Novel Neutrosophic Cubic Graphs Structures With Application in Decision Making Problems

Article in IEEE Access - June 2019
DOI: 10.1109/ACCESS.2019.2925040

CITATION
1

READS
159

5 authors, including:

Muhammad Gulistan
Hazara University
85 PUBLICATIONS 502 CITATIONS
SEE PROFILE

Mumtaz Ali
Deakin University
190 PUBLICATIONS 2,173 CITATIONS
SEE PROFILE

Muhammad Azhar
Hazara University
6 PUBLICATIONS 24 CITATIONS
SEE PROFILE

Seifedine Kadry
Beirut Arab University
409 PUBLICATIONS 1,491 CITATIONS
SEE PROFILE

Some of the authors of this publication are also working on these related projects:

Big Data in Finance View project
Project 1: Learning Analytics View project

All content following this page was uploaded by Muhammad Gulistan on 10 July 2019.
The user has requested enhancement of the downloaded file.
Novel Neutrosophic Cubic Graphs Structures with Application in Decision Making Problems

Muhammad Gulistan1, Muntaz Ali2, Muhammad Azhar1, Seungmin Raho3, Seifedine Kadry4

1Department of Mathematics and Statistics, Hazara University, Mansehra, Pakistan
2University of Southern Queensland, Toowoomba, QLD 4300, Australia
3Department of software, Sejong University, Seoul, Korea
4Department of Mathematics & Computer Science, Beirut Arab University, Lebanon

Corresponding author: Muhammad Gulistan, Seungmin Raho (gulistanmath@hu.edu.pk, smrhosejong.edu)

ABSTRACT: Graphs allows us to study the different patterns of inside the data by making a mental image. The aim of this paper is to develop neutrosophic cubic graph structure which is the extension of neutrosophic cubic graphs. As neutrosophic cubic graphs are defined for one set of edges between vertices while neutrosophic cubic graph structures are defined for more than one set of edges. Further, we defined some basic operations such as Cartesian product, composition, union, join, cross product, strong product and lexigraphic product of two neutrosophic cubic graph structures. Several types of other interesting properties of neutrosophic cubic graph structures are discussed in this paper. Finally, a decision-making algorithm based on the idea of neutrosophic cubic graph structures is constructed. The proposed decision-making algorithm is applied in a decision-making problem to check the validity.

INDEX TERMS: Neutrosophic Cubic Set, Neutrosophic Cubic graphs structures, application.

I. INTRODUCTION

Fuzzy sets: The extension of classical set theory in the form of fuzzy sets was given by Zadeh in 1965 in his seminal paper [1]. Further he introduced the interval-valued fuzzy sets in 1975 [2]. Atanassov use the notion of membership and non-membership of an element in a set X and gave the idea of intuitionistic fuzzy sets. Use of intuitionistic fuzzy sets is helpful in the introduction of additional degrees of freedom (non-membership and hesitation margins) into set description and is extensively used as a tool of intensive research by scholars and scientists from the over many years. Various theories like theory of probability, fuzzy set theory, intuitionistic fuzzy sets, rough set theory etc., are consistently being used as powerful constructive tools to deal with multiform uncertainties and imprecision enclosed in complex systems. But all these above theories do not model the modelled information adequately. Therefore, due to the existence of indeterminacy in various world problems, neutrosophy founds its way into the modern research. Neutrosophy is a generalization of fuzzy set, where the models represented by three types concepts that is truthfulness, falsehood and neutrality. Neutrosophy is a Latin word “neuter” - neutral, Greek "sophia" - skill/Wisdom). Neutrosophy is a branch of philosophy, introduced by FlorentinSmarandache which studies the origin, nature, and scope of neutralities, as well as theirinteractions with different ideational spectra. Neutrosophy considers a proposition, theory, event, concept, or entity, "A" in relation to its opposite, "Anti-A" and that which is not A, "Non-A", and that which is neither "A" nor "Anti-A", denoted by "Neut-A". Neutrosophy is the basis of neutrosophic logic, neutrosophic probability, neutrosophic set, and neutrosophic statistics. Inspired from the realities of real life phenomena like sport games (winning/ tie/ defeating), votes (yes/ NA no) and decision making (making a decision/ hesitating/ not making), Smarandache [3, 4] introduced a new concept of a neutrosophic set and neutrosophic logic (NS in short) in 1999, which is the generalization of a fuzzy sets and intuitionistic fuzzy set. NS is described by membership degree, indeterminate degree and non-membership degree. The idea of NS generates the theory of neutrosophic sets by giving representation to indeterminates. This theory is considered as complete representation of almost every model of all real-world problems. Therefore, if uncertainty is involved in a problem we use fuzzy theory while dealing indeterminacy, we need neutrosophic theory. In fact, this theory has several applications in many different fields like control theory, databases, medical diagnosis problem and decision-making problems. These sets models have been studied by many authors. Using Neutrosophic theory, many mathematicians introduced the concept of neutrosophic algebraic structures such as neutrosophic algebraic structures, neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophicbigroups, neutrosophic N-groups, neutrosophicbisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophicbiloops, neutrosophic N-loop, neutrosophic groupoids, neutrosophicbigroupoids and neutrosophic AG-groupoids. In 2012, Jun et al. gave the idea of cubic sets [5]. For more detail of cubic set one can cite [6, 7, 8, 9, 10, 11]. More recently Jun et al. combine neutrosophic set with cubic sets and gave the idea of Neutrosophic cubic set [12] and define different operations [13]. Further interval neutrosophic sets was introduced by Wang et al. [14]. Fuzzy Graphs: In 1975 Rosenfeld [15] extended the idea given by Kaufmann in 1973 [16] and initiate the concept of fuzzy graphs and considered the relations between fuzzy sets. In 1997 Bhattacharya explained some remarks on fuzzy graphs [17]. Mordeson and Nair explained the study of fuzzy graphs and fuzzy hypergraphs in their book in 2001 [18]. Akram et al. gave the idea of interval valued fuzzy graphs [19, 20], intuitionistic fuzzy graphs and bipolar fuzzy graphs [21, 22, 23]. Strong intuitionistic fuzzy graphs were presented by Akram and Davvaz [24]. Intuitionistic fuzzy sets were further generalized by Smarandache [4]. Cayley interval-valued fuzzy threshold graphs were studied by Borzooei and Rashmanlou [25]. Buckley gave the concept of self-centered graphs [26]. Further characterized g-self-centered fuzzy graphs was given by Sunitha et al. [27]. Mishra et al. [28] introduced the idea of coherent category of interval-valued intuitionistic fuzzy graphs. Pal et al. [29] and Pramanik et al. [30, 31] discussed some results to the theory of interval-valued fuzzy graphs. Parvathi et al. [32] defined operations on intuitionistic fuzzy graphs. The idea of product of intuitionistic fuzzy graphs was introduced by Sahoo and Pal [33]. Gulistan et al. [34] presented the idea of neutrophic cubic graphs with real life application in industry. The main role of neutrosophic cubic graph structure theory in computer application is the development of graph algorithms. These algorithms are used to those problems that are modeled in the form of graphs and the corresponding computer science applications problems. Theoretical concept of the neutrosophic cubic graphs structures are highly utilized by computer science application. Especially in...
research area of computer science such as data mining, image segmentation, clustering, image capturing and networking. The neutrosophic cubic graph structures are more flexible and compatible then fuzzy graphs due to the fact that they have many applications in networks.

Our approach: In this paper we initiate the idea of neutrosophic cubic graph structures which is extension of neutrosophic cubic graphs. Neutrosophic cubic graphs are defined for one set of edges between vertices while neutrosophic cubic graphs structures are defined for more than one set of edges. We also defined basic operations like Cartesian product, composition, union, join, cross product, strong product and lexicographic product of two neutrosophic cubic graph structures. At the end we discuss the application of neutrosophic cubic graphs in decision making problems.

II. Preliminaries
We briefly describe few fundamental concepts, ideas and preliminaries of neutrosophic sets, neutrosophic cubic sets and neutrosophic cubic graphs.

