Pythagorean neutrosophic Subring of a ring

Jansi R\textsuperscript{1}, Mohana K\textsuperscript{2}

\textsuperscript{1} Research Scholar, Nirmala College for Women, Coimbatore.
\textsuperscript{2} Assistant Professor, Nirmala College for Women, Coimbatore.

Abstract

In this paper, we study some of the properties of Pythagorean neutrosophic subring of a ring and prove some results on these. Using some basic definitions, we derive some important theorems. Intersection is applied into the Pythagorean neutrosophic subring of a ring.

Key words: Pythagorean neutrosophic set, Pythagorean neutrosophic subring.

1. Introduction

Fuzzy sets were introduced by Zadeh [19] and he discussed only membership function. After the extensions of fuzzy set theory Atanassov [6] generalized this concept and introduced a new concept called intuitionistic fuzzy set (IFS). Yager [16] familiarized the model of Pythagorean fuzzy set. IFS has its greatest use in practical multiple attribute decision making (MADM) problems, and the academic research have achieved great development [16, 17, 18]. However, in some practical problems, the sum of membership degree and non-membership degree to which an alternative satisfying attribute provided by decision maker (DM) may be bigger than 1, but their square sum is less than or equal to 1. Azriel Rosenfeld [7] was studied about fuzzy rings.

In 2006, F. Smarandache introduced, for the first time, the degree of dependence (and consequently the degree of independence) between the components of the fuzzy set, and also between the components of the neutrosophic set. In 2016, the refined neutrosophic set was generalized to the degree of dependence or independence of subcomponents [13]. There are three special cases in neutrosophic set. First, truth membership, falsity membership, indeterminacy are independent. Second, truth membership and falsity membership are dependent and indeterminacy is independent. Third, truth membership, falsity

\textsuperscript{1} mathematicsgasc@gmail.com, \textsuperscript{2} riyaraju1116@gmail.com
membership, indeterminacy are independent. We studied about second case. Jansi, Mohana and F. Smarandache [11] was introduced the concept of Pythagorean neutrosophic set [PN-set]. That is, if truth membership and falsity membership are dependent and indeterminacy is independent under the restriction that the sum of truth membership, falsity membership and indeterminacy does not exceed 2. Sometimes, we face many problems which cannot be handled by using this set, for example when \( T = 0.8, I = 0.9, F = 0.4 \). T and F are dependent that condition was \( T + F \leq 1 \). Here \( T + F \geq 1 \). Totally, the sum of truth membership, falsity membership and indeterminacy does exceed 2. We cannot used that set. At that time we use PN-set. Also PN-set includes truth membership, falsity membership and indeterminacy but under the restriction their square sum of truth membership, falsity membership and indeterminacy does not exceed 2. That is, \( T^2 + I^2 + F^2 \leq 2 \). We introduce the concept of Pythagorean neutrosophic subring and established some results.

2. Preliminaries

Definition 3.1 [19] Let \( X \) be a nonempty set. A fuzzy set \( A \) drawn from \( X \) is defined as \( A = \{ (x : \mu_A(x)) : x \in X \} \), where \( \mu_A : X \rightarrow [0, 1] \) is the membership function of the fuzzy set \( A \).

Definition 3.2 (Pythagorean Fuzzy Set) [16] Let \( X \) be a non-empty set and \( I \) the unit interval \([0, 1]\). A PF set \( S \) is an object having the form \( P = \{ (x, \mu_P(x), v_P(x)) : x \in X \} \) where the function \( \mu_P : X \rightarrow [0, 1] \) and \( v_P : X \rightarrow [0, 1] \) denote respectively the degree of membership and degree of non-membership of each element \( x \in X \) to the set \( P \), and \( 0 \leq (\mu_P(x))^2 + (v_P(x))^2 \leq 1 \) for each \( x \in X \).

