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Singular neutrosophic extended triplet groups and generalized groups

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Abstract

Neutrosophic extended triplet group (NETG) is an interesting extension of the concept of classical group, which can be used to express general symmetry. This paper further studies the structural characterizations of NETG. First, some examples are given to show that some results in literature are false. Second, the differences between generalized groups and neutrosophic extended triplet groups are investigated in detail. Third, the notion of singular neutrosophic extended triplet group (SNETG) is introduced, and some homomorphism properties are discussed and a Lagrange-like theorem for finite SNETG is proved. Finally, the following important result is proved: a semigroup is a singular neutrosophic extended triplet group (SNETG) if and only if it is a generalized group.
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1. Introduction and basic concepts

The theory of neutrosophic set was introduced by Smarandache, and it is applied to many fields (see Smarandache, 2005; Ye, 2014; Liu, Khan, Ye, & Mahmood, 2018; Zhang, Bo, Smarandache, & Dai, 2018; Zhang, Bo, Smarandache, & Park, 2018). In recent years, the ideology of neutrosophic set has been applicable in related algebraic structures. In particular, Smarandache and Ali (2018) introduced the notion of neutrosophic triplet group, which is a new extension of the concept of classical group. Now, this new algebraic structure has aroused scholars' interest, and some new research papers have been published one after another (see Smarandache, 2017; Zhang, Smarandache, & Liang, 2017; Jaiyeola and Smarandache, 2018; Smarandache, Şahin, & Kargin, 2018; Ali, Smarandache, & Khan, 2018; Zhang, Hu, Smarandache, & An, 2018). In fact, neutrosophic triplet structures are closely connected with related non-classical logic algebras (see Zhang, Wu, Smarandache, & Hu, 2018; Zhang, 2017; Zhang, Park, & Wu, 2018). In (Smarandache, 2017), the notion of neutrosophic extended triplet group (NETG) was introduced as a generalization of neutrosophic triplet group.

On the other hand, Molaei (Molaei, 1999) introduced the notion of generalized group, as a class of algebras of

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interest in physics. After that, some scholars studied the properties of generalized groups (see (Araujo and Konieczny, 2002; Akinmoyewa, 2009; Adeniran, Akinmoyewa, Solarin, & Jaiyeola, 2011). According to Araujo & Konieczny, 2002), generalized group is equivalent to the notion of completely simple semigroup.

Intuitively, as two generalizations of classical group, the notion of neutrosophic extended triplet group is very close to generalized group. However, the comparative analysis of the two kinds of algebraic structures is far from perfect. This paper will further analyze their connections and differences. First, we recall some basic concepts.

Definition 1 Smarandache, 2017. Let N be a set together with a binary operation $*$. Then, N is called a neutrosophic extended triplet set if for any $a \in N$, there exist a neutral of "a" denoted by $neut(a)$, and an opposite of "a" denoted by $anti(a)$, with $neut(a)$ and $anti(a)$ belonging to N, such that:

$$a * neut(a) = neut(a) * a = a;$$

$$a * anti(a) = anti(a) * a = neut(a).$$

The triple $a, neut(a)$ and $anti(a)$ is referred to as a neutrosophic triplet, and denoted by $(a, neut(a), anti(a))$.

Remark 1. The above definition is a generalization of original definition of a neutrosophic triplet set. For a neutrosophic extended triplet, the neutral of x is allowed to also be equal to the classical identity element as a special case.

Note that, for a neutrosophic triplet set $(N, *)$, $a \in N, anti(a)$ may not be unique. In order not to cause ambiguity, we use the following notations to distinguish:

$anti(a)$: denote any certain one of opposite of a;
$\{anti(a)\}$: denote the set of all opposite of a.

Definition 2 (*Smarandache and Ali, 2018; Smarandache, 2017*). Let $(N, *)$ be a neutrosophic extended triplet set. Then, N is called a neutrosophic extended triplet group, if the following conditions are satisfied:

(1) If $(N, *)$ is well-defined, i.e., for any $a, b \in N$, one has $a * b \in N$.
(2) If $(N, *)$ *is associative, i.e.*, $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$.

N is called a commutative neutrosophic extended triplet group if for all $a, b \in N, a * b = b * a$.

Remark 2. The most prominent character of neutrosophic extended triplet group (NETG), which is different from other algebraic structure features, is "triplet". According to the definition above, just as $a * b = b * a = a$, the b cannot be called a neutral element of a. It is only when there exists c, at the same time, such that $a * c = c * a = b$, the b can be called neutral element of a. Therefore, "neutral element" and "identity element" are two different concepts.

Here are some other related notions: let G be a non-empty set, define a binary operation $*$ on G. If $x * y \in G$, $\forall x, y \in G, (G, *)$ is called a groupoid. If the equations $a * x = b$ and $y * a = b$ have unique solutions relative to x and y respectively, then $(G, *)$ is called a quasigroup.

Definition 3 (*Molaei, 1999; Akinmoyewa, 2009*). A generalized group $(G, *)$ is a non-empty set admitting a binary operation $*$ called multiplication subject to the set of rules given below:

(i) $(x * y) * z = x * (y * z)$ for all $x, y, z \in G$.
(ii) For each $x \in G$, there exists a unique $e(x) \in G$ such that $x * e(x) = e(x) * x = x$.
(iii) For each $x \in G$, there exists $x^{-1} \in G$ such that $x * x^{-1} = x^{-1} * x = e(x)$.

