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Some Linguistic Neutrosophic Cubic Mean Operators and Entropy with Applications in a Corporation to Choose an Area Supervisor

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Abstract: In this paper, we combined entropy with linguisti neutrosophic cubic numbers and used it in daily life problems related to a corporation that is going to choose an area supervisor, which is the main target of our proposed model. For this, we first develop the theory of linguistic neutrosophic cubic numbers, which explains the indeterminate and incomplete information by truth, indeterminacy and falsity linguistic variables (LVs) for the past, present, as well as for the future time very effectively. After giving the definitions, we initiate some basic operations and properties of linguistic neutrosophic cubic numbers. We also define the linguistic neutrosophic cubic Hamy mean operator and weighted linguistic neutrosophic cubic Hamy mean (WLNCHM) operator with some properties, which can handle multi-input agents with respect to the different time frame. Finally, as an application, we give a numerical example in order to test the applicability of our proposed model.

Keywords: neutrosophic set; neutrosophic cubic set; linguistic neutrosophic cubic numbers; linguistic neutrosophic cubic weighted averaging operator; entropy of linguistic neutrosophic cubic numbers; application; multiple attribute decision making problem

1. Introduction

In 1965, Zadeh [1] introduced the notion of fuzzy sets, which became a significant tool of studying many vague and uncertain concepts. It has a large number of applications in social, medicine and computer sciences. Atanassov [2] generalized the theme of a fuzzy set (FS) by initiating the idea of intuitionistic fuzzy sets (IFS) by introducing the idea of non membership of an element in a certain set. Jun et al. [3] initiated the idea of cubic sets, in which there are two representations: one is used for the membership/certain value and the other one is used for the non membership/uncertain value. The membership function is handled in the form of an interval, and the non membership is handled by the ordinary fuzzy set. Cubic sets have been considered by many authors in other areas of mathematics, for instance KU subalgebras [4,5], graph theory [6], left almost Γ-semihypergroups [7], LA-semihypergroups [8–11], semigroups [12,13] and Hv-LA-semigroups [14,15]. Smarandache [16,17] presented the new idea of the neutrosophic set (NS) and neutrosophic logic, which the generalized fuzzy set and intuitionistic fuzzy set. The neutrosophic set (NS) is defined by truth, indeterminacy and falsity membership degrees. For applications in physical, technical and different engineering regions, Wang et al. [18] suggested the concept of a single-valued neutrosophic set (SVNS) in 2010. After this, many researchers used neutrosophic sets in different research directions such as De and Beg [19] and Gulistan et al. [20]. Jun et al. [21,22] extended the idea of cubic sets to neutrosophic cubic sets and defined different properties of external and internal neutrosophic cubic sets. Recently, Gulistan et al. [23] combined neutrosophic cubic sets with graphs. In multi-criteria decision making problems, the application of neutrosophic cubic sets was proposed by Zhan et al. [24]. In [25], Hashim et al. used neutrosophic bipolar fuzzy sets in the HOPE foundation with different types of similarity measures. For the aspects of real-life objectives, the human desire of judgment can be used for linguistic expression rather than numerical expression to better suit the thinking of people. Therefore, Zadeh [26] introduced the concept of linguistic variable and applied it to fuzzy reasoning. The idea of aggregation operators was presented by many researchers in decision making problems; see for example [27–29]. The concept of linguistic intuitionistic fuzzy numbers (LIFN) was introduced by Chen et al. [30]. After that, some researchers also gave the idea of linguistic intuitionistic multi-criteria group decision-making problems [31]. The theme of LNN_S was initiated by Fang et al. [32]. Besides, a multi-criteria decision making problem like the linguistic intuitionistic multi-criteria decision-making problem was also introduced [33]. Ye in 2016 presented the concept of an LNN_S and also gave the idea of different aggregation operators in multiple attribute group decision making problems [34]. Then, the concept of a linguistic neutrosophic number was proposed to solve multiple attribute group decision making problems by Li et al. in [35]. In [36], Hara et al. proposed some inequalities for certain bivariate means. A useful tool known as entropy is used to determine the uncertainty in sets, like the fuzzy set (FS) and intuitionistic fuzzy set (IFS), where LNCSis defined by managing uncertain information about truth, indeterminacy and falsity membership functions. In 1965, Zadeh [37] first defined the entropy of FS to determine the ambiguity in a quantitative manner. In the same way, the non-probabilistic entropy was axiomatized by De Luca-Termini [38]. He also analyzed mathematical properties of this functional and gave the considerations of and applicability to pattern analysis. A distance entropy measure was proposed by Kaufmann [39]. A new non-probabilistic entropy measure was introduced by Kosko [40]. In [41], Majumdar and Samanta introduced the notion of two single-valued neutrosophic sets, their properties and also defined the distance between these two sets. They also investigated the measure of entropy of a single-valued neutrosophic set. The entropy of IFSs was introduced by Szmidt and Kacprzyk [42]. This entropy measure was consistent with the considerations of fuzzy sets. Afterward, the measurement of fuzziness in terms of distance between the fuzzy set and its compliment was put forward by Yager [43]; see also [37,44] for more details. The entropy in terms of neutrosophic sets was discussed by Patrascu in [45]. The of linguistic neutrosophic numbers (LNNs) and the linguistic neutrosophic Hamy mean (HM) (LNHM) operator was investigated by Liu et al., in [46]. Ye discussed linguistic neutrosophic cubic numbers and their multiple attribute decision making method in [47].

The present study proposes a new notion of linguistic neutrosophic cubic numbers (LNCNs), where the undetermined LNN_S agrees with the truth, indeterminacy and falsity membership. Besides that, we define the different operations on LNCNs, the linguistic neutrosophic cubic Hamy mean operator and the weighted linguistic neutrosophic cubic Hamy mean (WLNCHM) operator with some properties that can handle multi-input agents with respect to the different time frames. We define score, accuracy and certain functions of LNCNs. At the end, we use the developed approach in a decision making problem related to a corporation choosing an area supervisor.

2. Preliminaries

In this section, we give some helpful material from the existing literature.

Definition 1. *[35] LNN_S (linguistic neutrosophic numbers): Let U be a universal set and $\mathring{p} = (\mathring{p}_0, \mathring{p}_1, \ldots, \mathring{p}_t)$ be a linguistic term set (LT^S). An LNS$\breve{A}$ in U is specified by the truth, indeterminacy and falsity membership functions $\mathring{\alpha}_{\mathring{A}}, \mathring{\beta}_{\mathring{A}}$ and $\mathring{\gamma}_{\mathring{A}}$, where $\mathring{\alpha}_{\mathring{A}}, \mathring{\beta}_{\mathring{A}}, \mathring{\gamma}_{\mathring{A}} : U \to [0, t]$, and $\forall\, u \in U$, $\mathring{g} = (\mathring{p}_{\mathring{\alpha}_A(u)}, \mathring{p}_{\mathring{\beta}_A(u)}, \mathring{p}_{\mathring{\gamma}_A(u)}) \in \mathring{A}$ is called an LNN of $\mathring{A}$.*

Remark 1. [35] *Let $\mathring{A}$ be the set of LNN$_S$, then its complement is represented by $\mathring{A}^C$, which is denoted as $\mathring{\alpha}_{\mathring{A}} = \mathring{\gamma}_{\mathring{A}}; \mathring{\beta}_{\mathring{A}} = t - \mathring{\beta}_{\mathring{A}}; \mathring{\gamma}_{\mathring{A}} = \mathring{\alpha}_{\mathring{A}}$.*

Definition 2. [35] *Let $\mathring{g} = (\mathring{p}_{\mathring{\alpha}}, \mathring{p}_{\mathring{\beta}}, \mathring{p}_{\mathring{\gamma}}), \mathring{g}_1 = (\mathring{p}_{\mathring{\alpha}_1}, \mathring{p}_{\mathring{\beta}_1}, \mathring{p}_{\mathring{\gamma}_1}), \mathring{g}_2 = (\mathring{p}_{\mathring{\alpha}_2}, \mathring{p}_{\mathring{\beta}_2}, \mathring{p}_{\mathring{\gamma}_2})$ be any LNN$_S$ and $\lambda > 0$. Then (i):*

$$\mathring{g}_1 \oplus \mathring{g}_2 = (\mathring{p}_{\mathring{\alpha}_1}, \mathring{p}_{\mathring{\beta}_1}, \mathring{p}_{\mathring{\gamma}_1}) \oplus (\mathring{p}_{\mathring{\alpha}_2}, \mathring{p}_{\mathring{\beta}_2}, \mathring{p}_{\mathring{\gamma}_2}) = \left(\mathring{p}_{\mathring{\alpha}_1+\mathring{\alpha}_2-\frac{\mathring{\alpha}_1\mathring{\alpha}_2}{t}}, \mathring{p}_{\frac{\mathring{\beta}_1\mathring{\beta}_2}{t}}, \mathring{p}_{\frac{\mathring{\gamma}_1\mathring{\gamma}_2}{t}}\right) \tag{1}$$

(ii):

$$\mathring{g}_1 \otimes \mathring{g}_2 = (\mathring{p}_{\mathring{\alpha}_1}, \mathring{p}_{\mathring{\beta}_1}, \mathring{p}_{\mathring{\gamma}_1}) \otimes (\mathring{p}_{\mathring{\alpha}_2}, \mathring{p}_{\mathring{\beta}_2}, \mathring{p}_{\mathring{\gamma}_2}) = \left(\mathring{p}_{\frac{\mathring{\alpha}_1\mathring{\alpha}_2}{t}}, \mathring{p}_{\mathring{\beta}_1+\mathring{\beta}-\frac{\mathring{\beta}_1\mathring{\beta}_2}{t}}, \mathring{p}_{\mathring{\gamma}_1+\mathring{\gamma}_2-\frac{\mathring{\gamma}_1\mathring{\gamma}_2}{t}}\right) \tag{2}$$

(iii):

$$\lambda\mathring{g} = \lambda(\mathring{p}_{\mathring{\alpha}}, \mathring{p}_{\mathring{\beta}}, \mathring{p}_{\mathring{\gamma}}) = \left(\mathring{p}_{t-t(1-\frac{\mathring{\alpha}}{t})^\lambda}, \mathring{p}_{t(\frac{\mathring{\beta}}{t})^\lambda}, \mathring{p}_{t(\frac{\mathring{\gamma}}{t})^\lambda}\right); \tag{3}$$

(iv)

$$\mathring{g}^\lambda = (\mathring{p}_{\mathring{\alpha}}, \mathring{p}_{\mathring{\beta}}, \mathring{p}_{\mathring{\gamma}})^\lambda = \left(\mathring{p}_{t(\frac{\mathring{\alpha}}{t})^\lambda}, \mathring{p}_{t-t(1-\frac{\mathring{\beta}}{t})^\lambda}, \mathring{p}_{t-t(1-\frac{\mathring{\gamma}}{t})^\lambda}\right). \tag{4}$$

Definition 3. [35] *Let $\mathring{g} = (\mathring{p}_{\mathring{\alpha}}, \mathring{p}_{\mathring{\beta}}, \mathring{p}_{\mathring{\gamma}})$ be an LNN. The following are the score and accuracy function of LNN,*

$$\hat{S}(\mathring{g}) = \frac{2t + \mathring{\alpha} - \mathring{\beta} - \mathring{\gamma}}{3t}, \tag{5}$$

$$\hat{H}(\mathring{g}) = \frac{\mathring{\alpha} - \mathring{\gamma}}{t}. \tag{6}$$

Definition 4. [35] *Let $\mathring{g}_1 = (\mathring{p}_{\mathring{\alpha}_1}, \mathring{p}_{\mathring{\beta}_1}, \mathring{p}_{\mathring{\gamma}_1}), \mathring{g}_2 = (\mathring{p}_{\mathring{\alpha}_2}, \mathring{p}_{\mathring{\beta}_2}, \mathring{p}_{\mathring{\gamma}_2})$ be LNNs. Then: (1) If $\hat{S}(\mathring{g}_1) < \hat{S}(\mathring{g}_2)$, then $\mathring{g}_1 \prec \mathring{g}_2$. (2) If $\hat{S}(\mathring{g}_1) = \hat{S}(\mathring{g}_2)$, (a) and $\hat{H}(\mathring{g}_1) < \hat{H}(\mathring{g}_2)$, then $\mathring{g}_1 \prec \mathring{g}_2$, (b) and $\hat{H}(\mathring{g}_1) = \hat{H}(\mathring{g}_2)$, then $\mathring{g}_1 \approx \mathring{g}_2$.*

Definition 5. [36] *Suppose $u_{\hat{\imath}}(\hat{\imath} = 1, 2, \ldots, n)$ is an assortment of non-negative real numbers and parameter $\mathring{k} = 1, 2, \ldots, n$. The Hamy mean (HM) is defined as:*

$$HM^{\mathring{k}}(x_1, x_2, \ldots, x_n) = \frac{\sum\limits_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(\prod\limits_{j=1}^{\mathring{k}} u_{\hat{\imath}j}\right)^{\frac{1}{\mathring{k}}}}{\binom{n}{\mathring{k}}} \tag{7}$$

where $(\hat{\imath}_1, \hat{\imath}_2, \ldots, \hat{\imath}_{\mathring{k}})$ navigate all the k-tuple arrangements of $(1, 2, \ldots, n)$, $\binom{n}{\mathring{k}}$ is the binomial coefficient and $\binom{n}{\mathring{k}} = \frac{n!}{\mathring{k}!(n-\mathring{k})!}$. The following are some properties of HM: (1) $HM^{(\mathring{k})}(0, 0, \ldots, 0) = 0, HM^{(\mathring{k})}(u, u, \ldots, u) = u$, (2) $HM^{(\mathring{k})}(u_1, u_2, \ldots, u_n) \le HM^{(\mathring{k})}(v_1, v_2, \ldots, v_n)$, if $u_{\hat{\imath}} \le v_{\hat{\imath}}$ for all $\hat{\imath}$, (3) $\min\{u_{\hat{\imath}}\} \le HM^{(\mathring{k})}(u_1, u_2, \ldots, u_n) \le \max\{u_{\hat{\imath}}\}$.

Definition 6. [17] *(Neutrosophic set) Let U be a non-empty set. A neutrosophic set in U is a structure of the form $A := \{u; A_{Tru}(u), A_{\hat{\imath}nd}(u), A_{Fal}(u) | u \in U\}$, is characterized by a truth membership Tru, indeterminacy membership $\hat{\imath}nd$ and falsity membership Fal, where $A_{Tru}, A_{\hat{\imath}nd}, A_{Fal} : U \to [0, 1]$.*

Definition 7. [21] *(Neutrosophic cubic set) Let Xbe a non-empty set; an NCSin U is defined in the form of a pair $\Omega = (\mathring{A}, \Lambda)$ where $\mathring{A} = \{(x; \mathring{A}_{\tilde{T}ru(u)}, \mathring{A}_{\tilde{I}nd(u)}, \mathring{A}_{\tilde{F}al(u)}) \mid u \in U\}$ is an interval neutrosophic set in U and $\Lambda = \{(u; \Lambda_{Tru(u)}, \Lambda_{\hat{\imath}nd(u)}, \Lambda_{Fal(u)}) \mid u \in U)\}$ is a neutrosophic set in U.*

3. Linguistic Neutrosophic Cubic Numbers and Operators

In this section, we define the linguistic neutrosophic cubic numbers and also discuss different operations and properties related to linguistic neutrosophic cubic numbers. We define the cubic Hamy mean operator, LNCHM operator and WLNCHM operator and discuss their properties.

Definition 8. *LNCNs (linguistic neutrosophic cubic numbers): Let U be a universal set and* $\mathring{p} = (\mathring{p}_0, \mathring{p}_1, \ldots, \mathring{p}_t)$ *be a* LT^S. *An LNCN* $\mathring{A}$ *in U is determined by truth membership function* $(\tilde{\alpha}_{\mathring{A}}, \mathring{\alpha}_{\mathring{A}})$, *an indeterminacy membership function* $(\tilde{\beta}_{\mathring{A}}, \mathring{\beta}_{\mathring{A}})$ *and a falsity membership function* $(\tilde{\gamma}_{\mathring{A}}, \mathring{\gamma}_{\mathring{A}})$, *where* $\tilde{\alpha}_{\mathring{A}}, \tilde{\beta}_{\mathring{A}}, \tilde{\gamma}_{\mathring{A}} : U \to D[0,t]$ *and* $\mathring{\alpha}_{\mathring{A}}, \mathring{\beta}_{\mathring{A}}, \mathring{\gamma}_{A} : U \to [0,t], \forall\ u \in U$, *and it is denoted by* $\mathring{g} = (\mathring{p}_{(\tilde{\alpha}_{\mathring{A}}, \mathring{\alpha}_{A})(u)}, \mathring{p}_{(\tilde{\beta}_{\mathring{A}}, \mathring{\beta}_{A})(u)}, \mathring{p}_{(\tilde{\gamma}_{\mathring{A}}, \mathring{\gamma}_{A})(u)}) \in \mathring{A}$.

