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Study on the Development of Neutrosophic Triplet Ring and Neutrosophic Triplet Field

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Abstract: Rings and fields are significant algebraic structures in algebra and both of them are based on the group structure. In this paper, we attempt to extend the notion of a neutrosophic triplet group to a neutrosophic triplet ring and a neutrosophic triplet field. We introduce a neutrosophic triplet ring and study some of its basic properties. Further, we define the zero divisor, neutrosophic triplet subring, neutrosophic triplet ideal, nilpotent integral neutrosophic triplet domain, and neutrosophic triplet ring homomorphism. Finally, we introduce a neutrosophic triplet field.

Keywords: ring; field; neutrosophic triplet; neutrosophic triplet group; neutrosophic triplet ring; neutrosophic triplet field

1. Introduction

The concept of a ring first arose from attempts to prove Fermat's last theorem [1], starting with Richard Dedekind in the 1880s. After contributions from other fields, mainly number theory, the notion of a ring was generalized and firmly established during the 1920s by Emmy Noether and Wolfgang Krull [2]. Modern ring theory, a very active mathematical discipline, studies rings in their own right. To explore rings, mathematicians have devised various notions to break rings into smaller, more understandable pieces, such as ideals, quotient rings, and simple rings. In addition to these abstract properties, ring theorists also make various distinctions between the theories of commutative rings and noncommutative rings, the former belonging to algebraic number theory and algebraic geometry. A particularly rich theory has been developed for a certain special class of commutative rings, known as fields, which lies within the realm of field theory. Likewise, the corresponding theory for noncommutative rings, that of noncommutative division rings, constitutes an active research interest for noncommutative ring theorists. Since the discovery of a mysterious connection between noncommutative ring theory and geometry during the 1980s by Alain Connes [3–5], noncommutative geometry has become a particularly active discipline in ring theory.

The foundation of the subject (i.e., the mapping from subfields to subgroups and vice versa) is set up in the context of an absolutely general pair of fields. In addition to the clarification that normally accompanies such a generalization, there are useful applications to infinite algebraic extensions and to the Galois Theory of differential equations [6]. There is also a logical simplicity to the procedure: everything hinges on a pair of estimates of field degrees and subgroup indices. One might describe it as a further step in the Dedekind–Artin linearization [7].

An early contributor to the theory of noncommutative rings was the Scottish mathematician Wedderburn who, in 1905, proved “Wedderburn's Theorem”, namely that every finite division ring is

commutative and so is a field [8]. It was only around the 1930s that the theories of commutative and noncommutative rings came together and that their ideas began to influence each other.

Neutrosophy is a new branch of philosophy which studies the nature, origin, and scope of neutralities as well as their interaction with ideational spectra. The concept of neutrosophic logic and a neutrosophic set was first introduced by Florentin Smarandache [9] in 1995, where each proposition in neutrosophic logic is approximated to have the percentage of truth in a subset T , the percentage of indeterminacy in a subset I , and the percentage of falsity in a subset F such that this neutrosophic logic is called an extension of fuzzy logic, especially to intuitionistic fuzzy logic [10]. The generalization of classical sets [9], fuzzy sets [11], and intuitionistic fuzzy sets [10], etc., is in fact the neutrosophic set. This mathematical tool is used to handle problems consisting of uncertainty, imprecision, indeterminacy, inconsistency, incompleteness, and falsity. By utilizing the idea of neutrosophic theory, Vasantha Kandasamy and Florentin Smarandache studied neutrosophic algebraic structures [12–14] by inserting a literal indeterminate element “ I ”, where $I^2 = I$, in the algebraic structure and then combining “ I ” with each element of the structure with respect to the corresponding binary operation, denoted $*$. They call it the neutrosophic element, and the generated algebraic structure is then termed as a neutrosophic algebraic structure. Some other neutrosophic algebraic structures can be seen as neutrosophic fields [15], neutrosophic vector spaces [16], neutrosophic groups [17], neutrosophic bigroups [17], neutrosophic N-groups [15], neutrosophic semigroups [12], neutrosophic bisemigroups [12], neutrosophic N-semigroups [12], neutrosophic loops [12], neutrosophic biloops [12], neutrosophic N-loop [12], neutrosophic groupoids [12] and neutrosophic bigroupoids [12] and so on.

