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New Similarity and Entropy Measures of Interval Neutrosophic Sets with Applications in Multi-Attribute Decision-Making

Han Yang, Xiaoman Wang and Keyun Qin *

College of Mathematics, Southwest Jiaotong University, Chengdu 610031, China; yanghan@163.net (H.Y.); xiaomanwang@aliyun.com (X.W.)

* Correspondence: keyunqin@263.net

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Abstract: Information measures play an important role in the interval neutrosophic sets (INS) theory. The main purpose of this paper is to study the similarity and entropy of INS and its application in multi-attribute decision-making. We propose a new inclusion relation between interval neutrosophic sets where the importance of the three membership functions may be different. Then, we propose the axiomatic definitions of the similarity measure and entropy of the interval neutrosophic set (INS) based on the new inclusion relation. Based on the Hamming distance, cosine function and cotangent function, some new similarity measures and entropies of INS are constructed. Finally, based on the new similarity and entropy, we propose a multi-attribute decision-making method and illustrate that these new similarities and entropies are reasonable and effective.

Keywords: interval neutrosophic sets; inclusion relationship; similarity measure; entropy; multi-attribute decision-making

1. Introduction

Zadeh [1,2] put forward the theory of fuzzy sets in 1965, which is an effective method to deal with fuzzy information, but only limited to the truth-membership function. In actual decision-making, because of the fuzziness of people's thinking and the complexity of objective things, it is difficult for decision-makers to evaluate only through truth-membership function. On this basis, Atanassov [3] proposed an intuitionistic fuzzy set, and added a falsity-membership function to the fuzzy set to represent uncertain information. That is to say, the intuitionistic fuzzy concentration has both truth-membership function $T_A(x)$ and falsity-membership function $F_A(x)$, and $T_A(x)$, $F_A(x) \in [0,1]$, $0 \leq T_A(x) + F_A(x) \leq 1$. However, intuitionistic fuzzy sets can only solve incomplete information, but can not deal with the uncertain information and inconsistent information in practical decision-making problems. For example, when voting, some agreed, some opposed and some abstained. Therefore, Smarandache [4] proposed the concept of neutrosophic sets. On the basis of the intuitionistic fuzzy set, a neutrosophic set is characterized independently by the truth-membership function $T_A(x)$, the falsity-membership function $F_A(x)$, and the indeterminacy-membership function $I_A(x)$. Wang et al. [5,6] proposed the concept of single-valued neutrosophic sets (SVNS) and interval neutrosophic sets (INS), which are the subclasses of neutrosophic sets, and the set-theoretic operators and various properties of SVNSs and INSs are given. In the interval neutrosophic sets, the truth-membership function $T_A(x)$, the indeterminacy-membership function $I_A(x)$, and the falsity-membership function $F_A(x)$ are all expressed in the form of interval numbers. Then, some researchers put forward some algorithms of SVNSs and INSs, and applied them to decision-making problems. The correlation coefficients and weighted correlation coefficients of single-valued neutrosophic sets are proposed by Ye [7,8]. It is proved that the cosine similarity under singular concentration is a special case of the correlation coefficients. Furthermore, a single-valued neutrosophic cross-entropy measurement method is proposed and applied to multi-attribute decision-making in single-valued neutrosophic environment. Chi and Liu [9] applied a TOPSIS (The Order Performance technique based on Similarity to Ideal Solution) method to classify interval neutrosophic multi-attribute decision-making problems to alternative levels. Ye [10] proposed the Hamming distance and the Euclidean distance in INSs and defined similarity measure based on distance, and applied them to multi-attribute decision-making with interval neutrosophic information. In addition, Ye [11] proposed the definition of a simplified neutrosophic set (SNS). It is a subclass of the neutrosophic set and includes an SVNS and an INS. Ye also proposed some aggregation operators, including a simplified neutrosophic weighted arithmetic average operator and a simplified neutrosophic weighted geometric average operator. Based on the two aggregation operators and cosine similarity measure for SNSs, a multicriteria decision-making method is established. Zhang et al. [12] further proposed the comparison rules on the basis of truth-membership function, indeterminacy-membership function, and falsity-membership function of interval neutrosophic number (INN). Based on the possibility degree of two interval numbers, a comparison method was proposed. Then, the interval neutrosophic number weighted averaging (INNWA) operator and interval neutrosophic number weighted geometry (INNWG) operator were developed and applied to interval neutrosophic multi-attribute decision-making problems.

In 1972, De Luca and Termin gave the axiomatization definition of fuzzy entropy to characterize the degree of uncertainty [13]. Similarity is mainly used to estimate the degree of similarity between two objects. Wang [14] proposes the definition of similarity based on distance. Ye [7,8,10,11,15,16], Ridvan Sahin et al. [17] studied the similarity and entropy of the interval neutrosophic sets from different angles. Zhang et al. [18,19] proposed a new inclusion relationship of single-valued neutrosophic sets (called type-3 inclusion relations), and gave the algebraic structure of the singular-valued neutrosophic set corresponding to the type-3 inclusion relationship. Based on the third inclusion relationship, Keyun Qin [20] proposed new similarity and entropy.

In this paper, we first give the definitions of the neutrosophic sets, interval neutrosophic sets, then define the new inclusion relationship of the interval neutrosophic sets, and then give the new similarity and entropy based on the new inclusion relationship. Then, apply it to multi-attribute decisions. In Section 2, we introduced the related concepts of the neutrosophic sets and redefined the inclusion relationship. In Section 3, we introduced the similarity and entropy of interval neutrosophic value, and based on the new inclusion relationship, we reestablish a similarity and entropy. Then, we extend the similarity and entropy of the interval neutrosophic value to the interval neutrosophic sets. In Section 4, we give an example of applying new similarities and entropies to multi-attribute decisions, and compare with other methods, the results show that the proposed similarity and entropy are reasonable and effective.

2. Preliminaries

In this section, we recall some fundamental notions and properties related to an interval neutrosophic set.

Definition 1. *(See [4]) Let X be an object set and x be an element in the object set X. A neutrosophic set A of X can be expressed as*

$$A = \{[x, (T_A(x), I_A(x), F_A(x))] | x \in X\},$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]0^-, 1^+[$, which represent truth-membership, indeterminacy-membership, and falsity-membership, respectively, $0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

In order to easily apply a neutrosophic set theory to science and engineering, Wang et al. [5] presented the concept of interval neutrosophic set (INS) as follows.

