# Single-valued neutrosophic entropy and similarity measures and their application in multi-attribute decision-making

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**Abstract:** The single-valued neutrosophic sets (SVNSs) are useful tools to describe uncertainty and inconsistent information that exist in real world. For SVNSs theory, one of the most important topics is single-valued neutrosophic information measure method. This paper investigates a multi-attribute decision-making (MADM) method by using single-valued neutrosophic entropy and similarity measure. First, the concepts of single-valued neutrosophic entropy and similarity measure are presented. Then, based on the trigonometric functions (i.e., sine function and cosine function), we construct some information measure formulas and prove that they satisfy the axiomatic requirements of the single-valued neutrosophic entropy and similarity measure, respectively. Furthermore, the inter-relationship between entropy and similarity measure as well as their mutual transformations are further studied. By using Lagrange Multiplier Method and closeness degree, we proposed a novel single-valued neutrosophic MADM method. Finally, a practical example of investment evaluation problem is provided to compare our method with the existing ones.

**Keywords:** Multi-attribute decision making (MADM); Single-valued neutrosophic sets (SVNSs); Entropy; Similarity measure; Trigonometric functions

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## 1. Introduction

As the ambiguity and uncertainty of attribute in MADM problems, the evaluation values cannot be described by crisp numbers, and it can be expressed by fuzzy value expression in more suitable occasions, such as fuzzy sets (FSs) [1], Pythagorean fuzzy sets (PFSs) [2,3], intuitionistic fuzzy sets (IFSs) [4-7], interval-valued intuitionistic fuzzy sets (IVIFSs) [8-10]. However, IFSs and IVIFSs can't handle inconsistent information and indeterminate information. Therefore, from philosophical point of view, Smarandache [11,12] originally introduced the generalization concept of traditional IFSs, called neutrosophic sets (NSs). The NSs simultaneously take into account the truth membership, the indeterminacy membership and the falsity membership, and they are independent. In order to facilitate practical application, the notion of SVNS [13] was presented, which is a subclass of NSs.

As two important research topics in the MADM theory, entropy and similarity measure have been studied by some researchers [14-16]. In order to measure the fuzziness of decision-making information, Zadeh [17] first defined the concept of fuzzy entropy. Luca and Termini [18] presented the axioms with which the fuzzy entropy should comply, and more formally defined the entropy of a FS. With the help of the ratio for intuitionistic fuzzy cardinalities, Szmidt and Kacprzyk [19] introduced several axiomatic requirements of intuitionistic fuzzy information entropy measure, and then they also constructed a non-probabilistic-type entropy measure under the intuitionistic fuzzy information environment. For the MADM problems, Ye [20] constructed two interval-valued intuitionistic fuzzy entropy measures, and then determined the attribute weights by using entropy weighted model. Based on the continuous ordered weighted averaging operator, Jin et al. [21] proposed a new entropy for IVIFSs, called interval-valued intuitionistic fuzzy continuous weighted entropy, and then an interval-valued intuitionistic fuzzy MADM method is investigated. Similarity measure is mainly used to measure the discrimination information. The concept of similarity measure for FSs was introduced by Liu [22]. Beliakov et al. [23] constructed a series of similarity measures for IFSs, and developed a new approach for intuitionistic fuzzy MADM problems. Based on cotangent function, Ye [24] proposed two cotangent similarity measures for SVNSs. In order to obtain the ranking order of the alternatives, Ye [25] also proposed three vector similarity measures for SVNSs by utilizing Jaccard, Dice, and cosine similarity measures in vector space. With respect to the drawbacks of similarity measure [25], Ye [26] constructed the modified cosine similarity measures for SVNSs to deal with single-valued neutrosophic MADM model, and applied to medical diagnosis problems.

It is known that how to design entropy and similarity measure to cope with uncertainty and vagueness are two challenging and significant issues [27]. Therefore, just similar to other fuzzy information environment, it is necessary to study the axiomatic notion of single-valued neutrosophic entropy and similarity measures. Recently, although Majumdar and Samant [27] introduced the axiomatic requirements for single-valued neutrosophic entropy, it might be having some drawbacks in some situations (details given in Example 1). In addition, under the single-valued neutrosophic information environment, there are few studies focused on the relationship between entropy and similarity measure. Therefore, the following research issues are studied in this paper:

(a) The axiomatic definitions of entropy and similarity measure for SVNSs are introduced;

(b) Based on trigonometric functions, some information measure formulas are constructed;

(c) The relationship between entropy and similarity measure for SVNSs is analyzed;

(d) We develop a single-valued neutrosophic MADM method, and apply to investment evaluation problem.

In order to do so, the rest of this paper is organized as follows. Section 2 reviews some basic concepts of SVNSs. In Section 3, the axiomatic notions of entropy and similarity measure for SVNSs are presented, and several measure formulas are constructed. In this section, we also analyze the relationship between entropy and similarity measure for SVNSs. Section 4 develops a new single-valued neutrosophic MADM method. In Section 5, a numerical example is provided to illustrate the application of the developed method. Conclusions and further research are contained in Section 6.

## 2. Preliminaries

Some basic concepts related to SVNSs are introduced .

**Definition 1** [11]. Let $X$ be a fixed set, with a generic element in $X$ denoted by $x$. A NS $A \subseteq X$ is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity- membership function $F_A(x)$, where $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $]^-0,1^+[$, such that $T_A(x)$: $X \to ]^-0,1^+[, I_A(x): X \to ]^-0,1^+[$ and $F_A(x): X \to ]^-0,1^+[$, and the sum of $T_A(x), I_A(x)$ and $F_A(x)$ satisfies the condition $^-0 \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$.

