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\textbf{\textit{\L}}-Single Valued Extremely Disconnected Ideal Neutrosophic Topological Spaces

Fahad Alsharari

Department of Mathematics, College of Science and Human Studies, Hotat Sudair, Majmaah University, Majmaah 11952, Saudi Arabia; f.alsharari@mu.edu.sa

\textbf{Abstract:} This paper aims to mark out new concepts of \textit{r}-single valued neutrosophic sets, called \textit{r}-single valued neutrosophic \(\mathbb{L}\)-closed and \(\mathbb{L}\)-open sets. The definition of \(\mathbb{L}\)-single valued neutrosophic irresolute mapping is provided and its characteristic properties are discussed. Moreover, the concepts of \(\mathbb{L}\)-single valued neutrosophic extremally disconnected and \(\mathbb{L}\)-single valued neutrosophic normal spaces are established. As a result, a useful implication diagram between the \(\mathbb{L}\)-single valued neutrosophic ideal open sets is obtained. Finally, some kinds of separation axioms, namely \(\mathbb{L}\)-single valued neutrosophic ideal-\(r\) \((\text{r-SVNIR}_{r},\) for short), where \(i = \{0,1,2,3\}\), and \(\mathbb{L}\)-single valued neutrosophic ideal-\(T\) \((\text{r-SVNIT}_{j},\) for short), where \(j = \{1,2,2,3,4\}\), are introduced. Some of their characterizations, fundamental properties, and the relations between these notions have been studied.

\textbf{Keywords:} \(r\)-single valued neutrosophic \(\mathbb{L}\)-closed; \(\mathbb{L}\)-single valued neutrosophic irresolute mapping; \(\mathbb{L}\)-single valued neutrosophic extremally disconnected; \(\mathbb{L}\)-single valued neutrosophic normal; r-SVNIR; r-SVNIT.

1. \textbf{Introduction}

In 1999, Smarandache introduced the concept of a neutrosophy \cite{1}. It has been used at various axes of mathematical theories and applications. In recent decades, the theory made an outstanding advancement in the field of topological spaces. Salama et al. \cite{2–6}, for example, among many others, wrote their works in fuzzy neutrosophic topological spaces (FNTS), following Chang \cite{7}'s discoveries in the way of fuzzy topological spaces (FTS).

\textbf{\textit{\L}}-fuzzy topological spaces. Yan, Wang, Nanjing, Liang, and Yan \cite{9,10} developed a parallel theory in the context of intuitionistic \(I\)-fuzzy topological spaces.

Śostak, in 1985 \cite{8}, marked out a new definition of fuzzy topology as a crisp subfamily of family of fuzzy sets, which seems to be a drawback in the process of fuzzification of the concept of topological spaces. Yan, Wang, Nanjing, Liang, and Yan \cite{9,10} developed a parallel theory in the context of intuitionistic \(I\)-fuzzy topological spaces.

The idea of “single-valued neutrosophic set” \cite{11} was set out by Wang in 2010. Gayyar \cite{12}, in his 2016 paper, foregrounded the concept of a “smooth neutrosophic topological spaces”. The ordinary single-valued neutrosophic topology was presented by Kim \cite{13}. Recently, Saber et al. \cite{14,15} familiarized the concepts of single-valued neutrosophic ideal open local function, single-valued neutrosophic topological space, and the connectedness and stratification of single-valued neutrosophic topological spaces.

Neutrosophy, and especially neutrosophic sets, are powerful, general, and formal frameworks that generalize the concept of the ordinary sets, fuzzy sets, and intuitionistic fuzzy sets from philosophical point of view. This paper sets out to introduce and examine a new class of sets called \(r\)-single valued \(\mathbb{L}\)-closed in the single valued neutrosophic topological spaces in Śostak’s sense. More precisely, different attributes, like \(\mathbb{L}\)-single valued neutrosophic irresolute mapping, \(\mathbb{L}\)-single valued neutrosophic extremally disconnected, \(\mathbb{L}\)-single valued neutrosophic normal spaces, and some kinds of separation axioms, were developed. It can be fairly claimed that we have achieved expressive definitions, distinguished theorems, important lemmas, and counterexamples to investigate, in-depth, our
consequences and to find out the best results. It is notable to say that different crucial notions in single valued neutrosophic topology were generalized in this article. Different attributes, like extremely disconnected and some kinds of separation axioms, which have a significant impact on the overall topology’s notions, were also studied.

It is notable to say that the application aspects to this area of research can be further pointed to. There are many applications of neutrosophic theories in many branches of sciences. Possible applications are to control engineering and to Geographical Information Systems, and so forth, and could be secured, as mentioned by many authors, such as Reference [16–20].

In this study, $\tilde{X}$ is assumed to be a nonempty set, $\xi = [0, 1]$ and $\xi_0 = (0, 1)$. For $a \in \xi$, $\tilde{a}(v) = a$ for all $v \in \tilde{X}$. The family of all single-valued neutrosophic sets on $\tilde{X}$ is denoted by $\tilde{X}^\ast$.

2. Preliminaries

This section is devoted to provide a complete survey and trace previous studies related to the idea of this research article.

Definition 1 ([21]). Let $\tilde{X}$ be a non-empty set. A neutrosophic set (briefly, NS) in $\tilde{X}$ is an object having the form

$$\sigma_n = \{ (\nu_n, \tilde{\rho}_n(v), \tilde{\eta}_n(v), \tilde{\gamma}_n(v)) : v \in \tilde{X}\},$$

where

$$\tilde{\rho} : \tilde{X} \rightarrow ]^{-0, 1^+}[, \tilde{\eta} : \tilde{X} \rightarrow ]^{-0, 1^+}[,$$

and

$$-0 \leq \tilde{\rho}_n(v) + \tilde{\eta}_n(v) + \tilde{\gamma}_n(v) \leq 3^+$$

represent the degree of membership (namely $\tilde{\rho}_n(v)$), the degree of indeterminacy (namely $\tilde{\eta}_n(v)$), and the degree of non-membership (namely $\tilde{\gamma}_n(v)$), respectively, of any $v \in \tilde{X}$ to the set $\sigma_n$.

Definition 2 ([11]). Let $\tilde{X}$ be a space of points (objects), with a generic element in $\tilde{X}$ denoted by $v$. Then, $\sigma_n$ is called a single valued neutrosophic set (briefly, SVNS) in $\tilde{X}$, if $\sigma_n$ has the form $\sigma_n = (\tilde{\rho}_n, \tilde{\eta}_n, \tilde{\gamma}_n)$, where $\tilde{\rho}_n, \tilde{\eta}_n, \tilde{\gamma}_n : \tilde{X} \rightarrow [0, 1]$. In this case, $\tilde{\rho}_n, \tilde{\eta}_n, \tilde{\gamma}_n$ are called truth membership function, indeterminacy membership function, and falsity membership function, respectively.

Let $\tilde{X}$ be a nonempty set and $\xi = [0, 1]$ and $\xi_0 = (0, 1)$. A single-valued neutrosophic set $\sigma_n$ on $\tilde{X}$ is a mapping defined as $\sigma_n = (\tilde{\rho}_n, \tilde{\eta}_n, \tilde{\gamma}_n) : \tilde{X} \rightarrow \xi$ such that $0 \leq \tilde{\rho}_n(v) + \tilde{\eta}_n(v) + \tilde{\gamma}_n(v) \leq 3$.

We denote the single-valued neutrosophic sets $\langle 0, 1, 1 \rangle$ and $\langle 1, 0, 0 \rangle$ by $\tilde{0}$ and $\tilde{1}$, respectively.

Definition 3 ([11]). Let $\sigma_n = (\tilde{\rho}_n, \tilde{\eta}_n, \tilde{\gamma}_n)$ be an SVNS on $\tilde{X}$. The complement of the set $\sigma_n$ (briefly $\sigma_n^\complement$) is defined as follows:

$$\tilde{\rho}_n^\complement(v) = \tilde{\eta}_n(v), \quad \tilde{\eta}_n^\complement(v) = [\tilde{\rho}_n]^{-}(v), \quad \tilde{\gamma}_n^\complement(v) = \tilde{\rho}_n(v).$$

Definition 4 ([22,23]). Let $\tilde{X}$ be a non-empty set and let $\sigma_n, \gamma_n \in \tilde{X}^\ast$ be given by $\sigma_n = (\tilde{\rho}_n, \tilde{\eta}_n, \tilde{\gamma}_n)$ and $\gamma_n = (\tilde{\rho}_n, \tilde{\eta}_n, \tilde{\gamma}_n)$. Then:

1. We say that $\sigma_n \subseteq \gamma_n$ if $\tilde{\rho}_n \leq \tilde{\rho}_n, \tilde{\eta}_n \geq \tilde{\eta}_n, \tilde{\gamma}_n \geq \tilde{\gamma}_n$.
2. The intersection of $\sigma_n$ and $\gamma_n$ denoted by $\sigma_n \cap \gamma_n$ is an SVNS and is given by

$$\sigma_n \cap \gamma_n = (\tilde{\rho}_n \cap \tilde{\rho}_n, \tilde{\eta}_n \cap \tilde{\eta}_n, \tilde{\gamma}_n \cap \tilde{\gamma}_n).$$

3. The union of $\sigma_n$ and $\gamma_n$ denoted by $\sigma_n \cup \gamma_n$ is an SVNS and is given by

$$\sigma_n \cup \gamma_n = (\tilde{\rho}_n \cup \tilde{\rho}_n, \tilde{\eta}_n \cup \tilde{\eta}_n, \tilde{\gamma}_n \cup \tilde{\gamma}_n).$$
For any arbitrary family \( \{ \sigma_n \}_{i \in \mathbb{I}} \subseteq \xi^\mathbb{X} \) of SVNS, the union and intersection are given by

\[ \bigcap_{i \in \mathbb{I}} [\sigma_n]_i = \{ \cap_{i \in \mathbb{I}} \overline{\sigma}_n \}, \quad \bigcup_{i \in \mathbb{I}} [\sigma_n]_i = \{ \cup_{i \in \mathbb{I}} \overline{\sigma}_n \}, \quad \bigcap_{i \in \mathbb{I}} [\sigma_n]_i = \{ \cap_{i \in \mathbb{I}} \overline{\sigma}_n \}. \]

**Definition 5** ([12]). A single-valued neutrosophic topological space is an ordered quadruple \((\tilde{X}, \tilde{t}, \tilde{c}, \tilde{e})\) where \(\tilde{t} \subseteq \mathbb{R} \times \tilde{t} \times \tilde{c} \times \tilde{e} : \tilde{X} \rightarrow \tilde{X}\) are mappings satisfying the following axioms:

- (SVNT1) \(\tilde{t}(0) = \tilde{t}(1) = 1\) and \(\tilde{t}(0) = \tilde{t}(1) = 0\).
- (SVNT2) \(\tilde{t}(\sigma_n \cap \gamma_n) \geq \tilde{t}(\sigma_n) \cap \tilde{t}(\gamma_n)\), \(\tilde{c}(\sigma_n \cap \gamma_n) \leq \tilde{c}(\sigma_n) \cup \tilde{c}(\gamma_n)\), \(\tilde{e}(\sigma_n \cap \gamma_n) \leq \tilde{e}(\sigma_n) \cap \tilde{e}(\gamma_n)\), for all \(\sigma_n, \gamma_n \in \xi^\mathbb{X}\).
- (SVNT3) \(\tilde{t}(\cup_{\eta \in \Gamma} [\sigma_n]) \geq \cup_{\eta \in \Gamma} \tilde{t}([\sigma_n])\), \(\tilde{e}(\cup_{\eta \in \Gamma} [\sigma_n]) \leq \cup_{\eta \in \Gamma} \tilde{e}([\sigma_n])\) for all \(\{ [\sigma_n], \eta \in \Gamma \} \in \xi^\mathbb{X}\).

The quadruple \((\tilde{X}, \tilde{t}, \tilde{c}, \tilde{e})\) is called a single-valued neutrosophic topological space (SVNTS, for short). We will occasionally write \(\tilde{t}^{\emptyset}_{\tilde{c}^{\emptyset}}\) for \((\tilde{t}, \tilde{c}, \tilde{e})\) and it will cause no ambiguity.

**Definition 6** ([14]). Let \((\tilde{X}, \tilde{t}, \tilde{c}, \tilde{e})\) be an SVNTS. Then, for every \(\sigma_n \in \xi^\mathbb{X}\) and \(r \in \mathbb{C}_{\emptyset}\), the single valued neutrosophic closure and the single valued neutrosophic interior of \(\sigma_n\) are defined by:

\[ \text{C}_{\tilde{t}^{\emptyset}}(\sigma_n, r) = \bigcap \{ \gamma_n \in \xi^\mathbb{X} : \sigma_n \leq \gamma_n, \tilde{c}(\gamma_n) \geq r, \tilde{c}(\gamma_n) \leq 1 - r, \tilde{e}(\gamma_n) \leq 1 - r \}, \]

\[ \text{int}_{\tilde{t}^{\emptyset}}(\sigma_n, r) = \bigcup \{ \gamma_n \in \xi^\mathbb{X} : \sigma_n \geq \gamma_n, \tilde{t}(\gamma_n) \geq r, \tilde{t}(\gamma_n) \leq 1 - r, \tilde{e}(\gamma_n) \leq 1 - r \}. \]

**Definition 7** ([24]). Let \((\tilde{X}, \tilde{t}^{\emptyset})\) be an SVNTS and \(r \in \mathbb{C}_{\emptyset}, \sigma_n \in \xi^\mathbb{X}\). Then,

1. \(\sigma_n\) is a \(r\)-single valued neutrosophic semiopen (r-SVNSO, for short) iff \(\sigma_n \leq \text{C}_{\tilde{t}^{\emptyset}}(\text{int}_{\tilde{t}^{\emptyset}}(\sigma_n, r), r)\).
2. \(\sigma_n\) is a \(r\)-single valued neutrosophic \(\beta\)-open (r-SV\(\beta\)O, for short) iff \(\sigma_n \leq \text{C}_{\tilde{t}^{\emptyset}}(\text{C}_{\tilde{t}^{\emptyset}}(\text{int}_{\tilde{t}^{\emptyset}}(\text{C}_{\tilde{t}^{\emptyset}}(\sigma_n, r), r), r)\).