Definition 2. *(See [5]) Let X be an object set, an interval neutrosophic set A of X can be expressed as*

$$A = \{[x, (T_A(x), I_A(x), F_A(x))] | x \in X\},$$

where, for each $x \in X$, $T_A(x)$, $I_A(x)$ and $F_A(x)$ respectively represent truth-membership, indeterminacy-membership, and falsity-membership, which are sub-sets belonging to [0, 1]. In addition, $((T_A(x), I_A(x), F_A(x)))$ is the interval neutrosophic value.

Definition 3. *(See [5]) The complement of A is defined as $A^c = \{[x, (T_{A^c}(x), I_{A^c}(x), F_{A^c}(x)] | x \in X\}$, among them $T_{A^c} = F_A(x) = [F_A^L(x), F_A^U(x)]$, $I_{A^c}(x) = [1 - I_A^U(x), 1 - I_A^L(x)]$, $F_{A^c} = T_A(x) = [T_A^L(x), T_A^U(x)]$.*

For the inclusion relation of single-valued neutrosophic sets, an original definition is proposed by Smarandache. It is called type-1 inclusion relation in [18,19]. Another one is called type-2 inclusion relation. Similarly, in the interval neutrosophic set, an original definition is proposed by Smarandache, we call it type-1 inclusion relation, and denoted by $\subseteq_1$.

Definition 4. *(See [5]) let A, B be the two interval neutrosophic sets, $A \subseteq_1 B$ if and only if $T_A^L(x) \le T_B^L(x)$, $T_A^U(x) \le T_B^U(x)$, $I_A^L(x) \ge I_B^L(x)$, $I_A^U(x) \ge I_B^U(x)$, $F_A^L(x) \ge F_B^L(x)$, $F_A^U(x) \ge F_B^U(x)$.*

Definition 5. *Suppose that intervals $[\alpha_1, \beta_1]$, $[\alpha_2, \beta_2] \in [0, 1]$, then*

(1) $[\alpha_1, \beta_1] \le [\alpha_2, \beta_2]$, *iff* $\alpha_1 \le \alpha_2$ *and* $\beta_1 \le \beta_2$;

(2) $[\alpha_1, \beta_1] < [\alpha_2, \beta_2]$, *iff* $[\alpha_1, \beta_1] \le [\alpha_2, \beta_2]$ *and* $\alpha_1 < \alpha_2$ *(or* $\beta_1 < \beta_2$*).*

In Definition 4, truth-membership, indeterminacy-membership, and falsity-membership are equally important, but, in some cases, people tend to pay more attention to true membership and false membership, so Zhang et al. [19] proposed a new kind of inclusion relation and examined the basic properties of the new kind of inclusion relation.

Definition 6. *(See [19]) Let A and B be two neutrosophic sets in the universe X. The type-3 inclusion relation is defined as follows: $A \subseteq_2 B$ if and only if $x \in X, (T_A(x) < T_B(x), F_A(x) < F_B(x))$, or $(T_A(x) = T_B(x), F_A(x) \ge F_B(x))$, or $(T_A(x) = T_B(x), F_A(x) = F_B(x) and I_A(x) > I_B(x))$.*

Based on [19], we define a new inclusion relation called type-2 in the same way, and denoted by $\subseteq_2$.

Definition 7. *Let $x = ([x_1^L, x_1^U], [x_2^L, x_2^U], [x_3^L, x_3^U])$ and $y = ([y_1^L, y_1^U], [y_2^L, y_2^U], [y_3^L, y_3^U])$ be the interval neutrosophic values. $x \le_2 y$ if and only if one of the following three conditions is true:*

(1) $[x_1^L, x_1^U] < [y_1^L, y_1^U]$ *and* $[x_3^L, x_3^U] \ge [y_3^L, y_3^U]$;
(2) $[x_1^L, x_1^U] = [y_1^L, y_1^U]$ *and* $[x_3^L, x_3^U] > [y_3^L, y_3^U]$;
(3) $[x_1^L, x_1^U] = [y_1^L, y_1^U]$ *and* $[x_3^L, x_3^U] = [y_3^L, y_3^U]$ *and* $[x_2^L, x_2^U] \ge [y_2^L, y_2^U]$.

The inclusion relations $\subseteq_2$ of interval neutrosophic sets are based on it. Let A, B be the two interval neutrosophic sets, $A \subseteq_2 B$ if and only if one of the following three conditions is true:

(1) $[T_A^L(x), T_A^U(x)] < [T_B^L(x), T_B^U(x)]$ *and* $[F_A^L(x), F_A^U(x)] \ge [F_B^L(x), F_B^U(x)]$;
(2) $[T_A^L(x), T_A^U(x)] = [T_B^L(x), T_B^U(x)]$ *and* $[F_A^L(x), F_A^U(x)] > [F_B^L(x), F_B^U(x)]$;
(3) $[T_A^L(x), T_A^U(x)] = [T_B^L(x), T_B^U(x)]$ *and* $[F_A^L(x), F_A^U(x)] = [F_B^L(x), F_B^U(x)]$ *and* $[I_A^L(x), I_A^U(x)] \ge [I_B^L(x), I_B^U(x)]$.

3. Similarity and Entropy of Interval Neutrosophic Sets

3.1. Similarity of Interval Neutrosophic Value

Let $D^* = \{x | x = ([x_1^L, x_1^U], [x_2^L, x_2^U], [x_3^L, x_3^U])\}$ be the set of interval neutrosophic values.

Definition 8. *(See [10]) Letting S: $D^* \times D^* \longrightarrow [0,1]$, the real function S is a similarity between interval neutrosophic values x and y, if S satisfies the following conditions:*

(P1) $0 \le S(x,y) \le 1$;
(P2) $S(x,y) = 1$ *if and only if* $x = y$;
(P3) $S(x,y) = S(y,x)$;
(P4) *For all* $x, y, z \in D^*$, *if* $x \le y \le z$, *then* $S(x,z) \le S(x,y)$, $S(x,z) \le S(y,z)$.