Definition 3.3 (Pythagorean neutrosophic set [PN]-sets) [11] Let \( X \) be a non-empty set (universe). A PN sets \( A = \{ (x, P_A(x), Q_A(x), R_A(x)) : x \in X \} \) where \( P_A : X \rightarrow [0, 1], Q_A : X \rightarrow [0, 1] \) and \( R_A : X \rightarrow [1, 0] \) are the mappings such that \( 0 \leq (P_A(x))^2 + (Q_A(x))^2 + (R_A(x))^2 \leq 2 \) and \( P_A(x) \) denote the membership degree, \( Q_A(x) \) denote the indeterminacy and \( R_A(x) \) denote the non-membership degree. Here \( T \) and \( F \) are dependent neutrosophic components and \( I \) is an independent neutrosophic component.

3. PN Subring of a Ring

Definition 4.1 Let \( A = \{ (X, P_A(x), Q_A(x), R_A(x)) : x \in X \} \) and

\[mathematicsgasc@gmail.com, riyaraju1116@gmail.com\]
Let $A = \{(x, P_A(x), Q_A(x), R_A(x)) / x \in X\}$ be two PN sets, then their operations are defined as follows:

(1) Type 1:
$A \cup B = \{(x, \max(P_A(x), P_B(x)), \min(Q_A(x), Q_B(x)), \min(R_A(x), R_B(x)) : x \in X\}$

Type 2:
$A \cup B = \{(x, \max(P_A(x), P_B(x)), \max(Q_A(x), Q_B(x)), \min(R_A(x), R_B(x)) : x \in X\}$

(2) Type 1:
$A \cap B = \{(x, \min(P_A(x), P_B(x)), \max(Q_A(x), Q_B(x)), \max(R_A(x), R_B(x)) : x \in X\}$

Type 2:
$A \cap B = \{(x, \min(P_A(x), P_B(x)), \min(Q_A(x), Q_B(x)), \max(R_A(x), R_B(x)) : x \in X\}$

**Definition 4.2** Let $A = \{(x, P_A(x), Q_A(x), R_A(x)) / x \in X\}$ be PN set, then the complement of $A$ is

Type 1: $A^c = \{(x, R_A(x), 1 - Q_A(x), P_A(x)) : x \in X\}$

Type 2: $A^c = \{(x, R_A(x), Q_A(x), P_A(x)) : x \in X\}$

**Definition 4.3** Let $A = \{(x, P_A(x), Q_A(x), R_A(x)) / x \in X\}$ and $B = \{(x, P_B(x), Q_B(x), R_B(x)) / x \in X\}$ be two PN sets, then $A \subseteq B$ if and only if $P_A(x) \leq P_B(x), Q_A(x) \geq Q_B(x), R_A(x) \geq R_B(x)$.

and $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

**Definition 4.4** Let $(R, +, \cdot)$ be a ring, A PN subset $A$ of $R$ is said to be a PN subring of $R$ if the following conditions are satisfied

(i) $P_A(x + y) \geq \min \{P_A(x), P_A(y)\}$

(ii) $P_A(xy) \geq \min \{P_A(x), P_A(y)\}$

(iii) $Q_A(x + y) \leq \max \{Q_A(x), Q_A(y)\}$

(iv) $Q_A(xy) \leq \max \{Q_A(x), Q_A(y)\}$

(v) $R_A(x + y) \leq \max \{R_A(x), R_A(y)\}$

(vi) $R_A(xy) \leq \max \{R_A(x), R_A(y)\}$

for all $x$ and $y$ in $R$.

**Theorem 4.5** Let $A = \{(x, P_A(x), Q_A(x), R_A(x)) / x \in R\}$ be a PN subring of a ring $R$. Then $P_A(-x) = P_A(x), Q_A(-x) = Q_A(x), R_A(-x) = R_A(x)$.

$P_A(x) \leq P_A(e), Q_A(x) \geq Q_A(e), R_A(x) \geq R_A(e)$ for all $x$ in $R$ and the identity element $e$ in $R$.