In this paper, we introduce the neutrosophic triplet ring. Further, we define the neutrosophic triplet zero divisor, neutrosophic triplet subring, neutrosophic triplet ideal, nilpotent neutrosophic triplet, integral neutrosophic triplet domain, and neutrosophic triplet ring homomorphism. Finally, we introduce a neutrosophic triplet field. The rest of the paper is organized as follows. After the literature review in Section 1 and basic concepts in Section 2, we introduce the neutrosophic triplet ring in Section 3. Section 4 is about the introduction of the integral neutrosophic triplet domain with some of its interesting properties, and is also where we develop the neutrosophic triplet ring homomorphism. In Section 5, we study neutrosophic triplet fields. Conclusions are given in Section 6.

2. Basic Concepts

In this section, all definitions and examples have been taken from [18] to provide some basic concepts about neutrosophic triplets and neutrosophic triplet groups.

Definition 1. Let N be a set together with a binary operation $*$. Then N is called a neutrosophic triplet set if for any $a \in N$, there exists a neutral of “ a ” called $neut(a)$, different from the classical algebraic unitary element, and an opposite of “ a ” called $anti(a)$, with $neut(a)$ and $anti(a)$ belonging to N , such that

$$a * neut(a) = neut(a) * a = a$$

and

$$a * anti(a) = anti(a) * a = neut(a).$$

The element a , $neut(a)$, and $anti(a)$ are collectively called a neutrosophic triplet and we denote it by $(a, neut(a), anti(a))$. By $neut(a)$, we mean the neutral of a , and a is just the first coordinate of a neutrosophic triplet and not a neutrosophic triplet [18].

For the same element “ a ” in N , there may be more than one neutral $neut(a)$ and more than one opposite $anti(a)$.

Definition 2. The element b in $(N, *)$ is the second component, denoted by $neut(\cdot)$, of a neutrosophic triplet, if there exist other elements a and c in N such that $a * b = b * a = a$ and $a * c = c * a = b$. The formed neutrosophic triplet is (a, b, c) [12].

Definition 3. The element c in $(N, *)$ is the third component, denoted by $anti(\cdot)$ of a neutrosophic triplet, if there exist other elements a and b in N such that $a * b = b * a = a$ and $a * c = c * a = b$. The formed neutrosophic triplet is (a, b, c) [12].

Example 1. Consider Z_6 under multiplication modulo 6, where

$$Z_6 = \{0, 1, 2, 3, 4, 5\}.$$

Then the element 2 gives rise to a neutrosophic triplet because $neut(2) = 4 \neq 1$, as $2 \times 4 = 4 \times 2 = 8 \equiv 2 \pmod{6}$. Also, $anti(2) = 2$ because $2 \times 2 = 4$. Thus $(2, 4, 2)$ is a neutrosophic triplet. Similarly 4 gives rise to a neutrosophic triplet because $neut(4) = anti(4) = 4$ So $(4, 4, 4)$ is a neutrosophic triplet. However, 3 does not give rise to a neutrosophic triplet as $neut(3) = 5$ but $anti(3)$ does not exist in Z_6 , and lastly, 0 gives rise to a trivial neutrosophic triplet as $neut(0) = anti(0) = 0$. The trivial neutrosophic triplet is denoted by $(0, 0, 0)$ [12].

Definition 4. Let $(N, *)$ be a neutrosophic triplet set. Then N is called a neutrosophic triplet group if the following conditions are satisfied [12].

1. If $(N, *)$ is well defined, i.e., for any $a, b \in N$, one has $a * b \in N$.
2. If $(N, *)$ is associative, i.e., $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$.

The neutrosophic triplet group, in general, is not a group in the classical algebraic sense. We consider the neutrosophic neutrals as replacing the classical unitary element, and the neutrosophic opposites as replacing the classical inverse elements.