In order to apply NS in science and engineering applications, Wang et al. [13] presented the concept of SVNSs, which is a special case of NSs.

**Definition 2** [13]. Let $X$ be a fixed set, with a generic element in $X$ denoted by $x$. A SVNS $A$ in the universe of discourseis $X$ is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$, then a SVNS $A$ can be defined as $A=\{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x\in X\}$, where $T_A(x), I_A(x), F_A(x)\in[0,1]$, and $T_A(x)+I_A(x)+F_A(x)\in[0,3]$.

For convenience, we refer to $\alpha=\langle T_\alpha, I_\alpha, F_\alpha\rangle$ as a single-valued neutrosophic value (SVNV), which is a basic unit of SVNS. Let $\tilde{\Omega}$ be the set of all the SVNVs in $X$.

**Definition 3** [28]. Let $\alpha=\langle T_\alpha, I_\alpha, F_\alpha\rangle$ be a SVNV, then the complement of $\alpha$ denotes $\alpha^c=\langle 1-T_\alpha, 1-I_\alpha, 1-F_\alpha\rangle$.

Let $\alpha=\langle T_\alpha, I_\alpha, F_\alpha\rangle \triangleq \langle \alpha_1,\alpha_2,\alpha_3\rangle$, then $\alpha^c=\langle 1-\alpha_1, 1-\alpha_2, 1-\alpha_3\rangle$, i.e., $\alpha_t^c=1-\alpha_t$, $t=1,2,3$.

For a SVNV $\alpha=\langle \alpha_1,\alpha_2,\alpha_3\rangle$, Majumdar and Samanta [27] axiomatized single-valued neutrosophic entropy measure.

**Definition 4** [27]. The entropy of a SVNS $\alpha=\langle \alpha_1,\alpha_2,\alpha_3\rangle$ is a function $\varepsilon:\alpha\to[0,1]$ which satisfies the following axioms:

(i) $\varepsilon(\alpha)=0$ if $\alpha$ is a crisp number;

(ii) $\varepsilon(\alpha)=1$ if $\langle \alpha_1,\alpha_2,\alpha_3\rangle=\langle 0.5,0.5,0.5\rangle$;

(iii) $\varepsilon(\alpha)=\varepsilon(\alpha^c)$;

(iv) $\varepsilon(\alpha)\ge\varepsilon(\beta)$, if $\alpha$ more uncertain than $\beta$, i.e.,

$$\alpha_1+\alpha_3\le\beta_1+\beta_3 \text{ and } |\alpha_2-\alpha_2^c|\le|\beta_2-\beta_2^c|.$$

However, one can find some drawbacks of Definition 4 in some situations. This is demonstrated in Example 1.

**Example 1.** Suppose that $\alpha=\langle 1,0,0\rangle$ and $\beta=\langle 0.5,0,0.6\rangle$ are two SVNVs, then $\alpha^c=\langle 0,1,1\rangle$ and $\beta^c=\langle 0.5,1,0.4\rangle$. According to the axiomatic requirement (iv) in Definition 4, since $\alpha_1+\alpha_3=1+0=1<1.1=0.5+0.6=\beta_1+\beta_3$ and $|\alpha_2-\alpha_2^c|=1=|\beta_2-\beta_2^c|$, which indicates that $\alpha$ is more uncertain than $\beta$. However, $\alpha=\langle 1,0,0\rangle$ is a crisp number, it means that the entropy of $\alpha$ is $\varepsilon(\alpha)=0$ and $\alpha$ less uncertain than $\beta$. Thus, the contradiction exists in the Definition 4, and Definition 4 is unreasonable in this case.

Therefore, to circumvent the aforesaid drawbacks of definition in [27], the concept of entropy for SVNVs needs to be improved.

## 3 Single-valued neutrosophic entropy and similarity measure

In this section, we present the axiomatic notions of single-valued neutrosophic entropy and similarity measure, and then construct two information measure formulas based on sine function and cosine function. The relationship between the single-valued neutrosophic entropy and similarity measure is also investigated.

### 3.1. Single-valued neutrosophic entropy

**Definition 5.** Assume that $\alpha = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ is a SVNV. A function $E : \tilde{\Omega} \to [0,1]$ is called a single-valued neutrosophic entropy on $\alpha$, if it satisfies the following axiomatic requirements:

(E1) $E(\alpha) = 0 \Leftrightarrow \alpha_t = 0$ or $\alpha_t = 1, t = 1,2,3$;

(E2) $E(\alpha) = 1 \Leftrightarrow \langle \alpha_1, \alpha_2, \alpha_3 \rangle = \langle 0.5, 0.5, 0.5 \rangle$;

(E3) $E(\alpha) = E(\alpha^c)$;

(E4) $E(\alpha) \le E(\beta)$, if $\beta$ more uncertain than $\alpha$, i.e.,

$\alpha_t \le \beta_t$ when $\beta_t - \beta_t^c \le 0, t = 1,2,3$, or $\alpha_t \ge \beta_t$ when $\beta_t - \beta_t^c \ge 0, t = 1,2,3$.

