Article

Study on the Algebraic Structure of Refined Neutrosophic Numbers

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Abstract: This paper aims to explore the algebra structure of refined neutrosophic numbers. Firstly, the algebra structure of neutrosophic quadruple numbers on a general field is studied. Secondly, The addition operator $\oplus$ and multiplication operator $\otimes$ on refined neutrosophic numbers are proposed and the algebra structure is discussed. We reveal that the set of neutrosophic refined numbers with an additive operation is an abelian group and the set of neutrosophic refined numbers with a multiplication operation is a neutrosophic extended triplet group. Moreover, algorithms for solving the neutral element and opposite elements of each refined neutrosophic number are given.

Keywords: neutrosophic extended triplet group; neutrosophic quadruple numbers; refined neutrosophic numbers; refined neutrosophic quadruple numbers; neutrosophic set

1. Introduction

The notion of neutrosophic set was proposed by F. Smarandache [1], which is an extension of fuzzy set and in order to solve real-world problems. A neutrosophic set has three membership functions, and each membership degree is a real standard or non-standard subset of the nonstandard unit interval $]0^-, 1^+[= 0^- \cup [0, 1] \cup 1^+$.

In recent years, the idea of neutrosophic set has been applicable in related algebraic structures. Among these algebraic structures, Smarandache and Ali [2] proposed the algebraic system neutrosophic triplet group (NTG), which is an extension of the classical group but the neutral element is different from the classical algebraic unit element. To regard the unit element as a special neutral element, the neutrosophic extended triplet group (NETG) has been proposed [3,4] and the classical group is regarded as a special case of a NETG. Moreover, some research papers have carried out in-depth research based on NTG (NETG). For example, the inclusion relations of neutrosophic sets [5], neutrosophic triplet coset [6], neutrosophic duplet semi-groups [7], generalized neutrosophic extended triplet group [8], AG-neutrosophic extended triplet loops [9,10], the neutrosophic set theory to pseudo-BCI algebras [11], neutrosophic triplet ring and a neutrosophic triplet field [12,13], neutrosophic triplet normed space [14], neutrosophic soft sets [15], neutrosophic vector spaces [16] and so on have been studied.

As an example of NETG, Ma [8] revealed that for each $n \in Z^+, n \geq 2, (Z_n, \otimes)$ is a commutative NETG if and only if the factorization of n is a product of single factors. As another example, Ma [17] showed that the set of neutrosophic quadruple numbers with a multiplication operation is a NETG. The concept of neutrosophic numbers of the form $a + bI$, where I is the indeterminacy with $I^n = I$, and, a and are real or complex numbers. If I into many types of indeterminacies $I_1, I_2, \cdots, I_q$, in [18], Smarandache extended the neutrosophic numbers $a + bI$ into refined neutrosophic numbers of the form $a + b_1 I_1 + b_2 I_2 + \cdots + b_n I_n$, where $a, b_1, b_2, \cdots, b_n$ are real or complex numbers and considered the refined neutrosophic set based on these refined neutrosophic numbers. The notion of

neutrosophic quadruple number, which has form: $NQ = a + bT + cI + dF$ where a, b, c, d are real (or complex) numbers; and T is the truth/membership/probability; I is the indeterminacy; and F is the false/membership/improbability are called Neutrosophic Quadruple (Real, respectively, Complex) Numbers. "a" is called the known part of NQ, while $bT + cI + dF$ is called the unknown part of NQ. Similar to refined neutrosophic numbers, if T can be split into many types of truths, $T_1, T_2, \cdots, T_p$, I into many types of indeterminacies, $I_1, I_2, \cdots, I_r$, and F into many types of falsities, $F_1, F_2, \cdots, F_r$, we can get the refined neutrosophic quadruple numbers. We know that the set of neutrosophic quadruple numbers with a multiplication operation is a NETG. In this paper, we explore the algebra structure of refined neutrosophic numbers (refined neutrosophic quadruple numbers) and give new examples of NETG. In fact, the solving method of the neutral element and opposite elements for each refined neutrosophic number is different from the solving method for each neutrosophic quadruple number.

The paper is organized as follows. Section 2 gives the basic concepts. In Section 3, we show that the set of neutrosophic quadruple numbers on the general field with a multiplication operation also consists of a NETG. In Section 4, the algebra structure of refined neutrosophic numbers and refined neutrosophic quadruple numbers are studied. Finally, the summary and future work is presented in Section 5.

2. Basic Concepts

In this section, we provide the related basic definitions and properties of NETG, neutrosophic quadruple numbers, and refined neutrosophic numbers (for details, see [3,4,18–20]).

Definition 1 ([3,4])**.** *Let N be a non-empty set together with a binary operation $*$. Then, N is called a neutrosophic extended triplet set if, for any $a \in N$, there exists a neutral of "a" (denote by $neut(a)$), and an opposite of "a"(denote by $anti(a)$), such that $neut(a) \in N$, $anti(a) \in N$ and:*

$$a * neut(a) = neut(a) * a = a, \quad a * anti(a) = anti(a) * a = neut(a).$$

The triplet $(a, neut(a), anti(a))$ is called a neutrosophic extended triplet.

Definition 2 ([3,4])**.** *Let $(N, *)$ be a neutrosophic extended triplet set. Then, N is called a neutrosophic extended triplet group (NETG), if the following conditions are satisfied:*

(1) *$(N, *)$ is well-defined, i.e., for any $a, b \in N$, one has $a * b \in N$.*
(2) *$(N, *)$ is associative, i.e., $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$.*

A NETG N is called a commutative NETG if for all $a, b \in N, a * b = b * a$.

Proposition 1 ([4])**.** *$(N, *)$ be a NETG. We have:*

(1) *$neut(a)$ is unique for any $a \in N$.*
(2) *$neut(a) * neut(a) = neut(a)$ for any $a \in N$.*
(3) *$neut(neut(a)) = neut(a)$ for any $a \in N$.*

Definition 3 ([18,19])**.** *A neutrosophic number is a number of the form (a, bI), where I is the indeterminacy with $I^2 = I$, and a and b are real or complex numbers. A refined neutrosophic number is a number of the form $(a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n)$, where $I_1, I_2, \cdots, I_n$ are different types of indeterminacies, and $a_0, a_1, a_2, \cdots, a_n$ are real or complex numbers. The set NN defined by*

$$NN = \{(a, bI) | a, b \in \mathbb{R} \text{ or } \mathbb{C}\}. \tag{1}$$

is called a neutrosophic set of neutrosophic numbers. The set RNN defined by

$$RNN = \{(a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n) | a_0, a_1, a_2, \cdots, a_n \in \mathbb{R} \text{ or } \mathbb{C}\}. \tag{2}$$

is called a neutrosophic set of refined neutrosophic numbers.

Definition 4 ([18,20])**.** *A neutrosophic quadruple number is a number of the form* (a, bT, cI, dF), *where* T, I *and* F *have their usual neutrosophic logic meanings, i.e., truth, indeterminacy and false, respectively, and* $a, b, c, d \in \mathbb{R}$ *or* $\mathbb{C}$. *The set* NQ *defined by*

$$NQ = \{(a, bT, cI, dF) | a, b, c, d \in \mathbb{R} \text{ or } \mathbb{C}\}. \tag{3}$$

is called a neutrosophic set of quadruple numbers. For a neutrosophic quadruple number (a, bT, cI, dF), a *is called the known part and* (bT, cI, dF) *is called the unknown part. The set* RNQ *defined by*

$$\begin{aligned} RNQ = \quad & \{(a, b_1 T_1, b_2 T_2, \cdots, b_p T_p, c_1 I_1, c_2 I_2, \cdots, c_q I_q, d_1 F_1, d_2 F_2, \cdots, d_r F_r) | \\ & a, b_1, b_2, \cdots, b_p, c_1, c_2, \cdots, c_q, d_1, d_2, \cdots, d_r \in \mathbb{R} \text{ or } \mathbb{C}\}. \end{aligned} \tag{4}$$

is called a neutrosophic set of refined neutrosophic quadruple numbers.

