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Positive implicative MBJ-neutrosophic ideals of BCK/BCI-algebras

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Abstract. The notion of positive implicative MBJ-neutrosophic ideal is introduced, and several properties are investigated. Relations between MBJ-neutrosophic ideal and positive implicative MBJ-neutrosophic ideal are discussed. Characterizations of positive implicative MBJ-neutrosophic ideal are displayed.

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1. Introduction

In order to handle uncertainties in many real applications, the fuzzy set was introduced by L.A. Zadeh [19] in 1965. In 1983, K. Atanassov introdued the notion of intuitionistic fuzzy set as a generalization of fuzzy set. As a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set, the notion of neutrosophic set is initiated by Smarandache ([14], [15] and [16]). Neutrosophic algebraic structures in BCK/BCI-algebras are discussed in the papers [1], [2], [4], [5], [6], [7], [8], [12], [13], [17] and [18]. In [10], the notion of MBJ-neutrosophic sets is introduced as another generalization of neutrosophic set, it is applied to BCK/BCI-algebras. Mohseni et al. [10] introduced the concept of MBJ-neutrosophic subalgebras in BCK/BCI-algebras, and investigated related properties. They gave a characterization of MBJ-neutrosophic subalgebra, and established a new MBJ-neutrosophic subalgebra by using an MBJ-neutrosophic subalgebra of a BCI-algebra. They considered the homomorphic inverse image of MBJ-neutrosophic subalgebra, and discussed translation of MBJ-neutrosophic subalgebra. Jun and Roh [11] applied the notion of MBJ-neutrosophic sets to ideals of BCK/BI-algebras, and introduce the concept of MBJ-neutrosophic ideals in BCK/BCI-algebras. They provided a condition for an

MBJ-neutrosophic subalgebra to be an MBJ-neutrosophic ideal in a BCK-algebra, and considered conditions for an MBJ-neutrosophic set to be an MBJ-neutrosophic ideal in a BCK/BCI-algebra. They discussed relations between MBJ-neutrosophic subalgebras, MBJ-neutrosophic $\circ$-subalgebras and MBJ-neutrosophic ideals. In a BCI-algebra, they provided conditions for an MBJ-neutrosophic ideal to be an MBJ-neutrosophic subalgebra, and considered a characterization of an MBJ-neutrosophic ideal in an (S)-BCK-algebra.

In this paper, we introduce the notion of positive implicative MBJ-neutrosophic ideal, and investigate several properties. We discuss relations between MBJ-neutrosophic ideal and positive implicative MBJ-neutrosophic ideal. We provide characterizations of positive implicative MBJ-neutrosophic ideal.

2. Preliminaries

By a BCI-algebra, we mean a set X with a binary operation $*$ and a special element 0 that satisfies the following conditions:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,
(III) $x * x = 0$,
(IV) $x * y = 0,\ y * x = 0 \ \Rightarrow\ x = y$,

for all $x, y, z \in X$. If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a BCK-algebra.

Every BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$

$$(\forall x, y, z \in X)\,(x \leq y \ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2.2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{2.4}$$

where $x \leq y$ if and only if $x * y = 0$.

A nonempty subset S of a BCK/BCI-algebra X is called a subalgebra of X if $x * y \in S$, for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called an ideal of X, if it satisfies:

$$0 \in I, \tag{2.5}$$

$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \ \Rightarrow\ x \in I)\,. \tag{2.6}$$

A subset I of a BCK-algebra X is called a positive implicative ideal of X (see [9]), if it satisfies (2.5) and

$$(\forall x, y, z \in X)(((x * y) * z \in I,\ y * z \in I \ \Rightarrow\ x * z \in I)\,. \tag{2.7}$$

By an interval number, we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of I, where $0 \leq a^- \leq a^+ \leq 1$. Denote by $[I]$ the set of all interval numbers. Let us define what is known as refined minimum (briefly, rmin) and refined maximum (briefly, rmax) of two elements in $[I]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[I]$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\begin{aligned}
&\operatorname{rmin}\{\tilde{a}_1, \tilde{a}_2\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right],\\
&\operatorname{rmax}\{\tilde{a}_1, \tilde{a}_2\} = \left[\max\left\{a_1^-, a_2^-\right\}, \max\left\{a_1^+, a_2^+\right\}\right],\\
&\tilde{a}_1 \succeq \tilde{a}_2 \ \Leftrightarrow\ a_1^- \geq a_2^-,\ a_1^+ \geq a_2^+,
\end{aligned}$$

and similarly, we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$), we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$). Let $\tilde{a}_i \in [I]$, where $i \in \Lambda$. We define

$$\operatorname*{rinf}_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

Let X be a nonempty set. A function $A : X \to [I]$ is called an interval-valued fuzzy set (briefly, an IVF set) in X. Let $[I]^X$ stand for the set of all IVF sets in X. For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the degree of membership of an element x to A, where $A^- : X \to I$ and $A^+ : X \to I$ are fuzzy sets in X which are called a lower fuzzy set and an upper fuzzy set in X, respectively. For simplicity, we denote $A = [A^-, A^+]$.