Example 2. Consider $(Z_{10}, \#)$, where $\#$ is defined as $a\#b = 3ab \pmod{10}$. Then $(Z_{10}, \#)$ is a neutrosophic triplet group under the binary operation $\#$, as shown in Table 1 [18].

Table 1. Cayley table of neutrosophic triplet group $(Z_{10}, \#)$.

#	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	3	6	9	2	5	8	1	4	7
2	0	6	2	8	4	0	6	2	8	4
3	0	9	8	7	6	5	4	3	2	1
4	0	2	4	6	8	0	2	4	6	8
5	0	5	0	5	0	5	0	5	0	5
6	0	8	6	4	2	0	8	6	4	2
7	0	1	2	3	4	5	6	7	8	9
8	0	4	8	2	6	0	4	8	2	6
9	0	7	4	1	8	5	2	9	6	3

It is also associative, i.e.,

$$(a\#b)\#c = a\#(b\#c).$$

Now we take the LHS to prove the RHS.

$$\begin{aligned} (a\#b)\#c &= (3ab)\#c \\ &= 3(3ab)c = 9abc \\ &= 3a(3bc) = 3a(b\#c) \\ &= a\#(b\#c) \end{aligned}$$

For each $a \in Z_{10}$, we have $neut(a)$ in Z_{10} .

That is, $neut(0) = 0, neut(1) = 7, neut(2) = 2, neut(3) = 7, neut(4) = 2$, and so on.

Similarly, for each $a \in Z_{10}$, we have $anti(a)$ in Z_{10} .

That is, $anti(0) = 0, anti(1) = 9, anti(2) = 2, anti(3) = 3, anti(4) = 1$, and so on. Thus $(Z_{10}, \#)$ is a neutrosophic triplet group with respect to $\#$ [12].

3. Neutrosophic Triplet Rings

In this section, we introduce neutrosophic triplet rings and study some of their basic properties and notions.

Notations 1. Since the neutrosophic triplet ring and the neutrosophic triplet field are algebraic structures endowed with two internal laws $*$ and $\#$, in order to avoid any confusion, we use the following notation: $neut * (x)$ and $anti * (x)$ for the neutrals and anti's, respectively, of the element x with respect to the law $*$ and $neut\#(x)$ and $anti\#(x)$ for the neutrals and anti's, respectively, of the element x with respect to the law $\#$.

Definition 5. Let $(NTR, *, \#)$ be a set together with two binary operations $*$ and $\#$. Then NTR is called a neutrosophic triplet ring if the following conditions hold:

1. $(NTR, *)$ is a commutative neutrosophic triplet group with respect to $*$;
2. $(NTR, \#)$ is well defined and associative;
3. $a\#(b * c) = (a\#b) * (a\#c)$ and $(b * c)\#a = (b\#a) * (c\#a)$ for all $a, b, c \in NTR$.

Remark 1. An NTR in general is not a classical ring.

Definition 6. Let $(NTR, *, \#)$ be a neutrosophic triplet ring and let $a \in NTR$. We call the structure a unitary neutrosophic triplet ring (UNTR) if each element a has a $neut\#(a)$.

Definition 7. Let $(NTR, *, \#)$ be a neutrosophic triplet ring. We call the structure a commutative unitary neutrosophic triplet ring if it is a UNTR and $\#$ is commutative.

Definition 8. Let $(NTR, *, \#)$ be a neutrosophic triplet ring and let $0 \neq a \in NTR$. If there exists a nonzero element $b \in NTR$ such that $b\#a = 0$, then b is called a left zero divisor of a . Similarly, an element $b \in NTR$ is called a right zero divisor of a if $a\#b = 0$.

A zero divisor of an element is one which is both a left zero divisor and a right zero divisor of that element.

Theorem 1. Let NTR be a commutative neutrosophic triplet ring and $a, b \in NTR$ such that $a, b, neut\#(a), neut\#(b), neut(a\#b)$, and $anti\#(a\#b)$ are cancellable and that $neut\#(a), neut\#(b)$ and $anti\#(a), anti\#(b)$ do exist in NTR. Then

1. $neut\#(a)\#neut\#(b) = neut\#(a\#b)$; and
2. $anti\#(a)\#anti\#(b) = anti\#(a\#b)$.

