

## VECTOR SIMILARITY MEASURES FOR SIMPLIFIED NEUTROSOPHIC HESITANT FUZZY SET AND THEIR APPLICATIONS

TAHIR MAHMOOD, JUN YE AND QAISAR KHAN

ABSTRACT. In this article we present three similarity measures between simplified neutrosophic hesitant fuzzy sets, which contain the concept of single valued neutrosophic hesitant fuzzy sets and interval valued neutrosophic hesitant fuzzy sets, based on the extension of Jaccard similarity measure, Dice similarity measure and Cosine similarity in the vector space. Then based on these three defined similarity measures we present a multiple attribute decision making method to solve the simplified neutrosophic hesitant fuzzy multiple criteria decision making problem, in which the evaluated values of the alternative with respect to the criteria is represented by simplified neutrosophic hesitant fuzzy elements. Further we applied the proposed similarity measures to pattern recognition. At the end a numerical examples are discussed to show the effectiveness of the proposed similarity measures.

### 1. INTRODUCTION

The evidence provided by people to examine issues is frequently fuzzy and crisp, and the fuzzy set ( $\widetilde{FS}$ ) has been verified to be very important in multi-criteria decision making.  $\widetilde{FS}$  theory was first presented by L. A. Zadeh in 1965[49] seeing degrees of membership to progressively assess the membership of elements in a set. After the introduction of  $\widetilde{FS}$ , Atanassov[1] presented the concept of intuitionistic fuzzy set ( $\widetilde{IFS}$ ) by considering membership and non-membership degrees of an element to the set. Atanassov et al. extended  $\widetilde{IFS}$  to interval valued intuitionistic fuzzy set ( $\widetilde{IVIFS}$ )[2]. After the introduction of  $\widetilde{IFS}$  a lot of similarity measures are proposed by many researchers as the extension of the similarity measures for  $\widetilde{FS}$ , as a similarity measure is an important tool for determining similarity between objects. Li and Cheng[20] presented similarity measure for  $\widetilde{IFS}$  which is the first to apply them to pattern recognition. S.K De et al. gave some application of  $\widetilde{IFS}$  to medical diagnosis[19]. Ye[36] presented cosine similarity measure for  $\widetilde{IFS}$  and applied it to medical diagnosis and pattern recognition. Ye[37, 38] also presented

---

2010 *Mathematics Subject Classification.* 90B50, 91B06, 91B10, 62C99.

*Key words and phrases.* Vector similarity measure; Jaccard similarity measure; Dice similarity measure; Cosine similarity measure; hesitant fuzzy sets; interval valued hesitant fuzzy sets; Single valued neutrosophic hesitant fuzzy sets; interval neutrosophic hesitant fuzzy sets; multi-criteria decision making; pattern recognition.

©2016 Universiteti i Prishtinës, Prishtinë, Kosovë.

Submitted May 1, 2016. Published November 17, 2016.

cosine similarity measure and vector similarity measures for  $\widetilde{IVIFS}$  and trapezoidal  $\widetilde{IFN}$  and applied them to multiple criteria decision making problem.

Neutrosophic set ( $\widetilde{NS}$ ) was first presented by F. Smarandache [25, 26] which simplifies the concept of crisp set,  $\widetilde{FS}$ ,  $\widetilde{IFS}$ ,  $\widetilde{IVIFS}$ , and so on. In  $\widetilde{NS}$  its indeterminacy is computed unambiguously and its truth-membership, indeterminacy-membership and falsity-membership are signified freely.  $\widetilde{NS}$  simplifies all the above defined sets from philosophical point of view. Several similarity measures for  $\widetilde{NS}$  has been defined by Broumi. S. et al. [3]. Broumi. S. et al. [6, 7, 8, 9, 10, 11] single valued neutrosophic graph, Bipolar Single Valued Neutrosophic Graphs, Isolated Single Valued Neutrosophic Graphs, Bipolar Single Valued Neutrosophic Graph Theory, Single Valued Neutrosophic Graphs: Degree, Order and Size, and An Introduction to Bipolar Single Valued Neutrosophic Graph Theory. Majumdar et al. [21] presented similarity and entropy for  $\widetilde{NS}$ .  $\widetilde{NS}$  is difficult to apply in real life and engineering problem. The concept of single valued neutrosophic sets  $\widetilde{SVNSs}$  was introduced for the first time by F. Smarandache in 1998 [25]. Wang et al. [30, 31] introduced the concept of interval neutrosophic sets ( $\widetilde{INSs}$ ) and provided basic operation and properties of  $\widetilde{SVNSs}$  and  $\widetilde{INSs}$ . The  $\widetilde{SVNSs}$  and  $\widetilde{INSs}$  is a subclass of  $\widetilde{NS}$ . Sahin R. et al. [24] defined submethod for  $\widetilde{SVNSs}$ . Zhang et al [50] also defined some operation for  $\widetilde{INS}$ . Ye [39, 40, 41, 42] defined correlation coefficient, improved correlation, entropy and similarity measure for  $\widetilde{SVNS}$  and  $\widetilde{INS}$  and gave their application in multiple attribute decision making problems ( $\widetilde{MCDM}$ ). Broumi. S. et al. [4, 5] presented correlation and cosine similarity measure for  $\widetilde{INS}$ . Ye [43] presented the concept of simplified neutrosophic sets ( $\widetilde{SNSs}$ ) which contained the concept of  $\widetilde{SVNS}$  and  $\widetilde{INS}$  and defined some basic operation for  $\widetilde{SNSs}$  and applied them to multi-criteria decision making problem. Ye [44, 45] also defined vector similarity measures for  $\widetilde{SNSs}$  and improved cosine similarity measures for  $\widetilde{SNSs}$ . Peng et al [18] find some drawbacks of the operation for  $\widetilde{SNS}$  defined by Ye and presented new operations for  $\widetilde{SNS}$ .

Hesitant fuzzy set ( $\widetilde{HFS}$ ) is another generalization of  $\widetilde{FS}$  proposed by Torra and Narukawa [28, 29]. The advantage of  $\widetilde{HFS}$  is that it authorizations the membership degree of an element to a given set with a limited different values, which can ascend in a group decision making problem. Recently a  $\widetilde{HFS}$  has acknowledged more and more consideration since its presence. Xia and Xu [32, 33, 34, 35] presented hesitant fuzzy information, similarity measures for  $\widetilde{HFS}$  and correlation for  $\widetilde{HFS}$ . Chen et al [13] presented some correlation coefficient for  $\widetilde{HFS}$  and applied them to clustering analysis. Ye [46] presented vector similarity measures for  $\widetilde{HFS}$ . Chen et al. [12] further extended HFS to interval valued hesitant fuzzy set. ( $\widetilde{IVHFS}$ ) Farhadinia. B [15, 16] defined information measures for  $\widetilde{HFS}$ , and  $\widetilde{IVHFS}$ . and distance measures for high order  $\widetilde{HFS}$ .

Dual hesitant fuzzy sets ( $\widetilde{DHFS}$ ) was presented by Zhu [51, 52] which is the generalization of  $\widetilde{HFS}$ ,  $\widetilde{FS}$ ,  $\widetilde{IFS}$  and fuzzy multi sets are special cases of  $\widetilde{DHFS}$ . Recently Su. Z [27] presented similarity measure for  $\widetilde{DHFSs}$  and applied it to

pattern recognition. Correlation of  $\widetilde{DHFS}$  was presented by Ye [47] and applied them to  $\widetilde{MCDM}$  under  $\widetilde{DHF}$  information. As mentioned above that hesitancy is the most common problem in decision making for which  $\widetilde{HFS}$  is a suitable means by allowing several possible values for an element to a set. However in  $\widetilde{HFS}$  they consider only one truth-membership function and it cannot express this problem with a few different values assigned by truth-membership hesitant degree, indeterminacy-membership degree, and falsity-membership degrees due to doubts of decision makers. and also in  $\widetilde{DHFS}$  they consider two functions that is membership and non-membership functions and can not consider indeterminacy-membership function. To overawed this problem and the decision maker can get more information Ye [48] presented the concept of single valued neutrosophic hesitant fuzzy sets ( $\widetilde{SVNHF}$ ) and defined some basic operations, aggregation operators and applied it to  $\widetilde{MCDM}$  under  $\widetilde{SVNHF}$  environment. Lui P. [22] presented the concept of interval neutrosophic hesitant fuzzy set ( $\widetilde{INHF}$ ) and defined some basic operations, aggregation operators and applied them to multicriteria decision making problem.

Since vector similarity measures played an important role in decision making, so we present vector similarity measures for  $\widetilde{SNHF}$  and then give its applications in  $\widetilde{MCDM}$  and pattern recognition.

The rest of the article is organized as follows. In section 2 we defined some basic definitions related to our work. In section 3 we defined vector similarity measures for simplified neutrosophic hesitant fuzzy sets. In section 4 we apply the proposed similarity measures to multi attribute decision making problem under simplified neutrosophic environment. in section 5 we apply the proposed similarity measures to pattern recognition. At the end we give some numerical examples to show the effectiveness of the proposed similarity measures, conclusion and references are given.

## 2. PRELIMINARIES

In this section we define some basic definitions about hesitant fuzzy sets, interval valued hesitant fuzzy set, single valued neutrosophic hesitant fuzzy set, interval neutrosophic hesitant fuzzy sets and the vector similarity measures, such as Jaccard similarity measure, Dice similarity measure, Cosine similarity measure.

**Definition 2.1.** [28, 29] Let  $\bar{U}$  be a fixed set, a hesitant fuzzy set  $\hat{M}$  on  $\bar{U}$  is defined in terms of a function  $f_{\hat{M}}(a)$  that when applied to  $\bar{U}$  returns a finite subsets of  $[0, 1]$ . Hesitant fuzzy set is mathematically represented as,

$$\hat{M} = \{ \langle a, f_{\hat{M}}(a) \mid a \in \bar{U} \rangle \},$$

Where  $f_{\hat{M}}(a)$  is a set of some different values in  $[0, 1]$ , denoting the possible membership degrees of the element  $a \in \bar{U}$  to  $\hat{M}$ .