Remark 2. *Suppose A is a set of LNCNs, then its complement is represented by* A^c *and defined as* $\{(\tilde{\alpha}_{\mathring{A}}, \mathring{\alpha}_{\mathring{A}})^c = (\tilde{\gamma}_{\mathring{A}}, \mathring{\gamma}_{\mathring{A}}), (\tilde{\beta}_{\mathring{A}}, \mathring{\beta}_{\mathring{A}})^c = (t - \tilde{\beta}_{\mathring{A}}, t - \mathring{\beta}_{\mathring{A}}), (\tilde{\gamma}_{\mathring{A}}, \mathring{\gamma}_{\mathring{A}})^c = (\tilde{\alpha}_{\mathring{A}}, \mathring{\alpha}_{\mathring{A}})\}$.

Definition 9. *Let* $\mathring{g} = \left(\mathring{p}_{(\tilde{\alpha},\mathring{\alpha})}, \mathring{p}_{(\tilde{\beta},\mathring{\beta})}, \mathring{p}_{(\tilde{\gamma},\mathring{\gamma})}\right)$, $\mathring{g}_1 = \left(\mathring{p}_{(\tilde{\alpha}_1,\mathring{\alpha}_1)}, \mathring{p}_{(\tilde{\beta}_1,\mathring{\beta}_1)}, \mathring{p}_{(\tilde{\gamma}_1,\mathring{\gamma}_1)}\right)$,
$\mathring{g}_2 = \left(\mathring{p}_{(\tilde{\alpha}_2,\mathring{\alpha}_2)}, \mathring{p}_{(\tilde{\beta}_2,\mathring{\beta}_2)}, \mathring{p}_{(\tilde{\gamma}_2,\mathring{\gamma}_2)}\right)$ *be any LNCNs and* $\lambda > 0$. *Then, we define:*
(i):

$$\begin{aligned}\mathring{g}_1 \oplus \mathring{g}_2 &= \left(\mathring{p}_{(\tilde{\alpha}_1,\mathring{\alpha}_1)}, \mathring{p}_{(\tilde{\beta}_1,\mathring{\beta}_1)}, \mathring{p}_{(\tilde{\gamma}_1,\mathring{\gamma}_1)}\right) \oplus \left(\mathring{p}_{(\tilde{\alpha}_2,\mathring{\alpha}_2)}, \mathring{p}_{(\tilde{\beta}_2,\mathring{\beta}_2)}, \mathring{p}_{(\tilde{\gamma}_2,\mathring{\gamma}_2)}\right) \\ &= \left(\mathring{p}_{(\tilde{\alpha}_1+\tilde{\alpha}_2, \mathring{\alpha}_1+\mathring{\alpha}_2)-\left(\frac{\tilde{\alpha}_1 \cdot \tilde{\alpha}_2}{t}, \frac{\mathring{\alpha}_1 \cdot \mathring{\alpha}_2}{t}\right)}, \mathring{p}_{\left(\frac{\tilde{\beta}_1 \cdot \tilde{\beta}_2}{t}, \frac{\mathring{\beta}_1 \cdot \mathring{\beta}_2}{t}\right)}, \mathring{p}_{\left(\frac{\tilde{\gamma}_1 \cdot \tilde{\gamma}_2}{t}, \frac{\mathring{\gamma}_1 \cdot \mathring{\gamma}_2}{t}\right)}\right);\end{aligned} \tag{8}$$

(ii):

$$\begin{aligned}\mathring{g}_1 \otimes \mathring{g}_2 &= \left(\mathring{p}_{(\tilde{\alpha}_1,\mathring{\alpha}_1)}, \mathring{p}_{(\tilde{\beta}_1,\mathring{\beta}_1)}, \mathring{p}_{(\tilde{\gamma}_1,\mathring{\gamma}_1)}\right) \otimes \left(\mathring{p}_{(\tilde{\alpha}_2,\mathring{\alpha}_2)}, \mathring{p}_{(\tilde{\beta}_2,\mathring{\beta}_2)}, \mathring{p}_{(\tilde{\gamma}_2,\mathring{\gamma}_2)}\right) \\ &= \left(\mathring{p}_{\left(\frac{\tilde{\alpha}_1 \cdot \tilde{\alpha}_2}{t}, \frac{\mathring{\alpha}_1 \cdot \mathring{\alpha}_2}{t}\right)}, \mathring{p}_{(\tilde{\beta}_1+\tilde{\beta}_2, \mathring{\beta}_1+\mathring{\beta}_2)-\left(\frac{\tilde{\beta}_1 \cdot \tilde{\beta}_2}{t}, \frac{\mathring{\beta}_1 \cdot \mathring{\beta}_2}{t}\right)}, \mathring{p}_{(\tilde{\gamma}_1+\tilde{\gamma}_2, \mathring{\gamma}_1+\mathring{\gamma}_2)-\left(\frac{\tilde{\gamma}_1 \cdot \tilde{\gamma}_2}{t}, \frac{\mathring{\gamma}_1 \cdot \mathring{\gamma}_2}{t}\right)}\right);\end{aligned} \tag{9}$$

(iii):

$$\begin{aligned}\lambda \mathring{g} &= \lambda\left(\left(\mathring{p}_{(\tilde{\alpha}_1,\mathring{\alpha}_1)}, \mathring{p}_{(\tilde{\beta}_1,\mathring{\beta}_1)}, \mathring{p}_{(\tilde{\gamma}_1,\mathring{\gamma}_1)}\right) \oplus \left(\mathring{p}_{(\tilde{\alpha}_2,\mathring{\alpha}_2)}, \mathring{p}_{(\tilde{\beta}_2,\mathring{\beta}_2)}, \mathring{p}_{(\tilde{\gamma}_2,\mathring{\gamma}_2)}\right)\right) \\ &= \left(\mathring{p}_{t-t\left(1-\frac{\tilde{\alpha}}{t}, 1-\frac{\mathring{\alpha}}{t}\right)^{\lambda}}, \mathring{p}_{t\left(\frac{\tilde{\beta}}{t}, \frac{\mathring{\beta}}{t}\right)^{\lambda}}, \mathring{p}_{t\left(\frac{\tilde{\gamma}}{t}, \frac{\mathring{\gamma}}{t}\right)^{\lambda}}\right)\end{aligned} \tag{10}$$

(iv):

$$\begin{aligned}\mathring{g}^{\lambda} &= \left(\left(\mathring{p}_{(\tilde{\alpha}_1,\mathring{\alpha}_1)}, \mathring{p}_{(\tilde{\beta}_1,\mathring{\beta}_1)}, \mathring{p}_{(\tilde{\gamma}_1,\mathring{\gamma}_1)}\right) \oplus \left(\mathring{p}_{(\tilde{\alpha}_2,\mathring{\alpha}_2)}, \mathring{p}_{(\tilde{\beta}_2,\mathring{\beta}_2)}, \mathring{p}_{(\tilde{\gamma}_2,\mathring{\gamma}_2)}\right)\right)^{\lambda} \\ &= \left(\mathring{p}_{t\left(\frac{\tilde{\alpha}}{t}, \frac{\mathring{\alpha}}{t}\right)^{\lambda}}, \mathring{p}_{t-t\left(1-\frac{\tilde{\beta}}{t}, 1-\frac{\mathring{\beta}}{t}\right)^{\lambda}}, \mathring{p}_{t-t\left(1-\frac{\tilde{\gamma}}{t}, 1-\frac{\mathring{\gamma}}{t}\right)^{\lambda}}\right)\end{aligned} \tag{11}$$

It is clear that these operational result are still LNCNs.

Definition 10. *Let* $\mathring{g} = \left(\mathring{p}_{(\tilde{\alpha},\mathring{\alpha})}, \mathring{p}_{(\tilde{\beta},\mathring{\beta})}, \mathring{p}_{(\tilde{\gamma},\mathring{\gamma})}\right)$, *be an LNCN that depends on* LT^S, $\mathring{p}$. *Then, the score function, accuracy function and certain function of the LNCN,* $\mathring{g}$, *are defined as follows:*

(i):

$$\varphi(\mathring{g}) = \varphi\left(\mathring{p}_{(\tilde{\alpha},\mathring{\alpha})}, \mathring{p}_{(\tilde{\beta},\mathring{\beta})}, \mathring{p}_{(\tilde{\gamma},\mathring{\gamma})}\right) = \frac{1}{9t}[\left(4t + \mathring{p}_{\tilde{\alpha}} - \mathring{p}_{\tilde{\beta}} - \mathring{p}_{\tilde{\gamma}}\right) + \left(2t + \mathring{p}_{\mathring{\alpha}} - \mathring{p}_{\mathring{\beta}} - \mathring{p}_{\mathring{\gamma}}\right)], \ \text{for } \varphi(\mathring{g}) \in [0,1] \tag{12}$$

(ii):

$$\Phi(\mathring{g}) = \frac{1}{3t}[(\mathring{p}_{\tilde{\alpha}} - \mathring{p}_{\tilde{\gamma}}) + (\mathring{p}_{\mathring{\alpha}} - \mathring{p}_{\mathring{\gamma}})], \text{for } \Phi(\mathring{g}) \in [-1,1] \tag{13}$$

(iii):

$$\Psi(\mathring{g}) = \frac{\mathring{p}_{\tilde{\alpha}} + \mathring{p}_{\mathring{\alpha}}}{3t} \ \text{for } \Psi(\mathring{g}) \in [0,1]. \tag{14}$$

Now, with the help of the above-defined function, we introduce a ranking method for these function.

Definition 11. *Let two LNCNs be* $\mathring{g}_1 = \left(\mathring{p}_{(\tilde{\alpha}_1,\mathring{\alpha}_1)}, \mathring{p}_{(\tilde{\beta}_1,\mathring{\beta}_1)}, \mathring{p}_{(\tilde{\gamma}_1,\mathring{\gamma}_1)}\right)$ *and* $\mathring{g}_2 = \left(\mathring{p}_{(\tilde{\alpha}_2,\mathring{\alpha}_2)}, \mathring{p}_{(\tilde{\beta}_2,\mathring{\beta}_2)}, \mathring{p}_{(\tilde{\gamma}_2,\mathring{\gamma}_2)}\right)$. *Then, their ranking method is defined as:*

1. *If* $\varphi(\mathring{g}_1) > \varphi(\mathring{g}_2)$, *then* $\mathring{g}_1 \succ \mathring{g}_2$,
2. *If* $\varphi(\mathring{g}_1) = \varphi(\mathring{g}_2)$ *and* $\Phi(\mathring{g}_1) > \Phi(\mathring{g}_2)$, *then* $\mathring{g}_1 \succ \mathring{g}_2$,
3. *If* $\varphi(\mathring{g}_1) = \varphi(\mathring{g}_2), \Phi(\mathring{g}_1) = \Phi(\mathring{g}_2)$ *and* $\Psi(\mathring{g}_1) > \Psi(\mathring{g}_2)$, *then* $\mathring{g}_1 \succ \mathring{g}_2$,
4. *If* $\varphi(\mathring{g}_1) = \varphi(\mathring{g}_2), \Phi(\mathring{g}_1) = \Phi(\mathring{g}_2)$ *and* $\Psi(\mathring{g}_1) = \Psi(\mathring{g}_2)$, *then* $\mathring{g}_1 \sim \mathring{g}_2$.

Example 1. *Let* $\mathring{g}_1 = \left(\mathring{p}_{(\tilde{\alpha}_1,\mathring{\alpha}_1)}, \mathring{p}_{(\tilde{\beta}_1,\mathring{\beta}_1)}, \mathring{p}_{(\tilde{\gamma}_1,\mathring{\gamma}_1)}\right)$, $\mathring{g}_2 = \left(\mathring{p}_{(\mathring{\alpha}_2,\mathring{\alpha}_2)}, \mathring{p}_{(\mathring{\beta}_2,\mathring{\beta}_2)}, \mathring{p}_{(\mathring{\gamma}_2,\mathring{\gamma}_2)}\right)$ *and* $\mathring{g}_3 = \left(\mathring{p}_{(\tilde{\alpha}_3,\mathring{\alpha}_3)}, \mathring{p}_{(\tilde{\beta}_3,\mathring{\beta}_3)}, \mathring{p}_{(\tilde{\gamma}_3,\tilde{\gamma}_3)}\right)$ *be three LNCNs in the linguistic term set* $\varphi = \{\varphi_{\mathring{g}} \mid \mathring{g} \in [0,8]\}$ *where* $\mathring{g}_1 = ([0.2,0.3], [0.4,0.5], [0.3,0.5], (0.1,0.2,0.3))$, $\mathring{g}_2 = ([0.3,0.4], [0.4,0.5], [0.5,0.6], (0.2,0.4,0.6))$, $\mathring{g}_3 = ([0.4,0.5], [0.4,0.6], [0.5,0.7], (0.2,0.3,0.5))$, *then we will find the values of their score, accuracy and certain function as follows:*

(i) Score functions:

$$\varphi(\mathring{g}) = \frac{1}{9t}[\left(4t + \mathring{p}_{\tilde{\alpha}} - \mathring{p}_{\tilde{\beta}} - \mathring{p}_{\tilde{\gamma}}\right) + \left(2t + \mathring{p}_{\mathring{\alpha}} - \mathring{p}_{\mathring{\beta}} - \mathring{p}_{\mathring{\gamma}}\right)], \text{ for } \varphi(\mathring{g}) \in [0,1]$$

$$\varphi(\mathring{g}_1) = \frac{[32 + 0.2 + 0.3 - (0.4 + 0.5 + 0.3 + 0.5) + 16 + 0.1 - (0.2 + 0.3)]}{72} = 0.644$$

$$\varphi(\mathring{g}_2) = \frac{[32 + 0.3 + 0.4 - (0.4 + 0.5 + 0.5 + 0.6) + 16 + 0.2 - (0.4 + 0.6)]}{72} = 0.6375$$

$$\varphi(\mathring{g}_3) = \frac{[32 + 0.4 + 0.5 - (0.4 + 0.6 + 0.5 + 0.7) + 16 + 0.2 - (0.3 + 0.5)]}{72} = 0.638$$

(ii) Accuracy functions:

$$\begin{aligned}
\Phi(\mathring{g}) &= \frac{1}{3t}[(\mathring{p}_{\tilde{\alpha}} - \mathring{p}_{\tilde{\gamma}}) + (\mathring{p}_{\mathring{\alpha}} - \mathring{p}_{\mathring{\gamma}})], \textit{ for } \Phi(\mathring{g}) \in [-1,1] \\
\Phi(\mathring{g}_1) &= \frac{[0.2+0.3-(0.3+0.5)+0.1-0.3]}{24} \\
&= -0.0208 \\
\Phi(\mathring{g}_2) &= \frac{[0.3+0.4-(0.5+0.6)+0.2-0.6]}{24} \\
&= -0.0333 \\
\Phi(\mathring{g}_3) &= \frac{[0.4+0.5-(0.6+0.7)+0.3-0.5]}{24} \\
&= -0.0292
\end{aligned}$$

(iii) Certain functions:

$$\begin{aligned}
\Psi(\mathring{g}) &= \frac{\mathring{p}_{\tilde{\alpha}} + \mathring{p}_{\mathring{\alpha}}}{3t} \textit{ for } \Psi(\mathring{g}) \in [0,1] \\
\Psi(\mathring{g}_1) &= \frac{[0.2+0.3+0.1]}{24} \\
&= 0.025 \\
\Psi(\mathring{g}_2) &= \frac{[0.3+0.4+0.2]}{24} \\
&= 0.0375 \\
\Psi(\mathring{g}_3) &= \frac{[0.4+0.5+0.2]}{24} \\
&= 0.0416
\end{aligned}$$

Definition 12. *Suppose* $(\tilde{u}_{\hat{\imath}}, u_{\hat{\imath}})$ *where* $\hat{\imath} = 1,2,\ldots,n$ *is an assortment of non-negative real numbers and parameter* $\mathring{k} = 1,2,\ldots,n$*. Then, the cubic Hamy mean (CHM) is defined as follows:*

$$CHM^{\mathring{k}}(\tilde{u}_{\hat{\imath}}, u_{\hat{\imath}}) = \frac{\sum\limits_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n} \left(\prod\limits_{j=1}^{\mathring{k}} \tilde{u}_{\hat{\imath}_j}, \prod\limits_{j=1}^{\mathring{k}} u_{\hat{\imath}_j} \right)^{\frac{1}{\mathring{k}}}}{\binom{n}{\mathring{k}}} \tag{15}$$

where $(\hat{\imath}_1, \hat{\imath}_2, \ldots, \hat{\imath}_{\mathring{k}})$ *navigate all the k-tuple arrangements of* $(1,2,\ldots,n.)$*,* $\binom{n}{\mathring{k}}$ *is the binomial coefficient and* $\binom{n}{\mathring{k}} = \frac{n!}{\mathring{k}!(n-\mathring{k})!}$.