Definition 5 ([18,20])**.** *Let* N *be a set, endowed with a total order* $a \prec b$, *named "a prevailed by b", "a less stronger than b" or "a less preferred than b". We consider* $a \preceq b$ *as "a prevailed by or equal to b", "a less stronger than or equal to b", or "a less preferred than or equal to b".*

For any elements $a, b \in N$, *with* $a \preceq b$, *one has the absorbance law:*

$$a \cdot b = b \cdot a = \text{ absorb}(a, b) = \max(a, b) = b. \tag{5}$$

which means that the bigger element absorbs the smaller element. Clearly,

$$a \cdot a = a^2 = \text{ absorb}(a, a) = \max(a, a) = a. \tag{6}$$

and

$$a_1 \cdot a_2 \cdots a_n = \max(a_1, a_2, \cdots, a_n). \tag{7}$$

Analogously, we say that "a $\succ$ *b" and we read: "a prevails to b", "a is stronger than b" or "a is preferred to b". In addition, a* $\succeq$ *b, and we read: "a prevails or is equal to b", "a is stronger than or equal to b", or "a is preferred or equal to b".*

Definition 6 ([18,20])**.** *Consider the set* $\{T, I, F\}$. *Suppose in an optimistic way we consider the prevalence order* $T \succ I \succ F$. *Then, we have:* $TI = IT = \max(T, I) = T$, $TF = FT = \max(T, F) = T$, $IF = FI = \max(I, F) = I$, $TT = T^2 = T$, $II = I^2 = I$, $FF = F^2 = F$.

Analogously, suppose in a pessimistic way we consider the prevalence order $T \prec I \prec F$. *Then, we have:* $TI = IT = \max(T, I) = I$, $TF = FT = \max(T, F) = F$, $IF = FI = \max(I, F) = F$, $TT = T^2 = T$, $II = I^2 = I$, $FF = F^2 = F$.

Definition 7 ([18,20])**.** *Let* $a = (a_1, a_2 T, a_3 I, a_4 F), b = (b_1, b_2 T, b_3 I, b_4 F) \in NQ$. *Suppose, in an pessimistic way, the neutrosophic expert considers the prevalence order* $T \prec I \prec F$. *Then, the multiplication operation is defined as follows:*

$$\begin{aligned} a * b = \quad & (a_1, a_2 T, a_3 I, a_4 F) * (b_1, b_2 T, b_3 I, b_4 F) \\ = \quad & (a_1 b_1, (a_1 b_2 + a_2 b_1 + a_2 b_2) T, (a_1 b_3 + a_2 b_3 + a_3 b_1 + a_3 b_2 + a_3 b_3) I, \\ & (a_1 b_4 + a_2 b_4 + a_3 b_4 + a_4 b_1 + a_4 b_2 + a_4 b_3 + a_4 b_4) F). \end{aligned} \tag{8}$$

Suppose in an optimistic way the neutrosophic expert considers the prevalence order $T \succ I \succ F$. *Then,*

$$\begin{aligned} a \star b = \quad & (a_1, a_2 T, a_3 I, a_4 F) \star (b_1, b_2 T, b_3 I, b_4 F) \\ = \quad & (a_1 b_1, (a_1 b_2 + a_2 b_1 + a_2 b_2 + a_3 b_2 + a_4 b_2 + a_2 b_3 + a_2 b_4) T, \\ & (a_1 b_3 + a_3 b_1 + a_3 b_3 + a_3 b_4 + a_4 b_3) I, (a_1 b_4 + a_4 b_1 + a_4 b_4) F). \end{aligned} \tag{9}$$

Proposition 2 ([18,20])**.** *Let* $NQ = \{(a, bT, cI, dF) : a, b, c, d \in \mathbb{R} \text{ or } \mathbb{C}\}$. *We have:*

(1) $(NQ, *)$ *is a commutative monoid.*
(2) $(NQ, \star)$ *is a commutative monoid.*

Theorem 1. *[17] For the algebra system* $(NQ, *)$*(or* $(NQ, \star)$*), for every element* $a \in NQ$*, there exists the neutral element* $neut(a)$ *and opposite element* $anti(a)$*, which means that the algebra system* $(NQ, *)$ *(or* $(NQ, \star)$*) is a NETG.*

3. The Algebra Structure of Neutrosophic Quadruple Numbers on General Field

From the above section, we can see that the neutrosophic quadruple numbers are defined on number field $\mathbb{R}$ or $\mathbb{C}$. In this section, the notions of the neutrosophic quadruple numbers on a general field are introduced and the algebra structure of the neutrosophic quadruple numbers on general field is explored.

Let $(\mathbb{F}, +, \cdot)$ be a field, and 0 and 1 are the unit elements for operator $+$ and $\cdot$, respectively. For every $a \in \mathbb{F}$, $-a$ is the inverse element of a for operator $+$, and a^{-1} is the inverse element of a for operator $\cdot$. In the following, field $(\mathbb{F}, +, \cdot)$ is denoted by $\mathbb{F}$ for short and $a \cdot b$ is denoted by ab.

Definition 8. *Let* $\mathbb{F}$ *be a field; a neutrosophic quadruple number is a number of the form* (a, bT, cI, dF)*, where* T, I, F *have their usual neutrosophic logic meanings, i.e., truth, indeterminacy and false, respectively, and* $a, b, c, d \in \mathbb{F}$*. The set NQF defined by*

$$NQF = \{(a, bT, cI, dF) | a, b, c, d \in \mathbb{F}\}. \tag{10}$$

is called a neutrosophic set of quadruple numbers on field $\mathbb{F}$*.*

Definition 9. *Let* $a = (a_1, a_2T, a_3I, a_4F), b = (b_1, b_2T, b_3I, b_4F) \in NQF$*, then the addition operator is defined as follows:*

$$a \oplus b = (a_1 + b_1, (a_2 + b_2)T, (a_3 + b_3)I, (a_4 + b_4)F). \tag{11}$$

Definition 10. *Let* $a = (a_1, a_2T, a_3I, a_4F), b = (b_1, b_2T, b_3I, b_4F) \in NQF$*. Suppose, in an pessimistic way, the neutrosophic expert considers the prevalence order* $T \prec I \prec F$*. Then, the multiplication operation is defined as follows:*

$$\begin{aligned}
a * b = \ & (a_1, a_2T, a_3I, a_4F) * (b_1, b_2T, b_3I, b_4F) \\
= \ & (a_1b_1, (a_1b_2 + a_2b_1 + a_2b_2)T, (a_1b_3 + a_2b_3 + a_3b_1 + a_3b_2 + a_3b_3)I, \\
& (a_1b_4 + a_2b_4 + a_3b_4 + a_4b_1 + a_4b_2 + a_4b_3 + a_4b_4)F).
\end{aligned} \tag{12}$$

Suppose in an optimistic way the neutrosophic expert considers the prevalence order $T \succ I \succ F$*. Then,*

$$\begin{aligned}
a \star b = \ & (a_1, a_2T, a_3I, a_4F) \star (b_1, b_2T, b_3I, b_4F) \\
= \ & (a_1b_1, (a_1b_2 + a_2b_1 + a_2b_2 + a_3b_2 + a_4b_2 + a_2b_3 + a_2b_4)T, \\
& (a_1b_3 + a_3b_1 + a_3b_3 + a_3b_4 + a_4b_3)I, (a_1b_4 + a_4b_1 + a_4b_4)F).
\end{aligned} \tag{13}$$

Theorem 2. $(NQF, \oplus)$ *is an abelian group.*

Proof. It is obvious. $\square$

Theorem 3. *For the algebra system* $(NQF, *)$ *(or* $(NQF, \star)$*), for every element* $a \in NQF$*, there exists the neutral element* $neut(a)$ *and opposite element* $anti(a)$*, thus the algebra system* $(NQF, *)$ *(or* $(NQF, \star)$*) is a NETG.*

The proof's method is similar to the proof when $\mathbb{F} = \mathbb{R}$ in [17]. The detailed proof is omitted. For algebra system $(NQF, *)$, Table 1 gives all the subsets which have the same neutral element, and the corresponding neutral element and opposite elements. In the following, from two examples, we show that how to solve the the neutral element and opposite elements of each element for algebra system $(NQF, *)$ on different fields.

