Positive implicative MBJ-neutrosophic ideals of
$BCK/BCI$-algebras

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ABSTRACT. The notion of positive implicative MBJ-neutrosophic ideal is introduced, and several properties are investigated. Relations between MBJ-neutrosophic ideal and positive implicative MBJ-neutrosophic ideal are discussed. Characterizations of positive implicative MBJ-neutrosophic ideal are displayed.

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1. Introduction

In order to handle uncertainties in many real applications, the fuzzy set was introduced by L.A. Zadeh [19] in 1965. In 1983, K. Atanassov introduced the notion of intuitionistic fuzzy set as a generalization of fuzzy set. As a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set, the notion of neutrosophic set is initiated by Smarandache ([14], [15] and [16]). Neutrosophic algebraic structures in $BCK/BCI$-algebras are discussed in the papers [1], [2], [4], [5], [6], [7], [8], [12], [13], [17] and [18]. In [10], the notion of MBJ-neutrosophic sets is introduced as another generalization of neutrosophic set, it is applied to $BCK/BCI$-algebras. Mohseni et al. [10] introduced the concept of MBJ-neutrosophic subalgebras in $BCK/BCI$-algebras, and investigated related properties. They gave a characterization of MBJ-neutrosophic subalgebra, and established a new MBJ-neutrosophic subalgebra by using an MBJ-neutrosophic subalgebra of a $BCI$-algebra. They considered the homomorphic inverse image of MBJ-neutrosophic subalgebra, and discussed translation of MBJ-neutrosophic subalgebra. Jun and Roh [11] applied the notion of MBJ-neutrosophic sets to ideals of $BCK/BI$-algebras, and introduce the concept of MBJ-neutrosophic ideals in $BCK/BCI$-algebras. They provided a condition for an
MBJ-neutrosophic subalgebra to be an MBJ-neutrosophic ideal in a BCK-algebra, and considered conditions for an MBJ-neutrosophic set to be an MBJ-neutrosophic ideal in a BCK/BCI-algebra. They discussed relations between MBJ-neutrosophic subalgebras, MBJ-neutrosophic ◦-subalgebras and MBJ-neutrosophic ideals. In a BCI-algebra, they provided conditions for an MBJ-neutrosophic ideal to be an MBJ-neutrosophic subalgebra, and considered a characterization of an MBJ-neutrosophic ideal in an (S)-BCK-algebra.

In this paper, we introduce the notion of positive implicative MBJ-neutrosophic ideal, and investigate several properties. We discuss relations between MBJ-neutrosophic ideal and positive implicative MBJ-neutrosophic ideal. We provide characterizations of positive implicative MBJ-neutrosophic ideal.

2. Preliminaries

By a BCI-algebra, we mean a set \( X \) with a binary operation \( * \) and a special element \( 0 \) that satisfies the following conditions:

(I) \( ((x * y) * (x * z)) * (z * y) = 0, \)

(II) \( (x * (x * y)) * y = 0, \)

(III) \( x * x = 0, \)

(IV) \( x * y = 0, y * x = 0 \Rightarrow x = y, \)

for all \( x, y, z \in X \). If a BCI-algebra \( X \) satisfies the following identity:

(V) \( (\forall x \in X) (0 * x = 0), \)

then \( X \) is called a BCK-algebra.

Every BCK/BCI-algebra \( X \) satisfies the following conditions:

(2.1) \( (\forall x \in X) (x * 0 = x), \)

(2.2) \( (\forall x, y, z \in X) (x \leq y \Rightarrow x * z \leq y * z, z * y \leq z * x), \)

(2.3) \( (\forall x, y, z \in X) ((x * y) * z = (x * z) * y), \)

(2.4) \( (\forall x, y, z \in X) ((x * z) * (y * z) \leq x * y) \)

where \( x \leq y \) if and only if \( x * y = 0. \)

A nonempty subset \( S \) of a BCK/BCI-algebra \( X \) is called a subalgebra of \( X \) if \( x * y \in S \), for all \( x, y \in S \). A subset \( I \) of a BCK/BCI-algebra \( X \) is called an ideal of \( X \), if it satisfies:

(2.5) \( 0 \in I, \)

(2.6) \( (\forall x \in X) (\forall y \in I) (x * y \in I \Rightarrow x \in I). \)

A subset \( I \) of a BCK-algebra \( X \) is called a positive implicative ideal of \( X \) (see [9]), if it satisfies (2.5) and

(2.7) \( (\forall x, y, z \in X)(((x * y) * z \in I, y * z \in I \Rightarrow x * z \in I). \)