**Definition 2.2.** [12] Let  $\bar{U}$  be a fixed set, an interval valued hesitant fuzzy set  $\hat{D}$  on  $\bar{U}$  is defined in terms of a function  $f_{\hat{D}}(a)$  that when applied to  $\bar{U}$  returns a finite set of subintervals of  $[0, 1]$ . An interval valued hesitant fuzzy set is mathematically represented as,

$$\hat{D} = \{ \langle a, f_{\hat{D}}(a) | a \in \bar{U} \rangle \},$$

Where  $f_{\hat{D}}(a)$  is a set of some different values in  $[0, 1]$ , denoting the possible membership degrees of the element  $a \in \bar{U}$  to  $\hat{D}$ .

**Definition 2.3.** [48] Let  $\bar{U}$  be a non empty fixed set. A single valued neutrosophic hesitant fuzzy set  $\hat{N}$  is the structure of the form:

$$\hat{N} = \{ \langle a, \check{t}(a), \check{i}(a), \check{f}(a) \rangle \}$$

In which  $\check{t}(a), \check{i}(a)$  and  $\check{f}(a)$  are three sets of some values in the real unit interval  $[0, 1]$ , representing the possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees and falsity-membership hesitant degree of the element  $a \in \bar{U}$  to the set  $\hat{N}$ , respectively, with the condition that  $0 \leq \varphi, \chi, \psi \leq 1$  and  $0 \leq \varphi^+ + \chi^+ + \psi^+ \leq 3$ , where  $\varphi \in \check{t}(a), \chi \in \check{i}(a), \psi \in \check{f}(a), \varphi^+ = \cup_{\varphi \in \check{t}} \max\{\varphi\}, \chi^+ = \cup_{\chi \in \check{i}} \max\{\chi\}, \psi^+ = \cup_{\psi \in \check{f}} \max\{\psi\}$ .

**Definition 2.4.** [22] Let  $\bar{U}$  be a non empty fixed set. A interval neutrosophic hesitant fuzzy set  $\hat{H}$  is structure of the form

$$\hat{H} = \{ \langle a, \check{t}(a), \check{i}(a), \check{f}(a) \rangle \}$$

In which  $\check{t}(a), \check{i}(a)$  and  $\check{f}(a)$  are three sets of some interval values in the set of real unit interval  $[0, 1]$ , representing the possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees and falsity-membership hesitant degree of the element  $a \in \bar{U}$  to the set  $\hat{H}$ , respectively, with the condition that  $\varphi = [\varphi^l, \varphi^u] \in \check{t}(a), \chi = [\chi^l, \chi^u] \in \check{i}(a), \psi = [\psi^l, \psi^u] \in \check{f}(a) \subseteq [0, 1]$  and  $0 \leq \sup \varphi^+ + \sup \chi^+ + \sup \psi^+ \leq 3, \varphi^+ = \cup_{\varphi \in \check{t}} \max\{\varphi\}, \chi^+ = \cup_{\chi \in \check{i}} \max\{\chi\}, \psi^+ = \cup_{\psi \in \check{f}} \max\{\psi\}$ .

**2.1. Vector similarity measures.** Let  $A = (c_1, c_2, \dots, c_m)$  and  $B = (e_1, e_2, \dots, e_m)$  be two vectors of length is  $m$  where all the coordinates are positive. The jaccard similarity measure of these two vectors [17] is given by

$$\tilde{J}(A, B) = \frac{A.B}{\|A\|_2^2 + \|B\|_2^2 - A.B} = \frac{\sum_{k=1}^m c_k e_k}{\sum_{k=1}^m c_k^2 + \sum_{k=1}^m e_k^2 - \sum_{k=1}^m c_k e_k}. \tag{2.1}$$

Where  $A.B = \sum_{k=1}^m c_k e_k$  is the inner product of the vectors  $A$  and  $B$  and  $\|A\|^2 = \sqrt{\sum_{k=1}^m c^2}$  and  $\|B\|^2 = \sqrt{\sum_{k=1}^m e^2}$  are the Euclidean norms of  $A$  and  $B$ .

Then the Dice similarity measure [14] is defined as follows:

$$\tilde{D}(A, B) = \frac{2A.B}{\|A\|_2^2 + \|B\|_2^2} = \frac{2 \sum_{k=1}^m c_k e_k}{\sum_{k=1}^m c_k^2 + \sum_{k=1}^m e_k^2} \tag{2.2}$$

and the Cosine similarity measure[23] is defined as follows:

$$\tilde{C}(A, B) = \frac{A \cdot B}{\|A\|_2 \cdot \|B\|_2} = \frac{\sum_{k=1}^m c_k e_k}{\sqrt{\sum_{k=1}^m c_k^2} \cdot \sqrt{\sum_{k=1}^m e_k^2}} \quad (2.3)$$

The Cosine similarity measure is nothing just the cosine angle between two vectors.

the above formulas are same in the sense each take the value in the unit interval  $[0, 1]$ . The jaccard and dice and cosine similarity measures are respectively undefined if  $c_k = e_k = 0$ , for  $k = 1, 2, \dots, m$  and  $c_k = 0$  or  $e_k = 0$ ,  $k = 1, 2, \dots, m$ . Now we assume that the cosine similarity measure equal to zero when  $c_k = 0$  or  $e_k = 0$ ,  $k = 1, 2, \dots, m$ .

These vector similarity measure satisfy the following properties for the vectors  $A$  and  $B$ .

- (1)  $J(A, B) = J(B, A)$ ,  $D(A, B) = D(B, A)$ , and  $C(A, B) = C(B, A)$
- (2)  $0 \leq J(A, B)$ ,  $D(A, B)$ ,  $C(A, B) \leq 1$
- (3)  $J(A, B) = D(A, B) = C(A, B) = 1$  if  $A = B$

### 3. VECTOR SIMILARITY MEASURE FOR SIMPLIFIED NEUTROSOPHIC HESITANT FUZZY SETS

If the simplified neutrosophic hesitant fuzzy set contains finite set of single points, that is single valued neutrosophic hesitant fuzzy set. Then the Jaccard, Dice and Cosine similarity measures are defined as follows:

**Definition 3.1.** Let  $\tilde{A}$  and  $\tilde{B}$  be two  $SV\widehat{NHF}$ Ss on the universal set  $\widehat{U} = \{u_1, u_2, \dots, u_m\}$ , respectively denoted by  $\tilde{A} = \{\langle u_j, \tau_{\tilde{A}}(u_j), \iota_{\tilde{A}}(u_j), \kappa_{\tilde{A}}(u_j) | u_j \in \widehat{U} \rangle\}$  and  $\tilde{B} = \{\langle u_j, \tau_{\tilde{B}}(u_j), \iota_{\tilde{B}}(u_j), \kappa_{\tilde{B}}(u_j) | u_j \in \widehat{U} \rangle\}$  for all  $j = 1, 2, \dots, m$ . Then we define the jaccard, Dice and cosine similarity measures for  $SV\widehat{NHF}$ Ss  $\tilde{A}$  and  $\tilde{B}$  as follows:

$$\widehat{J}_{SV\widehat{NHF}}(\tilde{A}, \tilde{B}) = \frac{1}{m} \sum_{j=1}^m \frac{\sum_{i=1}^{l_{e_j}} \bar{\tau}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{\tau}_{\tilde{B}\sigma(i)}(\tilde{u}_j) + \sum_{i=1}^{l_{e_j}} \bar{\iota}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{\iota}_{\tilde{B}\sigma(i)}(\tilde{u}_j) + \sum_{i=1}^{l_{e_j}} \bar{\kappa}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{\kappa}_{\tilde{B}\sigma(i)}(\tilde{u}_j)}{\sum_{i=1}^{l_{e_j}} [\bar{\tau}_{\tilde{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{l_{e_j}} [\bar{\iota}_{\tilde{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{l_{e_j}} [\bar{\kappa}_{\tilde{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{l_{e_j}} [\bar{\tau}_{\tilde{B}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{l_{e_j}} [\bar{\iota}_{\tilde{B}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{l_{e_j}} [\bar{\kappa}_{\tilde{B}\sigma(i)}(\tilde{u}_j)]^2 - \left( \sum_{i=1}^{l_{e_j}} \bar{\tau}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{\tau}_{\tilde{B}\sigma(i)}(\tilde{u}_j) + \sum_{i=1}^{l_{e_j}} \bar{\iota}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{\iota}_{\tilde{B}\sigma(i)}(\tilde{u}_j) + \sum_{i=1}^{l_{e_j}} \bar{\kappa}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{\kappa}_{\tilde{B}\sigma(i)}(\tilde{u}_j) \right)} \quad (1)$$

$$\widehat{D}_{SVNHF\widehat{S}}(\widetilde{A}, \widetilde{B}) = \frac{1}{m} \sum_{j=1}^m \frac{2 \left( \sum_{i=1}^{le_j} \bar{\tau}_{\widetilde{A}\sigma(i)}(\tilde{u}_j) \bar{\tau}_{\widetilde{B}\sigma(i)}(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{l}_{\widetilde{A}\sigma(i)}(\tilde{u}_j) \bar{l}_{\widetilde{B}\sigma(i)}(\tilde{u}_j) \right) + \sum_{i=1}^{le_j} \bar{\kappa}_{\widetilde{A}\sigma(i)}(\tilde{u}_j) \bar{\kappa}_{\widetilde{B}\sigma(i)}(\tilde{u}_j)}{\sum_{i=1}^{le_j} [\bar{\tau}_{\widetilde{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{\widetilde{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{\widetilde{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\tau}_{\widetilde{B}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{\widetilde{B}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{\widetilde{B}\sigma(i)}(\tilde{u}_j)]^2} \quad (2)$$

$$\widehat{C}_{SVNHF\widehat{S}}(\widetilde{A}, \widetilde{B}) = \frac{1}{m} \sum_{j=1}^m \frac{\sum_{i=1}^{le_j} \bar{\tau}_{\widetilde{A}\sigma(i)}(\tilde{u}_j) \bar{\tau}_{\widetilde{B}\sigma(i)}(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{l}_{\widetilde{A}\sigma(i)}(\tilde{u}_j) \bar{l}_{\widetilde{B}\sigma(i)}(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\kappa}_{\widetilde{A}\sigma(i)}(\tilde{u}_j) \bar{\kappa}_{\widetilde{B}\sigma(i)}(\tilde{u}_j)}{\sqrt{\sum_{i=1}^{le_j} [\bar{\tau}_{\widetilde{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{\widetilde{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{\widetilde{A}\sigma(i)}(\tilde{u}_j)]^2} + \sqrt{\sum_{i=1}^{le_j} [\bar{\tau}_{\widetilde{B}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{\widetilde{B}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{\widetilde{B}\sigma(i)}(\tilde{u}_j)]^2} \quad (3)$$

