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The Structure of Idempotents in Neutrosophic Rings and Neutrosophic Quadruple Rings

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Abstract: This paper aims to reveal the structure of idempotents in neutrosophic rings and neutrosophic quadruple rings. First, all idempotents in neutrosophic rings $\langle R \cup I \rangle$ are given when R is $\mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}$ or \mathbb{Z}_n . Secondly, the neutrosophic quadruple ring $\langle R \cup T \cup I \cup F \rangle$ is introduced and all idempotents in neutrosophic quadruple rings $\langle \mathbb{C} \cup T \cup I \cup F \rangle, \langle \mathbb{R} \cup T \cup I \cup F \rangle, \langle \mathbb{Q} \cup T \cup I \cup F \rangle, \langle \mathbb{Z} \cup T \cup I \cup F \rangle$ and $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ are also given. Furthermore, the algorithms for solving the idempotents in $\langle \mathbb{Z}_n \cup I \rangle$ and $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ for each nonnegative integer n are provided. Lastly, as a general result, if all idempotents in any ring R are known, then the structure of idempotents in neutrosophic ring $\langle R \cup I \rangle$ and neutrosophic quadruple ring $\langle R \cup T \cup I \cup F \rangle$ can be determined.

Keywords: neutrosophic rings; neutrosophic quadruple rings; idempotents; neutrosophic extended triplet group; neutrosophic set

1. Introduction

The notions of neutrosophic set and neutrosophic logic were proposed by Smarandache [1]. In neutrosophic logic, every proposition is considered by the truth degree T , the indeterminacy degree I , and the falsity degree F , where T, I and F are subsets of the nonstandard unit interval $]0^-, 1^+[= 0^- \cup [0, 1] \cup 1^+$.

Using the idea of neutrosophic set, some related algebraic structures have been studied in recent years. Among these algebraic structures, by extending classical groups, the neutrosophic triplet group (NTG) and the neutrosophic extended triplet group (NETG) have been introduced in refs. [2–4]. As an example, paper [5] shows that $(\mathbb{Z}_{p_1 p_2 \dots p_t}, \cdot)$ is not only a semigroup, but also a NETG, where \cdot the classical mod multiplication and p_1, p_2, \dots, p_t are distinct primes. After the notions were put forward, NTG and NETG have been carried out in-depth research. For example, the inclusion relations of neutrosophic sets [6], neutrosophic triplet coset [7], neutrosophic duplet semi-groups [8], AG-neutrosophic extended triplet loops [9,10], the neutrosophic set theory to pseudo-BCI algebras [11], neutrosophic triplet ring and a neutrosophic triplet field [12,13], neutrosophic triplet normed space [14], neutrosophic soft sets [15], neutrosophic vector spaces [16], and so on.

In contrast to the neutrosophic triplet ring, the neutrosophic ring $\langle R \cup I \rangle$, which is a ring generated by the ring R and the indeterminate element I ($I^2 = I$), was proposed by Vasantha and Smarandache in [17]. The concept of neutrosophic ring was further developed and studied in [18–20].

As a special kind of element in an algebraic system, the idempotent element plays a major role in describing the structure and properties of the algebra. For example, Boolean rings refer to rings in which all elements are idempotent, clean rings [21] refer to rings in which each element is clean (an

element in a ring is clean, if it can be written as the sum of an idempotent element and an invertible element), and Albel ring is a ring if each element in the ring is central. From these we can see that some rings can be characterized by idempotents. Thus, it is also quite meaningful to find all idempotents in a ring. In this paper, the idempotents in neutrosophic rings and neutrosophic quadruple rings will be studied in depth, and all idempotents in them can be obtained if the idempotents in R are known. In addition, the relationship between idempotents and neutral elements will be given. The elements of each NETG can be partitioned by neutrals [10]. Therefore, as an application, if $R = \mathbb{F}$, where \mathbb{F} is any field, we can divide the elements of $\langle R \cup I \rangle$ (or $\langle R \cup T \cup I \cup F \rangle$) by idempotents. As another application, in paper [22], the authors explore the idempotents and semi-idempotents in neutrosophic ring $\langle \mathbb{Z}_n \cup I \rangle$ and some open problems and conjectures are given. In this paper, we will answer partial open problems and conjectures in paper [22] and some further studies are discussed.

The outline of this paper is organized as follows. Section 2 gives the basic concepts. In Section 3, the idempotents in neutrosophic ring $\langle R \cup I \rangle$ will be explored. For neutrosophic rings $\langle \mathbb{Z}_n \cup I \rangle$, $\langle \mathbb{C} \cup I \rangle$, $\langle \mathbb{R} \cup I \rangle$, $\langle \mathbb{Q} \cup I \rangle$ and $\langle \mathbb{Z} \cup I \rangle$, all idempotents will be given. Moreover, the open problem and conjectures proposed in paper [22] about idempotents in neutrosophic ring $\langle \mathbb{Z}_n \cup I \rangle$ will be solved. In Section 4, the neutrosophic quadruple ring $\langle R \cup T \cup I \cup F \rangle$ is introduced and all idempotents in neutrosophic quadruple rings $\langle \mathbb{C} \cup T \cup I \cup F \rangle$, $\langle \mathbb{R} \cup T \cup I \cup F \rangle$, $\langle \mathbb{Q} \cup T \cup I \cup F \rangle$, $\langle \mathbb{Z} \cup T \cup I \cup F \rangle$ and $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ will be given. Finally, the summary and future work is presented in Section 5.

2. Basic Concepts

In this section, the related basic definitions and properties of neutrosophic ring $\langle R \cup I \rangle$ and NETG are provided, the details can be seen in [3,4,17,18].

Definition 1. ([17,18]) Let $(R, +, \cdot)$ be any ring. The set

$$\langle R \cup I \rangle = \{a + bI : a, b \in R\}$$

is called a neutrosophic ring generated by R and I . Let $a_1 + b_1I, a_2 + b_2I \in \langle R \cup I \rangle$, The operators \oplus and \otimes on $\langle R \cup I \rangle$ are defined as follows:

$$(a_1 + b_1I) \oplus (a_2 + b_2I) = (a_1 + a_2) + (b_1 + b_2)I,$$

$$(a_1 + b_1I) \otimes (a_2 + b_2I) = (a_1 \cdot a_2) + (a_1 \cdot b_2 + b_1 \cdot a_2 + b_1 \cdot b_2)I.$$

Remark 1. It is easy to verify that $(\langle R \cup I \rangle, \oplus, \otimes)$ is a ring, so $\langle R \cup I \rangle$ is named by a neutrosophic ring is reasonable.

Remark 2. It should be noted that the operators $+, \cdot$ are defined on ring R and \oplus, \otimes are defined on neutrosophic ring $\langle R \cup I \rangle$. For simplicity of notation, we also use $+, \cdot$ to replace \oplus, \otimes on ring $\langle R \cup I \rangle$. That is $a + b$ also means $a \oplus b$ if $a, b \in \langle R \cup I \rangle$. $a \cdot b$ also means $a \otimes b$ if $a, b \in \langle R \cup I \rangle$. For short $a \cdot b$ denoted by ab and $a \cdot a$ denoted by a^2 .

Example 1. $\langle \mathbb{Z} \cup I \rangle$, $\langle \mathbb{Q} \cup I \rangle$, $\langle \mathbb{R} \cup I \rangle$ and $\langle \mathbb{C} \cup I \rangle$ are neutrosophic rings of integer, rational, real and complex numbers, respectively. $\langle \mathbb{Z}_n \cup I \rangle$ is neutrosophic ring of modulo integers. Of course, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ and \mathbb{Z}_n are neutrosophic rings when $b = 0$.

Definition 2. ([17,18]) Let $\langle R \cup I \rangle$ be a neutrosophic ring. $\langle R \cup I \rangle$ is said to be commutative if

$$ab = ba, \forall a, b \in \langle R \cup I \rangle.$$

In addition, if there exists $1 \in \langle R \cup I \rangle$ such that $1 \cdot a = a \cdot 1 = a$ for all $a \in \langle R \cup I \rangle$ then we call $\langle R \cup I \rangle$ a commutative neutrosophic ring with unity.