Definition 2.1 [34] Neutrosophic sets is define as:

\[ A = (\{x, F_A(x), T_A(x), I_A(x): x \in X\} \]

where \( X \) is a universe of discoveries and \( A \) is characterized by a truth-membership function \( T_A: X \to [0, 1]^\ast \) an indeterminacy-membership function \( I_A: X \to [0, 1]^\ast \) and a falsity-membership function \( F_A: X \to [0, 1]^\ast \). There is not restriction on the sum of \( T_A(x), I_A(x), F_A(x) \).

Definition 2.2 [35] A single valued neutrosophic sets is define as:

\[ A_{NS} = (\{x, F_A(x), T_A(x), I_A(x): x \in X\} \]

where \( X \) is a universe of discoveries and \( A_{NS} \) is characterized by a truth-membership function \( T_A: X \to [0, 1]^\ast \) an indeterminacy-membership function \( I_A: X \to [0, 1]^\ast \) and a falsity-membership function \( F_A: X \to [0, 1]^\ast \). There is not restriction on the sum of \( T_A(x), I_A(x), F_A(x) \).

Definition 2.3 [35] Let us consider two single valued neutrosophic sets

\[ A_{NS} = (\{x, F_{A1}(x), T_{A1}(x), I_{A1}(x): x \in X\} \]

and

\[ B_{NS} = (\{x, F_{B1}(x), T_{B1}(x), I_{B1}(x): x \in X\} \]

then set theoretical operations for these two single valued neutrosophic sets are given as:

(i) \( A_{NS} \cap B_{NS} \)

(ii) \( A_{NS} \cup B_{NS} \)

(iii) \( A_{NS} \setminus B_{NS} \)

Definition 2.4 [2, 36] Let \( A_1 = (\{x, T_{A1}(x), I_{A1}(x), F_{A1}(x): x \in X\} \)

and \( A_2 = (\{x, T_{A2}(x), I_{A2}(x), F_{A2}(x): x \in X\} \)

be two single valued neutrosophic number. Then, the operations for NNS are defined as below:

\[ \lambda A_1 = (1 - (1 - T_{A1})), \lambda I_{A1}, \lambda F_{A1}, \]

\[ A_1 + A_2 = (T_{A1} + T_{A2}, I_{A1} + I_{A2}, F_{A1} + F_{A2}) \]

\[ A_1 \cdot A_2 = (T_{A1} \cdot T_{A2}, I_{A1} \cdot I_{A2}, F_{A1} \cdot F_{A2}) \]

\[ A_1 \setminus A_2 = (T_{A1} - T_{A2}, I_{A1} - I_{A2}, F_{A1} - F_{A2}) \]

\[ A_1 \setminus A_2 = (T_{A1} - T_{A2}, I_{A1} - I_{A2}, F_{A1} - F_{A2}) \]

\[ A_1 \cdot A_2 = (T_{A1} - T_{A2}, I_{A1} - I_{A2}, F_{A1} - F_{A2}) \]

\[ A_1 \setminus A_2 = (T_{A1} - T_{A2}, I_{A1} - I_{A2}, F_{A1} - F_{A2}) \]

(iii) \( (\hat{F}_u(v), \gamma) \leq \max(\hat{F}_u(v), \gamma) \)

Definition 2.7 [24] Let \( G = (V, E) \) be a graph and \( G = (M, N) \) be a Neutrosophic Cubic Graph on \( V \) is said to be truth-internal (T-internal) if the following conditions hold

\[ T_A(x) \leq T_A(x), \theta(x) \forall x \in V, T_0(e) \geq T_0(e), \theta(e) \forall e \in E \]

an indeterminacy-internal (I-internal) if the following conditions hold

\[ I_A(x) \leq I_A(x), \theta(x) \forall x \in V, I_0(e) \leq I_0(e), \theta(e) \forall e \in E \]

falsity-internal (F-internal) if the following conditions hold

\[ F_A(x) \leq F_A(x), \theta(x) \forall x \in V, F_0(e) \leq F_0(e), \theta(e) \forall e \in E \]

true-external (T-external) if the following conditions hold

\[ T_A(x) \leq T_A(x), \theta(x) \forall x \in V, T_0(e) \leq T_0(e), \theta(e) \forall e \in E \]

indeterminacy-internal (I-external) if the following conditions hold

\[ I_A(x) \leq I_A(x), \theta(x) \forall x \in V, I_0(e) \leq I_0(e), \theta(e) \forall e \in E \]

falsity-external (F-external) if the following conditions hold

\[ F_A(x) \leq F_A(x), \theta(x) \forall x \in V, F_0(e) \leq F_0(e), \theta(e) \forall e \in E \]

A neutrosophic cubic graph is said to be internal neutrosophic cubic graph if it is truth-internal, indeterminacy-internal and falsity-internal.

III. Neutrosophic Cubic Graph Structures
In this section we define the extension of neutrosophic cubic graphs to neutrosophic cubic graph structures.

Definition 3.1 Let \( G^* = (V, E_1, E_2, \ldots, E_n) \) be a graph structure. Then \( G^* = (M, N, N_1, \ldots, N_n) \) is said to be neutrosophic cubic graph structure of \( G^* \) where

\[ M = (A, B, \overline{A} = (\overline{T_A}, \overline{I_A}, \overline{F_A}, \overline{F_A} = B) \]

is the neutrosophic cubic set representation of \( V \) and

\[ \overline{N_i} = \left( \overline{C_i}, \overline{T_{C_i}}, \overline{I_{C_i}}, \overline{F_{C_i}}, \overline{F_{C_i}} \right) \]

are the neutrosophic cubic set representations of \( E_1, E_2, \ldots, E_n \) respectively, if the following conditions are satisfied:

(i) \( M \) is a neutrosophic cubic set on \( V \) such that \( \forall x \in V \)

\[ 0 \leq \overline{T_A}(x) + \overline{I_A}(x) + \overline{F_A}(x) \leq 3 \]

(ii) \( N_i \) is a neutrosophic cubic set on \( E_i \) such that \( \forall x \in E_i \)

\[ 0 \leq \overline{T_{C_i}}(x) + \overline{I_{C_i}}(x) + \overline{F_{C_i}}(x) \leq 3 \]

Expected: Let \( G^* = (V, E_1, E_2) \) be a graph structure where

\[ V = \{a, b, c, d\} \]

\[ E_1 = \{ab, ac, bc\} \]

\[ E_2 = \{ad, bc, bd\} \]

This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2019.2925040, IEEE Access
Definition 3.2 Let $\tilde{G}_{i3} = (M_i, N_{i1}, N_{i2}, \ldots, N_{iL})$ and $\tilde{G}_{i2} = (M_{i2}, N_{i2}, N_{i2}, \ldots, N_{i2})$ be two neurotrophic cubic graph structures defined on $\tilde{G}_3 = (V_3, E_{i1}, E_{i2}, \ldots, E_{i1})$ and $\tilde{G}_2 = (V_2, E_{i2}, E_{i2}, \ldots, E_{i2})$ respectively. The cartesian product of $\tilde{G}_1$ and $\tilde{G}_2$ is defined as

$$\tilde{G}_{12} = \tilde{G}_{13} \times \tilde{G}_{i2} = (M_i, N_{i1}, N_{i2}, \ldots, N_{iL}) \times (M_{i2}, N_{i2}, N_{i2}, \ldots, N_{i2})$$

$$(C_{i1}, D_{i1}) \times (C_{i2}, D_{i2}) \times \ldots \times (C_{i1}, D_{i1}) \times (C_{i2}, D_{i2})$$

where $C_{i1} = \{x, y, \tilde{y}, \hat{y}, \bar{y}, \underline{y}, \overline{y}, \tilde{x}, \hat{x}, \bar{x}, \underline{x}, \overline{x}\}$ and $D_{i1} = \{0, 0.4, 0.5, 0.6, 0.7\}$.