Example 2. *Let* $(\tilde{u}_{\hat{\imath}}, u_{\hat{\imath}}) = ((\tilde{u}_1, u_1), (\tilde{u}_2, u_2))$ $i = 1,2$ *and* $k = 1$*, where* $u_1 = ([0.2, 0.4], (0.6))$, $u_2 = ([0.3, 0.5], (0.7))$.

$$
\begin{aligned}
& CHM^1\left((\tilde{u}_1, u_1), (\tilde{u}_2, u_2)\right) \\
&= \frac{\sum\left(((\tilde{u}_{11}, u_{11})(\tilde{u}_{22}, u_{22}))\right)^1}{\binom{2}{1}} \\
&= \frac{\left(((\tilde{u}_{11}, u_{11})(\tilde{u}_{22}, u_{22}))\right)^1 + \left(((\tilde{u}_{11}, u_{11})(\tilde{u}_{22}, u_{22}))\right)^1}{\binom{2}{1}} \\
&= \frac{\sum\begin{pmatrix} (([0.2, 0.4], (0.6))\,([0.2, 0.4], (0.6))) \\ (([0.3, 0.5], (0.7))\,([0.3, 0.5], (0.7))) \end{pmatrix}^1}{\binom{2}{1}} \\
&= \frac{\begin{pmatrix} (([0.2, 0.4], (0.6))\,([0.2, 0.4], (0.6))) \\ (([0.3, 0.5], (0.7))\,([0.3, 0.5], (0.7))) \end{pmatrix}^1 + \begin{pmatrix} (([0.2, 0.4], (0.6))\,([0.2, 0.4], (0.6))) \\ (([0.3, 0.5], (0.7))\,([0.3, 0.5], (0.7))) \end{pmatrix}^1}{\binom{2}{1}} \\
&= \frac{(([0.04, 0.16], (0.84))\,([0.09, 0.25], (0.91))) + (([0.04, 0.16], (0.84))\,([0.09, 0.25], (0.91)))}{\binom{2}{1}} \\
&= \frac{\begin{pmatrix} ([0.004, 0.04], (0.98)) \\ +([0.004, 0.04], (0.98)) \end{pmatrix}}{\binom{2}{1}} \\
&= \frac{([0.008, 0.08], (0.96))}{\binom{2}{1}} \\
&= ([0.004, 0.04], (0.48))
\end{aligned}
$$

Definition 13. *Suppose* $(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}})$ *where* $\hat{\imath} = 1, 2, \ldots, n.$ *is an assortment of linguistic neutrosophic cubic numbers and parameter* $\mathring{k} = 1, 2, \ldots, n$. *Then, the LNCHM operator is defined as follows:*

$$
LNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) = \frac{\sum\limits_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n} \left(\prod\limits_{j=1}^{\mathring{k}} \mathring{g}_{\hat{\imath}_j}, \prod\limits_{j=1}^{\mathring{k}} \mathring{g}_{\hat{\imath}_j}\right)^{\frac{1}{\mathring{k}}}}{\binom{n}{\mathring{k}}} \tag{16}
$$

where $\left(\hat{\imath}_1, \hat{\imath}_2, \ldots, \hat{\imath}_{\mathring{k}}\right)$ *navigate all the k-tuple arrangements of* $(1, 2, \ldots, n.)$, $\binom{n}{\mathring{k}}$ *is the binomial coefficient and* $\binom{n}{\mathring{k}} = \frac{n!}{\mathring{k}!(n-\mathring{k})!}$.

Example 3. *Let* $(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) = ((\tilde{g}_1, \mathring{g}_2), (\tilde{g}_2, \mathring{g}_2))$ $i = 1, 2$ *and* $k = 1$, *where* $\tilde{g}_1 = ([0.2, 0.4], [0.3, 0.4], [0.4, 0.6], (0.6, 0.5, 0.8))$, $\tilde{g}_2 = ([0.3, 0.5], [0.4, 0.7], [0.2, 0.4], (0.7, 0.8, 0.6))$,

$$
\begin{aligned}
& LNCHM^{1}\left((\tilde{g}_1, \mathring{g}_2), (\tilde{g}_2, \mathring{g}_2)\right) \\
&= \frac{\sum \left(((\tilde{g}_{11}, \mathring{g}_{11}), (\tilde{g}_{22}, \mathring{g}_{22}))\right)^{1}}{\binom{2}{1}} \\
&= \frac{\left(((\tilde{g}_{11}, \mathring{g}_{11})(\tilde{g}_{22}, \mathring{g}_{22}))\right)^{1} + \left(((\tilde{g}_{11}, \mathring{g}_{11})(\tilde{g}_{22}, \mathring{g}_{22}))\right)^{1}}{\binom{2}{1}} \\
&= \frac{\sum \left(\begin{pmatrix} ([0.2, 0.4], [0.3, 0.4], [0.4, 0.6], (0.6, 0.5, 0.8)) \\ ([0.2, 0.4], [0.3, 0.4], [0.4, 0.6], (0.6, 0.5, 0.8)) \end{pmatrix} \begin{pmatrix} ([0.3, 0.5], [0.4, 0.7], [0.2, 0.4], (0.7, 0.8, 0.6)) \\ ([0.3, 0.5], [0.4, 0.7], [0.2, 0.4], (0.7, 0.8, 0.6)) \end{pmatrix} \right)^{1}}{\binom{2}{1}} \\
&= \frac{\left(\begin{pmatrix} ([0.2, 0.4], [0.3, 0.4], [0.4, 0.6], (0.6, 0.5, 0.8)) \\ ([0.2, 0.4], [0.3, 0.4], [0.4, 0.6], (0.6, 0.5, 0.8)) \end{pmatrix} \begin{pmatrix} ([0.3, 0.5], [0.4, 0.7], [0.2, 0.4], (0.7, 0.8, 0.6)) \\ ([0.3, 0.5], [0.4, 0.7], [0.2, 0.4], (0.7, 0.8, 0.6)) \end{pmatrix} \right)^{1} + \left(\begin{pmatrix} ([0.2, 0.4], [0.3, 0.4], [0.4, 0.6], (0.6, 0.5, 0.8)) \\ ([0.2, 0.4], [0.3, 0.4], [0.4, 0.6], (0.6, 0.5, 0.8)) \end{pmatrix} \begin{pmatrix} ([0.3, 0.5], [0.4, 0.7], [0.2, 0.4], (0.7, 0.8, 0.6)) \\ ([0.3, 0.5], [0.4, 0.7], [0.2, 0.4], (0.7, 0.8, 0.6)) \end{pmatrix} \right)^{1}}{\binom{2}{1}} \\
&= \frac{\begin{pmatrix} ([0.04, 0.16], [0.09, 0.16], [0.16, 0.36], (0.84, 0.75, 0.96)) \\ ([0.09, 0.25], [0.16, 0.49], [0.04, 0.16], (0.91, 0.96, 0.84)) \end{pmatrix} + \begin{pmatrix} ([0.04, 0.16], [0.09, 0.16], [0.16, 0.36], (0.84, 0.75, 0.96)) \\ ([0.09, 0.25], [0.16, 0.49], [0.04, 0.16], (0.91, 0.96, 0.84)) \end{pmatrix}}{\binom{2}{1}} \\
&= \frac{\begin{pmatrix} ([0.004, 0.04], [0.014, 0.08], [0.006, 0.06], (0.98, 0.99, 0.99)) \\ + ([0.004, 0.04], [0.014, 0.08], [0.006, 0.06], (0.98, 0.99, 0.99)) \end{pmatrix}}{\binom{2}{1}} \\
&= \frac{([0.008, 0.08], [0.03, 0.2], [0.012, 0.12], (0.96, 0.98, 0.98))}{\binom{2}{1}} \\
&= ([0.004, 0.04], [0.02, 0.1], [0.006, 0.06], (0.48, 0.49, 0.49))
\end{aligned}
$$

Theorem 1. *Let* $(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) = \left(\mathring{p}_{(\tilde{\alpha}_{\hat{\imath}}, \mathring{\alpha}_{\hat{\imath}})}, \mathring{p}_{(\tilde{\beta}_{\hat{\imath}}, \mathring{\beta}_{\hat{\imath}})}, \mathring{p}_{(\tilde{\gamma}_{\hat{\imath}}, \mathring{\gamma}_{\hat{\imath}})}\right)$ $(\hat{\imath} = 1, 2, \ldots, n)$ *be an arrangement of LNCNs, then the accumulated value from Definition 13 is obviously an LNCN, and:*

$$
\begin{aligned}
& LNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) \\
&= \left(\begin{array}{c}
\mathring{p}_{t - t\left(\prod_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n} \left(1 - \left(\frac{\prod_{j=1}^{\mathring{k}} \tilde{\alpha}_{\hat{\imath}j}}{t^{\mathring{k}}}, \frac{\prod_{j=1}^{\mathring{k}} \mathring{\alpha}_{\hat{\imath}j}}{t^{\mathring{k}}}\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \mathring{p}_{t\left(\prod_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n} \left(1 - \left(\prod_{j=1}^{\mathring{k}} \left(1 - \frac{\tilde{\beta}_{\hat{\imath}j}}{t}, 1 - \frac{\mathring{\beta}_{\hat{\imath}j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \\
\mathring{p}_{t\left(\prod_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n} \left(1 - \left(\prod_{j=1}^{\mathring{k}} \left(1 - \frac{\tilde{\gamma}_{\hat{\imath}j}}{t}, 1 - \frac{\mathring{\gamma}_{\hat{\imath}j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}
\end{array} \right)
\end{aligned} \tag{17}
$$

Proof. According to Equations (1)–(4), we have:

$$\left(\prod_{j=1}^{\mathring{k}} \tilde{g}_{\hat{\imath}_j}, \prod_{j=1}^{\mathring{k}} \mathring{g}_{\hat{\imath}_j}\right) = \left(\mathring{p}_{\frac{\prod_{j=1}^{\mathring{k}} \tilde{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}-1}}, \frac{\prod_{j=1}^{\mathring{k}} \mathring{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}-1}}}, \mathring{p}_{t-t\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\beta}_{\hat{\imath}_j}}{t}, 1-\frac{\mathring{\beta}_{\hat{\imath}_j}}{t}\right)}, \mathring{p}_{t-t\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\gamma}_{\hat{\imath}_j}}{t}, 1-\frac{\mathring{\gamma}_{\hat{\imath}_j}}{t}\right)} \right)$$

$$\left(\prod_{j=1}^{\mathring{k}} \tilde{g}_{\hat{\imath}_j}, \prod_{j=1}^{\mathring{k}} \mathring{g}_{\hat{\imath}_j}\right)^{\frac{1}{\mathring{k}}} = \left(\mathring{p}_{\left(\frac{\prod_{j=1}^{\mathring{k}} \tilde{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}}, \frac{\prod_{j=1}^{\mathring{k}} \mathring{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}}\right)^{\frac{1}{\mathring{k}}}}, \mathring{p}_{t-t\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\beta}_{\hat{\imath}_j}}{t}, 1-\frac{\mathring{\beta}_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}}, \mathring{p}_{t-t\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\gamma}_{\hat{\imath}_j}}{t}, 1-\frac{\mathring{\gamma}_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}} \right)$$

$$\sum_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n} \left(\prod_{j=1}^{\mathring{k}} \tilde{g}_{\hat{\imath}_j}, \prod_{j=1}^{\mathring{k}} \mathring{g}_{\hat{\imath}_j}\right)^{\frac{1}{\mathring{k}}}$$
$$= \left(\mathring{p}_{t-t\prod_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n}\left(1-\left(\frac{\prod_{j=1}^{\mathring{k}} \tilde{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}}, \frac{\prod_{j=1}^{\mathring{k}} \mathring{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}}\right)^{\frac{1}{\mathring{k}}}\right)}, \mathring{p}_{t\prod_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\beta}_{\hat{\imath}_j}}{t}, 1-\frac{\mathring{\beta}_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)}, \mathring{p}_{t\prod_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\gamma}_{\hat{\imath}_j}}{t}, 1-\frac{\mathring{\gamma}_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)} \right)$$

Then, we obtain:

$$\frac{1}{\binom{n}{\mathring{k}}} \sum_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n} \left(\prod_{j=1}^{\mathring{k}} \tilde{g}_{\hat{\imath}_j}, \prod_{j=1}^{\mathring{k}} \mathring{g}_{\hat{\imath}_j}\right)^{\frac{1}{\mathring{k}}}$$
$$= \left(\mathring{p}_{t-t\left(\prod_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n}\left(1-\left(\frac{\prod_{j=1}^{\mathring{k}} \tilde{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}}, \frac{\prod_{j=1}^{\mathring{k}} \mathring{\alpha}_{\hat{\imath}j}}{t^{\mathring{k}}}\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \mathring{p}_{t\left(\prod_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\beta}_{\hat{\imath}j}}{t}, 1-\frac{\mathring{\beta}_{\hat{\imath}j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \mathring{p}_{t\left(\prod_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\gamma}_{\hat{\imath}j}}{t}, 1-\frac{\mathring{\gamma}_{\hat{\imath}j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}} \right)$$

Therefore,

$$LNCHM^{\mathring{k}}(\mathring{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}})$$

$$= \left(\mathring{p}_{t-t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\frac{\prod_{j=1}^{\mathring{k}}\tilde{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}},\frac{\prod_{j=1}^{\mathring{k}}\mathring{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}}\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \mathring{p}_{t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\beta}_{\hat{\imath}_j}}{t},1-\frac{\mathring{\beta}_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \mathring{p}_{t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\gamma}_{\hat{\imath}_j}}{t},1-\frac{\mathring{\gamma}_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}} \right)$$

In addition, since:

$$0 \le t-t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\frac{\prod_{j=1}^{\mathring{k}}\tilde{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}},\frac{\prod_{j=1}^{\mathring{k}}\mathring{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}}\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}} \le t,$$

$$0 \le t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\beta}_{\hat{\imath}_j}}{t},1-\frac{\mathring{\beta}_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}} \le t,$$

$$0 \le t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\gamma}_{\hat{\imath}_j}}{t},1-\frac{\mathring{\gamma}_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}} \le t,$$

Therefore,

$$\left(\begin{array}{c} \mathring{p}_{t-t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\frac{\prod_{j=1}^{\mathring{k}}\tilde{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}},\frac{\prod_{j=1}^{\mathring{k}}\mathring{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}}\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \\ \mathring{p}_{t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\beta}_{\hat{\imath}_j}}{t},1-\frac{\mathring{\beta}_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \\ \mathring{p}_{t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\gamma}_{\hat{\imath}_j}}{t},1-\frac{\mathring{\gamma}_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}} \end{array} \right)$$

is also an LNCN. □

Example 4. *Let* $\mathring{p} = \{\mathring{p}_0, \mathring{p}_1, \mathring{p}_2, \mathring{p}_3, \mathring{p}_4\}$ *be an* LT^S *with odd cardinality* $t+1$ *and* $\mathring{g}_1 = (\mathring{p}_3, \mathring{p}_2, \mathring{p}_1), \mathring{g}_2 = (\mathring{p}_4, \mathring{p}_3, \mathring{p}_1,)$*, be two LNCNsbased on* $\mathring{p}$*. Then, we can use the suggested LNCHM operator to aggregate these* *two LNCNs (suppose $\mathring{k} = 2$) and to produce an inclusive value $LNCHM^{(\mathring{k})}(\mathring{g}_1, \mathring{g}_2) = \left(\mathring{p}_{(\tilde{\alpha},\mathring{\alpha})}, \mathring{p}_{(\tilde{\beta},\mathring{\beta})}, \mathring{p}_{(\tilde{\gamma},\mathring{\gamma})}\right)$ described as follows; where:*

$$(\mathring{g}_1, \mathring{g}_2) = \begin{pmatrix} ([0.2, 0.3], [0.2, 0.5], [0.2, 0.5], (0.9, 0.7, 0.9)), \\ ([0.4, 0.5], [0.3, 0.5], [0.3, 0.5], (0.8, 0.8, 0.6)) \end{pmatrix}$$