Definition 3. ([17,18]) An element a in a neutrosophic ring $\langle R \cup I \rangle$ is called an idempotent element if $a^2 = a$.

Definition 4. ([3,4]) Let N be a non-empty set together with a binary operation $*$. Then, N is called a neutrosophic extended triplet set if for any $a \in N$, there exists a neutral of “ a ” (denote by $\text{neut}(a)$), and an opposite of “ a ” (denote by $\text{anti}(a)$), such that $\text{neut}(a) \in N$, $\text{anti}(a) \in N$ and:

$$a * \text{neut}(a) = \text{neut}(a) * a = a, \quad a * \text{anti}(a) = \text{anti}(a) * a = \text{neut}(a).$$

The triplet $(a, \text{neut}(a), \text{anti}(a))$ is called a neutrosophic extended triplet.

Definition 5. ([3,4]) Let $(N, *)$ be a neutrosophic extended triplet set. Then, N is called a neutrosophic extended triplet group (NETG), if the following conditions are satisfied:

- (1) $(N, *)$ is well-defined, i.e., for any $a, b \in N$, one has $a * b \in N$.
- (2) $(N, *)$ is associative, i.e., $(a * b) * c = a * (b * c)$ for all $a, b, c \in N$.

A NETG N is called a commutative NETG if for all $a, b \in N$, $a * b = b * a$.

Proposition 1. ([4]) $(N, *)$ be a NETG. We have:

- (1) $\text{neut}(a)$ is unique for any $a \in N$.
- (2) $\text{neut}(a) * \text{neut}(a) = \text{neut}(a)$ for any $a \in N$.
- (3) $\text{neut}(\text{neut}(a)) = \text{neut}(a)$ for any $a \in N$.

Proposition 2. ([10]) Let $(N, *)$ is a NETG, denote the set of all different neutral element in N by $E(N)$. For any $e \in E(N)$, denote $N(e) = \{x | \text{neut}(x) = e, x \in N\}$. Then:

- (1) $N(e)$ is a classical group, and the unit element is e .
- (2) For any $e_1, e_2 \in E(N)$, $e_1 \neq e_2 \Rightarrow N(e_1) \cap N(e_2) = \emptyset$.
- (3) $N = \bigcup_{e \in E(N)} N(e)$. i.e., $\bigcup_{e \in E(N)} N(e)$ is a partition of N .

3. The Idempotents in Neutrosophic Rings

In this section, we will explore the idempotents in neutrosophic rings $\langle R \cup I \rangle$. If R is $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ or \mathbb{Z}_n , all idempotents in neutrosophic rings $\langle \mathbb{Z}_n \cup I \rangle, \langle \mathbb{C} \cup I \rangle, \langle \mathbb{R} \cup I \rangle, \langle \mathbb{Q} \cup I \rangle$ or $\langle \mathbb{Z} \cup I \rangle$ will be given. Moreover, we can also obtain all idempotents in neutrosophic ring $\langle R \cup I \rangle$ if all idempotents in any ring R are known. As an application, the open problem and conjectures about the idempotents of neutrosophic ring $\langle \mathbb{Z}_n \cup I \rangle$ in paper [22] will be solved. Moreover, an example is given to show how to use the idempotents to get a partition for a neutrosophic ring. The following proposition reveal the relation of a neutral element and an idempotent element.

Proposition 3. Let G be a non-empty set, $*$ is a binary operation on G . For each $a \in G$, a is idempotent iff it is a neutral element.

Proof. Necessity: If a is idempotent, i.e., $a * a = a$, from Definition 4, which shows that a has neutral element a and opposite element a , so a is a neutral element.

Sufficiency: If a is a neutral element, from Proposition 1(2), we have $a * a = a$, thus a is idempotent. \square

Theorem 1. The set of all idempotents in neutrosophic ring $\langle \mathbb{C} \cup I \rangle, \langle \mathbb{R} \cup I \rangle, \langle \mathbb{Q} \cup I \rangle$ or $\langle \mathbb{Z} \cup I \rangle$ is $\{0, 1, I, 1 - I\}$.

Proof. We just give the proof for $\langle \mathbb{R} \cup I \rangle$, and the same result can be obtained for $\langle \mathbb{C} \cup I \rangle, \langle \mathbb{Q} \cup I \rangle$ or $\langle \mathbb{Z} \cup I \rangle$.

Let $a + bI \in \langle \mathbb{R} \cup I \rangle$. If $a + bI$ is idempotent, so $(a + bI)^2 = a + bI$, which means

$$\begin{cases} a^2 = a \\ 2ab + b^2 = b \end{cases} \quad (1)$$

From $a^2 = a$, we can get $a = 0$ or $a = 1$. When $a = 0$, from $2ab + b^2 = b$, we can get $b = 0$ or $b = 1$. That is 0 and I are idempotents. When $a = 1$, from $2ab + b^2 = b$, we can get $b = 0$ or $b = -1$. That is 1 and $1 - I$ are idempotents. Thus, the set of all idempotents of neutrosophic ring $\langle R \cup I \rangle$ is $\{0, 1, I, 1 - I\}$. \square

The above theorem reveals that the set of all idempotents in neutrosophic ring $\langle R \cup I \rangle$ is $\{0, 1, I, 1 - I\}$ when R is $\mathbb{C}, \mathbb{R}, \mathbb{Q}$ or \mathbb{Z} . For any ring R , we have the following results.

Proposition 4. *If a is idempotent in any ring R , then aI is also idempotent in neutrosophic ring $\langle R \cup I \rangle$.*

Proof. If $a \in R$ is idempotent, i.e., $a^2 = a$, so $(aI)^2 = (0 + aI)(0 + aI) = a^2I = aI$, thus, aI is also idempotent in neutrosophic ring $\langle R \cup I \rangle$. \square

Proposition 5. *In neutrosophic ring $\langle R \cup I \rangle$, then $a - aI$ is idempotent iff a is idempotent.*

Proof. Necessity: If $a - aI$ is idempotent, i.e., $(a - aI)^2 = a - aI$, so $(a - aI)^2 = (a - aI)(a - aI) = a^2 - 2aI + a^2I = a^2 + (a^2 - 2a)I = a - aI$, which means $a^2 = a$ and $a^2 - 2a = -a$. Thus, we have $a^2 = a$, so a is idempotent.

Sufficiency: If a is idempotent, so $(a - aI)^2 = a^2 + (a^2 - 2a)I = a - aI$, thus $a - aI$ is idempotent. \square

Theorem 2. *In neutrosophic ring $\langle R \cup I \rangle$, let $a + bI \in \langle R \cup I \rangle$, then $a + bI$ is idempotent iff a is idempotent in R and $b = c - a$, where c is any idempotent element in R .*

Proof. Necessity: If $a + bI$ is idempotent, i.e., $(a + bI)^2 = a + bI$, so $(a + bI)^2 = a^2 + (2ab + b^2)I = a + bI$, which means $a^2 = a$ and $2ab + b^2 = b$. From $a^2 = a$, we can get a is idempotent. From $2ab + b^2 = b$ and $a^2 = a$, we can get $(b + a)^2 = b^2 + 2ab + a^2 = b + a$, so $b + a$ is also idempotent in R , denoted by c , so $b = c - a$.