Here  $le_j = \max(le(\bar{\tau}_{\widetilde{A}}(\tilde{u}_j), \bar{l}_{\widetilde{A}}(\tilde{u}_j), \bar{\kappa}_{\widetilde{A}}(\tilde{u}_j)), le(\bar{\tau}_{\widetilde{B}}(\tilde{u}_j), \bar{l}_{\widetilde{B}}(\tilde{u}_j), \bar{\kappa}_{\widetilde{B}}(\tilde{u}_j)))$  for all  $\tilde{u}_j \in \widehat{U}$ , where  $le(\bar{\tau}_{\widetilde{A}}(\tilde{u}_j), \bar{l}_{\widetilde{A}}(\tilde{u}_j), \bar{\kappa}_{\widetilde{A}}(\tilde{u}_j))$  and  $le(\bar{\tau}_{\widetilde{B}}(\tilde{u}_j), \bar{l}_{\widetilde{B}}(\tilde{u}_j), \bar{\kappa}_{\widetilde{B}}(\tilde{u}_j))$ , respectively represents the number of values in  $\bar{\tau}_{\widetilde{A}}(\tilde{u}_j), \bar{l}_{\widetilde{A}}(\tilde{u}_j), \bar{\kappa}_{\widetilde{A}}(\tilde{u}_j)$  and  $\bar{\tau}_{\widetilde{B}}(\tilde{u}_j), \bar{l}_{\widetilde{B}}(\tilde{u}_j), \bar{\kappa}_{\widetilde{B}}(\tilde{u}_j)$ . When the number of values in  $\bar{\tau}_{\widetilde{A}}(\tilde{u}_j), \bar{l}_{\widetilde{A}}(\tilde{u}_j), \bar{\kappa}_{\widetilde{A}}(\tilde{u}_j)$  and  $\bar{\tau}_{\widetilde{B}}(\tilde{u}_j), \bar{l}_{\widetilde{B}}(\tilde{u}_j), \bar{\kappa}_{\widetilde{B}}(\tilde{u}_j)$  are not equal that is  $le(\bar{\tau}_{\widetilde{A}}(\tilde{u}_j), \bar{l}_{\widetilde{A}}(\tilde{u}_j), \bar{\kappa}_{\widetilde{A}}(\tilde{u}_j)) \neq le(\bar{\tau}_{\widetilde{B}}(\tilde{u}_j), \bar{l}_{\widetilde{B}}(\tilde{u}_j), \bar{\kappa}_{\widetilde{B}}(\tilde{u}_j))$ . Then the pessimistic add the minimum value while the optimistic add the maximum value. This depend on the decision makers that was successfully applied for hesitant fuzzy sets by [34].

The above defined three vector similarity measures satisfy the condition defined in Ye[46].

$$(A_1) \widehat{J}_{SVNHF\widehat{S}}(\widetilde{A}, \widetilde{B}) = \widehat{J}_{SVNHF\widehat{S}}(\widetilde{B}, \widetilde{A}), \widehat{D}_{SVNHF\widehat{S}}(\widetilde{A}, \widetilde{B}) = \widehat{D}_{SVNHF\widehat{S}}(\widetilde{B}, \widetilde{A}), \widehat{C}_{SVNHF\widehat{S}}(\widetilde{A}, \widetilde{B}) = \widehat{C}_{SVNHF\widehat{S}}(\widetilde{B}, \widetilde{A}).$$

$$(A_2) 0 \leq \widehat{J}_{SVNHF\widehat{S}}(\widetilde{A}, \widetilde{B}), \widehat{D}_{SVNHF\widehat{S}}(\widetilde{A}, \widetilde{B}), \widehat{C}_{SVNHF\widehat{S}}(\widetilde{A}, \widetilde{B}) \leq 1.$$

$$(A_3) \widehat{J}_{SVNHF\widehat{S}}(\widetilde{A}, \widetilde{B}) = \widehat{D}_{SVNHF\widehat{S}}(\widetilde{A}, \widetilde{B}) = \widehat{C}_{SVNHF\widehat{S}}(\widetilde{A}, \widetilde{B}) = \langle 1, 0, 0 \rangle \text{ iff } \widetilde{A} = \widetilde{B}$$

*Proof.* (A1) obviously it is true.  $\square$

(A2) obviously the property is true due to the inequality  $\tilde{x}^2 + \tilde{y}^2 \geq 2\tilde{x}\tilde{y}$  for eq 1 and eq 2 and the cosine value for eq 3.

(A3) When  $\widetilde{A} = \widetilde{B}$  then  $\bar{\tau}_{\widetilde{A}}(\tilde{u}_j) = \bar{\tau}_{\widetilde{B}}(\tilde{u}_j)$ ,  $\bar{l}_{\widetilde{A}}(\tilde{u}_j) = \bar{l}_{\widetilde{B}}(\tilde{u}_j)$  and  $\bar{\kappa}_{\widetilde{A}}(\tilde{u}_j) = \bar{\kappa}_{\widetilde{B}}(\tilde{u}_j)$  for each  $\tilde{u}_j \in \widehat{U}$ ,  $j = 1, 2, \dots, m$ . So there are

$$\widehat{J}_{SVNHF\widehat{S}}(\widetilde{A}, \widetilde{B}) = 1, \widehat{D}_{SVNHF\widehat{S}}(\widetilde{A}, \widetilde{B}) = 1, \text{ and } \widehat{C}_{SVNHF\widehat{S}}(\widetilde{A}, \widetilde{B}) = 1.$$

In real life problem, the elements  $\tilde{u}_j$  ( $j = 1, 2, \dots, m$ ) in a universal set  $\widehat{U} = \{\tilde{u}_1, \tilde{u}_2, \dots, \tilde{u}_j\}$  have different weights. Let  $\widehat{\varpi} = (\varpi_1, \varpi_2, \dots, \varpi_j)^{\widehat{T}}$  be the weight vector of  $\tilde{u}_j$  ( $j = 1, 2, \dots, m$ ) with  $\varpi_j \geq 0, j = 1, 2, \dots, m$ , and  $\sum_{j=1}^m \varpi_j = 1$ . Therefore we

extend the above three similarity measures to weighted vector similarity measures for  $SV\widehat{NHF}Ss$ .

The weighted Jaccard similarity measure for  $SV\widehat{NHF}Ss$   $\tilde{A}$  and  $\tilde{B}$  as follows:

$$\hat{J}_{SV\widehat{NHF}S}(\tilde{A}, \tilde{B}) = \sum_{j=1}^m \varpi_j \frac{\sum_{i=1}^{le_j} \bar{\tau}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{\tau}_{\tilde{B}\sigma(i)}(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{l}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{l}_{\tilde{B}\sigma(i)}(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\kappa}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{\kappa}_{\tilde{B}\sigma(i)}(\tilde{u}_j)}{\sum_{i=1}^{le_j} [\bar{\tau}_{\tilde{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{\tilde{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{\tilde{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\tau}_{\tilde{B}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{\tilde{B}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{\tilde{B}\sigma(i)}(\tilde{u}_j)]^2 - \left( \sum_{i=1}^{le_j} \bar{\tau}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{\tau}_{\tilde{B}\sigma(i)}(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{l}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{l}_{\tilde{B}\sigma(i)}(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\kappa}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{\kappa}_{\tilde{B}\sigma(i)}(\tilde{u}_j) \right)} \quad (4)$$

The Dice similarity measure for  $SV\widehat{NHF}Ss$   $\tilde{A}$  and  $\tilde{B}$  as follows:

$$\hat{D}_{SV\widehat{NHF}S}(\tilde{A}, \tilde{B}) = \sum_{j=1}^m \varpi_j \frac{2 \left( \sum_{i=1}^{le_j} \bar{\tau}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{\tau}_{\tilde{B}\sigma(i)}(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{l}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{l}_{\tilde{B}\sigma(i)}(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\kappa}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{\kappa}_{\tilde{B}\sigma(i)}(\tilde{u}_j) \right)}{\sum_{i=1}^{le_j} [\bar{\tau}_{\tilde{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{\tilde{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{\tilde{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\tau}_{\tilde{B}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{\tilde{B}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{\tilde{B}\sigma(i)}(\tilde{u}_j)]^2} \quad (5)$$