The complement of \(r - \text{SVNSO}\) (resp. \(r - \text{SV\(\beta\)O}\)) is said to be an \(r - \text{SVNSC}\) (resp. \(r - \text{SV\(\beta\)C}\), respectively).

**Definition 8** ([14]). Let \(\tilde{X}\) be a nonempty set and \(v \in \tilde{X}\). If \(s \in (0, 1), t \in (0, 1)\) and \(p \in [0, 1]\),

\[ x_{s,t,p}(\kappa) = \begin{cases} (s, t, p), & \text{if } x = v, \\ (0, 1, 1), & \text{otherwise.} \end{cases} \]

We say \(x_{s,t,p} \in \sigma_n\) iff \(s < \tilde{c}_{\sigma_n}(v), t \geq \tilde{t}_{\sigma_n}(v)\) and \(p \geq \tilde{e}_{\sigma_n}(v)\). To avoid the ambiguity, we denote the set of all neutrosophic points by \(pt(\xi^\mathbb{X})\).

A single-valued neutrosophic set \(\sigma_n\) is said to be quasi-coincident with another single-valued neutrosophic set \(\gamma_n\), denoted by \(\sigma_n \sqsubset \gamma_n\), if there exists an element \(v \in \tilde{X}\) such that

\[ \tilde{c}_{\sigma_n}(v) + \tilde{c}_{\gamma_n}(v) > 1, \quad \tilde{t}_{\sigma_n}(v) + \tilde{t}_{\gamma_n}(v) \leq 1, \quad \tilde{e}_{\sigma_n}(v) + \tilde{e}_{\gamma_n}(v) \leq 1. \]

**Definition 9** ([14]). A mapping \(\mathcal{I}^{\emptyset} = \mathcal{I}_0, \mathcal{I}_c, \mathcal{I}_e : \xi^\mathbb{X} \rightarrow \xi^\mathbb{X}\) is called single-valued neutrosophic ideal (SVNI) on \(\tilde{X}\) if it satisfies the following conditions:

1. \(\mathcal{I}^{\emptyset}(0) = 1\) and \(\mathcal{I}^{\emptyset}(\tilde{0}) = \tilde{0}\).
2. If \(\sigma_n \leq \gamma_n\), then \(\mathcal{I}^{\emptyset}(\gamma_n) \leq \mathcal{I}^{\emptyset}(\sigma_n)\), \(\mathcal{I}_c(\gamma_n) \geq \mathcal{I}_c(\sigma_n)\), and \(\mathcal{I}_e(\gamma_n) \geq \mathcal{I}_e(\sigma_n)\), for \(\gamma_n, \sigma_n \in \xi^\mathbb{X}\).
3. \(\mathcal{I}^{\emptyset}(\sigma_n \cup \gamma_n) \geq \mathcal{I}^{\emptyset}(\sigma_n) \cap \mathcal{I}^{\emptyset}(\gamma_n), \mathcal{I}_c(\sigma_n \cup \gamma_n) \leq \mathcal{I}_c(\sigma_n) \cup \mathcal{I}_c(\gamma_n)\) and \(\mathcal{I}_e(\sigma_n \cup \gamma_n) \leq \mathcal{I}_e(\sigma_n) \cap \mathcal{I}_e(\gamma_n)\), for each \(\sigma_n, \gamma_n \in \xi^\mathbb{X}\).

The triple \((\tilde{X}, \mathcal{I}^{\emptyset}, \mathcal{I}^{\emptyset})\) is called a single valued neutrosophic ideal topological space in Šostak’s sense (SVNITS, for short).
Definition 10 ([14]). Let \((\tilde{X}, \tau^{\tilde{\mathbb{B}}}, I^{\tilde{\mathbb{B}}})\) be an SVNITS for each \(\sigma_n \in \xi^{\tilde{X}}\). Then, the single valued neutrosophic ideal open local function \([\sigma_n]_{\tilde{\mathbb{B}}}^{1,2}(\tau^{\tilde{\mathbb{B}}}, I^{\tilde{\mathbb{B}}})\) of \(\sigma_n\) is the union of all single-valued neutrosophic points \(x_{1,1,k}\) such that, if \(\gamma_n \in Q_{\tau^{\tilde{\mathbb{B}}}}(x_{1,1,k}, r)\) and \(I^{\tilde{\mathbb{B}}} (\xi_{n}) \leq 1 - r, \ I^{\tilde{\mathbb{B}}} (\xi_{n}) \leq 1 - r, \ \tilde{\sigma}_{n}(v) + \tilde{\eta}_{n}(v) - 1 \leq \tilde{\tau}_{n}(v), \) and \(\tilde{\sigma}_{n}(v) + \tilde{\tau}_{n}(v) - 1 \leq \tilde{\eta}_{n}(v).\) Occasionally, we will write \([\sigma_n]_{\tilde{\mathbb{B}}}^{1,2}\) for \([\sigma_n]_{\tilde{\mathbb{B}}}^{1,2}(\tau^{\tilde{\mathbb{B}}}, I^{\tilde{\mathbb{B}}})\), and it will cause no ambiguity.

Remark 1 ([14]). Let \((\tilde{X}, \tau^{\tilde{\mathbb{B}}}, I^{\tilde{\mathbb{B}}})\) be an SVNITS and \(\sigma_n \in \xi^{\tilde{X}}\). Then,

\[
\tilde{C}^{\tilde{\mathbb{B}}}(\sigma_n, r) = \sigma_n \cup [\sigma_n]^2_{\tilde{\mathbb{B}}}, \quad \tilde{C}^{\tilde{\mathbb{B}}}(\sigma_n, r) = \sigma_n \cap [\sigma_n]^2_{\tilde{\mathbb{B}}}. 
\]

It is clear that \(\tilde{C}^{\tilde{\mathbb{B}}}(\sigma_n)\) is a single-valued neutrosophic closure operator and \((\tau^{\tilde{\mathbb{B}}}(I^{\tilde{\mathbb{B}}}, \tau^{\tilde{\mathbb{B}}}(I^{\tilde{\mathbb{B}}}, \nu^{\tilde{\mathbb{B}}}(I^{\tilde{\mathbb{B}}})})\) is the single-valued neutrosophic topology generated by \(\tilde{C}^{\tilde{\mathbb{B}}}(\sigma_n)\), i.e.,

\[
\tau^{\tilde{\mathbb{B}}}((\sigma_n)) = \bigcup\{r | \tilde{C}^{\tilde{\mathbb{B}}}(\sigma_n, r) = \sigma_n\}. 
\]

Theorem 1 ([14]). Let \(\{([\sigma_n])_{i \in I} \subset \xi^{\tilde{X}}\) be a family of single-valued neutrosophic sets on \(\tilde{X}\) and \((\tilde{X}, \tau^{\tilde{\mathbb{B}}}, I^{\tilde{\mathbb{B}}})\) be an r-SVNITS. Then,

1. \((\bigcup([\sigma_n])_{i \in I})_{j} : i \in j \leq \bigcup\{i \in j : i \in j\}_{j},\)
2. \((\bigcap([\sigma_n])_{i \in I})_{j} \geq \bigcap\{i \in j : i \in j\}_{j}.)\)

Theorem 2 ([14]). Let \((\tilde{X}, \tau^{\tilde{\mathbb{B}}}, I^{\tilde{\mathbb{B}}})\) be an SVNITS and \(\sigma_n, \gamma_n \in \xi^{\tilde{X}}, \ r \in \xi_{0}.\) Then,

1. \(\int_{\tau^{\tilde{\mathbb{B}}}^{\tilde{\mathbb{B}}}}(\sigma_n \land \gamma_n, r) \leq \int_{\tau^{\tilde{\mathbb{B}}}^{\tilde{\mathbb{B}}}}(\sigma_n, r) \lor \int_{\tau^{\tilde{\mathbb{B}}}^{\tilde{\mathbb{B}}}}(\gamma_n, r),\)
2. \(\int_{\tau^{\tilde{\mathbb{B}}}^{\tilde{\mathbb{B}}}}(\sigma_n, r) \leq \int_{\tau^{\tilde{\mathbb{B}}}^{\tilde{\mathbb{B}}}}(\sigma, r) \leq \sigma_n \leq \tau_{\tau^{\tilde{\mathbb{B}}}^{\tilde{\mathbb{B}}}}(\sigma_n, r) \leq C_{\tau^{\tilde{\mathbb{B}}}^{\tilde{\mathbb{B}}}}(\sigma_n, r),\)
3. \(\int_{\tau^{\tilde{\mathbb{B}}}^{\tilde{\mathbb{B}}}}([\sigma_n]^{\mathbb{B}}) = \int_{\tau^{\tilde{\mathbb{B}}}^{\tilde{\mathbb{B}}}}(\sigma_n, r), \) and \(\int_{\tau^{\tilde{\mathbb{B}}}^{\tilde{\mathbb{B}}}}(\sigma_n, r) = \int_{\tau^{\tilde{\mathbb{B}}}^{\tilde{\mathbb{B}}}}(\sigma_n) r),\)
4. \(\int_{\tau^{\tilde{\mathbb{B}}}^{\tilde{\mathbb{B}}}}(\sigma_n \land \gamma_n, r) \leq \int_{\tau^{\tilde{\mathbb{B}}}^{\tilde{\mathbb{B}}}}(\sigma_n, r) \land \int_{\tau^{\tilde{\mathbb{B}}}^{\tilde{\mathbb{B}}}}(\gamma_n, r).\)

3. \(\tilde{\mathbb{L}}\)-Single Valued Neutrosophic Ideal Irresolute Mapping

This section provides the definitions of the \(r\)-single-valued neutrosophic \(\tilde{\mathbb{L}}\)-open set (SVNLO, for short), the \(r\)-single-valued neutrosophic \(\tilde{\mathbb{L}}\)-closed set (SVNL\(\mathbb{C}\), for short) and the \(\tilde{\mathbb{L}}\)-single valued neutrosophic ideal irresolute mapping (\(\tilde{\mathbb{L}}\)-SVNI-irresolute, for short) in the sense of Sostak. To understand the aim of this section, it is essential to clarify its content and elucidate the context in which the definitions, theorems, and examples are performed.

Some results follow.

Definition 11. Let \((\tilde{X}, \tau^{\tilde{\mathbb{B}}}, I^{\tilde{\mathbb{B}}})\) be an r-SVNITS for every \(\sigma_n \in \xi^{\tilde{X}}\) and \(r \in \xi_{0}.\) Then, \(\sigma_n\) is called r-SVNLC iff \(\tilde{C}^{\tilde{\mathbb{B}}}(\sigma_n, r) = \sigma_n.\) The complement of the r-SVNLC is called r-SVNLO.

Proposition 1. Let \((\tilde{X}, \tau^{\tilde{\mathbb{B}}}, I^{\tilde{\mathbb{B}}})\) be an r-SVNITS and \(\sigma_n \in \xi^{\tilde{X}}.\) Then,

1. \(\sigma_n\) is r-SVNLC iff \([\sigma_n]_{\tilde{\mathbb{B}}}^{1,2} \leq \sigma_n,\)
2. \(\sigma_n\) is r-SVNLO iff \([\sigma_n]_{\tilde{\mathbb{B}}}^{1,2} \geq [\sigma_n]_{\tilde{\mathbb{B}}}^{1,2}.\)
3. If \(\tau^{\tilde{\mathbb{B}}}([\sigma_n]_{\tilde{\mathbb{B}}}^{1,2}) \geq \sigma_n \geq \tau^{\tilde{\mathbb{B}}}([\sigma_n]_{\tilde{\mathbb{B}}}^{1,2}) \leq 1 - r,\)
4. If \(\tau^{\tilde{\mathbb{B}}}([\sigma_n]_{\tilde{\mathbb{B}}}^{1,2}) \leq \sigma_n \leq \tau^{\tilde{\mathbb{B}}}([\sigma_n]_{\tilde{\mathbb{B}}}^{1,2}) \leq 1 - r,\)
5. If \(\sigma_n\) is r-SVNLC (resp. r-SVNLC, then \(\int_{\tau^{\tilde{\mathbb{B}}}^{\tilde{\mathbb{B}}}}([\sigma_n]_{\tilde{\mathbb{B}}}^{1,2}) \leq \sigma_n\) (resp. \(\int_{\tau^{\tilde{\mathbb{B}}}^{\tilde{\mathbb{B}}}}([\sigma_n]_{\tilde{\mathbb{B}}}^{1,2}) \leq \sigma_n)\).

Proof. The proof of (1) and (2) are straightforward from Definition 11. (3) Let \(\tau^{\tilde{\mathbb{B}}}([\sigma_n]_{\tilde{\mathbb{B}}}^{1,2}) \geq \sigma_n \geq \tau^{\tilde{\mathbb{B}}}([\sigma_n]_{\tilde{\mathbb{B}}}^{1,2}) \leq 1 - r,\) then,

\[
\sigma_n = C_{\tau^{\tilde{\mathbb{B}}}^{\tilde{\mathbb{B}}}}(\sigma_n, r) \geq \tilde{C}^{\tilde{\mathbb{B}}}(\sigma_n, r) = \sigma_n \cup [\sigma_n]_{\tilde{\mathbb{B}}}^{1,2} \geq [\sigma_n]_{\tilde{\mathbb{B}}}^{1,2}. 
\]
(4) The proof is a direct consequence of (1).
(5) Let $\sigma_n$ be an r-SVNC. Then,
\[
\sigma_n \geq \text{int}_{\tilde{\varphi}^{\tilde{\eta}}} (\mathcal{C}_{\tilde{\varphi}^{\tilde{\eta}}} (\sigma_n, r), r) \geq \text{int}_{\tilde{\varphi}^{\tilde{\eta}}} (\text{Cl}^\xi_{\tilde{\varphi}^{\tilde{\eta}}} (\sigma_n, r), r) = \text{int}_{\tilde{\varphi}^{\tilde{\eta}}} (\left[ \sigma_n \cup [\sigma_n]_r^\xi \right], r) \geq \text{int}_{\tilde{\varphi}^{\tilde{\eta}}} (\left[ \sigma_n \right]_r^\xi, r).
\]

The other case is similarly proved. \[\square\]

**Example 1.** Suppose that $\tilde{X} = \{a, b\}$. Define $\epsilon_n, \gamma_n, \xi_n \in \tilde{X}$ as follows:

$\gamma_n = \langle (0.3, 0.3), (0.3, 0.3), (0.3, 0.3) \rangle$; \hspace{1em} $\epsilon_n = \langle (0.7, 0.7), (0.7, 0.7), (0.7, 0.7) \rangle$;

$\xi_n = \langle (0.2, 0.2), (0.2, 0.2), (0.2, 0.2) \rangle$.

