mathematics

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Some Generalized Dice Measures for Double-Valued Neutrosophic Sets and Their Applications

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Received: 10 June 2018; Accepted: 9 July 2018; Published: 10 July 2018

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Abstract: Neutrosophic sets (NSs) are used to illustrate uncertain, inconsistent, and indeterminate information existing in real-world problems. Double-valued neutrosophic sets (DVNSs) are an alternate form of NSs, in which the indeterminacy has two distinct parts: indeterminacy leaning toward truth membership, and indeterminacy leaning toward falsity membership. The aim of this article is to propose novel Dice measures and generalized Dice measures for DVNSs, and to specify Dice measures and asymmetric measures (projection measures) as special cases of generalized Dice measures via specific parameter values. Finally, the proposed generalized Dice measures and generalized weighted Dice measures were applied to pattern recognition and medical diagnosis to show their effectiveness.

Keywords: double-valued neutrosophic set; Dice similarity measure; pattern recognition; medical diagnosis

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1. Introduction

The fuzzy set (FS) theory introduced by Zadeh [1] is applied to various fields and has various successful applications. In FSs, the degree of membership of an element is a single value in the closed interval $[0, 1]$. However, in real situations, one may not always be confident that the degree of non-membership of an element in the FS is simply equal to one minus degree of membership. That is to say, there may be a degree of hesitation. For this purpose, the concept of intuitionistic fuzzy sets (IFSs) [2] was introduced by Atanassov as a generalization of FSs. The only limitation of IFSs is that the degree of hesitation is not defined independently. To overcome this shortcoming, Smarandache [3] proposed the concept of neutrosophic sets (NSs), which were the generalization of IFSs and FSs. After that, some researchers defined subclasses of NSs, such as single-valued neutrosophic sets (SVNSs) [4], interval neutrosophic sets (INSs) [5], and simplified neutrosophic sets (SNS) [6]. Zhang et al. [7] proposed some basic operational laws for cubic neutrosophic numbers, and defined some aggregation operators for its application to multiple attribute decision making (MADM). Ye et al. [8] proposed correlation co-efficients for normal neutrosophic numbers, and applied them to MADM. Liu et al. [9–11] proposed prioritized aggregation operators and power Heronian-mean aggregation operators for hesitant interval neutrosophic sets, hesitant intuitionistic fuzzy sets, and linguistic neutrosophic sets, and applied them to MADM and multiple attribute group decision making (MAGDM).

In recent years, distance and similarity measures gained much more attention from researchers, due to their wide applications in various fields such as data mining, pattern recognition, medical diagnosis, and decision making. For this reason, several distances and similarity measures were developed for IFSs [12–14]. De et al. [15] gave an application of IFSs in medical diagnosis. Dengfeng et al. [16] and Grzegorzewski et al. [17] developed some new similarity measures for IFSs based on the Hausdorff metric, and applied them to pattern recognition. Hwang et al. [18] and Khatibi et al. [19] proposed similarity measures for IFSs based on the Sugeno integral, and presented their application in pattern recognition. Tang et al. [20] developed generalized Dice similarity measures for IFSs, and gave their application in MADM. Ye [21,22] proposed cosine and Dice similarity measures for IFS and interval-valued intuitionistic fuzzy sets (IVIFSs). Similar to IFSs, several authors developed distance and similarity measures for NSs and its subclasses. Majumdar et al. [23] developed similarity and entropy measures for NSs. Ye [24–26] further proposed distance and vector similarity measures, and generalized Dice similarity measures for SNSs and INSs, and applied them to MADM. Some authors found drawbacks of the proposed cosine similarity measures for SNSs, and Ye [27] further proposed improved cosine similarity measures for SNSs, and gave their applications in medical diagnosis and pattern recognition.

Let us consider a situation where we ask someone about a statement; he/she may be sure that the possibility of the statement being true is 0.8, and that the possibility of the statement being false is 0.4. Additionally, the degree to which he/she is not sure but thinks it is true is 0.3, and the degree to which he/she is not sure but thinks it is false is 0.4. In order to deal with such kinds of information, Kandasamy [28] introduced the concept of double-valued neutrosophic sets (DVNSs) as an alternate form of NSs, providing more reliability and clarity to indeterminacy. In DVNSs, indeterminacy is empathized into two parts: indeterminacy leaning toward truth membership and indeterminacy leaning toward falsity membership. The first refinement of neutrosophic sets was done by Smarandache [29] in 2013, whereby the truth value (T) was refined into various types of sub-truths such as T_1, T_2, etc., and similarly, indeterminacy (I) was split/refined into various types of sub-indeterminacies such as I_1, I_2, etc., and the sub-falsehood (F) was split into F_1, F_2, etc. DVNSs are a special case of n-valued neutrosophic sets.

Currently, the research on DVNSs is rare, and it is necessary to study some basic theories about DVNSs. As such, the aims of this article were (1) to propose two forms of Dice measures [30] for DVNSs; (2) to propose two types of weighted Dice measures for DVNSs; (3) to propose weighted generalized Dice measures for DVNSs; and (4) to show the effectiveness of the proposed Dice measures in pattern recognition and medical diagnosis.

In order to do so, the remainder of this article is structured as follows: in Section 2, some basic concepts related to DVNSs and Dice similarity measures are reviewed; in Section 3, some Dice measures and weighted Dice measures for DVNSs are proposed; in Section 4, another form of Dice measure for DVNSs is proposed; in Section 5, some generalized Dice measures and generalized weighted Dice measures for DVNSs are proposed; in Section 6, applications of the proposed Dice measures for DVNSs in pattern recognition and medical diagnosis are discussed, using numerical examples. Finally, comparisons, discussions, conclusions, and references are given.

2. Preliminaries

In this section, some basic concepts related to DVNSs and Dice similarity measures are given.

2.1. Double-Valued Neutrosophic Sets and Their Operational Laws

Definition 1. *[28] Let $\widetilde{U}$ be the universe of the discourse set. A DVNS is an object of the form,*

$$D = \left\{ \left\langle u, \left(t_D(u), i_{t_D}(u), i_{f_D}(u), f_D(u) \right) \right\rangle \middle| u \in \widetilde{U} \right\}, \tag{1}$$

where $t_D(u), i_{t_D}(u), i_{f_D}(u)$ and $f_D(u)$ represent the truth membership, indeterminacy leaning toward truth membership, indeterminacy leaning toward falsity membership, and the falsity-membership functions of the element $u \in \widetilde{U}$, respectively, with the condition $0 \leq t_D(u) + i_{t_D}(u) + i_{f_D}(u) + f_D(u) \leq 4$.

Definition 2. *[28] The complement of a DVNS, D, is denoted by D^c, and is defined as*

$$t_{D^c}(u) = f_D(u),\ i_{t_{D^c}}(u) = 1 - i_{t_D}(u), i_{f_{D^c}}(u) = 1 - i_{f_D}(u), f_D(u) = t_D(u).$$

For all $u \in \widetilde{U}$.

Definition 3. *[28] Let D_1 and D_2 be two DVNSs. Then, we can say that $D_1 \subseteq D_2$, if and only if*

$$t_{D_1} \leq t_{D_2}, i_{t_{D_1}}(u) \geq i_{t_{D_2}}(u), i_{f_{D_1}}(u) \leq i_{f_{D_2}}(u), f_{D_1}(u) \geq f_{D_2}(u).$$

For all $u \in \widetilde{U}$.

Definition 4. *[28] Let D_1 and D_2 be two DVNSs. Then, we can say that $D_1 = D_2$, if and only if*

$$t_{D_1} = t_{D_2}, i_{t_{D_1}}(u) = i_{t_{D_2}}(u), i_{f_{D_1}}(u) = i_{f_{D_2}}(u), f_{D_1}(u) = f_{D_2}(u).$$

For all $u \in \widetilde{U}$.