Based on the first inclusion, many similarities have been proposed, such as [10,21]:

$$S_1(x,y) = 1 - \frac{1}{6}(|x_1^L - y_1^L| + |x_1^U - y_1^U| + |x_2^L - y_2^L| + |x_2^U - y_2^U| + |x_3^L - y_3^L| + |x_3^U - y_3^U|);$$
$$S_2(x,y) = 1 - \frac{1}{6}((x_1^L - y_1^L)^2 + (x_1^U - y_1^U)^2 + (x_2^L - y_2^L)^2 + (x_2^U - y_2^U)^2 + (x_3^L - y_3^L)^2 + (x_3^U - y_3^U)^2)^{\frac{1}{2}};$$
$$S_3(x,y) = 1 - \frac{1}{3}\{\max[|x_1^L - y_1^L|, |x_1^U - y_1^U|] + \max[|x_2^L - y_2^L|, |x_2^U - y_2^U|] + \max[|x_3^L - y_3^L|, |x_3^U - y_3^U|]\};$$
$$S_4(x,y) = 1 - \max[\frac{1}{2}(|x_1^L - y_1^L| + |x_1^U - y_1^U|), \frac{1}{2}(|x_2^L - y_2^L|, |x_2^U - y_2^U|) + \frac{1}{2}(|x_3^L - y_3^L| + |x_3^U - y_3^U|)].$$

The above similarity is based on the first inclusion to successfully solve many problems, but it is not suitable to the inclusion relationship in Definition 7. For example, x = ([0.3,0.4], [0.1,0.2], [0.8,0.9]), y = ([0.5,0.6], [0.7,0.9], [0.4,0.5]), z = ([0.5,0.7], [0.4,0.5], [0.2,0.3]), then $S_1(x,y) = \frac{35}{60} \approx 0.5833$, $S_1(y,z) = 0.8$, $S_1(x,z) = \frac{37}{60} \approx 0.6167$, so $S_1(x,y) < S_1(x,z)$. $S_2(x,y) \approx 0.8137$, $S_2(y,z) \approx 0.9028$, $S_2(x,z) \approx 0.8309$, so $S_2(x,y) < S_2(x,z)$. $S_3(x,y) = \frac{17}{30} \approx 0.5667$, $S_3(y,z) = \frac{23}{30} \approx 0.7667$, $S_3(x,z) = 0.6$, so $S_3(x,y) < S_3(x,z)$. $S_4(x,y) = 0.35$, $S_4(y,z) = 0.65$, $S_4(x,z) = 0.4$, so $S_4(x,y) < S_4(x,z)$.

Through the above analysis, we propose a new similarity based on $\subseteq_2$.

Definition 9. *Let* $x = ([x_1^L, x_1^U], [x_2^L, x_2^U], [x_3^L, x_3^U])$, $y = ([y_1^L, y_1^U], [y_2^L, y_2^U], [y_3^L, y_3^U])$, *We define the following similarity:*

$$S(x,y) = \begin{cases} 1 - \frac{|x_2^L - y_2^L| + |x_2^U - y_2^U|}{4}, & [x_1^L, x_1^U] = [y_1^L, y_1^U] \text{ and } [x_3^L, x_3^U] = [y_3^L, y_3^U]. \\ \frac{4 - |x_1^L - y_1^L| - |x_1^U - y_1^U| - |x_3^L - y_3^L| - |x_3^U - y_3^U|}{8}, & \text{else.} \end{cases} \quad (1)$$

Theorem 1. $S(x,y)$ *defined as formula (1) is a similarity between x and y.*

Proof. Let $x = ([x_1^L, x_1^U], [x_2^L, x_2^U], [x_3^L, x_3^U]) \in D^*$, $y = ([y_1^L, y_1^U], [y_2^L, y_2^U], [y_3^L, y_3^U]) \in D^*$, if $[x_1^L, x_1^U] = [y_1^L, y_1^U]$ and $[x_3^L, x_3^U] = [y_3^L, y_3^U]$, then $S(x,y) = 1 - \frac{|x_2^L - y_2^L| + |x_2^U - y_2^U|}{4}$, so $0.5 \le S(x,y) \le 1$; if $[x_1^L, x_1^U] \neq [y_1^L, y_1^U]$ and $[x_3^L, x_3^U] \neq [y_3^L, y_3^U]$, then $S(x,y) = \frac{4 - |x_1^L - y_1^L| - |x_1^U - y_1^U| - |x_3^L - y_3^L| - |x_3^U - y_3^U|}{8}$, so $0 \le S(x,y) \le 0.5$.

(P1) Obviously, $0 \le S(x,y) \le 1$.
(P2) $S(x,y) = 1$, if and only if $S(x,y) = 1 - \frac{|x_2^L - y_2^L| + |x_2^U - y_2^U|}{4}$, if and only if $[x_1^L, x_1^U] = [y_1^L, y_1^U]$, $[x_3^L, x_3^U] = [y_3^L, y_3^U]$ and $[x_2^L, x_2^U] = [y_2^L, y_2^U]$.
(P3) Obviously, $S(x,y) = S(y,x)$.
(P4) Let $x = ([x_1^L, x_1^U], [x_2^L, x_2^U], [x_3^L, x_3^U])$, $y = ([y_1^L, y_1^U], [y_2^L, y_2^U], [y_3^L, y_3^U])$, $z = ([z_1^L, z_1^U], [z_2^L, z_2^U], [z_3^L, z_3^U])$,

and $x \le y \le z$, then

(1) If $[x_1^L, x_1^U] < [y_1^L, y_1^U]$, $[x_3^L, x_3^U] \ge [y_3^L, y_3^U]$ and $[y_1^L, y_1^U] < [z_1^L, z_1^U]$, $[y_3^L, y_3^U] \ge [z_3^L, z_3^U]$, so $S(x,y) = \frac{4 - |x_1^L - y_1^L| - |x_1^U - y_1^U| - |x_3^L - y_3^L| - |x_3^U - y_3^U|}{8}$, $S(y,z) = \frac{4 - |y_1^L - z_1^L| - |y_1^U - z_1^U| - |y_3^L - z_3^L| - |y_3^U - z_3^U|}{8}$, $S(x,z) = \frac{4 - |x_1^L - z_1^L| - |x_1^U - z_1^U| - |x_3^L - z_3^L| - |x_3^U - z_3^U|}{8}$, also because $[x_1^L, x_1^U] < [y_1^L, y_1^U] < [z_1^L, z_1^U]$, $[x_3^L, x_3^U] \ge [y_3^L, y_3^U] \ge [z_3^L, z_3^U]$, so $S(x,z) \le S(x,y)$, $S(x,z) \le S(y,z)$.

(2) If $[x_1^L, x_1^U] < [y_1^L, y_1^U]$, $[x_3^L, x_3^U] \geq [y_3^L, y_3^U]$ and $[y_1^L, y_1^U] = [z_1^L, z_1^U]$, $[y_3^L, y_3^U] > [z_3^L, z_3^U]$, so $S(x,y) = \frac{4-|x_1^L-y_1^L|-|x_1^U-y_1^U|-|x_3^L-y_3^L|-|x_3^U-y_3^U|}{8}$,$S(y,z) = \frac{4-|y_1^L-z_1^L|-|y_1^U-z_1^U|-|y_3^L-z_3^L|-|y_3^U-z_3^U|}{8}$, $S(x,z) = \frac{4-|x_1^L-z_1^L|-|x_1^U-z_1^U|-|x_3^L-z_3^L|-|x_3^U-z_3^U|}{8}$, also because $[x_1^L, x_1^U] < [y_1^L, y_1^U] = [z_1^L, z_1^U]$, $[x_3^L, x_3^U] \geq [y_3^L, y_3^U] > [z_3^L, z_3^U]$, so $S(x,z) \leq S(x,y)$, $S(x,z) \leq S(y,z)$.