Define $\tilde{\tau}^{\tilde{\varphi}^{\tilde{\eta}}}, \tilde{\tau}^{\tilde{\varphi}^{\tilde{\eta}}}, \tilde{\varphi}^{\tilde{\eta}} : \tilde{X} \rightarrow \tilde{X}$ as follows:

\[
\tilde{\tau}^{\tilde{\varphi}^{\tilde{\eta}}} (\sigma_n) = \begin{cases} 1, & \text{if } \sigma_n = 0, \\ 1, & \text{if } \sigma_n = 1, \\ \frac{1}{3}, & \text{if } \sigma_n = \gamma_n; \\ \frac{1}{3}, & \text{if } \sigma_n = \epsilon_n; \\ 0, & \text{if } \sigma_n = \xi_n. \\ \end{cases}
\]

\[
\tilde{\tau}^{\tilde{\varphi}^{\tilde{\eta}}} (\sigma_n) = \begin{cases} 1, & \text{if } \sigma_n = 0, \\ 1, & \text{if } \sigma_n = 1, \\ \frac{1}{3}, & \text{if } \sigma_n = \gamma_n; \\ \frac{1}{3}, & \text{if } \sigma_n = \epsilon_n; \\ 1, & \text{if } \sigma_n = \xi_n. \\ \end{cases}
\]

\[
\tilde{\tau}^{\tilde{\varphi}^{\tilde{\eta}}} (\sigma_n) = \begin{cases} 0, & \text{if } \sigma_n = 0, \\ 0, & \text{if } \sigma_n = 1, \\ \frac{2}{3}, & \text{if } \sigma_n = \gamma_n; \\ \frac{2}{3}, & \text{if } \sigma_n = \epsilon_n; \\ 1, & \text{if } \sigma_n = \xi_n. \\ \end{cases}
\]

\[
\tilde{\tau}^{\tilde{\varphi}^{\tilde{\eta}}} (\sigma_n) = \begin{cases} 0, & \text{if } \sigma_n = 0, \\ 0, & \text{if } \sigma_n = 1, \\ \frac{2}{3}, & \text{if } \sigma_n = \gamma_n; \\ \frac{2}{3}, & \text{if } \sigma_n = \epsilon_n; \\ 1, & \text{if } \sigma_n = \xi_n. \\ \end{cases}
\]

Let $\tilde{X}, \tilde{\tau}^{\tilde{\varphi}^{\tilde{\eta}}}, \tilde{\tau}^{\tilde{\varphi}^{\tilde{\eta}}}, \tilde{\varphi}^{\tilde{\eta}}$ be an SVNITS. Then, we have the following.

(1) Every intersection of r-SVNC’s is r-SVNC.

(2) Every union of r-SVNEO’s is r-SVNEO.

**Proof.** (1) Let $\{[\sigma_n]_i\}_{i \in j}$ be a family of r-SVNC’s. Then, for every $i \in j$, we obtain $[\sigma_n]_i = \text{Cl}^\xi_{\tilde{\varphi}^{\tilde{\eta}}} ([\sigma_n]_i, r)$, and, by Theorem 1(2), we have

\[
\cap_{i \in [\sigma_n]} = \cap_{i \in j} \text{Cl}^\xi_{\tilde{\varphi}^{\tilde{\eta}}} ([\sigma_n]_i, r) = \cap_{i \in j} ([\sigma_n]_i \cup ([\sigma_n]_i)_r^\xi) \geq \cap_{i \in j} ([\sigma_n]_i \cup ([\sigma_n]_i)_r^\xi)
\]

Therefore, $\cap_{i \in [\sigma_n]}$ is r-SVNC.
(2) From Theorem 1(1). □

Lemma 2. Let \((\tilde{X}, \tilde{\tau}^{\tilde{0}}, \tilde{\tau}^{\tilde{0}})\) be an SVNITS for each \(r \in \xi_0\). Then,

1. For each \(r\)-SVNLE \(\sigma_n \in \tilde{\xi}^X\), \(\sigma_n \tilde{\tau}_n\) if \(\sigma_n \tilde{\tau}(\gamma_n, r)\).
2. \(x_{s,t,k} \tilde{\tau}^{\tilde{0}}(\gamma_n, r)\) if \(\sigma_n \tilde{\tau}_n\) for every \(r\)-SVNLE \(\sigma_n \in \tilde{\xi}^X\) with \(x_{s,t,k} \in \sigma_n\).

Proof. (1) Let \(\sigma_n\) be an \(r\)-SVNLE and \(\sigma_n \tilde{\tau}_n\). Then, for any \(v \in \tilde{X}\), we obtain

\[\tilde{\tilde{\tau}}_n(v) + \tilde{\tilde{\tau}}_n(v) > 1, \quad \tilde{\tilde{\tau}}_n(v) + \tilde{\tilde{\tau}}_n(v) \leq 1, \quad \tilde{\tilde{\tau}}_n(v) + \tilde{\tilde{\tau}}_n(v) \leq 1.\]

This implies that \(\tilde{\tilde{\tau}}_n \leq \tilde{\tilde{\tau}}_n\), \(\tilde{\tilde{\tau}}_n \geq \tilde{\tilde{\tau}}_n\), and \(\tilde{\tilde{\tau}}_n \geq \tilde{\tilde{\tau}}_n\); hence, \(\gamma_n \leq [\sigma_n]\). Since \(\sigma_n\) is \(r\)-SVNLE, \(\tilde{\tau}^{\tilde{0}}(\gamma_n, r) \leq \tilde{\tau}^{\tilde{0}}([\sigma_n], r) = [\sigma_n]\). It follows that \(\sigma_n \tilde{\tau}^{\tilde{0}}(\gamma_n, r)\).

(2) Let \(x_{s,t,k} \tilde{\tau}^{\tilde{0}}(\gamma_n, r)\). Then, \(\sigma_n \tilde{\tau}^{\tilde{0}}(\gamma_n, r)\) with \(x_{s,t,k} \in \sigma_n\). By (1), we have \(\gamma_n \pi_n\) for each \(r\)-SVNLE \(\sigma_n \in \tilde{\xi}^X\). On the other hand, let \(\sigma_n \tilde{\tau}_n\). Then, \(\gamma_n \leq [\sigma_n]\). Since \(\sigma_n\) is \(r\)-SVNLE,

\[\tilde{\tau}^{\tilde{0}}(\gamma_n, r) \leq \tilde{\tau}^{\tilde{0}}([\sigma_n], r) = [\sigma_n] \quad \text{and} \quad \sigma_n \tilde{\tau}^{\tilde{0}}(\gamma_n, r).\]

Since \(x_{s,t,k} \in \sigma_n\), we obtain \(x_{s,t,k} \tilde{\tau}^{\tilde{0}}(\gamma_n, r)\) □

Definition 12. Suppose that \(f : (\tilde{X}, \tilde{\tau}_1^{\tilde{0}}, \tilde{\tau}_2^{\tilde{0}}) \rightarrow (\tilde{Y}, \tilde{\tau}_1^{\tilde{0}}, \tilde{\tau}_2^{\tilde{0}})\) is a mapping. Then,

1. \(f\) is called \(\tilde{\tau}\)-SVNI-irresolute if \(f^{-1}(\sigma_n)\) is \(r\)-SVNLE in \(\tilde{X}\) for any \(r\)-SVNLE \(\sigma_n\) in \(\tilde{Y}\).
2. \(f\) is called \(\tilde{\tau}\)-SVNI-irresolute open if \(f(\sigma_n)\) is \(r\)-SVNLE in \(\tilde{Y}\) for any \(r\)-SVNLE \(\sigma_n\) in \(\tilde{X}\).
3. \(f\) is called \(\tilde{\tau}\)-SVNI-irresolute closed iff \(f(\sigma_n)\) is \(r\)-SVNLE in \(\tilde{Y}\) for any \(r\)-SVNLE \(\sigma_n\) in \(\tilde{X}\).

Theorem 3. Let \(f : (\tilde{X}, \tilde{\tau}_1^{\tilde{0}}, \tilde{\tau}_2^{\tilde{0}}) \rightarrow (\tilde{Y}, \tilde{\tau}_1^{\tilde{0}}, \tilde{\tau}_2^{\tilde{0}})\) be a mapping. Then, the following conditions are equivalent:

1. \(f\) is \(\tilde{\tau}\)-SVNI-irresolute,
2. \(f^{-1}(\sigma_n)\) is \(r\)-SVNLE, for each \(r\)-SVNLE \(\sigma_n\) in \(\tilde{Y}\),
3. \(f(\tilde{\tau}_1^{\tilde{0}}(\sigma_n, r)) \leq \tilde{\tau}_2^{\tilde{0}}(f(\sigma_n), r)\) for each \(\sigma_n \in \tilde{\xi}^X\), \(r \in \xi_0\),
4. \(\tilde{\tau}_2^{\tilde{0}}(f^{-1}(\gamma_n), r) \leq f^{-1}(\tilde{\tau}_2^{\tilde{0}}(\gamma_n, r))\) for each \(\gamma_n \in \tilde{\xi}^Y\), \(r \in \xi_0\).

Proof. (1)⇒(2): Let \(\sigma_n\) be an \(r\)-SVNLE in \(\tilde{Y}\). Then, \([\sigma_n]\) is \(r\)-SVNLE in \(\tilde{Y}\) by (1), we obtain \(f^{-1}([\sigma_n])\) is \(r\)-SVNLE. But, \(f^{-1}([\sigma_n]) = f^{-1}(\sigma_n)\). Then, \(f^{-1}(\sigma_n)\) is \(r\)-SVNLE in \(\tilde{X}\).

(2)⇒(3): For each \(\sigma_n \in \tilde{\xi}^X\) and \(r \in \xi_0\), since \(\tilde{\tau}_2^{\tilde{0}}(\tilde{\tau}_1^{\tilde{0}}(f(\sigma_n), r) = \tilde{\tau}_2^{\tilde{0}}(f(\sigma_n), r)).

From Definition 11, \(\tilde{\tau}_2^{\tilde{0}}(f(\sigma_n), r)\) is \(r\)-SVNLE in \(\tilde{Y}\). By (2), \(f^{-1}(\tilde{\tau}_2^{\tilde{0}}(f(\sigma_n), r))\) is \(r\)-SVNLE in \(\tilde{X}\). Since \(\sigma_n \leq f^{-1}(f(\sigma_n)) \leq f^{-1}(\tilde{\tau}_2^{\tilde{0}}(f(\sigma_n), r)),\)

by Definition 11, we get,

\[\tilde{\tau}_2^{\tilde{0}}(f(\sigma_n), r) \leq f^{-1}(\tilde{\tau}_2^{\tilde{0}}(f(\sigma_n), r)) = f^{-1}(\tilde{\tau}_2^{\tilde{0}}(f(\sigma_n), r)).\]

Hence,

\[f(\tilde{\tau}_2^{\tilde{0}}(f(\sigma_n), r)) = f^{-1}(\tilde{\tau}_2^{\tilde{0}}(f(\sigma_n), r)) \leq \tilde{\tau}_2^{\tilde{0}}(f(\sigma_n), r).\]

(3)⇒(4): For each \(\gamma_n \in \tilde{\xi}^Y\) and \(r \in \xi_0\), put \(\sigma_n = f^{-1}(\gamma_n)\). By (3),

\[f(\tilde{\tau}_2^{\tilde{0}}(f^{-1}(\gamma_n), r)) \leq \tilde{\tau}_2^{\tilde{0}}(f^{-1}(\gamma_n), r) \leq \tilde{\tau}_2^{\tilde{0}}(\gamma_n, r).\]
It implies that $\text{Cl}^L_{\tilde{t}^2_{\tilde{t}^2}}(f^{-1}(\gamma_n), r) \leq f^{-1}(\text{Cl}^L_{\tilde{t}^2_{\tilde{t}^2}}(\gamma_n, r))$.

(4)⇒(1): Let $\gamma_n$ be an r-SVNL set in $\tilde{Y}$. Then, $[\gamma_n]^c$ is an r-SVNL set in $\tilde{Y}$. Hence,

$$f^{-1}([\gamma_n]^c) = f^{-1}(\text{Cl}^L_{\tilde{t}^2_{\tilde{t}^2}}([\gamma_n]^c, r)) \geq \text{Cl}^L_{\tilde{t}^2_{\tilde{t}^2}}(f^{-1}(\gamma_n)^c, r).$$

On the other hand, $f^{-1}([\gamma_n]^c) \leq \text{Cl}^L_{\tilde{t}^2_{\tilde{t}^2}}(f^{-1}([\gamma_n]^c), r)$. Thus, $f^{-1}([\gamma_n]^c) = \text{Cl}^L_{\tilde{t}^2_{\tilde{t}^2}}(f^{-1}([\gamma_n]^c), r)$, that is $f^{-1}([\gamma_n]^c)$ is an r-SVNL set in $\tilde{X}$. Hence, $f^{-1}(\gamma_n)$ is an r-SVNL set in $\tilde{X}$.

Theorem 4. Let $f : (\tilde{X}, \tilde{t}^2_{\tilde{t}^2}, \tilde{I}^2_{\tilde{t}^2}) \rightarrow (\tilde{Y}, \tilde{t}^2_{\tilde{t}^2}, \tilde{I}^2_{\tilde{t}^2})$ be a mapping. Then, the following conditions are equivalent:

1. $f$ is $L$-SVNI-irresolute open,
2. $f(\text{int}^L_{\tilde{t}^2_{\tilde{t}^2}}(\sigma_n, r)) \leq \text{int}^L_{\tilde{t}^2_{\tilde{t}^2}}(f(\sigma_n), r)$ for each $\sigma_n \in \tilde{X}$, $r \in \tilde{c}_0$,
3. $\text{int}^L_{\tilde{t}^2_{\tilde{t}^2}}((f^{-1}(\gamma_n), r) \leq f^{-1}(\text{int}^L_{\tilde{t}^2_{\tilde{t}^2}}(\gamma_n, r))$ for each $\gamma_n \in \tilde{X}$, $r \in \tilde{c}_0$,
4. For any $\gamma_n \in \tilde{X}$ and any r-SVNL set $\sigma_n \in \tilde{X}$ with $f^{-1}(\gamma_n) \leq \sigma_n$, there exists an r-SVNL set $\gamma_n \in \tilde{X}$ with $f^{-1}(\gamma_n) \leq \sigma_n$.

