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The category of neutrosophic crisp sets

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Abstract. We introduce the category **NCSet** consisting of neutrosophic crisp sets and morphisms between them. And we study **NCSet** in the sense of a topological universe and prove that it is Cartesian closed over **Set**, where **Set** denotes the category consisting of ordinary sets and ordinary mappings between them.

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1. Introduction

In 1965, Zadeh [20] had introduced a concept of a fuzzy set as the generalization of a crisp set. In 1986, Atanassove [1] proposed the notion of intuitionistic fuzzy set as the generalization of fuzzy sets considering the degree of membership and non-membership. In 1998 Smarandache [19] introduced the concept of a neutrosophic set considering the degree of membership, the degree of indeterminacy and the degree of non-membership. Moreover, Salama et al. [15, 16, 18] applied the concept of neutrosophic crisp sets to topology and relation.

After that time, many researchers [2, 3, 4, 5, 7, 8, 10, 12, 13, 14] have investigated fuzzy sets in the sense of category theory, for instance, **Set**(**H**), $\mathbf{Set_f}$(**H**), $\mathbf{Set_g}$(**H**), **Fuz**(**H**). Among them, the category **Set**(**H**) is the most useful one as the "standard" category, because **Set**(**H**) is very suitable for describing fuzzy sets and mappings between them. In particular, Carrega [2], Dubuc [3], Eytan [4], Goguen [5], Pittes [12], Ponasse [13, 14] had studied **Set**(**H**) in topos view-point. However Hur et al. investigated **Set**(**H**) in topological view-point. Moreover, Hur et al. [8] introduced the category **ISet**(**H**) consisting of intuitionistic H-fuzzy sets and morphisms between them, and studied **ISet**(**H**) in the sense of topological universe. Recently, Lim et al [10] introduced the new category **VSet**(**H**) and investigated it in the sense of topological universe.

The concept of a topological universe was introduced by Nel [11], which implies a Cartesian closed category and a concrete quasitopos. Furthermore the concept has already been up to effective use for several areas of mathematics.

In this paper, first, we obtain some properties of neutrosophic crisp sets proposed by Salama and Smarandache [17] in 2015. Second, we introduce the category **NCSet** consisting of neutrosophic crisp sets and morphisms between them. And we prove that the category **NCSet** is topological and cotopological over **Set** (See Theorem 4.6 and Corollary 4.8), where **Set** denotes the category consisting of ordinary sets and ordinary mappings between them. Furthermore, we prove that final episinks in **NCSet** are preserved by pullbacks(See Theorem 4.10) and **NCSet** is Cartesian closed over **Set** (See Theorem 4.15).

2. Preliminaries

In this section, we list some basic definitions and well-known results from [6, 9, 11] which are needed in the next sections.

Definition 2.1 ([9]). Let **A** be a concrete category and $((Y_j, \xi_j))_J$ a family of objects in **A** indexed by a class J. For any set X, let $(f_j : X \to Y_j)_J$ be a source of mappings indexed by J. Then an **A**-structure ξ on X is said to be initial with respect to (in short, w.r.t.) $(X, (f_j), (Y_j, \xi_j))_J$, if it satisfies the following conditions:

(i) for each $j \in J$, $f_j : (X, \xi) \to (Y_j, \xi_j)$ is an **A**-morphism,

(ii) if (Z, ρ) is an **A**-object and $g : Z \to X$ is a mapping such that for each $j \in J$, the mapping $f_j \circ g : (Z, \rho) \to (Y_j, \xi_j)$ is an **A**-morphism, then $g : (Z, \rho) \to (X, \xi)$ is an **A**-morphism.

In this case, $(f_j : (X, \xi) \to (Y_j, \xi_j))_J$ is called an initial source in **A**.

Dual notion: cotopological category.

Result 2.2 ([9], Theorem 1.5). *A concrete category* **A** *is topological if and only if it is cotopological.*

Result 2.3 ([9], Theorem 1.6). *Let* **A** *be a topological category over Set, then it is complete and cocomplete.*

Definition 2.4 ([9]). Let **A** be a concrete category.

(i) The **A**-fibre of a set X is the class of all **A**-structures on X.

(ii) **A** is said to be properly fibred over **Set**, it satisfies the followings:

(a) (Fibre-smallness) for each set X, the **A**-fibre of X is a set,

(b) (Terminal separator property) for each singleton set X, the **A**-fibre of X has precisely one element,

(c) if ξ and η are **A**-structures on a set X such that $id : (X, \xi) \to (X, \eta)$ and $id : (X, \eta) \to (X, \xi)$ are **A**-morphisms, then $\xi = \eta$.

Definition 2.5 ([6]). A category **A** is said to be Cartesian closed, if it satisfies the following conditions:

(i) for each **A**-object A and B, there exists a product $A \times B$ in **A**,

(ii) exponential objects exist in A, i.e., for each **A**-object A, the functor $A \times - : A \to A$ has a right adjoint, i.e., for any **A**-object B, there exist an **A**-object B^A and a **A**-morphism $e_{A,B} : A \times B^A \to B$ (called the evaluation) such that for any

A-object C and any **A**-morphism $f : A\times C \to B$, there exists a unique **A**-morphism $\bar{f} : C \to B^A$ such that the diagram commutes:

Definition 2.6 ([6]). A category **A** is called a topological universe over **Set**, if it satisfies the following conditions:

(i) **A** is well-structured, i.e. (a) **A** is concrete category; (b) **A** satisfies the fibre-smallness condition; (c) **A** has the terminal separator property,

(ii) **A** is cotopological over **Set**,

(iii) final episinks in **A** are preserved by pullbacks, i.e., for any episink $(g_j : X_j \to Y)_J$ and any **A**-morphism $f : W \to Y$, the family $(e_j : U_j \to W)_J$, obtained by taking the pullback f and g_j, for each $j \in J$, is again a final episink.