(i):

$$\frac{1}{\binom{n}{\mathring{k}}} = \frac{\mathring{k}!(n-\mathring{k})!}{n!} = \frac{2!(2-2)!}{2!} = 1$$

(ii):

$$t - t\left(\prod_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n}\left(1 - \left(\frac{\prod_{j=1}^{\mathring{k}} \tilde{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}}, \frac{\prod_{j=1}^{\mathring{k}} \mathring{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}}\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}$$
$$= ([0.28, 0.39], 0.17)$$

(iii):

$$t\left(\prod_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n}\left(1 - \left(\prod_{j=1}^{\mathring{k}}\left(1 - \frac{\tilde{\beta}_{\hat{\imath}_j}}{t}, 1 - \frac{\mathring{\beta}_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}$$
$$= ([0.3, 0.5], 0.75)$$

(iv):

$$t\left(\prod_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n}\left(1 - \left(\prod_{j=1}^{\mathring{k}}\left(1 - \frac{\tilde{\gamma}_{\hat{\imath}_j}}{t}, 1 - \frac{\mathring{\gamma}_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}$$
$$= ([0.3, 0.5], 0.74)$$

Therefore, we get:

$$LNCM^2(\tilde{g}_1, \mathring{g}_2) = \left(\mathring{p}_{(\tilde{\alpha},\mathring{\alpha})}, \mathring{p}_{(\tilde{\beta},\mathring{\beta})}, \mathring{p}_{(\tilde{\gamma},\mathring{\gamma})}\right)$$
$$= ([0.28, 0.39], [0.3, 0.5], [0.3, 0.5], (0.17, 0.75, 0.74)).$$

Now, we will study some of the ideal properties of LNCNs.

Property 1. *(Idempotency) If $(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) = (\tilde{g}, \mathring{g}) = \left(\mathring{p}_{(\tilde{\alpha},\mathring{\alpha})}, \mathring{p}_{(\tilde{\beta},\mathring{\beta})}, \mathring{p}_{(\tilde{\gamma},\mathring{\gamma})}\right)$ $\forall$ $(\hat{\imath} = 1, 2, \ldots, n)$, then:*

$$LNCHM^{\mathring{k}}(\tilde{g}, \mathring{g}) = \left(\mathring{p}_{(\tilde{\alpha},\mathring{\alpha})}, \mathring{p}_{(\tilde{\beta},\mathring{\beta})}, \mathring{p}_{(\tilde{\gamma},\mathring{\gamma})}\right) \tag{18}$$

Proof. Since $(\tilde{g}, \mathring{g}) = \left(\mathring{p}_{(\tilde{\alpha},\mathring{\alpha})}, \mathring{p}_{(\tilde{\beta},\mathring{\beta})}, \mathring{p}_{(\tilde{\gamma},\mathring{\gamma})}\right)$, based on Theorem 1, we have:

$$
\begin{aligned}
&LNCHM^{\mathring{k}}(\tilde{g}, \mathring{g}) \\
&= \left(\begin{array}{c} \mathring{p}_{t-t\left(\prod\limits_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\frac{\tilde{\alpha}^{\mathring{k}}}{t^{\mathring{k}}},\frac{\mathring{\alpha}^{\mathring{k}}}{t^{\mathring{k}}}\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \mathring{p}_{t\left(\prod\limits_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\left(1-\frac{\tilde{\beta}}{t},1-\frac{\mathring{\beta}}{t}\right)^{\mathring{k}}\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \\ \mathring{p}_{t\left(\prod\limits_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\left(1-\frac{\tilde{\gamma}}{t},1-\frac{\mathring{\gamma}}{t}\right)^{\mathring{k}}\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}} \end{array} \right) \\
&= \left(\mathring{p}_{t-t\left(\left(1-\left(\frac{\tilde{\alpha}}{t},\frac{\mathring{\alpha}}{t}\right)\right)^{\binom{n}{\mathring{k}}}\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \mathring{p}_{t\left(\left(1-\left(1-\frac{\tilde{\beta}}{t},1-\frac{\mathring{\beta}}{t}\right)\right)^{\binom{n}{\mathring{k}}}\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \mathring{p}_{t\left(\left(1-\left(1-\frac{\tilde{\gamma}}{t},1-\frac{\mathring{\gamma}}{t}\right)\right)^{\binom{n}{\mathring{k}}}\right)^{\frac{1}{\binom{n}{\mathring{k}}}}} \right) \\
&= \left(\mathring{p}_{t-t\left(1-\left(\frac{\tilde{\alpha}}{t},\frac{\mathring{\alpha}}{t}\right)\right)}, \mathring{p}_{t\left(1-\left(1-\frac{\tilde{\beta}}{t},1-\frac{\mathring{\beta}}{t}\right)\right)}, \mathring{p}_{t\left(1-\left(1-\frac{\tilde{\gamma}}{t},1-\frac{\mathring{\gamma}}{t}\right)\right)} \right) \\
&= \left(\mathring{p}_{(\tilde{\alpha},\mathring{\alpha})}, \mathring{p}_{(\tilde{\beta},\mathring{\beta})}, \mathring{p}_{(\tilde{\gamma},\mathring{\gamma})}\right) = (\tilde{g}, \mathring{g})
\end{aligned}
$$

□

Property 2. *(Commutativity) Let* $(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}})$ *for all* $(\hat{\imath} = 1, 2, \ldots, n)$ *be an assortment of LNCNs and* $(\tilde{g}'_{\hat{\imath}}, \mathring{g}'_{\hat{\imath}})$ *be any permutation of* $(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}})$*, then:*

$$LNCHM^{\mathring{k}}(\tilde{g}'_{\hat{\imath}}, \mathring{g}'_{\hat{\imath}}) = LNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) \tag{19}$$

Proof. The conclusion is obvious, because Property 2 depends on Definition 13.

$$
\begin{aligned}
LNCHM^{\mathring{k}}(\tilde{g}'_{\hat{\imath}}, \mathring{g}'_{\hat{\imath}}) &= \frac{\sum\limits_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(\prod\limits_{j=1\hat{\imath}_j}^{\mathring{k}}\tilde{g}_{\hat{\imath}_j}, \prod\limits_{j=1}^{\mathring{k}}\mathring{g}'_{\hat{\imath}_j}\right)^{\frac{1}{\mathring{k}}}}{\binom{n}{\mathring{k}}} \\
&= \frac{\sum\limits_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(\prod\limits_{j=1}^{\mathring{k}}\tilde{g}_{\hat{\imath}_j}, \prod\limits_{j=1}^{\mathring{k}}\mathring{g}_{\hat{\imath}_j}\right)^{\frac{1}{\mathring{k}}}}{\binom{n}{\mathring{k}}} \\
&= LNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}})
\end{aligned}
$$

□

Property 3. *(Monotonicity) Let* $(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) = \left(\mathring{p}_{(\tilde{\alpha}_{\hat{\imath}},\mathring{\alpha}_{\hat{\imath}})}, \mathring{p}_{(\tilde{\beta}_{\hat{\imath}},\mathring{\beta}_{\hat{\imath}})}, \mathring{p}_{(\tilde{\gamma}_{\hat{\imath}},\mathring{\gamma}_{\hat{\imath}})}\right), (\tilde{f}_{\hat{\imath}}, f_{\hat{\imath}}) = \left(\mathring{p}_{(\tilde{q}_{\hat{\imath}},q_{\hat{\imath}})}, \mathring{p}_{(\tilde{r}_{\hat{\imath}},r_{\hat{\imath}})}, \mathring{p}_{(\tilde{s}_{\hat{\imath}},s_{\hat{\imath}})}\right) (\hat{\imath} = 1, 2, \ldots, n)$ *be two collections of LNCNs; if* $(\tilde{\alpha}_{\hat{\imath}}, \mathring{\alpha}_{\hat{\imath}}) \le (\tilde{q}_{\hat{\imath}}, q_{\hat{\imath}}), (\tilde{\beta}_{\hat{\imath}}, \mathring{\beta}_{\hat{\imath}}) \le (\tilde{r}_{\hat{\imath}}, r_{\hat{\imath}}), (\tilde{\gamma}_{\hat{\imath}}, \mathring{\gamma}_{\hat{\imath}}) \le (\tilde{s}_{\hat{\imath}}, s_{\hat{\imath}})$ *for all* $\hat{\imath}$*, then:*

$$LNCHM^{\mathring{k}}(\mathring{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) \le LNCHM^{\mathring{k}}\left(\tilde{f}_{\hat{\imath}}, f_{\hat{\imath}}\right) \tag{20}$$

Proof. Since $0 \le (\tilde{\alpha}_{\hat{\imath}}, \mathring{\alpha}_{\hat{\imath}}) \le (\tilde{q}_{\hat{\imath}}, q_{\hat{\imath}}), (\tilde{\beta}_{\hat{\imath}}, \mathring{\beta}_{\hat{\imath}}) \ge (\tilde{r}_{\hat{\imath}}, r_{\hat{\imath}}) \ge 0, (\tilde{\gamma}_{\hat{\imath}}, \mathring{\gamma}_{\hat{\imath}}) \ge (\tilde{s}_{\hat{\imath}}, s_{\hat{\imath}}) \ge 0, t \ge 0$ and according to Theorem 1, we get:

$$t - t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\frac{\prod_{j=1}^{\mathring{k}}\tilde{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}},\frac{\prod_{j=1}^{\mathring{k}}\mathring{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}}\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}$$

$$\le t - t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\frac{\prod_{j=1}^{\mathring{k}}\tilde{q}_{\hat{\imath}_j}}{t^{\mathring{k}}},\frac{\prod_{j=1}^{\mathring{k}}q_{\hat{\imath}_j}}{t^{\mathring{k}}}\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}},$$

$$-t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\beta}_{\hat{\imath}_j}}{t},1-\frac{\mathring{\beta}_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}$$

$$\le -t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{r}_{\hat{\imath}_j}}{t},1-\frac{r_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}},$$

$$-t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\gamma}_{\hat{\imath}_j}}{t},1-\frac{\mathring{\gamma}_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}$$

$$\le -t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{s}_{\hat{\imath}_j}}{t},1-\frac{s_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}.$$

□

Let $(\tilde{g},\mathring{g}) = LNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}},\mathring{g}_{\hat{\imath}})$, $(\tilde{f},f) = LNCHM^{\mathring{k}}\left(\tilde{f}_{\hat{\imath}},f_{\hat{\imath}}\right)$ and $\psi(\mathring{g})$ and $\Psi(f)$ be the score functions of $\mathring{g}$ and f. According to the score value in Definition 11 and the above inequality, we can simply have $\psi(\mathring{g}) \le \Psi(f)$. Then, in the following, we argue some cases:

1. If $\psi(\mathring{g}) \le \Psi(f)$, we can obtain $LNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}},\mathring{g}_{\hat{\imath}}) \le LNCHM^{\mathring{k}}\left(\tilde{f}_{\hat{\imath}},f_{\hat{\imath}}\right)$;
2. if $\psi(\mathring{g}) = \Psi(f)$, then:

$$\frac{\begin{array}{c}2t+t-t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\frac{\prod_{j=1}^{\mathring{k}}\tilde{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}},\frac{\prod_{j=1}^{\mathring{k}}\mathring{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}}\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}\\ -t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\beta}_{\hat{\imath}_j}}{t},1-\frac{\mathring{\beta}_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}\\ -t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{\gamma}_{\hat{\imath}_j}}{t},1-\frac{\mathring{\gamma}_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}\end{array}}{3t}$$

$$=\frac{\begin{array}{c}2t+t-t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\frac{\prod_{j=1}^{\mathring{k}}\tilde{q}_{\hat{\imath}_j}}{t^{\mathring{k}}},\frac{\prod_{j=1}^{\mathring{k}}q_{\hat{\imath}_j}}{t^{\mathring{k}}}\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}\\ -t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{r}_{\hat{\imath}_j}}{t},1-\frac{r_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}\\ -t\left(\prod_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\tilde{s}_{\hat{\imath}_j}}{t},1-\frac{s_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}\end{array}}{3t}$$

Since $0 \leq (\mathring{\alpha}_{\hat{\imath}}, \mathring{\alpha}_{\hat{\imath}}) \leq (\tilde{q}_{\hat{\imath}}, q_{\hat{\imath}}), (\mathring{\beta}_{\hat{\imath}}, \mathring{\beta}_{\hat{\imath}}) \geq (\tilde{r}_{\hat{\imath}}, r_{\hat{\imath}}) \geq 0, (\mathring{\gamma}_{\hat{\imath}}, \mathring{\gamma}_{\hat{\imath}}) \geq (\tilde{s}_{\hat{\imath}}, s_{\hat{\imath}}) \geq 0, t \geq 0$, we can assume that:

$$t - t\left(\prod_{1\leq \hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\leq n}\left(1-\left(\frac{\prod_{j=1}^{\mathring{k}}\widetilde{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}}, \frac{\prod_{j=1}^{\mathring{k}}\mathring{\alpha}_{\hat{\imath}_j}}{t^{\mathring{k}}}\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}$$

$$= t - t\left(\prod_{1\leq \hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\leq n}\left(1-\left(\frac{\prod_{j=1}^{\mathring{k}}\tilde{q}_{\hat{\imath}_j}}{t^{\mathring{k}}}, \frac{\prod_{j=1}^{\mathring{k}}q_{\hat{\imath}_j}}{t^{\mathring{k}}}\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}},$$

$$-t\left(\prod_{1\leq \hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\leq n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\widetilde{\gamma}_{\hat{\imath}_j}}{t}, 1-\frac{\mathring{\gamma}_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}$$

$$= -t\left(\prod_{1\leq \hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\leq n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\frac{\widetilde{s}_{\hat{\imath}_j}}{t}, 1-\frac{s_{\hat{\imath}_j}}{t}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}},$$

and based on the accuracy value in Definition 11, then $\Phi(\mathring{g}) = \Phi(f)$. Finally, we get:

$$LNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) \leq LNCHM^{\mathring{k}}\left(\tilde{f}_{\hat{\imath}}, f_{\hat{\imath}}\right)$$

Property 4. *(Boundedness) Let* $(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) = (\mathring{p}_{\tilde{\alpha}'_{\hat{\imath}}}, \mathring{p}_{\tilde{\beta}'_{\hat{\imath}}}, \mathring{p}_{\tilde{\gamma}'_{\hat{\imath}}}, \mathring{p}_{\mathring{\alpha}_{\hat{\imath}}}, \mathring{p}_{\mathring{\beta}_{\hat{\imath}}}, \mathring{p}_{\mathring{\gamma}_{\hat{\imath}}})(\hat{\imath} = 1, 2, \ldots, n)$ *be the collection of LNCNs and:*

$$\mathring{g}^{+} = \max(\mathring{p}_{\max(\tilde{\alpha}_{\hat{\imath}})}, \mathring{p}_{\min(\tilde{\beta}_{\hat{\imath}})}, \mathring{p}_{\min(\tilde{\gamma}_{\hat{\imath}})}, \mathring{p}_{\max(\mathring{\alpha}_{\hat{\imath}})}, \mathring{p}_{\min(\mathring{\beta}_{\hat{\imath}})}, \mathring{p}_{\min(\mathring{\gamma}_{\hat{\imath}})}),$$
$$\mathring{g}^{-} = \min(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) = (\mathring{p}_{\min(\tilde{\alpha}_{\hat{\imath}})}, \mathring{p}_{\max(\tilde{\beta}_{\hat{\imath}})}, \mathring{p}_{\max(\tilde{\gamma}_{\hat{\imath}})},$$
$$\mathring{p}_{\min(\mathring{\alpha}_{\hat{\imath}})}, \mathring{p}_{\max(\mathring{\beta}_{\hat{\imath}})}, \mathring{p}_{\max(\mathring{\gamma}_{\hat{\imath}})}),$$

then

$$\mathring{g}^{-} \leq LNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) \leq \mathring{g}^{+} \tag{21}$$

Proof. Based on Properties 1 and 3, we have:

$$LNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) \geq LNCHM^{\mathring{k}}(\tilde{g}^{-}_{\hat{\imath}}, \mathring{g}^{-}_{\hat{\imath}}) = \mathring{g}^{-}$$

$$LNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) \leq LNCHM^{\mathring{k}}(\tilde{g}^{+}_{\hat{\imath}}, \mathring{g}^{+}_{\hat{\imath}}) = \mathring{g}^{+}.$$

The proof is completed. □

In addition, we will deliberate about some desirable cases of the LNCHM operator for the parameter $\mathring{k}$.