**Proof:** Let $x$ be in $R$. Now $P_A(x) = P_A(-(-x)) \geq P_A(-x) \geq P_A(x)$. 
Therefore \( P_A(x) = P_A(-x) \) for all \( x \) in \( R \).
\( Q_A(x) = Q_A(-x) \leq Q_A(-x) \leq Q_A(x) \).
Therefore \( Q_A(x) = Q_A(-x) \) for all \( x \) in \( R \).
\( R_A(x) = R_A(-x) \leq R_A(-x) \leq R_A(x) \).
Therefore \( R_A(x) = R_A(-x) \) for all \( x \) in \( R \).
Also, \( P_A(e) = P_A(x-x) \geq min \{ P_A(x), P_A(x) \} = P_A(x) \).
Therefore \( P_A(e) \geq P_A(x) \) for all \( x \) in \( R \).
\( Q_A(e) = Q_A(x-x) \leq max \{ Q_A(x), Q_A(x) \} = Q_A(x) \).
Therefore \( Q_A(e) \leq Q_A(x) \) for all \( x \) in \( R \).
\( R_A(e) = R_A(x-x) \leq max \{ R_A(x), R_A(x) \} = R_A(x) \).
Therefore \( R_A(e) \leq R_A(x) \) for all \( x \) in \( R \).

**Theorem 4.6** Let \( A = \{(x, P_A(x), Q_A(x), R_A(x))/x \in R \} \) be a PN subring of a ring \( R \). Then
(i) \( P_A(x + y) = P_A(e) \) implies that \( P_A(x) = P_A(y) \) for \( x \) and \( y \) in \( R \).
(ii) \( Q_A(x + y) = Q_A(e) \) implies that \( Q_A(x) = Q_A(y) \) for \( x \) and \( y \) in \( R \).
(iii) \( R_A(x + y) = R_A(e) \) implies that \( R_A(x) = R_A(y) \) for \( x \) and \( y \) in \( R \).

**Proof:**
\( P_A(x) = P_A(x + y - y) \geq min \{ P_A(x + y), P_A(y) \} = min \{ P_A(e), P_A(y) \} = P_A(y) \).
\( P_A(y) = P_A(y + x - x) \geq min \{ P_A(y + x), P_A(x) \} = min \{ P_A(e), P_A(x) \} = P_A(x) \).
Therefore \( P_A(x) = P_A(y) \) for \( x \) and \( y \) in \( R \).
\( Q_A(x) = Q_A(x + y - y) \leq max \{ Q_A(x + y), Q_A(y) \} = max \{ Q_A(e), Q_A(y) \} = Q_A(y) \).
\( Q_A(y) = Q_A(y + x - x) \leq max \{ Q_A(y + x), Q_A(x) \} = max \{ Q_A(e), Q_A(x) \} = Q_A(x) \).
Therefore \( Q_A(x) = Q_A(y) \) for \( x \) and \( y \) in \( R \).
\( R_A(x) = R_A(x + y - y) \leq max \{ R_A(x + y), R_A(y) \} = max \{ R_A(e), R_A(y) \} = R_A(y) \).
\( R_A(y) = R_A(y + x - x) \leq max \{ R_A(y + x), R_A(x) \} = max \{ R_A(e), R_A(x) \} = R_A(x) \).
Therefore \( R_A(x) = R_A(y) \) for \( x \) and \( y \) in \( R \).

**Theorem 4.7** Let \( A = \{(x, P_A(x), Q_A(x), R_A(x))/x \in R \} \) be a PN subring of a ring \( R \).
(i) If \( P_A(x + y) = 1 \), then \( P_A(x) = P_A(y) \) for \( x \) and \( y \) in \( R \).
(ii) If \( Q_A(x + y) = 0 \), then \( Q_A(x) = Q_A(y) \) for \( x \) and \( y \) in \( R \).
(iii) If \( R_A(x + y) = 0 \), then \( R_A(x) = R_A(y) \) for \( x \) and \( y \) in \( R \).
Proof: Now \( P_A(x) = P_A(x + y - y) \geq \min \{P_A(x + y), P_A(y)\} = \min \{1, P_A(y)\} \)
\( = P_A(y) = P_A(-y) = P_A(x - x - y) \geq \min \{P_A(x), P_A(x + y)\} = \min \{P_A(x), 1\} \)
\( = P_A(x) . \)
Therefore \( P_A(x) = P_A(y) \) for \( x \) and \( y \) in \( R \).
Hence (i) is proved.
\( Q_A(x) = Q_A(x + y - y) \leq \max \{Q_A(x + y), Q_A(y)\} = \max \{0, Q_A(y)\} = Q_A(y) \)
\( = Q_A(-y) = Q_A(x - x - y) \leq \max \{Q_A(x), Q_A(x + y)\} = \max \{Q_A(x), 0\} \)
\( = Q_A(x) . \)
Therefore \( Q_A(x) = Q_A(y) \) for \( x \) and \( y \) in \( R \).
Hence (ii) is proved.
\( R_A(x) = R_A(x + y - y) \leq \max \{R_A(x + y), R_A(y)\} = \max \{0, R_A(y)\} = R_A(y) \)
\( = R_A(-y) = R_A(x - x - y) \leq \max \{R_A(x), R_A(x + y)\} = \max \{R_A(x), 0\} \)
\( = R_A(x) . \)
Therefore \( R_A(x) = R_A(y) \) for \( x \) and \( y \) in \( R \).
Hence (iii) is proved.
Hence (iv) is proved.