With the help of sine function and cosine function, we present an information measure formula for SVNV $\alpha$ as follows:

$$E_1(\alpha) = \frac{1}{3(\sqrt{2}-1)} \sum_{t=1}^{3} \left( \sin \frac{\alpha_t - \alpha_t^c + 1}{4}\pi + \cos \frac{\alpha_t - \alpha_t^c + 1}{4}\pi - 1 \right). \tag{1}$$

**Theorem 1.** For a SVNV $\alpha = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$, the mapping $E_1(\alpha)$, defined by Eq. (1), satisfy the axiomatic requirements (E1-E4) in Definition 5.

**Proof.** Let $f(x) = \frac{1}{\sqrt{2}-1} \left( \sin \frac{x}{4}\pi + \cos \frac{x}{4}\pi - 1 \right), x \in [0,2]$, then

$$f'(x) = \frac{df(x)}{dx} = \frac{\pi}{4(\sqrt{2}-1)} \left( \cos \frac{x}{4}\pi - \sin \frac{x}{4}\pi \right) = \frac{\sqrt{2}\pi}{4(\sqrt{2}-1)} \cos \frac{x+1}{4}\pi, x \in [0,2]. \tag{2}$$

It is easy know that $f'(x) \ge 0, x \in [0,1]$ and $f'(x) \le 0, x \in [1,2]$, thus, if $x \in [0,1]$, $f(x)$ is monotonically increasing function; if $x \in [1,2]$, $f(x)$ is monotonically decreasing function. In addition, we can obtain that $0 \le f(x) \le 1$, and $f_{\min}(x) = 0 \Leftrightarrow x = 0$ or $x = 2$; $f_{\max}(x) = 1 \Leftrightarrow x = 1$.

(E1) Assume that $E_1(\alpha)=0$. Since $0 \le \alpha_t \le 1, t=1,2,3$, then $\alpha_t - \alpha_t^c + 1 = \alpha_t - (1-\alpha_t) + 1 = 2\alpha_t \in [0,2]$. Therefore, from the above analysis, we have $0 \le E_1(\alpha) \le 1$, and $E_1(\alpha)=0$ if and only if $\alpha_t - \alpha_t^c + 1 = 2\alpha_t = 0$ or $\alpha_t - \alpha_t^c + 1 = 2\alpha_t = 2$, i.e., $E_1(\alpha)=0$ if and only if $\alpha_t = 0$ or $\alpha_t = 1, t=1,2,3$. On the other hand, if $\alpha_t = 0$ or $\alpha_t = 1, t=1,2,3$, then we have $\alpha_t - \alpha_t^c + 1 = 0$ or $2$, thus $E_1(\alpha)=0$.

(E2) Since $\alpha_t - \alpha_t^c + 1 = \alpha_t - (1-\alpha_t) + 1 = 2\alpha_t \in [0,2]$, then according to the above analysis, we obtain that $E_1(\alpha)=1 \Leftrightarrow \alpha_t - \alpha_t^c + 1 = 1, t=1,2,3$, i.e., $E_1(\alpha)=1 \Leftrightarrow \alpha_t = 0.5, t=1,2,3$.

(E3) As $\alpha_t^c = 1-\alpha_t, t=1,2,3$, then $(\alpha_t^c)^c = \alpha_t, t=1,2,3$, it follows that

$$\begin{aligned}
E_1(\alpha^c) &= \frac{1}{3(\sqrt{2}-1)} \sum_{t=1}^{3} \left( \sin \frac{\alpha_t^c - (\alpha_t^c)^c + 1}{4}\pi + \cos \frac{\alpha_t^c - (\alpha_t^c)^c + 1}{4}\pi - 1 \right) \\
&= \frac{1}{3(\sqrt{2}-1)} \sum_{t=1}^{3} \left( \sin \frac{\alpha_t^c - \alpha_t + 1}{4}\pi + \cos \frac{\alpha_t^c - \alpha_t + 1}{4}\pi - 1 \right) \\
&= \frac{1}{3(\sqrt{2}-1)} \sum_{t=1}^{3} \left( \sin \frac{2-(\alpha_t - \alpha_t^c + 1)}{4}\pi + \cos \frac{2-(\alpha_t - \alpha_t^c + 1)}{4}\pi - 1 \right) \\
&= \frac{1}{3(\sqrt{2}-1)} \sum_{t=1}^{3} \left( \cos \frac{\alpha_t - \alpha_t^c + 1}{4}\pi + \sin \frac{\alpha_t - \alpha_t^c + 1}{4}\pi - 1 \right) = E_1(\alpha).
\end{aligned} \tag{3}$$

(E4) Suppose that $\alpha_t \le \beta_t$ when $\beta_t - \beta_t^c \le 0, t=1,2,3$, then we have $\beta_t \le \beta_t^c = 1-\beta_t, t=1,2,3$, thus $\alpha_t \le \beta_t \le 0.5, t=1,2,3$. While $\alpha_t - \alpha_t^c + 1 = 2\alpha_t, \beta_t - \beta_t^c + 1 = 2\beta_t$, then

$$0 \le \alpha_t - \alpha_t^c + 1 \le \beta_t - \beta_t^c + 1 \le 1, t=1,2,3. \tag{4}$$

From the above analysis, we know that $f(x)$ is monotonically increasing for $x \in [0,1]$. Therefore, $E_1(\alpha) \le E_1(\beta)$.

Similarly, if $\alpha_t \ge \beta_t$ when $\beta_t - \beta_t^c \ge 0, t=1,2,3$, we can get that $E_1(\alpha) \le E_1(\beta)$. Thus, we complete the proof of Theorem 1. □

**Definition 6.** For a SVNV $\alpha = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$, then $E_1(\alpha)$, defined by Eq. (1), is called the single-valued neutrosophic entropy of SVNV $\alpha$.