Example 1. *Let* $\mathbb{F} = Z_5 = \{[0], [1], [2], [3], [4]\}$, *then* $(Z_5, +, \cdot)$ *is a field, where* $+$ *and* $\cdot$ *are the classical mod addition and multiplication, respectively. For algebra system* $(NQF, *)$, *if* $a = (a_1, a_2T, a_3I, a_4F) = ([2], [4]T, [3]I, [1]F)$, *i.e.,* $a_1 \neq [0], a_1 + a_2 \neq [0], a_1 + a_2 + a_3 \neq [0], a_1 + a_2 + a_3 + a_4 = [0]$, *then, from Table 1, we can get* $neut(a) = ([1], [0], [0], [4]F)$. *Let* $anti(a) = (c_1, c_2T, c_3I, c_4F)$, *so* $c_1 = a_1^{-1} = [3], c_2 = [3], c_3 = [3], c_4 \in Z_5$, *thus* $anti(a) = ([3], [3]T, [3]I, c_4F)$, *where* $c_4 \in Z_5$. *Thus, we can easily get the neutral element and opposite elements of each neutrosophic quadruple number on general field. For more examples, see the following:*

1. *Let* $b = ([1], [2]T, [1]I, [3]F)$, *then* $neut(b) = ([1], [0], [0], [0])$ *and* $anti(b) = ([1], [1]T, [2]I, [4]F)$.
2. *Let* $c = ([0], [0], [1]I, [4]F)$, *then* $neut(c) = ([0], [0], [1], [4])$ *and* $anti(c) = (c_1, c_2T, c_3I, c_4F)$, *where* $c_1 \oplus c_2 \oplus c_3 = [1], c_4 \in Z_5$.
3. *Let* $d = ([0], [1]T, [1]I, [1]F)$, *then* $neut(d) = ([0], [1]T, [0], [0])$ *and* $anti(d) = (c_1, c_2T, [2]I, [4]F)$, *where* $c_1 \oplus c_2 = [1]$.

Table 1. The corresponding neutral element and opposite elements for $(NQF, *)$.

The Subset of NQF	Neutral Element	Opposite Elements (c_1, c_2T, c_3I, c_4F)
$\{(0,0,0,0)\}$	$(0,0,0,0)$	$c_i \in \mathbb{F}$
$\{(0,0,0,a_4F)\|a_4 \neq 0\}$	$(0,0,0,F)$	$c_1 + c_2 + c_3 + c_4 = a_4^{-1}$
$\{(0,0,a_3I, -a_3F)\|a_3 \neq 0\}$	$(0,0,I,-F)$	$c_1 + c_2 + c_3 = a_3^{-1}, c_4 \in \mathbb{F}$
$\{(0,0,a_3I, a_4F)\|a_3 \neq 0, a_3 + a_4 \neq 0\}$	$(0,0,I,0)$	$c_1 + c_2 + c_3 = a_3^{-1}, c_4 = -(a_4 a_3^{-1}(a_3 + a_4)^{-1})$
$\{(0, a_2T, -a_2I, 0)\|a_2 \neq 0\}$	$(0,T,-I,0)\}$	$c_1 + c_2 = a_2^{-1}, c_3, c_4 \in \mathbb{F}$
$\{(0, a_2T, -a_2I, a_4F)\|a_2 \neq 0, a_4 \neq 0\}$	$(0,T,-I,F)$	$c_1 + c_2 = a_2^{-1}, c_3 + c_4 = a_4^{-1} + (-a_2^{-1})$
$\{(0, a_2T, a_3I, a_4F)\|a_2 \neq 0, a_2 + a_3 \neq 0, a_2 + a_3 + a_4 = 0\}$	$(0,T,0,-F)$	$c_1 + c_2 = a_2^{-1}, c_3 = -(a_3 a_2^{-1}(a_2 + a_3)^{-1}), c_4 \in \mathbb{F}$
$\{(0, a_2T, a_3I, a_4F)\|a_2 \neq 0, a_2 + a_3 \neq 0, a_2 + a_3 + a_4 \neq 0\}$	$(0,T,0,0)$	$c_1 + c_2 = a_2^{-1}, c_3 = -(a_3 a_2^{-1}(a_2 + a_3)^{-1}), c_4 = -(a_4(a_2 + a_3)^{-1}(a_2 + a_3 + a_4)^{-1})$
$\{(a_1, -a_1T, 0, 0)\|a_1 \neq 0\}$	$(1,-T,0,0)\}$	$c_1 = a_1^{-1}, c_2, c_3, c_4 \in \mathbb{F}$
$\{(a_1, -a_1T, 0, a_4F)\|a_1 \neq 0, a_4 \neq 0\}$	$(1,-T,0,F)$	$c_1 = a_1^{-1}, c_2 + c_3 + c_4 = a_4^{-1} + (-a_1^{-1})$
$\{(a_1, -a_1T, a_3I, -a_3F)\|a_1 \neq 0, a_3 \neq 0\}$	$(1,-T,I,-F)$	$c_1 = a_1^{-1}, c_2 + c_3 = a_3^{-1} - a_1^{-1}, c_4 \in \mathbb{F}$
$\{(a_1, -a_1T, a_3I, a_4F)\|a_1 \neq 0, a_3 \neq 0, a_3 + a_4 \neq 0\}$	$(1,-T,I,0)$	$c_1 = a_1^{-1}, c_2 + c_3 = a_3^{-1} + (-a_1^{-1}), c_4 = -(a_4 a_3^{-1}(a_3 + a_4)^{-1})$
$\{(a_1, a_2T, a_3I, 0)\|a_1 \neq 0, a_1 + a_2 \neq 0, a_1 + a_2 + a_3 = 0\}$	$(1,0,-I,0)$	$c_1 = a_1^{-1}, c_2 = -(a_2 a_1^{-1}(a_1 + a_2)^{-1}), c_3, c_4 \in \mathbb{F}$
$\{(a_1, a_2T, a_3I, a_4F)\|a_1 \neq 0, a_1 + a_2 \neq 0, a_1 + a_2 + a_3 = 0, a_4 \neq 0\}$	$(1,0,-I,F)$	$c_1 = a_1^{-1}, c_2 = -(a_2 a_1^{-1}(a_1 + a_2)^{-1}), c_3 + c_4 = a_4^{-1} + (-(a_1 + a_2)^{-1})$
$\{(a_1, a_2T, a_3I, a_4F)\|a_1 \neq 0, a_1 + a_2 \neq 0, a_1 + a_2 + a_3 \neq 0, a_1 + a_2 + a_3 + a_4 = 0\}$	$(1,0,0,-F)$	$c_1 = a_1^{-1}, c_2 = -(a_2 a_1^{-1}(a_1 + a_2)^{-1}), c_3 = -(a_3(a_1 + a_2)^{-1}(a_1 + a_2 + a_3)^{-1}), c_4 \in \mathbb{F}$
$\{(a_1, a_2T, a_3I, a_4F)\|a_1 \neq 0, a_1 + a_2 \neq 0, a_1 + a_2 + a_3 \neq 0, a_1 + a_2 + a_3 + a_4 \neq 0\}$	$(1,0,0,0)$	$c_1 = a_1^{-1}, c_2 = -(a_2 a_1^{-1}(a_1 + a_2)^{-1}), c_3 = -(a_3(a_1 + a_2)^{-1}(a_1 + a_2 + a_3)^{-1}), c_4 = -(a_4(a_1 + a_2 + a_3)^{-1}(a_1 + a_2 + a_3 + a_4)^{-1})$

Example 2. *Let* $\mathbb{F}_4 = \{0, 1, x, y\}$, *the operators* $+$ *and* $\cdot$ *on* $\mathbb{F}_4$ *is defined by Table 2.*

Table 2. The operators $+$ and $\cdot$ on $\mathbb{F}_4$.