Definition 4 (*Akinmoyewa, 2009; Adeniran et al., 2011*). Let $(G, *)$ be a generalized group. If $e(x * y) = e(x) * e(y)$ for all $x, y \in G$, then G is called normal generalized group.

Theorem 1 (*Araujo and Konieczny, 2002; Akinmoyewa, 2009; Adeniran et al., 2011*). *For each element x in a generalized group $(G, *)$, there exists a unique $x^{-1} \in G$.*

Theorem 2 (*Araujo and Konieczny, 2002; Akinmoyewa, 2009; Adeniran et al., 2011*). *Let $(G, *)$ be a generalized group. If $x * y = y * x$ for all $x, y \in G$, then G is a group.*

2. Some counterexamples on neutrosophic extended triplet groups

For a neutrosophic extended triplet group, the Ref. Jaiyeola and Smarandache (2018) gives some important research directions, but there were some errors. In this section, some counterexamples will be constructed.

Example 1. Let $X = \{a, b, c, d\}$. The operation $*$ on X is defined as Table 1. Then, $(X, *)$ is a non-commutative neutrosophic extended triplet group, and

$$neut(a) = a, anti(a) = a; neut(b) = b, anti(b) = b;$$

$$neut(c) = c, anti(c) = c; neut(d) = d, \{anti(d)\} = \{c, d\}.$$

(1) Obviously, each element x in X has a unique $neut(x)$, but $(X, *)$ is not a generalized group, since $d * d = d$, $c * d = d * c = d$. Thus, the condition in Definition 3 (ii) is not satisfied for $(X, *)$. It follows

Table 1
Neutrosophic extended triplet group $(X, *)$.

*	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	d	d	c	d
d	d	d	d	d

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that Lemma 1 (2) in (Jaiyeola & Smarandache, 2018) is not true.

(2) Since $\{anti(anti(d))\} = \{c, d\}$, so putting $anti(anti(d)) = c \in \{anti(anti(d))\}$, then $anti(anti(d)) \neq d$. This means that Theorem 1 in (Jaiyeola & Smarandache, 2018) is not true.

(3) Let $H = \{a, b, d\}$, then $(H, *)$ is a neutrosophic extended triplet group, that is, $(H, *)$ is a neutrosophic extended triplet subgroup of $(X, *)$. Consider $d \in H, c \in \{anti(d)\}$ but $c \notin H$. It follows that Lemma 2 (iii) in (Jaiyeola & Smarandache, 2018) is not true.

(4) Let $Y = \{1, 2, 3\}$. The operation $*$ on Y is defined as Table 2. Then, $(Y, *)$ is a non-commutative neutrosophic extended triplet group, and

$$neut(1) = 1, anti(1) = 1; neut(2) = 2, anti(2) = 2;$$
$$neut(3) = 3, \{anti(3)\} = \{2, 3\}.$$

Denote $f : X \to Y; a \mapsto 1, b \mapsto 1, c \mapsto 2, d \mapsto 3$. Then f is a homomorphism.

Putting

$$anti(d) = c \in \{anti(d)\},\ anti(3) = 3 \in \{anti(3)\},$$

Then

$$f(anti(d)) = f(c) = 2 \neq 3 = anti(3) = anti(f(d))$$

This means that Theorem 5 (2) in (Jaiyeola & Smarandache, 2018) is not true. Moreover, according to Definition 6 in (Jaiyeola & Smarandache, 2018) (about X_a, $\ker f_a$, $\ker f$), we have

$$X_a = \{a\},\ \ker f_a = \{a, b\};\ X_b = \{b\},\ \ker f_b = \{a, b\};$$
$$X_c = \{c\},\ \ker f_c = \{c\};\ X_d = \{d\},\ \ker f_d\{d\};$$
$$\ker f = \{a, b, c, d\}.$$

Thus,

$$|X_a| = 1 \neq 2 = [X_a : \ker f_a] \times |\ker f_a|,$$
$$|X_b| = 1 \neq 2 = [X_b : \ker f_b] \times |\ker f_b|;$$
$$\sum_{a \in X} [Xa : \ker f_a] \cdot |\ker f_a|$$
$$= 1 \times 2 + 1 \times 2 + 1 \times 1 + 1 \times 1 = 6.$$

Therefore,

$$|X| < \sum\nolimits_{a \in X} [X_a : \ker f_a] \cdot |\ker f_a|$$

It follows that Thereom 6 (6) and (8) in (Jaiyeola & Smarandache, 2018) are not true.

Table 2
Neutrosophic extended triplet group $(Y, *)$.

*	1	2	3
1	1	1	1
2	3	2	3
3	3	3	3

Moreover, in the proof of Thereom 6 (6) in (Jaiyeola & Smarandache, 2018), an assertion is used: if X is finite, $|\ker f_a| = |c * \ker f_a|$ for all c in X_a. In fact, it is not true, since in this example, $|\ker f_a| = 2 \neq 1 = |a * \ker f_a|$.