1. When $\mathring{k} = 1$, the LNCHM operator in (16) will be reduced to the LNCHA (linguistic neutrosophic cubic Hamy averaging) operator:

$$LNCHM^{1}(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) = \frac{\sum_{1 \le \hat{\imath}_1 \le n} \left(\prod_{j=1}^{1} \tilde{g}_{\hat{\imath}_j}, \prod_{j=1}^{1} \mathring{g}_{\hat{\imath}_j} \right)^{\frac{1}{1}}}{\binom{n}{1}}$$

$$= \left(\begin{array}{c} \mathring{p}_{t - t\left(\prod_{1 \le \hat{\imath}_1 \le n} \left(1 - \left(\prod_{j=1}^{1} \tilde{\alpha}_{\hat{\imath}_j}, \prod_{j=1}^{1} \mathring{\alpha}_{\hat{\imath}_j} \right)^{\frac{1}{1}} \right) \right)^{\frac{1}{\binom{n}{1}}}}, \mathring{p}_{t\left(\prod_{1 \le \hat{\imath}_1 \le n} \left(1 - \left(\prod_{j=1}^{1} \left(1 - \frac{\tilde{\beta}_{\hat{\imath}_j}}{t}, 1 - \frac{\mathring{\beta}_{\hat{\imath}_j}}{t} \right) \right)^{\frac{1}{1}} \right) \right)^{\frac{1}{\binom{n}{1}}}}, \\ \mathring{p}_{t\left(\prod_{1 \le \hat{\imath}_1 \le n} \left(1 - \left(\prod_{j=1}^{1} \left(1 - \frac{\tilde{\gamma}_{\hat{\imath}_j}}{t}, 1 - \frac{\mathring{\gamma}_{\hat{\imath}_j}}{t} \right) \right)^{\frac{1}{1}} \right) \right)^{\frac{1}{\binom{n}{1}}}} \end{array} \right) \tag{22}$$

$$= \left(\mathring{p}_{t - t\left(\prod_{j=1}^{1} (1 - \tilde{\alpha}_{\hat{\imath}}, 1 - \mathring{\alpha}_{\hat{\imath}}) \right)^{\frac{1}{n}}}, \mathring{p}_{t\left(\prod_{j=1}^{1} \left(\frac{\tilde{\beta}_{\hat{\imath}}}{t}, \frac{\mathring{\beta}_{\hat{\imath}}}{t} \right) \right)^{\frac{1}{n}}}, \mathring{p}_{t\left(\prod_{j=1}^{1} \left(\frac{\tilde{\gamma}_{\hat{\imath}}}{t}, \frac{\mathring{\gamma}_{\hat{\imath}}}{t} \right) \right)^{\frac{1}{n}}} \right)$$

$$(\text{let } \hat{\imath}_1 = \hat{\imath}) = \frac{1}{n} \sum_{\hat{\imath}=1}^{n} \mathring{g}_{\hat{\imath}} = LNCA(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}})$$

2. When $\mathring{k} = n$, the LNCHM operator in (16) will reduce to the LNCHA (linguistic neutrosophic cubic Hamy averaging) operator:

$$LNCM^{n}(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) = \frac{\sum_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n} \left(\prod_{j=1}^{n} \tilde{g}_{\hat{\imath}j}, \prod_{j=1}^{n} \mathring{g}_{\hat{\imath}j} \right)^{\frac{1}{n}}}{\binom{n}{n}}$$

$$= \left(\begin{array}{c} \mathring{p}_{t - t\left(\prod_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n} \left(1 - \left(\frac{\prod_{j=1}^{n} \tilde{\alpha}_{\hat{\imath}j}}{t^n}, \frac{\prod_{j=1}^{n} \mathring{\alpha}_{\hat{\imath}j}}{t^n} \right)^{\frac{1}{n}} \right) \right)^{\frac{1}{\binom{n}{n}}}}, \\ \mathring{p}_{t\left(\prod_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n} \left(1 - \left(\prod_{j=1}^{n} \left(1 - \frac{\tilde{\beta}_{\hat{\imath}j}}{t}, 1 - \frac{\mathring{\beta}_{\hat{\imath}j}}{t} \right) \right)^{\frac{1}{n}} \right) \right)^{\frac{1}{\binom{n}{n}}}}, \\ \mathring{p}_{t\left(\prod_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n} \left(1 - \left(\prod_{j=1}^{n} \left(1 - \frac{\tilde{\gamma}_{\hat{\imath}j}}{t}, 1 - \frac{\mathring{\gamma}_{\hat{\imath}j}}{t} \right) \right)^{\frac{1}{n}} \right) \right)^{\frac{1}{\binom{n}{n}}}} \end{array} \right) \tag{23}$$

$$= \left(\mathring{p}^{\,t - t\left(1 - \left(\frac{\prod_{j=1}^{n} \tilde{\alpha}_{\hat{\imath}j}}{t^n}, \frac{\prod_{j=1}^{n} \mathring{\alpha}_{\hat{\imath}j}}{t^n}\right)\right)^{\frac{1}{n}}}, \mathring{p}^{\,t\left(1 - \left(\prod_{j=1}^{n}\left(1 - \frac{\tilde{\beta}_{\hat{\imath}j}}{t}, 1 - \frac{\mathring{\beta}_{\hat{\imath}j}}{t}\right)\right)^{\frac{1}{n}}\right)}, \mathring{p}^{\,t\left(1 - \left(\prod_{j=1}^{n}\left(1 - \frac{\tilde{\gamma}_{\hat{\imath}j}}{t}, 1 - \frac{\mathring{\gamma}_{\hat{\imath}j}}{t}\right)\right)^{\frac{1}{n}}\right)} \right)$$

$$= \left(\mathring{p}^{\,t\left(\frac{\prod_{j=1}^{n} \tilde{\alpha}_{\hat{\imath}j}}{t^n}, \frac{\prod_{j=1}^{n} \mathring{\alpha}_{\hat{\imath}j}}{t^n}\right)^{\frac{1}{n}}}, \mathring{p}^{\,t - t\left(\prod_{j=1}^{n}\left(1 - \frac{\tilde{\beta}_{\hat{\imath}j}}{t}, 1 - \frac{\mathring{\beta}_{\hat{\imath}j}}{t}\right)\right)^{\frac{1}{n}}}, \mathring{p}^{\,t - t\left(\prod_{j=1}^{n}\left(1 - \frac{\tilde{\gamma}_{\hat{\imath}j}}{t}, 1 - \frac{\mathring{\gamma}_{\hat{\imath}j}}{t}\right)\right)^{\frac{1}{n}}} \right)$$

$$\text{let } \left(\hat{\imath}_j = \hat{\imath}\right) = \prod_{\hat{\imath}=1}^{n} \mathring{g}_{\hat{\imath}}^{\frac{1}{n}} = LNG(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}})$$

Definition 14. *Suppose* $(\mathring{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}})$ *where* $\hat{\imath} = 1, 2, \ldots, n$. *is an assortment of linguistic neutrosophic cubic numbers and parameter* $\mathring{k} = 1, 2, \ldots, n$. *and* $\mathring{w} = (\mathring{w}_1, \mathring{w}_2 \ldots, \mathring{w}_n)^T$ *the weight vector of* $\hat{\imath}_{\hat{\imath}}$ *with* $\mathring{w}_{\hat{\imath}} \in [0, 1]$ *and* $\sum_{\hat{\imath}=1}^{n} \mathring{w}_{\hat{\imath}} = 1$, *then the WLNCHM operator is defined as:*

$$WLNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) = \frac{\sum_{1 \le \hat{\imath}_1 < \ldots \hat{\imath}_{\mathring{k}} \le n} \left(\prod_{j=1}^{\mathring{k}} \mathring{w}_{\hat{\imath}j} \tilde{g}_{\hat{\imath}j}, \prod_{j=1}^{\mathring{k}} \mathring{w}_{\hat{\imath}j} \mathring{g}_{\hat{\imath}j} \right)^{\frac{1}{\mathring{k}}}}{\binom{n}{\mathring{k}}} \tag{24}$$

where $\left(\hat{\imath}_1, \hat{\imath}_2, \ldots, \hat{\imath}_{\mathring{k}}\right)$ *navigate all the k-tuple arrangements of* $(1, 2, \ldots, \mathring{n})$, $\binom{n}{\mathring{k}}$ *is the binomial coefficient and* $\binom{n}{\mathring{k}} = \frac{n!}{\mathring{k}!(n - \mathring{k})!}$.

Example 5. *Let* $(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) = ((\tilde{g}_1, \mathring{g}_1), (\tilde{g}_2, \mathring{g}_2))$ $i = 1, 2$ *and* $k = 1$, *where* $\tilde{g}_1 = ([0.2, 0.4], [0.3, 0.4], [0.4, 0.6], (0.6, 0.5, 0.8))$, $\tilde{g}_2 = ([0.3, 0.5], [0.4, 0.7], [0.2, 0.4], (0.7, 0.8, 0.6))$ *and* $\mathring{w} = (0.5, 0.5)$*:*

$$
\begin{aligned}
&WLNCHM^{1}\left((\tilde{g}_1, \mathring{g}_2), (\tilde{g}_2, \mathring{g}_2)\right) \\
&= \frac{\sum \left(\left((\mathring{w}_{11}\tilde{g}_{11}, \mathring{w}_{11}\mathring{g}_{11})(\mathring{w}_{22}\tilde{g}_{22}, \mathring{w}_{22}\mathring{g}_{22})\right)\right)^{1}}{\binom{2}{1}} \\
&= \frac{\left(\left((\mathring{w}_{11}\tilde{g}_{11}, \mathring{w}_{11}\mathring{g}_{11}), (\mathring{w}_{22}\tilde{g}_{22}, \mathring{w}_{22}\mathring{g}_{22})\right)\right)^{1} + \left(\left((\mathring{w}_{11}\tilde{g}_{11}, \mathring{w}_{11}\mathring{g}_{11}), (\mathring{w}_{22}\tilde{g}_{22}, \mathring{w}_{22}\mathring{g}_{22})\right)\right)^{1}}{\binom{2}{1}} \\
&= \frac{\sum \begin{pmatrix} (0.5)(0.5)\begin{pmatrix} ([0.2,0.4],[0.3,0.4],[0.4,0.6],(0.6,0.5,0.8)) \\ ([0.2,0.4],[0.3,0.4],[0.4,0.6],(0.6,0.5,0.8)) \end{pmatrix} \\ (0.5)(0.5)\begin{pmatrix} ([0.3,0.5],[0.4,0.7],[0.2,0.4],(0.7,0.8,0.6)) \\ ([0.3,0.5],[0.4,0.7],[0.2,0.4],(0.7,0.8,0.6)) \end{pmatrix} \end{pmatrix}^{1}}{\binom{2}{1}} \\
&= \frac{\begin{pmatrix} (0.5)(0.5)\begin{pmatrix} ([0.2,0.4],[0.3,0.4],[0.4,0.6],(0.6,0.5,0.8)) \\ ([0.2,0.4],[0.3,0.4],[0.4,0.6],(0.6,0.5,0.8)) \end{pmatrix} \\ (0.5)(0.5)\begin{pmatrix} ([0.3,0.5],[0.4,0.7],[0.2,0.4],(0.7,0.8,0.6)) \\ ([0.3,0.5],[0.4,0.7],[0.2,0.4],(0.7,0.8,0.6)) \end{pmatrix} \end{pmatrix}^{1} + \begin{pmatrix} (0.5)(0.5)\begin{pmatrix} ([0.2,0.4],[0.3,0.4],[0.4,0.6],(0.6,0.5,0.8)) \\ ([0.2,0.4],[0.3,0.4],[0.4,0.6],(0.6,0.5,0.8)) \end{pmatrix} \\ (0.5)(0.5)\begin{pmatrix} ([0.3,0.5],[0.4,0.7],[0.2,0.4],(0.7,0.8,0.6)) \\ ([0.3,0.5],[0.4,0.7],[0.2,0.4],(0.7,0.8,0.6)) \end{pmatrix} \end{pmatrix}^{1}}{\binom{2}{1}} \\
&= \frac{\begin{pmatrix} ([0.003,0.01],[0.006,0.01],[0.01,0.023],(0.3,0.23,0.2)) \\ ([0.006,0.02],[0.01,0.034],[0.03,0.01],(0.32,0.2,0.3)) \end{pmatrix} + \begin{pmatrix} ([0.003,0.01],[0.006,0.01],[0.01,0.023],(0.3,0.23,0.2)) \\ ([0.006,0.02],[0.01,0.034],[0.03,0.01],(0.32,0.2,0.3)) \end{pmatrix}}{\binom{2}{1}} \\
&= \frac{\begin{pmatrix} ([0.00002,0.0002],[0.00006,0.00034],[0.0003,0.0023],(0.52,0.4,0.44)) \\ +([0.00002,0.0002],[0.00006,0.00034],[0.0003,0.0023],(0.52,0.4,0.44)) \end{pmatrix}}{\binom{2}{1}} \\
&= \frac{([0.00004,0.0004],[0.00012,0.0007],[0.0006,0.005],(0.3,0.2,0.23))}{\binom{2}{1}} \\
&= ([0.00002,0.0002],[0.00006,0.0004],[0.0003,0.003],(0.2,0.1,0.12))
\end{aligned}
$$

Depending on the operations of LNCNs that were given in the above Equations (1)–(4), with the help of Equation (24), we can formulate the following theorem.