By an interval number, we mean a closed subinterval \( \tilde{a} = [a^-, a^+] \) of \( I \), where \( 0 \leq a^- \leq a^+ \leq 1 \). Denote by \([I]\) the set of all interval numbers. Let us define what is known as refined minimum (briefly, rmin) and refined maximum (briefly, rmax) of two elements in \([I]\). We also define the symbols \( \geq, \leq, = \) in case of two
elements in \([I]\). Consider two interval numbers \(\tilde{a}_1 := [a_{1-}, a_{1+}]\) and \(\tilde{a}_2 := [a_{2-}, a_{2+}]\). Then
\[
\begin{align*}
\text{rmin} \{\tilde{a}_1, \tilde{a}_2\} &= \left[\min \left\{a_{1-}, a_{2-}\right\}, \min \left\{a_{1+}, a_{2+}\right\}\right], \\
\text{rmax} \{\tilde{a}_1, \tilde{a}_2\} &= \left[\max \left\{a_{1-}, a_{2-}\right\}, \max \left\{a_{1+}, a_{2+}\right\}\right], \\
\tilde{a}_1 \succeq \tilde{a}_2 & \Leftrightarrow a_{1-} \geq a_{2-}, a_{1+} \geq a_{2+},
\end{align*}
\]
and similarly, we may have \(\tilde{a}_1 \preceq \tilde{a}_2\) and \(\tilde{a}_1 = \tilde{a}_2\). To say \(\tilde{a}_1 \succ \tilde{a}_2\) (resp. \(\tilde{a}_1 \prec \tilde{a}_2\)), we mean \(\tilde{a}_1 \geq \tilde{a}_2\) and \(\tilde{a}_1 \neq \tilde{a}_2\) (resp. \(\tilde{a}_1 \leq \tilde{a}_2\) and \(\tilde{a}_1 \neq \tilde{a}_2\)). Let \(\tilde{a}_i \in [I]\), where \(i \in \Lambda\). We define
\[
\begin{align*}
\inf_{i \in \Lambda} \tilde{a}_i &= \left[\inf_{i \in \Lambda} a_{1-}^i, \inf_{i \in \Lambda} a_{1+}^i\right] \quad \text{and} \quad \sup_{i \in \Lambda} \tilde{a}_i &= \left[\sup_{i \in \Lambda} a_{1-}^i, \sup_{i \in \Lambda} a_{1+}^i\right].
\end{align*}
\]

Let \(X\) be a nonempty set. A function \(A : X \to [I]\) is called an interval-valued fuzzy set (briefly, an IVF set) in \(X\). Let \([I]^X\) stand for the set of all IVF sets in \(X\). For every \(A \in [I]^X\) and \(x \in X\), \(A(x) = [A^{-}(x), A^{+}(x)]\) is called the degree of membership of an element \(x\) to \(A\), where \(A^{-} : X \to I\) and \(A^{+} : X \to I\) are fuzzy sets in \(X\) which are called a lower fuzzy set and an upper fuzzy set in \(X\), respectively. For simplicity, we denote \(A = [A^{-}, A^{+}]\).

Let \(X\) be a non-empty set. A neutrosophic set (NS) in \(X\) (see [15]) is a structure of the form:
\[
A := \{x; A_T(x), A_I(x), A_F(x) \mid x \in X\}
\]
where \(A_T : X \to [0, 1]\) is a truth membership function, \(A_I : X \to [0, 1]\) is an indeterminate membership function, and \(A_F : X \to [0, 1]\) is a false membership function.

We refer the reader to the books [3, 9] for further information regarding \(BCK/BCI\)-algebras, and to the site “http://fs.gallup.unm.edu/neutrosophy.htm” for further information regarding neutrosophic set theory.

Let \(X\) be a non-empty set. By an MBJ-neutrosophic set in \(X\) (see [10]), we mean a structure of the form:
\[
\mathcal{A} := \{x; M_A(x), \hat{B}_A(x), J_A(x) \mid x \in X\},
\]
where \(M_A\) and \(J_A\) are fuzzy sets in \(X\), which are called a truth membership function and a false membership function, respectively, and \(\hat{B}_A\) is an IVF set in \(X\) which is called an indeterminate interval-valued membership function.

For the sake of simplicity, we shall use the symbol \(\mathcal{A} = (M_A, \hat{B}_A, J_A)\) for the MBJ-neutrosophic set
\[
\mathcal{A} := \{x; M_A(x), \hat{B}_A(x), J_A(x) \mid x \in X\}.
\]

Let \(X\) be a \(BCK/BCI\)-algebra. An MBJ-neutrosophic set \(\mathcal{A} = (M_A, \hat{B}_A, J_A)\) in \(X\) is called an MBJ-neutrosophic subalgebra of \(X\) (see [10]), if it satisfies:
\[
(\forall x, y \in X) \begin{cases}
M_A(x \ast y) \geq \min\{M_A(x), M_A(y)\}, \\
\hat{B}_A(x \ast y) \geq \text{rmin}\{\hat{B}_A(x), \hat{B}_A(y)\}, \\
J_A(x \ast y) \leq \max\{J_A(x), J_A(y)\}.
\end{cases}
\]
Let $X$ be a $BCK/BCI$-algebra. An MBJ-neutrosophic set $A = (M_A, \tilde{B}_A, J_A)$ in $X$ is called an MBJ-neutrosophic ideal of $X$ (see [11]), if it satisfies:

$$(\forall x \in X) \left( M_A(0) \geq M_A(x), \tilde{B}_A(0) \geq \tilde{B}_A(x), J_A(0) \leq J_A(x) \right)$$

and

$$(\forall x, y \in X) \left( \begin{array}{l}
M_A(x) \geq \min\{M_A(x \ast y), M_A(y)\} \\
\tilde{B}_A(x) \geq \tilde{r}\min\{\tilde{B}_A(x \ast y), \tilde{B}_A(y)\} \\
J_A(x) \leq \max\{J_A(x \ast y), J_A(y)\} 
\end{array} \right).$$

3. Positive implicative MBJ-neutrosophic ideals

In what follows, let $X$ be a $BCK$-algebra unless otherwise specified.