Example 2. Let $X = \{a, b, c, d, e, f\}$. The operation $*$ on X is defined as Table 3. Then, $(X, *)$ is a non-commutative neutrosophic extended triplet group, and

$$neut(a) = a, \{anti(a)\} = \{a, c, d, e, f\}; neut(b) = b,$$
$$anti(b) = b; neut(c) = c, anti(c) = c; neut(d) = c,$$
$$anti(d) = d; neut(e) = e, anti(e) = e; neut(f) = e,$$
$$anti(f) = f.$$

(1) Obviously, each element $x$in X has a unique $neut(x)$, but $(X, *)$ is not a generalized group, since $a * a = a, c * a = a * c = a, d * a = a * d = a, e * a = a * e = a, f * a = a * f = a$. So, the condition in Definition 3 (ii) is not satisfied for $(X, *)$. It follows that Lemma 1 (2) in (Jaiyeola & Smarandache, 2018) is not true.

(2) Since $\{anti(anti(a))\} = \{a, c, d, e, f\}$, then putting $anti(anti(a)) = c \in \{anti(anti(a))\}$, we get $anti(anti(a)) \neq a$. This means that Theorem 1 in (Jaiyeola & Smarandache, 2018) is not true.

(3) Let $H = \{a, b, c, d\}$, then $(H, *)$ is a neutrosophic extended triplet group, that is, $(H, *)$ is a neutrosophic extended triplet subgroup of $(X, *)$. Consider $a \in H, e \in \{anti(a)\}$ but $e \notin H$. It follows that Lemma 2 (iii) in (Jaiyeola & Smarandache, 2018) is not true.

(4) According to Definition 6 in (Jaiyeola & Smarandache, 2018) (about X_a), we have $X_a = \{a\}$. Consider $d \in \{anti(a)\}$, we have $anti(a * a) = anti(a) = d \neq c = d * d = anti(a) * anti(a)$. It follows that Theorem 6 (4) in (Jaiyeola & Smarandache, 2018) is not true.

(5) Let $Y = \{1, 2, 3, 4\}$. The operation $*$ on Y is defined as Table 4. Then, $(Y, *)$ is a non-commutative neutrosophic extended triplet group, and

$$neut(1) = 1, \{anti(1)\} = \{1, 3, 4\}; neut(2) = 2, anti(2) = 2;$$
$$neut(3) = 3, anti(3) = 3; neut(4) = 4, anti(4) = 4.$$

Denote $f : X \to Y; a \mapsto 1, b \mapsto 2, c \mapsto 3, d \mapsto 3, e \mapsto 4, f \mapsto 4$. Then f is a homomorphism. Putting $anti(a) = a \in \{anti(a)\}$, and $anti(1) = 3 \in \{anti(1)\}$, then $f(anti(a)) = f(a) = 1 \neq 3 = anti(1) = anti(f(a))$. This means that Theorem 5 (2) in (Jaiyeola & Smarandache, 2018) is not true. Moreover, according to Definition 6 in (Jaiyeola & Smarandache, 2018) (about X_a, $\ker f_a$, $\ker f$), we have

Table 3
Neutrosophic extended triplet group $(X, *)$.

*	a	b	c	d	e	f
a	a	b	a	a	a	a
b	a	b	a	a	a	a
c	a	b	c	d	a	a
d	a	a	d	c	a	a
e	a	b	a	a	e	f
f	a	b	a	a	f	e

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Table 4
Neutrosophic extended triplet group $(Y, *)$.

*	1	2	3	4
1	1	2	1	1
2	1	2	1	1
3	1	2	3	1
4	1	2	1	4

$X_a = \{a\}$, $\ker f_a = \{a\}$; $X_b = \{b\}$, $\ker f_b = \{b\}$;
$X_c = \{c, d\}$, $\ker f_c = \{c, d\}$;
$X_d = \{d, c\}$, $\ker f_d = \{d, c\}$; $X_e = \{e, f\}$, $\ker f_e = \{e, f\}$;
$X_f = \{f, e\}$, $\ker f_f = \{f, e\}$; $\ker f = \{a, b, c, d, e, f\}$.

Thus,

$$\sum_{a \in X} [X_a : \ker f_a] \cdot |\ker f_a| = $$
$$= 1 \times 1 + 1 \times 1 + 1 \times 2 + 1 \times 2 + 1 \times 2 + 1 \times 2$$
$$= 10.$$

Therefore, $|X| < \sum_{a \in X} [X_a : \ker f_a] \cdot |\ker f_a|$. It follows that Theorem 6 (6) and (8) in (Jaiyeola & Smarandache, 2018) are not true.

3. The differences between neutrosophic extended triplet groups and generalized groups

In this section, the differences between neutrosophic extended triplet group (NETG) and generalized group (GG) are summarized.

Note 1. For a neutrosophic extended triplet group $(X, *)$, the opposite element of an element x in X may not be unique (see Example 1 and Example 2). But, for each element x in a generalized group $(G, *)$, there exists a unique $x^{-1} \in G$ (see Theorem 1).
Note 2. For a generalized group $(G, *)$, if x, a in G such that $a * x = x * a = x$, then $a = x^{-1}$. But, for a neutrosophic extended triplet group $(X, *)$, if x, a in X such that $a * x = x * a = x$, we cannot get that $a = neut(x)$ in general (see Remark 2, and in Example 2, $c * a = a * c = a$, but $c \neq neut(a) = a$).
Note 3. For a generalized group $(G, *)$, if $a$in G, then $e(a^{-1}) = e(a)$, see (Araujo and Konieczny, 2002; Akinmoyewa, 2009; Adeniran et al., 2011). But, for a neutrosophic extended triplet group $(X, *)$, if a in X, we cannot get that $neut(anti(a)) = neut(a)$ in general (in Example 1, putting $anti(d) = c$, we have $neut(anti(d)) = neut(c) = c \neq neut(d) = d$).
Note 4. There exists some commutative neutrosophic extended triplet groups which are not classical groups. But, every commutative generalized group is a classical group (see Theorem 2).
Note 5. For a generalized group $(G, *)$, if a in G, then $(a^{-1})^{-1} = a$, see Theorem 3.1 in (Adeniran, Akinmoyewa, Solarin, & Jaiyeola, 2011). But, for a neutrosophic extended triplet group $(X, *)$, if a in X, we cannot get that $anti(anti(a)) = a$ in general (see Example 1 (2) and Example 2 (2)).
Note 6. Let $f : G \to H$ be a homomorphism where G and H are two distinct generalized groups, then $f(a^{-1}) = (f(a))^{-1}$ for all a in G. But, for neutrosophic extended triplet groups, $f(anti(a)) \neq anti(f(a))$ in general (see Example 1 (4) and Example 2 (5)).