(3) If $[x_1^L, x_1^U] < [y_1^L, y_1^U]$, $[x_3^L, x_3^U] \geq [y_3^L, y_3^U]$ and $[y_1^L, y_1^U] = [z_1^L, z_1^U]$, $[y_3^L, y_3^U] = [z_3^L, z_3^U]$, $[y_2^L, y_2^U] \geq [z_2^L, z_2^U]$, so $S(x,y) = \frac{4-|x_1^L-y_1^L|-|x_1^U-y_1^U|-|x_3^L-y_3^L|-|x_3^U-y_3^U|}{8}$,$S(y,z) = 1 - \frac{|y_2^L-z_2^L|+|y_2^U-z_2^U|}{4}$, $S(x,z) = \frac{4-|x_1^L-z_1^L|-|x_1^U-z_1^U|-|x_3^L-z_3^L|-|x_3^U-z_3^U|}{8}$, also because $[x_1^L, x_1^U] < [y_1^L, y_1^U] = [z_1^L, z_1^U]$, $[x_3^L, x_3^U] \geq [y_3^L, y_3^U] = [z_3^L, z_3^U]$, $[y_2^L, y_2^U] \geq [z_2^L, z_2^U]$, so $S(x,z) \leq S(x,y)$, $S(x,z) < 0.5 \leq S(y,z)$.

(4) If $[x_1^L, x_1^U] = [y_1^L, y_1^U]$, $[x_3^L, x_3^U] > [y_3^L, y_3^U]$ and $[y_1^L, y_1^U] < [z_1^L, z_1^U]$, $[y_3^L, y_3^U] \geq [z_3^L, z_3^U]$, so $S(x,y) = \frac{4-|x_1^L-y_1^L|-|x_1^U-y_1^U|-|x_3^L-y_3^L|-|x_3^U-y_3^U|}{8}$,$S(y,z) = \frac{4-|y_1^L-z_1^L|-|y_1^U-z_1^U|-|y_3^L-z_3^L|-|y_3^U-z_3^U|}{8}$, $S(x,z) = \frac{4-|x_1^L-z_1^L|-|x_1^U-z_1^U|-|x_3^L-z_3^L|-|x_3^U-z_3^U|}{8}$, also because $[x_1^L, x_1^U] = [y_1^L, y_1^U] < [z_1^L, z_1^U]$, $[x_3^L, x_3^U] > [y_3^L, y_3^U] \geq [z_3^L, z_3^U]$, so $S(x,z) \leq S(x,y)$, $S(x,z) \leq S(y,z)$.

(5) If $[x_1^L, x_1^U] = [y_1^L, y_1^U]$, $[x_3^L, x_3^U] > [y_3^L, y_3^U]$ and $[y_1^L, y_1^U] = [z_1^L, z_1^U]$, $[y_3^L, y_3^U] > [z_3^L, z_3^U]$, so $S(x,y) = \frac{4-|x_1^L-y_1^L|-|x_1^U-y_1^U|-|x_3^L-y_3^L|-|x_3^U-y_3^U|}{8}$,$S(y,z) = \frac{4-|y_1^L-z_1^L|-|y_1^U-z_1^U|-|y_3^L-z_3^L|-|y_3^U-z_3^U|}{8}$, $S(x,z) = \frac{4-|x_1^L-z_1^L|-|x_1^U-z_1^U|-|x_3^L-z_3^L|-|x_3^U-z_3^U|}{8}$, also because $[x_1^L, x_1^U] = [y_1^L, y_1^U] = [z_1^L, z_1^U]$, $[x_3^L, x_3^U] > [y_3^L, y_3^U] > [z_3^L, z_3^U]$, so $S(x,z) \leq S(x,y)$, $S(x,z) \leq S(y,z)$.

(6) If $[x_1^L, x_1^U] = [y_1^L, y_1^U]$, $[x_3^L, x_3^U] > [y_3^L, y_3^U]$ and $[y_1^L, y_1^U] = [z_1^L, z_1^U]$, $[y_3^L, y_3^U] = [z_3^L, z_3^U]$, $[y_2^L, y_2^U] \geq [z_2^L, z_2^U]$, so $S(x,y) = \frac{4-|x_1^L-y_1^L|-|x_1^U-y_1^U|-|x_3^L-y_3^L|-|x_3^U-y_3^U|}{8}$,$S(y,z) = 1 - \frac{|y_2^L-z_2^L|+|y_2^U-z_2^U|}{4}$, $S(x,z) = \frac{4-|x_1^L-z_1^L|-|x_1^U-z_1^U|-|x_3^L-z_3^L|-|x_3^U-z_3^U|}{8}$, also because $[x_1^L, x_1^U] = [y_1^L, y_1^U] = [z_1^L, z_1^U]$, $[x_3^L, x_3^U] > [y_3^L, y_3^U] = [z_3^L, z_3^U]$, $[y_2^L, y_2^U] \geq [z_2^L, z_2^U]$, so $S(x,z) = S(x,y)$, $S(x,z) < 0.5 \leq S(y,z)$.

(7) If $[x_1^L, x_1^U] = [y_1^L, y_1^U]$, $[x_3^L, x_3^U] = [y_3^L, y_3^U]$, $[x_2^L, x_2^U] \geq [y_2^L, y_2^U]$ and $[y_1^L, y_1^U] < [z_1^L, z_1^U]$, $[y_3^L, y_3^U] \geq [z_3^L, z_3^U]$, so $S(x,y) = 1 - \frac{|x_2^L-y_2^L|+|x_2^U-y_2^U|}{4}$,$S(y,z) = \frac{4-|y_1^L-z_1^L|-|y_1^U-z_1^U|-|y_3^L-z_3^L|-|y_3^U-z_3^U|}{8}$, $S(x,z) = \frac{4-|x_1^L-z_1^L|-|x_1^U-z_1^U|-|x_3^L-z_3^L|-|x_3^U-z_3^U|}{8}$, also because $[x_1^L, x_1^U] = [y_1^L, y_1^U] < [z_1^L, z_1^U]$, $[x_3^L, x_3^U] = [y_3^L, y_3^U] \geq [z_3^L, z_3^U]$,$[x_2^L, x_2^U] \geq [y_2^L, y_2^U]$, so $S(x,z) < 0.5 \leq S(x,y)$, $S(x,z) \leq S(y,z)$.