Definition 5. *[28] Let D_1 and D_2 be two DVNSs. Then, the union and intersection of D_1 and D_2 is denoted and defined as follows:*

$$D_1 \cup D_2 = C = \left(t_C(u), i_{t_C}(u), i_{f_C}(u), f_C(u)\right) = \left(\max\left(t_{D_1}(u), t_{D_2}(u)\right), \max\left(i_{t_{D_1}}(u), i_{t_{D_2}}(u)\right),\right.$$
$$\left.\min\left(i_{f_{D_1}}(u), i_{f_{D_2}}(u)\right), \min\left(f_{D_1}(u), f_{D_2}(u)\right)\right)$$

and

$$D_1 \cap D_2 = D = \left(t_D(u), i_{t_D}(u), i_{f_D}(u), f_D(u)\right) = \left(\min\left(t_{D_1}(u), t_{D_2}(u)\right), \min\left(i_{t_{D_1}}(u), i_{t_{D_2}}(u)\right),\right.$$
$$\left.\max\left(i_{f_{D_1}}(u), i_{f_{D_2}}(u)\right), \max\left(f_{D_1}(u), f_{D_2}(u)\right)\right).$$

2.2. Some Dice Similarity Measures

In this subsection, the concept of Dice similarity measures is defined, adapted from [30].

Definition 6. *[30] Let $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_m\}$ be two vectors of length m, where all coordinates are positive real numbers. Then, the Dice measure is denoted and defined as*

$$\widetilde{D}(A, B) = \frac{2A.B}{\|A\|_2^2 + \|B\|_2^2} = \frac{2\sum_{i=1}^{m} a_i b_i}{\sum_{i=1}^{m}(a_i)^2 + \sum_{i=1}^{m}(a_i)^2}, \tag{2}$$

where $A.B = \sum_{i=1}^{m} a_i b_i$ is called the inner product of the vector A and B, and $\|A\|_2 = \sqrt{\sum_{i=1}^{m}(a_i)^2}$ and $\|B\|_2 = \sqrt{\sum_{i=1}^{m}(b_i)^2}$ are the L_2 norms of A and B (also called Euclidean norms).

The Dice similarity measures take a value in the closed interval $[0, 1]$. However, the Dice measure is undefined if $a_i = b_i = 0$ for $i = 1, 2, \ldots, m$. So, let us assume that the Dice measure is zero, whenever $a_i = b_i = 0$ for $i = 1, 2, \ldots, m$.

3. Dice Similarity Measures for DVNSs

In this section, we develop some Dice similarity measures for DVNSs, and the related properties are satisfied.

Definition 7. *Let $\widetilde{D}_1 = \{d_{11}, d_{12}, \ldots, d_{1m}\}$ and $\widetilde{D}_2 = \{d_{21}, d_{22}, \ldots, d_{2m}\}$ be two collections of DVNSs. If $d_{1i} = \left\langle t_{1i}, i_{t_{1i}}, i_{f_{1i}}, f_{1i} \right\rangle$ and $d_{2i} = \left\langle t_{2i}, i_{t_{2i}}, i_{f_{2i}}, f_{2i} \right\rangle$ are the $i-th$ DVNNs in $\widetilde{D}_1$ and $\widetilde{D}_2$, respectively, then the Dice distance measure between $\widetilde{D}_1$ and $\widetilde{D}_2$ is defined as*

$$D_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) = \frac{1}{m}\sum_{i=1}^{m} \frac{2 d_{1i}\cdot d_{2i}}{|d_{1i}|2 + |d_{2i}|^2} = \frac{1}{m}\sum_{i=1}^{m} \frac{2\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + \left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)}. \quad (3)$$

Obviously, the above-defined Dice similarity measure between DVNSs, $\widetilde{D}_1$ and $\widetilde{D}_2$, satisfies the following assertions:

(1) $0 \le D_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) \le 1$;
(2) $D_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) = D_{DVNS1}\left(\widetilde{D}_2, \widetilde{D}_1\right)$;
(3) $D_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) = 1,\ if\ \widetilde{D}_1 = \widetilde{D}_2$; *that is* $t_{1i} = t_{2i}, i_{t_{1i}} = i_{t_{2i}}, i_{f_{1i}} = i_{f_{2i}}, f_{1i} = f_{2i}, for\ i = 1, 2, \ldots, m.$

Proof:

(1) Let us assume the $i-$ th DVNN in the summation of Equation (3).

$$D_{DVNS1}(d_{1i}, d_{2i}) = \frac{2\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + \left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)}.$$

Obviously, $D_{DVNS1}(d_{1i}, d_{2i}) \ge 0$, and according to the inequality, $x^2 + y^2 \ge 2xy$, we have

$$\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + \left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right) \ge 2\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right).$$

Therefore, $0 \le D_{DVNS1}(d_{1i}, d_{2i}) \le 1$. Hence, from Equation (3), the summation of m *terms* is $0 \le D_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) \le 1$.

(2) Obviously, it is true.

(3) When $\widetilde{D}_1 = \widetilde{D}_2$, then $d_{1i} = d_{2i}$, so $t_{1i} = t_{2i}, i_{t_{1i}} = i_{t_{2i}}, i_{f_{1i}} = i_{f_{2i}}, f_{1i} = f_{2i},\ for\ i = 1, 2, \ldots, m$. So, we get

$$D_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) = \frac{1}{m}\sum_{i=1}^{m} \frac{2\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + \left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} = \frac{1}{m}\sum_{i=1}^{m} \frac{2\left(t_{1i}t_{1i} + i_{t_{1i}}i_{t_{1i}} + i_{f_{1i}}i_{f_{1i}} + f_{1i}f_{1i}\right)}{\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + \left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right)} = \frac{1}{m}\sum_{i=1}^{m} \frac{2\left(t_{1i}{}^2 + i_{t_{1i}}{}^2 + i_{f_{1i}}{}^2 + f_{1i}{}^2\right)}{2\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right)} = 1,$$

which completes the proof of (8). □

In real-life problems, one usually takes the importance degree of each element DVNN d_{zi} (z = 1,2;i = 1,2,. . . ,n) into account. Let $\widetilde{W} = (\varpi_1, \varpi_2, \ldots, \varpi_m)^T$ be the importance degree for

d_{zi} $(z = 1, 2;\ i = 1, 2, \ldots, n)$ with $\varpi_i \geq 0$ *and* $\sum_{i=1}^{n} \varpi_i = 1$. Then, based on Equation (3), we further proposed the concept of weighted Dice similarity measures of DVNSs as follows:

$$WD_{WDVNN1}(\widetilde{D}_1, \widetilde{D}_2) = \sum_{i=1}^{n} \varpi_i \frac{2 d_{1i} \cdot d_{2i}}{|d_{1i}|_2 + |d_{2i}|^2} = \sum_{i=1}^{n} \varpi_i \frac{2\left(t_{1i}t_{2i} + i_{t_{1i}} i_{t_{2i}} + i_{f_{1i}} i_{f_{2i}} + f_{1i} f_{2i}\right)}{\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + \left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)}. \quad (4)$$

In particular, if $\widetilde{W} = \left(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\right)^T$, then the weighted Dice similarity measure reduces to the Dice similarity measure defined in Equation (3).

