a WCNETG and endowed with a compatible partial order, then $(G, *, \leq)$ is called a partially ordered WCNETG (po-WCNETG). Hence, po-WCNETGs must be po-NETGs.

Remark 1. *Obviously, the properties of NETGs and WCNETGs are holding in po-NETGs and po-WCNETGs, respectively.*

In the following, we give an example of a po-NETG.

Example 1. *Let $G = \{0,\ a,\ b,\ c,\ 1\}$ with the Hasse diagram as shown in Figure 1, in which 0 denotes the bottom element (mean the element is smallest element w.r.t. to partial order) and 1 denotes the top element (mean the element is largest element w.r.t. to partial order) of G. Then G is a partially ordered set.*

Define multiplication $*$ on G as shown in Table 1 , where a, b, c to label the elements in the po-NETG and the multiplication $*$ among these elements.

Table 1. Multiplication $*$ on G.

*	0	a	b	c	1
0	0	0	0	0	0
a	0	b	c	a	1
b	0	c	a	b	1
c	0	a	b	c	1
1	0	1	1	1	1

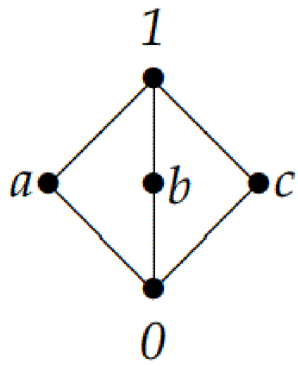

Figure 1. Hasse diagram.

We can verify that $(G, *)$ is a WCNETG. Moreover,

$$neut(0) = 0,\ \{anti(0)\} = \{0,\ a,\ b,\ c,\ 1\},\ 0^{-1} = 0;$$

$$neut(a) = c,\ \{anti(a)\} = \{b\},\ a^{-1} = b;$$

$$neut(b) = c,\ \{anti(b)\} = \{a\},\ b^{-1} = a;$$

$$neut(c) = c,\ \{anti(c)\} = \{c\},\ c^{-1} = c;$$

$$neut(1) = 1,\ \{anti(1)\} = \{a,\ b,\ c,\ 1\},\ 1^{-1} = 1.$$

It is easy to see that the partial order shown in Fig.1 is compatible with multiplication $*$. Hence, $(G, *, \leq)$ is a po-WCNETG.

Definition 6. *If $(G, *, \leq)$ is a po-NETG, then $a \in G$ is said to be a positive element if $neut(a) \leq a$; and a negative element if $a \leq neut(a)$. The subset P_G of all positive elements of G is called the positive cone of G, and the subset N_G of all negative elements the negative cone.*

Remark 2. *By Proposition 1, $\forall\ a \in G$, $neut(a) \in P_G \cap N_G$, so $P_G \cap N_G \neq \emptyset$.*

Lemma 1. *Let $(G, *)$ be an NETG. Then $\forall\ a \in G$,*

$$[neut(a)]^{-1} = neut(a) = neut(a^{-1}).$$

Proof. Let $a \in G$. Then

$$\begin{aligned} [neut(a)]^{-1} &= anti(neut(a)) * neut(neut(a)) \\ &= anti(neut(a)) * neut(a) \\ &= neut(neut(a)) \\ &= neut(a). \end{aligned}$$

On the other hand, by Proposition 2(3), we have $neut(a^{-1}) = neut(anti(a) * neut(a)) = neut(a)$. $\square$

Remark 3. *If G is a po-NETG and $P \subseteq G$, we shall use the notation*

$$P^{-1} = \{a^{-1} : a \in P\}.$$

Proposition 6. *Let $(G, *, \leq)$ be a po-NETG. Then $P_G \cap P_G^{-1} = \{a \in G : a = neut(a) = a^{-1}\}$.*