(8) If $[x_1^L, x_1^U] = [y_1^L, y_1^U]$, $[x_3^L, x_3^U] = [y_3^L, y_3^U]$, $[x_2^L, x_2^U] \geq [y_2^L, y_2^U]$ and $[y_1^L, y_1^U] = [z_1^L, z_1^U]$, $[y_3^L, y_3^U] > [z_3^L, z_3^U]$, so $S(x,y) = 1 - \frac{|x_2^L-y_2^L|+|x_2^U-y_2^U|}{4}$,$S(y,z) = \frac{4-|y_1^L-z_1^L|-|y_1^U-z_1^U|-|y_3^L-z_3^L|-|y_3^U-z_3^U|}{8}$, $S(x,z) = \frac{4-|x_1^L-z_1^L|-|x_1^U-z_1^U|-|x_3^L-z_3^L|-|x_3^U-z_3^U|}{8}$, also because $[x_1^L, x_1^U] = [y_1^L, y_1^U] = [z_1^L, z_1^U]$, $[x_3^L, x_3^U] = [y_3^L, y_3^U] > [z_3^L, z_3^U]$,$[x_2^L, x_2^U] \geq [y_2^L, y_2^U]$, so $S(x,z) < 0.5 \leq S(x,y)$, $S(x,z) \leq S(y,z)$.

(9) If $[x_1^L, x_1^U] = [y_1^L, y_1^U]$, $[x_3^L, x_3^U] = [y_3^L, y_3^U]$, $[x_2^L, x_2^U] \geq [y_2^L, y_2^U]$ and $[y_1^L, y_1^U] = [z_1^L, z_1^U]$, $[y_3^L, y_3^U] = [z_3^L, z_3^U]$, $[y_2^L, y_2^U] \geq [z_2^L, z_2^U]$, so $S(x,y) = 1 - \frac{|x_2^L-y_2^L|+|x_2^U-y_2^U|}{4}$,$S(y,z) = 1 - \frac{|y_2^L-z_2^L|+|y_2^U-z_2^U|}{4}$, $S(x,z) = 1 - \frac{|x_2^L-z_2^L|+|x_2^U-z_2^U|}{4}$, also because $[x_1^L, x_1^U] = [y_1^L, y_1^U] = [z_1^L, z_1^U]$, $[x_3^L, x_3^U] = [y_3^L, y_3^U] = [z_3^L, z_3^U]$,$[x_2^L, x_2^U] \geq [y_2^L, y_2^U] \geq [z_2^L, z_2^U]$, so $S(x,z) < 0.5 \leq S(x,y)$, $S(x,z) = S(y,z)$.

Therefore, as defined in formula (1), a similarity between x and y is defined. □

3.2. Entropy of Interval Neutrosophic Value

Since entropy is also an important means in the analysis of uncertainty information, we give the concept of entropy of interval neutrosophic value.

Definition 10. *(See [22]) Letting E: $D^* \longrightarrow [0,1]$, the real function E is an entropy of interval neutrosophic value, if E satisfies the following conditions:*

(N1) $E(x) = 0$ *if and only if* $[x_1^L, x_1^U] = [0,0]$ *or* $[1,1]$ *and* $[x_3^L, x_3^U] = [0,0]$ *or* $[1,1]$;

(N2) $E(x) = 1$ *if and only if* $[x_1^L, x_1^U] = [x_2^L, x_2^U] = [x_3^L, x_3^U] = [0.5, 0.5]$;

(N3) $E(x) = E(x^c)$;

(*N*4) *Let* $x = ([x_1^L, x_1^U], [x_2^L, x_2^U], [x_3^L, x_3^U]) \in D^*$, $y = ([y_1^L, y_1^U], [y_2^L, y_2^U], [y_3^L, y_3^U]) \in D^*$, *then* $y^c = ([y_3^L, y_3^U], [1 - y_2^U, 1 - y_2^L], [y_1^L, y_1^U])$, $E(x) \leq E(y)$, *that is, x is more ambiguous than y, if* $x \leq_2 y$, *when* $y \leq_2 y^c$, *or if* $y \leq_2 x$, *when* $y^c \leq_2 y$.

Entropy is usually calculated by the similarity of x and x^c, so we define the following entropy:

$$E(x) = S(x, x^c) = \begin{cases} 1 - \frac{|2x_2^L - 1| + |2x_2^U - 1|}{4}, & [x_1^L, x_1^U] = [x_3^L, x_3^U] = [0.5, 0.5], \\ \frac{4 - 2|x_1^L - x_3^L| - 2|x_1^U - x_3^U|}{8}, & \text{else.} \end{cases} \tag{2}$$

Furthermore, if interval neutrosophic values are a fuzzy set i.e.,

$$D^* = x | x = ([x_1^L, x_1^u], [x_1^L, x_1^u], [1 - x_1^L, 1 - x_1^u]),$$

then Definition 10 is equivalent to the definition of entropy measure given by de Luca and Termini [13]. If interval neutrosophic values are an intuitionistic fuzzy set i.e.,

$$D^* = x | x = ([x_1^L, x_1^u], [1 - x_1^u - x_3^u, 1 - x_1^L - x_3^L], [x_3^L, x_3^u]),$$

then Definition 10 is equivalent to the definition of entropy measure given in [23].

Theorem 2. *$E(x)$ defined as (2) is an entropy of x.*

Proof. If $[x_1^L, x_1^U] = [x_3^L, x_3^U] = [0.5, 0.5]$, then $E(x) = 1 - \frac{|2x_2^L - 1| + |2x_2^U - 1|}{4}$, so $0.5 \leq E(x) \leq 1$; otherwise, $E(x) = \frac{4 - 2|x_1^L - x_3^L| - 2|x_1^U - x_3^U|}{8}$, so $0 \leq E(x) \leq 0.5$.

(*N*1) $E(x) = 0$ if and only if $|x_1^L - x_1^U| = 1$ and $|x_3^L - x_3^U| = 1$, also because $[x_1^L, x_1^U] \in [0, 1]$ and $[x_3^L, x_3^U] \in [0, 1]$, so $[x_1^L, x_1^U] = [0, 0] or [1, 1]$, $[x_1^L, x_1^U] = [1, 1] or [0, 0]$, so x is a distinct set.

(*N*2) Obviously, $E(x) = 1$ if and only if $[x_1^L, x_1^U] = [x_2^L, x_2^U] = [x_3^L, x_3^U] = [0.5, 0.5]$,

(*N*3) Obviously, $E(x) = E(x^c)$.