The Cosine similarity measure for  $SV\widehat{NHF}Ss$   $\tilde{A}$  and  $\tilde{B}$  as follows:

$$\hat{C}_{SV\widehat{NHF}S}(\tilde{A}, \tilde{B}) = \sum_{j=1}^m \varpi_j \frac{\sum_{i=1}^{le_j} \bar{\tau}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{\tau}_{\tilde{B}\sigma(i)}(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{l}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{l}_{\tilde{B}\sigma(i)}(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\kappa}_{\tilde{A}\sigma(i)}(\tilde{u}_j) \bar{\kappa}_{\tilde{B}\sigma(i)}(\tilde{u}_j)}{\left( \sqrt{\sum_{i=1}^{le_j} [\bar{\tau}_{\tilde{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{\tilde{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{\tilde{A}\sigma(i)}(\tilde{u}_j)]^2} \right) \left( \sqrt{\sum_{i=1}^{le_j} [\bar{\tau}_{\tilde{B}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{\tilde{B}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{\tilde{B}\sigma(i)}(\tilde{u}_j)]^2} \right)} \quad (6)$$

If the  $SNHFS$  contain a set of interval values instead of a set of single points. Then

$$\begin{aligned}
\widehat{J}_{SVNHFS}(\tilde{A}, \tilde{B}) = \frac{1}{2m} \sum_{j=1}^m & \frac{\sum_{i=1}^{le_j} \bar{\tau}_{A\sigma(i)}^L(\tilde{u}_j) \bar{\tau}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\tau}_{A\sigma(i)}^U(\tilde{u}_j) \bar{\tau}_{B\sigma(i)}^U(\tilde{u}_j)}{\sum_{i=1}^{le_j} [\bar{\tau}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\tau}_{A\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{A\sigma(i)}^L(\tilde{u}_j)]^2 +} \\
& + \sum_{i=1}^{le_j} \bar{l}_{A\sigma(i)}^L(\tilde{u}_j) \bar{l}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{l}_{A\sigma(i)}^U(\tilde{u}_j) \bar{l}_{B\sigma(i)}^U(\tilde{u}_j) + \\
& \sum_{i=1}^{le_j} \bar{\kappa}_{A\sigma(i)}^L(\tilde{u}_j) \bar{\kappa}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\kappa}_{A\sigma(i)}^U(\tilde{u}_j) \bar{\kappa}_{B\sigma(i)}^U(\tilde{u}_j)}{\sum_{i=1}^{le_j} [\bar{l}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{A\sigma(i)}^U(\tilde{u}_j)]^2 +} \\
& \sum_{i=1}^{le_j} [\bar{\tau}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\tau}_{B\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \\
& \sum_{i=1}^{le_j} [\bar{l}_{B\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{B\sigma(i)}^U(\tilde{u}_j)]^2 -} \\
& \left( \sum_{i=1}^{le_j} \bar{\tau}_{A\sigma(i)}^L(\tilde{u}_j) \bar{\tau}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\tau}_{A\sigma(i)}^U(\tilde{u}_j) \bar{\tau}_{B\sigma(i)}^U(\tilde{u}_j) \right. \\
& + \sum_{i=1}^{le_j} \bar{l}_{A\sigma(i)}^L(\tilde{u}_j) \bar{l}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{l}_{A\sigma(i)}^U(\tilde{u}_j) \bar{l}_{B\sigma(i)}^U(\tilde{u}_j) + \\
& \left. \sum_{i=1}^{le_j} \bar{\kappa}_{A\sigma(i)}^L(\tilde{u}_j) \bar{\kappa}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\kappa}_{A\sigma(i)}^U(\tilde{u}_j) \bar{\kappa}_{B\sigma(i)}^U(\tilde{u}_j) \right)
\end{aligned} \tag{7}$$

The dice similarity measure for  $INHFS$ s

$$\begin{aligned}
\widehat{D}_{SVNHFS}(\tilde{A}, \tilde{B}) = \frac{1}{4m} \sum_{j=1}^m & \frac{2 \left( \begin{aligned} & \left( \sum_{i=1}^{le_j} \bar{\tau}_{A\sigma(i)}^L(\tilde{u}_j) \bar{\tau}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\tau}_{A\sigma(i)}^U(\tilde{u}_j) \bar{\tau}_{B\sigma(i)}^U(\tilde{u}_j) \right) \\ & + \left( \sum_{i=1}^{le_j} \bar{l}_{A\sigma(i)}^L(\tilde{u}_j) \bar{l}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{l}_{A\sigma(i)}^U(\tilde{u}_j) \bar{l}_{B\sigma(i)}^U(\tilde{u}_j) \right) \\ & + \left( \sum_{i=1}^{le_j} \bar{\kappa}_{A\sigma(i)}^L(\tilde{u}_j) \bar{\kappa}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\kappa}_{A\sigma(i)}^U(\tilde{u}_j) \bar{\kappa}_{B\sigma(i)}^U(\tilde{u}_j) \right) \end{aligned} \right)}{\sum_{i=1}^{le_j} [\bar{\tau}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\tau}_{A\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{A\sigma(i)}^L(\tilde{u}_j)]^2 +} \\
& \sum_{i=1}^{le_j} [\bar{l}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{A\sigma(i)}^U(\tilde{u}_j)]^2 + \\
& \sum_{i=1}^{le_j} [\bar{\tau}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\tau}_{B\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \\
& \sum_{i=1}^{le_j} [\bar{l}_{B\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{B\sigma(i)}^U(\tilde{u}_j)]^2}
\end{aligned} \tag{8}$$



The cosine similarity measure for *INHFSs*.

$$\widehat{C}_{SVNHFS}(\tilde{A}, \tilde{B}) = \frac{1}{2m} \sum_{j=1}^m \frac{\sum_{i=1}^{le_j} \tilde{\tau}_{A\sigma(i)}^L(\tilde{u}_j) \tilde{\tau}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \tilde{\tau}_{A\sigma(i)}^U(\tilde{u}_j) \tilde{\tau}_{B\sigma(i)}^U(\tilde{u}_j) + \sum_{i=1}^{le_j} \tilde{l}_{A\sigma(i)}^L(\tilde{u}_j) \tilde{l}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \tilde{l}_{A\sigma(i)}^U(\tilde{u}_j) \tilde{l}_{B\sigma(i)}^U(\tilde{u}_j) + \sum_{i=1}^{le_j} \tilde{\kappa}_{A\sigma(i)}^L(\tilde{u}_j) \tilde{\kappa}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \tilde{\kappa}_{A\sigma(i)}^U(\tilde{u}_j) \tilde{\kappa}_{B\sigma(i)}^U(\tilde{u}_j)}{\left( \sqrt{\frac{\sum_{i=1}^{le_j} [\tilde{\tau}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{\tau}_{A\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{l}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{l}_{A\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{\kappa}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{\kappa}_{A\sigma(i)}^U(\tilde{u}_j)]^2}{\sum_{i=1}^{le_j} [\tilde{\tau}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{\tau}_{B\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{l}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{l}_{B\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{\kappa}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{\kappa}_{B\sigma(i)}^U(\tilde{u}_j)]^2}} \right)} \quad (9)$$

The weighted Jaccard similarity measures for *INHFS*

$$\widehat{WJ}_{SVNHFS}(\tilde{A}, \tilde{B}) = \sum_{j=1}^m \varpi_j \frac{\sum_{i=1}^{le_j} \tilde{\tau}_{A\sigma(i)}^L(\tilde{u}_j) \tilde{\tau}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \tilde{\tau}_{A\sigma(i)}^U(\tilde{u}_j) \tilde{\tau}_{B\sigma(i)}^U(\tilde{u}_j) + \sum_{i=1}^{le_j} \tilde{l}_{A\sigma(i)}^L(\tilde{u}_j) \tilde{l}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \tilde{l}_{A\sigma(i)}^U(\tilde{u}_j) \tilde{l}_{B\sigma(i)}^U(\tilde{u}_j) + \sum_{i=1}^{le_j} \tilde{\kappa}_{A\sigma(i)}^L(\tilde{u}_j) \tilde{\kappa}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \tilde{\kappa}_{A\sigma(i)}^U(\tilde{u}_j) \tilde{\kappa}_{B\sigma(i)}^U(\tilde{u}_j)}{\sum_{i=1}^{le_j} [\tilde{\tau}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{\tau}_{A\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{l}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{l}_{A\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{\kappa}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{\kappa}_{A\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{\tau}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{\tau}_{B\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{l}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{l}_{B\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{\kappa}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\tilde{\kappa}_{B\sigma(i)}^U(\tilde{u}_j)]^2 - \left( \begin{aligned} &\sum_{i=1}^{le_j} \tilde{\tau}_{A\sigma(i)}^L(\tilde{u}_j) \tilde{\tau}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \tilde{\tau}_{A\sigma(i)}^U(\tilde{u}_j) \tilde{\tau}_{B\sigma(i)}^U(\tilde{u}_j) + \\ &\sum_{i=1}^{le_j} \tilde{l}_{A\sigma(i)}^L(\tilde{u}_j) \tilde{l}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \tilde{l}_{A\sigma(i)}^U(\tilde{u}_j) \tilde{l}_{B\sigma(i)}^U(\tilde{u}_j) + \\ &\sum_{i=1}^{le_j} \tilde{\kappa}_{A\sigma(i)}^L(\tilde{u}_j) \tilde{\kappa}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \tilde{\kappa}_{A\sigma(i)}^U(\tilde{u}_j) \tilde{\kappa}_{B\sigma(i)}^U(\tilde{u}_j) \end{aligned} \right)} \quad (10)$$