Theorem 4.8 Let \( A = \{ (x, P_A(x), Q_A(x), R_A(x)) / x \in R \} \) be a PN subring of a ring \( R \).
(i)\( P_A(xy^{-1}) = 0 \), then either \( P_A(x) = 0 \) or \( P_A(y) = 0 \), for \( x \) and \( y \) in \( R \).
(ii)\( Q_A(xy^{-1}) = 0 \), then either \( Q_A(x) = 0 \) or \( Q_A(y) = 0 \), for \( x \) and \( y \) in \( R \).
(iii)\( R_A(xy^{-1}) = 0 \), then either \( R_A(x) = 0 \) or \( R_A(y) = 0 \), for \( x \) and \( y \) in \( R \).
(iv)\( (xy^{-1}) = 0 \), then either \( (x) = 0 \) or \( (y) = 0 \), for \( x \) and \( y \) in \( R \).

Proof: Let \( x \) and \( y \) in \( R \).
(i)By the definition \( P_A(xy^{-1}) \geq \min \{P_A(x), P_A(y)\} \),
which implies that \( 0 \geq \min \{P_A(x), P_A(y)\} \).
Therefore, either \( P_A(x) = 0 \) or \( P_A(y) = 0 \).
(ii)By the definition \( Q_A(xy^{-1}) \leq \max \{Q_A(x), Q_A(y)\} \),
which implies that \( 0 \leq \max \{Q_A(x), Q_A(y)\} \).
Therefore, either \( Q_A(x) = 0 \) or \( Q_A(y) = 0 \).
(iii)By the definition \( R_A(xy^{-1}) \leq \max \{R_A(x), R_A(y)\} \),
which implies that \( 0 \leq \max \{R_A(x), R_A(y)\} \).
Therefore, either \( R_A(x) = 0 \) or \( R_A(y) = 0 \).