Proof. ($\Longrightarrow$) Let $a \in G$. By Proposition 1 and Lemma 1, we have

$$neut(a) \in \{a \in G : a = neut(a) = a^{-1}\},$$

so $\{a \in G : a = neut(a) = a^{-1}\} \neq \emptyset$. By Lemma 1, it is clear that

$$\{a \in G : a = neut(a) = a^{-1}\} \subseteq P_G \cap P_G^{-1}.$$

($\Longleftarrow$) Let $b \in P_G \cap P_G^{-1}$, then $neut(b) \leq b$ and $\exists\, c \in P_G$ such that $b = c^{-1}$, so

$$b = c^{-1} = anti(c) * neut(c) \leq anti(c) * c = neut(c) = neut(b^{-1}) = neut(b),$$

that is, $b \leq neut(b)$, whence $b = neut(b)$. Hence,

$$c = b^{-1} = [neut(b)]^{-1} = neut(b) = b.$$

Then we can conclude that $b \in \{a \in G : a = neut(a) = a^{-1}\}$, and so

$$P_G \cap P_G^{-1} \subseteq \{a \in G : a = neut(a) = a^{-1}\}.$$

Thus, $P_G \cap P_G^{-1} = \{a \in G : a = neut(a) = a^{-1}\}$. $\square$

Remark 4. *If $(G, *, \leq)$ is a po-NETG and $P \subseteq G$, then we shall use the notation*

$$P^2 = \{a * b : a,\ b \in P\}.$$

Proposition 7. *(1) If $(G, *, \leq)$ is a po-NETG, then $P_G \subseteq P_G^2$.*
(2) If $(G, *, \leq)$ is a po-WCNETG, then $P_G = P_G^2$.

Proof. (1) If $(G, *, \leq)$ is a po-NETG, then $\forall\, a \in P_G$, by $neut(a) \in P_G$, we have $a = a * neut(a) \in P_G^2$, and so $P_G \subseteq P_G^2$.

(2) If $(G, *, \leq)$ is a po-WCNETG, then $\forall\, a \in P_G$, $\forall b \in P_G$, by Propositions 3 and 4, we have $neut(a * b) = neut(b) * neut(a) = neut(a) * neut(b) \leq a * b$, and so $a * b \in P_G$, thus $P_G^2 \subseteq P_G$. Consequently, $P_G = P_G^2$. $\square$

Proposition 8. *Let $(G, *, \leq)$ be a po-WCNETG. Then $\forall\, a \in G$, $aP_Ga^{-1} \subseteq P_G$.*

Proof. Let $a \in G$ and $b \in P_G$, then by Propositions 3 and 4, we have $neut(a * b * a^{-1}) = neut(a^{-1}) * neut(a * b) = neut(a * b) * neut(a^{-1}) = [neut(b) * neut(a)] * neut(a^{-1}) = neut(b) * [neut(a) * neut(a^{-1})] = neut(b) * neut(a^{-1} * a) = neut(b) * neut(neut(a)) = neut(b) * neut(a) = neut(b) * (a * a^{-1}) = [neut(b) * a] * a^{-1} = [a * neut(b)] * a^{-1} \leq a * b * a^{-1}$, thus $aba^{-1} \in P_G$. Therefore, $aP_Ga^{-1} \subseteq P_G$. $\square$

Lemma 2. *Let $(G, *)$ be a WCNETG. Then $\forall\, a \in G$, $\forall b \in G$, $(a * b)^{-1} = b^{-1} * a^{-1}$.*

Proof. We know $a * b$ is an element of $G\ \forall\ a \in G,\ \forall b \in G$ and by Proposition 4, we have $anti(b) * anti(a) \in \{anti(a * b)\}$. Then using Propositions 1, 5 and Note 1 we get the following identities:

$$\begin{aligned}
b^{-1} * a^{-1} &= [anti(b) * neut(b)] * [anti(a) * neut(a)] \\
&= anti(b) * [neut(b) * anti(a)] * neut(a) && \text{(Because the multiplication } * \text{ is associative)} \\
&= anti(b) * [anti(a) * neut(b)] * neut(a) && \text{(Because } G \text{ is a WCNETG)} \\
&= [anti(b) * anti(a)] * [neut(b) * neut(a)] && \text{(Because the multiplication } * \text{ is associative)} \\
&= [anti(b) * anti(a)] * neut(a * b) && \text{(By Proposition 3)} \\
&= (a * b)^{-1}. \quad \square
\end{aligned}$$