$+$	0	1	x	y	$\cdot$	0	1	x	y
0	0	1	x	y	0	0	0	0	0
1	1	0	y	x	1	0	1	x	y
x	x	y	0	1	x	0	x	y	1
y	y	x	1	0	y	0	y	1	x

Then, $(\mathbb{F}_4, +, \cdot)$ *is a field. Set* $NQF = \{(a, bT, cI, dF)|a, b, c, d \in \mathbb{F}_4\}$. *We have:*

1. *Let* $a = (0, 0, xI, xF)$, *then* $neut(a) = (0, 0, I, F)$ *and* $anti(a) = (c_1, c_2 T, c_3 I, c_4 F)$, *where* $c_1 + c_2 + c_3 = y, c_4 \in \mathbb{F}_4$.
2. *Let* $b = (0, xT, xI, yF)$, *then* $neut(b) = (0, T, I, F)$ *and* $anti(b) = (c_1, c_2 T, c_3 I, c_4 F)$, *where* $c_1 + c_2 = y, c_3 + c_4 = 1$.
3. *Let* $c = (x, xT, 0, 0)$, *then* $neut(c) = (1, T, 0, 0)$ *and* $anti(c) = (y, c_2 T, c_3 I, c_4 F)$, *where* $c_2, c_3, c_4 \in \mathbb{F}_4$.

In the same way, for algebra system $(NQF, \star)$, Table 3 gives all the subsets which have the same neutral element, and the corresponding neutral element and opposite elements.

Table 3. The corresponding neutral element and opposite elements for $(NQF, \star)$.

The Subset of NQF	Neutral Element	Opposite Elements $(c_1, c_2 T, c_3 I, c_4 F)$
$\{(0,0,0,0)\}$	$(0,0,0,0)$	$c_i \in \mathbb{F}$
$\{(0, a_2 T, 0, 0)|a_2 \neq 0\}$	$(0, T, 0, 0)$	$c_1 + c_2 + c_3 + c_4 = a_2^{-1}$
$\{(0, -a_3 T, a_3 I, 0)|a_3 \neq 0\}$	$(0, -T, I, 0)$	$c_1 + c_3 + c_4 = a_3^{-1}, c_2 \in \mathbb{F}$
$\{(0, a_2 T, a_3 I, a_4 F)|a_3 \neq 0, a_2 + a_3 \neq 0\}$	$(0, 0, I, 0)$	$c_1 + c_3 + c_4 = a_3^{-1}$, $c_2 = -(a_2 a_3^{-1}(a_2 + a_3)^{-1})$
$\{(0, 0, -a_4 I, a_4 F)|a_4 \neq 0\}$	$(0, 0, -I, F)\}$	$c_1 + c_4 = a_4^{-1}, c_2, c_3 \in \mathbb{F}$
$\{(0, a_2 T, -a_4 I, a_4 F)|a_2 \neq 0, a_4 \neq 0\}$	$(0, T, -I, F)$	$c_1 + c_4 = a_4^{-1}, c_2 + c_3 = a_2^{-1} + (-a_4^{-1})$
$\{(0, a_2 T, a_3 I, a_4 F)|a_4 \neq 0, a_3 + a_4 \neq 0, a_2 + a_3 + a_4 = 0\}$	$(0, -T, 0, F)$	$c_1 + c_4 = a_4^{-1}, c_3 = -(a_3 a_4^{-1}(a_3 + a_4)^{-1})$, $c_2 \in \mathbb{F}$
$\{(0, a_2 T, a_3 I, a_4 F)|a_4 \neq 0, a_3 + a_4 \neq 0, a_2 + a_3 + a_4 \neq 0\}$	$(0, 0, 0, F)$	$c_1 + c_4 = a_4^{-1}, c_3 = -(a_3 a_4^{-1}(a_3 + a_4)^{-1})$, $c_2 = -(a_2(a_3 + a_4)^{-1}(a_2 + a_3 + a_4)^{-1})$
$\{(a_1, 0, 0, -a_1 F)|a_1 \neq 0\}$	$(1, 0, 0, -F)\}$	$c_1 = a_1^{-1}, c_2, c_3, c_4 \in \mathbb{F}$
$\{(a_1, a_2 T, 0, -a_1 F)|a_1 \neq 0, a_2 \neq 0\}$	$(1, T, 0, -F)$	$c_1 = a_1^{-1}, c_2 + c_3 + c_4 = a_2^{-1} + (-a_1^{-1})$
$\{(a_1, -a_3 T, a_3 I, -a_1 F)|a_1 \neq 0, a_3 \neq 0\}$	$(1, -T, I, -F)$	$c_1 = a_1^{-1}, c_3 + c_4 = a_3^{-1} + (-a_1^{-1}), c_4 \in \mathbb{F}$
$\{(a_1, a_2 T, a_3 I, -a_1 F)|a_1 \neq 0, a_3 \neq 0, a_2 + a_3 \neq 0\}$	$(1, 0, I, -F)$	$c_1 = a_1^{-1}, c_3 + c_4 = a_3^{-1} + (-a_1^{-1})$, $c_2 = -(a_2 a_3^{-1}(a_2 + a_3)^{-1})$
$\{(a_1, 0, a_3 I, a_4 F)|a_1 \neq 0, a_1 + a_4 \neq 0, a_1 + a_3 + a_4 = 0\}$	$(1, 0, -I, 0)$	$c_1 = a_1^{-1}, c_4 = -(a_4 a_1^{-1}(a_1 + a_4)^{-1})$, $c_2, c_3 \in \mathbb{F}$
$\{(a_1, a_2 T, a_3 I, a_4 F)|a_1 \neq 0, a_1 + a_4 \neq 0, a_1 + a_3 + a_4 = 0, a_2 \neq 0\}$	$(1, T, -I, 0)$	$c_1 = a_1^{-1}, c_4 = -(a_4 a_1^{-1}(a_1 + a_4)^{-1})$, $c_2 + c_3 = a_2^{-1} + (-(a_1 + a_4)^{-1})$
$\{(a_1, a_2 T, a_3 I, a_4 F)|a_1 \neq 0, a_1 + a_4 \neq 0, a_1 + a_3 + a_4 \neq 0, a_1 + a_2 + a_3 + a_4 = 0\}$	$(1, -T, 0, 0)$	$c_1 = a_1^{-1}, c_4 = -(a_4 a_1^{-1}(a_1 + a_4)^{-1})$, $c_3 = -(a_3(a_1 + a_4)^{-1}(a_1 + a_3 + a_4)^{-1})$, $c_2 \in \mathbb{F}$
$\{(a_1, a_2 T, a_3 I, a_4 F)|a_1 \neq 0, a_1 + a_4 \neq 0, a_1 + a_3 + a_4 \neq 0, a_1 + a_2 + a_3 + a_4 \neq 0\}$	$(1, 0, 0, 0)$	$c_1 = a_1^{-1}, c_4 = -(a_4 a_1^{-1}(a_1 + a_4)^{-1})$, $c_3 = -(a_3(a_1 + a_4)^{-1}(a_1 + a_3 + a_4)^{-1})$, $c_2 = -(a_2(a_1 + a_3 + a_4)^{-1}(a_1 + a_2 + a_3 + a_4)^{-1})$

4. The Algebra Structure of Refined Neutrosophic Numbers on General Field

In the above section, we reveal that the algebra structure of the neutrosophic quadruple numbers on general field. In this section, we explore the the algebra structure of the refined neutrosophic numbers (refined neutrosophic quadruple numbers) on general field.