3. Neutrosophic crisp sets

In [17], Salama and Smarandache introduced the concept of a neutrosophic crisp set in a set X and defined the inclusion between two neutrosophic crisp sets, the intersection [union] of two neutrosophic crisp sets, the complement of a neutrosophic crisp set, neutrosophic crisp empty [resp., whole] set as more than two types. And they studied some properties related to neutrosophic crisp set operations. However, by selecting only one type, we define the inclusion, the intersection [union], and neutrosophic crisp empty [resp., whole] set again and find some properties.

Definition 3.1. Let X be a non-empty set. Then A is called a neutrosophic crisp set (in short, NCS) in X if A has the form $A = (A_1, A_2, A_3)$,
where A_1, A_2, and A_3 are subsets of X,

The neutrosophic crisp empty [resp., whole] set, denoted by ϕ_N [resp., X_N] is an NCS in X defined by $\phi_N = (\phi, \phi, X)$ [resp., $X_N = (X, X, \phi)$]. We will denote the set of all NCSs in X as $NCS(X)$.

In particular, Salama and Smarandache [17] classified a neutrosophic crisp set as the followings.

A neutrosophic crisp set $A = (A_1, A_2, A_3)$ in X is called a:

(i) neutrosophic crisp set of Type 1 (in short, NCS-Type 1), if it satisfies

$$A_1 \cap A_2 = A_2 \cap A_3 = A_3 \cap A_1 = \phi,$$

(ii) neutrosophic crisp set of Type 2 (in short, NCS-Type 2), if it satisfies

$$A_1 \cap A_2 = A_2 \cap A_3 = A_3 \cap A_1 = \phi \text{ and } A_1 \cup A_2 \cup A_3 = X,$$

(iii) neutrosophic crisp set of Type 3 (in short, NCS-Type 3), if it satisfies

$$A_1 \cap A_2 \cap A_3 = \phi \text{ and } A_1 \cup A_2 \cup A_3 = X.$$

We will denote the set of all NCSs-Type 1 [resp., Type 2 and Type 3] as $NCS_1(X)$ [resp., $NCS_2(X)$ and $NCS_3(X)$].

Definition 3.2. Let $A = (A_1, A_2, A_3), B = (B_1, B_2, B_3) \in NCS(X)$. Then

(i) A is said to be contained in B, denoted by $A \subset B$, if

$$A_1 \subset B_1,\ A_2 \subset B_2 \text{ and } A_3 \supset B_3,$$

(ii) A is said to equal to B, denoted by $A = B$, if

$$A \subset B \text{ and } B \subset A,$$

(iii) the complement of A, denoted by A^c, is an NCS in X defined as:

$$A^c = (A_3, A_2^c, A_1),$$

(iv) the intersection of A and B, denoted by $A \cap B$, is an NCS in X defined as:

$$A \cap B = (A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3),$$

(v) the union of A and B, denoted by $A \cup B$, is an NCS in X defined as:

$$A \cup B = (A_1 \cup B_1, A_2 \cup B_2, A_3 \cap B_3).$$

Let $(A_j)_{j \in J} \subset NCS(X)$, where $A_j = (A_{j,1}, A_{j,2}, A_{j,3})$. Then

(vi) the intersection of $(A_j)_{j \in J}$, denoted by $\bigcap_{j \in J} A_j$ (simply, $\bigcap A_j$), is an NCS in X defined as:

$$\bigcap A_j = (\bigcap A_{j,1}, \bigcap A_{j,2}, \bigcup A_{j,3}),$$

(vii) the the union of $(A_j)_{j \in J}$, denoted by $\bigcup_{j \in J} A_j$ (simply, $\bigcup A_j$), is an NCS in X defined as:

$$\bigcup A_j = (\bigcup A_{j,1}, \bigcup A_{j,2}, \bigcap A_{j,3}).$$

The followings are the immediate results of Definition 3.2.

Proposition 3.3. *Let $A, B, C \in NCS(X)$. Then*
(1) $\phi_N \subset A \subset X_N$,
(2) *if* $A \subset B$ *and* $B \subset C$, *then* $A \subset C$,
(3) $A \cap B \subset A$ *and* $A \cap B \subset B$,
(4) $A \subset A \cup B$ *and* $B \subset A \cup B$,
(5) $A \subset B$ *if and only if* $A \cap B = A$,
(6) $A \subset B$ *if and only if* $A \cup B = B$.

Also the followings are the immediate results of Definition 3.2.

Proposition 3.4. *Let $A, B, C \in NCS(X)$. Then*
(1) (Idempotent laws): $A \cup A = A$, $A \cap A = A$,
(2) (Commutative laws): $A \cup B = B \cup A$, $A \cap B = B \cap A$,
(3) (Associative laws): $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$,
(4) (Distributive laws): $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$,
(5) (Absorption laws): $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$,
(6) (DeMorgan's laws): $(A \cup B)^c = A^c \cap B^c$, $(A \cap B)^c = A^c \cup B^c$,
(7) $(A^c)^c = A$,
(8) (8a) $A \cup \phi_N = A$, $A \cap \phi_N = \phi_N$,
(8b) $A \cup X_N = X_N$, $A \cap X_N = A$,
(8c) $X_N^c = \phi_N$, $\phi_N^c = X_N$,
(8d) *in general,* $A \cup A^c \neq X_N$, $A \cap A^c \neq \phi_N$.

Proposition 3.5. *Let $A \in NCS(X)$ and let $(A_j)_{j \in J} \subset NCS(X)$. Then*
(1) $(\bigcap A_j)^c = \bigcup A_j^c$, $(\bigcup A_j)^c = \bigcap A_j^c$,
(2) $A \cap (\bigcup A_j) = \bigcup (A \cap A_j)$, $A \cup (\bigcap A_j) = \bigcap (A \cup A_j)$.