Lemma 3. *Let $(G, *, \leq)$ be a po-NETG. Then $P_G = P_N^{-1}$ and $P_G^{-1} = P_N$.*

Proof. Let $a \in G$. If $a \in P_G$, then $neut(a) \leq a$, it follows by Lemma 1 that $a^{-1} = neut(a^{-1}) * a^{-1} = neut(a) * a^{-1} \leq a * a^{-1} = neut(a) = neut(a^{-1})$, and so $a^{-1} \in P_N$, whence $a = (a^{-1})^{-1} \in P_N^{-1}$. Hence, $P_G \subseteq P_N^{-1}$. Similarly, we can prove that if $a \in P_N$ then $a^{-1} \in P_G$, so $P_N^{-1} \subseteq P_G$. Consequently, $P_G = P_N^{-1}$. Similarly, $P_G^{-1} = P_N$. $\square$

Definition 7. *Let $(G, *)$ be a WCNETG. If $\forall a \in G,\ \forall b \in G,\ \forall c \in G,\ a * neut(c) = b * neut(c)$ implies $a = b$, then we say G satisfies neutrosophic cancellation law.*

Lemma 4. *Let $(G, *)$ be a WCNETG satisfying neutrosophic cancellation law and $P \subseteq G$ satisfy $\forall\ a \in P,\ a * a = a$. Then $\forall\ a \in G,\ \forall\ b \in G,\ a * neut(b) \in P$ implies $neut(a) = a = a^{-1}$.*

Proof. If $a * neut(b) \in P$, then $a * neut(b) = (a * neut(b)) * (a * neut(b)) = (a * a) * neut(b)$, and so $a * a = a$, whence $neut(a) = a\ \forall\ a \in G,\ \forall\ b \in G$. Then by Lemma 1, we get $a^{-1} = [neut(a)]^{-1} = neut(a) = a$. $\square$

Proposition 9. *Let $(G, *)$ be a WCNETG satisfying neutrosophic cancellation law and $P \subseteq G$ satisfy the following conditions:*

(1) $P^2 \subseteq P$;
(2) $P \cap P^{-1} = \{a \in G : neut(a) = a = a^{-1}\}$;
(3) $\forall\ a \in P,\ a * a = a$;
(4) $\forall\ a \in G,\ aPa^{-1} \subseteq P$,

then a compatible partial order on G exists such that P is the positive cone of G relative to it. Moreover, G is a chain with respect to this partial order if and only if $P \cup P^{-1} = G$.

Proof. Define the relation $\leq$ on G by

$$a \leq b \Leftrightarrow b * a^{-1} \in P.$$

By Proposition 1 and Lemma 1, we have $\forall\ a \in G,\ neut(a) \in P \cap P^{-1} \subseteq P$, and so $\leq$ is reflexive on G obviously.
If now $a \leq b$ and $b \leq a$, then $b * a^{-1} \in P$ and $a * b^{-1} \in P$. Since by Lemma 2 we know that

$$(a * b^{-1})^{-1} = (b^{-1})^{-1} * a^{-1} = b * a^{-1},$$

we conclude

$$b * a^{-1} \in P \cap P^{-1}.$$

It follows by (2) that $b * a^{-1} = neut(b * a^{-1})$. However, by Proposition 4 and Lemma 1,

$$neut(b * a^{-1}) = neut(a^{-1}) * neut(b) = neut(a) * neut(b),$$

thus

$b * neut(a) = b * a^{-1} * a = neut(b * a^{-1}) * a = [neut(a) * neut(b)] * a = neut(a) * [a * neut(b)] = [neut(a) * a] * neut(b) = a * neut(b)$, that is, $b * neut(a) = a * neut(b)$.