Example: Let $\tilde{G}_{i3} = (M_1, N_{i1}, N_{i2}, N_{i3})$ and $\tilde{G}_{i2} = (M_{i2}, N_{i2}, N_{i2}, N_{i2})$ be two neurotrophic cubic graph structures defined on $\tilde{G}_3$ and $\tilde{G}_2$ respectively, where

$M_1 = \{a, (0.0, 0.3, 0.4, 0.5), (0.5, 0.6, 0.7), (0.0, 0.4), (0.5, 0.6, 0.7)\}$

$N_{i1} = \{b, (0.0, 0.2, 0.3, 0.5), (0.5, 0.6, 0.7)\}$

$N_{i2} = \{c, (0.0, 0.2, 0.3, 0.5), (0.5, 0.6, 0.7)\}$

and

$M_2 = \{(x, (0.4, 0.5, 0.6, 0.7), (0.0, 0.2, 0.3, 0.4), (0.0, 0.1, 0.2, 0.5))\}$

$N_{i2} = \{(y, (0.0, 0.2, 0.3, 0.4), (0.0, 0.1, 0.2, 0.5))\}$

Then $\tilde{G}_{11} \times \tilde{G}_{i2}$ will be

$M_{12} = \{(a, (0.0, 0.3, 0.4, 0.5)), (0.5, 0.6, 0.7))\}$

$M_{12} = \{(x, (0.4, 0.5, 0.6, 0.7), (0.0, 0.2, 0.3, 0.4), (0.0, 0.1, 0.2, 0.5))\}$
\[ N_{21} \times N_{22} =\]

\[
\begin{align*}
&\{(a, y(a, x)), (0.2, 0.3, 0.8), (0.0, 0.6, 0.4), (0.8, 0.9, 0.3)\}, \\
&\{(b, x(b, z)), (0.1, 0.2, 0.8), (0.2, 0.3, 0.7), (0.8, 0.9, 0.1)\}, \\
&\{(c, x(c, z)), (0.4, 0.5, 0.8), (0.2, 0.3, 0.7), (0.8, 0.9, 0.1)\}, \\
&\{(d, y(d, z)), (0.2, 0.3, 0.8), (0.5, 0.6, 0.4), (0.8, 0.9, 0.2)\}.
\end{align*}
\]

\[
F_{\text{min}}(x, y) = \min(F_{\text{min}}(x), F_{\text{min}}(y)).
\]

\[
C \{ \{0, r_{\text{max}}, r_{\text{min}}\} \}
\]

Proposition 3.3 The cartesian product of two neutrosophic cubic graph structures is also a neutrosophic cubic graph structure.

**Proof.** Condition is obvious for \( M_1 \times M_2 \). Therefore we verify for \( N_{n_1} \times N_{n_2} \in \{1, 2, \ldots, n\} \), where

\[ N_{n_1} \times N_{n_2} = (\{T_{i_1} \times x_{i_2}, T_{o_1} \times o_2\}, \{(I_{i_1} \times x_{i_2}, I_{o_1} \times o_2)\}).\]

Let \( x \in V_1 \) and \( y_1, y_2 \in E_1 \). Then

\[
\begin{align*}
T_{i_1} \times x_{i_2} &\in (x_1, x_2) = \min(T_{i_1}, T_{o_1}, T_{o_2}), \\
\min(\min(T_{i_1}, T_{o_1}, T_{o_2}), \min(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\min(\min(T_{i_1}, T_{o_1}, T_{o_2}), \min(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\max(\max(T_{i_1}, T_{o_1}, T_{o_2}), \max(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\min(\min(T_{i_1}, T_{o_1}, T_{o_2}), \min(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\max(\max(T_{i_1}, T_{o_1}, T_{o_2}), \max(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\min(\min(T_{i_1}, T_{o_1}, T_{o_2}), \min(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\max(\max(T_{i_1}, T_{o_1}, T_{o_2}), \max(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\min(\min(T_{i_1}, T_{o_1}, T_{o_2}), \min(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\max(\max(T_{i_1}, T_{o_1}, T_{o_2}), \max(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\min(\min(T_{i_1}, T_{o_1}, T_{o_2}), \min(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\max(\max(T_{i_1}, T_{o_1}, T_{o_2}), \max(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\min(\min(T_{i_1}, T_{o_1}, T_{o_2}), \min(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\max(\max(T_{i_1}, T_{o_1}, T_{o_2}), \max(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\min(\min(T_{i_1}, T_{o_1}, T_{o_2}), \min(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\max(\max(T_{i_1}, T_{o_1}, T_{o_2}), \max(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\min(\min(T_{i_1}, T_{o_1}, T_{o_2}), \min(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\max(\max(T_{i_1}, T_{o_1}, T_{o_2}), \max(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\min(\min(T_{i_1}, T_{o_1}, T_{o_2}), \min(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\max(\max(T_{i_1}, T_{o_1}, T_{o_2}), \max(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\min(\min(T_{i_1}, T_{o_1}, T_{o_2}), \min(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\max(\max(T_{i_1}, T_{o_1}, T_{o_2}), \max(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\min(\min(T_{i_1}, T_{o_1}, T_{o_2}), \min(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]

\[
\begin{align*}
\max(\max(T_{i_1}, T_{o_1}, T_{o_2}), \max(T_{o_1}, T_{o_2}, T_{o_2})).
\end{align*}
\]
Example: Let \( \tilde{G}_{12} = (M_1, N_{12}) \) and \( \tilde{G}_{22} = (M_2, N_{22}) \) be two neutrosophic cubic graph structures defined on \( \tilde{G}_1 \) and \( \tilde{G}_2 \) respectively, where

\[
\begin{align*}
I_{M_1} &= \{(a, (0.3,0.4),0.7), (0.5,0.6),0.4), (0,4,0.5),0.4)\} \\
I_{M_2} &= \{(x, (0.4,0.5),0.2), (0.6,0.7),0.4), (0.2,0.3),0.6)\} \\
I_{N_{11}} &= \{(a, (0.3,0.4),0.7), (0.5,0.6),0.3), (0,4,0.5),0.3)\} \\
I_{N_{12}} &= \{(x, (0.4,0.5),0.2), (0.6,0.7),0.3), (0.2,0.3),0.3)\} \\
I_{N_{21}} &= \{(a, (0.3,0.4),0.7), (0.5,0.6),0.5), (0,4,0.5),0.5)\} \\
I_{N_{22}} &= \{(x, (0.4,0.5),0.2), (0.6,0.7),0.5), (0.2,0.3),0.5)\}
\end{align*}
\]

Then \( \tilde{G}_{12} \) will be

\[
\begin{align*}
\tilde{G}_{12} &= \{(a, (0.3,0.4),0.7), (0.5,0.6),0.4), (0,4,0.5),0.4)\} \\
\tilde{G}_{12} &= \{(x, (0.4,0.5),0.2), (0.6,0.7),0.4), (0.2,0.3),0.6)\} \\
\end{align*}
\]

**Proposition 3.5** The composition of two neutrosophic cubic graph structures is again a neutrosophic cubic graph structure.