Let X be a non-empty set. A neutrosophic set (NS) in X (see [15]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0, 1]$ is a truth membership function, $A_I : X \to [0, 1]$ is an indeterminate membership function, and $A_F : X \to [0, 1]$ is a false membership function.

We refer the reader to the books [3, 9] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

Let X be a non-empty set. By an MBJ-neutrosophic set in X (see [10]), we mean a structure of the form:

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\},$$

where M_A and J_A are fuzzy sets in X, which are called a truth membership function and a false membership function, respectively, and $\tilde{B}_A$ is an IVF set in X which is called an indeterminate interval-valued membership function.

For the sake of simplicity, we shall use the symbol $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ for the MBJ-neutrosophic set

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\}.$$

Let X be a BCK/BCI-algebra. An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is called an MBJ-neutrosophic subalgebra of X (see [10]), if it satisfies:

$$(2.8) \qquad (\forall x, y \in X) \left(\begin{array}{l} M_A(x * y) \geq \min\{M_A(x), M_A(y)\}, \\ \tilde{B}_A(x * y) \succeq \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\}, \\ J_A(x * y) \leq \max\{J_A(x), J_A(y)\}. \end{array}\right)$$

Let X be a BCK/BCI-algebra. An MBJ-neutrosophic set $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ in X is called an MBJ-neutrosophic ideal of X (see [11]), if it satisfies:

$$(\forall x \in X)\left(\; M_A(0) \geq M_A(x),\, \tilde{B}_A(0) \succeq \tilde{B}_A(x),\, J_A(0) \leq J_A(x) \;\right) \tag{2.9}$$

and

$$(\forall x, y \in X)\left(\begin{array}{l} M_A(x) \geq \min\{M_A(x * y), M_A(y)\} \\ \tilde{B}_A(x) \succeq \mathrm{rmin}\{\tilde{B}_A(x * y), \tilde{B}_A(y)\} \\ J_A(x) \leq \max\{J_A(x * y), J_A(y)\} \end{array}\right). \tag{2.10}$$

3. Positive implicative MBJ-neutrosophic ideals

In what follows, let X be a BCK-algebra unless otherwise specified.

Definition 3.1. An MBJ-neutrosophic set $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ in X is called a positive implicative MBJ-neutrosophic ideal of X, if it satisfies (2.9) and

$$(\forall x, y, z \in X)\left(\begin{array}{l} M_A(x * z) \geq \min\{M_A((x * y) * z), M_A(y * z)\} \\ \tilde{B}_A(x * z) \succeq \mathrm{rmin}\{\tilde{B}_A((x * y) * z), \tilde{B}_A(y * z)\} \\ J_A(x * z) \leq \max\{J_A((x * y) * z), J_A(y * z)\} \end{array}\right). \tag{3.1}$$

Example 3.2. Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the binary operation $*$ which is given in Table 1:

Table 1. Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	2	0	0	2
3	3	3	3	0	3
4	4	4	4	4	0

Let $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ be an MBJ-neutrosophic set in X defined by Table 2:

Table 2. MBJ-neutrosophic set $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$

X	$M_A(x)$	$\tilde{B}_A(x)$	$J_A(x)$
0	0.7	$[0.4, 0.9]$	0.2
1	0.6	$[0.3, 0.8]$	0.5
2	0.5	$[0.2, 0.6]$	0.5
3	0.4	$[0.1, 0.3]$	0.7
4	0.3	$[0.2, 0.5]$	0.9

It is routine to verify that $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X.

Theorem 3.3. *Every positive implicative MBJ-neutrosophic ideal is an MBJ-neutrosophic ideal.*

Proof. If we take $z = 0$ in (3.1) and use (2.1), then we have the condition (2.10). Thus every positive implicative MBJ-neutrosophic ideal is an MBJ-neutrosophic ideal. □

The converse of Theorem 3.3 is not true as seen in the following example.