(*N*4) Let $x = ([x_1^L, x_1^U], [x_2^L, x_2^U], [x_3^L, x_3^U]) \in D^*, y = ([y_1^L, y_1^U], [y_2^L, y_2^U], [y_3^L, y_3^U]) \in D^*$, then $y = ([y_3^L, y_3^U], [1 - y_2^U, 1 - y_2^L], [y_1^L, y_1^U])$, if $x \leq_2 y$, when $y \leq_2 y^c$, because

$$E(x) = \begin{cases} 1 - \frac{|2x_2^L - 1| + |2x_2^U - 1|}{4}, & [x_1^L, x_1^U] = [x_3^L, x_3^U] = [0.5, 0.5], \\ \frac{4 - 2|x_1^L - x_3^L| - 2|x_1^U - x_3^U|}{8}, & \text{else,} \end{cases}$$

$$E(y) = \begin{cases} 1 - \frac{|2y_2^L - 1| + |2y_2^U - 1|}{4}, & [y_1^L, y_1^U] = [y_3^L, y_3^U] = [0.5, 0.5]. \\ \frac{4 - 2|y_1^L - y_3^L| - 2|y_1^U - y_3^U|}{8}, & \text{else.} \end{cases}$$

(1) If $[y_1^L, y_1^U] < [y_3^L, y_3^U]$, $[y_3^L, y_3^U] \geq [y_1^L, y_1^U]$ and $[x_1^L, x_1^U] < [y_1^L, y_1^U]$, $[x_3^L, x_3^U] \geq [y_3^L, y_3^U]$, so $[x_1^L, x_1^U] < [y_1^L, y_1^U] < [y_3^L, y_3^U] \leq [x_3^L, x_3^U]$, therefore $|x_1^L - x_3^L| \geq |y_1^L - y_3^L|$, $|x_1^U - x_3^U| \geq |y_1^U - y_3^U|$, also because $E(x) = \frac{4 - 2|x_1^L - x_3^L| - 2|x_1^U - x_3^U|}{8}$, $E(y) = \frac{4 - 2|y_1^L - y_3^L| - 2|y_1^U - y_3^U|}{8}$, so $E(x) \leq E(y)$.

(2) If $[y_1^L, y_1^U] < [y_3^L, y_3^U]$, $[y_3^L, y_3^U] \geq [y_1^L, y_1^U]$ and $[x_1^L, x_1^U] = [y_1^L, y_1^U]$, $[x_3^L, x_3^U] > [y_3^L, y_3^U]$, so $[x_1^L, x_1^U] = [y_1^L, y_1^U] < [y_3^L, y_3^U] \leq [x_3^L, x_3^U]$, therefore $|x_1^L - x_3^L| \geq |y_1^L - y_3^L|$, $|x_1^U - x_3^U| \geq |y_1^U - y_3^U|$, also because $E(x) = \frac{4 - 2|x_1^L - x_3^L| - 2|x_1^U - x_3^U|}{8}$, $E(y) = \frac{4 - 2|y_1^L - y_3^L| - 2|y_1^U - y_3^U|}{8}$, so $E(x) \leq E(y)$.

(3) If $[y_1^L, y_1^U] < [y_3^L, y_3^U]$, $[y_3^L, y_3^U] \geq [y_1^L, y_1^U]$ and $[x_1^L, x_1^U] = [y_1^L, y_1^U]$, $[x_3^L, x_3^U] = [y_3^L, y_3^U]$, $[x_2^L, x_2^U] = [y_2^L, y_2^U]$, so $[x_1^L, x_1^U] = [y_1^L, y_1^U] < [y_3^L, y_3^U] = [x_3^L, x_3^U]$, therefore $|x_1^L - x_3^L| \geq |y_1^L - y_3^L|$, $|x_1^U - x_3^U| \geq |y_1^U - y_3^U|$, also because $E(x) = \frac{4 - 2|x_1^L - x_3^L| - 2|x_1^U - x_3^U|}{8}$, $E(y) = \frac{4 - 2|y_1^L - y_3^L| - 2|y_1^U - y_3^U|}{8}$, so $E(x) \leq E(y)$.

(4) If $[y_1^L, y_1^U] = [y_3^L, y_3^U]$, $[y_3^L, y_3^U] > [y_1^L, y_1^U]$, contradiction.

(5) If $[y_1^L, y_1^U] = [y_3^L, y_3^U]$, $[y_2^L, y_2^U] \geq [1 - y_2^U, 1 - y_1^L]$ and $[x_1^L, x_1^U] < [y_1^L, y_1^U]$, $[x_3^L, x_3^U] \geq [y_3^L, y_3^U]$, so $[x_1^L, x_1^U] < [y_1^L, y_1^U] = [y_3^L, y_3^U] \leq [x_3^L, x_3^U]$; if $[y_1^L, y_1^U] = [y_3^L, y_3^U] = [0.5, 0.5]$, then $E(x) =$

$\frac{4-2|x_1^L-x_3^L|-2|x_1^U-x_3^U|}{8}$, $E(y) = 1 - \frac{|2y_2^L-1|+|2y_2^U-1|}{4}$, so $E(x) \leq 0.5 \leq E(y)$; if $[y_1^L, y_1^U] = [y_3^L, y_3^U] \neq [0.5, 0.5]$, then $E(x) = \frac{4-2|x_1^L-x_3^L|-2|x_1^U-x_3^U|}{8}$, $E(y) = \frac{4-2|y_1^L-y_3^L|-2|y_1^U-y_3^U|}{8}$, also because $|x_1^L - x_3^L| \geq |y_1^L - y_3^L| = 0$, $|x_1^U - x_3^U| \geq |y_1^U - y_3^U| = 0$, so $E(x) \leq E(y)$. In summary, $E(x) \leq E(y)$.

(6) If $[y_1^L, y_1^U] = [y_3^L, y_3^U]$, $[y_2^L, y_2^U] \geq [1 - y_2^U, 1 - y_1^L]$ and $[x_1^L, x_1^U] = [y_1^L, y_1^U]$, $[x_3^L, x_3^U] > [y_3^L, y_3^U]$, so $[x_1^L, x_1^U] = [y_1^L, y_1^U] = [y_3^L, y_3^U] < [x_3^L, x_3^U]$; if $[y_1^L, y_1^U] = [y_3^L, y_3^U] = [0.5, 0.5]$, then $E(x) = \frac{4-2|x_1^L-x_3^L|-2|x_1^U-x_3^U|}{8}$, $E(y) = 1 - \frac{|2y_2^L-1|+|2y_2^U-1|}{4}$, so $E(x) \leq 0.5 \leq E(y)$; if $[y_1^L, y_1^U] = [y_3^L, y_3^U] \neq [0.5, 0.5]$, then $E(x) = \frac{4-2|x_1^L-x_3^L|-2|x_1^U-x_3^U|}{8}$, $E(y) = \frac{4-2|y_1^L-y_3^L|-2|y_1^U-y_3^U|}{8}$, also because $|x_1^L - x_3^L| \geq |y_1^L - y_3^L| = 0$, $|x_1^U - x_3^U| \geq |y_1^U - y_3^U| = 0$, so $E(x) \leq E(y)$. In summary, $E(x) \leq E(y)$.