Sufficiency: If a and c are any idempotents in R , let $b = c - a$, so $(a + bI)^2 = (a + (c - a)I)^2 = a^2 + (2a(c - a) + (c - a)^2)I = a^2 + (2ac - 2a^2 + c^2 - 2ac + a^2)I = a + (c - a)I = a + bI$, thus $a + bI$ is idempotent. \square

Theorem 3. *If the number of different idempotents in ring R is t , then the number of different idempotents in the neutrosophic ring $\langle R \cup I \rangle$ is t^2 .*

Proof. If the number of idempotents in R is t and let $a + bI \in \langle R \cup I \rangle$ is idempotent, so from Theorem 2, we can infer that a is idempotent in R , i.e., a has t different selections. When a is fixed, set $b = c - a$, where c is any idempotent in R and c also has t different selections, which means b has t different selections. Thus, $a + bI$ has $t \cdot t = t^2$ different selections, i.e., the number of all idempotents in $\langle R \cup I \rangle$ is t^2 . \square

From the above analysis, for any ring R , all idempotents in $\langle R \cup I \rangle$ can be determined if all idempotents in R are known. In the following, we will explore all idempotents in neutrosophic ring $\langle \mathbb{Z}_n \cup I \rangle$, i.e., when $R = \mathbb{Z}_n$.

Theorem 4. ([5]) *In the algebra system (\mathbb{Z}_n, \cdot) (see Appendix A), \cdot is the classical mod multiplication, for each $a \in \mathbb{Z}_n$, a has neut(a) and anti(a) iff $\gcd(\gcd(a, n), n/\gcd(a, n)) = 1$.*

Theorem 5. ([5]) For an algebra system (\mathbb{Z}_n, \cdot) and $n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$, where each $p_i (i = 1, 2, \dots, t)$ is a prime, then the number of different neutral elements in \mathbb{Z}_n is 2^t .

Remark 3. From Proposition 3 and Theorem 5, we can infer that the number of all idempotents in $\mathbb{Z}_{p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}}$ is also 2^t .

Example 2. For (\mathbb{Z}_{36}, \cdot) , $n = 36 = 2^2 3^2$. From Theorem 5, the number of different neutral elements in \mathbb{Z}_{36} is $2^2 = 4$. They are:

- (1) [0] has the neutral element [0].
- (2) [1], [5], [7], [11], [13], [17], [19], [23], [25], [29], [31] and [35] have the same neutral element [1].
- (3) [9] and [27] have the same neutral element [9] being $\gcd(9, 36) = \gcd(27, 36) = 9$.
- (4) [4] and [8] have the same neutral element being $\gcd(4, 36) = \gcd(8, 36) = 4$. In fact, [4], [8], [16], [20], [28] and [32] have the same neutral element, which is [28].

From Remark 3, the number of idempotents in \mathbb{Z}_{36} is also 4, which are [0], [1], [9] and [28].

From Theorems 2 and 3 and Remark 3, it follows easily that:

Corollary 1. In neutrosophic ring $\langle \mathbb{Z}_n \cup I \rangle$, let $a + bI \in \langle \mathbb{Z}_n \cup I \rangle$, then $a + bI$ is idempotent iff $a^2 = a$ and $b = c - a$, where c is any idempotent element in \mathbb{Z}_n .

Corollary 2. For an algebra system (\mathbb{Z}_n, \cdot) and $n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$, where each p_1, p_2, \dots , and p_k are distinct primes. Then the number of different idempotents in $\langle \mathbb{Z}_n \cup I \rangle$ is 2^{2^t} .

The solving process for $\langle \mathbb{Z}_n \cup I \rangle$ is given by Algorithm 1. Just only input n , then we can get all idempotents in $\langle \mathbb{Z}_n \cup I \rangle$. The MATLAB code is provided in the Appendix B.

Example 3. Solve all idempotents in $\langle \mathbb{Z}_{600} \cup I \rangle$.

Since $n = 600 = 2^3 \cdot 3 \cdot 5^2$, from Theorem 5, we can get the different neutral elements in \mathbb{Z}_{600} are $\text{neut}(1), \text{neut}(2^3), \text{neut}(3), \text{neut}(5^2), \text{neut}(2^3 \cdot 3), \text{neut}(2^3 \cdot 5^2), \text{neut}(3 \cdot 5^2)$ and $\text{neut}(0)$, i.e., the different idempotents in \mathbb{Z}_{600} are 1, 376, 201, 25, 576, 400, 225, 0. From Corollary 2, the number of different idempotents in neutrosophic ring $\langle \mathbb{Z}_{600} \cup I \rangle$ is $2^{2^3} = 64$.

From Algorithm 1, the set of all 64 idempotents in $\langle \mathbb{Z}_{600} \cup I \rangle$ is: $\{0, I, 25I, 201I, 225I, 376I, 400I, 576I, 1 + 599I, 1, 1 + 24I, 1 + 200I, 1 + 224I, 1 + 375I, 1 + 399I, 1 + 575I, 25 + 575I, 25 + 576I, 25, 25 + 176I, 25 + 200I, 25 + 351I, 25 + 375I, 25 + 551I, 201 + 399I, 201 + 400I, 201 + 424I, 201, 201 + 24I, 201 + 175I, 201 + 199I, 201 + 375I, 225 + 375I, 225 + 376I, 225 + 400I, 225 + 576I, 225, 225 + 151I, 225 + 175I, 225 + 351I, 376 + 224I, 376 + 225I, 376 + 249I, 376 + 425I, 376 + 449I, 376, 376 + 24I, 376 + 200I, 400 + 200I, 400 + 201I, 400 + 225I, 400 + 401I, 400 + 425I, 400 + 576I, 400, 400 + 176I, 576 + 24I, 576 + 25I, 576 + 49I, 576 + 225I, 576 + 249I, 576 + 400I, 576 + 424I, 576\}$.

Algorithm 1: Solving the different idempotents in $\langle \mathbb{Z}_n \cup I \rangle$ Input: n 1: Factorization of integer n , we can get $n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$.2: Computing the neutral element of $1, p_1^{k_1}, p_2^{k_2}, \dots, p_t^{k_t}, p_1^{k_1} p_2^{k_2}, \dots, p_1^{k_1} p_t^{k_t}, \dots, p_2^{k_2} p_3^{k_3} \cdots p_t^{k_t}$ and $p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$. So, we can get all idempotents in \mathbb{Z}_n , denoted by a_1, a_2, \dots, a_{2^t} .3: Let $ID = []$;4: for $i = 1 : 2^t$ 5: $a = a_i$ 6: for $j = 1 : 2^t$ 7: $b = \text{mod}(a_j - a, n)$;8: $ID = [ID; [a, b]]$;

9: end

10: end

Output: ID : all the idempotents in $\langle \mathbb{Z}_n \cup I \rangle$

In paper [22], the authors studied the idempotents and semi-idempotents in $\langle \mathbb{Z}_n \cup I \rangle$ and proposed some open problems and conjectures. We list partial open problems and conjectures about idempotents in $\langle \mathbb{Z}_n \cup I \rangle$ as follows and answer them.

Problem 1. ([22]) Let $S = \langle \mathbb{Z}_{pq}, +, \cdot \rangle$, where p and q are two distinct primes, be the neutrosophic ring. Can S have non-trivial idempotents other than the ones mentioned in (b) of the Theorem 6?

Conjecture 1. ([22]) Let $S = \langle \mathbb{Z}_n, +, \cdot \rangle$ be the neutrosophic ring $n = pqr$, where p, q and r are three distinct primes.

1. $\mathbb{Z}_n = \mathbb{Z}_{pqr}$ has only six non-trivial idempotents associated with it.
2. If m_1, m_2, m_3, m_4, m_5 and m_6 are the idempotents, then, associated with each real idempotent m_i , we have seven non-trivial neutrosophic idempotents associated with it, i.e., $\{m_i + n_j I, j = 1, 2, \dots, 7\}$, such that $m_i + n_j \equiv t$, where t_j takes the seven distinct values from the set $\{0, 1, m_k, k \neq i; k = 1, 2, 3, \dots, 6\}$. $i = 1, 2, \dots, 6$.