Theorem 4.9 If \( A = \{ (x, P_A(x), Q_A(x), R_A(x)) / x \in R \} \) be a PN subring of \( R \), then
(i)\( P_A(xy) = P_A(yx) \) if and only if \( P_A(x) = P_A(y^{-1}xy) \), for \( x \) and \( y \) in \( R \).
(ii)\( Q_A(xy) = Q_A(yx) \) if and only if \( Q_A(x) = Q_A(y^{-1}xy) \), for \( x \) and \( y \) in \( R \).
(iii)\( R_A(xy) = R_A(yx) \) if and only if \( R_A(x) = R_A(y^{-1}xy) \), for \( x \) and \( y \) in \( R \).
**Proof:** Let $x$ and $y$ be in $R$.
Assume that $P_A(xy) = P_A(yx)$, so $P_A(y^{-1}xy) = P_A(y^{-1}yx) = P_A(x)$.
Therefore $P_A(x) = P_A(y^{-1}xy)$, for $x$ and $y$ in $R$.
Conversely, assume that $P_A(x) = P_A(y^{-1}xy)$,
we get $P_A(xy) = P_A(xyxx^{-1}) = P_A(yx)$.
Therefore $P_A(xy) = P_A(yx)$, for $x$ and $y$ in $R$.
Hence $P_A(xy) = P_A(yx)$ if and only if $P_A(x) = P_A(y^{-1}xy)$, for $x$ and $y$ in $R$.
Also assume that $Q_A(xy) = Q_A(yx)$, so $Q_A(y^{-1}xy) = Q_A(y^{-1}yx) = Q_A(x)$.
Therefore $Q_A(x) = Q_A(y^{-1}xy)$, for $x$ and $y$ in $R$.
Conversely, assume that $Q_A(x) = Q_A(y^{-1}xy)$,
we get $Q_A(xy) = Q_A(xyxx^{-1}) = Q_A(yx)$.
Therefore $Q_A(xy) = Q_A(yx)$, for $x$ and $y$ in $R$.
Hence $P_A(xy) = P_A(yx)$ if and only if $Q_A(x) = Q_A(y^{-1}xy)$, for $x$ and $y$ in $R$.
Also assume that $R_A(xy) = R_A(yx)$, so $R_A(y^{-1}xy) = R_A(y^{-1}yx) = R_A(x)$.
Therefore $R_A(x) = R_A(y^{-1}xy)$, for $x$ and $y$ in $R$.
Conversely, assume that $R_A(x) = R_A(y^{-1}xy)$,
we get $R_A(xy) = R_A(xyxx^{-1}) = R_A(yx)$.
Therefore $R_A(xy) = R_A(yx)$, for $x$ and $y$ in $R$.
Hence $R_A(xy) = R_A(yx)$ if and only if $R_A(x) = R_A(y^{-1}xy)$, for $x$ and $y$ in $R$.

**Theorem 4.10** If $A = \{(x, P_A(x), Q_A(x), R_A(x))/x \in R\}$ be a PN subring of $R$,
then $R_2 = \{x \in R/P_A(x) = P_A(e), Q_A(x) = Q_A(e), R_A(x) = R_A(e)\}$ is a subring of $R$.

**Proof:** Here $R_2 = \{x \in R/P_A(x) = P_A(e), Q_A(x) = Q_A(e), R_A(x) = R_A(e)\}$, by
Theorem 2.4.4,
$P_A(x^{-1}) = P_A(e), Q_A(x^{-1}) = Q_A(e), R_A(x^{-1}) = R_A(e)$
and $(x^{-1}) = (x) = (e)$.
Therefore $x^{-1} \in R_2$.

Now $P_A(xy^{-1}) \geq \min \{P_A(x), P_A(y)\} = \min \{P_A(e), P_A(e)\} = P_A(e)$, and
$P_A(e) = P_A((xy^{-1})(xy^{-1})^{-1}) \geq \min \{P_A(xy^{-1}), P_A(xy^{-1})\} = P_A(xy^{-1})$.
Hence $P_A(e) = P_A(xy^{-1})$.

$Q_A(xy^{-1}) \leq \max \{Q_A(x), Q_A(y)\} = \max \{Q_A(e), Q_A(e)\} = Q_A(e)$,
and $Q_A(e) = Q_A((xy^{-1})(xy^{-1})^{-1}) \leq \max \{Q_A(xy^{-1}), Q_A(xy^{-1})\} = Q_A(xy^{-1})$.
Hence $Q_A(e) = Q_A(xy^{-1})$.

$R_A(xy^{-1}) \leq \max \{R_A(x), R_A(y)\} = \max \{R_A(e), R_A(e)\} = R_A(e)$, and
$R_A(e) = R_A((xy^{-1})(xy^{-1})^{-1}) \leq \max \{R_A(xy^{-1}), R_A(xy^{-1})\}$
$= R_A(xy^{-1})$.
Hence $R_A(e) = R_A(xy^{-1})$.

1mathematicsbgasc@gmail.com, 2riyaraju1116@gmail.com
\[(xy^{-1}) \geq \min \{ (x, y) \} = \min \{ (e, e) \} = (e).\]

**Theorem 4.11** Let \( R \) be a ring, if \( A = \{(x, P_A(x), Q_A(x), R_A(x))/x \in R\} \) be a PN subring of \( R \), then \( P_A(xy) = \min \{ P_A(x), P_A(y) \}, Q_A(xy) = \max \{ Q_A(x), Q_A(y) \}, R_A(xy) = \max \{ R_A(x), R_A(y) \} \) for each \( x, y \) in \( R \) with \( P_A(x) \neq P_A(y), Q_A(x) \neq Q_A(y), R_A(x) \neq R_A(y) \).