The weighted Dice similarity measure

$$\widehat{WD}_{SV\widehat{NHFS}}(\tilde{A}, \tilde{B}) = \sum_{j=1}^m \varpi_j \frac{\left( \begin{aligned} & \left( \sum_{i=1}^{le_j} \bar{\tau}_{A\sigma(i)}^L(\tilde{u}_j) \bar{\tau}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\tau}_{A\sigma(i)}^U(\tilde{u}_j) \bar{\tau}_{B\sigma(i)}^U(\tilde{u}_j) \right) \\ & + \left( \sum_{i=1}^{le_j} \bar{l}_{A\sigma(i)}^L(\tilde{u}_j) \bar{l}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{l}_{A\sigma(i)}^U(\tilde{u}_j) \bar{l}_{B\sigma(i)}^U(\tilde{u}_j) \right) \\ & + \left( \sum_{i=1}^{le_j} \bar{\kappa}_{A\sigma(i)}^L(\tilde{u}_j) \bar{\kappa}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\kappa}_{A\sigma(i)}^U(\tilde{u}_j) \bar{\kappa}_{B\sigma(i)}^U(\tilde{u}_j) \right) \end{aligned} \right)}{\begin{aligned} & \sum_{i=1}^{le_j} [\bar{\tau}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\tau}_{A\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \\ & \sum_{i=1}^{le_j} [\bar{l}_{A\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{A\sigma(i)}^U(\tilde{u}_j)]^2 + \\ & \sum_{i=1}^{le_j} [\bar{\tau}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\tau}_{B\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \\ & \sum_{i=1}^{le_j} [\bar{l}_{B\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{B\sigma(i)}^U(\tilde{u}_j)]^2 \end{aligned}} \quad (11)$$

The weighted Cosine measure

$$\widehat{C}_{SV\widehat{NHFS}}(\tilde{A}, \tilde{B}) = \sum_{j=1}^m \varpi_j \frac{\sum_{i=1}^{le_j} \bar{\tau}_{A\sigma(i)}^L(\tilde{u}_j) \bar{\tau}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\tau}_{A\sigma(i)}^U(\tilde{u}_j) \bar{\tau}_{B\sigma(i)}^U(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{l}_{A\sigma(i)}^L(\tilde{u}_j) \bar{l}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{l}_{A\sigma(i)}^U(\tilde{u}_j) \bar{l}_{B\sigma(i)}^U(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\kappa}_{A\sigma(i)}^L(\tilde{u}_j) \bar{\kappa}_{B\sigma(i)}^L(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\kappa}_{A\sigma(i)}^U(\tilde{u}_j) \bar{\kappa}_{B\sigma(i)}^U(\tilde{u}_j)}{\left( \sqrt{\begin{aligned} & \sum_{i=1}^{le_j} [\bar{\tau}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\tau}_{A\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \\ & \sum_{i=1}^{le_j} [\bar{l}_{A\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{A\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{A\sigma(i)}^U(\tilde{u}_j)]^2 \end{aligned}} + \sqrt{\begin{aligned} & \sum_{i=1}^{le_j} [\bar{\tau}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\tau}_{B\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{l}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \\ & \sum_{i=1}^{le_j} [\bar{l}_{B\sigma(i)}^U(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{B\sigma(i)}^L(\tilde{u}_j)]^2 + \sum_{i=1}^{le_j} [\bar{\kappa}_{B\sigma(i)}^U(\tilde{u}_j)]^2 \end{aligned}} \right)} \quad (12)$$

#### 4. MULTI ATTRIBUTE DECISION MAKING WITH SINGLE VALUED NEUTROSOPHIC HESITANT INFORMATION

In this section we present a multi-criteria decision making method adapted from Ye[46] to utilize the vector similarity measures for  $SV\widehat{NHFS}$ s and  $IN\widehat{HFS}$ s with  $SVNH\widehat{F}$  and  $IN\widehat{HF}$  information respectively.

For a  $\widehat{MCDM}$  problem with  $SV\widehat{NH\widehat{F}}$  and  $IN\widehat{HF}$  information respectively, let  $\check{A} = \{\check{A}_1, \check{A}_2, \dots, \check{A}_n\}$  be a set of alternatives and  $\check{C} = \{\check{C}_1, \check{C}_2, \dots, \check{C}_m\}$  be a set of attributes. If for the alternatives  $\check{A}_j$  (for  $j = 1, 2, \dots, n$ ) the decision makers provide several values under the criteria  $\check{C}_i$  (for  $i = 1, 2, \dots, m$ ), then each value is considered as a  $SV\widehat{NH\widehat{F}}$  element and  $IN\widehat{HF}$  element  $(\bar{\tau}_{ji}, \bar{l}_{ji}, \bar{\kappa}_{ji})$  (for  $j = 1, 2, \dots, n, i = 1, 2, \dots, m$ ).

Therefore we can stimulate a  $\widehat{SVNHF}$  and  $\widehat{INHF}$  decision matrix respectively  $\tilde{D} = (\bar{\tau}_{ji}, \bar{l}_{ji}, \bar{\kappa}_{ji})_{n \times m}$ , where  $(\bar{\tau}_{ji}, \bar{l}_{ji}, \bar{\kappa}_{ji})$  (for  $j = 1, 2, \dots, n, i = 1, 2, \dots, m$ ) is in the form of  $\widehat{SVNHF}$  elements or  $\widehat{INHF}$  elements.

For the selection of best alternatives in multiple-criteria decision making environments, the concept of ideal point has been used in the decision set. Although in real world the ideal point does not exist. The ideal point deliver a suitable theoretical hypothesis to evaluate alternatives. Therefore we define each value in each ideal  $\widehat{SVNHF}$  or  $\widehat{INHF}$  element  $(\bar{\tau}_j^*, \bar{l}_j^*, \bar{\kappa}_j^*)$  for the ideal alternative  $\check{A}^* = \{(\check{C}_i, (\bar{\tau}_j^*, \bar{l}_j^*, \bar{\kappa}_j^*)) | \check{C}_i \in \check{C}\}$  as  $\bar{\tau}_{j\sigma(k)}^* = 1, \bar{l}_{j\sigma(k)}^* = 0, \bar{\kappa}_{j\sigma(k)}^* = 0$ , or  $\bar{\tau}_{j\sigma(k)}^* = [1, 1], \bar{l}_{j\sigma(k)}^* = [0, 0], \bar{\kappa}_{j\sigma(k)}^* = [0, 0]$ , for  $k = 1, 2, \dots, l_{e_i}$ , where  $l_{e_i}$  is the number of values or interval values in  $(\bar{\tau}_{ji}, \bar{l}_{ji}, \bar{\kappa}_{ji})$  (for  $j = 1, 2, \dots, n, i = 1, 2, \dots, m$ ).

For the different importance of each criteria the weighting vector of criteria is given as  $\tilde{\omega} = (\omega_1, \omega_2, \dots, \omega_m)^T$ , where  $\omega_i \geq 0, i = 1, 2, \dots, m$ , and  $\sum_{i=1}^m \omega_i = 1$ .

Then we exploit the three weighted vector similarity measures for  $\widehat{SVNHF}$ s or  $\widehat{INHF}$ s for  $\widehat{MCDM}$  under  $\widehat{SVNHF}$  or  $\widehat{INHF}$  information, which can be defined as follows:

**Step 1.** Calculate one of the three vector similarity measures between the alternative  $\check{A}_j$  (for  $j = 1, 2, \dots, n$ ) and the ideal alternative  $\check{A}^*$  by using one of the three formulas:

$$\widehat{\omega J}_{\widehat{SVNHF}}(\check{A}_j, \check{A}^*) = \sum_{j=1}^m \omega_i \frac{\sum_{i=1}^{l_{e_j}} \bar{\tau}_{\check{A}\sigma(i)}(\tilde{u}_j) \bar{\tau}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j) + \sum_{i=1}^{l_{e_j}} \bar{l}_{\check{A}\sigma(i)}(\tilde{u}_j) \bar{l}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j) + \sum_{i=1}^{l_{e_j}} \bar{\kappa}_{\check{A}\sigma(i)}(\tilde{u}_j) \bar{\kappa}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j)}{\sum_{i=1}^{l_{e_j}} [\bar{\tau}_{\check{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{l_{e_j}} [\bar{l}_{\check{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{l_{e_j}} [\bar{\kappa}_{\check{A}\sigma(i)}(\tilde{u}_j)]^2 + \sum_{i=1}^{l_{e_j}} [\bar{\tau}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j)]^2 + \sum_{i=1}^{l_{e_j}} [\bar{l}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j)]^2 + \sum_{i=1}^{l_{e_j}} [\bar{\kappa}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j)]^2 - \left( \sum_{i=1}^{l_{e_j}} \bar{\tau}_{\check{A}\sigma(i)}(\tilde{u}_j) \bar{\tau}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j) + \sum_{i=1}^{l_{e_j}} \bar{l}_{\check{A}\sigma(i)}(\tilde{u}_j) \bar{l}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j) + \sum_{i=1}^{l_{e_j}} \bar{\kappa}_{\check{A}\sigma(i)}(\tilde{u}_j) \bar{\kappa}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j) \right)} \quad (13)$$

$$\widehat{\varpi D}(\check{A}_j, \check{A}^*) = \sum_{j=1}^m \varpi_i \frac{2 \left( \sum_{i=1}^{le_j} \bar{\tau}_{\check{A}\sigma(i)}(\tilde{u}_j) \bar{\tau}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{l}_{\check{A}\sigma(i)}(\tilde{u}_j) \bar{l}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\kappa}_{\check{A}\sigma(i)}(\tilde{u}_j) \bar{\kappa}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j) \right)}{\sum_{i=1}^{le_j} \left[ \bar{\tau}_{\check{A}\sigma(i)}(\tilde{u}_j) \right]^2 + \sum_{i=1}^{le_j} \left[ \bar{l}_{\check{A}\sigma(i)}(\tilde{u}_j) \right]^2 + \sum_{i=1}^{le_j} \left[ \bar{\kappa}_{\check{A}\sigma(i)}(\tilde{u}_j) \right]^2 + \sum_{i=1}^{le_j} \left[ \bar{\tau}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j) \right]^2 + \sum_{i=1}^{le_j} \left[ \bar{l}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j) \right]^2 + \sum_{i=1}^{le_j} \left[ \bar{\kappa}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j) \right]^2} \quad (14)$$

$$\widehat{\varpi C}(\check{A}_j, \check{A}^*) = \sum_{j=1}^m \varpi_i \frac{\sum_{i=1}^{le_j} \bar{\tau}_{\check{A}\sigma(i)}(\tilde{u}_j) \bar{\tau}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{l}_{\check{A}\sigma(i)}(\tilde{u}_j) \bar{l}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j) + \sum_{i=1}^{le_j} \bar{\kappa}_{\check{A}\sigma(i)}(\tilde{u}_j) \bar{\kappa}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j)}{\sqrt{\sum_{i=1}^{le_j} \left[ \bar{\tau}_{\check{A}\sigma(i)}(\tilde{u}_j) \right]^2 + \sum_{i=1}^{le_j} \left[ \bar{l}_{\check{A}\sigma(i)}(\tilde{u}_j) \right]^2 + \sum_{i=1}^{le_j} \left[ \bar{\kappa}_{\check{A}\sigma(i)}(\tilde{u}_j) \right]^2} \sqrt{\sum_{i=1}^{le_j} \left[ \bar{\tau}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j) \right]^2 + \sum_{i=1}^{le_j} \left[ \bar{l}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j) \right]^2 + \sum_{i=1}^{le_j} \left[ \bar{\kappa}_{\check{A}^*\sigma(i)}^*(\tilde{u}_j) \right]^2} \quad (15)$$