**Definition 3.1.** An MBJ-neutrosophic set $A = (M_A, \tilde{B}_A, J_A)$ in $X$ is called a positive implicative MBJ-neutrosophic ideal of $X$, if it satisfies (2.9) and

$$(\forall x, y, z \in X) \left( \begin{array}{l}
M_A(x \ast z) \geq \min\{M_A((x \ast y) \ast z), M_A(y \ast z)\} \\
\tilde{B}_A(x \ast z) \geq \tilde{r}\min\{\tilde{B}_A((x \ast y) \ast z), \tilde{B}_A(y \ast z)\} \\
J_A(x \ast z) \leq \max\{J_A((x \ast y) \ast z), J_A(y \ast z)\} 
\end{array} \right).$$

**Example 3.2.** Consider a $BCK$-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 1:

<table>
<thead>
<tr>
<th>*</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
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<td>3</td>
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<td>4</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Let $A = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in $X$ defined by Table 2:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$M_A(x)$</th>
<th>$\tilde{B}_A(x)$</th>
<th>$J_A(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.7</td>
<td>[0.4,0.9]</td>
<td>0.2</td>
</tr>
<tr>
<td>1</td>
<td>0.6</td>
<td>[0.3,0.8]</td>
<td>0.5</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>[0.2,0.6]</td>
<td>0.5</td>
</tr>
<tr>
<td>3</td>
<td>0.4</td>
<td>[0.1,0.3]</td>
<td>0.7</td>
</tr>
<tr>
<td>4</td>
<td>0.3</td>
<td>[0.2,0.5]</td>
<td>0.9</td>
</tr>
</tbody>
</table>

It is routine to verify that $A = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of $X$. 

4
Theorem 3.3. Every positive implicative MBJ-neutrosophic ideal is an MBJ-neutrosophic ideal.

Proof. If we take $z = 0$ in (3.1) and use (2.1), then we have the condition (2.10). Thus every positive implicative MBJ-neutrosophic ideal is an MBJ-neutrosophic ideal. □

The converse of Theorem 3.3 is not true as seen in the following example.

Example 3.4. Consider a $BCK$-algebra $X = \{0, a, b, c\}$ with the binary operation $*$ which is given in Table 3:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
<th>b</th>
<th>c</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
</tr>
<tr>
<td>b</td>
<td>b</td>
<td>a</td>
<td>0</td>
<td>b</td>
</tr>
<tr>
<td>c</td>
<td>c</td>
<td>c</td>
<td>c</td>
<td>0</td>
</tr>
</tbody>
</table>

Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in $X$ defined by Table 4:

<table>
<thead>
<tr>
<th>$X$</th>
<th>$M_A(x)$</th>
<th>$\tilde{B}_A(x)$</th>
<th>$J_A(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.7</td>
<td>[0.4, 0.9]</td>
<td>0.2</td>
</tr>
<tr>
<td>a</td>
<td>0.6</td>
<td>[0.3, 0.8]</td>
<td>0.6</td>
</tr>
<tr>
<td>b</td>
<td>0.6</td>
<td>[0.3, 0.8]</td>
<td>0.6</td>
</tr>
<tr>
<td>c</td>
<td>0.4</td>
<td>[0.1, 0.3]</td>
<td>0.4</td>
</tr>
</tbody>
</table>

It is routine to verify that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$. Since

$$M_A(b * a) = 0.6 < 0.7 = \min\{M_A((b * a) * a), M_A(a * a)\},$$

$\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is not a positive implicative MBJ-neutrosophic ideal of $X$.

Lemma 3.5 ([11]). Every MBJ-neutrosophic ideal of $X$ satisfies the following assertion.

(3.2) $(\forall x, y \in X) \left( x \leq y \Rightarrow M_A(x) \geq M_A(y), \tilde{B}_A(x) \geq \tilde{B}_A(y), J_A(x) \leq J_A(y) \right)$.

We provide conditions for an MBJ-neutrosophic ideal to be a positive implicative MBJ-neutrosophic ideal.
Theorem 3.6. An MBJ-neutrosophic set \( A = (M_A, \tilde{B}_A, J_A) \) in \( X \) is a positive implicative MBJ-neutrosophic ideal of \( X \) if and only if it is an MBJ-neutrosophic ideal of \( X \) satisfying the following condition.

(3.3) \[
(\forall x, y \in X) \quad \begin{cases} 
M_A(x * y) \geq M_A((x * y) * y), \\
\tilde{B}_A(x * y) \geq \tilde{B}_A((x * y) * y), \\
J_A(x * y) \leq J_A((x * y) * y).
\end{cases}
\]

Proof. Assume that \( A = (M_A, \tilde{B}_A, J_A) \) is a positive implicative MBJ-neutrosophic ideal of \( X \). If \( z \) is replaced by \( y \) in (3.1), then

\[
M_A(x * y) \geq \min\{M_A((x * y) * y), M_A(y * y)\} = \min\{M_A((x * y) * y), M_A(0)\} = M_A((x * y) * y),
\]

\[
\tilde{B}_A(x * y) \geq \text{rmin}\{\tilde{B}_A((x * y) * y), \tilde{B}_A(y * y)\} = \text{rmin}\{\tilde{B}_A((x * y) * y), \tilde{B}_A(0)\} = \tilde{B}_A((x * y) * y),
\]

and

\[
J_A(x * y) \leq \max\{J_A((x * y) * y), J_A(y * y)\} = \max\{J_A((x * y) * y), J_A(0)\} = J_A((x * y) * y),
\]

for all \( x, y \in X \).