**Proof.** Condition is obvious for \( M_1 \circ M_2 \). We will prove it for \( N_{12} \circ N_{22} \). Let \( x_1, x_2, y_1, y_2 \in V_1 \) and \( x_1, y_2 \in V_2 \). Then

\[
\begin{align*}
\tilde{T}_{c_{12}} &= \min(\tilde{T}_{c_1}(x_1, x_2)(y_1, y_2)) \\
\tilde{T}_{d_{12}} &= \max(\tilde{T}_{d_1}(x_1, y_2)(x_2, y_2)) \\
\tilde{I}_{c_{12}} &= \min(\tilde{I}_{c_1}(x_1, x_2)(y_1, y_2)) \\
\tilde{I}_{d_{12}} &= \max(\tilde{I}_{d_1}(x_1, y_2)(x_2, y_2)) \\
\end{align*}
\]

Then for \( (x_1, x_2) \) in \( V_1 \times V_1 \), \( (y_1, y_2) \in V_2 \times V_2 \), we have

\[
\begin{align*}
\tilde{T}_{c_{12}} &= \min(\tilde{T}_{c_1}(x_1, x_2)(y_1, y_2)) \\
\tilde{T}_{d_{12}} &= \max(\tilde{T}_{d_1}(x_1, y_2)(x_2, y_2)) \\
\tilde{I}_{c_{12}} &= \min(\tilde{I}_{c_1}(x_1, x_2)(y_1, y_2)) \\
\tilde{I}_{d_{12}} &= \max(\tilde{I}_{d_1}(x_1, y_2)(x_2, y_2)) \\
\end{align*}
\]
This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/ACCESS.2019.2925040, IEEE Access.
\[
\begin{align*}
&\{(T_{A_1}, T_{A_2}, T_{A_3}, T_{A_4}, T_{A_5}, T_{A_6}, T_{A_7}, T_{A_8}, T_{A_9}, T_{A_{10}})}, \\
&\{(T_{B_1}, T_{B_2}, T_{B_3}, T_{B_4}, T_{B_5}, T_{B_6}, T_{B_7}, T_{B_8}, T_{B_9}, T_{B_{10}})}
\end{align*}
\]

\[
\begin{align*}
(T_{A_i} \cup R \cap R)(x) &= \min(T_{A_i}(x), T_{R}(x)) \text{ for } i = 1, 2, \ldots, 10 \\
(T_{B_i} \cup R \cap R)(x) &= \min(T_{B_i}(x), T_{R}(x)) \text{ for } i = 1, 2, \ldots, 10
\end{align*}
\]

\[
\begin{align*}
\{\tilde{T}_{C_1}, \tilde{T}_{C_2}, \tilde{T}_{C_3}, \tilde{T}_{C_4}, \tilde{T}_{C_5}, \tilde{T}_{C_6}, \tilde{T}_{C_7}, \tilde{T}_{C_8}, \tilde{T}_{C_9}, \tilde{T}_{C_{10}}) &= \{\tilde{T}_{C_1}, \tilde{T}_{C_2}, \tilde{T}_{C_3}, \tilde{T}_{C_4}, \tilde{T}_{C_5}, \tilde{T}_{C_6}, \tilde{T}_{C_7}, \tilde{T}_{C_8}, \tilde{T}_{C_9}, \tilde{T}_{C_{10}})
\end{align*}
\]

\[
\begin{align*}
\{\tilde{T}_{D_1}, \tilde{T}_{D_2}, \tilde{T}_{D_3}, \tilde{T}_{D_4}, \tilde{T}_{D_5}, \tilde{T}_{D_6}, \tilde{T}_{D_7}, \tilde{T}_{D_8}, \tilde{T}_{D_9}, \tilde{T}_{D_{10}}) &= \{\tilde{T}_{D_1}, \tilde{T}_{D_2}, \tilde{T}_{D_3}, \tilde{T}_{D_4}, \tilde{T}_{D_5}, \tilde{T}_{D_6}, \tilde{T}_{D_7}, \tilde{T}_{D_8}, \tilde{T}_{D_9}, \tilde{T}_{D_{10}})
\end{align*}
\]

\[
\begin{align*}
\{\tilde{T}_{E_1}, \tilde{T}_{E_2}, \tilde{T}_{E_3}, \tilde{T}_{E_4}, \tilde{T}_{E_5}, \tilde{T}_{E_6}, \tilde{T}_{E_7}, \tilde{T}_{E_8}, \tilde{T}_{E_9}, \tilde{T}_{E_{10}}) &= \{\tilde{T}_{E_1}, \tilde{T}_{E_2}, \tilde{T}_{E_3}, \tilde{T}_{E_4}, \tilde{T}_{E_5}, \tilde{T}_{E_6}, \tilde{T}_{E_7}, \tilde{T}_{E_8}, \tilde{T}_{E_9}, \tilde{T}_{E_{10}})
\end{align*}
\]

\[
\begin{align*}
\{\tilde{T}_{F_1}, \tilde{T}_{F_2}, \tilde{T}_{F_3}, \tilde{T}_{F_4}, \tilde{T}_{F_5}, \tilde{T}_{F_6}, \tilde{T}_{F_7}, \tilde{T}_{F_8}, \tilde{T}_{F_9}, \tilde{T}_{F_{10}}) &= \{\tilde{T}_{F_1}, \tilde{T}_{F_2}, \tilde{T}_{F_3}, \tilde{T}_{F_4}, \tilde{T}_{F_5}, \tilde{T}_{F_6}, \tilde{T}_{F_7}, \tilde{T}_{F_8}, \tilde{T}_{F_9}, \tilde{T}_{F_{10}})
\end{align*}
\]

\[
\begin{align*}
\{\tilde{T}_{G_1}, \tilde{T}_{G_2}, \tilde{T}_{G_3}, \tilde{T}_{G_4}, \tilde{T}_{G_5}, \tilde{T}_{G_6}, \tilde{T}_{G_7}, \tilde{T}_{G_8}, \tilde{T}_{G_9}, \tilde{T}_{G_{10}}) &= \{\tilde{T}_{G_1}, \tilde{T}_{G_2}, \tilde{T}_{G_3}, \tilde{T}_{G_4}, \tilde{T}_{G_5}, \tilde{T}_{G_6}, \tilde{T}_{G_7}, \tilde{T}_{G_8}, \tilde{T}_{G_9}, \tilde{T}_{G_{10}})
\end{align*}
\]

\[
\begin{align*}
\{\tilde{T}_{H_1}, \tilde{T}_{H_2}, \tilde{T}_{H_3}, \tilde{T}_{H_4}, \tilde{T}_{H_5}, \tilde{T}_{H_6}, \tilde{T}_{H_7}, \tilde{T}_{H_8}, \tilde{T}_{H_9}, \tilde{T}_{H_{10}}) &= \{\tilde{T}_{H_1}, \tilde{T}_{H_2}, \tilde{T}_{H_3}, \tilde{T}_{H_4}, \tilde{T}_{H_5}, \tilde{T}_{H_6}, \tilde{T}_{H_7}, \tilde{T}_{H_8}, \tilde{T}_{H_9}, \tilde{T}_{H_{10}})
\end{align*}
\]

Example: Let \(\tilde{G}_{11} = (M_1, N_{11}, S_{11})\) and \(\tilde{G}_{12} = (M_2, N_{12}, S_{12})\) be two neutrosophic cubic graph structures defined on \(\tilde{G}_1\) and \(\tilde{G}_2\) respectively.
Proposition 3.7 The P-union of two neurostochastic cubic graph structures is again a neurostochastic cubic graph structure.

Proof. Let $G_{11} = (M_1, N_{11}, N_{21}, \ldots, N_{n1})$ and $G_{12} = (M_2, N_{12}, N_{22}, \ldots, N_{n2})$ be two neurostochastic cubic graph structures defined on $G_{11} = (V_1, E_{11}, E_{21}, \ldots, E_{1n})$ and $G_{12} = (V_2, E_{21}, E_{22}, \ldots, E_{2n})$ respectively. Since all the conditions for $N_{11}$ and $N_{21}$ are satisfied automatically hence, we only verify conditions for $N_{1i} \cup P M_{2i}$; $i \in 1, 2, \ldots, n$. Let $xy \in E_{11} \cap E_{21}$ then

$$
(\bar{\mathcal{T}}_{c_{1i}} \cup P \bar{\mathcal{T}}_{c_{2i}})(xy) \leq \max(\bar{\mathcal{T}}_{c_{1i}}(xy), \bar{\mathcal{T}}_{c_{2i}}(xy))
$$

P-Union of two neurostochastic cubic graph structures is again a neurostochastic cubic graph structure.

Proposition 3.8 R-Union of two neurostochastic cubic graph structures may not be a neurostochastic cubic graph structure as in above example $I_{11} \cup P I_{21}(ab) = 0.8 \leq \max(0.6, 0.4) = \max(I_{11} \cup P I_{21}(a), I_{11} \cup P I_{21}(b))$ so it is not a neurostochastic cubic graph structure.