Proof.

(1) Consider the left-hand side, with $neut\#(a)\#neut\#(b)$. Multiply by a to the left and by b to the right; then we have

$$a\#neut\#(a)\#neut\#(b)\#b = (a\#neut\#(a))\#(neut\#(b)\#b) = a\#b,$$

since # is associative Now we consider the right-hand side; we have $neut^\#(a\#b)$. Multiplying by a to the left and by b to the right, we have

$$a\#neut^\#(a\#b)\#b = (a\#b)\#neut^\#(a\#b) = a\#b,$$

since # is associative and commutative,

Thus, $LHS = a\#b = a\#b = RHS$.

(2) Considering the left-hand side, we have $anti^\#(a)\#anti^\#(b)$.

Multiplying by a to the left and by b to the right, we have

$$a\#anti^\#(a\#anti^\#(b)\#b) = (a\#anti^\#(a))\#(anti^\#(b)\#b) = a\#b.$$

Now consider the right-hand side, where we have $anti^\#(a\#b)$.

Multiplying by a to the left and by b to the right, we have $a\#anti^\#(a\#b)\#b = (a\#b)\#anti^\#(a\#b) = a\#b$, since # is associative and commutative, \square

Definition 9. Let $(NTR, *, \#)$ be a neutrosophic triplet ring and let S be a subset of NTR . Then S is called a neutrosophic triplet subring of NTR if $(S, *, \#)$ is a neutrosophic triplet ring.

Definition 10. Let $(NTR, *, \#)$ be a neutrosophic triplet ring and I be a subset of NTR . Then I is called a neutrosophic triplet ideal of NTR if the following conditions are satisfied.

1. $(I, *)$ is a neutrosophic triplet subgroup of $(NTR, *)$; and
2. For all $x \in I$ and $r \in NTR$, $x\#r \in I$ and $r\#x \in I$.

Theorem 2. Every neutrosophic triplet ideal is trivially a neutrosophic triplet subring, but the converse is not true in general.

Remark 2. Let $(NTR, *, \#)$ be a neutrosophic triplet ring and let $a \in NTR$. Then the following are true.

1. $neut^*(a)$ and $anti^*(a)$ in general are not unique in NTR .
2. $neut^\#(a)$ and $anti^\#(a)$ (if they exist for some element a) in general are not unique in NTR .

Definition 11. Let NTR be a neutrosophic triplet ring and let $a \in NTR$. Then a is called a nilpotent element if $a^n = 0$, for some positive integer $n > 1$.

Theorem 3. Let NTR be a commutative neutrosophic triplet ring and let $a \in NTR$. If a is a nilpotent, the following are true.

1. $(neut * (a))^n = neut * (0)$; and
2. $(anti * (a))^n = anti * (0)$.

Proof.

(1) Suppose that a is a nilpotent in a neutrosophic triplet ring NTR . Then, by definition, $a^n = 0$ for some positive integer $n > 1$.

We prove by mathematical induction.

We can show that $neut * (a) * neut * (a) = neut * (a * b)$ and $anti * (a) * anti * (a) = anti * (a * b)$ in the same way as we did in Theorem 1 above by just replacing the law $*$ by $\#$.

Now we make $a = b$, so we get $neut * (a)^2 = neut * (a) * neut * (a) = neut(a^2)$.

We assume, by mathematical induction, that our equality is true for any positive integer up to $n - 1$, and we need to prove it for n .

Now we consider left-hand side of 1:

$$(neut * (a))^n = (neut * (a)) * (neut * (a))^{n-1} = neut * (a * a^{n-1}) = neut * (a^n) = neut * (0).$$

This completes the proof.

The proof of (2) is similar to that of (1) \square

4. Integral Neutrosophic Triplet Domain and Neutrosophic Triplet Ring Homomorphism

Section 4 is about the introduction of the integral neutrosophic triplet domain and some of its interesting properties. Moreover, in this section, we develop a neutrosophic triplet ring homomorphism.