Conversely, let \( A = (M_A, \tilde{B}_A, J_A) \) be an MBJ-neutrosophic ideal of \( X \) satisfying the condition (3.3). Since

\[
((x * z) * z) * (y * z) \leq (x * z) * y = (x * y) * z
\]

for all \( x, y, z \in X \), it follows from Lemma 3.5 that

(3.4) \[
\begin{align*}
M_A((x * y) * z) & \leq M_A(((x * z) * z) * (y * z)), \\
\tilde{B}_A((x * y) * z) & \leq \tilde{B}_A(((x * z) * z) * (y * z)), \\
J_A((x * y) * z) & \geq J_A(((x * z) * z) * (y * z)),
\end{align*}
\]

for all \( x, y, z \in X \). Using (3.3), (2.10) and (3.4), we have

\[
\begin{align*}
M_A(x * z) & \geq M_A((x * z) * z) \geq \min\{M_A(((x * z) * z) * (y * z)), M_A(y * z)\} \\
& \geq \min\{M_A((x * y) * z), M_A(y * z)\}, \\
\tilde{B}_A(x * z) & \geq \tilde{B}_A((x * z) * z) \geq \text{rmin}\{\tilde{B}_A(((x * z) * z) * (y * z)), \tilde{B}_A(y * z)\} \\
& \geq \text{rmin}\{\tilde{B}_A((x * y) * z), \tilde{B}_A(y * z)\}, \\
J_A(x * z) & \leq J_A((x * z) * z) \leq \max\{J_A(((x * z) * z) * (y * z)), J_A(y * z)\} \\
& \leq \max\{J_A((x * y) * z), J_A(y * z)\},
\end{align*}
\]

and

\[
\begin{align*}
J_A(x * z) & \leq J_A((x * z) * z) \leq \max\{J_A(((x * z) * z) * (y * z)), J_A(y * z)\} \\
& \leq \max\{J_A((x * y) * z), J_A(y * z)\},
\end{align*}
\]

for all \( x, y, z \in X \). Then \( A = (M_A, \tilde{B}_A, J_A) \) is a positive implicative MBJ-neutrosophic ideal of \( X \). \( \square \)
Theorem 3.7. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic ideal of $X$. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is positive implicative if and only if it satisfies the following condition.

\begin{align}
(3.5) \hspace{1cm} & \forall x, y, z \in X \left( \begin{array}{l}
M_A((x \ast y) \ast (y \ast z)) \geq M_A((x \ast y) \ast z), \\
\tilde{B}_A((x \ast z) \ast (y \ast z)) \geq \tilde{B}_A((x \ast y) \ast z), \\
J_A((x \ast z) \ast (y \ast z)) \leq J_A((x \ast y) \ast z).
\end{array} \right)
\end{align}

Proof. Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$ by Theorem 3.3, and satisfies the condition (3.3) by Theorem 3.6. Since

\[(x \ast (y \ast z)) \ast z = (x \ast z) \ast (y \ast z) \leq (x \ast y) \ast z,\]

for all $x, y, z \in X$, it follows from Lemma 3.5 that

\begin{align}
(3.6) \hspace{1cm} & M_A((x \ast y) \ast z) \leq M_A(((x \ast (y \ast z)) \ast z) \ast z), \\
& \tilde{B}_A((x \ast y) \ast z) \leq \tilde{B}_A(((x \ast (y \ast z)) \ast z) \ast z), \\
& J_A((x \ast y) \ast z) \geq J_A(((x \ast (y \ast z)) \ast z) \ast z),
\end{align}

for all $x, y, z \in X$. Using (2.3), (3.3) and (3.6), we have

\begin{align*}
M_A((x \ast z) \ast (y \ast z)) &= M_A((x \ast (y \ast z)) \ast z) \\
& \geq M_A(((x \ast (y \ast z)) \ast z) \ast z), \\
& \geq M_A((x \ast y) \ast z),
\end{align*}

\begin{align*}
\tilde{B}_A((x \ast z) \ast (y \ast z)) &= \tilde{B}_A((x \ast (y \ast z)) \ast z) \\
& \geq \tilde{B}_A(((x \ast (y \ast z)) \ast z) \ast z), \\
& \geq \tilde{B}_A((x \ast y) \ast z),
\end{align*}

and

\begin{align*}
J_A((x \ast z) \ast (y \ast z)) &= J_A((x \ast (y \ast z)) \ast z) \\
& \leq J_A(((x \ast (y \ast z)) \ast z) \ast z), \\
& \leq J_A((x \ast y) \ast z).
\end{align*}

Hence (3.5) is valid.