Definition 5 Araujo & Konieczny, 2002. A semigroup $(S, *)$ is said to be completely simple if it satisfies the following conditions:

$(C1) S * a * S = S$ for every $a \in S$.
$(C2)$ if$e, f \in G$ are idempotents such that $e * f = f * e = e$, then $e = f$.

Theorem 3 Araujo & Konieczny, 2002. *Let $(S, *)$ be a semigroup. Then the following are equivalent*:

(1) $(S, *)$ is completely simple;
(2) $(S, *)$ is a generalized group.

The following example shows that there exists neutrosophic extended triplet group in which the conditions (C1) and (C2) are not satisfied.

Example 3. Let $X = \{1, 2, 3, 4\}$. The operation $*$ on X is defined as Table 5. Then, $(X, *)$ is a non-commutative neutrosophic extended triplet group, and

$neut(1) = 1, anti(1) = 1; neut(2) = 2, anti(2) = 2;$
$neut(3) = 3, anti(3) = 3; neut(4) = 4, \{anti(4)\} = \{1, 2, 4\}.$
$X * 4 * X = \{3, 4\} \neq X;$
$4 * 4 = 4, 2 * 2 = 2, 4 * 2 = 2 * 4 = 4,$ but $4 \neq 2.$

Note 7. Every generalized group is a completely simple semigroup, but there exists some neutrosophic extended triplet groups which are not completely simple semigroups. In fact, a generalized group is a special type of neutrosophic extended triplet group. Thus, neutrosophic extended triplet group is a generalization of generalized group.

Table 5
Neutrosophic extended triplet group $(X, *)$.

*	1	2	3	4
1	1	1	4	4
2	2	2	4	4
3	3	3	3	3
4	4	4	4	4

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4. Singular neutrosophic extended tripet groups

Theorem 4 Zhang, Hu, Smarandache, & An, 2018. *Let* $(N, *)$ *be a neutrosophic extended triplet group. Then*

(1) $neut(a)$ is unique for any a in N.
(2) $neut(a) * neut(a) = neut(a)$ for any a in N.

Theorem 5 Zhang, Hu et al., 2018. *Let* $(N, *)$ *be a neutrosophic extended triplet group. Then* $\forall a \in N, \forall anti(a) \in \{anti(a)\}$,

(1) $neut(a) * p = q * neut(a)$, for any $p, q \in \{anti(a)\}$;
(2) $neut(neut(a)) = neut(a)$;
(3) $anti(neut(a)) * anti(a) \in \{anti(a)\}$;
(4) $neut(a * a) * a = a * neut(a * a) = a$;
$neut(a * a) * neut(a) = neut(a) * neut(a * a) = neut(a)$;
(5) $neut(anti(a)) * a = a * neut(anti(a)) = a$;
$neut(anti(a)) * neut(a) = neut(a) * neut(anti(a)) = neut(a)$;
(6) $anti(neut(a)) * a = a * anti(neut(a)) = a$, for any $anti(neut(a)) \in \{anti(neut(a))\}$.
(7) $a \in \{anti(neut(a) * anti(a))\}$;
(8) $neut(a) * anti(a) \in \{anti(a)\}$;
$anti(a) * neut(a) \in \{anti(a)\}$;
(9) $a \in \{anti(anti(a))\}$, that is, there exists $p \in \{anti(a)\}$ such that $a \in \{anti(p)\}$;
(10) $neut(a) * anti(anti(a)) = a$.

Definition 6. A neutrosophic extended triplet group $(X, *)$ is said to be singular, if $anti(a)$ is unique for any $a \in X$.

Applying Theorem 5 we can get the following results.

Theorem 6. *Let* $(X, *)$ *be a singular neutrosophic extended triplet group. Then* $\forall a \in X$,

(1) $neut(a) * anti(a) = anti(a) * neut(a) = anti(a)$;
(2) $anti(neut(a)) = neut(a)$;
(3) $a = anti(anti(a))$;
(4) $neut(anti(a)) = neut(a)$.

Proof.