Obviously, the above-defined weighted Dice similarity measure between DVNSs, $\widetilde{D}_1$ and $\widetilde{D}_2$, satisfies the following assertions:

(1) $0 \leq WD_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) \leq 1$;
(2) $WD_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) = WD_{DVNS1}\left(\widetilde{D}_2, \widetilde{D}_1\right)$;
(3) $WD_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) = 1,\ if\ \widetilde{D}_1 = \widetilde{D}_2;\ that\ is\ t_{1i} = t_{2i}, i_{t_{1i}} = i_{t_{2i}}, i_{f_{1i}} = i_{f_{2i}}, f_{1i} = f_{2i},\ for\ i = 1, 2, \ldots, m.$

The proof of these properties is the same as above.

The above-defined similarity measures have the disadvantage of not being flexible. So, in the following section, we defined another form of the above Dice similarity measure.

4. Another Form of the Dice Similarity Measure for DVNSs

In this section, another form of the Dice similarity measure for DVNSs is proposed, which is defined below.

Definition 8. *Let* $\widetilde{D}_1 = \{d_{11}, d_{12}, \ldots, d_{1m}\}$ *and* $\widetilde{D}_2 = \{d_{21}, d_{22}, \ldots, d_{2m}\}$ *be two DVSSs. If* $d_{1i} = \left\langle t_{1i}, i_{t_{1i}}, i_{f_{1i}}, f_{1i}\right\rangle$ *and* $d_{2i} = \left\langle t_{2i}, i_{t_{2i}}, i_{f_{2i}}, f_{2i}\right\rangle$ *are the* $i - th$ *DVNNs in* $\widetilde{D}_1$ *and* $\widetilde{D}_2$*, respectively, then the Dice similarity measure between* $\widetilde{D}_1$ *and* $\widetilde{D}_2$ *is defined as*

$$D_{DVNN2}\left(\widetilde{D}_1, \widetilde{D}_2\right) = \frac{2\left(\widetilde{D}_1 . \widetilde{D}_2\right)}{\left|\widetilde{D}_1\right|_2 + \left|\widetilde{D}_2\right|^2} = \frac{2\sum_{i=1}^{n}\left(t_{1i}t_{2i} + i_{t_{1i}} i_{t_{2i}} + i_{f_{1i}} i_{f_{2i}} + f_{1i} f_{2i}\right)}{\sum_{i=1}^{n}\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + \sum_{i=1}^{n}\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)}. \quad (5)$$

Obviously, the above-defined Dice similarity measure in Equation (5) satisfies the following properties:

(1) $0 \leq D_{DVNS2}\left(\widetilde{D}_1, \widetilde{D}_2\right) \leq 1$;
(2) $D_{DVNS2}\left(\widetilde{D}_1, \widetilde{D}_2\right) = D_{DVNS2}\left(\widetilde{D}_2, \widetilde{D}_1\right)$;
(3) $D_{DVNS2}\left(\widetilde{D}_1, \widetilde{D}_2\right) = 1,\ if\ \widetilde{D}_1 = \widetilde{D}_2;\ that\ is\ t_{1i} = t_{2i}, i_{t_{1i}} = i_{t_{2i}}, i_{f_{1i}} = i_{f_{2i}}, f_{1i} = f_{2i},\ for\ i = 1, 2, \ldots, m.$

Proof: The proof is the same as previously shown proofs. □

For real applications, the importance degree of each element d_{zi} $(z = 1, 2;\ i = 1, 2, \ldots, n)$ is under consideration. Then, let $\widetilde{W} = (\varpi_1, \varpi_2, \ldots, \varpi_n)$ be the importance degree for

d_{zi} $(z = 1, 2;\ i = 1, 2, \ldots, n)$, $\varpi_i \geq 0$ *and* $\sum\limits_{i=1}^{n} \varpi_i = 1$. So, based on Equation (5), we further proposed the concept of weighted Dice similarity measures of DVNSs as follows:

$$WD_{DVNN2}\left(\widetilde{D}_1, \widetilde{D}_2\right) = \frac{2\left(\widetilde{D}_1.\widetilde{D}_2\right)_{\varpi}}{|\widetilde{D}_1|_{\varpi}{}^2 + |\widetilde{D}_2|_{\varpi}{}^2}$$
$$= \frac{2\sum\limits_{i=1}^{n} \varpi^2{}_i\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\sum\limits_{i=1}^{n} \varpi^2{}_i\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + \sum\limits_{i=1}^{n} \varpi^2{}_i\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)}. \quad (6)$$

In particular, if $\widetilde{W} = \left(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m}\right)^T$, then the weighted Dice similarity measure reduces to the Dice similarity measure defined in Equation (5).

Obviously, the above-defined weighted Dice similarity measures in Equation (6) satisfy the following properties:

(1) $0 \leq WD_{DVNS2}\left(\widetilde{D}_1, \widetilde{D}_2\right) \leq 1$;
(2) $WD_{DVNS2}\left(\widetilde{D}_1, \widetilde{D}_2\right) = WD_{DVNS2}\left(\widetilde{D}_2, \widetilde{D}_1\right)$;
(3) $WD_{DVNS2}\left(\widetilde{D}_1, \widetilde{D}_2\right) = 1$, *if* $\widetilde{D}_1 = \widetilde{D}_2$; *that is* $t_{1i} = t_{2i}, i_{t_{1i}} = i_{t_{2i}}, i_{f_{1i}} = i_{f_{2i}}, f_{1i} = f_{2i}$, *for* $i = 1, 2, \ldots, m$.

As discussed earlier, the above-defined similarity measures have the disadvantage of not being flexible. As such, in the following section, we defined a generalized Dice similarity measure to overcome the shortcoming of the above Dice similarity measures.

5. A Generalized Dice Similarity Measure of DVNSs

In this section, we propose a generalized Dice similarity measure for DVNSs, as a generalization of the above-defined Dice similarity measures.

Definition 9. *Let* $\widetilde{D}_1 = \{d_{11}, d_{12}, \ldots, d_{1m}\}$ *and* $\widetilde{D}_2 = \{d_{21}, d_{22}, \ldots, d_{2m}\}$ *be two DVSSs. If* $d_{1i} = \left\langle t_{1i}, i_{t_{1i}}, i_{f_{1i}}, f_{1i}\right\rangle$ *and* $d_{2i} = \left\langle t_{2i}, i_{t_{2i}}, i_{f_{2i}}, f_{2i}\right\rangle$ *are the* $i-th$ *DVNNs in* $\widetilde{D}_1$ *and* $\widetilde{D}_2$*, respectively, then the generalized Dice similarity measure between* $\widetilde{D}_1$ *and* $\widetilde{D}_2$ *is defined as*

$$GD_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) = \frac{1}{n}\sum_{i=1}^{n} \frac{d_{1i} \cdot d_{2i}}{\rho|d_{1i}|_2 + (1-\rho)|d_{2i}|^2}$$
$$= \frac{1}{n}\sum_{i=1}^{n} \frac{\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\rho\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-\rho)\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)}; \quad (7)$$

$$GD_{DVNS2}\left(\widetilde{D}_1, \widetilde{D}_2\right) = \frac{\widetilde{D}_1.\widetilde{D}_2}{\rho|\widetilde{D}_1|_2 + (1-\rho)|\widetilde{D}_2|^2}$$
$$= \frac{\sum\limits_{i=1}^{n}\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\rho\sum\limits_{i=1}^{n}\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-\rho)\sum\limits_{i=1}^{n}\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)}, \quad (8)$$

where ρ *is a positive parameter for* $0 \leq \rho \leq 1$.