Example 3.4. Consider a BCK-algebra $X = \{0, a, b, c\}$ with the binary operation $*$ which is given in Table 3:

TABLE 3. Cayley table for the binary operation "$*$"

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by Table 4:

TABLE 4. MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$

X	$M_A(x)$	$\tilde{B}_A(x)$	$J_A(x)$
0	0.7	$[0.4, 0.9]$	0.2
a	0.6	$[0.3, 0.8]$	0.6
b	0.6	$[0.3, 0.8]$	0.6
c	0.4	$[0.1, 0.3]$	0.4

It is routine to verify that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of X. Since

$$M_A(b * a) = 0.6 < 0.7 = \min\{M_A((b * a) * a), M_A(a * a)\},$$

$\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is not a positive implicative MBJ-neutrosophic ideal of X.

Lemma 3.5 ([11])**.** *Every MBJ-neutrosophic ideal of X satisfies the following assertion.*

$$(3.2) \quad (\forall x, y \in X)\left(x \leq y \;\Rightarrow\; M_A(x) \geq M_A(y), \tilde{B}_A(x) \succeq \tilde{B}_A(y), J_A(x) \leq J_A(y)\right).$$

We provide conditions for an MBJ-neutrosophic ideal to be a positive implicative MBJ-neutrosophic ideal.

Theorem 3.6. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is a positive implicative MBJ-neutrosophic ideal of X if and only if it is an MBJ-neutrosophic ideal of X satisfying the following condition.*

$$(3.3) \qquad (\forall x, y \in X) \left(\begin{array}{l} M_A(x * y) \geq M_A((x * y) * y), \\ \tilde{B}_A(x * y) \succeq \tilde{B}_A((x * y) * y), \\ J_A(x * y) \leq J_A((x * y) * y). \end{array} \right)$$

Proof. Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X. If z is replaced by y in (3.1), then

$$\begin{aligned} M_A(x * y) &\geq \min\{M_A((x * y) * y), M_A(y * y)\} \\ &= \min\{M_A((x * y) * y), M_A(0)\} = M_A((x * y) * y), \end{aligned}$$

$$\begin{aligned} \tilde{B}_A(x * y) &\succeq \text{rmin}\{\tilde{B}_A((x * y) * y), \tilde{B}_A(y * y)\} \\ &= \text{rmin}\{\tilde{B}_A((x * y) * y), \tilde{B}_A(0)\} = \tilde{B}_A((x * y) * y), \end{aligned}$$

and

$$\begin{aligned} J_A(x * y) &\leq \max\{J_A((x * y) * y), J_A(y * y)\} \\ &= \max\{J_A((x * y) * y), J_A(0)\} = J_A((x * y) * y), \end{aligned}$$

for all $x, y \in X$.

Conversely, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic ideal of X satisfying the condition (3.3). Since

$$((x * z) * z) * (y * z) \leq (x * z) * y = (x * y) * z$$

for all $x, y, z \in X$, it follows from Lemma 3.5 that

$$(3.4) \qquad \begin{array}{l} M_A((x * y) * z) \leq M_A(((x * z) * z) * (y * z)), \\ \tilde{B}_A((x * y) * z) \preceq \tilde{B}_A(((x * z) * z) * (y * z)), \\ J_A((x * y) * z) \geq J_A(((x * z) * z) * (y * z)), \end{array}$$

for all $x, y, z \in X$. Using (3.3), (2.10) and (3.4), we have

$$\begin{aligned} M_A(x * z) \geq M_A((x * z) * z) &\geq \min\{M_A(((x * z) * z) * (y * z)), M_A(y * z)\} \\ &\geq \min\{M_A((x * y) * z), M_A(y * z)\}, \end{aligned}$$

$$\begin{aligned} \tilde{B}_A(x * z) \succeq \tilde{B}_A((x * z) * z) &\succeq \text{rmin}\{\tilde{B}_A(((x * z) * z) * (y * z)), \tilde{B}_A(y * z)\} \\ &\succeq \text{rmin}\{\tilde{B}_A((x * y) * z), \tilde{B}_A(y * z)\}, \end{aligned}$$

and

$$\begin{aligned} J_A(x * z) \leq J_A((x * z) * z) &\leq \max\{J_A(((x * z) * z) * (y * z)), J_A(y * z)\} \\ &\leq \max\{J_A((x * y) * z), J_A(y * z)\}, \end{aligned}$$

for all $x, y, z \in X$. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X. □

Theorem 3.7. *Let $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ be an MBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ is positive implicative if and only if it satisfies the following condition.*

$$(3.5) \qquad (\forall x, y, z \in X) \left(\begin{array}{l} M_A((x*z)*(y*z)) \geq M_A((x*y)*z), \\ \tilde{B}_A((x*z)*(y*z)) \succeq \tilde{B}_A((x*y)*z), \\ J_A((x*z)*(y*z)) \leq J_A((x*y)*z). \end{array} \right)$$