Conjecture 2. ([22]) Given $\langle \mathbb{Z}_n \cup I \rangle$, where $n = p_1 p_2 \cdots p_t$; $t > 2$ and p_i s are all distinct primes, find:

1. the number of idempotents in \mathbb{Z}_n ;
2. the number of idempotents in $\langle \mathbb{Z}_n \cup I \rangle \setminus \mathbb{Z}_n$;

Conjecture 3. ([22]) Prove if $\langle \mathbb{Z}_n \cup I \rangle$ and $\langle \mathbb{Z}_m \cup I \rangle$ are two neutrosophic rings where $n > m$ and $n = p^t q$ ($t > 2$, and p and q two distinct primes) and $m = p_1 p_2 \cdots p_s$ where p_i s are distinct primes. $1 \leq i \leq s$, then

1. prove \mathbb{Z}_n has a greater number of idempotents than \mathbb{Z}_m ; and
2. prove $\langle \mathbb{Z}_n \cup I \rangle$ has a greater number of idempotents than $\langle \mathbb{Z}_m \cup I \rangle$.

Theorem 6. ([22]) Let $S = \langle \mathbb{Z}_{pq}, +, \cdot \rangle$ where p and q are two distinct primes:

- (a) There are two idempotents in \mathbb{Z}_{pq} say r and s .
- (b) $\{r, s, rI, sI, I, r + tI, s + tI | t \in \{\mathbb{Z}_{pq} \setminus \{0\}\}\}$ such that $r + t = s, 1$ or 0 and $s + t = 0, 1$ or r is the partial collection of idempotents of S .

For Problem 1, from Remark 3, there are four idempotents in \mathbb{Z}_{pq} , which are $\{1, \text{neut}(p), \text{neut}(q), \text{neut}(pq) = 0\}$. Let $r = \text{neut}(p), s = \text{neut}(q)$, so there are two non-trivial idempotents r, s in \mathbb{Z}_{pq} . From Corollary 1 and 2, the number of all idempotents in $\langle \mathbb{Z}_{pq} \cup I \rangle$ is $2^4 = 16$, they are $\{0 + (0 - 0)I = 0, 0 + (1 - 0)I = I, 0 + (r - 0)I = rI, 0 + (s - 0)I = sI, 1 + (0 - 1)I = 1 + (n - 1)I, 1 + (1 - 1)I = 1, 1 + (r - 1)I, 1 + (s - 1)I, r + (0 - r)I = r + (n - r)I, r + (1 - r)I = r +$

$(n + 1 - r)I, r + (r - r)I = r, r + (s - r)I, s + (0 - s)I = s + (n - r)s, s + (1 - s)I = s + (n + 1 - s)I, s + (r - s)I, s + (s - s)I = s$. So there are 14 non-trivial idempotents in $\langle \mathbb{Z}_{pq} \cup I \rangle$, but there are only include 11 non-trivial idempotents in (b) of the Theorem 6, missing $\{1 + (n - 1)I, 1 + (r - 1)I, 1 + (s - 1)I\}$.

For Conjecture 1, from Corollary 1 and 2, there are eight idempotents in \mathbb{Z}_{pqr} , which are $\{1 = m_0, neut(p) = m_1, neut(q) = m_2, neut(r) = m_3, neut(pq) = m_4, neut(pr) = m_5, neut(qr) = m_6, neut(pqr) = 0 = m_7\}$. There are six non-trivial idempotents in \mathbb{Z}_{pqr} . In $\langle \mathbb{Z}_n \cup I \rangle$, all idempotents are $\{m_i + (m_j - m_i)I | i, j = 0, 1, 2, \dots, 7\}$.

For Conjecture 2, from Remark 3, the number of idempotents in $\mathbb{Z}_{p_1 p_2 \dots p_t}$ is 2^t , and the number of idempotents in $\langle \mathbb{Z}_{p_1 p_2 \dots p_t} \cup I \rangle \setminus \mathbb{Z}_{p_1 p_2 \dots p_t}$ is $2^{2^t} - 2^t$.

For Conjecture 3, from Remark 3, the number of idempotents in \mathbb{Z}_n is 2^2 , and the number of idempotents in \mathbb{Z}_m is 2^s , where $n = p^t q, m = p_1 p_2 \dots p_s$. So, if $s > 2$, \mathbb{Z}_m is characterized by a larger number of idempotents than \mathbb{Z}_n . In similarly way, the number of idempotents in $\langle \mathbb{Z}_n \cup I \rangle$ is 2^4 , and the number of idempotents in $\langle \mathbb{Z}_m \cup I \rangle$ is 2^{2^s} . So, if $s > 2$, we can infer that $\langle \mathbb{Z}_m \cup I \rangle$ is characterized by a larger number of idempotents than $\langle \mathbb{Z}_n \cup I \rangle$.

As another application, we will use the idempotents to divide the elements of the neutrosophic rings $\langle R \cup I \rangle$ when $R = \mathbb{F}$.

For each NETG $(N, *)$, $a \in N$, from Proposition 1, the neutral element of a is uniquely determined. From Proposition 2, $\bigcup_{e \in E(N)} N(e)$ is a partition of N . Since the idempotents and neutral elements are same, we can use the idempotents to get a partition of N . Let us illustrate these with the following example.

Example 4. Let $R = \mathbb{Z}_3$, which is a field. Since $n = 3$, from Theorem 5, we can get the different neutral elements in \mathbb{Z}_3 are $neut(1)$ and $neut(0)$, i.e., the different idempotents in \mathbb{Z}_3 are $1, 0$. From Corollary 2, the number of different idempotents in neutrosophic ring $\langle \mathbb{Z}_3 \cup I \rangle$ is $2^{2-1} = 4$.

From Algorithm 1, the set of all 4 idempotents in $\langle \mathbb{Z}_3 \cup I \rangle$ is: $\{0, 1, I, 1 + 2I\}$. We have $E(0) = \{0\}, E(1) = \{1, 2, 1 + I, 2 + 2I\}, E(I) = \{I, 2I\}, E(1 + 2I) = \{1 + 2I, 2 + I\}$. So $\langle \mathbb{Z}_3 \cup I \rangle = E(0) \cup E(1) \cup E(I) \cup E(1 + 2I)$.

4. The Idempotents in Neutrosophic Quadruple Rings

In the above section, we explored the idempotents in $\langle R \cup I \rangle$. In neutrosophic logic, each proposition is approximated to represent respectively the truth (T), the falsehood (F), and the indeterminacy (I). In this section, according the idea of neutrosophic ring $\langle R \cup I \rangle$, the neutrosophic quadruple ring $\langle R \cup T \cup I \cup F \rangle$ is proposed and the idempotents are given in this section.

Definition 6. Let $(R, +, \cdot)$ be any ring. The set

$$\langle R \cup T \cup I \cup F \rangle = \{a_1 + a_2T + a_3I + a_4F : a_1, a_2, a_3, a_4 \in R\} \tag{2}$$

is called a neutrosophic quadruple ring generated by R and T, I, F . Consider the order $T \prec I \prec F$. Let $a = a_1 + a_2T + a_3I + a_4F, b = b_1 + b_2T + b_3I + b_4F \in \langle R \cup T \cup I \cup F \rangle$, the operators \oplus, \otimes on $\langle R \cup T \cup I \cup F \rangle$ are defined as follows:

$$\begin{aligned} a \oplus b &= (a_1 + a_2T + a_3I + a_4F) \oplus (b_1 + b_2T + b_3I + b_4F) \\ &= a_1 + b_1 + (a_2 + b_2)T + (a_3 + b_3)I + (a_4 + b_4)F. \end{aligned} \tag{3}$$

$$\begin{aligned} a * b &= (a_1 + a_2T + a_3I + a_4F) * (b_1, b_2T, b_3I, b_4F) \\ &= a_1b_1 + (a_1b_2 + a_2b_1 + a_2b_2)T + (a_1b_3 + a_2b_3 + a_3b_1 + a_3b_2 + a_3b_3)I \\ &\quad + (a_1b_4 + a_2b_4 + a_3b_4 + a_4b_1 + a_4b_2 + a_4b_3 + a_4b_4)F. \end{aligned} \tag{4}$$

Remark 4. It is easy to verify that $(\langle R \cup T \cup I \cup F \rangle, \oplus, *)$ is a ring, moreover, it also has the same algebra structure with neutrosophic quadruple numbers (see [23–25]), so the we call $\langle R \cup T \cup I \cup F \rangle$ is a neutrosophic quadruple ring is reasonable.