(7) If $[y_1^L, y_1^U] = [y_3^L, y_3^U]$, $[y_2^L, y_2^U] \geq [1 - y_2^U, 1 - y_1^L]$ and $[x_1^L, x_1^U] = [y_1^L, y_1^U]$, $[x_3^L, x_3^U] = [y_3^L, y_3^U]$, $[x_2^L, x_2^U] \geq [y_2^L, y_2^U]$, so $[x_1^L, x_1^U] = [y_1^L, y_1^U] = [y_3^L, y_3^U] = [x_3^L, x_3^U]$; if $[x_1^L, x_1^U] = [y_1^L, y_1^U] = [y_3^L, y_3^U] = [x_3^L, x_3^U] = [0.5, 0.5]$, then $E(x) = 1 - \frac{|2x_2^L-1|+|2x_2^U-1|}{4}$, $E(y) = 1 - \frac{|2y_2^L-1|+|2y_2^U-1|}{4}$, also because $[x_2^L, x_2^U] \geq [y_2^L, y_2^U]$, so $E(x) \leq 0.5 \leq E(y)$; if $[x_1^L, x_1^U] = [y_1^L, y_1^U] = [y_3^L, y_3^U] = [x_3^L, x_3^U] \neq [0.5, 0.5]$, then $E(x) = \frac{4-2|x_1^L-x_3^L|-2|x_1^U-x_3^U|}{8}$, $E(y) = \frac{4-2|y_1^L-y_3^L|-2|y_1^U-y_3^U|}{8}$, also because $|x_1^L - x_3^L| = |y_1^L - y_3^L| = 0$, $|x_1^U - x_3^U| = |y_1^U - y_3^U| = 0$, so $E(x) = E(y)$. In summary, $E(x) \leq E(y)$.

As the same reason, we can easily get the conclusion that if $y \leq_2 x$, when $y^c \leq_2 y$, then $E(x) \leq E(y)$. Therefore, as defined in formula (2), an entropy is defined. □

3.3. Similarity and Entropy of Interval Neutrosophic Sets

In this subsection, we extend the notions of similarity measure and entropy measure of interval nuetrosophic values to interval nuetrosophic sets.

Definition 11. *Let A, B be the two interval neutrosophic sets, the real function S is a similarity between interval neutrosophic sets A and B, if S satisfies the following conditions:*

(P1) $0 \leq S(A, B) \leq 1$;
(P2) $S(A, B) = 1$ *if and only if* $A = B$;
(P3) $S(A, B) = S(B, A)$;
(P4) *For all* A, B, $C \in INSs$, *if* $A \subseteq B \subseteq C$, *then* $S(A, C) \leq S(A, B)$, $S(A, C) \leq S(B, C)$.

Definition 12. *Let A be the an interval neutrosophic set, the real function E is the entropy of interval neutrosophic sets, if E satisfies the following conditions:*

(N1) $E(A) = 0$ *if and only if* $[T_A^L, T_A^U] = [0, 0] or [1, 1]$, $[F_A^L, F_A^U] = [0, 0] or [1, 1]$;
(N2) $E(A) = 1$ *if and only if* $[T_A^L, T_A^U] = [I_A^L, I_A^U] = [F_A^L, F_A^U] = [0.5, 0.5]$;
(N3) $E(A) = E(A^c)$;
(N4) *Let A, B be the two interval neutrosophic sets, $E(A) \leq E(B)$, that is, B is more ambiguous than A, if $A \subseteq_2 B$, when $B \subseteq_2 B^c$, or $B \subseteq_2 A$, when $B^c \subseteq_2 B$.*

By aggregating the similarities and entropies of interval neutrosophic values, we have the following similarity and entropy of interval neutrosophic sets.

Theorem 3. *Let $X = \{x_1, x_2, ...x_n\}$ be an interval neutrosophic set, $s : D^* \times D^* \to [0, 1]$ is the similarity of interval neutrosophic sets, $\forall A, B \subseteq X$, the similarity S of A and B is defined as follows:*

$$S(A, B) = \frac{1}{n} \sum_{i=1}^{n} s(A(x_i), B(x_i)). \tag{3}$$

Theorem 4. *Let $X = \{x_1, x_2, ...x_n\}$ be an interval neutrosophic seet, $e : D^* \to [0, 1]$ is the entropy of interval neutrosophic sets, $\forall A \subseteq X$, the similarity S of A is defined as follows:*

$$E(A) = \tfrac{1}{n} \textstyle\sum_{i=1}^{n} e(A(x_i)).$$

If the weights $W = (w_1, w_2, ..., w_n)$ is added, $w_i \in [0,1]$ and $\sum_{i=1}^n w_i = 1$, then the similarities of A and B and the entropy of A are defined as follows:

$$S(A,B) = \sum_{i=1}^n w_i \cdot s(A(x_i), B(x_i)),$$
$$E(A) = \sum_{i=1}^n w_i \cdot e(A(x_i)).$$

4. The Numerical Example

Let us consider the decision problem adapted from [10]. Suppose that there is a group with four possible alternatives to invest: (1) A_1 is a food company; (2) A_2 is a car company; (3) A_3 is a weapons company; (4) A_4 is a computer company. Investment companies must make decisions based on three criteria: (1) C_1 is growth analysis; (2) C_2 is risk analysis; and (3) C_3 is environmental impact analysis. By using interval-valued intuitionistic fuzzy information, decision makers evaluated four possible alternatives based on the above three criteria and the evaluation are expressed as three interval neutrosophic sets (Table 1).

Table 1. The evaluation of alternatives.

	C_1	C_2	C_3
A_1	([0.4,0.5], [0.2,0.3], [0.3,0.4])	([0.4,0.6], [0.1,0.3], [0.2,0.4])	([0.7,0.9], [0.2,0.3], [0.4,0.5])
A_2	([0.6,0.7], [0.1,0.2], [0.2,0.3])	([0.6,0.7], [0.1,0.2], [0.2,0.3])	([0.3,0.6], [0.3,0.5], [0.8,0.9])
A_3	([0.3,0.6], [0.2,0.3], [0.3,0.4])	([0.5,0.6], [0.2,0.3], [0.3,0.4])	([0.4,0.5], [0.2,0.4], [0.7,0.9])
A_4	([0.7,0.8], [0.0,0.1], [0.1,0.2])	([0.6,0.7], [0.1,0.2], [0.1,0.3])	([0.6,0.7], [0.3,0.4], [0.8,0.9])

4.1. Ye's Multi-Attributes Decision-Making Method with Analysis

Ye [10] presented a multi-attributes decision-making method by using a single-valued neutrosophic set. The approach can be described as follows. Ye presents the definitions of the Hamming and Euclidean distances between INSs and the similarity measures between INSs based on the distances, which can be used in real scientific and engineering applications.