However, by Proposition 3, we have

$$b * neut(a) = neut(b) * neut(a) * b = neut(a * b) * b,$$

and similarly,

$$a * neut(b) = neut(a) * neut(b) * a = [neut(b) * neut(a)] * a = neut(a * b) * a,$$

therefore,

$$neut(a * b) * b = neut(a * b) * a,$$

and by neutrosophic cancellation law, consequently $a = b$. Hence, $\leq$ is anti-symmetric.

To prove that $\leq$ is transitive, let $a \leq b$ and $b \leq c$. Then

$$b * a^{-1} \in P \text{ and } c * b^{-1} \in P.$$

It follows by (1) that

$$P \supseteq P^2 \ni (c * b^{-1}) * (b * a^{-1}) = c * (b^{-1} * b) * a^{-1} = c * neut(b) * a^{-1} = (c * a^{-1}) * neut(b).$$

By (3) and Lemma 4, we have

$$neut(c * a^{-1}) = c * a^{-1} = (c * a^{-1})^{-1},$$

and so

$$c * a^{-1} \in P \cap P^{-1} \subseteq P,$$

that is, $c * a^{-1} \in P$. Thus, $a \leq c$. Therefore, $\leq$ is a partial order on G.

To see that it is compatible, let $x \leq y$. Then $y * x^{-1} \in P$ and it follows by (1) and (4) that, for every $a \in G$,

$$(a * y) * (a * x)^{-1} = (a * y) * (x^{-1} * a^{-1}) = a * (y * x^{-1}) * a^{-1} \in P,$$

$$(y * a) * (x * a)^{-1} = y * (a * a^{-1}) * x^{-1} = y * neut(a) * x^{-1} = (y * x^{-1}) * neut(a) \in P^2 \subseteq P,$$

which shows that

$$a * x \leq a * y \text{ and } x * a \leq y * a.$$

It follows that $\leq$ is compatible.

Finally, note that $\forall\, a \in G$,

$$neut(a) \leq a \Leftrightarrow a * [neut(a)]^{-1} \in P \Leftrightarrow a * neut(a) \in P \Leftrightarrow a \in P,$$

so P is the associated positive cone. Suppose now that $(G, \leq)$ is a chain, then for every $a \in G$, we have either

$$neut(a) \leq a \text{ or } a \leq neut(a).$$

It follows by Lemma 3 that

$$a \in P \text{ or } a \in P^{-1}.$$

Thus $G = P \cup P^{-1}$. Conversely, if $G = P \cup P^{-1}$, then for all $a,\ b \in G$, we have

$$a * b^{-1} \in P \text{ or } a * b^{-1} \in P^{-1},$$

that is,

$$a * b^{-1} \in P \text{ or } b * a^{-1} = (a * b^{-1})^{-1} \in P.$$

Hence, we have either $b \leq a$ or $a \leq b$. Therefore, $(G, \leq)$ is a chain. $\square$

By the following example, we clarify the above proposition as:

Example 2. *Let $G = \{a,\ b,\ c\}$. Define multiplication $*$ on G as shown in Table 2, where a, b, c to label the elements in the po-NETG and the multiplication $*$ among these elements.*

Table 2. Multiplication $*$ on G.

*	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

It is easy to verify that $(G, *)$ is a WCNETG and $(G, *)$ satisfies neutrosophic cancellation law, in which

$$neut(a) = neut(b) = neut(c) = a,$$

$$\{anti(a)\} = \{a\},\ a^{-1} = a;$$

$$\{anti(b)\} = \{c\},\ b^{-1} = c;$$

$$\{anti(c)\} = \{b\},\ c^{-1} = b.$$

Let $P = \{a\}$, then P satisfies all conditions mentioned in Proposition 9. Define the relation $\leq$ on G by $x \leq y \Leftrightarrow y * x^{-1} \in P$, then $\leq$ is a partial order on G and $(G, \leq)$ is a antichain. Obviously, P is the positive cone of G with respect to this partial order $\leq$.