Proof. (1) $A_j = (A_{j,1}, A_{j,2}, A_{j,3})$. Then $\bigcap A_j = (\bigcap A_{j,1}, \bigcap A_{j,2}, \bigcup A_{j,3})$. Thus

$$(\bigcap A_j)^c = (\bigcup A_{j,3}, (\bigcap A_{j,2})^c, \bigcap A_{j,1}) = (\bigcup A_{j,3}, \bigcup A_{j,2}^c, \bigcap A_{j,1}) = \bigcup A_j^c.$$

Similarly, the second part is proved.

(2) Let $A = (A_1, A_2, A_3)$. Then

$$A \cup (\bigcap A_j) = (A_1 \cup (\bigcap A_{j,1}), A_2 \cup (\bigcap A_{j,2}), A_3 \cap (\bigcup A_{j,3}))$$

$= (\bigcap(A_1 \cup A_{j,1}), \bigcap(A_2 \cup A_{j,2}), \bigcup(A_3 \cap A_{j,3})$
$= \bigcap(A \cup A_j).$

Similarly, the first part is proved. □

Definition 3.6. Let $f : X \to Y$ be a mapping, and let $A = (A_1, A_2, A_3) \in NCS(X)$ and $B = (B_1, B_2, B_3) \in NCS(Y)$. Then

(i) the image of A under f, denoted by $f(A)$, is an NCS in Y defined as:

$$f(A) = (f(A_1), f(A_2), f(A_3)),$$

(ii) the preimage of B, denoted by $f^{-1}(B)$, is an NCS in X defined as:

$$f^{-1}(B) = (f^{-1}(B_1), f^{-1}(B_2), f^{-1}(B_3)).$$

Proposition 3.7. *Let $f : X \to Y$ be a mapping and let $A, B, C \in NCS(X)$, $(A_j)_{j\in J} \subset NCS(X)$ and $D, E, F \in NCS(Y)$, $(D_k)_{k\in K} \subset NCS(Y)$. Then the followings hold:*

(1) *if $B \subset C$, then $f(B) \subset f(C)$ and if $E \subset F$, then $f^{-1}(E) \subset f^{-1}(F)$.*
(2) *$A \subset f^{-1}f(A))$ and if f is injective, then $A = f^{-1}f(A))$,*
(3) *$f(f^{-1}(D)) \subset D$ and if f is surjective, then $f(f^{-1}(D)) = D$,*
(4) *$f^{-1}(\bigcup D_k) = \bigcup f^{-1}(D_k)$, $f^{-1}(\bigcap D_k) = \bigcap f^{-1}(D_k)$,*
(5) *$f(\bigcup A_j) = \bigcup f(A_j)$, $f(\bigcap A_j) \subset \bigcap f(A_j)$,*
(6) *$f(A) = \phi_N$ if and only if $A = \phi_N$ and hence $f(\phi_N) = \phi_N$, in particular if f is surjective, then $f(X_N) = Y_N$,*
(7) *$f^{-1}(Y_N) = Y_N$, $f^{-1}(\phi_N) = \phi$.*

Definition 3.8 ([17])**.** Let $A = (A_1, A_2, A_3) \in NCS(X)$, where X is a set having at least distinct three points. Then A is called a neutrosophic crisp point (in short, NCP) in X, if A_1, A_2 and A_3 are distinct singleton sets in X.

Let $A_1 = \{p_1\}$, $A_2 = \{p_2\}$ and $A_3 = \{p_3\}$, where $p_1 \neq p_2 \neq p_3 \in X$. Then $A = (A_1, A_2, A_3)$ is an NCP in X. In this case, we will denote A as $p = (p_1, p_2, p_3)$. Furthermore, we will denote the set of all NCPs in X as $NCP(X)$.

Definition 3.9. Let $A = (A_1, A_2, A_3) \in NCS(X)$ and let $p = (p_1, p_2, p_3) \in NCP(X)$. Then p is said to belong to A, denoted by $p \in A$, if $\{p_1\} \subset A_1$, $\{p_2\} \subset A_2$ and $\{p_3\}^c \supset A_3$, i.e., $p_1 \in A_1$, $p_2 \in A_2$ and $p_3 \in A_3^c$.

Proposition 3.10. *Let $A = (A_1, A_2, A_3) \in NCS(X)$. Then*

$$A = \bigcup\{p \in NCP(X) : p \in A\}.$$

Proof. Let $p = (p_1, p_2, p_3) \in NCP(X)$. Then

$\bigcup\{p \in NCP(X) : p \in A\}$
$= (\bigcup\{p_1 \in X : p_1 \in A_1\}, \bigcup\{p_2 \in X : p_2 \in A_2\}, \bigcap\{p_3 \in X : p_3 \in A_3^c\}$
$= A.$ □

Proposition 3.11. *Let $A = (A_1, A_2, A_3), B = (B_1, B_2, B_3) \in NCS(X)$. Then $A \subset B$ if and only if $p \in B$, for each $p \in A$.*

Proof. Suppose $A \subset B$ and let $p = (p_1, p_2, p_3) \in A$. Then

$$A_1 \subset B_1, A_2 \subset B_2, A_3 \supset B_3$$

and

$$p_1 \in A_1, p_2 \in A_2, p_3 \in A_3^c.$$

Thus $p_1 \in B_1, p_2 \in B_2, p_3 \in B_3^c$. So $p \in B$. □

Proposition 3.12. *Let $(A_j)_{j\in J} \subset NCS(X)$ and let $p \in NCP(X)$.*
(1) *$p \in \bigcap A_j$ if and only if $p \in A_j$ for each $j \in J$.*
(2) *$p \in \bigcup A_j$ if and only if there exists $j \in J$ such that $p \in A_j$.*