Definition 11. *Let $\mathbb{F}$ be a field; a refined n-ary neutrosophic number is a number of the form $(a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n)$, where $I_1, I_2, \cdots,$ and I_n are different types of indeterminacies, and $a_0, a_1, a_2, \cdots, a_n \in \mathbb{F}$. The set RNF_n defined by*

$$RNF_n = \{(a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n) | a_0, a_1, a_2, \cdots, a_n \in \mathbb{F}\}. \tag{14}$$

is called a refined neutrosophic set on field $\mathbb{F}$.

Definition 12. *Let $x = (a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n), y = (b_0, b_1 I_1, b_2 I_2, \cdots, b_n I_n) \in RNF_n$, then the addition operator on RNF_n is defined as follows:*

$$x \oplus y = (a_0 + b_0, (a_1 + b_1) I_1, (a_2 + b_2) I_2, \cdots, (a_n + b_n) I_n). \tag{15}$$

Definition 13. *Let $x = (a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n), y = (b_0, b_1 I_1, b_2 I_2, \cdots, b_n I_n) \in RNF_n$, the neutrosophic expert considers the prevalence order $I_1 \prec I_2 \prec \cdots \prec I_n$. Then, the multiplication operation is defined as follows:*

$$
\begin{aligned}
x * y = {}& (a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n) * (b_0, b_1 I_1, b_2 I_2, \cdots, b_n I_n) \\
= {}& (a_0 b_0, (a_0 b_1 + a_1 b_1 + a_1 b_2) I_1, (a_0 b_2 + a_1 b_2 + a_2 b_0 + a_2 b_1 + a_2 b_2) I_2, \\
& \cdots, (a_0 b_n + a_1 b_n + a_2 b_n + \cdots + a_{n-1} b_n + a_n b_0 + a_n b_1 + \cdots + a_n b_n) I_n).
\end{aligned} \tag{16}
$$

The neutrosophic expert considers the prevalence order $I_1 \succ I_2 \succ \cdots \succ I_n$. Then,

$$
\begin{aligned}
x \star y = {}& (a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n) \star (b_0, b_1 I_1, b_2 I_2, \cdots, b_n I_n) \\
= {}& (a_0 b_0, (a_0 b_1 + a_1 b_1 + \cdots + a_n b_1 + a_1 b_0 + a_1 b_2 + a_1 b_3 + \cdots + a_1 b_n) I_1, \cdots, \\
& (a_0 b_{n-1} + a_{n-1} b_0 + a_{n-1} b_{n-1} + a_{n-1} b_n + a_n b_{n-1}) I_{n-1}, (a_0 b_n + a_n b_0 + a_n b_n) I_n).
\end{aligned} \tag{17}
$$

Theorem 4. $(RNF_n, \oplus)$ *is an abelian group.*

Proof. The proof is obvious. □

Theorem 5. *For the algebra system $(RNF_n, *)$ (or $(RNF_n, \star)$), for every element $a \in RNF_n$, there exists the neutral element $neut(a)$ and opposite element $anti(a)$, thus the algebra system $(RNF_n, *)$ (or $(RNF_n, \star)$) is a NETG.*

Proof. We use applied mathematical induction for n and only discuss the algebra system $(RNF_n, *)$. The algebra system $(RNF_n, \star)$ has a similar proof.

If $n = 2$, for refined 2-ary neutrosophic set, which is same as neutrosophic binary numbers set in [17], from Theorem 7 in [17], we can see that for every element $a \in RNF_2$, there exists the neutral element $neut(a)$ and opposite element $anti(a)$, thus the algebra system $(RNF_2, *)$ is a NETG.

Assume that the refined n-ary neutrosophic set RNF_n is a NETG. That is, for every element $a \in RNF_n$, there exists the neutral element $neut(a)$ and opposite element $anti(a)$. In the following, we prove that for the refined $(n + 1)$-ary neutrosophic set, which is a NETG.

For each $a = (a_0, a_1 I_1, a_2 I_2, \cdots, a_{n+1} I_{n+1}) \in RNF_{n+1}$, let $a' = (a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n)$, being $a' \in RNF_n$, then, from the above assumption condition, $neut(a')$ and $anti(a')$ exist and let $neut(a') = (b_0, b_1 I_1, b_2 I_2, \cdots, b_n I_n)$, $anti(a') = (c_0, c_1 I_1, c_2 I_2, \cdots, c_n I_n)$. We prove for $a = (a_0, a_1 I_1, a_2 I_2, \cdots, a_{n+1} I_{n+1})$, $neut(a)$ and $anti(a)$ exist. We discuss from the different cases of a_{n+1}.

Case A: If $a_{n+1} = 0$, Being $(a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n, 0) * (b_0, b_1 I_1, b_2 I_2, \cdots, b_n I_n, b_{n+1} I_{n+1}) = (a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n, 0)$, that is $(a_0 + a_1 + \cdots + a_n) b_{n+1} = 0$, thus we discuss from $a_0 + a_1 + \cdots + a_n = 0$ or $a_0 + a_1 + \cdots + a_n \neq 0$.

Case A1: If $a_{n+1} = 0, a_0 + a_1 + \cdots + a_n = 0$, so $(a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n, 0) * (c_0, c_1 I_1, c_2 I_2, \cdots, c_n I_n, c_{n+1} I_{n+1}) = (b_0, b_1 I_1, b_2 I_2, \cdots, b_n I_n, b_{n+1} I_{n+1})$, that is $b_{n+1} = 0$ and c_{n+1} can be chosen arbitrarily in $\mathbb{F}$.

Case A2: If $a_{n+1} = 0, a_0 + a_1 + \cdots + a_n \neq 0$, so from $(a_0 + a_1 + \cdots + a_n) b_{n+1} = 0$, we have $b_{n+1} = 0$, from $(a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n, 0) * (c_0, c_1 I_1, c_2 I_2, \cdots, c_n I_n, c_{n+1} I_{n+1}) = (b_0, b_1 I_1, b_2 I_2, \cdots, b_n I_n, 0)$, thus $c_{n+1} = 0$.

Case B: If $a_{n+1} \neq 0$, being $(a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n, a_{n+1} I_{n+1}) * (b_0, b_1 I_1, b_2 I_2, \cdots, b_n I_n, b_{n+1} I_{n+1}) = (a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n, a_{n+1} I_{n+1})$, that is $(a_0 + a_1 + \cdots + a_{n+1}) b_{n+1} + a_{n+1}(b_0 + b_1 + \cdots + b_n) = a_{n+1}$, that is $(a_0 + a_1 + \cdots + a_{n+1}) b_{n+1} = a_{n+1}(1 - b_0 - b_1 - \cdots - b_n)$, we discuss from $a_0 + a_1 + \cdots + a_{n+1} = 0$ or $a_0 + a_1 + \cdots + a_{n+1} \neq 0$.

Case B1: If $a_{n+1} \neq 0, a_0 + a_1 + \cdots + a_{n+1} = 0$, we have $b_0 + b_1 + \cdots + b_n = 1$, so from $(a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n, a_{n+1} I_{n+1}) * (c_0, c_1 I_1, c_2 I_2, \cdots, c_n I_n, c_{n+1} I_{n+1}) = (b_0, b_1 I_1, b_2 I_2, \cdots, b_n I_n, b_{n+1} I_{n+1})$, that is $b_{n+1} = (c_0 + c_1 + \cdots + c_n) a_{n+1} = b_{n+1}$ and c_{n+1} can be chosen arbitrarily in $\mathbb{F}$.