We use the following formulas under the  $\widehat{SVNH F}$  information.  
Under  $\widehat{INHF}$  information, we use the following formulas:



$$\begin{aligned}
 \widehat{\varpi C}_{\widehat{INHFS}}(\tilde{A}, \tilde{B}) = & \sum_{j=1}^m \varpi_j \frac{\sum_{i=1}^{le_j} \tilde{\tau}_{\tilde{A}\sigma(i)}^L(\tilde{u}_j) \tilde{\tau}_{\tilde{A}^*\sigma(i)}^{*L}(\tilde{u}_j) + \sum_{i=1}^{le_j} \tilde{\tau}_{\tilde{A}\sigma(i)}^U(\tilde{u}_j) \tilde{\tau}_{\tilde{A}^*\sigma(i)}^{*U}(\tilde{u}_j) + \\
 & 2 \sum_{i=1}^{le_j} \tilde{l}_{\tilde{A}\sigma(i)}^L(\tilde{u}_j) \tilde{l}_{\tilde{A}^*\sigma(i)}^{*L}(\tilde{u}_j) + \sum_{i=1}^{le_j} \tilde{l}_{\tilde{A}\sigma(i)}^U(\tilde{u}_j) \tilde{l}_{\tilde{A}^*\sigma(i)}^{*U}(\tilde{u}_j) \\
 & + \sum_{i=1}^{le_j} \tilde{\kappa}_{\tilde{A}\sigma(i)}^L(\tilde{u}_j) \tilde{\kappa}_{\tilde{A}^*\sigma(i)}^{*L}(\tilde{u}_j) + \sum_{i=1}^{le_j} \tilde{\kappa}_{\tilde{A}\sigma(i)}^U(\tilde{u}_j) \tilde{\kappa}_{\tilde{A}^*\sigma(i)}^{*U}(\tilde{u}_j)}{\sqrt{\left[ \begin{aligned} & \sum_{i=1}^{le_j} \left[ \tilde{\tau}_{\tilde{A}\sigma(i)}^L(\tilde{u}_j) \right]^2 + \sum_{i=1}^{le_j} \left[ \tilde{\tau}_{\tilde{A}\sigma(i)}^U(\tilde{u}_j) \right]^2 + \\ & \sum_{i=1}^{le_j} \left[ \tilde{l}_{\tilde{A}\sigma(i)}^L(\tilde{u}_j) \right]^2 + \sum_{i=1}^{le_j} \left[ \tilde{l}_{\tilde{A}\sigma(i)}^U(\tilde{u}_j) \right]^2 \\ & + \sum_{i=1}^{le_j} \left[ \tilde{\kappa}_{\tilde{A}\sigma(i)}^L(\tilde{u}_j) \right]^2 + \sum_{i=1}^{le_j} \left[ \tilde{\kappa}_{\tilde{A}\sigma(i)}^U(\tilde{u}_j) \right]^2 \end{aligned} \right]} \sqrt{\left[ \begin{aligned} & \sum_{i=1}^{le_j} \left[ \tilde{\tau}_{\tilde{A}^*\sigma(i)}^{*L}(\tilde{u}_j) \right]^2 + \sum_{i=1}^{le_j} \left[ \tilde{\tau}_{\tilde{A}^*\sigma(i)}^{*U}(\tilde{u}_j) \right]^2 \\ & + \sum_{i=1}^{le_j} \left[ \tilde{l}_{\tilde{A}^*\sigma(i)}^{*L}(\tilde{u}_j) \right]^2 + \sum_{i=1}^{le_j} \left[ \tilde{l}_{\tilde{A}^*\sigma(i)}^{*U}(\tilde{u}_j) \right]^2 \\ & + \sum_{i=1}^{le_j} \left[ \tilde{\kappa}_{\tilde{A}^*\sigma(i)}^{*L}(\tilde{u}_j) \right]^2 + \sum_{i=1}^{le_j} \left[ \tilde{\kappa}_{\tilde{B}\sigma(i)}^{*U}(\tilde{u}_j) \right]^2 \end{aligned} \right]} \quad (18)
 \end{aligned}$$

4.1. **Practical example.** This example is adopted from Ye [46] is used as the demonstration of the effectiveness of the proposed decision making method in real life problem.

**For SVNH information**

There is an investment company, which wants to invest some money in the best option. There is a panel with four possible alternatives to invest the money:

- (1)  $A_1$  is a car company
  - (2)  $A_2$  is a food company
  - (3)  $A_3$  is a computers company
  - (4)  $A_4$  is an arms company.
- The following three criterions on which the investment company must take decision are (1)  $C_1$  is the risk, (2)  $C_2$  is the growth, (3)  $C_3$  is the environmental impact. The weight vector of the criteria is given as  $(0.35, 0.25, 0.4)^T$  the four possible alternative

The decision matrix for attributes and alternative the data is represented by  $\widehat{SVNHFN}$

$$\check{D} = \begin{bmatrix} & \tilde{C}_1 & \tilde{C}_2 & \tilde{C}_3 \\ \tilde{A}_1 & \left\{ \begin{array}{l} \{0.3, 0.4\}, \{0.2, 0.3\}, \\ \{0.5, 0.6\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.4, 0.5\}, \{0.1, 0.3\}, \\ \{0.4, 0.6\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.1, 0.2\}, \{0.3, 0.4\}, \\ \{0.6, 0.7\} \end{array} \right\} \\ \tilde{A}_2 & \left\{ \begin{array}{l} \{0.9, 1\}, \{0.05, 0.07\}, \\ \{0.01, 0.02\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.8, 0.9\}, \{0.1, 0.2\}, \\ \{0.1, 0.2\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.7, 0.9\}, \{0.1, 0.2\}, \\ \{0.01, 0.02\} \end{array} \right\} \\ \tilde{A}_3 & \left\{ \begin{array}{l} \{0.5, 0.6\}, \{0.2, 0.3\}, \\ \{0.1, 0.2\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.5, 0.7\}, \{0.1, 0.2\}, \\ \{0.2, 0.3\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.5, 0.7\}, \{0.1, 0.2\}, \\ \{0.2, 0.3\} \end{array} \right\} \\ \tilde{A}_4 & \left\{ \begin{array}{l} \{0.9, 1\}, \{0.1, 0.2\}, \\ \{0.15, 0.25\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.8, 0.9\}, \{0.1, 0.2\}, \\ \{0.1, 0.2\} \end{array} \right\} & \left\{ \begin{array}{l} \{0.4, 0.5\}, \{0.2, 0.4\}, \\ \{0.3, 0.5\} \end{array} \right\} \end{bmatrix}$$

**Table 1:** The values obtained through the defined weighted vector similarity measures for  $\widehat{SVNHFS}$

	$WJ_{SVNHFS}(\tilde{A}^*, \hat{A}_k)$	$WD_{\widehat{SVNHFS}}(\tilde{A}^*, \hat{A}_k)$	$WC_{\widehat{SVNHFS}}(\tilde{A}^*, \hat{A}_k)$
$\tilde{A}_1$	0.2254	0.3384	0.4099
$\tilde{A}_2$	0.9420	0.9762	0.8752
$\tilde{A}_3$	0.68125	0.81120	0.8812
$\tilde{A}_4$	0.73040	0.8205	0.8395
Ranking order	$\tilde{A}_2 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_1$		

The values obtained in table 1 shows that the Jaccard and Dice similarity measures have the same result that is  $\tilde{A}_2$  is the best alternative while the Cosine similarity measure shows that  $\tilde{A}_3$  is the best alternative. From table 1 we conclude that the Jaccard and Dice similarity measure are better to use in multi-criteria decision making.