Proof. Assume that $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ is an MBJ-neutrosophic ideal of X by Theorem 3.3, and satisfies the condition (3.3) by Theorem 3.6. Since

$$((x*(y*z))*z)*z = ((x*z)*(y*z))*z \leq (x*y)*z,$$

for all $x, y, z \in X$, it follows from Lemma 3.5 that

$$(3.6) \qquad \begin{array}{l} M_A((x*y)*z) \leq M_A(((x*(y*z))*z)*z), \\ \tilde{B}_A((x*y)*z) \preceq \tilde{B}_A(((x*(y*z))*z)*z), \\ J_A((x*y)*z) \geq J_A(((x*(y*z))*z)*z), \end{array}$$

for all $x, y, z \in X$. Using (2.3), (3.3) and (3.6), we have

$$\begin{aligned} M_A((x*z)*(y*z)) &= M_A((x*(y*z))*z) \\ &\geq M_A(((x*(y*z))*z)*z) \\ &\geq M_A((x*y)*z), \end{aligned}$$

$$\begin{aligned} \tilde{B}_A((x*z)*(y*z)) &= \tilde{B}_A((x*(y*z))*z) \\ &\succeq \tilde{B}_A(((x*(y*z))*z)*z) \\ &\succeq \tilde{B}_A((x*y)*z), \end{aligned}$$

and

$$\begin{aligned} J_A((x*z)*(y*z)) &= J_A((x*(y*z))*z) \\ &\leq J_A(((x*(y*z))*z)*z) \\ &\leq J_A((x*y)*z). \end{aligned}$$

Hence (3.5) is valid.

Conversely, let $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ be an MBJ-neutrosophic ideal of X which satisfies the condition (3.5). If we put $z = y$ in (3.5) and use (III) and (2.1), then we obtain the condition (3.3). Thus $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X by Theorem 3.6. □

Theorem 3.8. *Let $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ be an MBJ-neutrosophic set in X. Then $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X if and only if it satisfies the condition (2.9) and*

$$(3.7) \qquad (\forall x, y, z \in X) \left(\begin{array}{l} M_A(x*y) \geq \min\{M_A(((x*y)*y)*z), M_A(z)\}, \\ \tilde{B}_A(x*y) \succeq \mathrm{rmin}\{\tilde{B}_A(((x*y)*y)*z), \tilde{B}_A(z)\}, \\ J_A(x*y) \leq \max\{J_A(((x*y)*y)*z), J_A(z)\}. \end{array} \right)$$

Proof. Assume that $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ is a positive implicative MBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ is an MBJ-neutrosophic ideal of X (see Theorem 3.3), and so the condition (2.9) is valid. Using (2.10), (III), (2.1), (2.3) and (3.5), we have

$$\begin{aligned} M_A(x*y) &\geq \min\{M_A((x*y)*z), M_A(z)\} \\ &= \min\{M_A(((x*z)*y)*(y*y)), M_A(z)\} \\ &\geq \min\{M_A(((x*z)*y)*y), M_A(z)\} \\ &= \min\{M_A(((x*y)*y)*z), M_A(z)\}, \end{aligned}$$

$$\begin{aligned} \tilde{B}_A(x*y) &\succeq \text{rmin}\{\tilde{B}_A((x*y)*z), \tilde{B}_A(z)\} \\ &= \text{rmin}\{\tilde{B}_A(((x*z)*y)*(y*y)), \tilde{B}_A(z)\} \\ &\succeq \text{rmin}\{\tilde{B}_A(((x*z)*y)*y), \tilde{B}_A(z)\} \\ &= \text{rmin}\{\tilde{B}_A(((x*y)*y)*z), \tilde{B}_A(z)\}, \end{aligned}$$

and

$$\begin{aligned} J_A(x*y) &\leq \max\{J_A((x*y)*z), J_A(z)\} \\ &= \max\{J_A(((x*z)*y)*(y*y)), J_A(z)\} \\ &\leq \max\{J_A(((x*z)*y)*y), J_A(z)\} \\ &= \max\{J_A(((x*y)*y)*z), J_A(z)\}, \end{aligned}$$

for all $x, y, z \in X$.