### 3.2. Single-valued neutrosophic similarity measure

**Definition 7.** Assume that $\alpha = \langle \alpha_1, \alpha_2, \alpha_3 \rangle$ and $\beta = \langle \beta_1, \beta_2, \beta_3 \rangle$ are two SVNVs. A real-valued function $S: \tilde{\Omega} \times \tilde{\Omega} \to [0,1]$ is called a single-valued neutrosophic similarity measure between $\alpha$ and $\beta$, if it satisfies the following axiomatic requirements:

(S1) $S(\alpha,\beta)=0 \Leftrightarrow \alpha_t-\beta_t=1$ or $\beta_t-\alpha_t=1, t=1,2,3$;

(S2) $S(\alpha,\beta)=1 \Leftrightarrow \langle\alpha_1,\alpha_2,\alpha_3\rangle=\langle\beta_1,\beta_2,\beta_3\rangle$;

(S3) $S(\alpha,\beta)=S(\beta,\alpha)$;

(S4) $S(\alpha,\gamma)\le S(\alpha,\beta), S(\alpha,\gamma)\le S(\beta,\gamma)$, if $\alpha_t\le\beta_t\le\gamma_t$ or $\alpha_t\ge\beta_t\ge\gamma_t, t=1,2,3$.

Similarly, based on the trigonometric function, we construct an information measure formula for SVNVs $\alpha=\langle\alpha_1,\alpha_2,\alpha_3\rangle$ and $\beta=\langle\beta_1,\beta_2,\beta_3\rangle$ as follows:

$$S_1(\alpha,\beta)=\frac{1}{3(\sqrt{2}-1)}\sum_{t=1}^{3}\left(\sqrt{2}\sin\frac{\alpha_t-\beta_t+2}{4}\pi-1\right). \tag{5}$$

**Theorem 2.** Assume that there are two SVNVs $\alpha=\langle\alpha_1,\alpha_2,\alpha_3\rangle$ and $\beta=\langle\beta_1,\beta_2,\beta_3\rangle$. The mapping $S_1(\alpha,\beta)$, defined by Eq. (5), satisfy the axiomatic requirements (S1-S4) in Definition 7.

**Proof.** Let $g(x)=\frac{1}{\sqrt{2}-1}\left(\sqrt{2}\sin\frac{x+2}{4}\pi-1\right), x\in[-1,1]$, then

$$g'(x)=\frac{dg(x)}{dx}=\frac{\sqrt{2}\pi}{4(\sqrt{2}-1)}\cos\frac{x+2}{4}\pi, x\in[-1,1]. \tag{6}$$

It is know that $g'(x)\ge 0, x\in[-1,0]$ and $g'(x)\le 0, x\in[0,1]$. Therefore, when $x\in[-1,0]$, $g(x)$ is monotonically increasing function; when $x\in[0,1]$, $g(x)$ is monotonically decreasing function. In addition, we can obtain that $0\le g(x)\le 1$, and $g_{\min}(x)=0\Leftrightarrow x=-1$ or $x=1$; $g_{\max}(x)=1\Leftrightarrow x=0$.

(S1) As $0\le\alpha_t\le\beta_t\le 1, t=1,2,3$, then $-1\le\alpha_t-\beta_t\le 1, t=1,2,3$, thus $0\le S_1(\alpha,\beta)\le 1$ and every term in the summation of $S_1(\alpha,\beta)$ is non-negative. Therefore, according to the above analysis, one can obtain that

$$S_1(\alpha,\beta)=0\Leftrightarrow g(\alpha_t-\beta_t)=0, t=1,2,3\Leftrightarrow \alpha_t-\beta_t=-1 \text{ or } \alpha_t-\beta_t=1, t=1,2,3, \tag{7}$$

i.e.,

$$S(\alpha,\beta)=0\Leftrightarrow\alpha_t-\beta_t=1 \text{ or } \beta_t-\alpha_t=1, t=1,2,3.$$

(S2) From the above analysis, it is easy to obtain that

$$S_1(\alpha,\beta)=1\Leftrightarrow g(\alpha_t-\beta_t)=1, t=1,2,3\Leftrightarrow\alpha_t-\beta_t=0\Leftrightarrow\alpha_t=\beta_t, t=1,2,3. \tag{8}$$

(S3) Because $\sin\left(\frac{\pi}{2}-x\right)=\sin\left(\frac{\pi}{2}+x\right), \forall x\in R$, then

$$S_1(\alpha,\beta)=\frac{1}{3(\sqrt{2}-1)}\sum_{t=1}^{3}\left(\sqrt{2}\sin\frac{\alpha_t-\beta_t+2}{4}\pi-1\right)$$

$$=\frac{1}{3(\sqrt{2}-1)}\sum_{t=1}^{3}\left(\sqrt{2}\sin\left(\frac{\pi}{2}+\frac{\alpha_t-\beta_t}{4}\pi\right)-1\right)$$
$$=\frac{1}{3(\sqrt{2}-1)}\sum_{t=1}^{3}\left(\sqrt{2}\sin\left(\frac{\pi}{2}-\frac{\alpha_t-\beta_t}{4}\pi\right)-1\right)$$
$$=\frac{1}{3(\sqrt{2}-1)}\sum_{t=1}^{3}\left(\sqrt{2}\sin\frac{\beta_t-\alpha_t+2}{4}\pi-1\right)=S_1(\beta,\alpha). \quad (9)$$