Conversely, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic ideal of $X$ which satisfies the condition (3.5). If we put $z = y$ in (3.5) and use (III) and (2.1), then we obtain the condition (3.3). Thus $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of $X$ by Theorem 3.6. \qed

Theorem 3.8. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in $X$. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of $X$ if and only if it satisfies the condition (2.9) and

\begin{align}
(3.7) \hspace{1cm} & \forall x, y, z \in X \left( \begin{array}{l}
M_A(x \ast y) \geq \min\{M_A(((x \ast y) \ast y) \ast z), M_A(z)\}, \\
\tilde{B}_A(x \ast y) \geq \min\{\tilde{B}_A(((x \ast y) \ast y) \ast z), \tilde{B}_A(z)\}, \\
J_A(x \ast y) \leq \max\{J_A(((x \ast y) \ast y) \ast z), J_A(z)\}.
\end{array} \right)
\end{align}
Proof. Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of $X$. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$ (see Theorem 3.3), and so the condition (2.9) is valid. Using (2.10), (III), (2.1), (2.3) and (3.5), we have

$$M_A(x * y) \geq \min\{M_A((x * y) * z), M_A(z)\}$$
$$= \min\{M_A(((x * z) * y) * (y * y)), M_A(z)\}$$
$$\geq \min\{M_A(((x * z) * y) * y), M_A(z)\}$$
$$= \min\{M_A(((x * y) * y) * z), M_A(z)\},$$

$$\tilde{B}_A(x * y) \geq \min\{\tilde{B}_A((x * y) * z), \tilde{B}_A(z)\}$$
$$= \min\{\tilde{B}_A(((x * z) * y) * (y * y)), \tilde{B}_A(z)\}$$
$$\geq \min\{\tilde{B}_A(((x * z) * y) * y), \tilde{B}_A(z)\}$$
$$= \min\{\tilde{B}_A(((x * y) * y) * z), \tilde{B}_A(z)\},$$

and

$$J_A(x * y) \leq \max\{J_A((x * y) * z), J_A(z)\}$$
$$= \max\{J_A(((x * z) * y) * (y * y)), J_A(z)\}$$
$$\leq \max\{J_A(((x * z) * y) * y), J_A(z)\}$$
$$= \max\{J_A(((x * y) * y) * z), J_A(z)\},$$

for all $x, y, z \in X$.

Conversely, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in $X$ which satisfies conditions (2.9) and (3.7). Then

$$M_A(x) = M_A(x * 0) \geq \min\{M_A(((x * 0) * 0) * z), M_A(z)\}$$
$$= \min\{M_A(x * z), M_A(z)\},$$

$$\tilde{B}_A(x) = \tilde{B}_A(x * 0) \geq \min\{\tilde{B}_A(((x * 0) * 0) * z), \tilde{B}_A(z)\}$$
$$= \min\{\tilde{B}_A(x * z), \tilde{B}_A(z)\},$$

and

$$J_A(x) = J_A(x * 0) \leq \max\{J_A(((x * 0) * 0) * z), J_A(z)\}$$
$$= \max\{J_A(x * z), J_A(z)\},$$

for all $x, z \in X$. Thus $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$. Taking $z = 0$ in (3.7) and using (2.1) and (2.9) imply that

$$M_A(x * y) \geq \min\{M_A(((x * y) * y) * 0), M_A(0)\}$$
$$\geq \min\{M_A((x * y) * y), M_A(0)\}$$
$$= M_A((x * y) * y),$$
for all \( x, y \in X \). It follows from Theorem 3.6 that \( \mathcal{A} = (M_A, \tilde{B}_A, J_A) \) is a positive implicating MBJ-neutrosophic ideal of \( X \). \( \square \)

Lemma 3.9 ([11]). Let \( X \) be a BCK/BCI-algebra. Then every MBJ-neutrosophic ideal \( \mathcal{A} = (M_A, \tilde{B}_A, J_A) \) of \( X \) satisfies the following assertion.

\[
\begin{align*}
(3.8) \quad x \ast y \leq z \Rightarrow \begin{cases} 
M_A(x) \geq \min\{M_A(y), M_A(z)\}, \\
\tilde{B}_A(x) \geq \min\{\tilde{B}_A(y), \tilde{B}_A(z)\}, \\
J_A(x) \leq \max\{J_A(y), J_A(z)\}, 
\end{cases}
\end{align*}
\]

for all \( x, y, z \in X \).

Lemma 3.10. If an MBJ-neutrosophic set \( \mathcal{A} = (M_A, \tilde{B}_A, J_A) \) in \( X \) satisfies the condition (3.8), then \( \mathcal{A} = (M_A, \tilde{B}_A, J_A) \) is an MBJ-neutrosophic ideal of \( X \).

Proof. Since \( 0 \ast x \leq x \) and \( x \ast (x \ast y) \leq y \) for all \( x, y \in X \), it follows from (3.8) that

\[
M_A(0) \geq M_A(x), \quad \tilde{B}_A(0) \geq \tilde{B}_A(x), \quad J_A(0) \leq J_A(x)
\]

and

\[
M_A(x) \geq \min\{M_A(x \ast y), M_A(y)\}, \quad \tilde{B}_A(x) \geq \min\{\tilde{B}_A(x \ast y), \tilde{B}_A(y)\}
\]

and

\[
J_A(x) \leq \max\{J_A(x \ast y), J_A(y)\}.
\]

Then \( \mathcal{A} = (M_A, \tilde{B}_A, J_A) \) is an MBJ-neutrosophic ideal of \( X \). \( \square \)

Theorem 3.11. Let \( \mathcal{A} = (M_A, \tilde{B}_A, J_A) \) be an MBJ-neutrosophic set in \( X \). Then \( \mathcal{A} = (M_A, \tilde{B}_A, J_A) \) is a positive implicating MBJ-neutrosophic ideal of \( X \) if and only if it satisfies the following condition.

\[
(3.9) \quad ((x \ast y) \ast a) \ast b = 0 \Rightarrow \begin{cases} 
M_A(x \ast y) \geq \min\{M_A(a), M_A(b)\}, \\
\tilde{B}_A(x \ast y) \geq \min\{\tilde{B}_A(a), \tilde{B}_A(b)\}, \\
J_A(x \ast y) \leq \max\{J_A(a), J_A(b)\}, 
\end{cases}
\]

for all \( x, y, a, b \in X \).