**Proof:** Assume that \( P_A(x) > P_A(y), Q_A(x) > Q_A(y), R_A(x) > R_A(y) \) and \( Q_A(x) > Q_A(y) \).

Then
\[ P_A(y) = P_A(x^{-1}y) \geq \min \{ P_A(x^{-1}), P_A(xy) \} = \min \{ P_A(x), P_A(xy) \} \]
\[ = P_A(xy) \geq \min \{ P_A(x), P_A(y) \} = P_A(y). \]

Therefore \( P_A(xy) = P_A(y) = \min \{ P_A(x), P_A(y) \} \).

\[ Q_A(y) = Q_A(x^{-1}y) \leq \max \{ Q_A(x^{-1}), Q_A(xy) \} = \max \{ Q_A(x), Q_A(xy) \} \]
\[ = Q_A(xy) \leq \max \{ Q_A(x), Q_A(y) \} = Q_A(y). \]

Therefore \( Q_A(xy) = Q_A(y) = \max \{ Q_A(x), Q_A(y) \} \).

\[ R_A(y) = R_A(x^{-1}y) \leq \max \{ R_A(x^{-1}), R_A(xy) \} = \max \{ R_A(x), R_A(xy) \} \]
\[ = R_A(xy) \leq \max \{ R_A(x), R_A(y) \} = R_A(y). \]

Therefore \( R_A(xy) = R_A(y) = \max \{ R_A(x), R_A(y) \} \).

**Theorem 4.12** If \( A = \{(x, P_A(x), Q_A(x), R_A(x))/x \in R\} \) and \( B = \{(x, P_B(x), Q_B(x), R_B(x))/x \in R\} \) are two PN subring of a ring \( R \), then their intersection \( A \cap B \) is a PN subring of \( R \).

**Proof:** Let \( A = \{(x, P_A(x), Q_A(x), R_A(x))/x \in R\} \) and \( B = \{(X, P_B(x), Q_B(x), R_B(x)): x \in R\} \).

Let \( C = A \cap B \) and \( C = \{(x, P_C(x), Q_C(x), R_C(x))/x \in R\} \).

Now,
\[ P_C(xy^{-1}) = \min \{ P_A(xy^{-1}), P_B(xy^{-1}) \} \geq \min \{ \min \{ P_A(x), P_A(y) \}, \min \{ P_B(x), P_B(y) \} \} \]
\[ \geq \min \{ \min \{ P_A(x), P_B(x) \}, \min \{ P_A(y), P_B(y) \} \} = \min \{ P_C(x), P_C(y) \}. \]

Also,
\[ Q_C(xy^{-1}) = \max \{ Q_A(xy^{-1}), Q_B(xy^{-1}) \} \leq \max \{ \max \{ Q_A(x), Q_A(y) \}, \max \{ Q_B(x), Q_B(y) \} \} \]
\[ \leq \max \{ \max \{ Q_A(x), Q_B(x) \}, \max \{ Q_A(y), Q_B(y) \} \} = \max \{ Q_C(x), Q_C(y) \}. \]

\[ R_C(xy^{-1}) = \max \{ R_A(xy^{-1}), R_B(xy^{-1}) \} \leq \max \{ \max \{ R_A(x), R_A(y) \}, \max \{ R_B(x), R_B(y) \} \} \]
\[ \leq \max \{ \max \{ R_A(x), R_B(x) \}, \max \{ R_A(y), R_B(y) \} \} = \max \{ R_C(x), R_C(y) \}. \]

Hence \( A \cap B \) is a PN subring of \( R \).
4. Conclusion

In this paper, we define the Pythagorean neutrosophic subring of a ring and investigate the relationship among these Pythagorean neutrosophic subring of a ring. Some characterization theorems of Pythagorean neutrosophic subring of a ring are obtained.

References