Proof. (1)⇒(2): For every $\sigma_n \in \tilde{X}$, $r \in \tilde{c}_0$ and $\text{int}^L_{\tilde{t}^2_{\tilde{t}^2}}(\sigma_n, r) \leq \sigma_n$ from Theorem 2(2), we have $f(\text{int}^L_{\tilde{t}^2_{\tilde{t}^2}}(\sigma_n, r)) \leq f(\sigma_n)$. By (1), $f(\text{int}^L_{\tilde{t}^2_{\tilde{t}^2}}(\sigma_n, r))$ is r-SVNL in $\tilde{Y}$. Hence,

$$f(\text{int}^L_{\tilde{t}^2_{\tilde{t}^2}}(\sigma_n, r)) = \text{int}^L_{\tilde{t}^2_{\tilde{t}^2}}(f(\text{int}^L_{\tilde{t}^2_{\tilde{t}^2}}(\sigma_n, r))) \leq \text{int}^L_{\tilde{t}^2_{\tilde{t}^2}}(f(\sigma_n), r).$$

(2)⇒(3): For each $\gamma_n \in \tilde{X}$ and $r \in \tilde{c}_0$, put $\sigma_n = f^{-1}(\gamma_n)$ from (2),

$$f(\text{int}^L_{\tilde{t}^2_{\tilde{t}^2}}((f^{-1}(\gamma_n), r) \leq \text{int}^L_{\tilde{t}^2_{\tilde{t}^2}}(f(f^{-1}(\gamma_n), r) \leq \text{int}^L_{\tilde{t}^2_{\tilde{t}^2}}(\gamma_n, r).$$

It implies that

$$\text{int}^L_{\tilde{t}^2_{\tilde{t}^2}}((f^{-1}(\gamma_n), r) \leq f^{-1}(\text{int}^L_{\tilde{t}^2_{\tilde{t}^2}}(f^{-1}(\gamma_n), r))) \leq f^{-1}(\text{int}^L_{\tilde{t}^2_{\tilde{t}^2}}(\gamma_n, r)).$$

(3)⇒(4): Obvious.

(4)⇒(1): Let $\epsilon_n$ be an r-SVNL set in $\tilde{X}$. Put $\gamma_n = [f(\epsilon_n)]^c$ and $\sigma_n = [\epsilon_n]^c$ such that $\sigma_n$ is r-SVNL set in $\tilde{X}$. We obtain

$$f^{-1}(\epsilon_n) = f^{-1}([f(\epsilon_n)]^c) = [f^{-1}(f(\epsilon_n))]^c \leq [\epsilon_n]^c = \sigma_n.$$

From (4), there exists r-SVNL set $\gamma_n \in \tilde{X}$ with $\gamma_n \leq \sigma_n$ such that $f^{-1}(\gamma_n) \leq \sigma_n = [\epsilon_n]^c$.

It implies $\epsilon_n \leq [f^{-1}(\gamma_n)]^c = f^{-1}([\gamma_n]^c)$. Thus, $f(\epsilon_n) \leq f(f^{-1}([\gamma_n]^c)) \leq [\gamma_n]^c$. On the other hand, since $\gamma_n \leq \epsilon_n$, we have

$$f(\epsilon_n) = [\gamma_n]^c \geq [\epsilon_n]^c.$$

Hence, $f(\epsilon_n) = [\epsilon_n]^c$, that is, $f(\epsilon_n)$ is r-SVNL set in $\tilde{Y}$.

Theorem 5. Let $f : (\tilde{X}, \tilde{t}^2_{\tilde{t}^2}, \tilde{I}^2_{\tilde{t}^2}) \rightarrow (\tilde{Y}, \tilde{t}^2_{\tilde{t}^2}, \tilde{I}^2_{\tilde{t}^2})$ be a mapping. Then, the following conditions are equivalent:

1. $f$ is $L$-SVNI-irresolute closed,
2. $f(\text{Cl}^L_{\tilde{t}^2_{\tilde{t}^2}}(\gamma_n, r)) \leq \text{Cl}^L_{\tilde{t}^2_{\tilde{t}^2}}(f(\gamma_n), r)$ for each $\gamma_n \in \tilde{X}$, $r \in \tilde{c}_0$. 

Theorem 6. Let $f : (X, z^{0\theta\emptyset}, T^{0\theta\emptyset}) \rightarrow (Y, z^{2\theta\emptyset}, T^{2\theta\emptyset})$ be a bijective mapping. Then, the following conditions are equivalent:

1. $f$ is $\mathcal{E}$-SVNI-irresolute closed,
2. $\text{Cl}_{\mathcal{E}}(f^{-1}(\sigma_n), r) \leq f^{-1}(\text{Cl}_{\mathcal{E}}(\sigma_n, r))$ for each $\sigma_n \in \tilde{X}$, $r \in \tilde{\emptyset}$.

Proof. (1) $\Rightarrow$ (2) : Suppose that $f$ is an $\mathcal{E}$-SVNI-irresolute closed. From Theorem 5(2), we claim that, for each $\gamma_n \in \tilde{X}$ and $r \in \tilde{\emptyset}$,

$$f(\text{Cl}_{\mathcal{E}}(\gamma_n, r)) \leq \text{Cl}_{\mathcal{E}}(f(\gamma_n), r).$$

Now, for all $\sigma_n \in \tilde{X}$, $r \in \tilde{\emptyset}$, put $\gamma_n = f^{-1}(\sigma_n)$, since $f$ is onto, it implies that $f(f^{-1}(\sigma_n)) = \sigma_n$. Thus,

$$f(\text{Cl}_{\mathcal{E}}(f^{-1}(\sigma_n), r)) \leq \text{Cl}_{\mathcal{E}}(f(f^{-1}(\sigma_n), r)) = \text{Cl}_{\mathcal{E}}(\sigma_n, r).$$

Again, since $f$ is onto, it follows:

$$\text{Cl}_{\mathcal{E}}(f^{-1}(\sigma_n), r) = f^{-1}(f(\text{Cl}_{\mathcal{E}}(f^{-1}(\sigma_n), r))) \leq f^{-1}(\text{Cl}_{\mathcal{E}}(\sigma_n, r)).$$

(2) $\Rightarrow$ (1) : Put $\sigma_n = f(\gamma_n)$. By the injection of $f$, we get

$$\text{Cl}_{\mathcal{E}}(\gamma_n, r) = \text{Cl}_{\mathcal{E}}(f^{-1}(f(\gamma_n), r)) \leq f^{-1}(\text{Cl}_{\mathcal{E}}(f(\gamma_n), r)),$$

for the reason that $f$ is onto, which implies that

$$f(\text{Cl}_{\mathcal{E}}(\gamma_n, r)) \leq f(f^{-1}(\text{Cl}_{\mathcal{E}}(f(\gamma_n), r))) = \text{Cl}_{\mathcal{E}}(f(\gamma_n), r).$$

4. $\mathcal{E}$-Single Valued Neutrosophic Extremally Disconnected and $\mathcal{E}$-Single Valued Neutrosophic Normal

This section is devoted to introducing $\mathcal{E}$-single valued neutrosophic extremely disconnected ($\mathcal{E}$-SVNE-disconnected, for short) and $\mathcal{E}$-single valued neutrosophic normal ($\mathcal{E}$-SVN-normal, for short), in the sense of Šostak. These definitions and their components, together with a set of criteria for identifying the spaces, are provided to illustrate how the ideas are applied.

Definition 13. An SVNITS $(\tilde{X}, \tilde{r}^{\theta\emptyset}, \tilde{r}^{\theta\emptyset})$ is called $\mathcal{E}$-SVNE-disconnected if $\tilde{r}^{\theta}(\text{Cl}_{\mathcal{E}}(\sigma_n, r)) \geq r$, $\tilde{r}^{\theta}(\text{Cl}_{\mathcal{E}}(\sigma_n, r)) \leq 1 - r$, $\tilde{r}^{\theta}(\text{Cl}_{\mathcal{E}}(\sigma_n, r)) \leq 1 - r$ for each $\tilde{r}^{\theta}(\sigma_n) \geq r$, $\tilde{r}^{\theta}(\sigma_n) \leq 1 - r$, $\tilde{r}^{\theta}(\sigma_n) \leq 1 - r$.

Definition 14. Let $(\tilde{X}, \tilde{r}^{\theta\emptyset}, \tilde{r}^{\theta\emptyset})$ be an SVNITS and $r \in \tilde{\emptyset}$. Then, $\sigma_n \in \tilde{X}$ is said to be:

1. $r$-single valued neutrosophic semi-ideal open set ($r$-SVNSIO) iff $\sigma_n \leq \text{Cl}_{\mathcal{E}}(\text{int}_{\mathcal{E}}(\sigma_n, r), r)$,
2. $r$-single valued neutrosophic pre-ideal open set ($r$-SVNPIO) iff $\sigma_n \leq \text{int}_{\mathcal{E}}(\text{Cl}_{\mathcal{E}}(\sigma_n, r), r)$,
3. $r$-single valued neutrosophic $\alpha$-ideal open set ($r$-SVN$\alpha$IO) iff $\sigma_n \leq \text{int}_{\mathcal{E}}(\text{Cl}_{\mathcal{E}}(\sigma_n, r), r)$,
4. $r$-single valued neutrosophic $\beta$-ideal open set ($r$-SVN$\beta$IO) iff $\sigma_n \leq C_{\mathcal{E}}(\text{int}_{\mathcal{E}}(\text{Cl}_{\mathcal{E}}(\sigma_n, r), r), r)$,
5. $r$-single valued neutrosophic $\beta$-ideal open set ($r$-SVN$\beta$IO) iff $\sigma_n \leq \text{Cl}_{\mathcal{E}}(\text{int}_{\mathcal{E}}(\text{Cl}_{\mathcal{E}}(\sigma_n, r), r), r)$. 

(6) $r$-single valued neutrosophic regular ideal open set ($r$-SVNRIO) iff $\sigma_n = \text{int}_{\tilde{\Phi}}(\text{Cl}_{\tilde{T}}(\tilde{\sigma}_n, r), r)$.

The complement of $r$-SVNSIO (resp. $r$-SVNPIO, $r$-SVNαIO, $r$-SVNβIO, $r$-SVNSβIO, $r$-SVNRIO) are called $r$-SVNSIC (resp. $r$-SVNPIIC, $r$-SVNαIC, $r$-SVNβIC, $r$-SVNSβIC, $r$-SVNRIC).

**Remark 2.** The following diagram can be easily obtained from the above definition:

$$
\begin{array}{ccc}
r - \text{SVNaIO} & \Rightarrow & r - \text{SVNSIO} & \Rightarrow & r - \text{SVNSO} \\
\downarrow & & \downarrow & & \downarrow \\
r - \text{SVNRIO} & \Rightarrow & r - \text{SVNPIO} & \Rightarrow & r - \text{SVNβIO} \\
\downarrow & & \downarrow & & \downarrow \\
r - \text{SVNSIO} & \Rightarrow & r - \text{SVNSβIO} & \Rightarrow & r - \text{SVNβIO}.
\end{array}
$$

**Theorem 7.** Let $(\tilde{X}, \tilde{\tau}^\Phi, \tilde{T}^\Phi)$ be an SVNITS and $r \in \tilde{\xi}_0$. Then, the following properties are equivalent:

1. $(\tilde{X}, \tilde{\tau}^\Phi, \tilde{T}^\Phi)$ is $L$-SVNE-disconnected,
2. $\tilde{\tau}^\Phi([\text{int}_{\tilde{\Phi}}(\sigma_n, r)]^c) \geq r$, $\tilde{\tau}^\Phi([\text{int}_{\tilde{\Phi}}(\sigma_n, r)]^c) \leq 1 - r$, $\tilde{\tau}^\Phi([\text{int}_{\tilde{\Phi}}(\sigma_n, r)]^c) \leq 1 - r$ for each $\tilde{\tau}^\Phi([\sigma_n]^{\tilde{\tau}^\Phi}) \geq r$, $\tilde{\tau}^\Phi([\sigma_n]^{\tilde{\tau}^\Phi}) \leq 1 - r$, $\tilde{\tau}^\Phi([\sigma_n]^{\tilde{\tau}^\Phi}) \leq 1 - r$,
3. $\text{Cl}_{\tilde{T}^\Phi}^\Phi([\text{int}_{\tilde{\Phi}}(\sigma_n, r)]^c, r) \leq \text{int}_{\tilde{\Phi}}^\Phi(\text{Cl}_{\tilde{T}^\Phi}^\Phi(\sigma_n, r), r)$, for each $\sigma_n \in \tilde{\xi}_X$,
4. Every $r$-SVNSIO set is $r$-SVNPIO,
5. $\tilde{\tau}^\Phi(\text{Cl}_{\tilde{T}^\Phi}^\Phi(\sigma_n, r)) \geq r$, $\tilde{\tau}^\Phi(\text{Cl}_{\tilde{T}^\Phi}^\Phi(\sigma_n, r)) \leq 1 - r$, $\tilde{\tau}^\Phi(\text{Cl}_{\tilde{T}^\Phi}^\Phi(\sigma_n, r)) \leq 1 - r$ for each $r$-SVNSβIO $\sigma_n \in \tilde{\xi}_X$,
6. Every $r$-SVNSβIO set is $r$-SVNPIO,
7. For each $\sigma_n \in \tilde{\xi}_X$, $\sigma_n$ is $r$-SVNaIO set iff it is $r$-SVNSIO.