Obviously, the above-defined Dice similarity measure between DVNSs, $\widetilde{D}_1$ and $\widetilde{D}_2$, satisfies the following assertions:

(1) $0 \leq GD_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) \leq 1$;
(2) $GD_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) = GD_{DVNS1}\left(\widetilde{D}_2, \widetilde{D}_1\right)$;
(3) $GD_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) = 1$, *if* $\widetilde{D}_1 = \widetilde{D}_2$; *that is* $t_{1i} = t_{2i}, i_{t_{1i}} = i_{t_{2i}}, i_{f_{1i}} = i_{f_{2i}}, f_{1i} = f_{2i}$, *for* $i = 1, 2, \ldots, m$.

and

(1) $0 \leq GD_{DVNS2}\left(\widetilde{D}_1, \widetilde{D}_2\right) \leq 1$;
(2) $GD_{DVNS2}\left(\widetilde{D}_1, \widetilde{D}_2\right) = GD_{DVNS2}\left(\widetilde{D}_2, \widetilde{D}_1\right)$;
(3) $GD_{DVNS2}\left(\widetilde{D}_1, \widetilde{D}_2\right) = 1,\ if\ \widetilde{D}_1 = \widetilde{D}_2;\ that\ is\ t_{1i} = t_{2i}, i_{t_{1i}} = i_{t_{2i}}, i_{f_{1i}} = i_{f_{2i}}, f_{1i} = f_{2i},\ for\ i = 1, 2, \ldots, m.$

Now, we discuss some special cases of generalized Dice similarity measures for the parameter ρ.

(1) If $\rho = 0.5$, then the two generalized Dice similarity measures defined in Equation (7) and Equation (8) reduce to Dice similarity measures defined in Equation (3) and Equation (5):

$$\begin{aligned} GD_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) &= \frac{1}{m}\sum_{i=1}^{m} \frac{\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\rho\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-\rho)\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \\ &= \frac{1}{m}\sum_{i=1}^{m} \frac{\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{0.5\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-0.5)\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \\ &= \frac{1}{m}\sum_{i=1}^{m} \frac{2\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + \left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \end{aligned},$$

and

$$\begin{aligned} GD_{DVNS2}\left(\widetilde{D}_1, \widetilde{D}_2\right) &= \frac{\sum_{i=1}^{n}\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\rho\sum_{i=1}^{n}\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-\rho)\sum_{i=1}^{n}\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \\ &= \frac{\sum_{i=1}^{n}\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{0.5\sum_{i=1}^{n}\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-0.5)\sum_{i=1}^{n}\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \\ &= \frac{2\sum_{i=1}^{n}\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\sum_{i=1}^{n}\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + \sum_{i=1}^{n}\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \end{aligned}.$$

(2) When $\rho = 0, 1$, Equations (7) and (8) reduce to the following asymmetric similarity measures:

$$\begin{aligned} GD_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) &= \frac{1}{m}\sum_{i=1}^{m} \frac{d_{1i}\cdot d_{2i}}{\rho|d_{1i}|2 + (1-\rho)|d_{2i}|^2} \\ &= \frac{1}{m}\sum_{i=1}^{m} \frac{\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\rho\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-\rho)\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \\ &= \frac{1}{m}\sum_{i=1}^{m} \frac{\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{0\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-0)\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \\ &= \frac{1}{m}\sum_{i=1}^{m} \frac{\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \qquad for \rho = 0 \end{aligned}, \tag{9}$$

$$\begin{aligned} GD_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) &= \frac{1}{m}\sum_{i=1}^{m} \frac{d_{1i}\cdot d_{2i}}{\rho|d_{1i}|2 + (1-\rho)|d_{1i}|^2} \\ &= \frac{1}{m}\sum_{i=1}^{m} \frac{\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\rho\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-\rho)\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \\ &= \frac{1}{m}\sum_{i=1}^{m} \frac{\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{1\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-1)\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \\ &= \frac{1}{n}\sum_{i=1}^{n} \frac{\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right)} \qquad for\ \rho = 1 \end{aligned}, \tag{10}$$

$$
\begin{aligned}
GD_{DVNS2}\left(\widetilde{D}_1,\widetilde{D}_2\right) &= \frac{\widetilde{D}_1.\widetilde{D}_2}{\rho\left|\widetilde{D}_1\right|_2+(1-\rho)\left|\widetilde{D}_2\right|^2} \\
&= \frac{\sum_{i=1}^{m}\left(t_{1i}t_{2i}+i_{t_{1i}}i_{t_{2i}}+i_{f_{1i}}i_{f_{2i}}+f_{1i}f_{2i}\right)}{\rho\sum_{i=1}^{m}\left(t^2{}_{1i}+i^2{}_{t_{1i}}+i^2{}_{f_{1i}}+f^2{}_{1i}\right)+(1-\rho)\sum_{i=1}^{m}\left(t^2{}_{2i}+i^2{}_{t_{2i}}+i^2{}_{f_{2i}}+f^2{}_{2i}\right)} \\
&= \frac{\sum_{i=1}^{n}\left(t_{1i}t_{2i}+i_{t_{1i}}i_{t_{2i}}+i_{f_{1i}}i_{f_{2i}}+f_{1i}f_{2i}\right)}{0\sum_{i=1}^{n}\left(t^2{}_{1i}+i^2{}_{t_{1i}}+i^2{}_{f_{1i}}+f^2{}_{1i}\right)+(1-0)\sum_{i=1}^{n}\left(t^2{}_{2i}+i^2{}_{t_{2i}}+i^2{}_{f_{2i}}+f^2{}_{2i}\right)} \\
&= \frac{\sum_{i=1}^{n}\left(t_{1i}t_{2i}+i_{t_{1i}}i_{t_{2i}}+i_{f_{1i}}i_{f_{2i}}+f_{1i}f_{2i}\right)}{\sum_{i=1}^{n}\left(t^2{}_{2i}+i^2{}_{t_{2i}}+i^2{}_{f_{2i}}+f^2{}_{2i}\right)} \qquad for\ \rho=0
\end{aligned}, \tag{11}
$$

$$
\begin{aligned}
GD_{DVNS2}\left(\widetilde{D}_1,\widetilde{D}_2\right) &= \frac{\widetilde{D}_1.\widetilde{D}_2}{\rho\left|\widetilde{D}_1\right|_2+(1-\rho)\left|\widetilde{D}_2\right|^2} \\
&= \frac{\sum_{i=1}^{m}\left(t_{1i}t_{2i}+i_{t_{1i}}i_{t_{2i}}+i_{f_{1i}}i_{f_{2i}}+f_{1i}f_{2i}\right)}{\rho\sum_{i=1}^{m}\left(t^2{}_{1i}+i^2{}_{t_{1i}}+i^2{}_{f_{1i}}+f^2{}_{1i}\right)+(1-\rho)\sum_{i=1}^{m}\left(t^2{}_{2i}+i^2{}_{t_{2i}}+i^2{}_{f_{2i}}+f^2{}_{2i}\right)} \\
&= \frac{\sum_{i=1}^{n}\left(t_{1i}t_{2i}+i_{t_{1i}}i_{t_{2i}}+i_{f_{1i}}i_{f_{2i}}+f_{1i}f_{2i}\right)}{1\sum_{i=1}^{n}\left(t^2{}_{1i}+i^2{}_{t_{1i}}+i^2{}_{f_{1i}}+f^2{}_{1i}\right)+(1-1)\sum_{i=1}^{n}\left(t^2{}_{2i}+i^2{}_{t_{2i}}+i^2{}_{f_{2i}}+f^2{}_{2i}\right)} \\
&= \frac{\sum_{i=1}^{n}\left(t_{1i}t_{2i}+i_{t_{1i}}i_{t_{2i}}+i_{f_{1i}}i_{f_{2i}}+f_{1i}f_{2i}\right)}{\sum_{i=1}^{n}\left(t^2{}_{1i}+i^2{}_{t_{1i}}+i^2{}_{f_{1i}}+f^2{}_{1i}\right)} \qquad for\ \rho=1
\end{aligned}. \tag{12}
$$

From the above investigation, the four asymmetric similarity measures are the extension of the relative projection measure of interval numbers [31]. Therefore, the four asymmetric similarity measures can be assumed as the projection measures of DVNSs.