Remark 5. Similarly with Remark 2, for simplicity of notation, we use $+, \cdot$ to replace $\oplus, *$ on neutrosophic quadruple ring $\langle R \cup T \cup I \cup F \rangle$. That is $a + b$ also means $a \oplus b$ if $a, b \in \langle R \cup T \cup I \cup F \rangle$. and $a \cdot b$ also means $a * b$ if $a, b \in \langle R \cup T \cup I \cup F \rangle$. For short $a \cdot b$ denoted by ab and $a \cdot a$ denoted by a^2 .

Example 5. $\langle \mathbb{Z} \cup T \cup I \cup F \rangle, \langle \mathbb{Q} \cup T \cup I \cup F \rangle, \langle \mathbb{R} \cup T \cup I \cup F \rangle$ and $\langle \mathbb{C} \cup T \cup I \cup F \rangle$ are neutrosophic quadruple rings of integer, rational, real and complex numbers, respectively. $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ is neutrosophic quadruple ring of modulo integers. Of course, $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ and \mathbb{Z}_n are neutrosophic quadruple rings when coefficients of T, I and F equal zero.

Definition 7. Let $\langle R \cup T \cup I \cup F \rangle$ be a neutrosophic quadruple ring. $\langle R \cup T \cup I \cup F \rangle$ is commutative if

$$ab = ba, \forall a, b \in \langle R \cup T \cup I \cup F \rangle.$$

In addition, if there exists $1 \in \langle R \cup T \cup I \cup F \rangle$, such that $1 \cdot a = a \cdot 1 = a$ for all $a \in \langle R \cup T \cup I \cup F \rangle$, then $\langle R \cup T \cup I \cup F \rangle$ is called a commutative neutrosophic quadruple ring with unity.

Definition 8. An element a in a neutrosophic quadruple ring $\langle R \cup T \cup I \cup F \rangle$ is called an idempotent element if $a^2 = a$.

Theorem 7. The set of all idempotents of neutrosophic quadruple rings $\langle \mathbb{C} \cup T \cup I \cup F \rangle, \langle \mathbb{R} \cup T \cup I \cup F \rangle, \langle \mathbb{Q} \cup T \cup I \cup F \rangle$ and $\langle \mathbb{Z} \cup T \cup I \cup F \rangle$ is

$$\{(1, 0, 0, 0), (0, 0, 0, F), (0, 0, I, -F), (0, 0, I, 0), (0, T, -I, 0), (0, T, -I, F), (0, T, 0, -F), (0, T, 0, 0), \\ (1, -T, 0, 0), (1, -T, 0, F), (1, -T, I, -F), (1, -T, I, 0), (1, 0, -I, 0), (1, 0, -I, F), (1, 0, 0, -F), (1, 0, 0, 0)\}.$$

Proof. We only give the proof for $\langle \mathbb{R} \cup T \cup I \cup F \rangle$, and the same result can be obtained for $\langle \mathbb{C} \cup T \cup I \cup F \rangle, \langle \mathbb{Q} \cup T \cup I \cup F \rangle$ or $\langle \mathbb{Z} \cup T \cup I \cup F \rangle$.

Let $a = a_1 + a_2T + a_3I + a_4F$, if a is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$, so $a^2 = a$, i.e., $(a_1 + a_2T + a_3I + a_4F)^2 = (a_1 + a_2T + a_3I + a_4F)$, which means

$$\begin{cases} a_1^2 = a_1, \\ 2a_1a_2 + a_2^2 = a_2, \\ 2(a_1 + a_2)a_3 + a_3^2 = a_3, \\ 2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4. \end{cases}$$

Since $a_1 \in \mathbb{R}$, so from $a_1^2 = a_1$, we can get $a_1 = 0$ or $a_1 = 1$.

Case A: if $a_1 = 0$, then from $2a_1a_2 + a_2^2 = a_2$, we can infer $a_2^2 = a_2$, so $a_2 = 0$ or $a_2 = 1$.

Case A1: if $a_1 = 0$ and $a_2 = 0$, so from $2(a_1 + a_2)a_3 + a_3^2 = a_3$, we can infer $a_3^2 = a_3$, so $a_3 = 0$ or $a_3 = 1$.

Case A11: if $a_1 = 0, a_2 = 0$ and $a_3 = 0$, so from $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, we can infer $a_4^2 = a_4$, so $a_4 = 0$ or $a_4 = 1$.

Case A111: if $a_1 = a_2 = a_3 = a_4 = 0$, i.e., $(0, 0, 0, 0)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.

Case A112: if $a_1 = a_2 = a_3 = 0$ and $a_4 = 1$, i.e., $(0, 0, 0, F)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.

Case A12: if $a_1 = a_2 = 0$ and $a_3 = 1$, so from $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, we can infer $2a_4 + a_4^2 = a_4$, so $a_4 = 0$ or $a_4 = -1$.

- Case A121: if $a_1 = a_2 = 0, a_3 = 1$ and $a_4 = 0$, i.e., $(0, 0, I, 0)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.
- Case A122: if $a_1 = a_2 = 0, a_3 = 1$ and $a_4 = -1$, i.e., $(0, 0, I, -F)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.
- Case A2: if $a_1 = 0$ and $a_2 = 1$, so from $2(a_1 + a_2)a_3 + a_3^2 = a_3$, we can infer $2a_3 + a_3^2 = a_3$, so $a_3 = 0$ or $a_3 = -1$.
- Case A21: if $a_1 = 0, a_2 = 1$, and $a_3 = 0$, so from $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, we can infer $2a_4 + a_4^2 = a_4$, so $a_4 = 0$ or $a_4 = -1$.
- Case A121: if $a_1 = 0, a_2 = 1, a_3 = 0$ and $a_4 = 0$, i.e., $(0, T, 0, 0)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.
- Case A112: if $a_1 = 0, a_2 = 1, a_3 = 0$ and $a_4 = -1$, i.e., $(0, T, 0, -F)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.
- Case A22: if $a_1 = 0, a_2 = 1$ and $a_3 = -1$, so from $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, we can infer $a_4^2 = a_4$, so $a_4 = 0$ or $a_4 = 1$.
- Case A121: if $a_1 = 0, a_2 = 1, a_3 = -1$ and $a_4 = 0$, i.e., $(0, T, -I, 0)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.
- Case A112: if $a_1 = 0, a_2 = 1, a_3 = -1$ and $a_4 = 1$, i.e., $(0, T, -I, F)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.
- Case B: if $a_1 = 1$, then from $2a_1a_2 + a_2^2 = a_2$, we can infer $2a_2 + a_2^2 = a_2$, so $a_2 = 0$ or $a_2 = -1$.
- Case B1: if $a_1 = 1$ and $a_2 = 0$, so from $2(a_1 + a_2)a_3 + a_3^2 = a_3$, we can infer $2a_3 + a_3^2 = a_3$, so $a_3 = 0$ or $a_3 = -1$.
- Case B11: if $a_1 = 1, a_2 = 0$ and $a_3 = 0$, so from $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, we can infer $2a_4 + a_4^2 = a_4$, so $a_4 = 0$ or $a_4 = -1$.
- Case B111: if $a_1 = 1, a_2 = 0, a_3 = 0$ and $a_4 = 0$, i.e., $(1, 0, 0, 0)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.
- Case B112: if $a_1 = 1, a_2 = 0, a_3 = 0$ and $a_4 = -1$, i.e., $(1, 0, 0, -F)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.
- Case B12: if $a_1 = 1, a_2 = 0$ and $a_3 = -1$, so from $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, we can infer $a_4^2 = a_4$, so $a_4 = 0$ or $a_4 = 1$.
- Case B121: if $a_1 = 1, a_2 = 0, a_3 = -1$ and $a_4 = 0$, i.e., $(1, 0, -I, 0)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.
- Case B122: if $a_1 = 1, a_2 = 0, a_3 = -1$ and $a_4 = 1$, i.e., $(1, 0, -I, F)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.
- Case B2: if $a_1 = 1$ and $a_2 = -1$, so from $2(a_1 + a_2)a_3 + a_3^2 = a_3$, we can infer $a_3^2 = a_3$, so $a_3 = 0$ or $a_3 = 1$.
- Case B21: if $a_1 = 1, a_2 = -1$, and $a_3 = 0$, so from $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, we can infer $a_4^2 = a_4$, so $a_4 = 0$ or $a_4 = 1$.
- Case B121: if $a_1 = 1, a_2 = -1, a_3 = 0$ and $a_4 = 0$, i.e., $(1, -T, 0, 0)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.
- Case B112: if $a_1 = 1, a_2 = -1, a_3 = 0$ and $a_4 = 1$, i.e., $(1, -T, 0, F)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.
- Case B22: if $a_1 = 1, a_2 = -1$ and $a_3 = 1$, so from $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, we can infer $2a_4 + a_4^2 = a_4$, so $a_4 = 0$ or $a_4 = -1$.
- Case B121: if $a_1 = 1, a_2 = -1, a_3 = 1$ and $a_4 = 0$, i.e., $(1, -T, I, 0)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.
- Case B112: if $a_1 = 1, a_2 = -1, a_3 = 1$ and $a_4 = -1$, i.e., $(1, -T, I, -F)$ is idempotent in $\langle \mathbb{R} \cup T \cup I \cup F \rangle$.