Conversely, let $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ be an MBJ-neutrosophic set in X which satisfies conditions (2.9) and (3.7). Then

$$\begin{aligned} M_A(x) = M_A(x*0) &\geq \min\{M_A(((x*0)*0)*z), M_A(z)\} \\ &= \min\{M_A(x*z), M_A(z)\}, \end{aligned}$$

$$\begin{aligned} \tilde{B}_A(x) = \tilde{B}_A(x*0) &\succeq \text{rmin}\{\tilde{B}_A(((x*0)*0)*z), \tilde{B}_A(z)\} \\ &= \text{rmin}\{\tilde{B}_A(x*z), \tilde{B}_A(z)\}, \end{aligned}$$

and

$$\begin{aligned} J_A(x) = J_A(x*0) &\leq \max\{J_A(((x*0)*0)*z), J_A(z)\} \\ &= \max\{J_A(x*z), J_A(z)\}, \end{aligned}$$

for all $x, z \in X$. Thus $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ is an MBJ-neutrosophic ideal of X. Taking $z = 0$ in (3.7) and using (2.1) and (2.9) imply that

$$\begin{aligned} M_A(x*y) &\geq \min\{M_A(((x*y)*y)*0), M_A(0)\} \\ &= \min\{M_A((x*y)*y), M_A(0)\} \\ &= M_A((x*y)*y), \end{aligned}$$

$$\begin{aligned}\tilde{B}_A(x*y) &\succeq \text{rmin}\{\tilde{B}_A(((x*y)*y)*0), \tilde{B}_A(0)\} \\ &= \text{rmin}\{\tilde{B}_A((x*y)*y), \tilde{B}_A(0)\} \\ &= \tilde{B}_A((x*y)*y),\end{aligned}$$

and

$$\begin{aligned}J_A(x*y) &\leq \max\{J_A(((x*y)*y)*0), J_A(0)\} \\ &= \max\{J_A((x*y)*y), J_A(0)\} \\ &= J_A((x*y)*y),\end{aligned}$$

for all $x, y \in X$. It follows from Theorem 3.6 that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X. □

Lemma 3.9 ([11])**.** *Let X be a BCK/BCI-algebra. Then every MBJ-neutrosophic ideal $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ of X satisfies the following assertion.*

$$x*y \leq z \;\Rightarrow \begin{cases} M_A(x) \geq \min\{M_A(y), M_A(z)\}, \\ \tilde{B}_A(x) \succeq \text{rmin}\{\tilde{B}_A(y), \tilde{B}_A(z)\}, \\ J_A(x) \leq \max\{J_A(y), J_A(z)\}, \end{cases} \tag{3.8}$$

for all $x, y, z \in X$.

Lemma 3.10. *If an MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X satisfies the condition (3.8), then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of X.*

Proof. Since $0*x \leq x$ and $x*(x*y) \leq y$ for all $x, y \in X$, it follows from (3.8) that

$$M_A(0) \geq M_A(x),\ \tilde{B}_A(0) \succeq \tilde{B}_A(x),\ J_A(0) \leq J_A(x)$$

and $M_A(x) \geq \min\{M_A(x*y), M_A(y)\}$, $\tilde{B}_A(x) \succeq \text{rmin}\{\tilde{B}_A(x*y), \tilde{B}_A(y)\}$ and

$$J_A(x) \leq \max\{J_A(x*y), J_A(y)\}.$$

Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of X. □

Theorem 3.11. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X if and only if it satisfies the following condition.*

$$(((x*y)*y)*a)*b = 0 \;\Rightarrow \begin{cases} M_A(x*y) \geq \min\{M_A(a), M_A(b)\}, \\ \tilde{B}_A(x*y) \succeq \text{rmin}\{\tilde{B}_A(a), \tilde{B}_A(b)\}, \\ J_A(x*y) \leq \max\{J_A(a), J_A(b)\}, \end{cases} \tag{3.9}$$

for all $x, y, a, b \in X$.

Proof. Assume that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an MBJ-neutrosophic ideal of X (see Theorem 3.3). Let $a, b, x, y \in X$ be such that $(((x*y)*y)*a)*b = 0$. Then

$$M_A(x*y) \geq M_A((x*y)*y) \geq \min\{M_A(a), M_A(b)\},$$

$$\tilde{B}_A(x*y) \succeq \tilde{B}_A((x*y)*y) \succeq \text{rmin}\{\tilde{B}_A(a), \tilde{B}_A(b)\},$$

and $J_A(x*y) \leq J_A((x*y)*y) \leq \max\{J_A(a), J_A(b)\}$ by Theorem 3.6 and Lemma 3.9. Thus (3.9) is valid.