For real applications, the importance degree of each element d_{zi} $(z=1,2;\ i=1,2,\ldots,n)$ is under consideration. Then, let $\widetilde{W}=(\varpi_1,\varpi_2,\ldots,\varpi_n)$ be the importance degree for d_{zi} $(z=1,2;\ i=1,2,\ldots,n)$, $\varpi_i\geq 0$ *and* $\sum_{i=1}^{n}\varpi_i=1$. So, based on Equations (7) and (8), we further proposed the concept of weighted generalized Dice similarity measures of DVNSs, which are defined as follows:

$$
\begin{aligned}
WGD_{DVNS1}\left(\widetilde{D}_1,\widetilde{D}_2\right) &= \sum_{i=1}^{m}\varpi_i\frac{d_{1i}\cdot d_{2i}}{\rho|d_{1i}|_2+(1-\rho)|d_{2i}|^2} \\
&= \sum_{i=1}^{m}\varpi_i\frac{\left(t_{1i}t_{2i}+i_{t_{1i}}i_{t_{2i}}+i_{f_{1i}}i_{f_{2i}}+f_{1i}f_{2i}\right)}{\rho\left(t^2{}_{1i}+i^2{}_{t_{1i}}+i^2{}_{f_{1i}}+f^2{}_{1i}\right)+(1-\rho)\left(t^2{}_{2i}+i^2{}_{t_{2i}}+i^2{}_{f_{2i}}+f^2{}_{2i}\right)}
\end{aligned}, \tag{13}
$$

$$
\begin{aligned}
WGD_{DVNS2}\left(\widetilde{D}_1,\widetilde{D}_2\right) &= \frac{\left(\widetilde{D}_1.\widetilde{D}_2\right)_\varpi}{\rho\left|\widetilde{D}_1\right|_\varpi{}^2+(1-\rho)\left|\widetilde{D}_2\right|_\varpi{}^2} \\
&= \frac{\sum_{i=1}^{m}\varpi_i^2\left(t_{1i}t_{2i}+i_{t_{1i}}i_{t_{2i}}+i_{f_{1i}}i_{f_{2i}}+f_{1i}f_{2i}\right)}{\rho\sum_{i=1}^{m}\varpi_i^2\left(t^2{}_{1i}+i^2{}_{t_{1i}}+i^2{}_{f_{1i}}+f^2{}_{1i}\right)+(1-\rho)\sum_{i=1}^{m}\varpi_i^2\left(t^2{}_{2i}+i^2{}_{t_{2i}}+i^2{}_{f_{2i}}+f^2{}_{2i}\right)}
\end{aligned}. \tag{14}
$$

In particular, if $\widetilde{W}=\left(\frac{1}{m},\frac{1}{m},\ldots,\frac{1}{m}\right)^T$, then the weighted Dice similarity measure reduces to the Dice similarity measure defined in Equation (7) and Equation (8).

Now, similar to the generalized Dice similarity measures defined in Equation (7) and Equation (8), the weighted generalized similarity measures defined above also have some special cases according to the parameter ρ.

(1) If $\rho = 0.5$, then

$$\begin{aligned} WGD_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) &= \sum_{i=1}^{m} \varpi_i \frac{d_{1i} \cdot d_{2i}}{\rho |d_{1i}|2 + (1-\rho)|d_{2i}|^2} \\ &= \sum_{i=1}^{m} \varpi_i \frac{\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\rho\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-\rho)\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \\ &= \sum_{i=1}^{m} \varpi_i \frac{\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{0.5\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-0.5)\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \\ &= \sum_{i=1}^{m} \varpi_i \frac{2\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + \left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \end{aligned} . \quad (15)$$

(2) If $\rho = 0, 1$, then Equation (13) reduces to the following asymmetric weighted generalized Dice similarity measures:

$$\begin{aligned} WGD_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) &= \sum_{i=1}^{m} \varpi_i \frac{d_{1i} \cdot d_{2i}}{\rho |d_{1i}|2 + (1-\rho)|d_{2i}|^2} \\ &= \sum_{i=1}^{m} \varpi_i \frac{\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\rho\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-\rho)\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \\ &= \sum_{i=1}^{m} \varpi_i \frac{\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{0\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-0)\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \\ &= \sum_{i=1}^{n} \varpi_i \frac{2\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \qquad for\ \rho = 0 \end{aligned} , \quad (16)$$

$$\begin{aligned} WGD_{DVNS1}\left(\widetilde{D}_1, \widetilde{D}_2\right) &= \sum_{i=1}^{m} \varpi_i \frac{d_{1i} \cdot d_{2i}}{\rho |d_{1i}|2 + (1-\rho)|d_{2i}|^2} \\ &= \sum_{i=1}^{m} \varpi_i \frac{\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\rho\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-\rho)\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \\ &= \sum_{i=1}^{m} \varpi_i \frac{\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{1\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-1)\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \\ &= \sum_{i=1}^{m} \varpi_i \frac{\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\rho\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right)} \qquad for\ \rho = 1 \end{aligned} . \quad (17)$$

Similarly, when $\rho = 0.5$ in Equation (14), then

$$\begin{aligned} WGD_{DVNS2}\left(\widetilde{D}_1, \widetilde{D}_2\right) &= \frac{\left(\widetilde{D}_1 . \widetilde{D}_2\right)_{\varpi}}{\rho\left|\widetilde{D}_1\right|_{\varpi}{}^2 + (1-\rho)\left|\widetilde{D}_2\right|_{\varpi}{}^2} \\ &= \frac{\sum_{i=1}^{m} \varpi_i^2\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{\rho\sum_{i=1}^{m} \varpi_i^2\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-\rho)\sum_{i=1}^{m} \varpi_i^2\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \\ &= \frac{\sum_{i=1}^{m} \varpi_i^2\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)}{0.5\sum_{i=1}^{m} \varpi_i^2\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + (1-0.5)\sum_{i=1}^{m} \varpi_i^2\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \\ &= \frac{2\left(\sum_{i=1}^{m} \varpi_i^2\left(t_{1i}t_{2i} + i_{t_{1i}}i_{t_{2i}} + i_{f_{1i}}i_{f_{2i}} + f_{1i}f_{2i}\right)\right)}{\sum_{i=1}^{m} \varpi_i^2\left(t^2{}_{1i} + i^2{}_{t_{1i}} + i^2{}_{f_{1i}} + f^2{}_{1i}\right) + \sum_{i=1}^{m} \varpi_i^2\left(t^2{}_{2i} + i^2{}_{t_{2i}} + i^2{}_{f_{2i}} + f^2{}_{2i}\right)} \end{aligned} . \quad (18)$$