From the above analysis, we can get the set of all idempotents in neutrosophic quadruple ring $\langle \mathbb{R} \cup T \cup I \cup F \rangle$ are $\{(1, 0, 0, 0), (0, 0, 0, F), (0, 0, I, -F), (0, 0, I, 0), (0, T, -I, 0), (0, T, -I, F), (0, T, 0, -F), (0, T, 0, 0), (1, -T, 0, 0), (1, -T, 0, F), (1, -T, I, -F), (1, -T, I, 0), (1, 0, -I, 0), (1, 0, -I, F), (1, 0, 0, -F), (1, 0, 0, 0)\}$. \square

The above theorem reveals that the idempotents in neutrosophic quadruple ring $\langle \mathbb{R} \cup T \cup I \cup F \rangle$ is fixed when R is $\mathbb{C}, \mathbb{R}, \mathbb{Q}$ or \mathbb{Z} . For any ring R , we have the following results.

Theorem 8. For neutrosophic quadruple ring $\langle \mathbb{R} \cup T \cup I \cup F \rangle$, $a = a_1 + a_2T + a_3I + a_4F$ is idempotent in neutrosophic quadruple ring $\langle \mathbb{R} \cup T \cup I \cup F \rangle$ iff a_1 is idempotent in R , $a_2 = c - a_1$, $a_3 = d - (a_1 + a_2)$ and $a_4 = e - (a_1 + a_2 + a_3)$, where c, d and e are any idempotents in R .

Proof. Necessity: If $a = a_1 + a_2T + a_3I + a_4F$ is idempotent, i.e., $(a_1 + a_2T + a_3I + a_4F)^2 = a_1 + a_2T + a_3I + a_4F$, which means

$$\begin{cases} a_1^2 = a_1, \\ 2a_1a_2 + a_2^2 = a_2, \\ 2(a_1 + a_2)a_3 + a_3^2 = a_3, \\ 2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4. \end{cases}$$

Since $a_1 \in R$, from $a_1^2 = a_1$, we can get a_1 is idempotent in R .

From $2a_1a_2 + a_2^2 = a_2$ and $a_1^2 = a_1$, we can get $(a_1 + a_2)^2 = a_1^2 + 2a_1a_2 + a_2^2 = a_1 + a_2$, so $a_1 + a_2$ is also idempotent in R , denoted by c , so $a_2 = c - a_1$.

From $2(a_1 + a_2)a_3 + a_3^2 = a_3$, and $(a_1 + a_2)^2 = a_1 + a_2$, we can get $(a_1 + a_2 + a_3)^2 = (a_1 + a_2)^2 + 2(a_1 + a_2)a_3 + a_3^2 = a_1 + a_2 + a_3$, so $a_1 + a_2 + a_3$ is also idempotent in R , denoted by d , so $a_3 = d - a_1 - a_2$.

From $2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_4$, and $(a_1 + a_2 + a_3)^2 = a_1 + a_2 + a_3$, we can get $(a_1 + a_2 + a_3 + a_4)^2 = (a_1 + a_2 + a_3)^2 + 2(a_1 + a_2 + a_3)a_4 + a_4^2 = a_1 + a_2 + a_3 + a_4$, so $a_1 + a_2 + a_3 + a_4$ is also idempotent in R , denoted by e , so $a_4 = e - a_1 - a_2 - a_3$.

Sufficiency: If a_1, c, d and e are arbitrary idempotents in R , let $a_2 = c - a_1$, $a_3 = d - (a_1 + a_2)$ and $a_4 = e - (a_1 + a_2 + a_3)$. so $(a_1 + a_2T + a_3I + a_4F)^2 = (a_1 + (c - a_1)T + (d - a_1 - a_2)I + (e - a_1 - a_2 - a_3)F)^2 = a_1^2 + (2(c - a_1)a_1 + (c - a_1)^2)T + (2c(d - a_1 - a_2) + (d - a_1 - a_2)^2)I + (2d(e - a_1 - a_2 - a_3) + (e - a_1 - a_2 - a_3)^2)F = a_1 + (c - a_1)T + (d - a_1 - a_2)I + (e - a_1 - a_2 - a_3)F$. Thus, $a = a_1 + a_2T + a_3I + a_4F$ is idempotent. \square

Theorem 9. If the number of different idempotents in R is t , then the number of different idempotents in neutrosophic quadruple ring $\langle R \cup T \cup I \cup F \rangle$ is t^4 .

Proof. If the number of different idempotents in R is t , let $a_1 + a_2T + a_3I + a_4F \in \langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ is idempotent, so a_1 is idempotent in R , i.e., a_1 has t different selections. When a_1 is selected, $a_2 = c - a_1$, where c is idempotent, which also has t different selections. When a_1, a_2 are selected, $a_3 = d - a_1 - a_2$, where d is idempotent, which also has t different selections. When a_1, a_2, a_3 is selected, $a_4 = e - a_1 - a_2 - a_3$, where e is idempotent, which also has t different selections. Thus, the number of all selections is $t \cdot t \cdot t \cdot t = t^4$, i.e., the number of different idempotents in $\langle R \cup T \cup I \cup F \rangle$ is t^4 . \square

From Theorems 8 and 9 and Remark 3, it follows easily that:

Corollary 3. In neutrosophic quadruple ring $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$, $a = a_1 + a_2T + a_3I + a_4F$ is idempotent in neutrosophic quadruple ring $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ iff a_1 is idempotent in \mathbb{Z}_n , $a_2 = c - a_1$, $a_3 = d - (a_1 + a_2)$ and $a_4 = e - (a_1 + a_2 + a_3)$, where c, d and e are any idempotents in \mathbb{Z}_n .

Corollary 4. The number of different idempotents in neutrosophic quadruple ring $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ is 2^{4t} .

The solving process for neutrosophic quadruple ring $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ is given by Algorithm 2. Just only input n , we can get all idempotents in $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$. The MATLAB code is provided in the Appendix C.