Proposition 10. *Let $(G, *)$ be a po-WCNETG. Then $\forall\ x \in G,\ \forall y \in G,\ x \leq y$ implies $y * x^{-1} \in P_G$.*

Proof. Let $\forall\ x \in G,\ \forall y \in G$. If $x \leq y$, then $neut(x) = x * x^{-1} \leq y * x^{-1}$, hence, by Proposition 4 and Lemma 1, we have $neut(y * x^{-1}) = neut(x^{-1}) * neut(y) = neut(x) * neut(y) \leq (y * x^{-1}) * neut(y) = neut(y) * (y * x^{-1}) = (neut(y) * y) * x^{-1} = y * x^{-1}$. Thus, $y * x^{-1} \in P_G$. $\square$

4. Homomorphisms and Quotient Sets of po-NETGs

Definition 8. *Let $(G, *, \leq_1)$ and $(T, \cdot, \leq_2)$ be two po-NETGs. The map $f : G \to T$ is called a po-NETG homomorphism of po-NETGs, if f satisfies: $\forall a \in G,\ \forall b \in G$*

(1) $f(a * b) = f(a) \cdot f(b)$;
(2) $a \leq_1 b$ *implies* $f(a) \leq_2 f(b)$.

Proposition 11. *Let $(G, *, \leq_1)$ and $(T, \cdot, \leq_2)$ be two po-NETGs, and let $f : G \to T$ be a po-NETG homomorphism of po-NETGs. The following properties hold:*

(1) $\forall a \in G, f(neut(a)) = neut(f(a))$;
(2) $\forall a \in G, \{f(b) : b \in \{anti(a)\}\} \subseteq \{anti(f(a))\}$, *and if f is bijective, then* $\{f(b) : b \in \{anti(a)\}\} = \{anti(f(a))\}$;
(3) $\forall a \in G, [f(a)]^{-1} = f(a^{-1})$;
(4) $\forall a \in P_G, f(a) \in P_T$;
(5) $\forall a \in N_G, f(a) \in N_T$.

Proof.

(1) $\forall\, a \in G,\ \forall\, b \in \{anti(a)\}$, since

$$f(a)\cdot f(neut(a)) = f(a * neut(a)) = f(a) = f(neut(a) * a) = f(neut(a))\cdot f(a),$$

$$f(a)\cdot f(b) = f(a*b) = f(neut(a)) = f(b*a) = f(b)\cdot f(a),$$

then we obtain $f(neut(a)) = neut(f(a))$.

(2) From the proof of (1), we can get that

$$\forall\, a \in G,\ \forall\, b \in \{anti(a)\},\ f(b) \in \{anti(f(a))\},$$

and so

$$\{f(b) : b \in \{anti(a)\}\} \subseteq \{anti(f(a))\}.$$

If f is bijective, then $\forall\, d \in \{anti(f(a))\}$, $\exists\, c \in G$ such that $f(c) = d$. Since

$$f(c*a) = f(c)\cdot f(a) = d\cdot f(a) = neut(f(a)) = f(neut(a)),$$

we have $c * a = neut(a)$. Similarly, we can get $a * c = neut(a)$. Thus, $c \in anti(a)$ and so

$$d = f(c) \in \{f(b) : b \in \{anti(a)\}\}.$$

By the arbitrariness of d, we have

$$\{anti(f(a))\} \subseteq \{f(b) : b \in \{anti(a)\}\}.$$

Then,

$$\{f(b) : b \in \{anti(a)\}\} = \{anti(f(a))\}.$$

(3) Let $a \in G$ and $b \in \{anti(a)\}$. By (2), $f(b) \in \{anti(f(a))\}$. Then by (1), we have

$$[f(a)]^{-1} = anti(f(a))\cdot neut(f(a)) = f(b)\cdot f(neut(a)) = f(b * neut(a)) = f(a^{-1}).$$

(4) Since $\forall\, a \in P_G,\ neut(a) \leq_1 a$, we have $neut(f(a)) = f(neut(a)) \leq_2 f(a)$, and so $f(a) \in P_T$.

(5) It is similar to (4).