Proof. Assume that \( \mathcal{A} = (M_A, \tilde{B}_A, J_A) \) is a positive implicating MBJ-neutrosophic ideal of \( X \). Then \( \mathcal{A} = (M_A, \tilde{B}_A, J_A) \) is an MBJ-neutrosophic ideal of \( X \) (see Theorem 3.3). Let \( a, b, x, y \in X \) be such that \(((x \ast y) \ast a) \ast b = 0\). Then

\[
M_A(x \ast y) \geq M_A((x \ast y) \ast y) \geq \min\{M_A(a), M_A(b)\},
\]

\[
\tilde{B}_A(x \ast y) \geq \tilde{B}_A((x \ast y) \ast y) \geq \min\{\tilde{B}_A(a), \tilde{B}_A(b)\},
\]

and

\[
J_A(x \ast y) \leq J_A((x \ast y) \ast y) \leq \max\{J_A(a), J_A(b)\}
\]

by Theorem 3.6 and Lemma 3.9. Thus (3.9) is valid.
Conversely, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in $X$ which satisfies the condition (3.9). Let $x, a, b \in X$ be such that $x * a \leq b$. Then

$$(((x * 0) * 0) * a) * b = 0,$$

and so

$$M_A(x) = M_A(x * 0) \geq \min\{M_A(a), M_A(b)\},$$

$$\tilde{B}_A(x) = \tilde{B}_A(x * 0) \geq \min\{\tilde{B}_A(a), \tilde{B}_A(b)\},$$

and

$$J_A(x) = J_A(x * 0) \leq \max\{J_A(a), J_A(b)\}$$

by (2.1) and (3.9). Thus $A = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$ by Lemma 3.10. Since $\min\{(x * y) * (x * y)\} = 0$ and $(x * y) * (x * y) = 0$ for all $x, y \in X$, we have

$$M_A(x * y) \geq \min\{M_A((x * y) * y), M_A(0)\} = M_A((x * y) * y),$$

$$\tilde{B}_A(x * y) \geq \min\{\tilde{B}_A((x * y) * y), \tilde{B}_A(0)\} = \tilde{B}_A((x * y) * y),$$

and

$$J_A(x * y) \leq \max\{J_A((x * y) * y), J_A(0)\} = J_A((x * y) * y)$$

by (3.9). It follows from Theorem 3.6 that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of $X$.

**Theorem 3.12.** Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in $X$. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of $X$ if and only if it satisfies the following condition.

$$M_A((x * z) * (y * z)) \geq \min\{M_A(a), M_A(b)\},$$

$$\tilde{B}_A((x * z) * (y * z)) \geq \min\{\tilde{B}_A(a), \tilde{B}_A(b)\},$$

$$J_A((x * z) * (y * z)) \leq \max\{J_A(a), J_A(b)\},$$

for all $x, y, z, a, b \in X$ with $(((x * y) * z) * a) * b = 0$.

**Proof.** Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of $X$. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of $X$ (see Theorem 3.3). Let $a, b, x, y, z \in X$ be such that $(((x * y) * z) * a) * b = 0$. Using Theorem 3.7 and Lemma 3.9, we have

$$M_A((x * z) * (y * z)) \geq M_A((x * y) * z) \geq \min\{M_A(a), M_A(b)\},$$

$$\tilde{B}_A((x * z) * (y * z)) \geq \tilde{B}_A((x * y) * z) \geq \min\{\tilde{B}_A(a), \tilde{B}_A(b)\},$$

and

$$J_A((x * z) * (y * z)) \leq J_A((x * y) * z) \leq \max\{J_A(a), J_A(b)\}.$$

Conversely, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in $X$ which satisfies the condition (3.10). Let $x, y, a, b \in X$ be such that $(((x * y) * y) * a) * b = 0$. Then

$$M_A(x * y) = M_A((x * y) * (y * y)) \geq \min\{M_A(a), M_A(b)\},$$
\[ \hat{B}_A(x * y) = \hat{B}_A((x * y) * (y * y)) \leq \min\{\hat{B}_A(a), \hat{B}_A(b)\}, \]

and

\[ J_A(x * y) = J_A((x * y) * (y * y)) \leq \max\{J_A(a), J_A(b)\} \]

by (2.1), (III) and (3.10). It follows from Theorem 3.11 that \( A = (M_A, \hat{B}_A, J_A) \) is a positive implicative MBJ-neutrosophic ideal of \( X \).