(1) Since $(X, *)$ is a singular neutrosophic extended triplet group, using Definition 6 and Theorem 5 (1) and (8), we have $neut(a) * anti(a) = anti(a) * neut(a) = anti(a)$ for all a in X.
(2) *Applying* (1),

$$neut(neut(a)) * anti(neut(a)) = anti(neut(a)), \forall a \in X.$$

By Theorem 5 (2), $neut(neut(a)) = neut(a)$. It follows that

$$neut(a) * anti(neut(a)) = anti(neut(a)), \forall a \in X.$$

On the other hand, using Definition 1, $neut(a) * anti(neut(a)) = neut(neut(a))$. Applying Theorem 5 (2) again, $neut(a) * anti(neut(a)) = neut(neut(a)) = neut(a)$. Therefore,

$$anti(neut(a)) = neut(a) * anti(neut(a)) = neut(a), \forall a \in X.$$

(3) Since $(X, *)$ is a singular neutrosophic extended triplet group, using Definition 6 and Theorem 5 (9), we get $a = anti(anti(a))$ for all a in X.
(4) Since $(X, *)$ is a singular neutrosophic extended triplet group, applying Definition 1,

$$anti(a) * anti(anti(a)) = neut(anti(a)), \forall a \in X.$$

Using (3), $anti(anti(a)) = a$, thus

$$anti(a) * a = anti(a) * anti(anti(a)) = neut(anti(a)), \forall a \in X.$$

On the other hand, $anti(a) * a = neut(a)$ (from Definition 1). Therefore,

$$neut(a) = anti(a) * a = anti(a) * anti(anti(a)) = neut(anti(a)), \forall a \in X.$$

Definition 7 Zhang, Hu et al., 2018. Let $(X, *)$ be a neutrosophic extended triplet group and H be a non-empty subset of X. Then H is called a NT-subgroup of X if

(1) $a * b \in H$ for all $a, b \in H$;
(2) there exists $anti(a) \in \{anti(a)\}$ such that $anti(a) \in H$ for all $a \in H$, where $\{anti(a)\}$ is the set of opposite element of a in $(X, *)$.

Proposition 1. *If* H *is a NT- subgroup of a neutrosophic extended triplet group* $(X, *)$, *then* $neut(a) \in H$ *for all* $a \in H$, *where* $neut(a)$ *is the neutral element of* a *in* $(X, *)$.

By Definition 6, Definition 7 and Proposition 1 we have

Proposition 2.. *If* H *is a non-empty subset of a singular neutrosophic extended triplet group* $(X, *)$, *then* H *is a NT-subgroup of* $(X, *)$ *if and only if it satisfies*

(1) $a * b \in H$ *for all* $a, b \in H$;
(2) $anti(a) \in H$ *for all* $a \in H$.

Definition 8 Jaiyeola & Smarandache, 2018. Let $(X, *)$ be a neutrosophic extended triplet group. Whenever $neut(a * b) = neut(a) * neut(b)$ for all $a, b \in X$, then X is referred to as a normal neutrosophic extended triplet group. Let $H \subseteq X$, if H is a NT-subgroup of X, then the relation of H and X can be denoted by $H \triangleleft X$. Whence, for any fixed $a \in X$,H is called a-normal NT-subgroup of X, written by $H \triangleleft_a X$, if $a * y * anti(a) \in H$ for all $y \in H$.

Definition 9 Jaiyeola & Smarandache, 2018. Let $f : X \rightarrow Y$ be a mapping such that X and Y are two neutrosophic extended triplet groups. Then f is referred to as a neutrosophic extended triplet group homomorphism if $f(c * d) = f(c) * f(d)$ for all $c, d \in X$. The kernel of f at $a \in X$ is defined by

$\ker f_a = \{x \in X : f(x) = neut(f(a))\}$

The kernel of f is defined by

$\ker f = \bigcup_{a \in X} \ker f_a$

where $X_a = \{x \in X : neut(x) = neut(a)\}$.

Theorem 7. *Let $f: X \to Y$ be a homomorphism, where X and Y are two singular neutrosophic extended triplet groups.*

(1) For all $a \in X, f(neut(a)) = neut(f(a))$ and $f(anti(a)) = anti(f(a))$.
(2) If H is a NT-subgroup of X, then $f(H)$ is a NT-subgroup of Y.
(3) If K is a NT-subgroup of Y and $f^{-1}(K) \neq \emptyset$, then $f^{-1}(K)$ is a NT-subgroup of X.
(4) If X is a normal neutrosophic extended triplet group and the set $X_f = \{(neut(a), f(a)) : a \in X\}$ with the product $(neut(a), f(a)) * (neut(b), f(b)) := (neut(a * b), f(a * b))$, then X_f is a neutrosophic extended triplet group.

Proof.

(1) For all a in X,
$f(neut(a)) * f(a) = f(neut(a) * a) = f(a)$,
$f(a) * f(neut(a)) = f(a * neut(a)) = f(a)$.

On the other hand,

$f(anti(a)) * f(a) = f(anti(a) * a) = f(neut(a))$,

$f(a) * f(anti(a)) = f(a * anti(a)) = f(neut(a))$.

Combining above facts, we get that $f(neut(a)) = neut(f(a))$. Thus,

$f(anti(a)) * f(a) = f(anti(a) * a) = f(neut(a)) = neut(f(a))$,

$f(a) * f(anti(a)) = f(a * anti(a)) = f(neut(a)) = neut(f(a))$.

Since X is singular, so $anti(f(a))$ is unique. It follows that $f(anti(a)) = anti(f(a))$.

(2) For any $f(h_1), f(h_2) \in f(H) = \{f(h) : h \in H\}$, where $h_1, h_2 \in H$. Since H is a NT-subgroup of X, then $h_1 * h_2 \in H$. Thus, $f(h_1) * f(h_2) = f(h_1 * h_2) \in f(H)$.