Proof. Let $A_j = (A_{j,1}, A_{j,2}, A_{j,3})$ for each $j \in J$ and let $p = (p_1, p_2, p_3)$.
(1) Suppose $p \in \bigcap A_j$. Then $p_1 \in \bigcap A_{j,1}, p_2 \in \bigcap A_{j,2}, p_3 \in \bigcup A_{j,3}^c$. Thus
$p_1 \in A_{j,1}, p_2 \in A_{j,2}, p_3 \in A_{j,3}^c$, for each $j \in J$. So $p \in A_j$ for each $j \in J$.
We can easily see that the sufficient condition holds.
(2) suppose the necessary condition holds. Then there exists $j \in J$ such that

$$p_1 \in A_{j,1}, p_2 \in A_{j,2}, p_3 \in A_{j,3}^c.$$

Thus $p_1 \in \bigcup A_{j,1}, p_2 \in \bigcup A_{j,2}, p_3 \in (\bigcap A_{j,3})^c$. So $p \in \bigcup A_j$.
We can easily prove that the necessary condition holds. □

Definition 3.13. Let $f : X \to Y$ be an injective mapping, where X, Y are sets having at least distinct three points. Let $p = (p_1, p_2, p_3) \in NCP(X)$. Then the image of p under f, denoted by $f(p)$, is an NCP in Y defined as:

$$f(p) = (f(p_1), f(p_2), f(p_3)).$$

Remark 3.14. In Definition 3.13, if either X or Y has two points, or f is not injective, then $f(p)$ is not an NCP in Y.

Definition 3.15 ([17]). Let $A = (A_1, A_2, A_3) \in NCS(X)$ and $B = (B_1, B_2, B_3) \in NCS(Y)$. Then the Cartesian product of A and B, denoted by $A \times B$, is an NCS in $X \times Y$ defined as: $A \times B = (A_1 \times B_1, A_2 \times B_2, A_3 \times B_3)$.

4. Properties of **NCSet**

Definition 4.1. A pair (X, A) is called a neutrosophic crisp space (in short, NCSp), if $A \in NCS(X)$.

Definition 4.2. A pair (X, A) is called a neutrosophic crisp space-Type j (in short, NCSp-Type j), if $A \in NCS_j(X)$, $j = 1, 2, 3$.

Definition 4.3. Let (X, A_X), (Y, A_Y) be two NCSps or NCSps-Type j, $j = 1, 2, 3$ and let $f : X \to Y$ be a mapping. Then $f : (X, A_X) \to (Y, A_Y)$ is called a morphism, if $A_X \subset f^{-1}(A_Y)$, equivalently,
$A_{X,1} \subset f^{-1}(A_{Y,1})$, $A_{X,2} \subset f^{-1}(A_{Y,2})$ and $A_{X,3} \supset f^{-1}(A_{Y,3})$,
where $A_X = (A_{X,1}, A_{X,2}, A_{X,3})$ and $A_Y = (A_{Y,1}, A_{Y,2}, A_{Y,3})$.
In particular, $f : (X, A_X) \to (Y, A_Y)$ is called an epimorphism [resp., a monomorphism and an isomorphism], if it is surjective [resp., injective and bijective].

From Definitions 3.9, 4.3 and Proposition 3.11, it is obvious that
$f : (X, A_X) \to (Y, A_Y)$ is a morphism
if and only if
$p = (p_1, p_2, p_3) \in f^{-1}(A_Y)$, for each $p = (p_1, p_2, p_3) \in A_X$, i.e.,
$f(p_1) \in A_{Y,1}$, $f(p_2) \in A_{Y,2}$, $f(p_3) \notin A_{Y,3}$, i.e.,

$$f(p) = (f(p_1), f(p_2), f(p_3)) \in A_Y.$$

The following is an immediate result of Definitions 4.3.

Proposition 4.4. *For each NCSp or each NCSps-Type j (X, A_X), $j = 1, 2, 3$, the identity mapping $id : (X, A_X) \to (X, A_X)$ is a morphism.*

Proposition 4.5. *Let (X, A_X), (Y, A_Y), (Z, A_Z) be NCSps or NCSps-Type j, $j = 1, 2, 3$ and let $f : X \to Y$, $g : Y \to Z$ be mappings. If $f : (X, A_X) \to (Y, A_Y)$ and $f : (Y, A_Y) \to (Z, A_Z)$ are morphisms, then $g \circ f : (X, A_X) \to (Z, A_Z)$ is a morphism.*

Proof. Let $A_X = (A_{X,1}, A_{X,2}, A_{X,3})$, $A_Y = (A_{Y,1}, A_{Y,2}, A_{Y,3})$ and $A_Z = (A_{Z,1}, A_{Z,2}, A_{Z,3})$. Then by the hypotheses, $A_X \subset f^{-1}(A_Y)$ and $A_Y \subset g^{-1}(A_Z)$. Thus by Definition 4.3,

$$A_{X,1} \subset f^{-1}(A_{Y,1}),\ A_{X,2} \subset f^{-1}(A_{Y,2}),\ A_{X,3} \supset f^{-1}(A_{Y,3})$$

and

$$A_{Y,1} \subset g^{-1}(A_{Z,1}),\ A_{Y,2} \subset g^{-1}(A_{Z,2}),\ A_{Y,3} \supset g^{-1}(A_{Z,3}).$$

So $A_{X,1} \subset f^{-1}(g^{-1}(A_{Z,1}))$, $A_{X,2} \subset f^{-1}(g^{-1}(A_{Z,2}))$, $A_{X,3} \supset f^{-1}(g^{-1}(A_{Z,3}))$.
Hence $A_{X,1} \subset (g \circ f)^{-1}(A_{Z,1})$, $A_{X,2} \subset (g \circ f)^{-1}(A_{Z,2})$, $A_{X,3} \supset (g \circ f)^{-1}(A_{Z,2})$.
Therefore $g \circ f$ is a morphism. □

From Propositions 4.4 and 4.5, we can form the concrete category **NCSet** [resp., $\mathbf{NCSet_j}$] consisting of NCSs [resp., -Type j, $j = 1, 2, 3$] and morphisms between them. Every **NCSet** [resp., $\mathbf{NCSet_j}$, $j = 1, 2, 3$]-morphism will be called a **NCSet** [resp., $\mathbf{NCSet_j}$, $j = 1, 2, 3$]-mapping.