**Theorem 3.13.** Let \( A = (M_A, \hat{B}_A, J_A) \) be an MBJ-neutrosophic set in \( X \). Then \( A = (M_A, \hat{B}_A, J_A) \) is a positive implicative MBJ-neutrosophic ideal of \( X \) if and only if it satisfies the following condition.

\[
\begin{align*}
M_A(x * z) & \geq \min\{M_A(a_1), M_A(a_2), \ldots, M_A(a_n)\}, \\
\hat{B}_A((x * z) * (y * z)) & \geq \min\{\hat{B}_A(a_1), \hat{B}_A(a_2), \ldots, \hat{B}_A(a_n)\}, \\
J_A((x * z) * (y * z)) & \leq \max\{J_A(a_1), J_A(a_2), \ldots, J_A(a_n)\},
\end{align*}
\]

for all \( x, y, a_1, a_2, \ldots, a_n \in X \) with \( \cdots(((x * y) * y) * a_1) \cdots * a_n = 0 \).

**Proof.** It is similar to the proof of Theorem 3.11. \( \square \)

**Theorem 3.14.** Let \( A = (M_A, \hat{B}_A, J_A) \) be an MBJ-neutrosophic set in \( X \). Then \( A = (M_A, \hat{B}_A, J_A) \) is a positive implicative MBJ-neutrosophic ideal of \( X \) if and only if it satisfies the following condition.

\[
\begin{align*}
M_A(x * z) & \geq \min\{M_A(a_1), M_A(a_2), \ldots, M_A(a_n)\}, \\
\hat{B}_A((x * z) * (y * z)) & \geq \min\{\hat{B}_A(a_1), \hat{B}_A(a_2), \ldots, \hat{B}_A(a_n)\}, \\
J_A((x * z) * (y * z)) & \leq \max\{J_A(a_1), J_A(a_2), \ldots, J_A(a_n)\},
\end{align*}
\]

for all \( x, y, z, a_1, a_2, \ldots, a_n \in X \) with \( \cdots(((x * y) * z) * a_1) \cdots * a_n = 0 \).

**Proof.** It is similar to the proof of Theorem 3.12. \( \square \)

Given an MBJ-neutrosophic set \( A = (M_A, \hat{B}_A, J_A) \) in \( X \), we consider the following sets.

\[
\begin{align*}
U(M_A; \alpha) & := \{x \in X \mid M_A(x) \geq \alpha\}, \\
U(\hat{B}_A; [\delta_1, \delta_2]) & := \{x \in X \mid \hat{B}_A(x) \geq [\delta_1, \delta_2]\}, \\
L(J_A; \beta) & := \{x \in X \mid J_A(x) \leq \beta\},
\end{align*}
\]

where \( \alpha, \beta \in [0, 1] \) and \([\delta_1, \delta_2] \in [I]\).

**Theorem 3.15.** An MBJ-neutrosophic set \( A = (M_A, \hat{B}_A, J_A) \) in \( X \) is a positive implicative MBJ-neutrosophic ideal of \( X \) if and only if the non-empty sets \( U(M_A; \alpha) \), \( U(\hat{B}_A; [\delta_1, \delta_2]) \) and \( L(J_A; \beta) \) are positive implicative ideals of \( X \), for all \( \alpha, \beta \in [0, 1] \) and \([\delta_1, \delta_2] \in [I]\).

**Proof.** Let \( A = (M_A, \hat{B}_A, J_A) \) be a positive implicative MBJ-neutrosophic ideal of \( X \). Let \( \alpha, \beta \in [0, 1] \) and \([\delta_1, \delta_2] \in [I]\) be such that \( U(M_A; \alpha) \), \( U(\hat{B}_A; [\delta_1, \delta_2]) \) and \( L(J_A; \beta) \) are non-empty. Obviously, \( 0 \in U(M_A; \alpha) \cap U(\hat{B}_A; [\delta_1, \delta_2]) \cap L(J_A; \beta) \). For any \( x, y, z, a, b, c, u, v, w \in X \), if \( (x * y) * z \in U(M_A; \alpha) \), \( y * z \in U(M_A; \alpha) \).
(a * b) * c ∈ U(\(\hat{B}_A; [\delta_1, \delta_2]\)), b * c ∈ U(\(\hat{B}_A; [\delta_1, \delta_2]\)), (u * v) * w ∈ L(J_A; \beta) and v * w ∈ L(J_A; \beta), then

\[ M_A(x * z) \geq \min \{ M_A((x * y) * z), M_A(y * z) \} \geq \min \{ \alpha, \alpha \} = \alpha, \]

\[ \hat{B}_A(a * c) \geq \min \{ \hat{B}_A((a * b) * c), \hat{B}_A(b * c) \} \geq \min \{ [\delta_1, \delta_2], [\delta_1, \delta_2] \} = [\delta_1, \delta_2], \]

\[ J_A(u * w) \leq \max \{ J_A((u * v) * w), J_A(v * w) \} \leq \min \{ \beta, \beta \} = \beta, \]

and so \( x * z \in U(M_A; \alpha) \), \( a * c \in U(\hat{B}_A; [\delta_1, \delta_2]) \) and \( u * w \in L(J_A; \beta) \). Therefore \( U(M_A; \alpha) \), \( U(\hat{B}_A; [\delta_1, \delta_2]) \) and \( L(J_A; \beta) \) are positive implicative ideals of \( X \).