(1) The Hamming distance

$d_1(A,B) = \frac{1}{6}\sum_{i=1}^n (|T_A^L(x_i) - T_B^L(x_i)| + |T_A^U(x_i) - T_B^U(x_i)| + |I_A^L(x_i) - I_B^L(x_i)| + |I_A^U(x_i) - I_B^U(x_i)| + |F_A^L(x_i) - F_B^L(x_i)| + |F_A^U(x_i) - F_B^U(x_i)|);$

(2) The Euclidean distance

$d_2(A,B) = \frac{1}{6}\sum_{i=1}^n ((T_A^L(x_i) - T_B^L(x_i))^2 + (T_A^U(x_i) - T_B^U(x_i))^2 + (I_A^L(x_i) - I_B^L(x_i))^2 + (I_A^U(x_i) - I_B^U(x_i))^2 + (F_A^L(x_i) - F_B^L(x_i))^2 + (F_A^U(x_i) - F_B^U(x_i))^2)^{\frac{1}{2}};$

Thus, the similarity measures between INSs are based on the distances as follows:

$S_1(A,B) = 1 - \frac{1}{6}\sum_{i=1}^n (|T_A^L(x_i) - T_B^L(x_i)| + |T_A^U(x_i) - T_B^U(x_i)| + |I_A^L(x_i) - I_B^L(x_i)| + |I_A^U(x_i) - I_B^U(x_i)| + |F_A^L(x_i) - F_B^L(x_i)| + |F_A^U(x_i) - F_B^U(x_i)|);$

$S_2(A,B) = 1 - \frac{1}{6}\sum_{i=1}^n ((T_A^L(x_i) - T_B^L(x_i))^2 + (T_A^U(x_i) - T_B^U(x_i))^2 + (I_A^L(x_i) - I_B^L(x_i))^2 + (I_A^U(x_i) - I_B^U(x_i))^2 + (F_A^L(x_i) - F_B^L(x_i))^2 + (F_A^U(x_i) - F_B^U(x_i))^2)^{\frac{1}{2}}.$

From the interval neutrosophic decision matrix, Ye obtain the following ideal alternative:

$$A = (([0.7,0.8],[0.0,0.1],[0.1,0.2]),([0.6,0.7],[0.1,0.2],[0.1,0.3]),([0.3,0.6],[0.3,0.5],[0.8,0.9])).$$

Thus, $S_1(A_1, A) = 0.7667$, $S_1(A_2, A) = 0.9542$, $S_1(A_3, A) = 0.8625$, $S_1(A_4, A) = 0.9600$. Therefore, $S_1(A_4, A) > S_1(A_2, A) > S_1(A_3, A) > S_1(A_1, A)$, so A_4 is the best choice. Meanwhile, $S_2(A_1, A) = 0.7370$, $S_2(A_2, A) = 0.9323$, $S_2(A_3, A) = 0.8344$, $S_2(A_4, A) = 0.9034$. Therefore, $S_2(A_2, A) > S_2(A_4, A) > S_2(A_3, A) > S_4(A_1, A)$, so A_4 is the best choice.

4.2. Multi-Attributes Decision Making Based on a New Similarity Measure

Next, we use the newly proposed similarity and entropy to get the best alternative. The best choice is $A = ([1, 1], [1, 1], [0, 0])$ because it is the largest in the second type of inclusion relationship, so it is optimal. For convenience, we use A_{ij} that indicates the neutrosophic value in line i column j. It is available from (1), $S(A_{11}, A) = 0.275$, $S(A_{12}, A) = 0.3$, $S(A_{13}, A) = 0.3375$, $S(A_{21}, A) = 0.35$, $S(A_{22}, A) = 0.35$, $S(A_{23}, A) = 0.15$, $S(A_{31}, A) = 0.275$, $S(A_{32}, A) = 0.225$, $S(A_{33}, A) = 0.1625$, $S(A_{41}, A) = 0.4$, $S(A_{42}, A) = 0.3625$, $S(A_{43}, A) = 0.2$. Thus, by (3), we can obtain that $S(A_1, A) = \frac{1}{3} \times 0.275 + \frac{1}{3} \times 0.3 + \frac{1}{3} \times 0.3375 \approx 0.3042$, for the same reason, we can obtain that $S(A_2, A) \approx 0.2833$, $S(A_3, A) \approx 0.2208$, $S(A_4, A) \approx 0.3208$. Therefore, $S(A_4, A) > S(A_1, A) > S(A_2, A) > S(A_3, A)$, so A_4 is the best choice.

In order to validate the feasibility of the proposed decision-making methods, a comparative study was conducted with other methods as follows.

In the similarity measures in [10], due to the differences in inclusion relation and in the best choice, we have different conclusions. Moreover, in [10], the distances between INSs are first calculated and any difference is then amplified in the results using criteria weights, which cause a distortion in the similarity between an alternative and the ideal alternative. Meanwhile, in [17], Sahin defined the interval neutrosophic cross-entropy in two different ways, which are based on extension of fuzzy cross-entropy and single-valued neutrosophic cross-entropy. Additionally, two multi-criteria decision-making methods using the interval neutrosophic cross-entropy between an alternative and the ideal alternative are developed in order to determine the order of the alternatives and choose the most preferred one(s). In addition, Sahin ranked the alternatives as $A_4 > A_1 > A_2 > A_3$. For this example, by using the new similarity measure proposed in this paper, we obtained the same ranking order of alternatives as in [17]. It shows that the new similarity measures proposed in this paper are effective and efficient.

5. Discussion and Conclusions

Based on the existing inclusion relation of the wisdom set in the interval, this paper combines the new inclusion relation type-3 of the single-valued neutrosophic sets proposed by Zhang [18,19], and gives the inclusion relationship of the new interval neutrosophic sets. The existing similarity measure is mainly designed for the original inclusion relationship, and does not apply to the new inclusion relationship, then we propose the similarity and entropy of the interval neutrosophic set. The new similarity and entropy also prove that the defined similarity and entropy satisfy the corresponding axiomatization definition, and finally apply it to the corresponding multi-attribute decision-making. The results show that the proposed similarity and entropy are reasonable and effective.

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