Theorem 2. *Let* $(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) = (\mathring{p}_{(\tilde{\alpha},\mathring{\alpha})}, \mathring{p}_{(\tilde{\beta},\mathring{\beta})}, \mathring{p}_{(\tilde{\gamma},\mathring{\gamma})})(\hat{\imath} = 1,2,\ldots,n)$ *be the collection of LNCNs,* $\mathring{w} = (\mathring{w}_1, \mathring{w}_2 \ldots, \mathring{w}_n)^T$ *be the weight vector of* $\hat{\imath}_{\hat{\imath}}$ *with* $\mathring{w}_{\hat{\imath}} \in [0,1], \hat{\imath} = 1,2,\ldots,n$ *and* $\sum_{\hat{\imath}=1}^{n} \mathring{w}_{\hat{\imath}} = 1$*. Then, the accumulated value acquired from the WLNCM operator in Equation (24) is obviously an LNCN, and:*

$$
\begin{aligned}
&WLNCM(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) \\
&= \left(\begin{array}{l}
\mathring{p}_{t-t\left(\prod\limits_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod\limits_{j=1}^{\mathring{k}}\left(1-\left(1-\frac{\tilde{\alpha}_{\hat{\imath}j}}{t},1-\frac{\mathring{\alpha}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \\
\mathring{p}_{t\left(\prod\limits_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod\limits_{j=1}^{\mathring{k}}\left(1-\left(\frac{\tilde{\beta}_{\hat{\imath}j}}{t},\frac{\mathring{\beta}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \\
\mathring{p}_{t\left(\prod\limits_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod\limits_{j=1}^{\mathring{k}}\left(1-\left(\frac{\tilde{\gamma}_{\hat{\imath}j}}{t},\frac{\mathring{\gamma}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}
\end{array} \right)
\end{aligned} \tag{25}
$$

Proof. According to the operational law of LNCNs, we have:

$$
\mathring{w}_{\hat{\imath}j}\mathring{g}_{\hat{\imath}j} = \left(\mathring{p}_{t-t\left(1-\frac{\tilde{\alpha}_{\hat{\imath}j}}{t},1-\frac{\mathring{\alpha}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}}, \mathring{p}_{t\left(\frac{\tilde{\beta}_{\hat{\imath}j}}{t},\frac{\mathring{\beta}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}}, \mathring{p}_{t\left(\frac{\tilde{\gamma}_{\hat{\imath}j}}{t},\frac{\mathring{\gamma}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}} \right),
$$

$$
\begin{aligned}
&\prod_{j=1}^{\mathring{k}} \mathring{w}_{\hat{\imath}j}\mathring{g}_{\hat{\imath}j} \\
&= \left(\mathring{p}_{t\prod\limits_{j=1}^{\mathring{k}}\left(1-\left(1-\frac{\tilde{\alpha}_{\hat{\imath}j}}{t},1-\frac{\mathring{\alpha}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)}, \mathring{p}_{t-t\prod\limits_{j=1}^{\mathring{k}}\left(1-\left(\frac{\tilde{\beta}_{\hat{\imath}j}}{t},\frac{\mathring{\beta}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)}, \mathring{p}_{t-t\prod\limits_{j=1}^{\mathring{k}}\left(1-\left(\frac{\tilde{\gamma}_{\hat{\imath}j}}{t},\frac{\mathring{\gamma}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)} \right)
\end{aligned}
$$

and:

$$
\begin{aligned}
&\left(\prod_{j=1}^{\mathring{k}} \mathring{w}_{\hat{\imath}j}\mathring{g}_{\hat{\imath}j}\right)^{\frac{1}{\mathring{k}}} \\
&= \left(\mathring{p}_{t\left(\prod\limits_{j=1}^{\mathring{k}}\left(1-\left(1-\frac{\mathring{\alpha}_{\hat{\imath}j}}{t},1-\frac{\mathring{\alpha}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)\right)^{\frac{1}{\mathring{k}}}}, \mathring{p}_{t-t\left(\prod\limits_{j=1}^{\mathring{k}}\left(1-\left(\frac{\mathring{\beta}_{\hat{\imath}j}}{t},\frac{\mathring{\beta}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)\right)^{\frac{1}{\mathring{k}}}}, \mathring{p}_{t-t\left(\prod\limits_{j=1}^{\mathring{k}}\left(1-\left(\frac{\mathring{\gamma}_{\hat{\imath}j}}{t},\frac{\mathring{\gamma}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)\right)^{\frac{1}{\mathring{k}}}} \right)
\end{aligned}
$$

then:

$$
\begin{aligned}
&\sum_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n} \left(\prod_{j=1}^{\mathring{k}} \mathring{g}_{\hat{\imath}j}\right)^{\frac{1}{\mathring{k}}} \\
&= \left(\begin{array}{c}
\mathring{p}_{t-t\prod\limits_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod\limits_{j=1}^{\mathring{k}}\left(1-\left(1-\frac{\tilde{\alpha}_{\hat{\imath}j}}{t},1-\frac{\mathring{\alpha}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)\right)^{\frac{1}{\mathring{k}}}\right)}, \mathring{p}_{t\prod\limits_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod\limits_{j=1}^{\mathring{k}}\left(1-\left(\frac{\tilde{\beta}_{\hat{\imath}j}}{t},\frac{\mathring{\beta}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)\right)^{\frac{1}{\mathring{k}}}\right)}, \\
\mathring{p}_{t\prod\limits_{1\le\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\le n}\left(1-\left(\prod\limits_{j=1}^{\mathring{k}}\left(1-\left(\frac{\tilde{\gamma}_{\hat{\imath}j}}{t},\frac{\mathring{\gamma}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)\right)^{\frac{1}{\mathring{k}}}\right)}
\end{array} \right)
\end{aligned}
$$

$$\frac{1}{\binom{n}{\mathring{k}}}\sum_{1\leq\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\leq n}\left(\prod_{j=1}^{\mathring{k}}\mathring{g}_{\hat{\imath}j}\right)^{\frac{1}{\mathring{k}}}$$

$$= \left(\begin{array}{l} \mathring{p}_{t-t\left(\prod_{1\leq\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\leq n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\left(1-\frac{\tilde{\alpha}_{\hat{\imath}j}}{t},1-\frac{\mathring{\alpha}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \\ \mathring{p}_{t\left(\prod_{1\leq\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\leq n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\left(\frac{\tilde{\beta}_{\hat{\imath}j}}{t},\frac{\mathring{\beta}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \\ \mathring{p}_{t\left(\prod_{1\leq\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\leq n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\left(\frac{\tilde{\gamma}_{\hat{\imath}j}}{t},\frac{\mathring{\gamma}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}} \end{array} \right)$$

Therefore,

$$WLNCHM(\tilde{g}_{\hat{\imath}},\mathring{g}_{\hat{\imath}})$$

$$= \left(\begin{array}{l} \mathring{p}_{t-t\left(\prod_{1\leq\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\leq n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\left(1-\frac{\tilde{\alpha}_{\hat{\imath}j}}{t},1-\frac{\mathring{\alpha}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \\ \mathring{p}_{t\left(\prod_{1\leq\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\leq n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\left(\frac{\tilde{\beta}_{\hat{\imath}j}}{t},\frac{\mathring{\beta}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}}, \\ \mathring{p}_{t\left(\prod_{1\leq\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\leq n}\left(1-\left(\prod_{j=1}^{\mathring{k}}\left(1-\left(\frac{\tilde{\gamma}_{\hat{\imath}j}}{t},\frac{\mathring{\gamma}_{\hat{\imath}j}}{t}\right)^{\mathring{w}_{\hat{\imath}j}}\right)\right)^{\frac{1}{\mathring{k}}}\right)\right)^{\frac{1}{\binom{n}{\mathring{k}}}}} \end{array} \right)$$

which proves Theorem. □

According to the operating rules of the *LNCN*s, the WLNCHM operators also have the same properties in the following:

Property 5. *(Commutativity) Let* $(\mathring{g}_{\hat{\imath}},\mathring{g}_{\hat{\imath}})$ *for all* $(\hat{\imath}=1,2,\ldots,n)$ *, be an assortment of LNCNs and* $(\tilde{g}'_{\hat{\imath}},\mathring{g}'_{\hat{\imath}})$ *be any permutation of* $(\tilde{g}_{\hat{\imath}},\mathring{g}_{\hat{\imath}})$*, then:*

$$WLNCHM^{\mathring{k}}(\tilde{g}'_{\hat{\imath}},\mathring{g}'_{\hat{\imath}}) = LNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}},\mathring{g}_{\hat{\imath}}) \tag{26}$$

Based on Definition (13), the conclusion is obvious,

$$WLNCHM^{\mathring{k}}(\mathring{w}_{\hat{\imath}j}\tilde{g}'_{\hat{\imath}},\mathring{w}_{\hat{\imath}j}\mathring{g}'_{\hat{\imath}})$$

$$= \frac{\sum_{1\leq\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\leq n}\left(\prod_{j=1}^{\mathring{k}}\mathring{w}_{\hat{\imath}j}\tilde{g}'_{\hat{\imath}j},\prod_{j=1}^{\mathring{k}}\mathring{w}_{\hat{\imath}j}\mathring{g}'_{\hat{\imath}j}\right)^{\frac{1}{\mathring{k}}}}{\binom{n}{\mathring{k}}}$$

$$= \frac{\sum_{1\leq\hat{\imath}_1<\ldots\hat{\imath}_{\mathring{k}}\leq n}\left(\prod_{j=1}^{\mathring{k}}\mathring{w}_{\hat{\imath}j}\tilde{g}_{\hat{\imath}j},\prod_{j=1}^{\mathring{k}}\mathring{w}_{\hat{\imath}j}\mathring{g}_{\hat{\imath}j}\right)^{\frac{1}{\mathring{k}}}}{\binom{n}{\mathring{k}}}$$

$$= WLNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}},\mathring{g}_{\hat{\imath}})$$

Property 6. *(Monotonicity) Let* $(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) = (\mathring{p}_{(\tilde{\alpha},\mathring{\alpha})}, \mathring{p}_{(\tilde{\beta},\mathring{\beta})}, \mathring{p}_{(\tilde{\gamma},\mathring{\gamma})}), (\tilde{f}_{\hat{\imath}}, f_{\hat{\imath}}) = \left(\mathring{p}_{(\tilde{p}_{\hat{\imath}},p_{\hat{\imath}})}, \mathring{p}_{(\tilde{q}_{\hat{\imath}},q_{\hat{\imath}})}, \mathring{p}_{(\tilde{r}_{\hat{\imath}},r_{\hat{\imath}})}\right)$ $(\hat{\imath} = 1, 2, \ldots, n)$ *be two collections of LNCNs; if* $\tilde{\alpha}_{\hat{\imath}} \le \tilde{p}_{\hat{\imath}}, \tilde{\beta}_{\hat{\imath}} \le \tilde{q}_{\hat{\imath}}, \tilde{\gamma}_{\hat{\imath}} \le \tilde{r}_{\hat{\imath}}$*, and* $\mathring{\alpha}_{\hat{\imath}} \le \mathring{p}_{\hat{\imath}}, \mathring{\beta}_{\hat{\imath}} \le q_{\hat{\imath}}, \mathring{\gamma}_{\hat{\imath}} \le r_{\hat{\imath}}$ *for all* $\hat{\imath}$*, then:*

$$WLNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) \le WLNCHM^{\mathring{k}}\,(\tilde{f}_{\hat{\imath}}, f_{\hat{\imath}}) \tag{27}$$

Property 7. *(Idempotency) If* $(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) = (\tilde{g}, \mathring{g}) = (\mathring{p}_{(\tilde{\alpha},\mathring{\alpha})}, \mathring{p}_{(\tilde{\beta},\mathring{\beta})}, \mathring{p}_{(\tilde{\gamma},\mathring{\gamma})})$ *for all* $(\hat{\imath} = 1, 2, \ldots, n)$*, then:*

$$WLNCHM^{\mathring{k}}(\mathring{g}, \mathring{g}) = (\mathring{p}_{(\tilde{\alpha},\mathring{\alpha})}, \mathring{p}_{(\tilde{\beta},\mathring{\beta})}, \mathring{p}_{(\tilde{\gamma},\mathring{\gamma})}) \tag{28}$$

Property 8. *(Boundedness) Let* $(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}})(\hat{\imath} = 1, 2, \ldots, n)$ *be an assortment of LNCNs and* $\mathring{g}^{+} = \max(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}), \mathring{g}^{-} = \min(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}})$*, then:*

$$\mathring{g}^{-} \le WLNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) \le \mathring{g}^{+} \tag{29}$$

Based on Properties 5 and 6, we have,

$$WLNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) \ge WLNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}}^{-}, \mathring{g}_{\hat{\imath}}^{-}) = \mathring{g}^{-}$$

$$WLNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}}, \mathring{g}_{\hat{\imath}}) \le WLNCHM^{\mathring{k}}(\tilde{g}_{\hat{\imath}}^{+}, \mathring{g}_{\hat{\imath}}^{+}) = \mathring{g}^{+}.$$

4. Entropy of LNCSs

Entropy is used to control the unpredictability in different sets like the fuzzy set (FS), intuitionistic fuzzy set (IFS), etc. In 1965, Zadeh [37] first defined the entropy of FS to determine the ambiguity in a quantitative manner. This notion of fuzziness plays a significant role in system optimization, pattern classification, control and some other areas. He also gave some points of its effects in system theory. Recently, the non-probabilistic entropy was axiomatized by Luca et al. [38]. The intuitionistic fuzzy sets are intuitive and have been widely used in the fuzzy literature. The entropy G of a fuzzy set H satisfies the following conditions,

1. $G(H) = 0$ if and only if $H \in 2^{x}$;
2. $G(H) = 1$ if and only if $\mu_A(x) = 0.5, \forall x \in X$;
3. $G(H) \le G(\hat{\imath})$ if and only if H is less fuzzy than $\hat{\imath}$, i.e., if $\mu_H(x) \le \mu_{\hat{\imath}}(x) \le 0.5, \forall x \in X$ or if $\mu_H(x) \ge \mu_{\hat{\imath}}(x) \ge 0.5, \forall x \in X$;
4. $G(H^{C}) = G(H)$.

Axioms 1–4 were expressed for fuzzy sets (known only by their membership functions), and they are stated for the intuitionistic fuzzy sets as follows:

1. $G(H) = 0$ if and only if $H \in 2^{x}$; (H non-fuzzy)
2. $G(H) = 1$ if and only if $\mu_H(x) = \nu_H(x), \forall x \in X$;
3. $G(H) \le G(\hat{\imath})$ if and only if H is less than $\hat{\imath}$, i.e., if $\mu_H(x) \le \mu_{\hat{\imath}}(x)$ and $\nu_H(x) \ge \nu_i(x)$ for $\mu_{\hat{\imath}}(x) \le \nu_i(x)$ or if $\mu_H(x) \ge \mu_{\hat{\imath}}(x)$ and $\nu_H(x) \le \nu_i(x)$ for $\mu_{\hat{\imath}}(x) \ge \nu_i(x)$,
4. $G(H^{C}) = G(H)$.

Differences occur in Axiom 2 and 3.

Kaufmann [39] suggested a distance measure of soft entropy. A new non-probabilistic entropy measure was introduced by Kosko [40]. In [41] Majumdar and Samanta introduced the notion of two single-valued neutrosophic sets, their properties and also defined the distance between these two sets. They also investigated the measure of entropy of a single-valued neutrosophic set. The entropy of IFSs was introduced by Szmidt and Kacprzyk [42]. The fuzziness measure in terms of distance between the fuzzy set and its compliment was put forward by Yager [43].

The $LNCS$ was examined by managing undetermined data with the truth, indeterminacy and falsity membership function. For the neutrosophic entropy, we will trace the Kosko idea for fuzziness calculation [40]. Kosko proposed to measure this information feature by a similarity function between the distance to the nearest crisp element and the distance to the farthest crisp element. For neutrosophic information, we refer the work by Patrascu [45] where he has given the following definition including from Equation (30) to (33). It states that: the two crisp elements are $(1,0,0)$ and $(0,0,1)$. We consider the following vector: $B = (\mu - \nu, \mu + \nu - 1, w)$. For $(1,0,0)$ and $(0,0,1)$, it results in $B_{Tru} = (1,0,0)$ and $B_{Fal} = (-1,0,0)$. We will now compute the distances as follows:

$$D(B, B_{Tru}) = |\mu - \nu - 1| + |\mu + \nu - 1| + w \tag{30}$$

$$D(B, B_{Fal}) = |\mu - \nu + 1| + |\mu + \nu - 1| + w \tag{31}$$

The neutrosophic entropy will be defined by the similarity between these two distances. The similarity E_c and neutrosophic entropy V_c are defined as follows:

$$E_c = 1 - \frac{|D(B, B_{Tru}) - D(B, B_{Fal})|}{D(B, B_{Tru}) + D(B, B_{Fal})} \tag{32}$$

$$V_c = 1 - \frac{|\mu - \nu|}{1| + |\mu + \nu - 1| + w} \tag{33}$$

Definition 15. *Suppose that* $H = \left\{ \left(x_{\hat{\imath}}, \mathring{p}_{(\tilde{\alpha}H, \mathring{\alpha}H)(x_{\hat{\imath}})}, \mathring{p}_{\left(\tilde{\beta}H, \mathring{\beta}H\right)(x_{\hat{\imath}})}, \mathring{p}_{(\tilde{\gamma}H, \mathring{\gamma}H)(x_{\hat{\imath}})} \right) \mid x_{\hat{\imath}} \in X \right\}$ *is an LNCS; we define the entropy of LNCS as a function* $G_{\mathring{k}} : \mathring{k}(X) \to [0, t]$, *where* t *is an odd cardinality with* $t+1$. *The following are some conditions.*

1. $G_{\mathring{k}}(H) = 0$ *îf* H *is a crisp set;*
2. $G_{\mathring{k}}(H) = [1,1]$ *if and only if* $\frac{\tilde{\mathring{\alpha}}H(x)}{t} = \frac{\tilde{\mathring{\beta}}H(x)}{t} = \frac{\tilde{\mathring{\gamma}}H(x)}{t} = [0.5, 0.5]$ *and* $G_{\mathring{k}}(H) = 1$ *if and only if* $\frac{\mathring{\alpha}H(x)}{t} = \frac{\mathring{\beta}H(x)}{t} = \frac{\mathring{\gamma}H(x)}{t} = 0.5, \forall x \in X$ *;*
3. $G_{\mathring{k}}(H) \leq G_{\mathring{k}}(\hat{\imath})$ *if and only if* H *is less indeterminable than* $\hat{\imath}$, *i.e., if* $\frac{\tilde{\mathring{\alpha}}H(x)}{t} + \frac{\tilde{\mathring{\gamma}}H(x)}{t} \geq \frac{\tilde{\mathring{\alpha}}\hat{\imath}(x)}{t} + \frac{\tilde{\mathring{\gamma}}\hat{\imath}(x)}{t}$, $\frac{\mathring{\alpha}H(x)}{t} + \frac{\mathring{\gamma}H(x)}{t} \geq \frac{\mathring{\alpha}\hat{\imath}(x)}{t} + \frac{\mathring{\gamma}\hat{\imath}(x)}{t}$ *and* $\left| \frac{\tilde{\mathring{\beta}}_H(x)}{t} - \frac{\tilde{\mathring{\beta}}_{H^C}(x)}{t} \right| \geq \left| \frac{\tilde{\mathring{\beta}}_{\hat{\imath}}(x)}{t} - \frac{\tilde{\mathring{\beta}}_{\hat{\imath}^C}(x)}{t} \right|$, $\left| \frac{\mathring{\beta}_H(x)}{t} - \frac{\mathring{\beta}_{H^C}(x)}{t} \right| \geq \left| \frac{\mathring{\beta}_{\hat{\imath}}(x)}{t} - \frac{\mathring{\beta}_{\hat{\imath}^C}(x)}{t} \right|$*;*
4. $G_{\mathring{k}}(H^C) = G_{\mathring{k}}(H)$.