Conversely, assume that the non-empty sets \( U(M_A; \alpha) \), \( U(\hat{B}_A; [\delta_1, \delta_2]) \) and \( L(J_A; \beta) \) are positive implicative ideals of \( X \) for all \( \alpha, \beta \in [0, 1] \) and \( [\delta_1, \delta_2] \in [I] \). Assume that \( M_A(0) < M_A(a) \), \( \hat{B}_A(0) < \hat{B}_A(a) \) and \( J_A(0) > J_A(a) \), for some \( a \in X \). Then \( 0 \notin U(M_A; M_A(a)) \cap U(\hat{B}_A; \hat{B}_A(a)) \cap L(J_A; J_A(a)) \), which is a contradiction. Thus \( M_A(0) \geq M_A(x) \), \( \hat{B}_A(0) \geq \hat{B}_A(x) \) and \( J_A(0) \leq J_A(x) \), for all \( x \in X \). If

\[ M_A(a_0 * c_0) < \min \{ M_A((a_0 * b_0) * c_0), M_A(b_0 * c_0) \}, \]

for some \( a_0, b_0, c_0 \in X \), then \( (a_0 * b_0) * c_0 \in U(M_A; t_0) \) and \( b_0 * c_0 \in U(M_A; t_0) \) but \( a_0 * c_0 \notin U(M_A; t_0) \) for \( t_0 : = \min \{ M_A((a_0 * b_0) * c_0), M_A(b_0 * c_0) \} \). This is a contradiction, and thus

\[ M_A(a * c) \geq \min \{ M_A((a * b) * c), M_A(b * c) \}, \]

for all \( a, b, c \in X \).

Similarly, we can show that \( J_A(a * c) \leq \max \{ J_A((a * b) * c), J_A(b * c) \} \), for all \( a, b, c \in X \). Suppose that \( \hat{B}_A(a_0 * c_0) < \min \{ \hat{B}_A((a_0 * b_0) * c_0), \hat{B}_A(b_0 * c_0) \} \), for some \( a_0, b_0, c_0 \in X \). Let \( \hat{B}_A((a_0 * b_0) * c_0) = [\lambda_1, \lambda_2] \), \( \hat{B}_A(b_0 * c_0) = [\lambda_3, \lambda_4] \) and \( \hat{B}_A(a_0 * c_0) = [\delta_1, \delta_2] \). Then

\[ [\delta_1, \delta_2] < \min \{ [\lambda_1, \lambda_2], [\lambda_3, \lambda_4] \} = \min \{ \lambda_1, \lambda_3 \}, \min \{ \lambda_2, \lambda_4 \}, \]

and so \( \delta_1 < \min \{ \lambda_1, \lambda_3 \} \) and \( \delta_2 < \min \{ \lambda_2, \lambda_4 \} \). Taking

\[ \gamma_1, \gamma_2 : = \frac{1}{2} \left( \hat{B}_A(a_0 * c_0) + \min \{ \hat{B}_A((a_0 * b_0) * c_0), \hat{B}_A(b_0 * c_0) \} \right) \]

implies that

\[ \gamma_1, \gamma_2 = \frac{1}{2} (\delta_1 + \delta_2 + \min \{ \lambda_1, \lambda_3 \}, \min \{ \lambda_2, \lambda_4 \}) \]

\[ = \left[ \frac{1}{2} (\delta_1 + \min \{ \lambda_1, \lambda_3 \}), \frac{1}{2} (\delta_2 + \min \{ \lambda_2, \lambda_4 \}) \right]. \]

It follows that

\[ \min \{ \lambda_1, \lambda_3 \} > \gamma_1 = \frac{1}{2} (\delta_1 + \min \{ \lambda_1, \lambda_3 \}) > \delta_1 \]

and

\[ \min \{ \lambda_2, \lambda_4 \} > \gamma_2 = \frac{1}{2} (\delta_2 + \min \{ \lambda_2, \lambda_4 \}) > \delta_2. \]

Thus \( \min \{ \lambda_1, \lambda_3 \}, \min \{ \lambda_2, \lambda_4 \} \geq [\gamma_1, \gamma_2] \geq [\delta_1, \delta_2] = \hat{B}_A(a_0 * c_0). \) So

\[ a_0 * c_0 \notin U(\hat{B}_A; [\gamma_1, \gamma_2]). \]

On the other hand,

\[ \hat{B}_A((a_0 * b_0) * c_0) = [\lambda_1, \lambda_2] \geq \min \{ \lambda_1, \lambda_3 \}, \min \{ \lambda_2, \lambda_4 \} \geq [\gamma_1, \gamma_2]. \]
and
\[ \tilde{B}_A(b_0 \ast c_0) = [\lambda_3, \lambda_4] \supseteq [\min\{\lambda_1, \lambda_3\}, \min\{\lambda_2, \lambda_4\}] \supset [\gamma_1, \gamma_2], \]
that is, \((a_0 \ast b_0) \ast c_0, b_0 \ast c_0 \in U(\tilde{B}_A; [\gamma_1, \gamma_2])\). This is a contradiction. Hence
\[ \tilde{B}_A(x \ast z) \supseteq \min\{\tilde{B}_A((x \ast y) \ast z), \tilde{B}_A(y \ast z)\}, \]
for all \(x, y, z \in X\). Consequently, \(A = (M_A, \tilde{B}_A, J_A)\) is a positive implicative MBJ-neutrosophic ideal of \(X\). \(\square\)

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