Conversely, let $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ be an MBJ-neutrosophic set in X which satisfies the condition (3.9). Let $x, a, b \in X$ be such that $x * a \leq b$. Then

$$(((x * 0) * 0) * a) * b = 0,$$

and so

$$M_A(x) = M_A(x * 0) \geq \min\{M_A(a), M_A(b)\},$$

$$\tilde{B}_A(x) = \tilde{B}_A(x * 0) \succeq \operatorname{rmin}\{\tilde{B}_A(a), \tilde{B}_A(b)\},$$

and

$$J_A(x) = J_A(x * 0) \leq \max\{J_A(a), J_A(b)\}$$

by (2.1) and (3.9). Thus $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ is an MBJ-neutrosophic ideal of X by Lemma 3.10. Since $(((x * y) * y) * ((x * y) * y)) * 0 = 0$ for all $x, y \in X$, we have

$$M_A(x * y) \geq \min\{M_A((x * y) * y), M_A(0)\} = M_A((x * y) * y),$$

$$\tilde{B}_A(x * y) \succeq \operatorname{rmin}\{\tilde{B}_A((x * y) * y), \tilde{B}_A(0)\} = \tilde{B}_A((x * y) * y),$$

and

$$J_A(x * y) \leq \max\{J_A((x * y) * y), J_A(0)\} = J_A((x * y) * y)$$

by (3.9). It follows from Theorem 3.6 that $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ is a positive implicative MBJ-neutrosophic ideal of X. □

Theorem 3.12. *Let $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ be an MBJ-neutrosophic set in X. Then $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ is a positive implicative MBJ-neutrosophic ideal of X if and only if it satisfies the following condition.*

$$\begin{array}{l} M_A((x * z) * (y * z)) \geq \min\{M_A(a), M_A(b)\}, \\ \tilde{B}_A((x * z) * (y * z)) \succeq \operatorname{rmin}\{\tilde{B}_A(a), \tilde{B}_A(b)\}, \\ J_A((x * z) * (y * z)) \leq \max\{J_A(a), J_A(b)\}, \end{array} \tag{3.10}$$

for all $x, y, z, a, b \in X$ with $(((x * y) * z) * a) * b = 0$.

Proof. Assume that $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ is a positive implicative MBJ-neutrosophic ideal of X. Then $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ is an MBJ-neutrosophic ideal of X (see Theorem 3.3). Let $a, b, x, y, z \in X$ be such that $(((x * y) * z) * a) * b = 0$. Using Theorem 3.7 and Lemma 3.9, we have

$$M_A((x * z) * (y * z)) \geq M_A((x * y) * z) \geq \min\{M_A(a), M_A(b)\},$$

$$\tilde{B}_A((x * z) * (y * z)) \succeq \tilde{B}_A((x * y) * z) \succeq \operatorname{rmin}\{\tilde{B}_A(a), \tilde{B}_A(b)\},$$

and

$$J_A((x * z) * (y * z)) \leq J_A((x * y) * z) \leq \max\{J_A(a), J_A(b)\}.$$

Conversely, let $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ be an MBJ-neutrosophic set in X which satisfies the condition (3.10). Let $x, y, a, b \in X$ be such that $(((x * y) * y) * a) * b = 0$. Then

$$M_A(x * y) = M_A((x * y) * (y * y) \geq \min\{M_A(a), M_A(b)\},$$

$$\tilde{B}_A(x*y) = \tilde{B}_A((x*y)*(y*y) \succeq \mathrm{rmin}\{\tilde{B}_A(a), \tilde{B}_A(b)\},$$

and

$$J_A(x*y) = J_A((x*y)*(y*y) \leq \max\{J_A(a), J_A(b)\}$$

by (2.1), (III) and (3.10). It follows from Theorem 3.11 that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X. □

Theorem 3.13. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X if and only if it satisfies the following condition.*

$$\begin{aligned} &M_A(x*y) \geq \min\{M_A(a_1), M_A(a_2), \cdots, M_A(a_n)\}, \\ &\tilde{B}_A(x*y) \succeq \mathrm{rmin}\{\tilde{B}_A(a_1), \tilde{B}_A(a_2), \cdots, \tilde{B}_A(a_n)\}, \\ &J_A(x*y) \leq \max\{J_A(a_1), J_A(a_2), \cdots, J_A(a_n)\}, \end{aligned} \tag{3.11}$$

for all $x, y, a_1, a_2, \cdots, a_n \in X$ with $(\cdots(((x*y)*y)*a_1)*\cdots)*a_n = 0$.