**Proof.** (1) $\Rightarrow$ (2): The proof is a direct consequence of Definition 14.

(2)$\Rightarrow$(3): For each $\sigma_n \in \tilde{\xi}_X$, $\tilde{\tau}^\Phi([\text{int}_{\tilde{\Phi}}(\sigma_n, r)]^c) \geq r$, $\tilde{\tau}^\Phi([\text{int}_{\tilde{\Phi}}(\sigma_n, r)]^c) \leq 1 - r$, and, by (2), we have

$$
\tilde{\tau}^\Phi([\text{int}_{\tilde{\Phi}}(\sigma_n, r)]^c) \geq r, \quad \tilde{\tau}^\Phi([\text{int}_{\tilde{\Phi}}(\sigma_n, r)]^c) \leq 1 - r,
$$

$$
\tilde{\tau}^\Phi([\text{int}_{\tilde{\Phi}}(\sigma_n, r)]^c) \leq 1 - r.
$$

Thus,

$$
\tilde{\tau}^\Phi(\text{Cl}_{\tilde{T}^\Phi}^\Phi(\text{int}_{\tilde{T}^\Phi}(\sigma_n, r), r)) \geq r, \quad \tilde{\tau}^\Phi(\text{Cl}_{\tilde{T}^\Phi}^\Phi(\text{int}_{\tilde{T}^\Phi}(\sigma_n, r), r)) \leq 1 - r, \quad \tilde{\tau}^\Phi(\text{Cl}_{\tilde{T}^\Phi}^\Phi(\text{int}_{\tilde{T}^\Phi}(\sigma_n, r), r)) \leq 1 - r;
$$

hence,

$$
\text{Cl}_{\tilde{T}^\Phi}^\Phi([\text{int}_{\tilde{T}^\Phi}(\sigma_n, r), r]) = \text{int}_{\tilde{T}^\Phi}^\Phi(\text{Cl}_{\tilde{T}^\Phi}^\Phi(\text{int}_{\tilde{T}^\Phi}(\sigma_n, r), r), r) \leq \text{int}_{\tilde{T}^\Phi}^\Phi(\text{Cl}_{\tilde{T}^\Phi}^\Phi(\sigma_n, r), r).
$$

(3)$\Rightarrow$(4): Let $\sigma_n$ be an $r$-SVNSIO set. Then, by (4), we have

$$
\sigma_n \leq \text{Cl}_{\tilde{T}^\Phi}^\Phi(\text{int}_{\tilde{T}^\Phi}(\sigma_n, r), r) \leq \text{int}_{\tilde{T}^\Phi}^\Phi(\text{Cl}_{\tilde{T}^\Phi}^\Phi(\sigma_n, r), r).
$$

Thus, $\sigma_n$ is an $r$-SVNPIO set.

(4)$\Rightarrow$(5): Since $\sigma_n$ is an $r$-SVNSβIO set, $\sigma_n \leq \text{Cl}_{\tilde{T}^\Phi}^\Phi(\text{int}_{\tilde{T}^\Phi}^\Phi(\text{Cl}_{\tilde{T}^\Phi}^\Phi(\sigma_n, r), r), r)$. Then, $\text{Cl}_{\tilde{T}^\Phi}^\Phi(\sigma_n, r)$ is $r$-SVNSIO, and, by (4), $\text{Cl}_{\tilde{T}^\Phi}^\Phi(\sigma_n, r) \leq \text{int}_{\tilde{T}^\Phi}^\Phi(\text{Cl}_{\tilde{T}^\Phi}^\Phi(\sigma_n, r), r)$; hence,

$$
\tilde{\tau}^\Phi(\text{Cl}_{\tilde{T}^\Phi}^\Phi(\sigma_n, r)) \geq r, \quad \tilde{\tau}^\Phi(\text{Cl}_{\tilde{T}^\Phi}^\Phi(\sigma_n, r)) \leq 1 - r, \quad \tilde{\tau}^\Phi(\text{Cl}_{\tilde{T}^\Phi}^\Phi(\sigma_n, r)) \leq 1 - r.
$$

(5)$\Rightarrow$(6): Let $\sigma_n$ be an $r$-SVNβIO set, then, by (5), $\text{Cl}_{\tilde{T}^\Phi}^\Phi(\sigma_n, r) \leq \text{int}_{\tilde{T}^\Phi}^\Phi(\text{Cl}_{\tilde{T}^\Phi}^\Phi(\sigma_n, r), r)$.

Thus,

$$
\sigma_n \leq \text{Cl}_{\tilde{T}^\Phi}^\Phi(\sigma_n, r) \leq \text{int}_{\tilde{T}^\Phi}^\Phi(\text{Cl}_{\tilde{T}^\Phi}^\Phi(\sigma_n, r), r).
$$
Therefore, $\sigma_n$ is an r-SVNPIO set.

(6) $\Rightarrow$ (7): Let $\sigma_n$ be an r-SVNSIO. Then, $\sigma_n$ is r-SVNSβIO, by (6), $\sigma_n$ is an r-SVNPIO set. Since $\sigma_n$ is r-SVNSIO and r-SVNPIO, $\sigma_n$ is r-SVNaIO.

(7) $\Rightarrow$ (1): Suppose that $\tau^\beta(\sigma_n) \geq r$, $\tau^\delta(\sigma_n) \leq 1 - r$, $\tau^\theta(\sigma_n) \leq 1 - r$ and then $\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r)$ is r-SVNSIO, and, by (7), $\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r)$ is r-SVNaIO. Hence,

$$\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r) \leq \text{int}_{\tau^\rho_{\eta\rho}}(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r), r, r, r, r) = \text{int}_{\tau^\rho_{\eta\rho}}(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r), r) \leq \text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r).$$

Hence,

$$\tau^\beta(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r)) \geq r, \quad \tau^\delta(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r)) \leq 1 - r, \quad \tau^\theta(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r)) \leq 1 - r.$$

Thus, $(\tilde{X}, \tilde{\tau}^{\rho\delta\theta}, \tilde{\tau}^{\rho\delta\theta})$ is L-SVNE-disconnected.

**Theorem 8.** Let $(\tilde{X}, \tilde{\tau}^{\rho\delta\theta}, \tilde{\tau}^{\rho\delta\theta})$ be an SVNITS $r \in \xi_0$ and $\sigma_n \in \tilde{\xi}_X$. Then, the following are equivalent:

1. $(\tilde{X}, \tilde{\tau}^{\rho\delta\theta}, \tilde{\tau}^{\rho\delta\theta})$ is L-SVNE-disconnected,
2. $\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r) \neg \text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\gamma_n, r)$, for every $\tau^\rho(\sigma_n) \geq r$, $\tau^\delta(\sigma_n) \leq 1 - r$ and $\tau^\theta(\sigma_n) \leq 1 - r$ and every r-SVNLIO $\gamma_n \in \tilde{\xi}_X$ with $\sigma_n \neg \gamma_n$,
3. $\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\text{int}_{\tau^\rho_{\eta\rho}}(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r), r, r) \neg \text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\gamma_n, r)$, for every $\sigma_n \in \tilde{\xi}_X$ and every r-SVNLIO $\gamma_n \in \tilde{\xi}_X$ with $\sigma_n \neg \gamma_n$.

**Proof.** (1) $\Rightarrow$ (2): Let $\tau^\rho(\sigma_n) \geq r$, $\tau^\delta(\sigma_n) \leq 1 - r$, $\tau^\theta(\sigma_n) \leq 1 - r$. Then, by (1),

$$\tau^\rho(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r)) \geq r, \quad \tau^\delta(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r)) \leq 1 - r, \quad \tau^\theta(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r)) \leq 1 - r.$$

Since $[\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r)]^c$ is an r-SVNLIO and $\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r) \neg \text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r)^c$, it implies that

$$\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r) \neg \text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r)^c, r).$$

(2) $\Rightarrow$ (1): Let $\tau^\rho(\sigma_n) \geq r$, $\tau^\delta(\sigma_n) \leq 1 - r$, $\tau^\theta(\sigma_n) \leq 1 - r$. Since $[\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r)]^c$ is an r-SVNLIO, then, by (2),

$$\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r) \neg \text{CI}^\xi_{\tau^\rho_{\eta\rho}}([\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r)]^c, r).$$

This implies that $\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r) \leq \text{int}_{\tau^\rho_{\eta\rho}}(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r), r) \leq \text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r)$, so

$$\tau^\rho(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r)) \geq r, \quad \tau^\delta(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r)) \leq 1 - r, \quad \tau^\theta(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r)) \leq 1 - r.$$

(2) $\Rightarrow$ (3): Suppose that $\sigma_n \in \tilde{\xi}_X$ and $\gamma_n$ is an r-SVNLIO with $\sigma_n \neg \gamma_n$. Since $\tau^\rho(\text{int}_{\tau^\rho_{\eta\rho}}(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r), r, r) \leq 1 - r, \quad \tau^\delta(\text{int}_{\tau^\rho_{\eta\rho}}(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r), r)) \leq 1 - r, \quad \tau^\theta(\text{int}_{\tau^\rho_{\eta\rho}}(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r), r)) \leq 1 - r$.

By (2), we have $\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\text{int}_{\tau^\rho_{\eta\rho}}(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r), r, r) \neg \text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\gamma_n, r))$.

(3) $\Rightarrow$ (2): Let $\tau^\rho(\sigma_n) \geq r$, $\tau^\delta(\sigma_n) \leq 1 - r$, $\tau^\delta(\sigma_n) \leq 1 - r$ and $\gamma_n$ be an r-SVNLIO with $\sigma_n \neg \gamma_n$. Then, by (3), we obtain $\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\text{int}_{\tau^\rho_{\eta\rho}}(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r), r, r) \neg \text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\gamma_n, r))$. Since

$$\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r) \leq \text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\text{int}_{\tau^\rho_{\eta\rho}}(\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r), r, r), r) \neg \text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\gamma_n, r)),$$

then, we have $\text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\sigma_n, r) \neg \text{CI}^\xi_{\tau^\rho_{\eta\rho}}(\gamma_n, r))$.  □

**Definition 15.** An SVNITS $(\tilde{X}, \tilde{\tau}^{\rho\delta\theta}, \tilde{\tau}^{\rho\delta\theta})$ is called L-SVNE-normal if, for every $[\sigma_n] \neg [\sigma_n]_2$ with $\tau^\rho([\sigma_n]_1) \geq r$, $\tau^\delta([\sigma_n]_1) \leq 1 - r$, $\tau^\theta([\sigma_n]_1) \leq 1 - r$ and $[\sigma_n]_2$ is r-SVNLIO, there exists
\[\gamma_n \in \xi X, \text{ for } j = \{1, 2\} \text{ with } \tau \hat{\delta}(\gamma_n)[j] \geq r, \tau \hat{\delta}(\gamma_n) \leq 1 - r, \tau \hat{\delta}(\gamma_n) \leq 1 - r, \gamma_n[2] \text{ is r-SVENC such that } \gamma_n[2] \leq \gamma_n[1], \gamma_n[1] \leq \gamma_n[2] \text{ and } \gamma_n[1] \equiv \gamma_n[2].\]

**Theorem 9.** Let \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}, \tilde{\theta}}, \tilde{S}^{\tilde{\rho}, \tilde{\theta}})\) be an SVNITS; then, the following are equivalent:

1. \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}, \tilde{\theta}}, \tilde{S}^{\tilde{\rho}, \tilde{\theta}})\) is an L-SVN-normal.
2. \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}, \tilde{\theta}}, \tilde{S}^{\tilde{\rho}, \tilde{\theta}})\) is an L-SVN-disconnected.

**Proof.** (1)⇒(2): Let \(\tau \hat{\delta}(\sigma_n) \geq r, \tau \hat{\delta}(\sigma_n) \leq 1 - r, \tau \hat{\delta}(\sigma_n) \leq 1 - r \text{ and } \text{Cl}^{\tilde{\tau}_n}(\sigma_n, r)\) be an r-SVN\(\slash\)LO. Then, \(\sigma_n \in \text{Cl}^{\tilde{\tau}_n}(\sigma_n, r)\). By the L-SVN-normality of \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}, \tilde{\theta}}, \tilde{S}^{\tilde{\rho}, \tilde{\theta}})\), there exist \(\gamma_n \in \xi X, \text{ for } i = \{1, 2\} \) with

\[\tau \hat{\delta}(\gamma_n)[i] \geq r, \tau \hat{\delta}(\gamma_n)[i] \leq 1 - r, \tau \hat{\delta}(\gamma_n)[i] \leq 1 - r, \text{ and } \gamma_n[2] \text{ r-SVENC such that } \text{Cl}^{\tilde{\tau}_n}(\sigma_n, r) \equiv \gamma_n[2]. \]

Since \(\text{Cl}^{\tilde{\tau}_n}(\sigma_n, r) \equiv \gamma_n[2] \text{ and } \gamma_n[1] \leq \gamma_n[2] \text{ and } \gamma_n[1] \equiv \gamma_n[2]. \)

Thus, \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}, \tilde{\theta}}, \tilde{S}^{\tilde{\rho}, \tilde{\theta}})\) is an L-SVN-disconnected.