□

Definition 9. *Let $(G, *, \leq)$ be a po-NETG and θ be an equivalence relation on G. If θ satisfies*

$$\forall a \in G,\ \forall b \in G,\ \forall c \in G,\ \forall d \in G,\ (a, b) \in \theta \ \&\ (c, d) \in \theta \Rightarrow (a * c, b * d) \in \theta,$$

then θ is called a congruence on G.

Obviously, $\theta_1 = \{(a, a) : a \in G\}$ and $\theta_2 = \{(a, b) : \forall\, a,\ b \in G\}$ are both congruences on G, and they are called identity congruence on G and pure congruence on G, respectively.

Definition 10. *Let $(G, *, \leq)$ be a po-NETG and θ be a congruence on G. A multiplication $\circ$ on the quotient set $G/\theta = \{[a]_\theta : a \in G\}$ is defined by*

$$[a]_\theta \circ [b]_\theta = [a * b]_\theta.$$

Proposition 12. *Let a relation $\preceq$ on $(G/\theta, \circ)$ be defined by*

$$\forall\ [a]_\theta \in G/\theta,\ \forall [b]_\theta \in G/\theta,\ [a]_\theta \preceq [b]_\theta \Leftrightarrow a \leq b.$$

Then, $(G/\theta, \circ, \preceq)$ is a po-NETG.

Proof. We can verify that $\circ$ is associative. Let $[a]_\theta \in G/\theta$ (see Definition 10), since

$$[neut(a)]_\theta \circ [a]_\theta = [neut(a) * a]_\theta = [a]_\theta = [a * neut(a)]_\theta = [a]_\theta \circ [neut(a)]_\theta,$$

and

$$[anti(a)]_\theta \circ [a]_\theta = [anti(a) * a]_\theta = [neut(a)]_\theta = [a * anti(a)]_\theta = [a]_\theta \circ [anti(a)]_\theta,$$

we conclude that $(G/\theta, \circ)$ is a NETG, in which $\forall\ [a]_\theta \in G/\theta,\ neut([a]_\theta) = [neut(a)]_\theta$ and $[anti(a)]_\theta \in \{anti([a]_\theta)\}$. Then it is easy to see that $\preceq$ is a partial order on $(G/\theta, \circ)$. Moreover, $\forall\ [a]_\theta \in G/\theta,\ \forall [b]_\theta \in G/\theta,\ \forall [c]_\theta \in G/\theta$, if $[a]_\theta \preceq [b]_\theta$, then $a \leq b$, so we have $a * c \leq b * c$, and $c * a \leq c * b$. Thus,

$$[a]_\theta \circ [c]_\theta = [a * c]_\theta \preceq [b * c]_\theta = [b]_\theta \circ [c]_\theta$$

and

$$[c]_\theta \circ [a]_\theta = [c * a]_\theta \preceq [c * b]_\theta = [c]_\theta \circ [b]_\theta.$$

Thus, $(G/\theta, \circ, \preceq)$ is a po-NETG. □

In the following, we give an example to illustrate Proposition 12.

Example 3. *Consider the po-NETG $(G, *, \leq)$ is given in Example 1. Now we define a relation θ on G by*

$$\theta = \{(0,0),\ (a,a),\ (b,b),\ (c,c),\ (1,1),\ (a,b),\ (b,a),\ (a,c),\ (c,a),\ (b,c),\ (c,b)\}.$$

Then we can verify that θ is a congruence on G with the following blocks:

$$[0]_\theta = \{0\},\ [a]_\theta = \{a,\ b,\ c\},\ [1]_\theta = \{1\}.$$

So the quotient set $G/\theta = \{[0]_\theta,\ [a]_\theta,\ [1]_\theta\}$. By Proposition 12, we know $(G/\theta, \circ, \preceq)$ is a po-NETG, in which $neut([0]_\theta) = [0]_\theta,\ neut([a]_\theta) = [c]_\theta = [a]_\theta,\ neut([1]_\theta) = [1]_\theta,\ \{anti([0]_\theta)\} = \{[0]_\theta,\ [a]_\theta,\ [1]_\theta\},\ \{anti([a]_\theta)\} = \{[a]_\theta\},\ \{anti([1]_\theta)\} = \{[a]_\theta,\ [1]_\theta\}$, and then G/θ is a chain, because $[0]_\theta \preceq [a]_\theta \preceq [1]_\theta$.