Moreover, for all $f(h) \in f(H)$, where $h \in H$. Since H is a NT-subgroup of X, then $anti(h) \in H$, by Proposition 2. Thus, applying (1) we have

$anti(f(h)) = f(anti(h)) \in f(H)$.

Therefore, using Proposition 2, we know that $f(H)$ is a NT-subgroup of Y.

(3) Suppose that K is a NT-subgroup of Y and $f^{-1}(K) \neq \emptyset$. For any $a, b \in f^{-1}(K)$, there exists $k_1, k_2 \in K$ such that $f(a) = k_1, f(b) = k_2$. Since K is a NT-subgroup of Y, then (by Proposition 2) $k_1, k_2 \in K$, $anti(k_1) \in K$, and (applying Definition 9 and (1))

$f(a * b) = f(a) * f(b) = k_1 * k_2 \in K$;
$f(anti(a)) = anti(f(a)) = anti(k_1) \in K$.

Thus, $a, b \in f^{-1}(K), anti(a) \in f^{-1}(K)$. Therefore, by Proposition 2, we get that $f^{-1}(K)$ is a NT-subgroup of X.

(4) Applying Theorem 6 and (1), we can verify that (4) holds. □

Theorem 8. *Let $f: X \to Y$ be a neutrosophic extended triplet group homomorphism, where X and Y are two singular neutrosophic extended triplet groups. Then*

(1) $a \in X_a$ for all a in X and $X = \bigcup_{a \in X} X_a$.
(2) for all a and b belong to X, $X_a \cap X_b = \emptyset$ or $X_a = X_b$.
(3) for all a in X, $neut(a) \in X_a$ and $anti(a) \in X_a$.
(4) if $x \in X_a$, then $neut(x) \in X_a$ and $anti(x) \in X_a$.
(5) if $x, y \in X_a$, then $x * y \in X_a$ and $anti(x * y) = anti(y) * anti(x)$.
(6) for all a in X, $(X_a, *)$ is a NT-subgroup of X and it is a classical group.
(7) $X_a \lhd_a X$.
(8) X_a is a normal neutrosophic extended triplet group.
(9) $\ker f_a \lhd_a X$.
(10) the binary relation $\approx$ is an equivalence relation on X_a, where $\approx$ is defined as follows: for $x, y \in X_a$, $x \approx y$ if and only if $anti(x) * y \in \ker f_a$.
(11) $X_a = \bigcup_{c \in X_a} (c * \ker f_a), \forall a \in X$.
(12) If X is finite, $|X_a| = \sum_{c*\ker f_a,\ c \in X_a} |c * \ker f_a| = [X_a : \ker f_a] \cdot |c * \ker f_a|$, for all $a \in X$, where $[X_a : \ker f_a]$ is the number of distinct $c * \ker f_a$ in X_a, and the summation is done for all the different $c * \ker f_a$.
(13) If X is finite, $|X| = \sum_{X_a,\ a \in X,\ c \in X_a} [X_a : \ker f_a] \cdot |c * \ker f_a|$, the summation is done for all the different X_a and corresponding $\ker f_a$. That is, only one calculation coincides with X_a and X_b.

Proof.

(1) For any a in X, by the definition of X_a, we know that $a \in X_a$. Then, $X \subseteq \bigcup_{a \in X} X_a$. Moreover, obviously, $\bigcup_{a \in X} X_a \subseteq X$. Thus, $X = \bigcup_{a \in X} X_a$.
(2) For any a, b in X, if $neut(a) = neut(b)$, then $X_a = X_b$.

If $neut(a) \neq neut(b)$, then $X_a \cap X_b = \emptyset$. In fact, assuming that there exists $x \in X_a \cap X_b$, then we have $neut(x) = neut(a)$ and $neut(x) = neut(b)$. From this and using Theorem 4 (1), $neut(a) = neut(b) = neut(x)$, this is a contradiction with $neut(a) \neq neut(b)$.

(3) By Theorem 5 (2), $neut(neut(a)) = neut(a)$, then $neut(a) \in X_a$. Applying Theorem 6 (4),$neut(anti(a)) = neut(a)$, it follows that $anti(a) \in X_a$.

(4) Assume $x \in X_a$, then $neut(x) = neut(a)$. Using Theorem 5 (2) and Theorem 6 (4) we have

$$neut(neut(x)) = neut(x) = neut(a),$$
$$neut(anti(x)) = neut(x) = neut(a).$$

Therefore, $neut(x) \in X_a$, $anti(x) \in X_a$.

(5) *Suppose* $x, y \in X_a$, *then*$neut(x) = neut(y) = neut(a)$. *Thus,*

$$neut(a) * (x * y) = neut(x) * (x * y) = (neut(x) * x) * y = x * y,$$

$$(x * y) * neut(a) = (x * y) * neut(y) = x * (y * neut(y)) = x * y.$$

On the other hand,

$$(x * y) * (anti(y) * anti(x)) = x * (y * anti(y)) * anti(x) = x * neut(y) * anti(x)$$
$$= x * neut(x) * anti(x) = x * anti(x) = neut(x) = neut(a).$$
$$(anti(y) * anti(x)) * (x * y) = anti(y) * (anti(x) * x) * y$$
$$= anti(y) * neut(x) * y = anti(y) * neut(y) * y = anti(y) * y = neut(y) = neut(a).$$

Therefore, $neut(x * y) = neut(a)$, *and* $anti(x * y) = anti(y) * anti(x)$. *It follows that* $x * y \in X_a$.