Algorithm 2: Solving the different idempotents in $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$

Input: n

- 1: Factorization of integer n , we can get $n = p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$.
- 2: Computing the neutral element of $1, p_1^{k_1}, p_2^{k_2}, \dots, p_t^{k_t}, p_1^{k_1} p_2^{k_2}, \dots, p_1^{k_1} p_2^{k_2}, \dots, p_2^{k_2} p_3^{k_3} \cdots p_t^{k_t}$ and $p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$. So, we can get all idempotents in \mathbb{Z}_n , denoted by c_1, c_2, \dots, c_{2^t} .
- 3: Let $ID = []$;
- 4: for $i = 1 : 2^t$
- 5: $a_1 = c_i$
- 6: for $j = 1 : 2^t$
- 7: $a_2 = \text{mod}(c_j - a_1, n)$;
- 8: for $m = 1 : 2^t$
- 9: $a_3 = \text{mod}(c_m - a_1 - a_2, n)$;
- 10: for $q = 1 : 2^t$
- 11: $a_4 = \text{mod}(c_q - a_1 - a_2 - a_3, n)$;
- 12: $ID = [ID; [a_1, a_2, a_3, a_4]]$;
- 13: end
- 14: end
- 15: end
- 16: end

Output: ID: all the idempotents in $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$

Example 6. Solve all idempotents in $\langle \mathbb{Z}_{12} \cup T \cup I \cup F \rangle$.

Since $n = 12 = 2^2 \cdot 3$, from Theorems 4 and 5, we can get the different neutral elements in \mathbb{Z}_{12} are $\text{neut}(1), \text{neut}(2^2), \text{neut}(3), \text{neut}(2^3 \cdot 3)$ and $\text{neut}(0)$, i.e., the different idempotents in \mathbb{Z}_{12} are 1, 4, 9, 0. From Corollary 4, the number of different idempotents in neutrosophic quadruple ring $\langle \mathbb{Z}_{12} \cup T \cup I \cup F \rangle$ is $2^{4 \cdot 2} = 256$.

From Algorithm 2, the set of all 256 idempotents in $\langle \mathbb{Z}_{12} \cup T \cup I \cup F \rangle$ is: $\{0, 1F, 4F, 9F, I + 11F, I, I + 3F, I + 8F, 4I + 8F, 4I + 9F, 4I + 5F, 9I + 3F, 9I + 4F, 9I + 7F, 9I, T + 11I, T + 11I + F, T + 11I + 4F, T + 11I + 9F, T + 11F, T, T + 3F, T + 8F, T + 3I + 8F, T + 3I + 9F, T + 3I, T + 3I + 5F, T + 8I + 3F, T + 8I + 4F, T + 8I + 7F, T + 8, 4T + 8I, 4T + 8I + F, 4T + 8I + 4F, 4T + 8I + 9F, 4T + 9I + 11F, 4T + 9I, 4T + 9I + 3F, 4T + 9I + 8F, 4T + 8F, 4T + 9F, 4T, 4T + 5F, 4T + 5I + 3F, 4T + 5I + 4F, 4T + 5I + 7F, 4T + 5I, 9T + 3I, 9T + 3I + F, 9T + 3I + 4F, 9T + 3I + 9F, 9T + 4I + 11F, 9T + 4I, 9T + 4I + 3F, 9T + 4I + 8F, 9T + 7I + 8F, 9T + 7I + 9F, 9T + 7I, 9T + 7I + 5F, 9T + 3F, 9T + 4F, 9T + 7F, 9T, 1 + 11T, 1 + 11T + F, 1 + 11T + 4F, 1 + 11T + 9F, 1 + 11T + I + 11F, 1 + 11T + I, 1 + 11T + I + 3F, 1 + 11T + I + 8F, 1 + 11T + 4I + 8F, 1 + 11T + 4I + 9F, 1 + 11T + 4I, 1 + 11T + 4I + 5F, 1 + 11T + 9I + 3F, 1 + 11T + 9I + 4F, 1 + 11T + 9I + 7F, 1 + 11T + 9I, 1 + 11I, 1 + 11I + F, 1 + 11I + 4F, 1 + 11I + 9F, 1 + 11F, 1, 1 + 3F, 1 + 8F, 1 + 3I + 8F, 1 + 3I + 9F, 1 + 3I, 1 + 3I + 5F, 1 + 8I + 3F, 1 + 8I + 4F, 1 + 8I + 7F, 1 + 8I, 1 + 3T + 8I, 1 + 3T + 8I + F, 1 + 3T + 8I + 4F, 1 + 3T + 8I + 9F, 1 + 3T + 9I + 11F, 1 + 3T + 9I, 1 + 3T + 9I + 3F, 1 + 3T + 9I + 8F, 1 + 3T + 8F, 1 + 3T + 9F, 1 + 3T, 1 + 3T + 5F, 1 + 3T + 5I + 3F, 1 + 3T + 5I + 4F, 1 + 3T + 5I + 7F, 1 + 3T + 5I, 1 + 8T + 3I, 1 + 8T + 3I + F, 1 + 8T + 3I + 4F, 1 + 8T + 3I + 9F, 1 + 8T + 4I + 11F, 1 + 8T + 4I, 1 + 8T + 4I + 3F, 1 + 8T + 4I + 8F, 1 + 8T + 7I + 8F, 1 + 8T + 7I + 9F, 1 + 8T + 7I, 1 + 8T + 7I + 5F, 1 + 8T + 3F, 1 + 8T + 4F, 1 + 8T + 7F, 1 + 8T, 4 + 8T, 4 + 8T + F, 4 + 8T + 4F, 4 + 8T + 9F, 4 + 8T + I + 11F, 4 + 8T + I, 4 + 8T + I + 3F, 4 + 8T + I + 8F, 4 + 8T + 4I + 8F, 4 + 8T + 4I + 9F, 4 + 8T + 4I, 4 + 8T + 4I + 5F, 4 + 8T + 9I + 3F, 4 + 8T + 9I + 4F, 4 + 8T + 9I + 7F, 4 + 8T + 9I, 4 + 9T + 11I, 4 + 9T + 11I + F, 4 + 9T + 11I + 4F, 4 + 9T + 11I + 9F, 4 + 9T + 11F, 4 + 9T, 4 + 9T + 3F, 4 + 9T + 8F, 4 + 9T + 3I + 8F, 4 + 9T + 3I + 9F, 4 + 9T + 3I, 4 + 9T + 3I + 5F, 4 + 9T + 8I + 3F, 4 + 9T + 8I + 4F, 4 + 9T + 8I + 7F, 4 + 9T + 8I, 4 + 8I, 4 + 8I + F, 4 + 8I + 4F, 4 + 8I + 9F, 4 + 9I + 11F, 4 + 9I, 4 + 9I + 3F, 4 + 9I + 8F, 4 + 8F, 4 + 9F, 4, 4 + 5F, 4 + 5I + 3F, 4 + 5I + 4F, 4 + 5I + 7F, 4 + 5I, 4 + 5T + 3I, 4 + 5T + 3I + F, 4 + 5T + 3I + 4F, 4 + 5T + 3I + 9F, 4 + 5T + 4I + 11F, 4 + 5T + 4I, 4 + 5T + 4I + 3F, 4 + 5T + 4I + 8F, 4 + 5T + 7I + 8F, 4 + 5T + 7I + 9F, 4 + 5T + 7I, 4 + 5T + 7I + 5F, 4 + 5T + 3F, 4 +$

$$5T + 4F, 4 + 5T + 7F, 4 + 5T, 9 + 3T, 9 + 3T + F, 9 + 3T + 4F, 9 + 3T + 9F, 9 + 3T + I + 11F, 9 + 3T + I, 9 + 3T + I + 3F, 9 + 3T + I + 8F, 9 + 3T + 4I + 8F, 9 + 3T + 4I + 9F, 9 + 3T + 4I, 9 + 3T + 4I + 5F, 9 + 3T + 9I + 3F, 9 + 3T + 9I + 4F, 9 + 3T + 9I + 7F, 9 + 3T + 9I, 9 + 4T + 11I, 9 + 4T + 11I + F, 9 + 4T + 11I + 4F, 9 + 4T + 11I + 9F, 9 + 4T + 11F, 9 + 4T, 9 + 4T + 3F, 9 + 4T + 8F, 9 + 4T + 3I + 8F, 9 + 4T + 3I + 9F, 9 + 4T + 3I, 9 + 4T + 3I + 5F, 9 + 4T + 8I + 3F, 9 + 4T + 8I + 4F, 9 + 4T + 8I + 7F, 9 + 4T + 8I, 9 + 7T + 8I, 9 + 7T + 8I + F, 9 + 7T + 8I + 4F, 9 + 7T + 8I + 9F, 9 + 7T + 9I + 11F, 9 + 7T + 9I, 9 + 7T + 9I + 3F, 9 + 7T + 9I + 8F, 9 + 7T + 8F, 9 + 7T + 9F, 9 + 7T, 9 + 7T + 5F, 9 + 7T + 5I + 3F, 9 + 7T + 5I + 4F, 9 + 7T + 5I + 7F, 9 + 7T + 5I, 9 + 3I, 9 + 3I + F, 9 + 3I + 4F, 9 + 3I + 9F, 9 + 4I + 11F, 9 + 4I, 9 + 4I + 3F, 9 + 4I + 8F, 9 + 7I + 8F, 9 + 7I + 9F, 9 + 7I, 9 + 7I + 5F, 9 + 3F, 9 + 4F, 9 + 7F, 9.}$$

Similarly, we will use the idempotents to divide the elements of the neutrosophic rings $\langle R \cup T \cup I \cup F \rangle$ when $R = \mathbb{F}$. Let us illustrate these with the following example.