Proposition 13. *Let $(G, *, \leq)$ be a po-NETG and θ be a congruence on G. Then the natural mapping $\natural_\theta : (G, *, \leq) \to (G/_\theta, \circ, \preceq)$ given by $\natural_\theta(a) = [a]_\theta$ is a po-NETG homomorphism of po-NETGs.*

Proof. As $\natural_\theta(a * b) = [a * b]_\theta = [a]_\theta \circ [b]_\theta = \natural_\theta(a) \circ \natural_\theta(b)\ \forall\ a \in G,\ \forall b \in G$. If $a \leq b$, then $[a]_\theta \preceq [b]_\theta$ which implies $\natural_\theta(a) \preceq \natural_\theta(b)$. Thus, the natural mapping $\natural_\theta : (G, *, \leq) \to (G/_\theta, \circ, \preceq)$ is a po-NETG homomorphism of po-NETGs. □

Next, we give an example to explain Proposition 13.

Example 4. *From Example 3, we consider the natural mapping $\natural_\theta : (G, *, \leq) \to (G/_\theta, \circ, \preceq)$. Thus, $\natural_\theta(0) = [0]_\theta,\ \natural_\theta(a) = \natural_\theta(b) = \natural_\theta(c) = [a]_\theta,\ \natural_\theta(1) = [1]_\theta$. It is easy to verify that $\natural_\theta$ is a po-NETG homomorphism of po-NETGs.*

Proposition 14. *Let $(G, *, \leq_1)$ and $(T, \cdot, \leq_2)$be two po-NETGs and $f : (G, *, \leq_1) \to (T, \cdot, \leq_2)$ be a po-NETG homomorphism of po-NETGs. We shall use the notation*

$$Kerf = \{(a, b) \in G \times G : f(a) = f(b)\},$$

then we can get the following properties:

(1) $Kerf$ is a congruence on G;
(2) f is a injective po-NETG homomorphism of po-NETGs if and only if $kerf$ is an identity congruence on G;
(3) There exists an injective po-NETG homomorphism of po-NETGs $g : (G/Kerf, \circ, \preceq) \to (T, \cdot, \leq_2)$ such that $f = g \circ \natural_{Kerf}$.

Proof.

(1) Obviously, $Kerf$ is an equivalence relation on G. Let $\forall\ a \in G, \forall\ b \in G, \forall\ c \in G, \forall\ d \in G$, if $(a, b) \in Kerf$ and $(c, d) \in Kerf$, then $f(a) = f(b)$ and $f(c) = f(d)$. Since f is a po-NETG homomorphism of po-NETGs, we have $f(a * c) = f(a) \cdot f(c) = f(b) \cdot f(d) = f(b * d)$, and so $(a * c, b * d) \in Kerf$. Thus, $Kerf$ is a congruence on G.

(2) If f is an injective po-NETG homomorphism of po-NETGs and if $(a, b) \in kerf$ then $f(a) = f(b)$. Therefore, we get $a = b$. Hence, by the arbitrariness of (a, b), we obtain $kerf$ is an identity congruence on G.

Conversely, suppose that $kerf$ is an identity congruence on G. $\forall\ a \in G, \forall\ b \in G$, if $f(a) = f(b)$, then $(a, b) \in kerf$, so $a = b$. Therefore, f is an injective po-NETG homomorphism of po-NETGs.