(S4) If $\alpha_t \le \beta_t \le \gamma_t, t=1,2,3$, then

$$-1\le \alpha_t-\gamma_t \le \beta_t-\gamma_t \le 0, -1\le \alpha_t-\gamma_t \le \alpha_t-\beta_t \le 0, t=1,2,3.$$

While $g(x)$ is monotonically increasing for $x\in[-1,0]$, it follows that

$$S_1(\alpha,\gamma)\le S_1(\alpha,\beta), S_1(\alpha,\gamma)\le S_1(\beta,\gamma).$$

Similarly, if $\alpha_t \ge \beta_t \ge \gamma_t, t=1,2,3$, we also can obtain that $S_1(\alpha,\gamma)\le S_1(\alpha,\beta)$, $S_1(\alpha,\gamma)\le S_1(\beta,\gamma)$. This completes the proof of Theorem 2. □

**Definition 8.** For two SVNVs $\alpha=\langle\alpha_1,\alpha_2,\alpha_3\rangle$ and $\beta=\langle\beta_1,\beta_2,\beta_3\rangle$, then $S_1(\alpha,\beta)$, defined by Eq. (5), is called the single-valued neutrosophic similarity measure between $\alpha$ and $\beta$.

### 3.3. The relationship between single-valued neutrosophic entropy and similarity measure

In this subsection, the relationship between the single-valued neutrosophic entropy and similarity measure is discussed in details.

**Theorem 3.** Suppose that $\alpha$ is a SVNV, then $E(\alpha)=S(\alpha,\alpha^c)$.

**Proof**. In what follows, we verify that $S(\alpha,\alpha^c)$ satisfy the requirements (E1)-(E4) listed in Definition 5.

(E1) $E(\alpha)=0 \Leftrightarrow S(\alpha,\alpha^c)=0 \Leftrightarrow \alpha_t-\alpha_t^c=1$ or $\alpha_t^c-\alpha_t=1, t=1,2,3$, i.e.,

$$\alpha_t-(1-\alpha_t)=1 \text{ or } (1-\alpha_t)-\alpha_t=1, t=1,2,3. \quad (10)$$

Hence, Eq. (10) holds, if and only if $\alpha_t=0$ or $\alpha_t=1, t=1,2,3$.

(E2) $E(\alpha)=1 \Leftrightarrow S(\alpha,\alpha^c)=1 \Leftrightarrow \langle\alpha_1,\alpha_2,\alpha_3\rangle=\langle\alpha_1^c,\alpha_2^c,\alpha_3^c\rangle$

$$\Leftrightarrow \langle\alpha_1,\alpha_2,\alpha_3\rangle=\langle 1-\alpha_1,1-\alpha_2,1-\alpha_3\rangle$$
$$\Leftrightarrow \alpha_t=1-\alpha_t, t=1,2,3 \Leftrightarrow \alpha_t=0.5, t=1,2,3$$
$$\Leftrightarrow \langle\alpha_1,\alpha_2,\alpha_3\rangle=\langle 0.5,0.5,0.5\rangle. \quad (11)$$

(E3) $E(\alpha^c)=S(\alpha^c,(\alpha^c)^c)=S(\alpha^c,\alpha)=S(\alpha,\alpha^c)=E(\alpha)$.

(E4) If $\alpha_t \le \beta_t$ when $\beta_t - \beta_t^c \le 0, t = 1,2,3$, this implies $\beta_t - (1-\beta_t) \le 0, t = 1,2,3$, i.e., $\beta_t \le 1-\beta_t, t = 1,2,3$, then $\alpha_t \le \beta_t \le 1-\beta_t \le 1-\alpha_t, t = 1,2,3$. Hence $\alpha_t \le \beta_t \le \beta_t^c \le \alpha_t^c, t = 1,2,3$. Based on the axiomatic requirement (S4) in Definition 7, we have $S(\alpha,\alpha^c) \le S(\beta,\alpha^c) \le S(\beta,\beta^c)$, i.e., $E(\alpha) \le E(\beta)$.

With the same reason, if $\alpha_t \ge \beta_t$ when $\beta_t - \beta_t^c \ge 0, t = 1,2,3$, we can also prove that $E(\alpha) \le E(\beta)$. This completes the proof of Theorem 3. □

The following corollary can be obtained in accordance with Theorem 3.

**Corollary 1.** Suppose that $\alpha$ is a SVNV, then $E_1(\alpha) = S_1(\alpha,\alpha^c)$.

## 4. MADM method with single-valued neutrosophic entropy and similarity measure

Assume that there is a MADM problem with single-valued neutrosophic information. Let $X = \{X_1, X_2, \cdots, X_m\}$ and $C = \{C_1, C_2, \cdots, C_n\}$ be the set of alternatives and attributes, respectively. Suppose that $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$ is the weight vector of the attributes, where $0 \le w_j \le 1, j = 1,2,\cdots,n, \sum_{j=1}^{n} w_j = 1$. Due to expert's limited knowledge and complexity of practical problems, the information about attribute weights is completely unknown. The DMs evaluate the alternative $X_i$ over the attribute $C_j$ by SVNV $\alpha_{ij} = \langle \alpha_1^{ij}, \alpha_2^{ij}, \alpha_3^{ij} \rangle$, and then all the SVNVs can be constructed as a single-valued neutrosophic decision matrix $D = (\alpha_{ij})_{m\times n}$.