Example 7. Let $R = \mathbb{Z}_3$, which is a field. From Example 4, the different idempotents in \mathbb{Z}_3 are 1, 0. From Corollary 4, the number of different idempotents in neutrosophic quadruple ring $\langle \mathbb{Z}_3 \cup T \cup I \cup F \rangle$ is $2^4 = 16$.

From Algorithm 2, the set of all 16 idempotents in $\langle \mathbb{Z}_3 \cup I \rangle$ is: $E = \{0, F, I + 2F, I, T + 2I, T + 2I + F, T + 2F, T, 1 + 2T, 1 + 2T + F, 1 + 2T + I + 2F, 1 + 2T + I, 1 + 2I, 1 + 2I + F, 1 + 2F, 1\}$. We have $E(0) = \{0\}$, $E(F) = \{F, 2F\}$, $E(I + 2F) = \{I + 2F, 2I + F\}$, $E(I) = \{I, I + F, 2I, 2I + 2F\}$, $E(T + 2I) = \{T + 2I, 2T + I\}$, $E(T + 2I + F) = \{T + 2I + F, T + 2I + 2F, 2T + I + F, 2T + I + 2F\}$, $E(T + 2F) = \{T + 2F, T + I + F, 2T + F, 2T + 2I + 2F\}$, $E(T) = \{T + F, T, T + I, T + I + 2F, 2T, 2T + 2F, 2T + 2I, 2T + 2I + F\}$, $E(1 + 2T) = \{1 + 2T, 2 + T\}$, $E(1 + 2T + F) = \{1 + 2T + F, 1 + 2T + 2F, 2 + T + F, 2 + T + 2F\}$, $E(1 + 2T + I + 2F) = \{1 + 2T + I + 2F, 1 + 2T + 2I + F, 2 + T + I + 2F, 2 + T + 2I + F\}$, $E(1 + 2T + I) = \{1 + 2T + I, 1 + 2T + I + F, 1 + 2T + 2I, 1 + 2T + 2I + 2F, 2 + T + I, 2 + T + I + F, 2 + T + 2I, 2 + T + 2I + 2F\}$, $E(1 + 2I) = \{1 + 2I, 1 + T + I, 2 + I, 2 + 2T + 2I\}$, $E(1 + 2I + F) = \{1 + 2I + F, 1 + 2I + 2F, 1 + T + I + F, 1 + T + I + 2F, 2 + I + F, 2 + I + 2F, 2 + 2T + 2I + F, 2 + 2T + 2I + 2F\}$, $E(1 + 2F) = \{1 + 2F, 1 + I + F, 1 + T + F, 1 + T + 2I + 2F, 2 + F, 2 + 2I + 2F, 2 + 2T + 2F, 2 + 2T + I + F\}$, $E(1) = \{1, 1 + F, 1 + I, 1 + I + 2F, 1 + T, 1 + T + 2F, 1 + T + 2I, 1 + T + 2I + F, 2, 2 + 2F, 2 + 2I, 2 + 2I + F, 2 + 2T, 2 + 2T + F, 2 + 2T + I, 2 + 2T + I + 2F\}$. So $\langle \mathbb{Z}_3 \cup T \cup I \cup F \rangle = \bigcup_{e \in E} E(e)$.

5. Conclusions

In this paper, we study the idempotents in neutrosophic ring $\langle R \cup I \rangle$ and neutrosophic quadruple ring $\langle R \cup T \cup I \cup F \rangle$. We not only solve the open problem and conjectures in paper [22] about idempotents in neutrosophic ring $\langle \mathbb{Z}_n \cup I \rangle$, but also give algorithms to obtain all idempotents in $\langle \mathbb{Z}_n \cup I \rangle$ and $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ for each n . Furthermore, if $R = \mathbb{F}$, then the neutrosophic rings (neutrosophic quadruple rings) can be viewed as a partition divided by the idempotents. As a general result, if all idempotents in ring R are known, then all idempotents in $\langle R \cup I \rangle$ and $\langle R \cup T \cup I \cup F \rangle$ can be obtained too. Moreover, if the number of all idempotents in ring R is t , then the numbers of all idempotents in $\langle R \cup I \rangle$ and $\langle R \cup T \cup I \cup F \rangle$ are t^2 and t^4 respectively. In the following, on the one hand, we will explore semi-idempotents in neutrosophic rings, on the other hand, we will study the algebra properties of neutrosophic rings and neutrosophic quadruple rings.

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Appendix A. The MATLAB code for solving the idempotents in (\mathbb{Z}_n, \cdot)

```

function neut = solve_neut(n)

% n: nonnegative integer
% neut: all idempotents in  $\mathbb{Z}_n$ 

B=[];
digits(32);
for i=1:n
    for j=1:n
        A1(i,j)=mod((i-1)*(j-1),n);
    end
end
a1=factor(n);
a2=unique(a1);
for i=1:length(a2)
    b=length(find(a1==a2(i)));
    B(i)=a2(i)^b;
end
D=[1];
for i=1:length(a2)
    C=combnk(B,i);
    A=prod(C,2);
    D=[D;A];
end
D=mod(D,n);
for i=1:length(D)
    if D(i)==1
        neut(i)=1;
    elseif D(i)==0
        neut(i)=0;
    else
        for j=1:n
            if mod(D(i)*j,n)==D(i)
                for k=1:n
                    if mod(D(i)*k,n)==j
                        neut(i)=j;
                        break
                    end
                end
            end
        end
    end
end
neut=sort(neut);

```

Appendix B. The MATLAB code for solving the idempotents in $(\mathbb{Z}_n \cup I)$

```

function ID = Idempotents_ZR(n)
% n: nonnegative integer

```

% ID: all idempotents in in neutrosophic ring $\langle \mathbb{Z}_n \cup I \rangle$

```
neut = solve_neut(n);

neutall=[];
for i=1:length(neut)
    for j=1:length(neut)
        c1=mod(neut(j)-neut(i),n);
        neutall=[neutall; [neut(i), c1]];
    end
end

ID=sortrows(neutall',1)';
```

Appendix C. The MATLAB code for solving the idempotents in $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$

```
function ID = Idempotents_ZRTIF(n)
% n: nonnegative integer
% ID: all idempotents in in neutrosophic quadruple ring  $\langle \mathbb{Z}_n \cup T \cup I \cup F \rangle$ 

neut = solve_neut(n);
neutall=[];
for i=1:length(neut)
    a1=neut(i);
    for j=1:length(neut)
        a2=mod(neut(j)-a1,n);
        for m=1:length(neut)
            a3=mod(neut(m)-a1-a2,n);
            for q=1:length(neut)
                a4=mod(neut(q)-a1-a2-a3,n);
                neutall=[neutall; [a1 a2 a3 a4]];
            end
        end
    end
end

ID=sortrows(neutall',1)';
```

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