Proposition 3.9 Let $G' = (V_1 \cup P V_2, E_{11} \cup P E_{21}, E_{21} \cup P E_{22}, \ldots, E_{n1} \cup P E_{n2})$ be the P-union of $G_1 = (V_1, E_{11}, E_{21}, \ldots, E_{1n})$ and $G_2 = (V_2, E_{21}, E_{22}, \ldots, E_{2n})$. Then every neurostochastic cubic graph structure $G_{11}$ and $G_{12}$ have a common $G_i$.

Proof. We define $M_1, M_2, N_{1i}$ and $N_{2i}$ for $i = 1, 2, \ldots, n$. For $x \in V_1$

$$
\bar{T}_{i_1}(x) = \bar{T}_{i_2}(x), \bar{T}_{j_1}(x) = \bar{T}_{j_2}(x)
$$

If $x \in V_2$

$$
\bar{T}_{i_1}(x) = \bar{T}_{i_2}(x), \bar{T}_{j_1}(x) = \bar{T}_{j_2}(x)
$$

If $xy \in E_{11}$

$$
\bar{T}_{i_1}(x) = \bar{T}_{j_1}(x), \bar{T}_{i_2}(x) = \bar{T}_{j_2}(x)
$$

If $xy \in E_{12}$

$$
\bar{T}_{i_1}(x) = \bar{T}_{j_1}(x), \bar{T}_{i_2}(x) = \bar{T}_{j_2}(x)
$$

so that $M_1, M_2, N_{1i}$ and $N_{2i}$ are neurostochastic cubic sets on $V_1, V_2, E_{11}$, and $E_{12}$ also $M = M_1 \cup P M_2$ and $N_{1i} = N_{11} \cup P N_{2i}$ for $i = 1, 2, \ldots, n$. Now for $x \in E_{11}$ and $t = 1, 2, \ldots, n$, we have

$$
\bar{T}_{i_1}(x) = \bar{T}_{j_1}(x), \bar{T}_{i_2}(x) = \bar{T}_{j_2}(x)
$$
Similarly we can prove it for $(I, I)$ and $(F, F)$. So $G_{ij} = (M_j, N_{j1}, N_{j2}, \ldots, N_{jn})$ is a neutrosophic cubic graph structure of $G_i$; $i = 1, 2$. Thus a neutrosophic cubic graph structure of $G_i$ is $G_{ij} \cup G_{ij}^c$ is the P-union of the neutrosophic cubic graph structures $G_i^c$ and $G_i$. This completes the proof.

**Definition 3.10:** Let $\bar{G}_{51} = (M_1, N_{11}, N_{12}, \ldots, N_{1n})$ and $\bar{G}_{52} = (M_2, N_{21}, N_{22}, \ldots, N_{2n})$ be two neutrosophic cubic graph structures defined on $G_i^c = (V_1, E_1, E_2, E_{31})$ and $G_i = (V_2, E_1, E_2, E_{32})$ respectively. P-join is denoted by $G_{51} \cup G_{52}$ and is defined by $G_{51} \cup G_{52} = (M_1, N_{11}, N_{12}, \ldots, N_{1n}) \cup (M_2, N_{21}, N_{22}, \ldots, N_{2n})$.

\[
(M_1 \cup \bar{M}_1, M_2 \cup \bar{M}_2, M_1 \cup \bar{M}_2, M_2 \cup \bar{M}_1),
\]

\[
\{((T_{A_{1i}} + T_{A_{2i}}), (T_{B_{1i}} + T_{B_{2i}}), (T_{C_{1i}} + T_{C_{2i}}), (T_{D_{1i}} + T_{D_{2i}}), (T_{E_{1i}} + T_{E_{2i}}), (T_{F_{1i}} + T_{F_{2i}}), (T_{G_{1i}} + T_{G_{2i}})),
\]

where $i \in I_1 \cup I_2$.

(ii) if $xy \in E_1 \cup E_{22}$; $i = 1, 2, \ldots, n$

\[
(T_{c_{1i}} + T_{c_{2i}})(xy) = (T_{c_{1i}})(xy) + (T_{c_{2i}})(xy),
\]

\[
(F_{c_{1i}} + F_{c_{2i}})(xy) = (F_{c_{1i}})(xy) + (F_{c_{2i}})(xy),
\]

(iii) if $xy \in E_i^c$, where $E_i$ is the set of all edges joining the vertices of $V_1$ and $V_2$; $i = 1, 2, \ldots, n$

\[
(T_{c_{1i}} + T_{c_{2i}})(xy) = \text{rmin}((T_{c_{1i}})(xy), (T_{c_{2i}})(xy)),
\]

\[
(F_{c_{1i}} + F_{c_{2i}})(xy) = \text{rmin}((F_{c_{1i}})(xy), (F_{c_{2i}})(xy)).
\]

**Definition 3.11** Let $\bar{G}_{51} = (M_1, N_{11}, N_{12}, \ldots, N_{1n})$ and $\bar{G}_{52} = (M_2, N_{21}, N_{22}, \ldots, N_{2n})$ be two neutrosophic cubic graph structures defined on $G_i^c = (V_1, E_{11}, E_{12}, E_{13})$ and $G_i = (V_2, E_{11}, E_{12}, E_{13})$ respectively. R-join is denoted by $G_{51} \otimes G_{52}$ and is defined by $G_{51} \otimes G_{52} = (M_1, N_{11}, N_{12}, \ldots, N_{1n}) \otimes (M_2, N_{21}, N_{22}, \ldots, N_{2n})$.

\[
(M_1 \otimes M_2, M_1 \otimes N_{11} \otimes N_{12} \otimes N_{1n} \otimes M_2, \ldots, N_{2n})
\]

\[
\{((T_{A_{1i}} + T_{A_{2i}}), (T_{B_{1i}} + T_{B_{2i}}), (T_{C_{1i}} + T_{C_{2i}}), (T_{D_{1i}} + T_{D_{2i}}), (T_{E_{1i}} + T_{E_{2i}}), (T_{F_{1i}} + T_{F_{2i}}), (T_{G_{1i}} + T_{G_{2i}})),
\]

where $i \in I_1 \times I_2$.

(ii) if $x \in X_1$ and $y \in X_2$; $i = 1, 2, \ldots, n$

\[
(T_{c_{1i}} + T_{c_{2i}})(xy) = \text{rmax}((T_{c_{1i}})(xy), (T_{c_{2i}})(xy)),
\]

\[
(F_{c_{1i}} + F_{c_{2i}})(xy) = \text{rmin}((F_{c_{1i}})(xy), (F_{c_{2i}})(xy)).
\]

**Proposition 3.12** The P-join of two neutrosophic cubic graph structures is again a neutrosophic cubic graph structure.

**Proof.** Straightforward.

**Definition 3.13** Let $\bar{G}_{51} = (M_1, N_{11}, N_{12}, \ldots, N_{1n})$ and $\bar{G}_{52} = (M_2, N_{21}, N_{22}, \ldots, N_{2n})$ be two neutrosophic cubic graph structures defined on $G_i^c = (V_1, E_{11}, E_{12}, E_{13})$ and $G_i = (V_2, E_{11}, E_{12}, E_{13})$ respectively. The cross product is denoted by $G_{51} \times G_{52}$ and is defined by $G_{51} \times G_{52} = (M_1 \times M_2, N_{11} \times N_{21}, \ldots, N_{1n} \times N_{2n})$.

\[
\{((T_{A_{1i}} + T_{A_{2i}}), (T_{B_{1i}} + T_{B_{2i}}), (T_{C_{1i}} + T_{C_{2i}}), (T_{D_{1i}} + T_{D_{2i}}), (T_{E_{1i}} + T_{E_{2i}}), (T_{F_{1i}} + T_{F_{2i}}), (T_{G_{1i}} + T_{G_{2i}})),
\]

where $i \in I_1 \times I_2$.