(2)⇒(1): Suppose that \(\tau \hat{\delta}(\sigma_n) \geq r, \tau \hat{\delta}(\sigma_n) \leq 1 - r, \tau \hat{\delta}(\sigma_n) \leq 1 - r \text{ and } \gamma_n \text{ is an r-SVNLO with } \sigma_n \equiv \gamma_n. \)

By the L-SVN-disconnected of \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}, \tilde{\theta}}, \tilde{S}^{\tilde{\rho}, \tilde{\theta}})\), we have

\[\tau \hat{\delta}(\text{Cl}^{\tilde{\tau}_n}(\sigma_n, r)) \geq r, \tau \hat{\delta}(\text{Cl}^{\tilde{\tau}_n}(\sigma_n, r)) \leq 1 - r, \tau \hat{\delta}(\text{Cl}^{\tilde{\tau}_n}(\sigma_n, r)) \leq 1 - r, \text{ and } \gamma_n \in \text{Cl}^{\tilde{\tau}_n}(\sigma_n, r) \equiv \gamma_n. \]

Thus, \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}, \tilde{\theta}}, \tilde{S}^{\tilde{\rho}, \tilde{\theta}})\) is an L-SVN-normal. \(\square\)

**Theorem 10.** Let \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}, \tilde{\theta}}, \tilde{S}^{\tilde{\rho}, \tilde{\theta}})\) be an SVNITS, \(\sigma_n \in \xi X \text{ and } r \in \xi 0. \)

Then, the following properties are equivalent:

1. \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}, \tilde{\theta}}, \tilde{S}^{\tilde{\rho}, \tilde{\theta}})\) is an L-SVN-disconnected.
2. If \(\sigma_n \text{ is r-SVNRIIO, then } \sigma_n \text{ is r-SVENC.}\)
3. If \(\sigma_n \text{ is r-SVNRIC, then } \sigma_n \text{ is r-SVENLO.}\)

**Proof.** (1)⇒(2): Let \(\sigma_n \text{ be an r-SVNRIIO. Then, } \sigma_n = \text{int}^{\tilde{\tau}_n}(\text{Cl}^{\tilde{\tau}_n}(\sigma_n, r), r) \text{ and } \tau \hat{\delta}(\sigma_n) \geq r, \tau \hat{\delta}(\sigma_n) \leq 1 - r, \tau \hat{\delta}(\sigma_n) \leq 1 - r. \)

By (1),

\[\tau \hat{\delta}(\text{Cl}^{\tilde{\tau}_n}(\sigma_n, r)) \geq r, \tau \hat{\delta}(\text{Cl}^{\tilde{\tau}_n}(\sigma_n, r)) \leq 1 - r, \tau \hat{\delta}(\text{Cl}^{\tilde{\tau}_n}(\sigma_n, r)) \leq 1 - r. \]

Hence \(\sigma_n = \text{int}^{\tilde{\tau}_n}(\text{Cl}^{\tilde{\tau}_n}(\sigma_n, r), r) = \text{Cl}^{\tilde{\tau}_n}(\sigma_n, r). \)

(2)⇒(1): Suppose that \(\sigma_n = \text{int}^{\tilde{\tau}_n}(\text{Cl}^{\tilde{\tau}_n}(\sigma_n, r), r), \text{ then } \tau \hat{\delta}(\sigma_n) \geq r, \tau \hat{\delta}(\sigma_n) \leq 1 - r, \tau \hat{\delta}(\sigma_n) \leq 1 - r. \)

By (2), \(\sigma_n \text{ is r-SVENC. This implies that } \text{Cl}^{\tilde{\tau}_n}(\sigma_n, r) \leq \text{Cl}^{\tilde{\tau}_n}(\text{int}^{\tilde{\tau}_n}(\text{Cl}^{\tilde{\tau}_n}(\sigma_n, r), r), r) = \text{int}^{\tilde{\tau}_n}(\text{Cl}^{\tilde{\tau}_n}(\sigma_n, r), r) \leq \text{Cl}^{\tilde{\tau}_n}(\sigma_n, r). \)

Thus,

\[\tau \hat{\delta}(\text{Cl}^{\tilde{\tau}_n}(\sigma_n, r)) \geq r, \tau \hat{\delta}(\text{Cl}^{\tilde{\tau}_n}(\sigma_n, r)) \leq 1 - r, \tau \hat{\delta}(\text{Cl}^{\tilde{\tau}_n}(\sigma_n, r)) \leq 1 - r, \]

then \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}, \tilde{\theta}}, \tilde{S}^{\tilde{\rho}, \tilde{\theta}})\) is an L-SVN-disconnected.

(2)⇔(3): Obvious. \(\square\)
Remark 3. The union of two r-SVNRIO sets need not to be an r-SVNRIO.

Theorem 11. If \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}00}, \tilde{\eta}^{\tilde{\rho}00})\) is \(\mathcal{L}\)-SVNE-disconnected and \(\sigma_n, \gamma_n \in \tilde{\xi}_r^{\tilde{\eta}}, r \in \tilde{\xi}_0\). Then, the following properties hold:

1. If \(\sigma_n\) and \(\gamma_n\) are r-SVNRIC, then \(\sigma_n \wedge \gamma_n\) is r-SVNRIC.
2. If \(\sigma_n\) and \(\gamma_n\) are r-SVNRIO, then \(\sigma_n \vee \gamma_n\) is r-SVNRIO.

Proof. Let \(\sigma_n\) and \(\gamma_n\) be r-SVNRIC. Then, \(\tilde{\tau}^\circ([\sigma_n]) \geq r, \tilde{\tau}^\circ([\gamma_n]) \geq 1 - r, \tilde{\tau}^\circ([\sigma_n]) \leq 1 - r \) and \(\tilde{\tau}^\circ([\gamma_n]) \leq 1 - r\), by Theorem 7, we have

\[
\tilde{\tau}^\circ([\text{int}_{\tilde{\tau}}^\circ(\sigma_n, r)]) \geq r, \quad \tilde{\tau}^\circ([\text{int}_{\tilde{\tau}}^\circ(\sigma_n, r)]) \leq 1 - r, \quad \tilde{\tau}^\circ([\text{int}_{\tilde{\tau}}^\circ(\gamma_n, r)]) \leq 1 - r.
\]

And

\[
\tilde{\tau}^\circ([\text{int}_{\tilde{\tau}}^\circ(\gamma_n, r)]) \geq r, \quad \tilde{\tau}^\circ([\text{int}_{\tilde{\tau}}^\circ(\gamma_n, r)]) \leq 1 - r, \quad \tilde{\tau}^\circ([\text{int}_{\tilde{\tau}}^\circ(\gamma_n, r)]) \leq 1 - r.
\]

This implies that

\[
\sigma_n \wedge \gamma_n = C_{\tilde{\tau}^{\tilde{\rho}00}} \left(\text{int}_{\tilde{\tau}^{\tilde{\rho}00}}(\sigma_n, r), r\right) \wedge C_{\tilde{\tau}^{\tilde{\rho}00}} \left(\text{int}_{\tilde{\tau}^{\tilde{\rho}00}}(\gamma_n, r), r\right)
\]

\[
= \text{int}_{\tilde{\tau}^{\tilde{\rho}00}}(\sigma_n, r) \wedge \text{int}_{\tilde{\tau}^{\tilde{\rho}00}}(\gamma_n, r)
\]

\[
\leq C_{\tilde{\tau}^{\tilde{\rho}00}} \left(\text{int}_{\tilde{\tau}^{\tilde{\rho}00}}(\sigma_n \wedge \gamma_n, r), r\right).
\]

On the other hand,

\[
C_{\tilde{\tau}^{\tilde{\rho}00}} \left(\text{int}_{\tilde{\tau}^{\tilde{\rho}00}}(\sigma_n \wedge \gamma_n, r), r\right) = C_{\tilde{\tau}^{\tilde{\rho}00}} \left(\text{int}_{\tilde{\tau}^{\tilde{\rho}00}}(\sigma_n, r) \wedge \text{int}_{\tilde{\tau}^{\tilde{\rho}00}}(\gamma_n, r), r\right)
\]

\[
\leq C_{\tilde{\tau}^{\tilde{\rho}00}} \left(\text{int}_{\tilde{\tau}^{\tilde{\rho}00}}(\sigma_n, r), r\right) \wedge C_{\tilde{\tau}^{\tilde{\rho}00}} \left(\text{int}_{\tilde{\tau}^{\tilde{\rho}00}}(\gamma_n, r), r\right)
\]

\[
= \sigma_n \wedge \gamma_n.
\]

Thus, \(C_{\tilde{\tau}^{\tilde{\rho}00}} \left(\text{int}_{\tilde{\tau}^{\tilde{\rho}00}}(\sigma_n \wedge \gamma_n, r), r\right) = \sigma_n \wedge \gamma_n\). Therefore, \(\sigma_n \wedge \gamma_n\) is an r-SVNRIC.

(2) The proof is similar to that of (1). \(\blacksquare\)

Theorem 12. Let \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}00}, \tilde{\eta}^{\tilde{\rho}00})\) be an SVNITS and \(r \in \tilde{\xi}_0\). Then, the following properties are equivalent:

1. \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}00}, \tilde{\eta}^{\tilde{\rho}00})\) is \(\mathcal{L}\)-SVNE-disconnected,
2. \(\tilde{\tau}^\circ(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r, \tilde{\tau}^\circ(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\gamma_n, r)) \leq 1 - r, \tilde{\tau}^\circ(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \leq 1 - r\) for every r-SVNSIO \(\sigma_n \in \tilde{\xi}_r^{\tilde{\tau}^\circ}\)
3. \(\tilde{\tau}^\circ(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r, \tilde{\tau}^\circ(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\gamma_n, r)) \leq 1 - r, \tilde{\tau}^\circ(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\gamma_n, r)) \leq 1 - r\) for every r-SVNPIO \(\sigma_n \in \tilde{\xi}_r^{\tilde{\tau}^\circ}\)
4. \(\tilde{\tau}^\circ(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r, \tilde{\tau}^\circ(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \leq 1 - r, \tilde{\tau}^\circ(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\gamma_n, r)) \leq 1 - r\) for every r-SVNRIO \(\sigma_n \in \tilde{\xi}_r^{\tilde{\tau}^\circ}\).

Proof. (1) \(\Rightarrow\) (2) and (1) \(\Rightarrow\) (3). Let \(\sigma_n\) be an r-SVNSIO (r-SVNPIO). Then, \(\sigma_n\) is r-SVNSIO, and, by Theorem 7, we have,

\[
\tilde{\tau}^\circ(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r, \tilde{\tau}^\circ(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\gamma_n, r)) \leq 1 - r, \tilde{\tau}^\circ(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \leq 1 - r.
\]

(2) \(\Rightarrow\) (4) and (3) \(\Rightarrow\) (4). Let \(\sigma_n\) be an r-SVNIO. Then, \(\sigma_n\) is r-SVNPIO and r-SVNSIO. Thus,

\[
\tilde{\tau}^\circ(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \geq r, \quad \tilde{\tau}^\circ(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \leq 1 - r, \quad \tilde{\tau}^\circ(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r)) \leq 1 - r.
\]

(4) \(\Rightarrow\) (1). Suppose that

\[
\tilde{\tau}^\circ(\text{int}_{\tilde{\tau}^{\tilde{\rho}}}(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r), r)) \geq r, \quad \tilde{\tau}^\circ(\text{int}_{\tilde{\tau}^{\tilde{\rho}}}(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\sigma_n, r), r)) \geq r, \quad \tilde{\tau}^\circ(\text{int}_{\tilde{\tau}^{\tilde{\rho}}}(\text{Cl}_{\tilde{\tau}^{\tilde{\rho}}}(\gamma_n, r), r)) \geq r.
\]
Then, by (4), we have
\[
\tilde{\tau}^d(C\tilde{\tau}^d(\text{int}_{\tilde{\tau}^d}(C\tilde{\tau}^d(\varsigma_n,r),r),r)) \geq r, \quad \tilde{\tau}^d(C\tilde{\tau}^d(\text{int}_{\tilde{\tau}^d}(C\tilde{\tau}^d(\varsigma_n,r),r),r)) \geq r,
\]
Thus, \(\tilde{\tau}^d(C\tilde{\tau}^d(\varsigma_n,r)) \geq r, \tilde{\tau}^d(C\tilde{\tau}^d(\varsigma_n,r)) \leq 1 - r, \tilde{\tau}^d(C\tilde{\tau}^d(\varsigma_n,r)) \leq 1 - r;\) hence, \((X, \tilde{\tau}^{\tilde{\rho}}, \tilde{\tau}^{\tilde{\rho}})\) is an L-SVNE-disconnected. \(\square\)

**Definition 16.** Let \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}}, \tilde{\tau}^{\tilde{\rho}})\) be an SVNITS. Then, \(\varsigma_n\) is said to be an \(r\)-SVNLSO if \(\varsigma_n \leq C_{\tilde{\tau}^{\tilde{\rho}}}(\text{int}_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r),r)\).

**Definition 17.** Let \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}}, \tilde{\tau}^{\tilde{\rho}})\) be an SVNITS for each \(r \in \xi_0, \varsigma_n \in \tilde{\xi}^X\) and \(x_{s,t,p} \in \text{Pt}(\tilde{\xi}^X)\). Then, \(x_{s,t,p}\) is called an \(r\)-SVNLS\(\delta\)\-cluster point of \(\varsigma_n\) if, for every \(\gamma_n \in Q_{\tilde{\tau}^{\tilde{\rho}}}(x_{s,t,p},r)\), we have \(\varsigma_n \leq \text{int}_{\tilde{\tau}^{\tilde{\rho}}}(C\tilde{\tau}^{\tilde{\rho}}(\gamma_n,r),r)\).

**Definition 18.** Let \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}}, \tilde{\tau}^{\tilde{\rho}})\) be an SVNITS for each \(r \in \xi_0, \varsigma_n \in \tilde{\xi}^X\) and \(x_{s,t,p} \in \text{Pt}(\tilde{\xi}^X)\). Then, the single-valued neutrosophic \(\delta\-\text{closure operator} is a mapping} C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r) = \text{int}_{\tilde{\tau}^{\tilde{\rho}}}(C\tilde{\tau}^{\tilde{\rho}}(\varsigma_n,r),r).

**Lemma 3.** Let \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}}, \tilde{\tau}^{\tilde{\rho}})\) be an SVNITS. Then, \(\varsigma_n\) is \(r\)-SVNLSO iff \(C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r) = C_{\tilde{\tau}^{\tilde{\rho}}}(\text{int}_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r),r)\).

**Proof.** Obvious. \(\square\)

**Lemma 4.** Let \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}}, \tilde{\tau}^{\tilde{\rho}})\) be an SVNITS for each \(\varsigma_n \in \tilde{\xi}^X\) and \(r \in \xi_0\). Then, \(C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r) \leq C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r)\).

**Proof.** Obvious. \(\square\)

**Lemma 5.** Let \((\tilde{X}, \tilde{\tau}^{\tilde{\rho}}, \tilde{\tau}^{\tilde{\rho}})\) be an SVNITS and \(\varsigma_n\) be an \(r\)-SVNLSO. Then, \(C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r) = C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r)\).

**Proof.** We show that \(C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r) \leq C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r)\). Suppose that \(C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r) \not\geq C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r)\); then, there exist \(v \in \tilde{X}\) and \(s, t, p \in \xi_0\) such that
\[
\tilde{\rho}_{C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r)}(v) < s \leq \tilde{\rho}_{C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r)}(v), \quad \tilde{\delta}_{C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r)}(v) \geq t > \tilde{\delta}_{C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r)}(v),
\]
\[
\tilde{\eta}_{C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r)}(v) \geq p > \tilde{\eta}_{C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r)}(v).
\]
By the definition of \(C_{\tilde{\tau}^{\tilde{\rho}}}\), there exists \(\tilde{\tau}^d(\gamma_n^c) \geq r, \tilde{\tau}^d(\gamma_n^c) \leq 1 - r, \tilde{\tau}^d(\gamma_n^c) \leq 1 - r\) with \(\varsigma_n \leq \gamma_n\) such that
\[
\tilde{\rho}_{C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r)}(v) \leq \tilde{\rho}_{\gamma_n}(v) < s < \tilde{\rho}_{C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r)}(v), \quad \tilde{\delta}_{C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r)}(v) \geq \tilde{\delta}_{\gamma_n}(v) > t > \tilde{\delta}_{C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r)}(v),
\]
\[
\tilde{\eta}_{C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r)}(v) \geq \tilde{\eta}_{\gamma_n}(v) > p > \tilde{\eta}_{C_{\tilde{\tau}^{\tilde{\rho}}}(\varsigma_n,r)}(v).
\]
Then, \( [\gamma_n]^c \in Q_{\pi^n}(x_{s,t,r,p}, r) \) and
\[
[\sigma_n]^c \geq [\gamma_n]^c \implies Cl_{\pi^n}(\sigma_n, r) \leq Cl_{\pi^n}(\gamma_n, r).
\]
Thus, \( \text{int}_{\pi^n}(\sigma_n, r) \subset [\gamma_n]^c \). Hence, \( \text{int}_{\pi^n}(\text{Cl}_{\pi^n}(\gamma_n, r)) \subset C_{\pi^n}(\text{Cl}_{\pi^n}(\sigma_n, r), r) \).