Theorem 4.6. *The category* **NCSet** *is topological over* **Set**.

Proof. Let X be any set and let $((X_j, A_j))_{j \in J}$ be any families of NCSps indexed by a class J. Suppose $(f_j : X \to (X_j, A_j))_J$ is a source of ordinary mappings. We define the NCS A_X in X by $A_X = \bigcap f_j^{-1}(A_j)$ and $A_X = (A_{X,1}, A_{X,2}, A_{X,3})$.
Then clearly, $A_{X,1} = \bigcap f_j^{-1}(A_{j,1})$, $A_{X,2} = \bigcap f_j^{-1}(A_{j,2})$, $A_{X,3} = \bigcup f_j^{-1}(A_{j,3})$.
Thus (X, A_X) is an NCSp and $A_{X,1} \subset f_j^{-1}(A_{j,1})$, $A_{X,2} \subset f_j^{-1}(A_{j,2})$ and $A_{X,3} \supset f_j^{-1}(A_{j,3})$. So each $f_j : (X, A_X) \to (X_j, A_j)$ is an **NCSet**-mapping.

Now let (Y, A_Y) be any NCSp and suppose $g : Y \to X$ is an ordinary mapping for which $f_j \circ g : (Y, A_Y) \to (X_j, A_j)$ is a **NCSet**-mapping for each $j \in J$. Then for each $j \in J$, $A_Y \subset (f_j \circ g)^{-1}(A_j) = g^{-1}(f_j^{-1}(A_j))$. Thus

$$A_Y \subset (f_j \circ g)^{-1}(A_j) = g^{-1}(\bigcap f_j^{-1}(A_j)) = g^{-1}(A_X).$$

So $g : (Y, A_Y) \to (X, A_X)$ is an **NCSet**-mapping. Hence $(f_j : (X, A_X) \to (X_j, A_j)_J$ is an initial source in **NCSet**. This completes the proof. □

Example 4.7. (1) Let X be a set, let (Y, A_Y) be an NCSp and let $f : X \to Y$ be an ordinary mapping. Then clearly, there exists a unique NCS A_X in X for which $f : (X, A_X) \to (Y, A_Y)$ is an **NCSet**-mapping. In fact, $A_X = f^{-1}(A_Y)$.

In this case, A_X is called the inverse image under f of the NCS structure A_Y.

(2) Let $((X_j, A_j))_{j\in J}$ be any family of NCSps and let $X = \Pi_{j\in J} X_j$. For each $j \in J$, let $pr_j : X \to X_j$ be the ordinary projection. Then there exists a unique NCS A_X in X for which $pr_j : (X, A_X \to (X_j, A_j)$ is an **NCSet**-mapping for each $j \in J$.

In this case, A_X is called the product of $(A_j)_{j\in J}$, denoted by

$$A_X = \Pi A_j = (\Pi A_{j,1}, \Pi A_{j,2}, \Pi A_{j,3})$$

and $(\Pi X_j, \Pi A_j)$ is called the product NCSp of $((X_j, A_j))_{j\in J}$.

In fact, $A_X = \bigcap_{j\in J} pr_j^{-1}(A_j)$.

In particular, if $J = \{1, 2\}$, then $A_1 \times A_2 = (A_{1,1} \times A_{2,1}, A_{1,2} \times A_{2,2}, A_{1,3} \times A_{2,3})$, where $A_1 = (A_{1,1}, A_{1,2}, A_{1,3}) \in NCS(X_1)$ and $A_2 = (A_{2,1}, A_{2,2}, A_{2,3}) \in NCS(X_2)$.

The following is obvious from Result 2.2. But we show directly it.

Corollary 4.8. *The category* **NCSet** *is cotopological over* **Set**.

Proof. Let X be any set and let $((X_j, A_j))_J$ be any family of NCSps indexed by a class J. Suppose $(f_j : X_j \to X)_J$ is a sink of ordinary mappings. We define A_X as $A_X = \bigcup f_j(A_j)$, where $A_X = (A_{X,1}, A_{X,2}, A_{X,3})$ and $A_j = (A_{j,1}, A_{j,2}, A_{j,3})$. Then clearly, $A_X \in NCS(X)$ and each $f_j : (X_j, A_j) \to (X, A_X)$ is an **NCSet**-mapping.

Now for each NCSp (Y, A_Y), let $g : X \to Y$ be an ordinary mapping for which each $g\circ f_j : (X_j, A_j) \to (Y, A_Y)$ is an **NCSet**-mapping. Then clearly for each $j \in J$,

$$A_j \subset (g \circ f_j)^{-1}(A_Y), \text{ i.e., } A_j \subset f_j^{-1}(g^{-1}(A_Y)).$$

Thus $\bigcup A_j \subset \bigcup f_j^{-1}(g^{-1}(A_Y))$. So $f_j(\bigcup A_j) \subset f_j(\bigcup f_j^{-1}(g^{-1}(A_Y)))$. By Proposition 3.7 and the definition of A_X,

$$f_j(\bigcup A_j) = \bigcup f_j(A_j) = A_X$$

and

$$f_j(\bigcup f_j^{-1}(g^{-1}(A_Y))) = \bigcup(f_j \circ f_j^{-1})(g^{-1}(A_Y)) = g^{-1}(A_Y).$$

Hence $A_X \subset g^{-1}(A_Y)$. Therefore $g : (X, A_X) \to (Y, A_Y)$ is an **NCSet**-mapping. This completes the proof. □

The following is proved similarly as the proof of Theorem 4.6.