(ii) if $x \in X_1$ and $y \in X_2$; $i = 1, 2, \ldots, n$

\[
(T_{c_{1i}} + T_{c_{2i}})(xy) = \text{rmax}((T_{c_{1i}})(xy), (T_{c_{2i}})(xy)),
\]

\[
(F_{c_{1i}} + F_{c_{2i}})(xy) = \text{rmin}((F_{c_{1i}})(xy), (F_{c_{2i}})(xy)).
\]
\[(\overline{F}_{c1} \ast \overline{F}_{c2})(x_1y_1)(x_2y_2) = \max(\overline{F}_{c1}(x_1y_1), \overline{F}_{c2}(x_2y_2))\]
\[(F_{d_1} \ast F_{d_2})(x_1y_1)(x_2y_2) = \min(F_{d_1}(x_1y_1), F_{d_2}(x_2y_2))\]

Example: Let \(G_{c1} = (M_1, N_{c11}, N_{c12})\) and \(G_{c2} = (M_2, N_{c21}, N_{c22})\) be two neutrosophic cubic graph structures defined on \(G_1\) and \(G_2\) respectively, where
\[lM_1 = \{(a, 0.4, 0.5, 0.3), (0.3, 0.4, 0.6), (0.6, 0.7, 0.5)\}
\[\{b, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3)\}\]
\[= (\{a, b\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]
\[= (\{b, a\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]
\[= (\{c, a\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]
\[= (\{d, a\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]
\[= (\{e, a\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]
\[= (\{f, a\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]
\[= (\{g, a\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]
\[= (\{h, a\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]
\[= (\{i, a\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]

Then \(G_{c1} \ast G_{c2}\) will be
\[lM_2 = \{(a, 0.4, 0.5, 0.3), (0.3, 0.4, 0.6), (0.6, 0.7, 0.5)\}
\[\{b, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3)\}\]
\[= (\{a, b\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]
\[= (\{b, a\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]
\[= (\{c, a\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]
\[= (\{d, a\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]
\[= (\{e, a\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]
\[= (\{f, a\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]
\[= (\{g, a\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]
\[= (\{h, a\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]
\[= (\{i, a\}, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.1, 0.2, 0.3))\]

Proposition 3.14 The cross product of two neutrosophic cubic graph structures is again a neutrosophic cubic graph structure.

Proof: Let \(G_{c1} = (M_1, N_{c11}, N_{c12}, \ldots, N_{c1n})\) and \(G_{c2} = (M_2, N_{c21}, N_{c22}, \ldots, N_{c2n})\) be two neutrosophic cubic graph structures defined on \(G_1\) and \(G_2\) respectively.

Condition is obvious for \(M_1 \ast M_2\). Therefore we verify for \(N_{c11} \ast N_{c22}; n = 1, 2, \ldots, n\)
\[N_{c11} \ast N_{c22} = (\{(T_{c11}, c_{12}), (T_{c12}, c_{12}), (T_{c13}, c_{12}), (T_{c14}, c_{12})\})\]
This work is licensed under a Creative Commons Attribution 3.0 License. For more information, see http://creativecommons.org/licenses/by/3.0/.
\[ T_{01\Delta T_{02}}(x,y)(x,y) = (T_{01} \circledast T_{02})(x,y)(x,y) = \text{max}(T_{01}(x,x), T_{02}(y)) \]

\[ I_{01\Delta T_{02}}(x,y)(x,y) = (I_{01} \circledast I_{02})(x,y)(x,y) = \text{rmin}(I_{01}(x,x), I_{02}(y)) \]

\[ \tilde{F}_{01\Delta T_{02}}(x,y)(x,y) = (\tilde{F}_{01} \circledast \tilde{F}_{02})(x,y)(x,y) = \text{rmax}(\tilde{F}_{01}(x,x), \tilde{F}_{02}(y)) \]

\[ F_{01\Delta T_{02}}(x,y)(x,y) = (F_{01} \circledast F_{02})(x,y)(x,y) = \text{min}(F_{01}(x,x), F_{02}(y)) \]

(iv) if \( x_1, x_2 \in E_{11} \) and \( y_1, y_2 \in E_{22}, i = 1, 2, \ldots, n \)

\[ \tilde{T}_{01\Delta T_{02}}(x_1, y_1)(x_2, y_2) = (\tilde{T}_{01} \circledast \tilde{T}_{02})(x_1, y_1)(x_2, y_2) = \text{rmin}(\tilde{T}_{01}(x_1, y_1), \tilde{T}_{02}(x_2, y_2)) \]

\[ T_{01\Delta T_{02}}(x_1, y_1)(x_2, y_2) = (T_{01} \circledast T_{02})(x_1, y_1)(x_2, y_2) = \text{max}(T_{01}(x_1, x_2), T_{02}(y_1, y_2)) \]

Example: Let \( \tilde{G}_{12} = (M_{11}, N_{12}, N_{22}) \) and \( \tilde{G}_{21} = (M_{21}, N_{22}, N_{12}) \) be two neutrosophic cubic graph structures defined on \( \tilde{G}_{12} \) and \( \tilde{G}_{21} \) respectively, where

\[ l_{M} = \{(a, (0.4, 0.5, 0.3), (0.3, 0.4, 0.6), (0.6, 0.7, 0.5)), (b, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (1, 0.1, 0.3)), (c, (0.4, 0.6, 0.3), (0.5, 0.6, 0.2), (0.7, 0.8, 0.2)), (d, (0.2, 0.3, 0.6), (0.3, 0.4, 0.6), (0.6, 0.7, 0.3)), (e, (0.1, 0.2, 0.5), (0.3, 0.4, 0.6), (0.7, 0.8, 0.2))\} \]

\[ l_{M_2} = \{\{(x, (0.2, 0.3, 0.5), (0.6, 0.7, 0.4)), (y, (0.5, 0.6, 0.2), (0.7, 0.8, 0.3), (0.1, 0.2, 0.5))\} \]

Then \( \tilde{G}_{21} \circledast \tilde{G}_{21} \) will be

\[ l_{M_2} = \{(a, (0.2, 0.3, 0.5), (0.3, 0.4, 0.6), (0.6, 0.7, 0.4)), (b, (0.2, 0.3, 0.6), (0.4, 0.5, 0.2), (0.6, 0.7, 0.5)), (c, (0.2, 0.3, 0.5), (0.5, 0.6, 0.3), (0.7, 0.8, 0.2)), (d, (0.2, 0.3, 0.6), (0.3, 0.4, 0.6), (0.6, 0.7, 0.3)), (e, (0.2, 0.3, 0.5), (0.3, 0.4, 0.6), (0.6, 0.7, 0.4))\} \]

**Proposition 3.16** The strong product of two neutrosophic cubic graph structures is again a neutrosophic cubic graph structure.

Proof. Let \( \tilde{G}_{12} = (M_{11}, N_{12}, N_{22}, \ldots, N_{n_2}) \) and \( \tilde{G}_{21} = (M_{21}, N_{22}, N_{12}, \ldots, N_{n_1}) \) be two neutrosophic cubic graph structures defined on \( \tilde{G}_{12} = (V_{11}, E_{11}, L_{11}, I_{11}, R_{11}) \) and \( \tilde{G}_{21} = (V_{21}, E_{21}, L_{21}, I_{21}, R_{21}) \) respectively. Condition is obvious for \( M_{1}, \tilde{M}_{2} \). Therefore we verify for \( N_{11} \circledast N_{12}, n = 1, 2, \ldots, n, \) where

\[ N_{11} \circledast N_{12} = \{(\tilde{T}_{01\Delta T_{02}}, \tilde{I}_{01\Delta T_{02}}), (\tilde{I}_{01\Delta T_{02}}, \tilde{T}_{01\Delta T_{02}}), (\tilde{F}_{01\Delta T_{02}}, \tilde{F}_{01\Delta T_{02}})\} \]