**For INS information**

Now we consider the above example for  $INHFSs$ ,

The decision matrix for attributes and alternative the data is represented by  $\widehat{INHFN}$

$$\check{D} = \begin{bmatrix} & \tilde{C}_1 & \tilde{C}_2 & \tilde{C}_3 \\ \tilde{A}_1 & \left\{ \begin{array}{l} \{[0.3, 0.4], [0.4, 0.5]\}, \\ \{[0.2, 0.3], [0.3, 0.4]\}, \\ \{[0.5, 0.6], [0.6, 0.7]\} \end{array} \right\} & \left\{ \begin{array}{l} \{[0.4, 0.5], [0.5, 0.6]\}, \\ \{[0.1, 0.2], [0.3, 0.4]\}, \\ \{[0.4, 0.5], [0.5, 0.6]\} \end{array} \right\} & \left\{ \begin{array}{l} \{[0.1, 0.2], [0.2, 0.3]\}, \\ \{[0.3, 0.4], [0.4, 0.5]\}, \\ \{[0.6, 0.7], [0.7, 0.8]\} \end{array} \right\} \\ \tilde{A}_2 & \left\{ \begin{array}{l} \{[0.8, 0.9], [0.9, 1]\}, \\ \{[0.01, 0.02], [0.02, 0.03]\}, \\ \{[0.01, 0.02], [0.02, 0.03]\} \end{array} \right\} & \left\{ \begin{array}{l} \{[0.8, 0.9], [0.9, 1]\}, \\ \{[0.1, 0.2], [0.2, 0.3]\}, \\ \{[0.1, 0.2], [0.1, 0.2]\} \end{array} \right\} & \left\{ \begin{array}{l} \{[0.7, 0.8], [0.9, 1]\}, \\ \{[0.05, 0.15], [0.15, 0.25]\}, \\ \{[0.01, 0.02], [0.05, 0.09]\} \end{array} \right\} \\ \tilde{A}_3 & \left\{ \begin{array}{l} \{[0.5, 0.6], [0.6, 0.7]\}, \\ \{[0.2, 0.3], [0.3, 0.4]\}, \\ \{[0.1, 0.2], [0.2, 0.3]\} \end{array} \right\} & \left\{ \begin{array}{l} \{[0.4, 0.5], [0.5, 0.7]\}, \\ \{[0.1, 0.2], [0.2, 0.3]\}, \\ \{[0.2, 0.3], [0.3, 0.4]\} \end{array} \right\} & \left\{ \begin{array}{l} \{[0.4, 0.5], [0.5, 0.7]\}, \\ \{[0.1, 0.2], [0.2, 0.3]\}, \\ \{[0.2, 0.3], [0.3, 0.4]\} \end{array} \right\} \\ \tilde{A}_4 & \left\{ \begin{array}{l} \{[0.8, 0.9], [0.9, 1]\}, \\ \{[0.1, 0.2], [0.2, 0.3]\}, \\ \{[0.15, 0.25], [0.25, 0.35]\} \end{array} \right\} & \left\{ \begin{array}{l} \{[0.7, 0.8], [0.9, 1]\}, \\ \{[0.1, 0.2], [0.2, 0.3]\}, \\ \{[0.1, 0.2], [0.2, 0.3]\} \end{array} \right\} & \left\{ \begin{array}{l} \{[0.4, 0.5], [0.5, 0.6]\}, \\ \{[0.2, 0.3], [0.3, 0.5]\}, \\ \{[0.3, 0.4], [0.4, 0.5]\} \end{array} \right\} \end{bmatrix}$$

**Table 2:** The values obtained through using weighted vector similarity measures for  $\widehat{INHFN}$

	$WJ_{SVNHFS}()$	$WD_{SVNHFS}()$	$WC_{SVNHFS}()$
$\tilde{A}_1$	0.2825	0.4238	0.4357
$\tilde{A}_2$	0.9437	0.9708	0.9765
$\tilde{A}_3$	0.6100	0.7568	0.7297
$\tilde{A}_4$	0.7171	0.8197	0.8337
Ranking order	$\tilde{A}_2 \succ \tilde{A}_4 \succ \tilde{A}_3 \succ \tilde{A}_1$		

The values obtained from table 2 shows that the best alternatives is  $\tilde{A}_2$ .

5. PATTERN RECOGNITION

In this section we applied the above defined vector similarity measures to pattern recognition. The method use here is adapted from Ye[36].

**Example 5.1.** We are given three sample pattern  $\tilde{A}_1, \tilde{A}_2$ , and  $\tilde{A}_3$  respectively which are represented by the following  $\widehat{SVNHFS}$ s in the finite universe  $\tilde{U} = \{a_1, a_2, a_3\}$ .

We are also given an unknown pattern  $\check{Q}$ . Our aim is to classify  $\check{Q}$  in one of the given pattern. The principle of recognizing pattern is that of the maximum degree of similarity between  $\widehat{SVNHFS}$ s is described by

$$N = \arg \text{Max}_{1 \leq k \leq 3} \left\{ \widetilde{C}_{\widehat{SVNHFS}}(\tilde{A}_k, \check{Q}) \right\}$$

$$\begin{aligned} \tilde{A}_1 &= \{ \langle a_1, \{0.6, 0.7\}, \{0.2, 0.3\}, \{0.3, 0.4\} \rangle, \langle a_2, \{0.4, 0.5\}, \{0.3, 0.4\}, \{0.4, 0.5\} \rangle, \\ &\quad \langle a_3, \{0.3, 0.4\}, \{0.2, 0.4\}, \{0.5, 0.7\} \rangle \} \end{aligned}$$

$$\begin{aligned} \tilde{A}_2 &= \{ \langle a_1, \{0.4, 0.5\}, \{0.3, 0.5\}, \{0.5, 0.6\} \rangle, \langle a_2, \{0.6, 0.7\}, \{0.2, 0.3\}, \{0.3, 0.4\} \rangle, \\ &\quad \langle a_3, \{0.5, 0.6\}, \{0.2, 0.3\}, \{0.2, 0.4\} \rangle \} \end{aligned}$$

$$\begin{aligned} \tilde{A}_3 &= \{ \langle a_1, \{0.2, 0.3\}, \{0.3, 0.4\}, \{0.6, 0.7\} \rangle, \langle a_2, \{0.4, 0.5\}, \{0.3, 0.4\}, \{0.4, 0.5\} \rangle, \\ &\quad \langle a_3, \{0.6, 0.7\}, \{0.2, 0.3\}, \{0.3, 0.4\} \rangle \} \end{aligned}$$

$$\begin{aligned} \check{Q} &= \{ \langle a_1, \{0.1, 0.2\}, \{0.3, 0.4\}, \{0.7, 0.8\} \rangle, \langle a_2, \{0.2, 0.3\}, \{0.3, 0.4\}, \{0.6, 0.7\} \rangle, \\ &\quad \langle a_3, \{0.4, 0.5\}, \{0.3, 0.5\}, \{0.5, 0.6\} \rangle \} \end{aligned}$$

$$(\tilde{A}_1, \check{Q}) = 0.3501, (\tilde{A}_2, \check{Q}) = 0.4317, (\tilde{A}_3, \check{Q}) = \mathbf{0.4723}$$

One can observe that the pattern should be classified in  $\tilde{A}_3$  according to the pattern recognition principle. This result is the same obtained by Ye[36].

For  $\widehat{INHFS}$

**Example 5.2.** For  $\widehat{INHFS}$  we use the same example defined above just the patterns are represented by  $\widehat{INHFS}$ s.

$$\begin{aligned} \tilde{A}_1 &= \{ \langle a_1, \{[0.4, 0.5], [0.6, 0.7]\}, \{[0.1, 0.2], [0.2, 0.3]\}, \{[0.2, 0.3], [0.3, 0.4]\} \rangle, \\ &\quad \langle a_2, \{[0.3, 0.4], [0.4, 0.5]\}, \{[0.3, 0.4], [0.3, 0.4]\}, \{[0.3, 0.4], [0.4, 0.5]\} \rangle, \\ &\quad \langle a_3, \{[0.2, 0.3], [0.3, 0.4]\}, \{[0.1, 0.3], [0.2, 0.4]\}, \{[0.4, 0.6], [0.5, 0.7]\} \rangle \} \end{aligned}$$

$$\begin{aligned} \tilde{A}_2 &= \{ \langle a_1, \{[0.3, 0.4], [0.4, 0.5]\}, \{[0.2, 0.3], [0.4, 0.5]\}, \{[0.3, 0.5], [0.4, 0.6]\} \rangle, \\ &\quad \langle a_2, \{[0.4, 0.6], [0.5, 0.7]\}, \{[0.2, 0.3], [0.3, 0.4]\}, \{[0.3, 0.4], [0.4, 0.5]\} \rangle, \\ &\quad \langle a_3, \{[0.4, 0.5], [0.6, 0.7]\}, \{[0.1, 0.2], [0.2, 0.3]\}, \{[0.1, 0.3], [0.2, 0.4]\} \rangle \} \end{aligned}$$

$$\begin{aligned} \tilde{A}_3 &= \{ \langle a_1, \{[0.1, 0.2], [0.2, 0.3]\}, \{[0.2, 0.3], [0.3, 0.4]\}, \{[0.5, 0.6], [0.6, 0.7]\} \rangle, \\ &\quad \langle a_2, \{[0.3, 0.4], [0.4, 0.5]\}, \{[0.2, 0.3], [0.3, 0.4]\}, \{[0.3, 0.4], [0.4, 0.5]\} \rangle, \\ &\quad \langle a_3, \{[0.5, 0.6], [0.6, 0.7]\}, \{[0.1, 0.2], [0.2, 0.3]\}, \{[0.2, 0.3], [0.3, 0.4]\} \rangle \} \end{aligned}$$

$$\begin{aligned} \check{Q} &= \{ \langle a_1, \{[0.1, 0.3], [0.2, 0.4]\}, \{[0.2, 0.3], [0.3, 0.4]\}, \{[0.6, 0.7], [0.7, 0.8]\} \rangle, \\ &\quad \langle a_2, \{[0.1, 0.2], [0.2, 0.3]\}, \{[0.2, 0.4], [0.4, 0.5]\}, \{[0.5, 0.7], [0.6, 0.8]\} \rangle, \\ &\quad \langle a_3, \{[0.2, 0.4], [0.4, 0.6]\}, \{[0.3, 0.4], [0.4, 0.5]\}, \{[0.4, 0.5], [0.5, 0.6]\} \rangle \} \end{aligned}$$

$$(\tilde{A}_1, \check{Q}) = 0.1954, (\tilde{A}_2, \check{Q}) = 0.1960, (\tilde{A}_3, \check{Q}) = \mathbf{0.2150}$$

One can observe that the pattern should be classified in  $\tilde{A}_3$  according to the pattern recognition principle. This result is the same obtained by Ye[36].