(3) We define a map $g : G/Kerf \to T$ by $\forall\ [a]_{Kerf} \in G/Kerf$, $g([a]_{Kerf}) = f(a)$, then g is injective. $\forall\ [a]_{Kerf}, [b]_{Kerf} \in G/Kerf$, we have $g([a]_{Kerf} \circ [b]_{Kerf}) = g([a * b]_{Kerf}) = f(a * b) = f(a) \cdot f(b) = g([a]_{Kerf}) \cdot g([b]_{Kerf})$, and if $[a]_{Kerf} \preceq [b]_{Kerf}$, then $a \leq_1 b$, thus, $f(a) \leq_2 f(b)$, that is, $g([a]_{Kerf}) \leq_2 g([b]_{Kerf})$. Hence, g is an injective po-NETG homomorphism of po-NETGs.

$$\begin{tikzcd} G \arrow[rr, "f"] \arrow[dr, "\natural_{Kerf}"'] & & T \\ & G/Kerf \arrow[ur, "g"'] & \end{tikzcd}$$

$\forall\ a \in G$, $(g \circ \natural_{Kerf})(a) = g(\natural_{Kerf}(a)) = g([a]_{Kerf}) = f(a)$, that is, $f = g \circ \natural_{Kerf}$.
□

In the following, we present an example to illustrate Proposition 14.

Example 5. *Consider $(G, *, \leq_1)$ be the po-NETG is given in Example 1, in which the partial order $\leq_1$ is the same as the partial order $\leq$ in Example 1. Assume that $T = \{m, n, p, q, r\}$ be a bounded lattice with a partial order $\leq_2$ with the Hasse diagram shown as in Figure 2 whose multiplication $\cdot$ is defined as $\wedge$.*

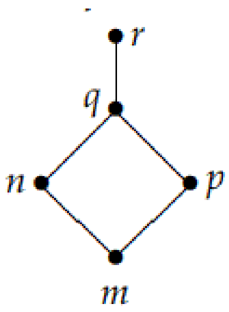

Figure 2. Hasse diagram.

We can verify that $(T, \cdot, \leq_2)$ is a po-NETG, in which $\forall\ x \in T$, $neut(x) = x$, $\{anti(m)\} = \{m,\ n,\ p,\ q,\ r\}$, $\{anti(n)\} = \{n,\ q,\ r\}$, $\{anti(p)\} = \{p,\ q,\ r\}$, $\{anti(q)\} = \{q,\ r\}$, $\{anti(r)\} = \{r\}$. Now, we define a map $f : G \to T$ by $f(0) = m$, $f(a) = f(b) = f(c) = f(1) = r$, then f is a po-NETG homomorphism of po-NETGs, and $Kerf = \{(0,0), (a,a), (b,b), (c,c), (1,1), (a,b), (a,c), (a,1), (b,a), (b,c), (b,1), (c,a), (c,b), (c,1), (1,a), (1,b), (1,c)\}$. Obviously, $Kerf$ is a congruence on G. f is not injective, and of course, $kerf$ is not an identity congruence on G.

5. Conclusions

In this paper, inspired by the research work in algebraic structures equipped with a partial order, we proposed the concepts of po-NETGs, deeply studied the relationships between po-NETGs and their positive cones, and characterized the positive cone of a WCNETG after defining a partial order relation on it. Moreover, we found that the quotient set of a po-NETG can construct another po-NETG by defining a special multiplication and a partial order on the quotient set, and we also achieved the interrelation of homomorphisms and congruences of po-NETGs. All these results are useful for exploring the structure characterization (for example, category properties) of po-NETGs. As a direction of future research, we will consider the application of the fuzzy set theory and the rough set theory to the research of algebraic structure of po-NETGs. Furthermore, we will discuss the relation between the homomorphisms and congruences of po-NETG and the morphisms of ordered lattice ringoids [30]. Finally, in the next paper, we will study sub-structures of po-NETGs and we give some examples using constructions such as central extensions or direct products related to sub-structures of po-NETGs.

Author Contributions: The authors contributed equally to writing this article. All authors read and approved the final manuscript.

Funding: This work was supported by the National Natural Science Foundation of China (Grant no. 11501435) and supported by the PhD research start-up fund (Grant no. BS1529).

Acknowledgments: Authors thank the editors and the anonymous reviewers for their insightful comments which improved the quality of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

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