Corollary 4.9. *The category* $\mathbf{NCSet_j}$ *is topological over* **Set** *for* $j = 1, 2, 3$.

The following is proved similarly as the proof of Corollary 4.8.

Corollary 4.10. *The category* $\mathbf{NCSet_j}$ *is cotopological over* **Set** *for* $j = 1, 2, 3$.

Theorem 4.11. *Final episinks in* **NCSet** *are prserved by pullbacks.*

Proof. Let $(g_j : (X_j, A_j) \to (Y, A_Y))_J$ be any final episink in **NCSet** and let $f : (W, A_W) \to (Y, A_Y)$ be any **NCSet**-mapping. For each $j \in J$, let

$$U_j = \{(w, x_j) \in W \times X_j : f(w) = g_j(x_j)\}.$$

For each $j \in J$, we define the NCS $A_{U_j} = (A_{U_{j,1}}, A_{U_{j,2}}, A_{U_{j,3}})$ in U_j by:

$$A_{U_{j,1}} = A_{W,1} \times A_{j,1}, A_{U_{j,2}} = A_{W,2} \times A_{j,2}, A_{U_{j,3}} = A_{W,3} \times A_{j,3}.$$

For each $j \in J$, let $e_j : U_j \to W$ and $p_j : U_j \to X_j$ be ordinary projections of U_j. Then clearly,

$$A_{U_{j,1}} \subset e_j^{-1}(A_{W,1}), A_{U_{j,2}} \subset e_j^{-1}(A_{W,2}), A_{U_{j,3}} \supset e_j^{-1}(A_{W,3})$$

and

$$A_{U_{j,1}} \subset p_j^{-1}(A_{j,1}), A_{U_{j,2}} \subset p_j^{-1}(A_{j,2}), A_{U_{j,3}} \supset p_j^{-1}(A_{j,3}).$$

Thus $A_{U_j} \subset e_j^{-1}(A_W)$ and $A_{U_j} \subset p_j^{-1}(A_j)$. So $e_j : (U_j, A_{U_j}) \to (W, A_W)$ and $p_j : (U_j, A_{U_j}) \to (X_j, A_j)$ are **NCSet**-mappings. Moreover, $g_h \circ p_h = f \circ e_j$ for each $j \in J$, i.e., the diagram is a pullback square in **NCSet**:

$$\begin{array}{ccc} (U_j, A_{U_j}) & \xrightarrow{\quad p_j \quad} & (X_j, A_j) \\ {\scriptstyle e_j}\downarrow & & \downarrow{\scriptstyle g_j} \\ (W, A_W) & \xrightarrow[\quad f \quad]{} & (Y, A_Y). \end{array}$$

Now in order to prove that $(e_j)_J$ is an episink in **NCSet**, i.e., each e_j is surjective, let $w \in W$. Since $(g_j)_J$ is an episink, there exists $j \in J$ such that $g_j(x_j) = f(w)$ for some $x_j \in X_j$. Thus $(w, x_j) \in U_j$ and $w = e_j(w, x_j)$. So $(e_j)_J$ is an episink in **NCSet**.

Finally, let us show that $(e_j)_J$ is final in **NCSet**. Let A^*_W be the final structure in W w.r.t. $(e_j)_J$ and let $w = (w_1, w_2, w_3) \in A_W$. Since $f : (W, A_W) \to (Y, A_Y)$ is an **NCSet**-mapping, by Definition 3.9,

$w_1 \in A_{W,1} \cap f^{-1}(A_{Y,1})$, $w_2 \in A_{W,2} \cap f^{-1}(A_{Y,2})$ and $w_3 \in A^c_{W,3} \cap (f^{-1}(A_{Y,3}))^c$.

Thus

$w_1 \in A_{W,1}, f(w_1) \in A_{Y,1}$, $w_2 \in A_{W,2}, f(w_2) \in A_{Y,2}$ and $w_3 \in A^c_{W,3}, f(w_3) \in A^c_{Y,3}$.

Since $(g_j)_J$ is final,

$$w_1 \in A_{W,1}, x_{j,1} \in \bigcup_J \bigcup_{x_{j,1} \in g_j^{-1}(f(w))} A_{j,1},$$

$$w_2 \in A_{W,2}, x_{j,2} \in \bigcup_J \bigcup_{x_{j,2} \in g_j^{-1}(f(w))} A_{j,2}$$

and

$$w_3 \in A^c_{W,3}, x_{j,3} \in (\bigcap_J \bigcap_{x_{j,3} \in g_j^{-1}(f(w))} A_{j,3})^c.$$

So $(w_1, x_{j,1}) \in A_{U_{j,1}}$, $(w_2, x_{j,2}) \in A_{U_{j,2}}$ and $(w_3, x_{j,3}) \in A^c_{U_{j,1}}$. Since A^*_W is the final structure in W w.r.t. $(e_j)_J$, $w \in A^*_W$, i.e., $A_W \subset A^*_W$. On the other hand, since $(e_j : (U_j, A_{U_j}) \to (W, A_W))_J$ is final, $1_W : (W, A^*_W) \to (W, A_W)$ is an **NCSet**-mapping and thus $A^*_W \subset A_W$. Hence $A^*_W = A_W$. Therefore $(e_j)_J$ is final. This completes the proof. □

The following is proved similarly as the proof of Theorem 4.9.

Corollary 4.12. *Final episinks in* $\mathbf{NCSet_j}$ *are prserved by pullbacks, for* $J = 1, 2, 3$.

For any singleton set $\{a\}$, NCS $A_{\{a\}}$ [resp., NCS-Type j $A_{\{a\},j}$, for $j = 1, 2, 3$] on $\{a\}$ is not unique, the category **NCSet** [resp., $\mathbf{NCSet_j}$, for $j = 1, 2, 3$] is not properly fibred over **Set**. Then by Definition 2.6, Corollary 4.8 and Theorem 4.11 [resp., Corollaries 4.10 and 4.12], we have the following result.