We need to consider three factors for the uncertain measure of $LNCS$; one is the truth membership and false membership, and the other is the indeterminacy term. We define the entropy measure of $G_{\mathring{k}}$ of an $LNCS$ H, which depends on the following terms:

$$G_{\mathring{k}}(H) = 1 - \frac{1}{n} \sum_{x \in X} \left(\frac{\tilde{\mathring{\alpha}}H(x)}{t} + \frac{\tilde{\mathring{\gamma}}H(x)}{t} \right) \cdot \left| \frac{\tilde{\mathring{\beta}}_H(x)}{t} - \frac{\tilde{\mathring{\beta}}_{H^C}(x)}{t} \right| \tag{34}$$

Then, we prove that Equation (34) can meet the condition of Definition 15.

Proof. 1. For a crisp set H, there is no indeterminacy function for any $LNCN$ of H. Hence, $G_{\mathring{k}}(H) = 0$ is satisfied.

2. If H is such that $\frac{\tilde{\mathring{\alpha}}H(x)}{t} = \frac{\tilde{\mathring{\beta}}H(x)}{t} = \frac{\tilde{\mathring{\gamma}}H(x)}{t} = [0.5, 0.5]$, $\frac{\mathring{\alpha}H(x)}{t}, \frac{\mathring{\beta}H(x)}{t}, \frac{\mathring{\gamma}H(x)}{t} = 0.5, \forall x \in X$, then $\frac{\tilde{\mathring{\alpha}}H(x)}{t} + \frac{\tilde{\mathring{\gamma}}H(x)}{t} = [1,1]$, $\frac{\mathring{\alpha}H(x)}{t} + \frac{\mathring{\gamma}H(x)}{t} = 1$ and $\frac{\tilde{\mathring{\beta}}_H(x)}{t} - \frac{\tilde{\mathring{\beta}}_{H^C}(x)}{t} = [0.5, 0.5] - [0.5, 0.5] = 0$, $\frac{\mathring{\beta}_H(x)}{t} - \frac{\mathring{\beta}_{H^C}(x)}{t} = 0.5 - 0.5 = 0, \forall x \in X \Rightarrow G_{\mathring{k}}(H) = 1$.

3. H is less uncertain than I; we assume $\frac{\tilde{\mathring{\alpha}}H(x)}{t}+\frac{\tilde{\mathring{\gamma}}H(x)}{t}\geq\frac{\tilde{\mathring{\alpha}}\hat{\imath}(x)}{t}+\frac{\tilde{\mathring{\gamma}}\hat{\imath}(x)}{t}$, $\frac{\mathring{\alpha}H(x)}{t}+\frac{\mathring{\gamma}H(x)}{t}\geq\frac{\mathring{\alpha}\hat{\imath}(x)}{t}+\frac{\mathring{\gamma}\hat{\imath}(x)}{t}$ and $\left|\frac{\tilde{\mathring{\beta}}_H(x)}{t}-\frac{\tilde{\mathring{\beta}}_{H^C}(x)}{t}\right|\geq\left|\frac{\tilde{\mathring{\beta}}_{\hat{\imath}}(x)}{t}-\frac{\tilde{\mathring{\beta}}_{\hat{\imath}^C}(x)}{t}\right|$, $\left|\frac{\mathring{\beta}_H(x)}{t}-\frac{\mathring{\beta}_{H^C}(x)}{t}\right|\geq\left|\frac{\mathring{\beta}_{\hat{\imath}}(x)}{t}-\frac{\mathring{\beta}_{\hat{\imath}^C}(x)}{t}\right|$. Depending on the entropy value in Equation (34), we can obtain $G_{\mathring{k}}(H)\leq G_{\mathring{k}}(\hat{\imath})$.
4. $H^C=\left\{\left(x_{\hat{\imath}},\mathring{p}_{\tilde{\gamma}_H(x_{\hat{\imath}})},\mathring{p}_{t-\tilde{\beta}_H(x_{\hat{\imath}})},\mathring{p}_{\tilde{\alpha}_H(x_{\hat{\imath}})},\mathring{p}_{\mathring{\gamma}_H(x_{\hat{\imath}})},\mathring{p}_{t-\mathring{\beta}_H(x_{\hat{\imath}})},\mathring{p}_{\mathring{\alpha}_H(x_{\hat{\imath}})}\right)\mid x_{\hat{\imath}}\in X\right\}$,

$G_{\mathring{k}}(H^C)=1-\frac{1}{n}\sum_{x\in X}\left(\frac{\widetilde{\mathring{\gamma}_H(x)}}{t}+\frac{\widetilde{\mathring{\alpha}_H(x)}}{t}\right)\cdot\left|\frac{\tilde{\mathring{\beta}}_{H^C}(x)}{t}-\frac{\tilde{\mathring{\beta}}_H(x)}{t}\right|=G_{\mathring{k}}(H)$.

□

Example 6. *Let $\mathring{p}=\{\mathring{p}_0,\mathring{p}_1,\mathring{p}_2,\mathring{p}_3,\mathring{p}_4\}$ be a linguistic term set with cardinality $t+1$, $\mathring{g}_1=(\mathring{p}_3,\mathring{p}_2,\mathring{p}_1)$, $\mathring{g}_2=(\mathring{p}_4,\mathring{p}_3,\mathring{p}_1,)$, be two LNCNs based on $\mathring{p}$ and U be the universal set where:*

$$H=\left\{\begin{array}{c}([0.1,0.3],[0.4,0.5],[0.4,0.6],(0.4,0.6,0.7)),\\ ([0.1,0.2],[0.2,0.5],[0.1,0.4],(0.4,0.6,0.5))\end{array}\right\}$$

is an LNCS in U. Then, the entropy of U will be:

$$\begin{aligned}G_{\mathring{k}}(H)&=1-\frac{1}{2}\left(\begin{array}{c}\left(\frac{[0.1,0.3]}{5}+\frac{[0.4,0.6]}{5}\right)\cdot\left|\frac{[0.4,0.5]}{5}-\frac{5-[0.4,0.5]}{5}\right|\\ +\left(\frac{[0.1,0.2]}{5}+\frac{[0.1,0.4]}{5}\right)\cdot\left|\frac{[0.1,0.4]}{5}-\frac{5-[0.1,0.4]}{5}\right|\end{array}\right)\\ &=[0.89,0.93]\end{aligned}$$

5. The Method for MAGDM Based on the WLNCHM Operator

In this section, we discuss MAGDM, based on the WLNCHM operator with $LNCN$.

Let $U=\{U_1,U_2,\ldots,U_m\}$ be the set of alternatives, $V=\{V_1,V_2,\ldots,V_n\}$ be the set of attributes and $\mathring{w}=(\mathring{w}_1,\mathring{w}_2,\ldots,\mathring{w}_n)^T$ be the weight vector. Then, by $LNCN$s and from the predefined linguistic term set $\varphi=\{\varphi_j\mid j\in[0,t]\}$ (where $t+1$ is an odd cardinality), the decision makers are invited to evaluate the alternatives $U_{\hat{\imath}}(\hat{\imath}=1,2,\ldots,m)$ over the attributes $V_j(j=1,2,\ldots,n)$. The DMs can assign the uncertain LT^S to the truth, indeterminacy and falsity linguistic terms and the certain LT^S to the truth, indeterminacy and falsity linguistic terms in each $LNCN$s, which is based on the LT^S in the evaluation process of the linguistic evaluation of each attribute $V_j(j=1,2,\ldots,n)$ on each alternative $U_{\hat{\imath}}(\hat{\imath}=1,2,\ldots,m)$. Thus, we obtain the decision matrix $S=(s_{\hat{\imath}_j})m\times n$, $\left(\tilde{\mathring{g}}_{\hat{\imath}_j},\mathring{g}_{\hat{\imath}_j}\right)=(\mathring{p}_{\tilde{\mathring{\alpha}}_{\hat{\imath}_j}},\mathring{p}_{\tilde{\mathring{\beta}}_{\hat{\imath}_j}},\mathring{p}_{\tilde{\mathring{\gamma}}_{\hat{\imath}_j}},\mathring{p}_{\mathring{\alpha}_{\hat{\imath}_j}},\mathring{p}_{\mathring{\beta}_{\hat{\imath}_j}},\mathring{p}_{\mathring{\gamma}_{\hat{\imath}_j}})$ $(\hat{\imath}=1,2,\ldots,m;j=1,2,\ldots,n)$ as an $LNCN$.

Based on the above information, the MAGDM on the WLNCM operator is described as follows:

Step 1: Regulate the decision making problem.

Step 2: Calculate $\mathring{g}_{\hat{\imath}_j}=WLNCM(s_{\hat{\imath}1},s_{\hat{\imath}2},\ldots,s_{\hat{\imath}n})$ to obtain the collective approximation value for alternatives $U_{\hat{\imath}}$ with respect to attribute V_j.

Step 3: In this step, we operate the entropy of LNCSs to find out the weight of the elements. $\mathring{g}_j=(\mathring{p}_{(\tilde{\alpha}_j,\mathring{\alpha}_j)},\mathring{p}_{(\tilde{\beta}_j,\mathring{\beta}_j)},\mathring{p}_{(\tilde{\gamma}_j,\mathring{\gamma}_j)})$

$$G_{\mathring{k}}(\mathring{g}_j)=1-\frac{1}{m}\sum_{x\in X}\left(\frac{\tilde{\mathring{\alpha}}_{R_j}(x)}{t}+\frac{\tilde{\mathring{\gamma}}_{R_j}(x)}{t}\right)\cdot\left|\frac{\tilde{\mathring{\beta}}_{R_j}(x)}{t}-\frac{\tilde{\mathring{\beta}}_{R_j^C}(x)}{t}\right|$$

$$\varpi=G_{\mathring{k}}(\mathring{g}_j)/\sum_{j=1}^{n}G_{\mathring{k}}(\mathring{g}_j)\tag{35}$$

Step 4: In this step, we calculate the values of the score function $\varphi(S)$, accuracy function $\Phi(S)$ and certain function $\Psi(S)$ based on Equations (12)–(14).

Step 5: In this step, we find out the sequence of the alternatives $U_{\hat{\imath}}(\hat{\imath} = 1, 2, \ldots, m)$. According to the ranking order of Definition 8, with a greater score function φ(S), the ranking order of alternatives $U_{\hat{\imath}}$ is the best. If the score functions are the same, then the accuracy function of alternatives $U_{\hat{\imath}}$ is larger, and then, the ranking order of alternatives $U_{\hat{\imath}}$ is better. Furthermore, if the score and accuracy function both are the same, then the certain function of alternatives $U_{\hat{\imath}}$ is larger, and then, the ranking order of alternatives $U_{\hat{\imath}}$ is best.

Step 6: End.

6. Numerical Applications

A corporation intends to choose one person to be the area supervisor from five candidates $(U_1 - U_4)$, to be further evaluated according to the three attributes, which are shown as follows: ideological and moral quality (V_1), professional ability (V_2) and creative ability (V_3). The weights of the indicators are $\mathring{w} = (0.5, 0.3, 0.2)$.

6.1. Procedure

Case 1: If the weights of the element are absolutely unidentified, then we use the suggested technique to solve the above problem in which the decision making steps are as follows:

Step 1: Let $U = \{U_1, U_2, \ldots, U_4\}$ be a set of alternatives and $V = \{V_1, V_2, V_3\}$ be a set of attributes. Let $S = (s_{\hat{\imath}\hat{\jmath}})_{4\times 3}$ be a set of decision matrices. A decision matrix evaluates each alternative based on the given attributes;

$$S_1 = \begin{array}{c} \\ U_1 \\ U_2 \\ U_3 \\ U_4 \end{array} \begin{array}{ccc} V_1 & V_2 & V_3 \\ \left\{\begin{array}{c}([0.4,0.5],\\ [0.1,0.2],\\ [0.3,0.6],\\ (0.6,0.3,0.7))\end{array}\right\} & \left\{\begin{array}{c}([0.3,0.5],\\ [0.6,0.7],\\ [0.4,0.6],\\ (0.6,0.8,0.7))\end{array}\right\} & \left\{\begin{array}{c}([0.2,0.5],\\ [0.4,0.7],\\ [0.7,0.8],\\ (0.6,0.8,0.9))\end{array}\right\} \\ \left\{\begin{array}{c}([0.4,0.7],\\ [0.7,0.8],\\ [0.4,0.8],\\ (0.8,0.9,0.9))\end{array}\right\} & \left\{\begin{array}{c}([0.4,0.7],\\ [0.7,0.8],\\ [0.1,0.5],\\ (0.8,0.9,0.7))\end{array}\right\} & \left\{\begin{array}{c}([0.1,0.4],\\ [0.1,0.7],\\ [0.7,0.9],\\ (0.5,0.8,1.0))\end{array}\right\} \\ \left\{\begin{array}{c}([0.2,0.7],\\ [0.5,0.7],\\ [0.1,0.8],\\ (0.8,0.8,0.9))\end{array}\right\} & \left\{\begin{array}{c}([0.5,0.5],\\ [0.4,0.6],\\ [0.3,0.8],\\ (0.6,0.7,0.9))\end{array}\right\} & \left\{\begin{array}{c}([0.1,0.5],\\ [0.4,0.9],\\ [0.2,0.8],\\ (0.6,1.0,0.9))\end{array}\right\} \\ \left\{\begin{array}{c}([0.4,0.9],\\ [0.3,0.7],\\ [0.4,0.9],\\ (1.0,0.8,1.1))\end{array}\right\} & \left\{\begin{array}{c}([0.1,0.3],\\ [0.2,0.7],\\ [0.7,0.7],\\ (0.4,0.8,0.9))\end{array}\right\} & \left\{\begin{array}{c}([0.2,0.6],\\ [0.2,0.7],\\ [0.1,0.8],\\ (0.7,0.8,0.9))\end{array}\right\} \end{array}$$