(6) *By* (4) *and* (5), *applying Proposition* 2 *we know that*$(X_a, *)$ *is a NT-subgroup of* X. *Since* X *is singular, and for all*$x \in X_a$, $neut(x) = neut(a)$, *so*$(X_a, *)$ *is a classical group.*

(7) *From* (6),$(X_a, *)$ *is a NT-subgroup of* X. *If*$x \in X_a$, *then*$neut(x) = neut(a)$, *and* $a * x * anti(a) \in X_a$, *by* (1), (3) *and* (5). *Thus, by Theorem* 6(1),

$$(a * x * anti(a)) * neut(a) = (a * x) * (anti(a) * neut(a)) = a * x * anti(a),$$
$$neut(a) * (a * x * anti(a)) = (neut(a) * a) * (x * anti(a)) = a * x * anti(a).$$

It follows that $neut(a) = neut(a * x * anti(a))$, since $(X_a, *)$ is a classical group. Therefore, $a * x * anti(a) \in X_a$, by Definition 8, we get that $X_a \lhd_a X$.

(8) *For all a in* X, *by* (6),$(X_a, *)$ *is a neutrosophic extended triplet group. If* $x, y \in X_a$, *then*$neut(x) = neut(y) = neut(a)$. *From the proof of* (5), *we have*$neut(x * y) = neut(a)$. *Applying Theorem* 4 (2),$neut(a) * neut(a) = neut(a)$. *Thus,*

$$neut(x*y) = neut(a) = neut(a)*neut(a) = neut(x)*neut(y).$$

According to Definition 8, *we get that* X_a *is a normal neutrosophic extended triplet group.*

(9) *Suppose* $x, y \in \ker f_a$, *then* $f(x) = f(y) = neut(f(a))$. *Thus, by Theorem* 4 (2), *Theorem* 7 (1), *and Theorem* 6 (2),

$$f(x * y) = f(x) * f(y) = neut(f(a)) * neut(f(a)) = neut(f(a)).$$
$$f(anti(x)) = anti(f(x)) = anti(neut(f(a))) = neut(f(a)).$$

It follows that $x * y \in \ker f_a$, $anti(x) \in \ker f_a$. *Applying Proposition* 2 *we get that* X_a *is a NT-subgroup.*

If $x \in \ker f_a$, *then* $f(x) = neut(f(a))$. *Considering* $a * x * anti(a)$, *we have*

$$\begin{aligned} f(a * x * anti(a)) &= f(a) * f(x) * f(anti(a)) \\ &= f(a) * neut(f(a)) * f(anti(a)) \\ &= f(a) * f(anti(a)) \\ &= f(a * anti(a)) \\ &= f(neut(a)) \\ &= neut(f(a)). \end{aligned}$$

Thus, $a * x * anti(a) \in \ker f_a$, *according to Definition* 8, *we get that* $\ker f_a \lhd_a X$.

(10) *Define the binary relation* $\approx$ *on* X_a *as follows*:

for $x, y \in X_a$, $x \approx y$ *if and only if* $anti(x) * y \in \ker f_a$. *Then,*

(i) *for any* x *in* X_a, $x \approx x$. *In fact,* $f(anti(x) * x) = f(neut(x)) = f(neut(a)) = neut(f(a))$, *that is,* $anti(x) * x \in \ker f_a$.

(ii) *if*$x \approx y$ *then* $y \approx x$. *In fact, if* $x \approx y$, *then* $anti(x) * y \in \ker f_a$. *Using* (9), *we have* $anti(anti(x) * y) \in \ker f_a$. *On the other hand,*

$$(anti(y) * x) * (anti(x) * y) = anti(y) * (x * anti(x)) * y$$
$$= anti(y) * neut(x) * y = anti(y) * neut(y) * y = neut(y) = neut(a).$$
$$(anti(x) * y) * (anti(y) * x) = anti(x) * (y * anti(y)) * x$$
$$= anti(x) * neut(y) * x = anti(x) * neut(x) * x = neut(x) = neut(a).$$

From this, applying (6),

$$anti(y) * x = anti(anti(x) * y).$$

Thus, $anti(y) * x = anti(anti(x) * y) \in \ker f_a$. *That is,*$y \approx x$.

(iii) *If* $x \approx y$ *and* $y \approx z$, *then* $x \approx z$. *In fact, from* $x \approx y$ *and* $y \approx z$, *we have* $anti(x) * y \in \ker f_a$, $anti(y) * z \in \ker f_a$. *Using* (6) *and* (9), *we have*

$$anti(x) * z = anti(x) * neut(z) * z = anti(x) * neut(y) * z$$
$$= anti(x) * (y * anti(y)) * z = (anti(x) * y) * (anti(y) * z) \in \ker f_a.$$

That is, $x \approx z$.

(11) *Now, we prove that the equivalence class* $[c]_{\approx}$ *of c is equal to* $c * \ker f_a$, *for all c in* X_a.