Case B2: If $a_{n+1} \neq 0, a_0 + a_1 + \cdots + a_{n+1} \neq 0$, we have $b_{n+1} = a_{n+1}(1 - b_0 - b_1 - \cdots - b_n)(a_0 + a_1 + a_2 \cdots a_n)^{-1}$, so from $(a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n, a_{n+1} I_{n+1}) * (c_0, c_1 I_1, c_2 I_2, \cdots, c_n I_n, c_{n+1} I_{n+1}) = (b_0, b_1 I_1, b_2 I_2, \cdots, b_n I_n, b_{n+1} I_{n+1})$, that is and $c_{n+1} = (b_{n+1} - a_{n+1}(c_0 + c_1 + \cdots + c_n))(a_0 + a_1 + \cdots + a_{n+1})^{-1}$.

From the above analysis, we can see that, for each $a \in RNF_n$, which has the neutral element $neut(a)$ and opposite element $anti(a)$, from the mathematical induction method, we can obtain that the algebra system $(RNF, *)$ is a NETG. $\square$

For algebra system $(RNF_{n+1}, *)$, if $a = (a_0, a_1 I_1, a_2 I_2, \cdots, a_{n+1} I_{n+1})$, let $a' = (a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n) \in RNF_n$, if we have $neut(a') = (b_0, b_1 I_1, b_2 I_2, \cdots, b_n I_n)$, $anti(a') = (c_0, c_1 I_1, c_2 I_2, \cdots, c_n I_n)$. Then, the corresponding neutral element and opposite elements of a are given in Table 4 according to the different cases of a_{n+1}.

Table 4. The corresponding neutral element and opposite elements for $(RNF_{n+1}, *)$

The Subset $\{(a_0, a_1 I_1, \cdots, a_{n+1} I_{n+1})\}$	Neutral Element $(b_0, b_1 I_1, \cdots, b_n I_n, b_{n+1} I_{n+1})$	Opposite Elements $(c_0, c_1 I_1, \cdots, c_n I_n, c_{n+1} I_{n+1})$
$a_{n+1} = 0$, $a_0 + a_1 + \cdots + a_n = 0$	$b_{n+1} = 0$	$c_{n+1} \in \mathbb{F}$
$a_{n+1} = 0$, $a_0 + a_1 + \cdots + a_n \neq 0$	$b_{n+1} = 0$	$c_{n+1} = 0$
$a_{n+1} \neq 0$, $a_0 + a_1 + \cdots + a_n + a_{n+1} = 0$	$b_{n+1} = (c_0 + c_1 + \cdots + c_n) a_{n+1}$	$c_{n+1} \in \mathbb{F}$
$a_{n+1} \neq 0$, $a_0 + a_1 + \cdots + a_n + a_{n+1} \neq 0$	$b_{n+1} = a_{n+1}(1 - b_0 - b_1 - \cdots - b_n)(a_0 + a_1 + a_2 + \cdots + a_n)^{-1}$	$c_{n+1} = (b_{n+1} - a_{n+1}(c_0 + c_1 + \cdots + c_n))(a_0 + a_1 + \cdots + a_{n+1})^{-1}$

For algebra system $(RNF_n, *)$, according to the results in Tables 3 and 4 in [17], we can easily obtain the neutral element and opposite elements when $n = 1, 2$ on general fields. In Table 1, we can get the neutral element and opposite elements when $n = 3$ on general field. Thus, from Theorem 5, we can get the neutral element and opposite elements of each element in RNF_n step-by-step. The solving method is given by Algorithm 1 and the following example is used to explain the algorithm.

Algorithm 1 Solving the neutral element and opposite elements of each element in $(RNF_n, *)$.

Input: $a = (a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n), n \geq 3$

1: $i = 3$;

2: For $i = 3 : n$

3:     $A_i = (a_0, a_1 I_1, a_2 I_2, \cdots, a_i I_i)$;

4:     If $i == 3$

5:         Obtain $neut(A_i), anti(A_i)$ by Table 1;

6:     else

7:         Obtain $neut(A_i), anti(A_i)$ by Table 4 combining the values of $neut(A_{i-1})$ and $anti(A_{i-1})$;

8:     end

9:     Save $neut(A_i), anti(A_i)$;

10: end

Output: $neut(a) = neut(A_n), anti(a) = anti(A_n)$

Example 3. *For algebra system* $(RNF_2, *)$, *and set* $\mathbb{F} = \mathbb{R}$. *If* $a = (a_0, a_1 I_1) = (0, -I_1)$, *from Table 3 in [17], we can get* $neut(a) = (0, I_1)$ *and* $anti(a) = (c_0, c_1 I_1)$, *where* $c_0 + c_1 = -1$.

In the following, we use two methods to solve the the neutral element and opposite elements of $b = (a_0, a_1 I_1, a - 2I_2) = (0, -I_1, I_2) \in RNF_3$ *and we get the same results.*

1. *Algorithm 1: From Table 2, being* $a_3 \neq 0$ *and* $a_0 + a_1 + a_2 = 0$, *thus* $b_2 = (c_0 + c_1) \cdot 1 = -1$, *that is* $neut(b) = (0, I_1, -I_2)$ *and* $anti(a) = (c_0, c_1 I_1, c_2 I_2)$, *where* $c_0 + c_1 = -1$ *and* c_2 *can be chosen arbitrarily in* $\mathbb{R}$.

2. *Rsults from Table 4 in [17]: Being* $a_2 \neq 0$ *and* $a_0 + a_1 + a_2 = 0$, *thus* $neut(b) = (0, I_1, -I_2)$ *and* $anti(a) = (c_0, c_1 I_1, c_2 I_2)$, *where* $c_0 + c_1 = -1$ *and* c_2 *can be chosen arbitrarily in* $\mathbb{R}$.

Example 4. *For algebra system* $(RNF_3, *)$, *and set* $\mathbb{F} = \mathbb{R}$. *If* $a = (a_0, a_1 I_1, a_2 I_2) = (1, -I_1, I_2) \in RNF_3$, *from Table 4 in [17], we can get* $neut(a) = (1, -I_1, I_2)$ *and* $anti(a) = (1, c_1 I_1, c_2 I_2)$, *where* $c_1 + c_2 = 0$.

In the same way, we use two methods to solve the the neutral element and opposite elements of $b = (a_0, a_1 I_1, a_2 I_2, a_3 I_3) = (1, -I_1, I_2, I_3) \in RNF_4$.

1. *Algorithm 1: From Table 2, being* $a_3 \neq 0$ *and* $a_0 + a_1 + a_2 + a_3 \neq 0$, *thus* $b_3 = 0$, *that is* $neut(b) = (0, -I_1, I_2, 0)$ *and* $c_3 = (0 - 1 \cdot 1 \cdot \frac{1}{2}) = -\frac{1}{2}$, *thus* $anti(a) = (1, c_1 I_1, c_2 I_2, -\frac{1}{2} I_3)$, *where* $c_1 + c_2 = 0$.

2. *Results from Table 1 in [17]: Being* $a_0 \neq 0, a_2 \neq 0$ *and* $a_2 + a_3 \neq 0$, *thus* $neut(b) = (1, -I_1, I_2, 0)$ *and* $c_0 = 1, c_1 + c_2 = 0, c_3 = -\frac{1}{1 \cdot (1+1)} = -\frac{1}{2}$, *thus* $anti(a) = (1, c_1 I_1, c_2 I_2, -\frac{1}{2} I_3)$, *where* $c_1 + c_2 = 0$.

For algebra system $(RNF_{n+1}, \star)$, set $a = (a_0, a_1 I_1, a_2 I_2, \cdots, a_{n+1} I_{n+1})$, being the order $I_1 \succ I_2 \succ \cdots \succ I_n$, thus we should obtain the neutral element and opposite elements of $a' = (a_0, a_2 I_2, \cdots, a_{n+1} I_{n+1})$. Knowing that $neut(a') = (b_0, b_2 I_2, b_3 I_3, \cdots, b_n I_n)$, $anti(a') = (c_0, c_2 I_2, c_3 I_3, \cdots, c_n I_n)$, then the corresponding neutral element and opposite elements of a are given in Table 5 according to the different cases of a_1.