$$S_2 = \begin{array}{cccc}
 & V_1 & V_2 & V_3 \\
U_1 & \left\{\begin{array}{c}([0.4,0.6],\\ [0.1,0.3],\\ [0.3,0.5],\\ (0.7,0.4,0.6))\end{array}\right\} & \left\{\begin{array}{c}([0.3,0.4],\\ [0.6,0.7],\\ [0.5,0.6],\\ (0.5,0.8,0.7))\end{array}\right\} & \left\{\begin{array}{c}([0.2,0.3],\\ [0.4,0.5],\\ [0.7,0.8],\\ (0.4,0.6,0.9))\end{array}\right\} \\
U_2 & \left\{\begin{array}{c}([0.3,0.7],\\ [0.7,0.8],\\ [0.6,0.8],\\ (0.8,0.9,1.0))\end{array}\right\} & \left\{\begin{array}{c}([0.4,0.5],\\ [0.7,0.9],\\ [0.1,0.4],\\ (0.6,1.0,0.8))\end{array}\right\} & \left\{\begin{array}{c}([0.3,0.4],\\ [0.1,0.8],\\ [0.6,0.9],\\ (0.5,0.9,1.0))\end{array}\right\} \\
U_3 & \left\{\begin{array}{c}([0.2,0.4],\\ [0.5,0.6],\\ [0.1,0.3],\\ (0.5,0.7,0.7))\end{array}\right\} & \left\{\begin{array}{c}([0.5,0.6],\\ [0.3,0.6],\\ [0.3,0.7],\\ (0.7,0.8,0.9))\end{array}\right\} & \left\{\begin{array}{c}([0.1,0.3],\\ [0.4,0.6],\\ [0.2,0.5],\\ (0.4,0.7,0.6))\end{array}\right\} \\
U_4 & \left\{\begin{array}{c}([0.4,0.7],\\ [0.3,0.5],\\ [0.4,0.6],\\ (0.8,0.6,0.7))\end{array}\right\} & \left\{\begin{array}{c}([0.1,0.4],\\ [0.2,0.6],\\ [0.6,0.7],\\ (0.5,0.7,0.8))\end{array}\right\} & \left\{\begin{array}{c}([0.2,0.7],\\ [0.2,0.8],\\ [0.1,0.5],\\ (0.8,0.9,0.7))\end{array}\right\}
\end{array}$$

$$S_3 = \begin{array}{cccc}
 & V_1 & V_2 & V_3 \\
U_1 & \left\{\begin{array}{c}([0.4,0.5],\\ [0.1,0.2],\\ [0.3,0.6],\\ (0.6,0.3,0.7))\end{array}\right\} & \left\{\begin{array}{c}([0.3,0.4],\\ [0.5,0.7],\\ [0.4,0.5],\\ (0.5,0.8,0.6))\end{array}\right\} & \left\{\begin{array}{c}([0.2,0.4],\\ [0.4,0.6],\\ [0.7,0.9],\\ (0.5,0.7,1.0))\end{array}\right\} \\
U_2 & \left\{\begin{array}{c}([0.4,0.5],\\ [0.7,0.9],\\ [0.4,0.9],\\ (0.6,1.0,1.1))\end{array}\right\} & \left\{\begin{array}{c}([0.4,0.6],\\ [0.7,0.9],\\ [0.1,0.4],\\ (0.7,1.0,0.5))\end{array}\right\} & \left\{\begin{array}{c}([0.1,0.4],\\ [0.1,0.7],\\ [0.7,0.8],\\ (0.5,0.8,0.9))\end{array}\right\} \\
U_3 & \left\{\begin{array}{c}([0.2,0.6],\\ [0.5,0.8],\\ [0.1,0.7],\\ (0.7,0.9,0.8))\end{array}\right\} & \left\{\begin{array}{c}([0.5,0.6],\\ [0.4,0.6],\\ [0.6,0.8],\\ (0.7,0.8,0.9))\end{array}\right\} & \left\{\begin{array}{c}([0.1,0.4],\\ [0.4,0.8],\\ [0.6,0.8],\\ (0.5,0.9,1.0))\end{array}\right\} \\
U_4 & \left\{\begin{array}{c}([0.3,0.9],\\ [0.4,0.7],\\ [0.5,0.9],\\ (1.1,0.8,1.0))\end{array}\right\} & \left\{\begin{array}{c}([0.1,0.2],\\ [0.2,0.5],\\ [0.6,0.7],\\ (0.3,0.6,0.8))\end{array}\right\} & \left\{\begin{array}{c}([0.2,0.5],\\ [0.2,0.4],\\ [0.1,0.9],\\ (0.7,0.8,1.0))\end{array}\right\}
\end{array}$$

Step 2: Calculate $s_{\hat{\imath}j} = WLNCHM(s_{\hat{\imath}1}, s_{\hat{\imath}2}, \ldots, s_{\hat{\imath}n})$ to obtain the overall assessment value for alternatives $U_{\hat{\imath}}$ with respect to attribute V_j.

$$\begin{array}{cccc}
 & V_1 & V_2 & V_3 \\
U_1 & \left\{\begin{array}{c}([0.110, 0.127],\\ [0.055, 0.084],\\ [0.095, 0.131],\\ (0.139, 0.101,\\ 0.142))\end{array}\right\} & \left\{\begin{array}{c}([0.101, 0.119],\\ [0.115, 0.127],\\ [0.110, 0.135],\\ (0.127, 0.156,\\ 0.142))\end{array}\right\} & \left\{\begin{array}{c}([0.078, 0.110],\\ [0.110, 0.135],\\ [0.146, 0.159],\\ (0.123, 0.110,\\ 0.169))\end{array}\right\} \\
U_2 & \left\{\begin{array}{c}([0.105, 0.139],\\ [0.146, 0.159],\\ [0.119, 0.159],\\ (0.149, 0.169,\\ 0.175))\end{array}\right\} & \left\{\begin{array}{c}([0.110, 0.135],\\ [0.146, 0.162],\\ [0.055, 0.115],\\ (0.146, 0.172,\\ 0.142))\end{array}\right\} & \left\{\begin{array}{c}([0.071, 0.110],\\ [0.055, 0.149],\\ [0.142, 0.162],\\ (0.123, 0.159,\\ 0.172))\end{array}\right\} \\
U_3 & \left\{\begin{array}{c}([0.078, 0.131],\\ [0.123, 0.146],\\ [0.055, 0.135],\\ (0.142, 0.156,\\ 0.156))\end{array}\right\} & \left\{\begin{array}{c}([0.123, 0.131],\\ [0.105, 0.135],\\ [0.123, 0.153],\\ (0.142, 0.153,\\ 0.165))\end{array}\right\} & \left\{\begin{array}{c}([0.055, 0.110],\\ [0.110, 0.153],\\ [0.101, 0.110],\\ (0.123, 0.162,\\ 0.159))\end{array}\right\} \\
U_4 & \left\{\begin{array}{c}([0.105, 0.159],\\ [0.101, 0.139],\\ [0.115, 0.156],\\ (0.172, 0.149,\\ 0.169))\end{array}\right\} & \left\{\begin{array}{c}([0.055, 0.095],\\ [0.078, 0.135],\\ [0.139, 0.146],\\ (0.110, 0.146,\\ 0.159))\end{array}\right\} & \left\{\begin{array}{c}([0.078, 0.135],\\ [0.078, 0.139],\\ [0.055, 0.149],\\ (0.149, 0.159,\\ 0.165))\end{array}\right\}
\end{array}$$

Step 3: We utilize the entropy of LNCSs to calculate the weight of the attributes, i.e., let $s_j = (\mathring{p}_{(\tilde{\alpha}_j, \mathring{\alpha}_j)}, \mathring{p}_{(\tilde{\beta}_j, \mathring{\beta}_j)}, \mathring{p}_{(\tilde{\gamma}_j, \mathring{\gamma}_j)})$ be the $LNCN$ and $G_{\mathring{k}}(s_j)$ be the weight of attributes, i.e.,

$$G_{\mathring{k}}(s_j) = 1 - \frac{1}{m}\sum_{x\in X}\left(\frac{\tilde{\mathring{\alpha}}_{S_j}(x)}{t} + \frac{\tilde{\mathring{\gamma}}_{S_j}(x)}{t}\right)\cdot\left|\frac{\tilde{\mathring{\beta}}_{S_j}(x)}{t} - \frac{\tilde{\mathring{\beta}}_{S_j^C}(x)}{t}\right|$$

$$G_{\mathring{k}}(s_1) = 1 - \frac{1}{4}\begin{pmatrix} \left(\frac{[0.110,0.127]}{7} + \frac{[0.095,0.131]}{7}\right)\cdot\left|\frac{[0.055,0.084]}{7} - \frac{7-[0.055,0.084]}{7}\right| \\ +\left(\frac{[0.105,0.139]}{7} + \frac{[0.119,0.159]}{7}\right)\cdot\left|\frac{[0.146,0.159]}{7} - \frac{7-[0.146,0.159]}{7}\right| \\ +\left(\frac{[0.078,0.131]}{7} + \frac{[0.055,0.135]}{7}\right)\cdot\left|\frac{[0.123,0.146]}{7} - \frac{7-[0.123,0.146]}{7}\right| \\ +\left(\frac{[0.105,0.159]}{7} + \frac{[0.115,0.156]}{7}\right)\cdot\left|\frac{[0.101,0.139]}{7} - \frac{7-[0.101,0.139]}{7}\right| \end{pmatrix}$$
$$= [0.975, 0.976]$$

$$G_{\mathring{k}}(s_2) = 1 - \frac{1}{4}\begin{pmatrix} \left(\frac{[0.101,0.119]}{7} + \frac{[0.110,0.135]}{7}\right)\cdot\left|\frac{[0.115,0.127]}{7} - \frac{7-[0.115,0.127]}{7}\right| \\ +\left(\frac{[0.110,0.135]}{7} + \frac{[0.055,0.115]}{7}\right)\cdot\left|\frac{[0.146,0.162]}{7} - \frac{7-[0.146,0.162]}{7}\right| \\ +\left(\frac{[0.123,0.131]}{7} + \frac{[0.123,0.153]}{7}\right)\cdot\left|\frac{[0.105,0.135]}{7} - \frac{7-[0.105,0.135]}{7}\right| \\ +\left(\frac{[0.055,0.095]}{7} + \frac{[0.139,0.146]}{7}\right)\cdot\left|\frac{[0.078,0.135]}{7} - \frac{7-[0.078,0.135]}{7}\right| \end{pmatrix}$$
$$= [0.975, 0.994]$$

$$G_{\mathring{k}}(s_3) = 1 - \frac{1}{4}\begin{pmatrix} \left(\frac{[0.078,0.110]}{7} + \frac{[0.146,0.159]}{7}\right)\cdot\left|\frac{[0.110,0.135]}{7} - \frac{7-[0.110,0.135]}{7}\right| \\ +\left(\frac{[0.071,0.110]}{7} + \frac{[0.142,0.162]}{7}\right)\cdot\left|\frac{[0.055,0.149]}{7} - \frac{7-[0.055,0.149]}{7}\right| \\ +\left(\frac{[0.055,0.110]}{7} + \frac{[0.101,0.110]}{7}\right)\cdot\left|\frac{[0.110,0.153]}{7} - \frac{7-[0.110,0.153]}{7}\right| \\ +\left(\frac{[0.078,0.135]}{7} + \frac{[0.055,0.149]}{7}\right)\cdot\left|\frac{[0.078,0.139]}{7} - \frac{7-[0.078,0.139]}{7}\right| \end{pmatrix}$$
$$= [0.935, 0.982]$$

$$\varpi = G_{\mathring{k}}(s_j) / \sum_{j=1}^{n} G_{\mathring{k}}(s_j)$$
$$\varpi_1 = \frac{[0.957, 0.976]}{[2.883, 2.952]} = [0.338.0.330]$$
$$\varpi_2 = \frac{[0.973, 0.994]}{[2.883, 2.952]} = [0.337, 0.336]$$
$$\varpi_3 = \frac{[0.935, 0.982]}{[2.883, 2.952]} = [0.324, 0.332]$$

Step 4: By the WLNCHM operator, we calculate the comprehensive evaluation value of each alternative as:

$$U_1 = ([0.132, 0.182], [0.140, 0.174], [0.127, 0.192], (0.199, 0.189, 0.212))$$
$$U_2 = ([0.128, 0.186], [0.147, 0.184], [0.141, 0.187], (0.174, 0.207, 0.199))$$
$$U_3 = ([0.093, 0.153], [0.117, 0.190], [0.147, 0.191], (0.200, 0.195, 0.205))$$
$$U_4 = ([0.103, 0.121], [0.133, 0.162], [0.152, 0.171], (0.160, 0.181, 0.175))$$

Step 5: We find the values of score function $\varphi(S)$ as:

$$\varphi(S) = \frac{1}{9t}[\left(4t + \mathring{\alpha} - \mathring{\beta} - \mathring{\gamma}\right) + \left(2t + \mathring{\alpha} - \mathring{\beta} - \mathring{\gamma}\right)], \text{for } \varphi(S) \in [0,1]$$

$$\begin{aligned}\varphi(S_1) &= \frac{1}{45}[20 + 0.13 + 0.2 - (0.14 + 0.2 + 0.13 + 0.2) \\ &+ 10 + 0.2 - (0.2 + 0.21)] \\ &= 654\end{aligned}$$

$$\begin{aligned}\varphi(S_2) &= \frac{1}{45}[20 + 0.2 + 0.2 - (0.15 + 0.2 + 0.14 + 0.2) \\ &+ 10 + 0.2 - (0.2 + 0.2)] \\ &= 0.656\end{aligned}$$

$$\begin{aligned}\varphi(S_3) &= \frac{1}{45}[20 + 0.1 + 0.2 - (0.12 + 0.2 + 0.15 + 0.2) \\ &+ 10 + 0.2 - (0.2 + 0.21)] \\ &= 0.653\end{aligned}$$

$$\begin{aligned}\varphi(S_4) &= \frac{1}{45}[20 + 0.1 + 0.1 - (0.1 + 0.2 + 0.2 + 0.2) \\ &+ 10 + 0.2 - (0.2 + 0.2)] \\ &= 0.657\end{aligned}$$

Step 6: According to the value of the score function, the ranking of the candidates can be confirmed, i.e., $S_4 \succ S_2 \succ S_1 \succ S_{3.}$, so S_4 is the best alternatives.

Case 2: If the DM gives the information about the attributes and weight and the weight vector is $\mathring{w} = (0.1, 0.5, 0.4)$, then the score function $\varphi(S_{\hat{\imath}})(\hat{\imath} = 1, 2, 3, 4)$ of Case 2 can be obtained as follows; $\varphi(S_1) = 0.451$, $\varphi(S_2) = 0.435$, $\varphi(S_3) = 0.504$, $\varphi(S_4) = 0.492$. The ranking of these score functions is $S_3 \succ S_4 \succ S_1 \succ S_2$. Thu,s due to the diverse weights of attributes, the ranking of Case 2 is different from that of Case 1.

In the MADM method, the attribute weights can return relative values in the decision method. However, due to the issues such as data loss, time pressure and incomplete field knowledge of the DMs, the information about attribute weights is not fully known or completely unknown. Through some methods, we should derive the weight vector of attributes to get possible alternatives. In Case 2, the attribute weights are usually determined based on DMs' opinions or preferences, while Case 1 uses the entropy concepts to determine weight values of attributes to successfully balance the manipulation of subjective factors. Therefore, the entropy of LNCS is applied in the decision process to give each attribute a more objective and reasonable weight.

6.2. Comparison Analysis

From the comparison analysis, one can see that the advanced method is more appropriate for articulating and handling the indeterminate and inconsistent information in linguistic decision making problems to overcome the insufficiency of several linguistic decision making methods in the existing work. In fact, most of the decision making problems based on different linguistic variables in the literature not only express inconsistent and indeterminate linguistic results, but the linguistic method suggested in the study is a generalization of existing linguistic methods and can handle and represent linguistic decision making problems with LNN information. We also see that the advanced method has much more information than the existing method in [26,32,44]. In addition, the literature [26,32,44] is the same as the best and worst and different from our methods. The reason for the difference between the given literature and our method may be the decision thought process.

Some initial information may be missing during the aggregation process. Moreover, the conclusions are different. Different aggregation operators may appear [32], and our methods are consistent with the aggregation operator and receive a different order. However, [32] may have some limitations because of the attributes. The weight vector is given directly, and the positive and negative ideal solutions are absolute. Other than this, the ranking in the literature [26,32,44] is different from the proposed method. The reason for the difference may be uncertainty in LNN membership since the information is inevitably distorted in LIFN. Our method develops the neutrosophic cubic theory and decision making method under a linguistic environment and provides a new way for solving linguistic MAGDM problems with indeterminate and inconsistent information.

7. Conclusions

In this paper, we work out the idea of $LNCN$s, their operational laws and also some properties and define the score, accuracy and certain functions of $LNCN$s for ranking $LNCN$s. Then, we define the LNCHM and WLNCHM operators. After that, we demonstrate the entropy of $LNCN$s and relate it to determine the weights. Next, we develop MAGDM based on WLNCHM operators to solve multi-attribute group decision making problems with $LNCN$ information. Finally, we provide an example of the developed method.

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