Theorem 4.13. *The category* **NCSet** *[resp.,* $\mathbf{NCSet_j}$*, for* $j = 1, 2, 3$*] satisfies all the conditions of a topological universe over* **Set** *except the terminal separator property.*

The following is an immediate result of Definitions 3.9 and 3.15.

Proposition 4.14. *Let* $p = (p_1, p_2, p_3), q = (q_1, q_2, q_3) \in NCP(X)$ *and let* $A = (A_1, A_2, A_3), B = (B_1, B_2, B_3) \in NCS(X)$*. Then* $(p, q) \in A \times B$ *if and only if* $(p_1, q_1) \in A_1 \times B_1$*,* $(p_2, q_2) \in A_2 \times B_2$ *and* $(p_3, q_3) \in (A_2 \times B_2)^c$*, i.e.,* $p_3 \in A_3^c$ *or* $q_3 \in B_3^c$*.*

Theorem 4.15. *The category* **NCSet** *is Cartesian closed over* **Set***.*

Proof. It is clear that **NCSet** has products by Theorem 4.6. Then it is sufficient to see that **NCSet** has exponential objects.

For any NCSps $\mathbf{X} = (X, A_X)$ and $\mathbf{Y} = (Y, A_Y)$, let Y^X be the set of all ordinary mappings from X to Y. We define the NCS $A_{Y^X} = (A_{Y^X,1}, A_{Y^X,2}, A_{Y^X,3})$ in Y^X by: for each $f = (f_1, f_2, f_3) \in Y^X$, $f \in A_{Y^X}$ if and only if $f(x) \in A_Y$, for each $x = (x_1, x_2, x_3) \in NCP(X)$, i.e.,

$$f_1 \in A_{Y^X,1}, f_2 \in A_{Y^X,2}, f_3 \notin A_{Y^X,3}$$

if and only if

$$f_1(x_1) \in A_{Y,1}, f_2(x_2) \in A_{Y,2}, f_3(x_3) \notin A_{Y,3}.$$

In fact,

$$A_{Y^X,1} = \{f_1 \in Y^X : f_1(x_1) \in A_{Y,1} \text{ for each } x_1 \in X\},$$

$$A_{Y^X,2} = \{f_2 \in Y^X : f_2(x_2) \in A_{Y,2} \text{ for each } x_2 \in X\},$$

$$A_{Y^X,3} = \{f_3 \in Y^X : f_3(x_3) \notin A_{Y,3} \text{ for some } x_3 \in X\}.$$

Then clearly, (Y^X, A_{Y^X}) is an NCSp.

Let $\mathbf{Y^X} = (Y^X, A_{Y^X})$. Then by the definition of A_{Y^X},

$$A_{Y^X,1} \subset f^{-1}(A_{Y,1}),\ A_{Y^X,2} \subset f^{-1}(A_{Y,2}) \text{ and } A_{Y^X,3} \supset f^{-1}(A_{Y,3}).$$

We define $e_{X,Y} : X \times Y^X \to Y$ by $e_{X,Y}(x, f) = f(x)$, for each $(x, f) \in X \times Y^X$. Let $(x, f) \in A_X \times A_{Y^X}$, where $x = (x_1, x_2, x_3)$, $f = (f_1, f_2, f_3)$. Then by Proposition 4.14 and the definition of $e_{X,Y}$,

$$(x_1, f_1) \in A_{X,1} \times A_{Y^X,1},\ (x_2, f_2) \in A_{X,2} \times A_{Y^X,2},\ (x_3, f_3) \in (A_{X,3} \times A_{Y^X,3})^c$$

and

$$e_{X,Y}(x_1, f_1) = f_1(x_1),\ e_{X,Y}(x_2, f_2) = f_2(x_2),\ e_{X,Y}(x_3, f_3) = f_3(x_3).$$

Thus by the definition of A_{Y^X},

$$(x_1, f_1) \in f^{-1}(A_{Y,1}) \times f^{-1}(A_{Y,1}),$$

$$(x_2, f_2) \in f^{-1}(A_{X,2}) \times f^{-1}(A_{X,2}),$$

$$(x_3, f_3) \in (f^{-1}(A_{X,3}) \times (f^{-1}(A_{X,3}))^c.$$

So $(x_1, f_1) \in e^{-1}_{X,Y}(A_{Y,1})$, $(x_2, f_2) \in e^{-1}_{X,Y}(A_{Y,2})$ and $(x_3, f_3) \in (e^{-1}_{X,Y}(A_{Y,3}))^c$. Hence $A_X \times A_{Y^X} \subset e^{-1}_{X,Y}(A_Y)$. Therefore $e_{X,Y} : \mathbf{X} \times \mathbf{Y^X} \to \mathbf{Y}$ is an **NCSet**-mapping.

For any $\mathbf{Z} = (Z, A_Z) \in$ **NCSet**, let $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ be an **NCSet**-mapping. We define $\bar{h} : Z \to Y^X$ by $[\bar{h}(z)](x) = h(x, z)$, for each $z \in Z$ and each $x \in X$. Let $(x, z) \in A_X \times A_Z$, where $x = (x_1, x_2, x_3)$ and $z = (z_1, z_2, z_3)$. Since $h : \mathbf{X} \times \mathbf{Z} \to \mathbf{Y}$ is an **NCSet**-mapping,

$A_{X,1} \times A_{Z,1} \subset h^{-1}(A_{Y,1})$, $A_{X,2} \times A_{Z,2} \subset h^{-1}(A_{Y,2})$, $A_{X,3} \times A_{Z,3} \supset h^{-1}(A_{Y,1})$.
Then by Proposition 4.14,

$(x_1, z_1) \in h^{-1}(A_{Y,1})$, $(x_2, z_2) \in h^{-1}(A_{Y,2})$, $(x_3, z_3) \in (h^{-1}(A_{Y,3}))^c$.