Since \( \sigma_n \) is an \( r \)-SVN\&SO, we have \( \text{int}_{\pi^n}(\text{Cl}_{\pi^n}(\gamma_n, r), r) \parallel \gamma_n \). So, \( x_{s,t,r,p} \) is not an \( r \)-SVN\&SO-cluster point of \( \sigma_n \). It is a contradiction for equation 3. Thus, \( C_{\pi^n}(\sigma_n, r) \geq C_{\Delta_1\pi^n}(\sigma_n, r) \).

By Lemma 4, we have \( C_{\pi^n}(\sigma_n, r) = C_{\Delta_1\pi^n}(\sigma_n, r) \).

**Theorem 13.** Let \( (X, E^{\pi^n}, E^{\pi^n}) \) be an SVNITS. Then, the following properties are equivalent:

1. \( (X, E^{\pi^n}, E^{\pi^n}) \) is \( L \)-SVN\&SO-disconnected,
2. If \( \sigma_n \) and \( \gamma_n \) are \( r \)-SVN\&SO, then \( Cl_{\pi^n}(\sigma_n, r) \land Cl_{\pi^n}(\gamma_n, r) \leq C_{\pi^n}(\sigma_n \land \gamma_n) \),
3. If \( \sigma_n \) is \( r \)-SVNSIO and \( \gamma_n \) is \( r \)-SVN\&SO, then \( Cl_{\pi^n}(\sigma_n, r) \land Cl_{\pi^n}(\gamma_n, r) \leq C_{\pi^n}(\sigma_n \land \gamma_n) \),
4. If \( \sigma_n \) is an \( r \)-SVN\&SO and \( \gamma_n \) is an \( r \)-SVNSIO, then \( Cl_{\pi^n}(\sigma_n, r) \land Cl_{\pi^n}(\gamma_n, r) \leq C_{\pi^n}(\sigma_n \land \gamma_n) \),
5. If \( \sigma_n \) is an \( r \)-SVN\&PIO and \( \gamma_n \) is an \( r \)-SVN\&SO, then \( Cl_{\pi^n}(\sigma_n, r) \land Cl_{\pi^n}(\gamma_n, r) \leq C_{\pi^n}(\sigma_n \land \gamma_n) \).

**Proof.**
1. \( \Rightarrow \) (2): Let \( \sigma_n \) be an \( r \)-SVN\&IO and \( \gamma_n \) be an \( r \)-SVN\&SO, by Theorem 7, \( \tau^0(\text{Cl}_{\pi^n}(\sigma_n, r)) \geq r, \tau^0(\text{Cl}_{\pi^n}(\gamma_n, r)) \leq 1 - r \). Then,
\[
\text{Cl}_{\pi^n}(\sigma_n, r) \land \text{Cl}_{\pi^n}(\gamma_n, r) \leq \text{Cl}_{\pi^n}(\text{Cl}_{\pi^n}(\sigma_n, r), r) \land \text{Cl}_{\pi^n}(\text{Cl}_{\pi^n}(\gamma_n, r), r) \leq \text{Cl}_{\pi^n}(\text{Cl}_{\pi^n}(\sigma_n \land \gamma_n), r).
\]
2. \( \Rightarrow \) (3): It follows from the fact that every \( r \)-SVN\&SO set is an \( r \)-SVN\&SOIO.
3. \( \Rightarrow \) (4): Clear.
4. \( \Rightarrow \) (1): Let \( \sigma_n \) be an \( r \)-SVN\&SO. Since \( \text{Cl}_{\pi^n}(\sigma_n, r)^c \leq \text{Cl}_{\pi^n}(\text{int}_{\pi^n}(\sigma_n, r), r, r) \), we have, \( \text{Cl}_{\pi^n}(\sigma_n, r)^c \) is an \( r \)-SVN\&SO. Then, by (4), \( \text{Cl}_{\pi^n}(\sigma_n, r) \subset C_{\pi^n}(\text{Cl}_{\pi^n}(\sigma_n, r, r)) \).

Therefore, \( \tau^0(\text{Cl}_{\pi^n}(\sigma_n, r)) \geq r, \tau^0(\text{Cl}_{\pi^n}(\gamma_n, r)) \leq 1 - r \). Thus, by Theorem 12, \( (X, E^{\pi^n}, E^{\pi^n}) \) is \( L \)-SVN\&SO-disconnected.

**Corollary 1.** Let \( (X, E^{\pi^n}, E^{\pi^n}) \) be an SVNITS. Then, the following properties are equivalent:

1. \( (X, E^{\pi^n}, E^{\pi^n}) \) is \( L \)-SVN\&SO-disconnected.
2. If \( \sigma_n \) is an \( r \)-SVN\&SO and \( \gamma_n \) is an \( r \)-SVN\&SO, then \( Cl_{\pi^n}(\sigma_n, r) \land Cl_{\pi^n}(\gamma_n, r) \leq C_{\pi^n}(\sigma_n \land \gamma_n) \),
3. If \( \sigma_n \) is an \( r \)-SVN\&IO and \( \gamma_n \) is an \( r \)-SVN\&SO, then \( Cl_{\pi^n}(\sigma_n, r) \land Cl_{\pi^n}(\gamma_n, r) \leq C_{\pi^n}(\sigma_n \land \gamma_n) \),
4. If \( \sigma_n \) is an \( r \)-SVN\&PIO and \( \gamma_n \) is an \( r \)-SVN\&SO, then \( Cl_{\pi^n}(\sigma_n, r) \land Cl_{\pi^n}(\gamma_n, r) \leq C_{\pi^n}(\sigma_n \land \gamma_n) \).

**Proof.** It follows directly from Lemma 3 and 5.
5. Some Types of Separation Axioms

In this section, some kinds of separation axioms, namely r-single valued neutrosophic ideal-R, r-SVNIR, for short, where $i = \{0, 1, 2, 3\}$, and r-single valued neutrosophic ideal-I, r-SVNIT, for short, where $i = \{1, 2, 2\frac{1}{2}, 3, 4\}$, in the sense of Sostak are defined. Some of their characterizations, fundamental properties, and the relations between these notions have been studied.

Definition 19. Let $(\tilde{X}, \tilde{\tau}^{i\theta}, \tilde{T}^{i\theta})$ be an SVNITS and $r \in \xi_0$. Then, $\tilde{X}$ is called:

1. r-SVNIR$_0$ if $x_{s,t,p}^{\tilde{\tau}^{i\theta}}(y_{s_1,t_1,p_1}, r)$ implies $y_{s_1,t_1,p_1}^{\tilde{\tau}^{i\theta}}(x_{s,t,p}, r)$ for any $x_{s,t,p} \neq y_{s_1,t_1,p_1}$.
2. r-SVNIR$_1$ if $x_{s,t,p}^{\tilde{\tau}^{i\theta}}(y_{s_1,t_1,p_1}, r)$ implies that there exist r-SVN£O sets $\sigma_n, \gamma_n \in \xi^X$ such that $x_{s,t,p} \in \sigma_n, y_{s_1,t_1,p_1} \in \gamma_n$ and $\sigma_n \supseteq \gamma_n$.
3. r-SVNIR$_2$ if $x_{s,t,p}^{\tilde{\tau}^{i\theta}}(y_{s_1,t_1,p_1}, r)$ implies that there exist r-SVN£O sets $\sigma_n, \gamma_n \in \xi^X$ such that $x_{s,t,p} \in \sigma_n, y_{s_1,t_1,p_1} \in \gamma_n$.
4. r-SVNIR$_3$ if $x_{s,t,p}^{\tilde{\tau}^{i\theta}}(y_{s_1,t_1,p_1}, r)$ implies that there exist r-SVN£O sets $\sigma_n, \gamma_n \in \xi^X$ such that $x_{s,t,p} \in \sigma_n, y_{s_1,t_1,p_1} \in \gamma_n$.

Theorem 14. Let $(\tilde{X}, \tilde{\tau}^{i\theta}, \tilde{T}^{i\theta})$ be an SVNITS and $r \in \xi_0$. Then, the following statements are equivalent:

1. $(\tilde{X}, \tilde{\tau}^{i\theta}, \tilde{T}^{i\theta})$ is r-SVNIR$_0$.
2. If $x_{s,t,p}^{\tilde{\tau}^{i\theta}} \sigma_n = \tilde{\tau}^{i\theta}(\sigma_n, r)$, then there exists r-SVN£O $\gamma_n \in \xi^X$ such that $x_{s,t,p} \supseteq \gamma_n$ and $\sigma_n \supseteq \gamma_n$.
3. If $x_{s,t,p}^{\tilde{\tau}^{i\theta}} \sigma_n = \tilde{\tau}^{i\theta}(\sigma_n, r)$, then $\tilde{\tau}^{i\theta}(x_{s,t,p}, r) = \tilde{\tau}^{i\theta}(\sigma_n, r)$.
4. If $x_{s,t,p}^{\tilde{\tau}^{i\theta}}(y_{s_1,t_1,p_1}, r)$, then $\tilde{\tau}^{i\theta}(y_{s_1,t_1,p_1}, r)$.

Proof. (1)⇒(2): Let $x_{s,t,p}^{\tilde{\tau}^{i\theta}}(\sigma_n, r)$. Then,

$$s + \rho_{\sigma_n}(v) < 1, \quad t + \rho_{\sigma_n}(v) \geq 1, \quad p + \eta_{\sigma_n}(v) \geq 1,$$

for every $y_{s_1,t_1,p_1} \in \sigma_n$, we have $s < \rho_{\sigma_n}(v), \ t \geq \rho_{\sigma_n}(v), \ p \geq \eta_{\sigma_n}(v)$. Thus, $x_{s,t,p}^{\tilde{\tau}^{i\theta}}(y_{s_1,t_1,p_1}, r)$. Since $(\tilde{X}, \tilde{\tau}^{i\theta}, \tilde{T}^{i\theta})$ is an r-SVNIR$_0$, we obtain $y_{s_1,t_1,p_1}^{\tilde{\tau}^{i\theta}}(y_{s_1,t_1,p_1}, r)$. By Lemma 2(2), there exists an r-SVN£O $\gamma_n \in \xi^X$ such that $x_{s,t,p} \supseteq \gamma_n$ and $y_{s_1,t_1,p_1} \supseteq \gamma_n$. Let

$$\gamma_n = \bigvee_{y_{s_1,t_1,p_1} \in \sigma_n} \{\xi_n : x_{s,t,p}^{\tilde{\tau}^{i\theta}}(y_{s_1,t_1,p_1}, \xi_n) \in \xi_{\gamma_n}\}.$$

From Lemma 1(1), $\gamma_n$ is an r-SVN£O. Then, $x_{s,t,p}^{\tilde{\tau}^{i\theta}}(\sigma_n, r), \sigma_n \supseteq \gamma_n$.

(2)⇒(3): Let $x_{s,t,p}^{\tilde{\tau}^{i\theta}}(\sigma_n, r)$. Then, there exists an r-SVN£O $\gamma_n \in \xi^X$ such that $x_{s,t,p} \supseteq \gamma_n$ and $\sigma_n \supseteq \gamma_n$. Since for every $v \in \tilde{X}$,

$$s < \rho_{\gamma_n}(v), \quad t \geq \rho_{\gamma_n}(v), \quad p \geq \eta_{\gamma_n}(v),$$

we obtain

$$\tilde{\tau}^{i\theta}(x_{s,t,p}, r) \leq \tilde{\tau}^{i\theta}([\gamma_n], r) = [\gamma_n] \leq [\sigma_n].$$
Therefore, $\text{Cl}_{\tau \pi \rho \eta}^\ell (x_{s,t,p}, r) \bar{\pi}_n = \text{Cl}_{\tau \pi \rho \eta}^\ell (\sigma_n, r)$.

$(3) \Rightarrow (4)$: Let $x_{s,t,p} \bar{\pi}_n \text{Cl}_{\tau \pi \rho \eta}^\ell (y_{s_1,t_1,p_1}, r)$. Then, $x_{s,t,p} \bar{\pi}_n \text{Cl}_{\tau \pi \rho \eta}^\ell (y_{s_1,t_1,p_1}, r) = \text{Cl}_{\tau \pi \rho \eta}^\ell (\text{Cl}_{\tau \pi \rho \eta}^\ell (y_{s_1,t_1,p_1}, r))$. By $(3)$, $s_1, t_1, p_1 (x_{s,t,p}, r) \bar{\pi}_n \text{Cl}_{\tau \pi \rho \eta}^\ell (y_{s_1,t_1,p_1}, r)$.