(3) If $\rho = 0, 1$, then Equation (14) reduces to the following asymmetric similarity measures:

$$\begin{aligned} WGD_{DVNS2}\left(\widetilde{D}_1, \widetilde{D}_2\right) &= \frac{\left(\widetilde{D}_1.\widetilde{D}_2\right)_{\varpi}}{\rho\left|\widetilde{D}_1\right|_{\varpi}{}^2+(1-\rho)\left|\widetilde{D}_2\right|_{\varpi}{}^2} \\ &= \frac{\sum\limits_{i=1}^{m} \varpi_i^2\left(t_{1i}t_{2i}+i_{t_{1i}}i_{t_{2i}}+i_{f_{1i}}i_{f_{2i}}+f_{1i}f_{2i}\right)}{\rho\sum\limits_{i=1}^{m} \varpi_i^2\left(t^2{}_{1i}+i^2{}_{t_{1i}}+i^2{}_{f_{1i}}+f^2{}_{1i}\right)+(1-\rho)\sum\limits_{i=1}^{m} \varpi_i^2\left(t^2{}_{2i}+i^2{}_{t_{2i}}+i^2{}_{f_{2i}}+f^2{}_{2i}\right)} \\ &= \frac{\sum\limits_{i=1}^{m} \varpi_i^2\left(t_{1i}t_{2i}+i_{t_{1i}}i_{t_{2i}}+i_{f_{1i}}i_{f_{2i}}+f_{1i}f_{2i}\right)}{0\sum\limits_{i=1}^{m} \varpi_i^2\left(t^2{}_{1i}+i^2{}_{t_{1i}}+i^2{}_{f_{1i}}+f^2{}_{1i}\right)+(1-0)\sum\limits_{i=1}^{m} \varpi_i^2\left(t^2{}_{2i}+i^2{}_{t_{2i}}+i^2{}_{f_{2i}}+f^2{}_{2i}\right)} \\ &= \frac{\sum\limits_{i=1}^{n} \varpi_i^2\left(t_{1i}t_{2i}+i_{t_{1i}}i_{t_{2i}}+i_{f_{1i}}i_{f_{2i}}+f_{1i}f_{2i}\right)}{\sum\limits_{i=1}^{n} \varpi_i^2\left(t^2{}_{2i}+i^2{}_{t_{2i}}+i^2{}_{f_{2i}}+f^2{}_{2i}\right)} \qquad for\ \rho = 0 \end{aligned}, \tag{19}$$

$$\begin{aligned} WGD_{DVNS2}\left(\widetilde{D}_1, \widetilde{D}_2\right) &= \frac{\left(\widetilde{D}_1.\widetilde{D}_2\right)_{\varpi}}{\rho\left|\widetilde{D}_1\right|_{\varpi}{}^2+(1-\rho)\left|\widetilde{D}_2\right|_{\varpi}{}^2} \\ &= \frac{\sum\limits_{i=1}^{m} \varpi_i^2\left(t_{1i}t_{2i}+i_{t_{1i}}i_{t_{2i}}+i_{f_{1i}}i_{f_{2i}}+f_{1i}f_{2i}\right)}{\rho\sum\limits_{i=1}^{m} \varpi_i^2\left(t^2{}_{1i}+i^2{}_{t_{1i}}+i^2{}_{f_{1i}}+f^2{}_{1i}\right)+(1-\rho)\sum\limits_{i=1}^{m} \varpi_i^2\left(t^2{}_{2i}+i^2{}_{t_{2i}}+i^2{}_{f_{2i}}+f^2{}_{2i}\right)} \\ &= \frac{\sum\limits_{i=1}^{m} \varpi_i^2\left(t_{1i}t_{2i}+i_{t_{1i}}i_{t_{2i}}+i_{f_{1i}}i_{f_{2i}}+f_{1i}f_{2i}\right)}{1\sum\limits_{i=1}^{m} \varpi_i^2\left(t^2{}_{1i}+i^2{}_{t_{1i}}+i^2{}_{f_{1i}}+f^2{}_{1i}\right)+(1-1)\sum\limits_{i=1}^{m} \varpi_i^2\left(t^2{}_{2i}+i^2{}_{t_{2i}}+i^2{}_{f_{2i}}+f^2{}_{2i}\right)} \\ &= \frac{2\left(\sum\limits_{i=1}^{n} \varpi_i^2\left(t_{1i}t_{2i}+i_{t_{1i}}i_{t_{2i}}+i_{f_{1i}}i_{f_{2i}}+f_{1i}f_{2i}\right)\right)}{\sum\limits_{i=1}^{n} \varpi_i^2\left(t^2{}_{1i}+i^2{}_{t_{1i}}+i^2{}_{f_{1i}}+f^2{}_{1i}\right)} \qquad for\ \rho = 1 \end{aligned}. \tag{20}$$

From the above investigation, the four asymmetric similarity measures are the extension of the relative projection measure of interval numbers [31]. Therefore, the four asymmetric similarity measures can be assumed as the projection measures of DVNSs.

6. Applications of Generalized Dice Similarity Measures for DVNSs

6.1. Pattern Recognition

In order to show the effectiveness of the proposed generalized Dice measures for DVNSs in pattern recognition, we present an example in this subsection.

Example 1. *Let us suppose that we have three patterns $\widetilde{R}_1, \widetilde{R}_2$ and $\widetilde{R}_3$. Then, the patterns are represented by the following DVNSs on $\widetilde{U} = \{\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3, \widetilde{u}_4\}$.*

$$\begin{aligned} \widetilde{R}_1 &= \{\langle\widetilde{u}_1, (0.5, 0.2, 0.2, 0.4)\rangle, \langle\widetilde{u}_2, (0.7, 0.1, 0.2, 0.1)\rangle, \langle\widetilde{u}_3, (0.4, 0.3, 0.2, 0.5)\rangle, \langle\widetilde{u}_4, (0.6, 0.2, 0.3, 0.4)\rangle\}, \\ \widetilde{R}_2 &= \{\langle\widetilde{u}_1, (0.5, 0.1, 0.3, 0.4)\rangle, \langle\widetilde{u}_2, (0.6, 0.1, 0.1, 0.2)\rangle, \langle\widetilde{u}_3, (0.2, 0.2, 0.3, 0.7)\rangle, \langle\widetilde{u}_4, (0.7, 0.2, 0.3, 0.3)\rangle\}, \\ \widetilde{R}_3 &= \{\langle\widetilde{u}_1, (0.6, 0.3, 0.2, 0.3)\rangle, \langle\widetilde{u}_2, (0.8, 0.1, 0.1, 0.2)\rangle, \langle\widetilde{u}_3, (0.4, 0.2, 0.3, 0.5)\rangle, \langle\widetilde{u}_4, (0.7, 0.2, 0.1, 0.2)\rangle\} \end{aligned}$$

and the unknown pattern, $\widetilde{P}$, is given as follows:

$$\widetilde{P} = \{\langle\widetilde{u}_1, (0.4, 0.3, 0.3, 0.5)\rangle, \langle\widetilde{u}_2, (0.8, 0.1, 0.2, 0.1)\rangle, \langle\widetilde{u}_3, (0.3, 0.2, 0.1, 0.6)\rangle, \langle\widetilde{u}_4, (0.7, 0.1, 0.1, 0.3)\rangle\}.$$

The aim was to find out to which known pattern the unknown pattern, $\widetilde{P}$, belonged. To show this, the generalized Dice distance measures between the known and unknown patterns were calculated, and then, the unknown pattern, $\widetilde{P}$, was assigned to one of the known patterns using the following formula:

$$Z^* = \arg\max_l\left\{D\left(\widetilde{R}_l, \widetilde{P}\right)\right\} \ for\ l = 1, 2, 3.$$

Furthermore, if the weight is considered, then we used the following formula:

$$\widetilde{Z}^* = \arg\max_l \left\{ GWD\left(\widetilde{R}_l, \widetilde{P}\right) \right\} \; for \; l = 1, 2, 3.$$

In Table 1, the generalized Dice measures between the unknown and known patterns are shown, calculated using Equation (7).

Table 1. Generalized Dice measures.

ρ	$GD_{DVNS1}\left(\widetilde{R}_1, \widetilde{P}\right)$	$GD_{DVNS1}\left(\widetilde{R}_2, \widetilde{P}\right)$	$GD_{DVNS1}\left(\widetilde{R}_3, \widetilde{P}\right)$
0	0.9376	0.9471	0.9619
0.3	0.9528	0.9445	0.9598
0.6	0.9721	0.9561	0.9580
0.8	0.9876	0.9734	0.9569
1	**1.0060**	**1.0001**	**0.9560**

The generalized Dice measures calculated using Equation (8) are given in Table 2.