Table 5. The corresponding neutral element and opposite elements for $(RNF, \star)$.

The Subset $\{(a_0, a_1 I_1, \cdots, a_{n+1} I_{n+1})\}$	Neutral Element $(b_0, b_1 I_1, \cdots, b_n I_n, b_{n+1} I_{n+1})$	Opposite Elements $(c_0, c_1 I_1, \cdots, c_n I_n, c_{n+1} I_{n+1})$
$a_1 = 0$, $a_0 + a_2 + \cdots + a_{n+1} = 0$	$b_1 = 0$	$c_1 \in \mathbb{F}$
$a_1 = 0$, $a_0 + a_2 + \cdots + a_{n+1} \neq 0$	$b_1 = 0$	$c_1 = 0$
$a_1 \neq 0$, $a_0 + a_1 + \cdots + a_n + a_{n+1} = 0$	$b_1 = (c_0 + c_2 + \cdots + c_{n+1})a_1$	$c_1 \in \mathbb{F}$
$a_1 \neq 0$, $a_0 + a_1 + \cdots + a_n + a_{n+1} \neq 0$	$b_1 = a_1(1 - (b_0 + b_2 + \cdots + b_{n+1}))(a_0 + a_1 + \cdots + a_{n+1})^{-1}$	$c_1 = (b_1 - a_1(c_0 + c_2 + \cdots + c_{n+1}))(a_0 + a_1 + \cdots + a_{n+1})^{-1}$

Similarly, we also can get the neutral element and opposite elements of each element in $(RNF_n, \star)$ step-by-step. The solving method is given by Algorithm 2 and the following example is used to explain the algorithm.

Algorithm 2 Solving the neutral element and opposite elements of each element in $(RNF_n, \star)$.

Input: $a = (a_0, a_1 I_1, a_2 I_2, \cdots, a_n I_n), n \geq 3$
1: $i = n - 2$;
2: While $i \geq 1$
3:   $A_i = (a_0, a_i I_i, \cdots, a_{n-1} I_{n-1}, a_n I_n)$;
4:   If $i == n - 2$
5:     Obtain $neut(A_i), anti(A_i)$ by Table 3;
6:   else
7:     Obtain $neut(A_i), anti(A_i)$ by Table 5 combining the values of $neut(A_{i+1})$ and $anti(A_{i+1})$;
8:   end
9:   Save $neut(A_i), anti(A_i)$;
10:   $i = i - 1$;
11: end
Output: $neut(a) = neut(A_1), anti(a) = anti(A_1)$

Example 5. *For algebra system $(RNF_6, \star)$, and set $\mathbb{F} = \mathbb{R}$, $a = (0, 0, -2I_2, -I_3, I_4, 0)$, solve the neutral element and opposite elements of a.*

According Algorithm 2 for algebra system $(RNF_6, \star)$: Firstly, we solve the neutral element and opposite elements of $a' = (0, -I_3, I_4, 0)$ from Table 3, and then solve the neutral element and opposite elements of $a'' = (0, -2I_2, -I_3, I_4, 0)$ from Table 5, lastly, we solve the neutral element and opposite elements of a from Table 5.

1.   *From Table 3: $neut(a') = (0, -I_3, I_4, 0)$ and $anti(a') = (c_0, c_3 I_3, c_4 I_4, c_5 I_5)$, where $c_0 + c_4 + c_5 = 1$, $c_3 \in \mathbb{R}$.*
2.   *From Table 5 and combining the results of the above step: Being $-2 \neq 0$ and $0 + (-2) + (-1) + 1 + 0 \neq 0$, thus $neut(a'') = (0, I_2, -I_3, I_4, 0)$ and $anti(a'') = (c_0, c_2 I_2, c_3 I_3, c_4 I_4, c_5 I_5)$, where $c_0 + c_4 + c_5 = 1$, $c_2 + c_3 = -\frac{3}{2}$.*
3.   *From Table 5 and combining the results of the above step: Being $0 = 0$ and $0 + 0 + (-2) + (-1) + 1 + 0 \neq 0$, thus $neut(a) = (0, 0, I_2, -I_3, I_4, 0)$ and $anti(a) = (c_0, 0, c_2 I_2, c_3 I_3, c_4 I_4, c_5 I_5)$, where $c_0 + c_4 + c_5 = 1$, $c_2 + c_3 = -\frac{3}{2}$.*

Similarly, we explore the the algebra structure of the refined neutrosophic quadruple numbers on general field in the following.

Definition 14. *Let $\mathbb{F}$ be a field; a refined neutrosophic quadruple number is a number of the form $(a, b_1 T_1, b_2 T_2, \cdots, b_p T_p, c_1 I_1, c_2 I_2, \cdots, c_q I_q, d_1 F_1, d_2 F_2, \cdots, d_r F_r)$, where $a, b_1, b_2, \cdots, b_p, c_1, c_2, \cdots, c_q, d_1, d_2, \cdots, d_r \in \mathbb{F}$. The set $RNQF_{pqr}$ defined by*

$$RNQF_{pqr} = \{(a, b_1 T_1, b_2 T_2, \cdots, b_p T_p, c_1 I_1, c_2 I_2, \cdots, c_q I_q, d_1 F_1, d_2 F_2, \cdots, d_r F_r) \mid$$
$$a, b_1, b_2, \cdots, b_p, c_1, c_2, \cdots, c_q, d_1, d_2, \cdots, d_r \in \mathbb{F}\}. \quad (18)$$

is called a refined neutrosophic quadruple set on field $\mathbb{F}$.

Definition 15. *Let $x = (a, b_1 T_1, b_2 T_2, \cdots, b_p T_p, c_1 I_1, c_2 I_2, \cdots, c_q I_q, d_1 F_1, d_2 F_2, \cdots, d_r F_r)$, $y = (e, f_1 T_1, f_2 T_2, \cdots, f_p T_p, g_1 I_1, g_2 I_2, \cdots, g_q I_q, h_1 F_1, h_2 F_2, \cdots, h_r F_r) \in RNQF_{pqr}$, then the addition operator is defined as follows:*

$$x \oplus y = (a + e, (b_1 + f_1) T_1, (b_2 + f_2) T_2, \cdots, (b_p + f_p) T_p, (c_1 + g_1) I_1, (c_2 + g_2) I_2, \cdots,$$
$$(c_q + g_q) I_q, (d_1 + h_1) F_1, (d_2 + h_2) F_2, \cdots, (d_r + h_r) F_r). \quad (19)$$

Definition 16. *Let $x = (a, b_1 T_1, b_2 T_2, \cdots, b_p T_p, c_1 I_1, c_2 I_2, \cdots, c_q I_q, d_1 F_1, d_2 F_2, \cdots, d_r F_r)$, $y = (e, f_1 T_1, f_2 T_2, \cdots, f_p T_p, g_1 I_1, g_2 I_2, \cdots, g_q I_q, h_1 F_1, h_2 F_2, \cdots, h_r F_r) \in RNQF_{pqr}$; the neutrosophic expert considers the prevalence order $T_1 \prec T_2 \prec \cdots \prec T_p \prec I_1 \prec I_2 \prec \cdots \prec I_q \prec F_1 \prec F_2 \prec \cdots \prec F_r$. Then, the multiplication operation is defined as follows:*

$$x * y = (a, b_1 T_1, b_2 T_2, \cdots, b_p T_p, c_1 I_1, c_2 I_2, \cdots, c_q I_q, d_1 F_1, d_2 F_2, \cdots, d_r F_r)$$
$$* (e, f_1 T_1, f_2 T_2, \cdots, f_p T_p, g_1 I_1, g_2 I_2, \cdots, g_q I_q, h_1 F_1, h_2 F_2, \cdots, h_r F_r)$$
$$= (ae, (af_1 + b_1 e + b_1 f_1) T_1, (af_2 + b_1 f_2 + b_2 e + b_2 f_1 + b_2 f_2) T_2,$$
$$\cdots, (ah_r + b_1 h_r + b_2 h_r + \cdots + d_{r-1} h_r + d_r e + d_r f_1 + \cdots + d_r h_r) F_r). \quad (20)$$