Proof. It is similar to the proof of Theorem 3.11. □

Theorem 3.14. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is a positive implicative MBJ-neutrosophic ideal of X if and only if it satisfies the following condition.*

$$\begin{aligned} &M_A((x*z)*(y*z)) \geq \min\{M_A(a_1), M_A(a_2), \cdots, M_A(a_n)\}, \\ &\tilde{B}_A((x*z)*(y*z)) \succeq \mathrm{rmin}\{\tilde{B}_A(a_1), \tilde{B}_A(a_2), \cdots, \tilde{B}_A(a_n)\}, \\ &J_A((x*z)*(y*z)) \leq \max\{J_A(a_1), J_A(a_2), \cdots, J_A(a_n)\}, \end{aligned} \tag{3.12}$$

for all $x, y, z, a_1, a_2, \cdots, a_n \in X$ with $(\cdots(((x*y)*z)*a_1)*\cdots)*a_n = 0$.

Proof. It is similar to the proof of Theorem 3.12. □

Given an MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X, we consider the following sets.

$$\begin{aligned} &U(M_A;\alpha) := \{x \in X \mid M_A(x) \geq \alpha\}, \\ &U(\tilde{B}_A;[\delta_1,\delta_2]) := \{x \in X \mid \tilde{B}_A(x) \succeq [\delta_1,\delta_2]\}, \\ &L(J_A;\beta) := \{x \in X \mid J_A(x) \leq \beta\}, \end{aligned}$$

where $\alpha, \beta \in [0,1]$ and $[\delta_1, \delta_2] \in [I]$.

Theorem 3.15. *An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is a positive implicative MBJ-neutrosophic ideal of X if and only if the non-empty sets $U(M_A;\alpha)$, $U(\tilde{B}_A;[\delta_1,\delta_2])$ and $L(J_A;\beta)$ are positive implicative ideals of X, for all $\alpha, \beta \in [0,1]$ and $[\delta_1, \delta_2] \in [I]$.*

Proof. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be a positive implicative MBJ-neutrosophic ideal of X. Let $\alpha, \beta \in [0,1]$ and $[\delta_1, \delta_2] \in [I]$ be such that $U(M_A;\alpha)$, $U(\tilde{B}_A;[\delta_1,\delta_2])$ and $L(J_A;\beta)$ are non-empty. Obviously, $0 \in U(M_A;\alpha) \cap U(\tilde{B}_A;[\delta_1,\delta_2]) \cap L(J_A;\beta)$. For any $x, y, z, a, b, c, u, v, w \in X$, if $(x*y)*z \in U(M_A;\alpha)$, $y*z \in U(M_A;\alpha)$,

$(a*b)*c \in U(\tilde{B}_A;[\delta_1,\delta_2])$, $b*c \in U(\tilde{B}_A;[\delta_1,\delta_2])$, $(u*v)*w \in L(J_A;\beta)$ and $v*w \in L(J_A;\beta)$, then

$$M_A(x*z) \geq \min\{M_A((x*y)*z), M_A(y*z)\} \geq \min\{\alpha,\alpha\} = \alpha,$$

$$\tilde{B}_A(a*c) \succeq \mathrm{rmin}\{\tilde{B}_A((a*b)*c), \tilde{B}_A(b*c)\} \succeq \mathrm{rmin}\{[\delta_1,\delta_2],[\delta_1,\delta_2]\} = [\delta_1,\delta_2],$$

$$J_A(u*w) \leq \max\{J_A((u*v)*w), J_A(v*w)\} \leq \min\{\beta,\beta\} = \beta,$$

and so $x*z \in U(M_A;\alpha)$, $a*c \in U(\tilde{B}_A;[\delta_1,\delta_2])$ and $u*w \in L(J_A;\beta)$. Therefore $U(M_A;\alpha)$, $U(\tilde{B}_A;[\delta_1,\delta_2])$ and $L(J_A;\beta)$ are positive implicative ideals of X.

Conversely, assume that the non-empty sets $U(M_A;\alpha)$, $U(\tilde{B}_A;[\delta_1,\delta_2])$ and $L(J_A;\beta)$ are positive implicative ideals of X for all $\alpha,\beta \in [0,1]$ and $[\delta_1,\delta_2] \in [I]$. Assume that $M_A(0) < M_A(a)$, $\tilde{B}_A(0) \prec \tilde{B}_A(a)$ and $J_A(0) > J_A(a)$, for some $a \in X$. Then $0 \notin U(M_A;M_A(a)) \cap U(\tilde{B}_A;\tilde{B}_A(a)) \cap L(J_A;J_A(a)$, which is a contradiction. Thus $M_A(0) \geq M_A(x)$, $\tilde{B}_A(0) \succeq \tilde{B}_A(x)$ and $J_A(0) \leq J_A(x)$, for all $x \in X$. If