Definition 12. Let $(NTR, *, \#)$ be a neutrosophic triplet ring. Then NTR is called a commutative neutrosophic triplet ring if $a\#b = b\#a$ for all $a, b \in NTR$.

Definition 13. A commutative neutrosophic triplet ring NTR is called an integral neutrosophic triplet domain if for all $a, b \in NTR$, $a\#b = 0$ implies $a = 0$ or $b = 0$.

Theorem 4. Let NTR be an integral neutrosophic triplet domain. Then the following are true for all $a, b \in NTR$.

1. If $neut^\#(a)$ and $neut^\#(b)$ do exist, then $neut^\#(a)\#neut^\#(b) = 0$ implies $neut^\#(a) = 0$ or $neut^\#(b) = 0$;
2. If $anti^\#(a)$ and $anti^\#(b)$ do exist, then $anti^\#(a)\#anti^\#(b) = 0$ implies $anti^\#(a) = 0$ or $anti^\#(b) = 0$.

Proof.

(1) Obvious, since NTR is an integral neutrosophic triplet domain, and $neut^\#(a)$ and $neut^\#(b)$ belong to NTR .

(2) Obvious, since NTR is an integral neutrosophic triplet domain, and $anti^\#(a)$ and $anti^\#(b)$ belong to NTR . \square

Proposition 1. A commutative neutrosophic triplet ring NTR is an integral neutrosophic triplet domain if, and only if, whenever $a, b, c \in NTR$ such that $a\#b = a\#c$ and $a \neq 0$, then $b = c$.

Proof. Suppose that NTR is an integral neutrosophic triplet domain and let $a, b, c \in NTR$. Since $a \neq 0$ and $a \in NTR$, a is not a zero divisor, so a is cancellable, i.e.,

$$a\#b = a\#c \Rightarrow b = c.$$

Reciprocally, let $a \in NTR$, such that $a \neq 0$; then, by hypothesis, a is cancellable, so a is not a zero divisor. NTR is an integral neutrosophic triplet domain. \square

Definition 14. Let $(NTR_1, *, \#)$ and (NTR_2, \oplus, \otimes) be two neutrosophic triplet rings. Let $f : NTR_1 \rightarrow NTR_2$ be a mapping. Then f is called a neutrosophic triplet ring homomorphism if the following conditions are true.

1. $f(a * b) = f(a) \oplus f(b)$, for all $a, b \in NTR_1$.
2. $f(a\#b) = f(a) \otimes f(b)$, for all $a, b \in NTR_1$.
3. $f(neut * (a)) = neut^\oplus(f(a))$, for all $a \in NTR_1$.
4. $f(anti * (a)) = anti^\oplus(f(a))$, for all $a \in NTR_1$.

5. Neutrosophic Triplet Fields

In this section, we study neutrosophic triplet fields and some of their interesting properties.

Definition 15. Let $(NTR, *, \#)$ be a neutrosophic triplet set together with two binary operations $*$ and $\#$. Then $(NTR, *, \#)$ is called a neutrosophic triplet field if the following conditions hold.

1. $(NTR, *)$ is a commutative neutrosophic triplet group with respect to $*$.
2. $(NTR, \#)$ is a neutrosophic triplet group with respect to $\#$.
3. $a\#(b * c) = (a\#b) * (a\#c)$ and $(b * c)\#a = (b\#a) * (c\#a)$ for all $a, b, c \in NTF$.

Example 3. Let X be a set and let $P(X)$ be the power set of X . Then $(P(X), \cup, \cap)$ is a neutrosophic triplet field since $neut(A) = A$ and $anti(A) = A$ for all $A \in P(X)$ with respect to both \cup and \cap .

Proposition 2. A neutrosophic triplet field NTF always has an $anti(a)$ for every $a \in NTF$ with respect to both laws $*$ and $\#$.

Proof. The proof is straightforward. \square

Theorem 5. A neutrosophic triplet ring is not in general a neutrosophic triplet field.