The neutrosophic expert considers the prevalence order $T_1 \succ T_2 \succ \cdots \succ T_p \succ I_1 \succ I_2 \succ \cdots \succ I_q \succ F_1 \succ F_2 \succ \cdots \succ F_r$. Then,

$$x \star y = (a, b_1 T_1, b_2 T_2, \cdots, b_p T_p, c_1 I_1, c_2 I_2, \cdots, c_q I_q, d_1 F_1, d_2 F_2, \cdots, d_r F_r)$$
$$\star (e, f_1 T_1, f_2 T_2, \cdots, f_p T_p, g_1 I_1, g_2 I_2, \cdots, g_q I_q, h_1 F_1, h_2 F_2, \cdots, h_r F_r)$$
$$= (ae, (af_1 + b_1 f_1 + \cdots + d_r f_1 + b_1 e + b_1 f_2 + b_1 f_3 + \cdots + b_1 h_r) T_1,$$
$$\cdots, (ah_{r-1} + d_{r-1} e + d_{r-1} h_{r-1} + d_{r-1} h_r + d_r h_{r-1}) F_{r-1}, (ah_r + d_r e + d_r h_r) F_r). \quad (21)$$

Similarly, we also have the following results.

Theorem 6. *$(RNQF_{pqr}, \oplus)$ is an abelian group.*

Theorem 7. *For the algebra system $(RNQF_{pqr}, *)$ (or $(RNQF_{pqr}, \star)$), for every element $a \in RNQF_{pqr}$, there exists the neutral element $neut(a)$ and opposite element $anti(a)$, thus the algebra system $(RNQF_{pqr}, *)$ (or $(RNQF_{pqr}, \star)$) is a NETG.*

Example 6. *For algebra system $(RNQF_{213}, *)$, and set $\mathbb{F} = \mathbb{R}$, $a = (1, 0, 2T_2, -3I_1, 2F_1, 0, -2F_3)$, solve the neutral element and opposite elements of a.*

According Algorithm 1 for algebra system $(RNF_{213}, *)$, firstly, we solve the neutral element and opposite elements of $a' = (1, 0, 2T_2, -3I_1)$ from Table 1. We then solve the neutral element and opposite elements of $a'' = (1, 0, 2T_2, -3I_1, 2F_1)$ from Table 4. Next we solve the neutral element and opposite elements of $a''' = (1, 0, 2T_2, -3I_1, 2F_1, 0)$ from Table 4. Finally, we solve the neutral element and opposite elements of a from Table 4.

1. *From Table 1, $neut(a') = (1, 0, 0, -I_1)$ and $anti(a') = (1, 0, -\frac{2}{3}T_2, c_4 I_1)$, where $c_4 \in \mathbb{R}$.*
2. *From Table 4 and combining the results of the above step: Being $2 \neq 0$ and $1 + 0 + 2 + (-3) + 2 \neq 0$, thus $neut(a'') = (1, 0, 0, -I_1, F_1)$ and $anti(a'') = (1, 0, -\frac{2}{3}T_2, c_4 I_1, c_5 F_1)$, where $c_4 + c_5 = \frac{1}{6}$.*

3. *From Table 4 and combining the results of the above step: Being $0 = 0$ and $1 + 0 + 2 + (-3) + 2 + 0 \neq 0$, thus $neut(a''') = (1, 0, 0, -I_2, F_1, 0)$ and $anti(a''') = (1, 0, -\frac{2}{3}T_2, c_4 I_1, c_5 F_1, 0)$, where $c_4 + c_5 = \frac{1}{6}$.*

4. *From Table 4 and combining the results of the above step: Being $-2 \neq 0$ and $1 + 0 + 2 + (-3) + 2 + 0 + (-2) = 0$, thus $neut(a) = (1, 0, 0, -I_1, F_1, 0, -F_3)$ and $anti(a) = (1, 0, -\frac{2}{3}T_2, c_4 I_1, c_5 F_1, 0, c_7 F_3)$, where $c_4 + c_5 = \frac{1}{6}, c_7 \in \mathbb{R}$.*

Example 7. *For algebra system $(RNQF_{213}, *)$, and set $\mathbb{F} = Z_5 = \{[0], [1], [2], [3], [4]\}$, $a = ([1], [0], [2]T_2, [2]I_1, [2]F_1, 0, [3]F_3)$, solve the neutral element and opposite elements of a.*

Similar to Example 6, according Algorithm 1 for algebra system $(RNF_{213}, *)$, firstly, we solve the neutral element and opposite elements of $a' = ([1], [0], [2]T_2, [2]I_1)$ from Table 1. We then solve the neutral element and opposite elements of $a'' = ([1], [0], [2]T_2, [2]I_1, [2]F_1)$ from Table 4. Next, we solve the neutral element and opposite elements of $a''' = ([1], [0], [2]T_2, [2]I_1, [2]F_1, [0])$ from Table 4. Finally, we solve the neutral element and opposite elements of a from Table 4.

1. *From Table 1, $neut(a') = ([1], [0], [0], [4]I_1)$ and $anti(a') = ([1], [0], [1]T_2, c_4 I_1)$, where $c_4 \in \mathbb{F}$.*

2. *From Table 4 and combining the results of the above step: Being $[2] \neq [0]$ and $[1] + [0] + [2] + [2] + [2] \neq [0]$, thus $neut(a'') = ([1], [0], [0], [4]I_1, [1]F_1)$ and $anti(a'') = ([1], [0], [1]T_2, c_4 I_1, c_5 F_1)$, where $c_4 + c_5 = [1]$.*

3. *From Table 4 and combining the results of the above step: Being $[0] = [0]$ and $[1] + [0] + [2] + [2] + [2] + [0] \neq 0$, thus $neut(a''') = ([1], [0], [0], [4]I_1, [1]F_1, [0])$ and $anti(a''') = ([1], [0], [1]T_2, c_4 I_1, c_5 F_1, [0])$, where $c_4 + c_5 = [1]$.*

4. *From Table 4 and combining the results of the above step: Being $[3] \neq [0]$ and $[1] + [0] + [2] + [2] + [2] + [0] + [3] = 0$, thus $neut(a) = ([1], [0], [0], [4]I_1, [1]F_1, [0], [4]F_3)$ and $anti(a) = ([1], [0], [1]T_2, c_4 I_1, c_5 F_1, [0], c_7 F_3)$, where $c_4 + c_5 = [1], c_7 \in \mathbb{F}$.*

5. Conclusions

In this paper, we study the algebra structure of $(NQF(RNF_n, RNQF_n), \oplus)$, $(NQF(RNF_n, RNQF_n), *)$ and $(NQF(RNF_n, RNQF_n), \star)$, and we prove that $(NQF(RNF_n, RNQF_n), *)$ (or $(NQF(RNF_n, RNQF_n), \star)$) is a neutrosophic extended triplet group, and provide new examples of neutrosophic extended triplet group and the neutral element and opposite elements of each refined *n*-ary neutrosophic number (refined neutrosophic quadruple number) can be obtained by given algorithms. In the following, we can explore the algebra structure of $(NQF(RNF_n, RNQF_n), \oplus, *)$ or $(NQF(RNF_n, RNQF_n), \oplus, \star)$. We can also explore the relation of neutrosophic quadruple numbers and other algebra systems in papers [21–23]. Moreover, on the one hand, we will discuss the neutrosophic quadruple numbers based on some particular ring which can form a neutrosophic extended triplet group, while, on the other hand, we will introduce a new operation $\circ$ in order to guarantee $(NQF(RNF_n, RNQF_n), *, \circ)$ is a neutrosophic triplet ring.

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