$$M_A(a_0*c_0) < \min\{M_A((a_0*b_0)*c_0), M_A(b_0*c_0)\},$$

for some $a_0,b_0,c_0 \in X$, then $(a_0*b_0)*c_0 \in U(M_A;t_0)$ and $b_0*c_0 \in U(M_A;t_0)$ but $a_0*c_0 \notin U(M_A;t_0)$ for $t_0 := \min\{M_A((a_0*b_0)*c_0), M_A(b_0*c_0)\}$. This is a contradiction, and thus

$$M_A(a*c) \geq \min\{M_A((a*b)*c), M_A(b*c)\},$$

for all $a,b,c \in X$.

Similarly, we can show that $J_A(a*c) \leq \max\{J_A((a*b)*c), J_A(b*c)\}$, for all $a,b,c \in X$. Suppose that $\tilde{B}_A(a_0*c_0) \prec \mathrm{rmin}\{\tilde{B}_A((a_0*b_0)*c_0), \tilde{B}_A(b_0*c_0)\}$, for some $a_0,b_0,c_0 \in X$. Let $\tilde{B}_A((a_0*b_0)*c_0) = [\lambda_1,\lambda_2]$, $\tilde{B}_A(b_0*c_0) = [\lambda_3,\lambda_4]$ and $\tilde{B}_A(a_0*c_0) = [\delta_1,\delta_2]$. Then

$$[\delta_1,\delta_2] \prec \mathrm{rmin}\{[\lambda_1,\lambda_2],[\lambda_3,\lambda_4]\} = [\min\{\lambda_1,\lambda_3\}, \min\{\lambda_2,\lambda_4\}],$$

and so $\delta_1 < \min\{\lambda_1,\lambda_3\}$ and $\delta_2 < \min\{\lambda_2,\lambda_4\}$. Taking

$$[\gamma_1,\gamma_2] := \tfrac{1}{2}\left(\tilde{B}_A(a_0*c_0) + \mathrm{rmin}\{\tilde{B}_A((a_0*b_0)*c_0), \tilde{B}_A(b_0*c_0)\}\right)$$

implies that

$$\begin{aligned}[\gamma_1,\gamma_2] &= \tfrac{1}{2}\left([\delta_1,\delta_2] + [\min\{\lambda_1,\lambda_3\}, \min\{\lambda_2,\lambda_4\}]\right) \\ &= \left[\tfrac{1}{2}(\delta_1 + \min\{\lambda_1,\lambda_3\}), \tfrac{1}{2}(\delta_2 + \min\{\lambda_2,\lambda_4\})\right].\end{aligned}$$

It follows that

$$\min\{\lambda_1,\lambda_3\} > \gamma_1 = \tfrac{1}{2}(\delta_1 + \min\{\lambda_1,\lambda_3\}) > \delta_1$$

and

$$\min\{\lambda_2,\lambda_4\} > \gamma_2 = \tfrac{1}{2}(\delta_2 + \min\{\lambda_2,\lambda_4\}) > \delta_2.$$

Thus $[\min\{\lambda_1,\lambda_3\}, \min\{\lambda_2,\lambda_4\}] \succ [\gamma_1,\gamma_2] \succ [\delta_1,\delta_2] = \tilde{B}_A(a_0*c_0)$. So

$$a_0*c_0 \notin U(\tilde{B}_A;[\gamma_1,\gamma_2]).$$

On the other hand,

$$\tilde{B}_A((a_0*b_0)*c_0) = [\lambda_1,\lambda_2] \succeq [\min\{\lambda_1,\lambda_3\}, \min\{\lambda_2,\lambda_4\}] \succ [\gamma_1,\gamma_2]$$

and

$$\tilde{B}_A(b_0 * c_0) = [\lambda_3, \lambda_4] \succeq [\min\{\lambda_1, \lambda_3\}, \min\{\lambda_2, \lambda_4\}] \succ [\gamma_1, \gamma_2],$$

that is, $(a_0 * b_0) * c_0, b_0 * c_0 \in U(\tilde{B}_A; [\gamma_1, \gamma_2])$. This is a contradiction. Hence

$$\tilde{B}_A(x * z) \succeq \mathrm{rmin}\{\tilde{B}_A((x * y) * z), \tilde{B}_A(y * z)\},$$

for all $x, y, z \in X$. Consequently, $\mathcal{A} = (M_A,\ \tilde{B}_A,\ J_A)$ is a positive implicative MBJ-neutrosophic ideal of X. □

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