Counterexample:

$$NTR = (\{1, 2\}, *, \#)$$

	*	1	2
1		2	1
2		1	2

Neutrosophic triplets: $(1, 2, 1), (2, 2, 2), (\{1, 2\}, *)$ is a commutative NTG.

	#	1	2
1		1	1
2		1	1

$(\{1, 2\}, \#)$ is well defined, associative, and commutative.

For the element 2 there is no $neut^\#(2)$ and, consequently, no $anti^\#(2)$.

Therefore, $NTR = (\{1, 2\}, \#)$ is a neutrosophic triplet commutative semigroup, but not a neutrosophic triplet group.

In conclusion, $NTR = ([1], *, \#)$ is a neutrosophic triplet commutative ring, but it is not a neutrosophic triplet field.

Theorem 6. A neutrosophic triplet field NTF is not in general an integral neutrosophic triplet domain NTD.

Proof. Consider the NTF $N = (\{0, 5\}, *, \#)$, where $0 * 0 = 0, 0 * 5 = 5 * 0 = 5, 5 * 5 = 5$. The neutrosophic triplets with respect to $*$ are $(0, 0, 0)$ and $(5, 0, 5)$. Hence, we get $5 * 5 = 0$.

Also $0\#0 = 0\#5 = 5\#0 = 5$ and $5\#5 = 0$. The neutrosophic triplets with respect to $\#$ are $(0, 5, 0)$ and $(5, 0, 5)$.

As we can see, $5\#5 = 0$.

Therefore, this is a NTF which is not an integral neutrosophic triplet domain. \square

Theorem 7. Assume that $f : NTR_1 \rightarrow NTR_2$ is a neutrosophic triplet ring homomorphism. The following then hold.

1. If S is a neutrosophic triplet subring $NTR_1(*, \#)$, then $f(S)$ is a neutrosophic triplet subring of $NTR_2(\oplus, \otimes)$.
2. If U is a neutrosophic triplet subring of NTR_2 , then $f^{-1}(U)$ is a neutrosophic triplet subring of NTR_1 .
3. If I is a neutrosophic triplet ideal of NTR_2 , then $f^{-1}(I)$ is a neutrosophic triplet ideal of NTR_1 .
4. If f is onto, and J is an ideal of NTR_1 , then $f(j)$ is an ideal of NTR_2 .

Proof.

(1) If S is a neutrosophic triplet subring $NTR_1(*, \#)$, then $f(S)$ is a neutrosophic triplet subring of $NTR_2(\oplus, \otimes)$.

Let $a, b \in S$, then $a * b \in S, neut * (a) \in S, anti * (a) \in S$.

Then $f(a), f(b) \in f(S)$ and $f(a * b) \in f(S)$, but $f(a * b) = f(a) \oplus f(b)$, since f is a homomorphism.

Thus, we have proved that if $f(a), f(b) \in f(S)$, then $f(a) \oplus f(b) \in f(S)$.

Since $neut^*(a)$ and $anti^*(a) \in S, f(neut(a))$ and $f(anti(a)) \in f(S)$ since f is a homomorphism.

But $f(neut^*(a)) = neut^\oplus f(a)$, and $f(anti^*(a)) = anti^\oplus f(a)$.

Therefore, if $f(a) \in f(S)$, then $neut^\oplus f(a) = f(neut * (a)) \in f(S)$ and, similarly,

$$anti^\oplus f(a) = f(anti * (a)) \in f(S).$$

Now, if $a, b \in S$, then $a \# b \in S$. Since $a \# b \in S, f(a \# b) \in f(S)$.

But $f(a \# b) = f(a) \otimes f(b)$.

Therefore, if $f(a), f(b) \in f(S)$, then $f(a) \otimes f(b) = f(a \# b) \in f(S)$.