Table 2. Generalized Dice measures.

ρ	$GD_{DVNS2}\left(\widetilde{R}_1, \widetilde{P}\right)$	$GD_{DVNS2}\left(\widetilde{R}_2, \widetilde{P}\right)$	$GD_{DVNS2}\left(\widetilde{R}_3, \widetilde{P}\right)$
0	0.9331	0.9330	0.9623
0.3	0.9522	0.9437	0.9821
0.6	0.9721	0.9546	1.0030
0.8	0.9858	0.9620	1.0170
1	**1.0000**	**0.9696**	**1.0314**

Let us assume that the weight vector of $\widetilde{u}_1$, $\widetilde{u}_2$, $\widetilde{u}_3$ and $\widetilde{u}_4$ is $\varpi = (0.3, 0.25, 0.25, 0.2)^T$. Then, the weighted Dice measures, calculated using Equation (13), are given in Table 3.

Table 3. Generalized weighted Dice measures.

ρ	$GWD_{DVNS1}\left(\widetilde{R}_1, \widetilde{P}\right)$	$GWD_{DVNS1}\left(\widetilde{R}_2, \widetilde{P}\right)$	$GWD_{DVNS1}\left(\widetilde{R}_3, \widetilde{P}\right)$
0	0.9325	0.9387	**0.9593**
0.3	0.9512	0.9407	**0.9570**
0.6	**0.9743**	0.9568	0.9549
0.8	**0.9925**	0.9770	0.9537
1	**1.0130**	1.0070	0.9526

Additionally, the generalized weighted Dice measures, calculated using Equation (14), are given in Table 4.

Table 4. Generalized weighted Dice measures.

ρ	$GWD_{DVNS2}\left(\widetilde{R}_1, \widetilde{P}\right)$	$GWD_{DVNS2}\left(\widetilde{R}_2, \widetilde{P}\right)$	$GWD_{DVNS2}\left(\widetilde{R}_3, \widetilde{P}\right)$
0	0.9231	0.9172	**0.9569**
0.3	0.9490	0.9362	**0.9982**
0.6	0.9765	0.9560	**1.0431**
0.8	0.9957	0.9697	**1.0754**
1	1.0150	0.9838	**1.1098**

From Table 1, we can see that, when the parameter $\rho = 0, 0.3$, then the unknown pattern, $\widetilde{P}$, belonged to pattern $\widetilde{R}_3$. When the value of parameter ρ was greater than 0.3, that is $\rho = 0.6, 0.8, 1$, then the unknown pattern, $\widetilde{P}$, belonged to pattern $\widetilde{R}_1$. Furthermore, from Table 2, we can see that, if the values of the parameter ρ were changed, the unknown pattern, $\widetilde{P}$, belonged to pattern $\widetilde{R}_3$.

Similarly, from Tables 3 and 4, the weighted generalized measures displayed the same situation discussed above.

6.2. Medical Diagnoses

In this subsection, we show the effectiveness of the proposed generalized Dice measures in medical diagnoses, using an example.

Example 2. *Let us assume that there are four patients, and the names of the patients are Al, Bob, Jeo, and Ted. Their symptoms are temperature, headache, stomach pain, cough, and chest pain. The set of patients and symptoms are denoted by* $H = \{Al, Bob, Jeo, Ted\}$ *and* $L = \{Temprature, Headache, Stomach\ pain, cough, Chest\ pain\}$. *Let us assume that the set of diagnoses under consideration is defined and denoted by* $Y = \{Malaria, viral\ fever, Typhoid, Stomach\ problem, Heart\ problem\}$. *The characteristic information of* H, L *and* Y *is represented in the form of DVNSs, given in Tables 5 and 6 below.*

Table 5. Symptom characteristics for the patient.

Patients	Temperature	Headache	Stomach pain	Cough	Chest Pain
Al	$\langle 0.7, 0.1, 0.1, 0.1\rangle$	$\langle 0.4, 0.2, 0.1, 0.1\rangle$	$\langle 0.2, 0.3, 0.2, 0.4\rangle$	$\langle 0.4, 0.2, 0.1, 0.1\rangle$	$\langle 0.1, 0.1, 0.3, 0.4\rangle$
Bob	$\langle 0.1, 0.2, 0.3, 0.5\rangle$	$\langle 0.3, 0.2, 0.3, 0.4\rangle$	$\langle 0.4, 0.2, 0.1, 0.1\rangle$	$\langle 0.1, 0.2, 0.2, 0.5\rangle$	$\langle 0.1, 0.3, 0.4, 0.5\rangle$
Jeo	$\langle 0.7, 0.1, 0.1, 0.1\rangle$	$\langle 0.7, 0.1, 0.1, 0.1\rangle$	$\langle 0.1, 0.1, 0.2, 0.4\rangle$	$\langle 0.2, 2, 0.3, 0.4\rangle$	$\langle 0.1, 0.1, 0.3, 0.3\rangle$
Ted	$\langle 0.4, 0.2, 0.1, 0.1\rangle$	$\langle 0.5, 0.1, 0.2, 0.5\rangle$	$\langle 0.3, 0.2, 0.1, 0.4\rangle$	$\langle 0.6, 0.1, 0.1, 0.2\rangle$	$\langle 0.3, 0.1, 0.2, 0.3\rangle$

Table 6. Symptom characteristics for the diagnoses.

Diagnoses	Temperature	Headache	Stomach pain	Cough	Chest Pain
Malaria	$\langle 0.3, 0.1, 0.1, 0.1\rangle$	$\langle 0.2, 0.3, 0.2, 0.4\rangle$	$\langle 0.1, 0.3, 0.4, 0.5\rangle$	$\langle 0.1, 0.2, 0.4, 0.4\rangle$	$\langle 0.1, 0.3, 0.4, 0.5\rangle$
Viral fever	$\langle 0.6, 0.1, 0.1, 0.1\rangle$	$\langle 0.2, 0.1, 0.3, 0.5\rangle$	$\langle 0.1, 0.1, 0.4, 0.7\rangle$	$\langle 0.6, 0.1, 0.1, 0.1\rangle$	$\langle 0.1, 0.1, 0.3, 0.7\rangle$
Typhoid	$\langle 0.3, 0.1, 0.1, 0.3\rangle$	$\langle 0.5, 0.1, 0.2, 0.1\rangle$	$\langle 0.2, 0.1, 0.2, 0.6\rangle$	$\langle 0.2, 0.1, 0.3, 0.5\rangle$	$\langle 0.1, 0.1, 0.4, 0.8\rangle$
Stomach problem	$\langle 0.1, 0.1, 0.3, 0.6\rangle$	$\langle 0.2, 0.2, 0.2, 0.5\rangle$	$\langle 0.8, 0.1, 0.1, 0.1\rangle$	$\langle 0.2, 0.2, 0.3, 0.7\rangle$	$\langle 0.2, 0.2, 0.3, 0.7\rangle$
Heart problem	$\langle 0.1, 0.1, 0.3, 0.6\rangle$	$\langle 0.1, 0.2, 0.4, 0.8\rangle$	$\langle 0.2, 0.1, 0.3, 0.7\rangle$	$\langle 0.2, 0.1, 0.4, 0.6\rangle$	$\langle 0.8, 0.1, 0.1, 0.1\rangle$

Subsequently, using Equations (7) and (8), we calculated the generalized Dice measures for the parameter $\rho = 0$, and the results are given in Tables 7 and 8.

Table 7. Generalized Dice measures.