Thus, the MADM method based on the single-valued neutrosophic entropy and similarity measure is described by the following decision steps:

**Step 1** Normalization of the SVN decision matrix $D = (\alpha_{ij})_{m\times n}$ by the following transformation approach:

$$\tilde{\alpha}_{ij} = \begin{cases} \alpha_{ij}, & \textit{for benefit attribute } C_j \\ \alpha_{ij}^c, & \textit{for cost attribute } C_j \end{cases}, i = 1,2,\cdots,m,\ j = 1,2,\cdots,n, \tag{12}$$

Then, we can obtain a normalized single-valued neutrosophic decision matrix $\tilde{D} = (\tilde{\alpha}_{ij})_{m\times n}$.

**Step 2** Calculate the weight vector of attribute $w = (w_1, w_2, \cdots, w_n)^{\mathrm{T}}$. It is known that the entropy of an attribute is smaller across alternatives, and the attribute should be assigned a larger weight [21]. Therefore, the following optimization model can be established to determine the attribute weights:

$$\min E_W = \sum_{j=1}^{n} E(C_j) w_j^2$$

$$s.t. \begin{cases} 0 \le w_j \le 1, j = 1, 2, \cdots, n, \\ \sum_{j=1}^{n} w_j = 1. \end{cases} \tag{13}$$

where $E(C_j) = \frac{1}{m} \sum_{i=1}^{m} E_1(\alpha_{ij})$, and $E_1(\alpha_{ij})$ can be calculated by Eq. (1). By using Lagrange Multiplier Method, one can obtain that

$$w_j = \frac{\left(E(C_j)\right)^{-1}}{\sum_{j=1}^{n} \left(E(C_j)\right)^{-1}}, j = 1, 2, \cdots, n. \tag{14}$$

**Step 3** Let $X^+ = \{\alpha_1^+, \alpha_2^+, \cdots, \alpha_n^+\}$ and $X^- = \{\alpha_1^-, \alpha_2^-, \cdots, \alpha_n^-\}$ be the positive alternative and negative alternative, respectively, where $\alpha_j^+ = \langle 1, 0, 0 \rangle, \alpha_j^- = \langle 0, 1, 1 \rangle, j = 1, 2, \cdots, n$. Then, we calculate the similarity measures between the alternative $X_i$ and $X^+$ or $X^-$ by using

$$S_i^+ = \sum_{j=1}^{n} w_j S_1(\tilde{\alpha}_{ij}, \alpha_j^+), i = 1, 2, \cdots, m, \tag{15}$$

$$S_i^- = \sum_{j=1}^{n} w_j S_1(\tilde{\alpha}_{ij}, \alpha_j^-), i = 1, 2, \cdots, m, \tag{16}$$

where $S_1(\tilde{\alpha}_{ij}, \alpha_j^+)$ and $S_1(\tilde{\alpha}_{ij}, \alpha_j^-)$ can be derived by Eq. (5);

**Step 4** Determine the closeness degree between alternative $X_i$ and the ideal alternatives by the following equations:

$$T_i = \frac{S_i^+}{S_i^+ + S_i^-}, i = 1, 2, \cdots, m. \tag{17}$$

**Step 5** Get the priority of the alternatives $X_i (i = 1, 2, \cdots, m)$ in accordance with $T_i (i = 1, 2, \cdots, m)$, and choose the best alternative that is the one with $\max_i T_i$;

## 5. Illustrative example

Considering the investment problem. An investment company wants to invest a sum of money in the best option [30]. There are five companies can be selected: car company ($X_1$), food company ($X_2$), computer company ($X_3$), game company ($X_4$), arms company ($X_5$). The DMs evaluate these companies by means of four main attributes, i.e., $C_1$: risk, $C_2$: the cost of investment, $C_3$: environmental impact and $C_4$: rate of return. The DMs utilize SVNVs $\alpha_{ij} = \langle \alpha_1^{ij}, \alpha_2^{ij}, \alpha_3^{ij} \rangle$ to express the evaluation information of five companies $X_i (i = 1,2,3,4,5)$ with respect to the above four attributes $C_j (j = 1,2,3,4)$, and then a SVN decision matrix $D = (\alpha_{ij})_{5\times4}$ can be constructed as follows [26]:

$$D = \begin{pmatrix} \langle 0.4,0.6,0.0\rangle & \langle 0.3,0.2,0.5\rangle & \langle 0.1,0.3,0.7\rangle & \langle 0.6,0.7,0.7\rangle \\ \langle 0.7,0.3,0.0\rangle & \langle 0.2,0.2,0.6\rangle & \langle 0.0,0.1,0.9\rangle & \langle 0.9,0.9,0.2\rangle \\ \langle 0.1,0.2,0.7\rangle & \langle 0.2,0.4,0.4\rangle & \langle 0.8,0.2,0.3\rangle & \langle 0.8,0.7,0.4\rangle \\ \langle 0.2,0.1,0.8\rangle & \langle 0.2,0.4,0.5\rangle & \langle 0.8,0.1,0.4\rangle & \langle 0.8,0.8,0.3\rangle \\ \langle 0.3,0.4,0.3\rangle & \langle 0.6,0.3,0.1\rangle & \langle 0.2,0.1,0.7\rangle & \langle 0.8,0.8,0.4\rangle \end{pmatrix}.$$

Now we use the proposed method to select the best company, which is described by the following decision steps:

**Step 1** By using Eq. (12), we convert all the cost attribute values to the benefit attribute values, and then we can obtain the normalized single-valued neutrosophic decision matrix $\tilde{D} = (\tilde{\alpha}_{ij})_{5\times4}$ as follows:

$$\tilde{D} = \begin{pmatrix} \langle 0.6,0.4,1.0\rangle & \langle 0.7,0.9,0.5\rangle & \langle 0.9,0.7,0.3\rangle & \langle 0.6,0.7,0.7\rangle \\ \langle 0.3,0.7,1.0\rangle & \langle 0.8,0.8,0.4\rangle & \langle 1.0,0.9,0.1\rangle & \langle 0.9,0.9,0.2\rangle \\ \langle 0.9,0.8,0.3\rangle & \langle 0.8,0.6,0.6\rangle & \langle 0.2,0.8,0.7\rangle & \langle 0.8,0.7,0.4\rangle \\ \langle 0.8,0.9,0.2\rangle & \langle 0.8,0.6,0.5\rangle & \langle 0.2,0.9,0.6\rangle & \langle 0.8,0.8,0.3\rangle \\ \langle 0.7,0.6,0.7\rangle & \langle 0.4,0.7,0.9\rangle & \langle 0.8,0.9,0.3\rangle & \langle 0.8,0.8,0.4\rangle \end{pmatrix}.$$

**Step 2** By model (13) and Eq. (14), we obtain the attribute weights:

$$w_1 = 0.1923, w_2 = 0.3208, w_3 = 0.1260, w_4 = 0.36.9.$$

**Step 3** Determine the similarity measures between the alternative $X_i$ and $X^+$ or $X^-$ by applying Eqs. (15) and (16):

$$S_1^+ = 0.3220, S_2^+ = 0.4011, S_3^+ = 0.5407, S_4^+ = 0.4617, S_5^+ = 0.3794,$$

$$S_1^- = 0.4619, S_2^- = 0.3894, S_3^- = 0.2915, S_4^- = 0.3533, S_5^- = 0.4308.$$

**Step 4** Computer the closeness degree $T(X_i)(i = 1,2,3,4,5)$ by using Eq. (17):

$$T_1 = 0.4108, T_2 = 0.5074, T_3 = 0.6497, T_4 = 0.5665, T_5 = 0.4683.$$

**Step 5** As $T_3 > T_4 > T_2 > T_5 > T_1$, then the ranking order of the five companies is $X_3 \succ X_4 \succ X_2 \succ X_5 \succ X_1$. Thus, $X_3$ is the best choice among the five companies.

In the following, in order to validate the effectiveness of the proposed MADM method, we compare our proposed method with other existing method proposed by Ye and Fu [29] to select the best company, which is described as follows:

First, based on the decision matrix $\tilde{D}=(\tilde{\alpha}_{ij})_{5\times 4}$, by using the following similarity measure formula in [29]:

$$T(A,B)=1-\frac{1}{n}\sum_{j=1}^{n}\tan\left(\frac{\pi}{4}\cdot\max\left\{\left|T_A(x_j)-T_B(x_j)\right|,\left|I_A(x_j)-I_B(x_j)\right|,\left|F_A(x_j)-F_B(x_j)\right|\right\}\right), \quad (18)$$

we determine the similarity measures $T(X_i,X^+)(i=1,2,3,4,5)$ as follows:

$$T(X_1,X^+)=0.2301, T(X_2,X^+)=0.1413, T(X_3,X^+)=0.3561,$$
$$T(X_4,X^+)=0.2639, T(X_5,X^+)=0.2381.$$

Then, we have $T(X_3,X^+)>T(X_4,X^+)>T(X_1,X^+)>T(X_5,X^+)>T(X_2,X^+)$. Therefore, the ranking of five companies is $X_3\succ X_4\succ X_1\succ X_5\succ X_2$, and the best company is $X_3$.

From the decision results, we can know that although the developed method and that of Ye and Fu [29] produce the same result, there exist a little different ranking of five companies between two methods. Actually, according to the original normalized single-valued neutrosophic decision matrix $\tilde{D}=(\tilde{\alpha}_{ij})_{5\times 4}$, we have $\tilde{\alpha}_{21}<\tilde{\alpha}_{11},\tilde{\alpha}_{22}>\tilde{\alpha}_{12},\tilde{\alpha}_{23}>\tilde{\alpha}_{13},\tilde{\alpha}_{24}>\tilde{\alpha}_{14}$ and $\tilde{\alpha}_{21}<\tilde{\alpha}_{51},\tilde{\alpha}_{22}>\tilde{\alpha}_{52},\tilde{\alpha}_{23}>\tilde{\alpha}_{53},\tilde{\alpha}_{24}>\tilde{\alpha}_{54}$, it implies $X_2$ is better than $X_1$ and $X_5$. Thus, the proposed MADM approach in this paper is more reasonable than that of Ye and Fu [29] in this case.

On the other hand, in the process of decision-making, our method takes all the single-valued neutrosophic information into account, and then the ranking results are determined. However, due to the Hausdorff distance formula is used in [29], in which some middle values are ignored, thus, the decision-making method proposed by Ye and Fu [29] leads to information loss. Therefore, our MADM method can derive the more accurate results.

## 6. Conclusions

In this paper, we present two axiomatic definitions of entropy and similarity measure for single-valued neutrosophic information. Then, based on sine function and cosine function, two information measure formulas are established, including single-valued neutrosophic entropy and similarity measure. The relationship between single-valued neutrosophic entropy and similarity measure is studied. In addition, we develop a novel method to cope with single-valued neutrosophic MADM problems with Lagrange Multiplier Method and closeness degree. Finally, the comparative analysis demonstrated the effectiveness and rationality of the proposed single-valued neutrosophic MADM method. It enriches and develops the single-valued neutrosophic theory and method.