(i) Let \( x \in V_{1}, y \in V_{2}, i = 1, 2, \ldots, n \)

\[ \tilde{T}_{c_{1\Delta c_{2}}}(x,y)(x,y) = \text{rmin}((\tilde{T}_{c_{1}}(x), \tilde{T}_{c_{2}}(y))) \]

\[ \tilde{T}_{b_{1\Delta b_{2}}}(x,y)(x,y) = \text{rmin}((\tilde{T}_{b_{1}}(x), \tilde{T}_{b_{2}}(y))) \]

\[ \tilde{T}_{a_{1\Delta a_{2}}}(x,y)(x,y) = \text{rmin}((\tilde{T}_{a_{1}}(x), \tilde{T}_{a_{2}}(y))) \]

\[ \tilde{T}_{a_{1\Delta a_{2}}}(x,y)(x,y) = \text{rmax}((\tilde{T}_{a_{1}}(x), \tilde{T}_{a_{2}}(y))) \]

(ii) Let \( x_1, x_2 \in E_{11} \) and \( y_1, y_2 \in E_{22} \)

\[ \tilde{T}_{c_{1\Delta c_{2}}}(x_1, y_1)(x_2, y_2) = \text{rmin}((\tilde{T}_{c_{1}}(x_1, x_2), \tilde{T}_{c_{2}}(y_1, y_2))) \]

\[ \tilde{T}_{b_{1\Delta b_{2}}}(x_1, y_1)(x_2, y_2) = \text{rmax}((\tilde{T}_{b_{1}}(x_1, x_2), \tilde{T}_{b_{2}}(y_1, y_2))) \]

\[ \tilde{T}_{a_{1\Delta a_{2}}}(x_1, y_1)(x_2, y_2) = \text{rmin}((\tilde{T}_{a_{1}}(x_1, x_2), \tilde{T}_{a_{2}}(y_1, y_2))) \]

\[ \tilde{T}_{a_{1\Delta a_{2}}}(x_1, y_1)(x_2, y_2) = \text{rmax}((\tilde{T}_{a_{1}}(x_1, x_2), \tilde{T}_{a_{2}}(y_1, y_2))) \]
(iii) Let \(x_1, x_2 \in E_{M_i}^i\) and \(y_1, y_2 \in E_{G_2}^i\), \(i = 1, 2, \ldots, n\)
\[
\min\left(\tilde{T}_{c_1}^i(x_1, x_2), \tilde{T}_{c_2}^i(y_1, y_2)\right)
\]
\[
\min\min\left(\tilde{T}_{c_1}^i(x_1, x_2), \tilde{T}_{c_2}^i(y_1, y_2)\right)
\]

\[
\text{Example:} \text{ Let } \mathcal{G}_1 \text{ and } \mathcal{G}_2 \text{ be two neurostrophic cubic graph structures as shown in figure 13. Then their lexicographic product will be}
\]
\[
M_1 \cdot M_2
\]
\[
\{(a, x), ([0.2,0.3],[0.5],[0.3,0.4],[0.6]), ([0.6,0.7],[0.4]),
\]
\[
\{(b, y), ([0.4,0.5],[0.3],[0.3,0.4],[0.6]), ([0.6,0.7],[0.5])\}
\]

\[
\mathcal{G}_1 \text{ and } \mathcal{G}_2 \text{ are two neurostrophic cubic graph structures defined on}
\]
\[
\mathcal{G}_1 = (V_1, E_{M_1}^i, E_{G_1}^i) \text{ and } \mathcal{G}_2 = (V_2, E_{M_2}^i, E_{G_2}^i) \text{ respectively. The lexicographic product is denoted by}
\]
\[
M_1 \cdot M_2
\]
\[
G_1 \cdot G_2
\]

\[
\text{Proposition 3.17} \text{ The lexicographic product of two neurostrophic cubic graph structures is again a neurostrophic cubic graph structure.}
\]

\[
\text{Proof.} \text{ Let } \mathcal{G}_1 = (M_1, N_{M_1}, N_{G_1}, \ldots, N_{M_1}) \text{ and } \mathcal{G}_2 = (M_2, N_{M_2}, N_{G_2}, \ldots, N_{M_2}) \text{ be two neurostrophic cubic graph structures defined on}
\]
\[
\mathcal{G}_1 = (V_1, E_{M_1}^i, E_{G_1}^i) \text{ and } \mathcal{G}_2 = (V_2, E_{M_2}^i, E_{G_2}^i) \text{ respectively. Condition is obvious for}
\]
\[
\mathcal{M}_1 \cdot \mathcal{M}_2
\]
\[
\text{Therefore we verify for } N_{1, \ldots, n} = 1, 2, \ldots, n, \text{ where}
\]
\[
\mathcal{G}_1 \cdot \mathcal{G}_2 = (V_1 \times V_2, E_{M_1}^i \cdot E_{M_2}^i, E_{G_1}^i \cdot E_{G_2}^i)
\]

\[
\text{(i)} \text{ Let } x \in V_1 \text{ and } y \in V_2 \text{ such that } i = 1, 2, \ldots, n
\]
\[
\hat{T}_{c_1}^i(x_1, y_2) = \min\left(\hat{T}_{c_1}^i(x_1, y_2), \hat{T}_{c_1}^i(x_1, y_2)\right)
\]
\[
\hat{T}_{c_1}^i(x_1, y_2) = \min\left(\hat{T}_{c_1}^i(x_1, y_2), \hat{T}_{c_1}^i(x_1, y_2)\right)
\]

\[
\text{(ii) Let } x \in V_1 \text{ and } y \in V_2 \text{ such that } i = 1, 2, \ldots, n
\]
\[
\hat{T}_{c_1}^i(x_1, y_2) = \min\left(\hat{T}_{c_1}^i(x_1, y_2), \hat{T}_{c_1}^i(x_1, y_2)\right)
\]
\[
\hat{T}_{c_1}^i(x_1, y_2) = \min\left(\hat{T}_{c_1}^i(x_1, y_2), \hat{T}_{c_1}^i(x_1, y_2)\right)
\]
rmin(rmin(\(\mathcal{T}_A(x_1), \mathcal{T}_A(y_1)\)), rmin(\(\mathcal{T}_A(x_2), \mathcal{T}_A(y_2)\)))

\[ T_{D_1}(x_1,y_1)(x_2,y_2) = \max(T_{D_1}(x_1,x_2), T_{D_1}(y_1,y_2)) \]

Similarly we can show it for \(\mathcal{F}_{D_1}\) and \(\mathcal{F}_{D_2}\). This completes the proof.

IV. Application in Multiple Attribute Group Decision Making Problem

In this section we discuss a multiple attribute group decision making problem and developed an algorithm.

Graphs are very important in daily life and allow us to study the behavior of something quickly. Graphs allow us to make a mental image of the data, so we can say that graphs help us to build a bridge between the abstract and the real. For too long we as humans have taken too much work upon our shoulders, its time to simplify our life and to use the best tool for the job. Graphing is one of these tools that might be used in such circumstances. Graphs are used in everyday life, from the local newspaper to the magazine stand. In computer science graphs are used to represent the flow of computation, used to measure the trafficking to a site, also used in fraud detection etc. So it is one of these skills that you simply cannot do without the help of graphs. Graphs can help us and make our life simpler from student to professionals. Fuzzy graph theory has been used in the world of Mathematics due to its effective applications.

We first provide an algorithm and then we discuss an example.

Algorithm:
1. Select the set \(V = \{A_1, A_2, \ldots, A_n\}\) of alternatives as a vertex set from the problem which is under study and select the membership grade for each element in the vertex set based on certain attributes.
2. Select the set \(E = \{E_{11}, E_{12}, E_{21}, \ldots, E_{n1}\}\) of attributes or criteria as the set of edges.
3. Use the Definition 3.1 of neutrosophic cubic graphs structures for finding the membership grade of each \(E_{ii}\) for \(i = 1,2,3, \ldots, n\).
4. After having the values of \(V\) and \(E\), draw the graph.
5. Find the strength of each edge using the following definition and compare them.