(i) *for c in* X_a, $c \in c * \ker f_a$. *In fact, by the definition of* $\ker f_a$, $neut(a) \in \ker f_a$. *From this, applying* (6), *we have*

$$c = c * neut(c) = c * neut(a) \in c * \ker f_a.$$

(ii) *for any x in* $\ker f_a$, $c * x \approx c$, *that is,* $c * \ker f_a \subseteq [c]_{\approx}$. *Since,*

$$anti(c) * (c * x) = (anti(c) * c) * x = neut(c) * x$$
$$= neut(a) * x \in \ker f_a.$$

That is, $c * x \approx c$.

(iii) *for any a in* $[c]_{\approx}$, *there exists* $x \in \ker f_a$ *such that* $a = c * x$, *that is,* $[c]_{\approx} \subseteq c * \ker f_a$. *Since* $a \approx c$, *then* $anti(c) * a \in \ker f_a$. *Denote* $x = anti(c) * a$, *then* $x \in \ker f_a$ *and*

$$a = neut(a) * a = neut(c) * a = c * anti(c) * a = c * x \in c * \ker f_a.$$

Therefore, $[c]_{\approx} \subseteq c * \ker f_a$, *and* $X_a = \bigcup_{c \in X_a} (c * \ker f_a)$.

(12) *It follows from* (10) *and* (11).
(13) *It follows from* (1), (2) *and* (12).

Theorem 9. *Let X and Y be two singular neutrosophic extended triplet groups,* $a \in X$ *and* $f : X \to Y$ *be a neutrosophic extended triplet group homomorphism. Then f is a monomorphism if and only if* $\ker f_a = \{neut(a)\}$ *for all* $a \in X$.

Proof.

(1) Assuming that f is a monomorphism, then for any x in $\ker f_a$ we have $f(x) = neut(f(a)) = f(neut(a))$, thus $x = neut(a)$. That is, $\ker f_a = \{neut(a)\}$.

Conversely, assuming $\ker f_a = \{neut(a)\}$ for all a in X, if $f(x) = f(y), x, y \in X$, then

$$f(anti(x) * y) = f(anti(x)) * f(y) = f(anti(x)) * f(x)$$
$$= f(anti(x) * x) = f(neut(x)) = neut(f(x)).$$
$$f(y * anti(x)) = f(y) * f(anti(x)) = f(x) * f(anti(x))$$
$$= f(x * anti(x)) = f(neut(x)) = neut(f(x)).$$

Thus, $anti(x) * y \in \ker f_x$ and $y * anti(x) \in \ker f_x$. Since $\ker f_x = \{neut(x)\}$, so $anti(x) * y = neut(x)$, $y * anti(x) = neut(x)$. Applying Theorem 6 (4), we get

$$anti(x) * y = neut(x) = neut(anti(x)),\ y * anti(x) = neut(x)$$
$$= neut(anti(x)).$$

This means that is the opposite element of $anti(x)$. Since X is singular, the opposite element is unique, thus

$$y = anti(anti(x)).$$

Using Theorem 6 (3), we have $y = anti(anti(x)) = x$. That is, f is a monomorphism. □

Theorem 10. $(X, *)$ *is a singular neutrosophic extended triplet group if and only if* $(X, *)$ *is a generalized group.*

Proof. By Definition 6, we know that every generalized group is a singular neutrosophic extended triplet group.

Conversely, assuming that $(X, *)$ is a singular neutrosophic extended triplet group, then, we only need to prove that

for any $x, a \in X$, $a * x = x * a = x$ implies $a = neut(x)$.

In fact, from $a * x = x$, and applying Theorem 5 (2), we have

$$a * neut(x) = a * (x * anti(x)) = (a * x) * anti(x)$$
$$= x * anti(x) = neut(x) = neut(neut(x)).$$

Similarly, we can get $neut(x) * a = neut(neut(x))$. That is,

$$a * neut(x) = neut(x) * a = neut(neut(x)).$$

This means that a is a opposite element of $neut(x)$. Since $(X, *)$ is a singular, it follows that $a = anti(neut(x))$. Using Theorem 6(2), we obtain $anti(neut(x)) = neut(x)$.

Therefore, $a = anti(neut(x)) = neut(x)$. From this, we know that $(X, *)$ is a generalized group. □

5. Conclusions

In this paper, neutrosophic extended triplet group (NETG) is discussed in depth, thereby some erroneous conclusions in the literature are corrected, and the differences between NETG and generalized group highlighted. From the results of this paper, the following algebraic structures: generalized group (GG), singular neutrosophic extended triplet group (SNETG) and completely simple semigroup are equivalent to each other. Therefore, it is discovered that a generalized group is a special type of neutrosophic extended triplet group, and NETG is a more extensive algebraic system than group and generalized group. On one hand, NETG preserves some properties of group (such as the Lagrange-like theorem obtained in this article). On the other hand, NETG has many characteristics different from group and generalized group. In the future, it is needful to do a more detailed and in-depth study to reveal its structural characteristics, and we will expand our research on some new developments in algebras and neutrosophic sets (see Zhang, Borzooei, & Jun, 2018; Liu, Zhang, Liu & Wang, 2016; Liu and Tang, 2016; Liu and Shi, 2017; Liu and Zhang, 2018; Ye, 2018).

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Author contributions

The contributions of the authors are roughly equal. Xiaohong Zhang and Xuejiao Wang initiated the research and wrote the draft, Florentin Smarandache and Tèmítópé Gbóláhàn Jaíyéolá gave some guidance and revised relevant paragraphs, Tieyan Lian and Xiaohong Zhang completed final version.

Conflicts of interest

The authors declare no conflict of interest.

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