$(4) \Rightarrow (1)$: Clear. □

**Theorem 15.** Let $(\bar{X}, \bar{\pi}^{\rho \eta}, \bar{\pi}^{\theta \vartheta})$ be an SVNITs and $r \in \xi_0$. Then, if $X$ is

(1) $[r \text{-SVNIR}_3 \text{ and } r \text{-SVNIR}_0] \Rightarrow (a) r \text{-SVNIR}_2 \Rightarrow (b) r \text{-SVNIR}_1 \Rightarrow (c) r \text{-SVNIR}_0.$

(2) $r \text{-SVNIT}_3 \Rightarrow r \text{-SVNIT}_1.$

(3) $r \text{-SVNIT}_3 \Rightarrow r \text{-SVNIT}_2.$

(4) $r \text{-SVNIT}_4 \Rightarrow r \text{-SVNIT}_2.$

(5) $r \text{-SVNIT}_4 \Rightarrow (a) r \text{-SVNIT}_3 \Rightarrow (b) r \text{-SVNIT}_{12} \Rightarrow (c) r \text{-SVNIT}_2 \Rightarrow (d) r \text{-SVNIT}_1.$

**Proof.** $(1a)$. Let $x_{s,t,p} \bar{\pi}_n = \text{Cl}_{\tau \pi \rho \eta}^\ell (\xi_n, r)$, by Theorem 14 $(3)$, $\text{Cl}_{\tau \pi \rho \eta}^\ell (x_{s,t,p}, r) \bar{\pi}_n = \text{Cl}_{\tau \pi \rho \eta}^\ell (\xi_n, r)$. Since $(\bar{X}, \bar{\pi}^{\rho \eta}, \bar{\pi}^{\theta \vartheta})$ is $r$-SVNIT3 and $\text{Cl}_{\tau \pi \rho \eta}^\ell (x_{s,t,p}, r) = \text{Cl}_{\tau \pi \rho \eta}^\ell (\text{Cl}_{\tau \pi \rho \eta}^\ell (y_{s_1,t_1,p_1}, r), r)$, there exist $r$-SVNLs sets $\sigma_n, \gamma_n \in \xi^X$ such that $x_{s,t,p} \in \text{Cl}_{\tau \pi \rho \eta}^\ell (y_{s_1,t_1,p_1}, r) \subseteq \sigma_n, \xi_n \subseteq \gamma_n$ and $\sigma_n \bar{\pi}_n \gamma_n$. Hence, $(\bar{X}, \bar{\pi}^{\rho \eta}, \bar{\pi}^{\theta \vartheta})$ is $r$-SVNIT2.

$(1b)$. For each $x_{s,t,p} \bar{\pi}_n \text{Cl}_{\tau \pi \rho \eta}^\ell (y_{s_1,t_1,p_1}, r)$, by $r$-SVNIT2 of $X$, there exist $r$-SVNLs sets $\sigma_n, \gamma_n \in \xi^X$ such that $x_{s,t,p} \in \sigma_n, y_{s_1,t_1,p_1} \in \gamma_n$ and $\sigma_n \bar{\pi}_n \gamma_n$. Thus, $(\bar{X}, \bar{\pi}^{\rho \eta}, \bar{\pi}^{\theta \vartheta})$ is $r$-SVNIT3.

$(1c)$. Let $(\bar{X}, \bar{\pi}^{\rho \eta}, \bar{\pi}^{\theta \vartheta})$ be $r$-SVNIT3. Then, for every $x_{s,t,p} \bar{\pi}_n \text{Cl}_{\tau \pi \rho \eta}^\ell (y_{s_1,t_1,p_1}, r)$ and $x_{s,t,p} \not\in y_{s_1,t_1,p_1}$, there exist $r$-SVNLs sets $\sigma_n, \gamma_n \in \xi^X$ such that $x_{s,t,p} \in \sigma_n, \gamma_n \not\in \gamma_n$ and $\sigma_n \bar{\pi}_n \gamma_n$. Hence, $x_{s,t,p} \in \sigma_n \subseteq \gamma_n$. Since $\gamma_n$ is an $r$-SVNL set, we obtain $\text{Cl}_{\tau \pi \rho \eta}^\ell (x_{s,t,p}, r) \subseteq \text{Cl}_{\tau \pi \rho \eta}^\ell ([\gamma_n]^c, r) = [\gamma_n]^c \subseteq [y_{s_1,t_1,p_1}]^c$. Thus, $y_{s_1,t_1,p_1} \bar{\pi}_n \text{Cl}_{\tau \pi \rho \eta}^\ell (x_{s,t,p}, r)$ and $(\bar{X}, \bar{\pi}^{\rho \eta}, \bar{\pi}^{\theta \vartheta})$ is $r$-SVNIT0.

$(2)$. Let $x_{s,t,p} \bar{\pi}_n y_{s_1,t_1,p_1}, r$. Then, $x_{s,t,p} \bar{\pi}_n y_{s_1,t_1,p_1}, r$. By $r$-SVNIT2 of $X$, there exist $r$-SVNLs sets $\sigma_n, \gamma_n \in \xi^X$ such that $x_{s,t,p} \in \sigma_n, y_{s_1,t_1,p_1} \in \gamma_n$ and $\sigma_n \bar{\pi}_n \gamma_n$. Hence, $(\bar{X}, \bar{\pi}^{\rho \eta}, \bar{\pi}^{\theta \vartheta})$ is $r$-SVNIT1.

$(3)$ and $(4)$ The proofs are direct consequence of $(2)$. $(5a)$. The proof is direct consequence of $(1)$.

$(5b)$. For each $x_{s,t,p} \bar{\pi}_n y_{s_1,t_1,p_1}$, since $X$ is both $r$-SVNIR2 and $r$-SVNIT1, then there exists an $r$-SVNL set $\xi_n \in \xi^X$ such that $x_{s,t,p} \in \xi_n$ and $y_{s_1,t_1,p_1} \bar{\pi}_n \xi_n$. Then,

$x_1 \in \xi_n = \text{int} \text{Cl}_{\tau \pi \rho \eta}^\ell (\xi_n, r) \leq \text{int} \text{Cl}_{\tau \pi \rho \eta}^\ell (y_{s_1,t_1,p_1}, r) = \text{Cl}_{\tau \pi \rho \eta}^\ell (y_{s_1,t_1,p_1}, r)^c.$

Hence, $x_{s,t,p} \bar{\pi}_n \text{Cl}_{\tau \pi \rho \eta}^\ell (y_{s_1,t_1,p_1}, r)$. By $r$-SVNIR2 of $X$, there exist $r$-SVNLs sets $\sigma_n, \gamma_n \in \xi^X$ such that $x_{s,t,p} \in \sigma_n, \text{Cl}_{\tau \pi \rho \eta}^\ell (y_{s_1,t_1,p_1}, r) \subseteq \gamma_n$ and $\sigma_n \bar{\pi}_n \gamma_n$. Thus, $\sigma_n \subseteq [\gamma_n]^c$, so

$\text{Cl}_{\tau \pi \rho \eta}^\ell (\sigma_n, r) \leq \text{Cl}_{\tau \pi \rho \eta}^\ell ([\gamma_n]^c, r) = [\gamma_n]^c \subseteq \text{Cl}_{\tau \pi \rho \eta}^\ell (y_{s_1,t_1,p_1}, r)^c.$

It implies $\text{Cl}_{\tau \pi \rho \eta}^\ell (\sigma_n, r) \bar{\pi}_n \text{Cl}_{\tau \pi \rho \eta}^\ell (y_{s_1,t_1,p_1}, r)$ with $x_{s,t,p} \in \sigma_n$ and $y_{s_1,t_1,p_1} \in \text{Cl}_{\tau \pi \rho \eta}^\ell (y_{s_1,t_1,p_1}, r)$. Thus, $(\bar{X}, \bar{\pi}^{\rho \eta}, \bar{\pi}^{\theta \vartheta})$ is $r$-SVNIT2.$^2$

$(5c)$. Let $x_{s,t,p} \bar{\pi}_n y_{s_1,t_1,p_1}$. Then, by $r$-SVNIT2 of $X$, there exist $r$-SVNLs sets $\sigma_n, \gamma_n \in \xi^X$ such that $x_{s,t,p} \in \sigma_n, y_{s_1,t_1,p_1} \in \gamma_n$ and $\text{Cl}_{\tau \pi \rho \eta}^\ell (\sigma_n, r) \bar{\pi}_n \text{Cl}_{\tau \pi \rho \eta}^\ell (\gamma_n, r)$, which implies that $\sigma_n \bar{\pi}_n \gamma_n$. Thus, $(\bar{X}, \bar{\pi}^{\rho \eta}, \bar{\pi}^{\theta \vartheta})$ is $r$-SVNIT2.$^2$

$(5d)$. Similar to the proof of $(5c)$. □

**Theorem 16.** Let $(\bar{X}, \bar{\pi}^{\rho \eta}, \bar{\pi}^{\theta \vartheta})$ be an SVNITs and $r \in \xi_0$. Then, the following statements are equivalent:

(1) $(\bar{X}, \bar{\pi}^{\rho \eta}, \bar{\pi}^{\theta \vartheta})$ is $r$-SVNIR2.
(2) If \( x_{s,t,p} \in \sigma_0 \) and \( \sigma_0 \) is r-SVNLEO set, then there exists r-SVNLEO set \( \gamma_n \in \xi^X \) such that \( x_{s,t,p} \in \gamma_n \leq C_{r_\overline{\xi}_{\inf}}(\gamma_n,r) \leq \sigma_0 \).

(3) If \( x_{s,t,p} \in \sigma_0 \), then there exists r-SVNLEO set \( [\gamma_n]_j \in \xi^X, j = \{1,2\} \) such that \( x_{s,t,p} \in [\gamma_n]_1, \sigma_0 \leq [\gamma_n]_2 \) and \( C_{r_\overline{\xi}_{\inf}}([\gamma_n]_1, r) \overline{\sigma_0} C_{r_\overline{\xi}_{\inf}}([\gamma_n]_2, r) \).

**Proof.** Similar to the proof of Theorem 14. \( \square \)

**Theorem 17.** Let \((\hat{X}, \tau_{\overline{\xi}_{\inf}}, I_{\overline{\xi}_{\inf}})\) be an SVNITS and \( r \in \xi_0 \). Then, the following statements are equivalent:

1. \((\hat{X}, \tau_{\overline{\xi}_{\inf}}, I_{\overline{\xi}_{\inf}})\) is r-SVNIR3.
2. If \([\sigma_0]_1 \overline{\tau}_n \) and \([\sigma_0]_2 \overline{\tau}_n \) are r-SVNLC sets, then there exists r-SVNLEO set \( \gamma_n \in \xi^X \) such that \( [\sigma_0]_1 \leq \gamma_n \) and \( C_{r_\overline{\xi}_{\inf}}(\gamma_n,r) \leq [\sigma_0]_2 \).
3. For any \([\sigma_0]_1 \leq [\sigma_0]_2 \), where \([\sigma_0]_1 \) is an r-SVNLEO set, and \([\sigma_0]_2 \) is an r-SVNEC set, then there exists an r-SVNLEO set \( \gamma_n \in \xi^X \) such that \( [\sigma_0]_1 \leq \gamma_n \leq C_{r_\overline{\xi}_{\inf}}(\gamma_n,r) \leq [\sigma_0]_2 \).

**Proof.** Similar to the proof of Theorem 15. \( \square \)

**Theorem 18.** Let \( f : (\hat{X}, \tau_{\overline{\xi}_{\inf}}, I_{\overline{\xi}_{\inf}}) \rightarrow (\hat{Y}, \tau_{\overline{\xi}_{\inf}}, I_{\overline{\xi}_{\inf}}) \) be a \( \lambda \)-SVNI-irresolute, bijective, \( \lambda \)-SVNI-irresolute open mapping and \((\hat{X}, \tau_{\overline{\xi}_{\inf}}, I_{\overline{\xi}_{\inf}})\) is r-SVNIR2. Then, \((\hat{Y}, \tau_{\overline{\xi}_{\inf}}, I_{\overline{\xi}_{\inf}})\) is r-SVNIR2.

**Proof.** Let \( y_{s,t,p} \in \overline{\gamma}_n = \overline{C}_r(\gamma_n,r) \). Then, by Definition 11, \( \gamma_n \) is an r-SVNLC set in \( \hat{Y} \). By Theorem 3(2), \( f^{-1}(\gamma_n) \) is an r-SVNLC set in \( \hat{X} \). Put \( \tilde{y}_{s,t,p} = f(x_{s,t,p}) \). Then, \( x_{s,t,p} \overline{f} f^{-1}(\gamma_n) \). By r-SVNIR2 of \( \hat{X} \), there exist r-SVNLEO sets \( \sigma_0, \gamma_n \in \xi^X \) such that \( x_{s,t,p} \in \sigma_0, f^{-1}(\gamma_n) \leq \gamma_n \) and \( \sigma_0 \overline{\gamma}_n \). Since \( f \) is bijective and \( \lambda \)-SVNI-irresolute open, \( y_{s,t,p} \in f(\sigma_0), \gamma_n \leq f(f^{-1}(\gamma_n)) \leq f(\gamma_n) \) and \( f(\sigma_0) \overline{f} f(\gamma_n) \). Thus, \((\hat{Y}, \tau_{\overline{\xi}_{\inf}}, I_{\overline{\xi}_{\inf}})\) is r-SVNIR2. \( \square \)

**Theorem 19.** Let \( f : (\hat{X}, \tau_{\overline{\xi}_{\inf}}, I_{\overline{\xi}_{\inf}}) \rightarrow (\hat{Y}, \tau_{\overline{\xi}_{\inf}}, I_{\overline{\xi}_{\inf}}) \) be an \( \lambda \)-SVNI-irresolute, bijective, \( \lambda \)-SVNI-irresolute open mapping and \((\hat{X}, \tau_{\overline{\xi}_{\inf}}, I_{\overline{\xi}_{\inf}})\) be an r-SVNIR2. Then, \((\hat{Y}, \tau_{\overline{\xi}_{\inf}}, I_{\overline{\xi}_{\inf}})\) is r-SVNIR3.

**Proof.** Similar to the proof of Theorem 18. \( \square \)

**6. Conclusions**

In summary, we have introduced the definition of the r-single valued neutrosophic \( \lambda \)-closed and r-single valued neutrosophic \( \lambda \)-open sets over single valued neutrosophic ideal topology space in Sostak’s sense. Many consequences have been arisen up to show that how far topological structures are preserved by these r-single valued neutrosophic \( \lambda \)-closed. We also have provided some counterexamples where such properties fail to be preserved. The most important contribution to this area of research is that we have introduced the notion of \( \lambda \)-single valued neutrosophic irresolute mapping, \( \lambda \)-single valued neutrosophic extremely disconnected spaces, \( \lambda \)-single valued neutrosophic normal spaces and that we defined some kinds of separation axioms, namely r-SVNIR\(_i\), where \( i = \{0,1,2,3\} \), and r-SVNI\(_j\), where \( j = \{1,2,3,4\} \), in the sense of Sostak. Some of their characterizations, fundamental properties, and the relations between these notions have been studied.

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Discussion for Further Works: The theory in this article can be extended in the following natural ways. One may study the properties of neutrosophic metric topological spaces using the concepts defined through this paper.

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