	Malaria	Viral Fever	Typhoid	Stomach Problem	Heart Problem
Al	**0.9018**	0.7191	0.7813	0.4406	0.3675
Bob	**0.8756**	0.5695	0.8100	0.7899	0.5693
Jeo	**0.9702**	0.6765	0.9216	0.4808	0.4314
Ted	**0.8422**	0.7779	0.7781	0.5780	0.5084

Table 8. Generalized Dice measures.

	Malaria	Viral Fever	Typhoid	Stomach Problem	Heart Problem
Al	**0.7337**	0.7008	0.6912	0.4382	0.3574
Bob	**0.8152**	0.5574	0.7512	0.7633	0.5486
Jeo	**0.7826**	0.6516	0.7788	0.4735	0.4138
Ted	**0.7337**	0.7295	0.6866	0.5406	0.5266

Similarly, for other values of parameter ρ using two types of generalized Dice measures, the results are given in Tables 9 and 10.

Table 9. Generalized Dice measures for different values of parameter ρ.

		Malaria	Viral Fever	Typhoid	Stomach Problem	Heart Problem
Al	$\rho = 0.2$	**0.7844**	0.7571	0.7500	0.4835	0.4042
	$\rho = 0.6$	0.7899	**0.8817**	0.8259	0.6146	0.5205
	$\rho = 0.8$	0.8419	**0.9826**	0.9057	0.7228	0.6219
	$\rho = 1$	0.9226	**1.1286**	1.0438	0.8933	0.7960
Bob	$\rho = 0.2$	0.8148	0.5856	0.7916	**0.8350**	0.6159
	$\rho = 0.6$	0.7817	0.6299	0.7990	**0.9644**	0.7569
	$\rho = 0.8$	0.7886	0.6643	0.8234	**1.0660**	0.8460
	$\rho = 1$	0.8125	0.7196	0.8649	**1.2243**	0.9911
Jeo	$\rho = 0.2$	0.8328	0.6921	**0.8664**	0.5088	0.4724
	$\rho = 0.6$	0.8153	0.7770	**0.8777**	0.6149	0.5996
	$\rho = 0.8$	0.8726	0.8718	**0.9495**	0.7175	0.7076
	$\rho = 1$	0.9877	1.0543	**1.1239**	0.9029	0.8871
Ted	$\rho = 0.2$	**0.8005**	0.8004	0.7678	0.6007	0.5595
	$\rho = 0.6$	0.7794	**0.8926**	0.7948	0.6980	0.7071
	$\rho = 0.8$	0.7996	**0.9777**	0.8447	0.7904	0.8221
	$\rho = 1$	0.8474	**1.1140**	0.9544	0.9441	0.8739

Table 10. Generalized Dice measures for different values of parameter ρ.

		Malaria	Viral Fever	Typhoid	Stomach Problem	Heart Problem
Al	$\rho = 0.2$	**0.7567**	0.7553	0.7324	0.4814	0.3980
	$\rho = 0.6$	0.8074	0.7125	**0.8315**	0.5996	0.5154
	$\rho = 0.8$	0.8354	**0.9850**	0.8918	0.6836	0.6045
	$\rho = 1$	0.8654	**1.0962**	0.9615	0.7949	0.7308
Bob	$\rho = 0.2$	0.8152	0.5862	0.7747	**0.8207**	0.5993
	$\rho = 0.6$	0.8152	0.5296	0.8266	**0.9660**	0.7353
	$\rho = 0.8$	0.8152	0.6939	0.8552	**1.0599**	0.8294
	$\rho = 1$	0.8152	0.7391	0.8859	**1.1739**	0.9511
Jeo	$\rho = 0.2$	0.7869	0.6883	**0.8071**	0.5111	0.4536
	$\rho = 0.6$	0.7956	0.7756	**0.8702**	0.6074	0.5617
	$\rho = 0.8$	0.8000	0.8281	**0.9057**	0.6707	0.6377
	$\rho = 1$	0.8045	0.8883	**0.9441**	0.7486	0.7374
Ted	$\rho = 0.2$	0.7434	**0.7753**	0.7163	0.5867	0.5801
	$\rho = 0.6$	0.7636	**0.8865**	0.7842	0.7070	0.7279
	$\rho = 0.8$	0.7741	**0.9549**	0.8232	0.7878	0.8342
	$\rho = 1$	0.7849	**1.0349**	0.8663	0.8895	0.9767

From Tables 7 and 8, we can see that all patients suffered from malaria when the value of parameter $\rho = 0$. On the other hand, from Table 9, we can see that, when $\rho = 0.2$, Al suffered from malaria, Bob suffered from a stomach problem, Jeo suffered from typhoid, and Ted suffered from malaria. When the values of parameter $\rho = 0.6, 0.8, 1$, then Al and Ted suffered from a viral fever, and Bob suffered from a stomach problem, while Jeo suffered from typhoid. From Table 10, we can see that, when $\rho = 0.2, 0.6$, Al suffered from malaria and typhoid, while Bob, Jeo, and Ted suffered from a stomach problem, malaria, and a viral fever. When $\rho = 0.8, 1$, then Al suffered from a viral fever, while Bob, Jeo, and Ted suffered from a stomach problem, malaria, and a viral fever, respectively.

7. Comparison and Discussion

A DVN set is a generalization of the neutrosophic set, intuitionistic fuzzy set, and fuzzy set. A DVNS is an illustration of the NS, which provides more perfection and clarity with regards to representing the existing indeterminate, vague, insufficient, and inconsistent information. A DVNS has the additional characteristic of being able to relate, with more sensitivity, the indeterminate and inconsistent information. While an SVNS can handle indeterminate and inconsistent information, it cannot relate the existing indeterminacy.

If we take Example 1 and use the distance measure defined by Kandasamy [28] for DVNSs, then the Hamming distance and Euclidean distance with known and unknown patterns are given in Table 11.

Table 11. Hamming and Euclidean distance measures.

	$Dis\left(\widetilde{R}_1, P\right)$	$Dis\left(\widetilde{R}_2, P\right)$	$Dis\left(\widetilde{R}_3, P\right)$
Hamming Distance	0.863	**0.095**	0.085
Euclidean Distance	0.491	**0.0602**	0.565

From Table 11, we can see that the unknown pattern, P, belonged to known pattern $\widetilde{R}_2$. When calculating our proposed Dice measure, we can see from Table 1, that when parameter $\rho = 0, 0.3$, then the unknown pattern, $\widetilde{P}$, belonged to pattern $\widetilde{R}_3$. When the value of parameter ρ was greater than 0.3, that is $\rho = 0.6, 0.8, 1$, then the unknown pattern, $\widetilde{P}$, belonged to pattern $\widetilde{R}_1$. Furthermore, from Table 2, we can see that, if the values of parameter ρ were changed, the unknown pattern, $\widetilde{P}$, belonged to pattern $\widetilde{R}_3$.

Thus, our proposed Dice similarity measure is more suitable for use in pattern recognition or medical diagnosis.

8. Conclusions

Neutrosophic sets (NSs) are used to illustrate uncertain, inconsistent, and indeterminate information which exists in real-world problems. Double-valued neutrosophic sets (DVNSs) are an alternate form of neutrosophic sets, in which the indeterminacy has two distinct parts: indeterminacy leaning toward truth membership, and indeterminacy leaning toward falsity membership. The aim of this article was to propose novel Dice measures and generalized Dice measures for DVNSs, and to specify that the Dice measures and the asymmetric measures (projection measures) were special cases of the generalized Dice measures using specific parameter values. Finally, the proposed generalized Dice measures and generalized weighted Dice measures were applied to pattern recognition and medical diagnosis to show their effectiveness.

Author Contributions: All authors contributed equally.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare that they have no competing interest.

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