Thus $h((x_1, z_1)) \in A_{Y,1}$, $h((x_2, z_2)) \in A_{Y,2}$, $h((x_3, z_3)) \in (A_{Y,3})^c$.
By the definition of $\bar{h}$,

$[\bar{h}(z_1)](x_1) \in A_{Y,1}$, $[\bar{h}(z_2)](x_2) \in A_{Y,2}$, $[\bar{h}(z_3)](x_3) \in (A_{Y,3})^c$.

By the definition of A_{Y^X},

$[\bar{h}(z_1)](A_{Z,1}) \subset A_{Y^X,1}$, $[\bar{h}(z_2)](A_{Z,2}) \subset A_{Y^X,2}$, $[\bar{h}(z_3)](A_{Z,3}) \supset A_{Y^X,3}$.

So $A_Z \subset \bar{h}^{-1}(A_{Y^X})$. Hence $\bar{h} : \mathbf{Z} \to \mathbf{Y}^{\mathbf{X}}$ is an **NCSet**-mapping. Furthermore, $\bar{h}$ is the unique **NCSet**-mapping such that $e_{X,Y} \circ (1_X \times \bar{h}) = h$. This completes the proof. □

The following is proved similarly as the proof of Theorem 4.15.

Corollary 4.16. *The category* $\mathbf{NCSet_j}$ *is Cartesian closed over* **Set** *for* $j = 1, 2, 3$.

5. Conclusions

For a non-empty set X, by defining a neutrosophic crisp set $A = (A_1, A_2, A_3)$ and an intuitionistic crisp set $A = (A_1, A_2)$ in X, respectively as follows:

(i) $A_1 \subset X, A_2 \subset X, A_3 \subset X$,

(ii) $A_1 \subset A_3^c, A_3 \subset A_2^c$,

and

(i) $A_1 \subset X, A_2 \subset X$,

(ii) $A_1 \subset A_2^c$,

we can form another categories $\mathbf{NCSet}_*$ and **ICSet**. Furthermore, we will study them in view points of a topological universe and obtain some relationship between them.

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References

[1] K. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems 20 (1986) 87–96.
[2] J. C. Carrega, The category $Set(H)$ and $Fzz(H)$, Fuzzy sets and systems 9 (1983) 327–332.
[3] E. J. Dubuc, Concrete quasitopoi Applications of Sheaves, Proc. Dunham1977, Lect. Notes in Math. 753 (1979) 239–254.
[4] M. Eytan, Fuzzy sets:a topological point of view, Fuzzy sets and systems 5 (1981) 47–67.
[5] J. A. Goguen, Categories of V-sets, Bull. Amer. Math. Soc. 75 (1969) 622–624.
[6] H. Herrlich, Catesian closed topological categories, Math. Coll. Univ. Cape Town 9 (1974) 1–16.
[7] K. Hur, A Note on the category $Set(H)$, Honam Math. J. 10 (1988) 89–94.
[8] K. Hur, H. W. Kang and J. H. Ryou, Intutionistic H-fuzzy sets, J. Korea Soc. Math. Edu. Ser. B:Pure Appl. Math. 12 (1) (2005) 33–45.
[9] C. Y. Kim, S. S. Hong, Y. H. Hong and P. H. Park, Algebras in Cartesian closed topological categories, Lecture Note Series Vol. 1 1985.
[10] P. K. Lim, S. R. Kim and K. Hur, The category $VSet(H)$, International Journal of Fuzzy Logic and Intelligent Systems 10 (1) (2010) 73–81.
[11] L. D. Nel, Topological universes and smooth Gelfand Naimark duality, mathematical applications of category theory, Prc. A. M. S. Spec. Sessopn Denver,1983, Contemporary Mathematics 30 (1984) 224–276.

[12] A. M. Pittes, Fuzzy sets do not form a topos, Fuzzy sets and Systems 8 (1982) 338–358.
[13] D. Ponasse, Some remarks on the category $Fuz(H)$ of M. Eytan, Fuzzy sets and Systems 9 (1983) 199–204.
[14] D. Ponasse, Categorical studies of fuzzy sets, Fuzzy sets and Systems 28 (1988) 235–244.
[15] A. A. Salama, Said Broumi and Florentin Smarandache, Neutrosophic Crisp Open Set and Neutrosophic Crisp Continuity via Neutrosophic Crisp Ideals, in Neutrosophic Theory and Its Applications. Collected Papers, Vol. I, EuropaNova asbl, pp. 199-205, Brussels, EU 2014. See http://fs.gallup.unm.edu/NeutrosophicTheoryApplications.pdf
[16] A. A. Salama, Said Broumi and Florentin Smarandache, Some Types of Neutrosophic Crisp Sets and Neutrosophic Crisp Relations, in Neutrosophic Theory and Its Applications. Collected Papers, Vol. I, EuropaNova asbl, pp. 378-385, Brussels, EU 2014.
[17] A. A. Salama and F. Smarandache, Neutrosophic Crisp Set Theory, The Educational Publisher Columbus, Ohio 2015.
[18] A. A. Salama, Florentin Smarandache and Valeri Kroumov, Neutrosophic Crisp Sets and Neutrosophic Crisp Topological Spaces, in Neutrosophic Theory and Its Applications. Collected Papers, Vol. I, EuropaNova asbl, pp. 206-212, Brussels, EU 2014.
[19] F. Smarandache, Neutrosophy Neutrisophic Property, Sets, and Logic, Amer Res Press, Rehoboth, USA 1998.
[20] L. A. Zadeh, Fuzzy sets, Information and Control 8 (1965) 338–353.

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