(2) Let $c, d \in U$. Then $f^{-1}(c), f^{-1}(d) \in f^{-1}(U)$. Also $c \oplus d \in U$, hence

$$f^{-1}(c \oplus d) \in f^{-1}(U),$$

$$f^{-1}(c) * f^{-1}(d) \in f^{-1}(U).$$

But

$$f^{-1}(c) * f^{-1}(d) = f^{-1}(c \oplus d),$$

because if we apply f on both sides we get

$$f\left(f^{-1}(c) * f^{-1}(d)\right) = f\left(f^{-1}(c \oplus d)\right),$$

or

$$f\left(f^{-1}(c)\right) \oplus f\left(f^{-1}(d)\right) = c \oplus d$$

or

$$c \oplus d = c \oplus d.$$

Similarly,

$$f^{-1}(c) \# f^{-1}(d) \in f^{-1}(U).$$

But

$$f^{-1}(c) \# f^{-1}(d) = f^{-1}(c \otimes d),$$

because if we apply f on both sides, we get

$$f\left(f^{-1}(c) \# f^{-1}(d)\right) = f\left(f^{-1}(c \otimes d)\right),$$

$$\text{or } f\left(f^{-1}(c)\right) \otimes f\left(f^{-1}(d)\right) = c \otimes d,$$

$$c \otimes d = c \otimes d.$$

Since $c \in U$, we have $neut^\oplus(c)$ and $anti^\oplus(c) \in U$, $f^{-1}(neut^\oplus(c)) = neut^*(f^{-1}(c))$ and $f^{-1}(anti^\oplus(c)) = anti^*(f^{-1}(c))$.

We prove them by applying f on both sides for each equality.

$$f(f^{-1}(neut^\oplus(c))) = f(neut^*(f^{-1}(c))),$$

$$\text{or } neut^\oplus(c) = neut^\oplus\left(f\left(f^{-1}(c)\right)\right),$$

$$\text{or } neut^\oplus(c) = neut^\oplus(c).$$

Similarly,

$$f(f^{-1}(anti^\oplus(c))) = f(anti^*(f^{-1}(c))),$$

or

$$anti^\oplus(c) = anti^\oplus\left(f\left(f^{-1}(c)\right)\right)$$

or

$$anti^\oplus(c) = anti^\oplus(c)$$

(3) Let $i \in I$ and $r \in NTR_2$. Then, $i \oplus r \in I$, and therefore, $f^{-1}(i \oplus r) \in f^{-1}(I)$.

$$f^{-1}(i) \in f^{-1}(I) \text{ and } f^{-1}(r) \in NTR_1.$$

We prove that

$$f^{-1}(i) * f^{-1}(r) = f^{-1}(i \oplus r).$$

Applying f to both sides, we get

$$f(f^{-1}(i) * f^{-1}(r)) = f\left(f^{-1}(i \oplus r)\right);$$

$$f(f^{-1}(i)) \oplus f\left(f^{-1}(r)\right) = i \oplus r;$$

$$i \oplus r = i + r.$$

Therefore, if $i \in I$, $r \in NTR_2$, then $i \oplus r \in f^{-1}(I)$.

(4) Let $j \in f(J)$ and $r \in NTR_2$. Since f is onto, then $\exists h \in J \subset NTR_1$ such that $f(h) = j$ and $\exists s \in NTR_1$ such that $f(s) = r$. We need to prove that $j \oplus r \in f(J)$.

Applying f^{-1} to both sides, we get

$$f^{-1}(j \oplus r) \in f^{-1}(f(J)),$$

or

$$f^{-1}(j) * f^{-1}(r) \in J$$

or

$$h * s \in J$$

which is true, since $h \in J$, which is an ideal in NTR_1 , while $s \in NTR_1$. \square

6. Conclusions

In this paper, we presented the neutrosophic triplet ring. Further, we presented the zero divisor, neutrosophic triplet subring, neutrosophic triplet ideal, nilpotent, integral neutrosophic triplet domain, and neutrosophic triplet ring homomorphism. Finally, we presented the neutrosophic triplet field. In the future, we can develop neutrosophic triplet vector spaces, neutrosophic modules, and neutrosophic triplet near rings, and so on.

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Author Contributions: Mumtaz Ali defined and studied the properties of neutrosophic triplet rings. Florentin Smarandache defined and studied the properties of the neutrosophic triplet fields. Mohsin Khan provided examples and the neutrosophic triplet ring homomorphism and organized the paper.

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