Definition 4.1 Let \(E = \{E_{ij}\}\) be a edge having neutrosophic cubic value and we define strength of edge as

\[ S(E) = M = \left\{ \begin{array}{ll} A, ([0.3,0.4],[0.6],[0.0,0.1],[0.8,0.9]) \\ B, ([0.4,0.5],[0.2],[0.7,0.8],[0.0,0.0]) \\ C, ([0.5,0.6],[0.4],[0.2,0.3],[0.6,0.5]) \\ D, ([0.1,0.2],[0.5],[0.3,0.4],[0.9,0.0]) \end{array} \right. \]

where

\[ E_{11} = (C_{11}, D_{11}) \]

\[ = (\mathcal{T}_{C_{11}}, \mathcal{T}_{D_{11}}), (\mathcal{F}_{C_{11}}, \mathcal{F}_{D_{11}}) \]

\[ = \{ \text{religious beliefs, religious festivals, effects of religion on society} \} \]

\[ E_{21} = (C_{21}, D_{21}) \]

\[ = (\mathcal{T}_{C_{21}}, \mathcal{T}_{D_{21}}), (\mathcal{F}_{C_{21}}, \mathcal{F}_{D_{21}}) \]

\[ = \{ \text{import, export, exchange} \} \]

\[ E_{31} = (C_{31}, D_{31}) \]

\[ = (\mathcal{T}_{C_{31}}, \mathcal{T}_{D_{31}}), (\mathcal{F}_{C_{31}}, \mathcal{F}_{D_{31}}) \]

\[ = \{ \text{Army, boundaries, intelligence agencies} \} \]

As the above given factors highly effect the relations among countries. These factors are responsible for the peace or war between two countries.

3. Using the Definition 3.1 we have

\[ N_{11} = \{ \mathcal{T}_{AB}, ([0.3,0.4],[0.6],[0.0,0.1],[0.8,0.9]) \}

\[ \{ \mathcal{T}_{CD}, ([0.4,0.5],[0.2],[0.7,0.8],[0.0,0.0]) \}

\[ \{ \mathcal{T}_{AD}, ([0.5,0.6],[0.4],[0.2,0.3],[0.6,0.5]) \}

\[ \{ \mathcal{T}_{BD}, ([0.1,0.2],[0.5],[0.3,0.4],[0.9]) \} \]

\[ N_{21} = \{ \mathcal{T}_{BC}, ([0.4,0.5],[0.2],[0.7,0.8],[0.0,0.0]) \}

\[ \{ \mathcal{T}_{AD}, ([0.5,0.6],[0.4],[0.2,0.3],[0.6,0.5]) \}

\[ \{ \mathcal{T}_{BD}, ([0.1,0.2],[0.5],[0.3,0.4],[0.9]) \} \]

\[ N_{31} = \{ \mathcal{T}_{AC}, ([0.3,0.4],[0.6],[0.2,0.3],[0.6,0.5]) \}

\[ \{ \mathcal{T}_{BD}, ([0.1,0.2],[0.5],[0.3,0.4],[0.9]) \} \]

4. Draw the graph as under:

Strength of edges is as under using the Definition 4.1, we have

\[ S(AB) = 0.4 \]

\[ S(AC) = 0.6 \]

\[ S(AD) = 0.3 \]

\[ S(BD) = 0.9 \]

\[ S(DC) = 0.9 \]

\[ S(BC) = 1.3 \]

It is shown in the following figure:

\[ S(BC) > S(DC) = S(BD) > S(AC) > S(AD) > S(AB) \]

Thus we can concluded that the countries \(B\) and \(C\) have strong relations between each other.
V. Comparative Analysis and Conclusions:
All versions of neutrosophic sets like, single valued neutrosophic set, interval valued neutrosophic set and neutrosophic cubic set are used in literature so far for the applications of neutrosophic sets. But neutrosophic cubic sets are a more generalized tool to handle imprecision and vagueness and all other versions of neutrosophic sets are the special cases of it. On the other sides we have the comparison between the different types of graphs as shown the following table:

<table>
<thead>
<tr>
<th>Type of Graph</th>
<th>Advantages and Limitations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crisp Graphs</td>
<td>These can handle only exact information</td>
</tr>
<tr>
<td>Fuzzy Graphs</td>
<td>These can handle imprecise and vague information but only can handle only the positive aspects.</td>
</tr>
<tr>
<td>Intuitionistic Fuzzy Graphs</td>
<td>These can handle both positive and negative aspects, but it is not always possible to assign a single membership and non-membership value.</td>
</tr>
<tr>
<td>Single values Neutrosophic Graphs</td>
<td>These can handle positive, negative and hesitant information’s in a much better way as compared to previous ones. But like intuitionistic fuzzy graphs it is not always possible to assign a single membership and non-membership value.</td>
</tr>
<tr>
<td>Interval-valued Neutrosophic Graphs</td>
<td>It can handle many problems as compared to previous. Yet have some limitations which can be handled through the hybrid version of neutrosophic cubic graphs.</td>
</tr>
<tr>
<td>Neutrosophic Cubic Graphs</td>
<td>This is the most generalized version of fuzzy graphs and it can handle many imprecise and vague problems. But in Neutrosophic Cubic Graphs the number of the set of edges is the only one. When the number of edges is more than one then we need the concept of neutrosophic cubic structures.</td>
</tr>
</tbody>
</table>

So, we used the concept of neutrosophic cubic sets in this paper with the concept of neutrosophic cubic structures. We have observed that by increasing the set of edges we can find more insight of the problem which is not possible through a single set of edges. In this paper we discussed the idea of neutrosophic cubic graph structures, and different operations on it such as Cartesian product, composition, P-union, R-union, P-join, R-join, cross product, strong product and lexicographic product. We provided different examples and results related to these operations. We also observed that R-union of two neutrosophic cubic graph structures may not be a neutrosophic cubic graph structure. Further we provided applications of neutrosophic cubic graph structures. In future we will try to different kinds of neutrosophic cubic graphs structures and will explore more results related with the application in real life.

Acknowledgment
This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2016R1D1A1A09919551).

REFERENCES
[9]. M Gulistan, H Wahab, F Smarandache, S Khan, S Shah, Some linguistic neutrosophic cubic mean operators and entropy with applications in a corporation to choose an area supervisor, Symmetry 10 (10), 428.

VOLUME XX, 2018


Seungmin Rho received his M.S. and Ph.D degrees in Computer Science from Ajou University, Korea in 2003 and 2008, respectively. He visited Multimedia Systems and Networking Lab. in Univ. of Texas at Dallas from Dec. 2003 to March 2004. Before he joined the Computer Sciences Department of Ajou University, he spent two years in industry. In 2008–2009, he was a Postdoctoral Research Fellow at the Computer Music Lab of the School of Computer Science in Carnegie Mellon University. He had been working as a Research Professor at School of Electrical Engineering in Korea University during 2009–2011. In 2012, he was an assistant professor at Division of Information and Communication in Baekseok University. Now he is currently an assistant professor at Department of Media Software at Sungkyul University. His current research interests include database, big data analysis, music retrieval, multimedia systems, machine learning, knowledge management as well as computational intelligence. He has published 100 papers in refereed journals and conference proceedings in these areas. He has been involved in more than 20 conferences and workshops as various chairs and more than 30 conferences/workshops as a program committee member. He has edited a number of international journal special issues as a guest editor, such as Multimedia Systems, Information Fusion, Engineering Applications of Artificial Intelligence, New Review of Hypermedia and Multimedia, Multimedia Tools and Applications, Personal and Ubiquitous Computing, Telecommunication Systems, Ad Hoc & Sensor Wireless Networks and etc. He has received a few awards including Who’s Who in America, Who’s Who in Science and Engineering, and Who’s Who in the World in 2007 and 2008, respectively.

SEIFEDINE KADRY received the bachelor's degree in applied mathematics from Lebanese University, in 1999, the M.S. degree in computation from Reims University, France, and EPFL, Lausanne, in 2002, the Ph.D. degree in applied statistics from Blaise Pascal University, France, in 2007, and the HDR degree from Rouen University, in 2017. He is currently an ABET Program Evaluator. His research interests include education using technology, system prognostics, stochastic systems, and probability and reliability analyses.