In the further, we will focus on investigate single-valued neutrosophic linguistic information measures and apply the single-valued neutrosophic information measures to solve practical applications in other areas such as pattern recognition, information fusion system, and image processing.

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## Conflicts of Interest

The authors declare no conflict of interest.

## References

[1] Zadeh L A. Fuzzy sets. Information and Control, 1965, 8: 338-353.

[2] Ren P, Xu Z, Gou X. Pythagorean fuzzy TODIM approach to multi-criteria decision making. Applied Soft Computing, 2016, 42: 246-259.

[3] Garg H. A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making. International Journal of Intelligent Systems, 2016, 31(9): 886-920.

[4] Atanassov K. Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 1986, 20: 87-96.

[5] Atanassov K. Two theorems for intuitionistic fuzzy sets. Fuzzy Sets and Systems,.2000, 110: 267-269.

[6] Jin F F, Ni Z W, Chen H Y, et al. Approaches to group decision making with intuitionistic fuzzy preference relations based on multiplicative consistency. Knowledge-Based Systems, 2016, 97: 48-59.

[7] Atanassov K. More on intuitionistic fuzzy sets. Fuzzy Sets and Systems, 1989, 33: 37-46.

[8] Atanassov K, Gargov G. Interval-valued intuitionistic fuzzy sets. Fuzzy Sets and Systems,1989, 31: 343-349.

[9] Li D F. TOPSIS-based nonlinear-programming methodology for multiattribute decision making with interval-valued intuitionistic fuzzy sets. IEEE Transactions on Fuzzy Systems, 2010, 18: 299-311.

[10] Afzali A, Rafsanjani M K, Saeid A B. A fuzzy multi-objective linear programming model based on interval-valued intuitionistic fuzzy sets for supplier selection. International Journal of Fuzzy Systems, 2016, 18(5): 864-874.

[11] Smarandache F. A unifying field in logics. neutrosophy: Neutrosophic probability, set and logic. American Research Press, Rehoboth, 1999.

[12] Smarandache F. A unifying field in logics neutrosophic logic. Neutrosophy, neutrosophic set, neutrosophic probability. American Research Press, 2003

[13] Wang H, Smarandache F, Zhang Y Q, Sunderraman R. Single-valued neutrosophic sets. Multispace and Multistructure, 2010, 4: 410-413.

[14] Jin F F, Ni Z W, Chen H Y, et al. Multiple attribute group decision making based on interval-valued hesitant fuzzy information measures. Computers & Industrial Engineering, 2016, 101: 103-115.

[15] Xu Z S, Xia M M. Hesitant fuzzy entropy and cross-entropy and their use in multi-attribute decision-making. International Journal of Intelligent Systems, 2012, 27: 799-822.

[16] Onar S C, Oztaysi B, Otay İ, Kahraman C. Multi-expert wind energy technology selection using interval-valued intuitionistic fuzzy sets. Energy, 2015, 90: 274-285.

[17] Zadeh L A. Probability measures of fuzzy events. Journal of Mathematical Analysis and Applications 1968, 23: 421-427.

[18] Luca D A, Termini S. A definition of nonprobabilistic entropy in the setting of fuzzy sets theory. Information and Control, 1972, 20: 301-312.

[19] Szmidt E, Kacprzyk J. Entropy for intuitionistic fuzzy sets. Fuzzy Sets and Systems, 2001, 118: 467-477.

[20] Ye J. Multicriteria fuzzy decision-making method using entropy weights-based correlation coefficients of interval-valued intuitionistic fuzzy sets. Applied Mathematical Modelling, 2010, 34: 3864-3870.

[21] Jin F F, Pei L D, Chen H Y, et al. Interval-valued intuitionistic fuzzy continuous weighted entropy and its application to multicriteria fuzzy group decision-making. Knowledge-Based Systems, 2014, 59: 132-141.

[22] Liu X C. Entropy, distance measure and similarity measure of fuzzy sets and their relations. Fuzzy Sets and Systems 1992, 52: 305-318.

[23] Beliakov G, Pagola M, Wilkin T. Vector valued similarity measures for Atanassov's intuitionistic fuzzy sets. Information Sciences, 2014, 280: 352-367.

[24] Ye J. Single-valued neutrosophic similarity measures based on cotangent function and their application in the fault diagnosis of steam turbine. Soft Computing, 2017, 21(3): 817-825.

[25] Ye J. Vector similarity measures of simplified neutrosophic sets and their application in multicriteria decision-making. International Journal of Fuzzy Systems, 2014, 16(2): 204-215.

[26] Ye J. Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses. Artificial intelligence in medicine, 2015, 63(3): 171-179.

[27] Majumdar P, Samanta S K. On similarity and entropy of neutrosophic sets. Journal of Intelligent & Fuzzy Systems, 2014, 26(3): 1245-1252.

[28] Peng J J, Wang J Q, Zhang H Y, Chen X H. An outranking approach for multi-criteria decision-making problems with simplified neutrosophic sets. Applied Soft Computing, 2014, 25: 336-346.

[29] Ye J, Fu J. Multi-period medical diagnosis method using a single-valued neutrosophic similarity measure based on tangent function. Computer Methods and Programs in Biomedicine, 2016, 123: 142-149.

[30] Ye J. An extended TOPSIS method for multiple attribute group decision making based on single valued neutrosophic linguistic numbers. Journal of Intelligent & Fuzzy Systems, 2015, 28(1): 247-255.