## 6. DISCUSSION

As simplified neutrosophic hesitant fuzzy set is an important extension of the Fuzzy set ( $\widetilde{FS}$ ), Intuitionistic fuzzy set ( $\widetilde{IFS}$ ), Single valued neutrosophic set ( $\widetilde{SVNS}$ ), interval neutrosophic set ( $\widetilde{INS}$ ), Hesitant fuzzy set ( $\widetilde{HFS}$ ) and Dual hesitant fuzzy set ( $\widetilde{DHFS}$ ), single valued neutrosophic hesitant fuzzy set ( $\widetilde{SVNHFS}$ ), interval neutrosophic hesitant fuzzy set ( $\widetilde{INHFS}$ ). As mentioned above that hesitancy is the most common problem in decision making for which  $\widetilde{HFS}$  is a suitable means by allowing several possible values for an element to a set. However in  $\widetilde{HFS}$  they consider only one truth-membership function and it cannot express this problem with a few different values assigned by truth-membership hesitant degree, indeterminacy-membership degree, and falsity-membership degrees due to doubts of decision makers. and also in  $\widetilde{DHFS}$  they consider two functions that is membership and non-membership functions and can not consider indeterminacy-membership function. Therefore simplified neutrosophic sets is the more suitable for decision making and can handle incomplete, inconsistency and indeterminate information which occur in decision making. The other advantage of the simplified neutrosophic set is that it contains the concept of single valued neutrosophic hesitant fuzzy sets and as well as interval neutrosophic hesitant fuzzy sets. As comparing to other sets the simplified neutrosophic hesitant fuzzy set contains more information and the decision makers can get more information to take their decision.

## 7. CONCLUSION

As simplified neutrosophic hesitant fuzzy set ( $\widetilde{SNHFS}$ ) is a new extension which consists of the concept of Fuzzy set ( $\widetilde{FS}$ ), Intuitionistic fuzzy set ( $\widetilde{IFS}$ ), Single valued neutrosophic set ( $\widetilde{SVNS}$ ), interval neutrosophic set ( $\widetilde{INS}$ ), Hesitant fuzzy set ( $\widetilde{HFS}$ ) and Dual hesitant fuzzy set ( $\widetilde{DHFS}$ ), single valued neutrosophic set ( $\widetilde{SVNHFS}$ ), interval neutrosophic hesitant fuzzy set ( $\widetilde{INHFS}$ ). In this article we defined vector similarity measures for simplified neutrosophic hesitant fuzzy set and applied them to multi-criteria decision making and pattern recognition. We also observe that the Jaccard and Dice similarity measures are better to apply in multi-criteria decision making problem than the Cosine similarity measure. In future we shall apply the above distance measures to medical diagnosis and also defined for Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset.

**Compliance with Ethical Structure**

We declare that we have no commercial or associative interest that represents a conflict of interest in connection with work submitted.

**Acknowledgments.** The authors would like to thank the anonymous referee for his/her comments that helped us improve this article.

## REFERENCES

- [1] K. T. Atanassov, "Intuitionistic fuzzy sets." *Fuzzy sets and Systems* 20.1 (1986): 87-96.
- [2] K. T. Atanassov, and Georgi Gargov. "Interval valued intuitionistic fuzzy sets." *Fuzzy sets and systems* 31.3 (1989): 343-349.
- [3] S. Broumi and F. Smarandache. "Several similarity measures of neutrosophic sets." *Neutrosophic Sets and Systems* 1.1 (2013): 54-62.

- [4] S. Broumi & F. Smarandache (2013). Correlation coefficient of interval neutrosophic set. In *Applied Mechanics and Materials* Vol. 436, pp. 511-517.
- [5] S. Broumi & F. Smarandache Cosine similarity measure of interval valued neutrosophic sets. *Neutrosophic Theory and Its Applications.*, 74
- [6] S. Broumi, M. Talea, A. Bakali, F. Smarandache, "Single Valued Neutrosophic Graphs," *Journal of New Theory*, N 10, 2016, pp. 86-101.
- [7] S. Broumi, M. Talea, A. Bakali, F. Smarandache, "On Bipolar Single Valued Neutrosophic Graphs," *Journal of New Theory*, N11, 2016, pp.84-102.
- [8] S. Broumi, A. Bakali, M. Talea, and F. Smarandache, Isolated Single Valued Neutrosophic Graphs. *Neutrosophic Sets and Systems*, Vol. 11, 2016, pp.74-78
- [9] S. Broumi, F. Smarandache, M. Talea and A. Bakali, An Introduction to Bipolar Single Valued Neutrosophic Graph Theory. *Applied Mechanics and Materials*, vol.841,2016, 184-191.
- [10] S. Broumi, M. Talea, F. Smarandache and A. Bakali, Single Valued Neutrosophic Graphs: Degree, Order and Size, *IEEE World Congress on Computational Intelligence*, 2016, pp.2444-2451.
- [11] S. Broumi, F. Smarandache, M. Talea and A. Bakali, An Introduction to Bipolar Single Valued Neutrosophic Graph Theory. *Applied Mechanics and Materials*, vol.841,2016, 184-191.
- [12] N. Chen, Shu-wei, and Li-na CAI. "Interval-valued Hesitant Fuzzy Sets." *Fuzzy Systems and Mathematics* 6 (2013): 007.
- [13] N. Chen, Z. Xu, & M. Xia, (2013). Correlation coefficients of hesitant fuzzy sets and their applications to clustering analysis. *Applied Mathematical Modelling*, 37(4), 2197-2211.
- [14] L. R. Dice, (1945). Measures of the amount of ecologic association between species. *Ecology*, 26(3), 297-302.
- [15] B. Farhadinia, (2013). Information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets. *Information Sciences*, 240, 129-144.
- [16] B. Farhadinia, (2014). Distance and similarity measures for higher order hesitant fuzzy sets. *Knowledge-Based Systems*, 55, 43-48.
- [17] P. Jaccard (1901), Distribution de la flore alpine dans le Bassin des Drouces et dans quelques regions voisines; *Bulletin de la Société Vaudoisedes Sciences Naturelles*, volume 37, no. 140, 241–272;
- [18] P. J. Juan, J. Q. Wang, J. Wang, H. Zhang, and X. Chen. "Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems." *International Journal of Systems Science ahead-of-print* (2014): 1-17
- [19] D. S. Kumar, R. Biswas, and A. R. Roy. "An application of intuitionistic fuzzy sets in medical diagnosis." *Fuzzy sets and Systems* 117.2 (2001): 209-213.
- [20] D. Li, and C. Chuntian. "New similarity measures of intuitionistic fuzzy sets and application to pattern recognitions." *Pattern Recognition Letters* 23.1 (2002): 221-225.
- [21] P. Majumdar & S. K. Samanta (2014). On similarity and entropy of neutrosophic sets. *Journal of Intelligent and fuzzy Systems*, 26(3), 1245-1252.
- [22] L. Peide, and L. Shi. "The generalized hybrid weighted average operator based on interval neutrosophic hesitant set and its application to multiple attribute decision making." *Neural Computing and Applications* 26.2 (2015): 457-471.
- [23] G. Salton, & M. J. McGill (1986). Introduction to modern information retrieval.
- [24] R. Sahin & A. Kucuk (2014). Subsethood measure for single valued neutrosophic sets. *Journal of Intelligent and Fuzzy Systems*.
- [25] F. Smarandache, *Neutrosophy. Neutrosophic Probability, Set, and Logic*, Amer. Res. Press, Rehoboth, USA, p.7-8, 1998.
- [26] F. Smarandache "A Unifying Field in Logics: Neutrosophic Logic." *Philosophy* (1999): 1-141.
- [27] Z. Su, Z. Xu, H. Liu, & S. Liu, (2015). Distance and similarity measures for dual hesitant fuzzy sets and their applications in pattern recognition. *Journal of Intelligent & Fuzzy Systems*, 29(2), 731-745.
- [28] V. Torra & Y. Narukawa (2009). On hesitant fuzzy sets and decision. In *Fuzzy Systems*, 2009. FUZZ-IEEE 2009. IEEE International Conference on pp. 1378-1382.
- [29] V. Torra (2010). Hesitant fuzzy sets. *International Journal of Intelligent Systems*, 25(6), 529-539.
- [30] H. Wang, F. Smarandache, Y.-Q. Zhang, R. Sunderraman; (2010) "Single valued neutrosophic sets." *Review P. 10*.

- [31] H. Wang, F. Smarandache, Y.-Q. Zhang, R. Sunderraman (2005); Interval Neutrosophic Sets and Logic: Theory and Applications in Computing, Hexis, p.87.
- [32] M. Xia, and Z. Xu. (2011) "[Hesitant fuzzy information aggregation in decision making.](#)" *International Journal of Approximate Reasoning* 52.3 395-407
- [33] M. Xia,Z. Xu and N. Chen. "[Some hesitant fuzzy aggregation operators with their application in group decision making.](#)" *Group Decision and Negotiation* 22.2 (2013): 259-279.
- [34] Z. Xu,& M. Xia (2011). Distance and similarity measures for hesitant fuzzy sets. *Information Sciences*, 181(11), 2128-2138.
- [35] Z. Xu & M. Xia (2011). On distance and correlation measures of hesitant fuzzy information. *International Journal of Intelligent Systems*, 26(5), 410-425.
- [36] J. Ye (2011). [Cosine similarity measures for intuitionistic fuzzy sets and their applications.](#) *Mathematical and Computer Modelling*, 53(1), 91-97.
- [37] J. Ye (2012). Multicriteria group decision-making method using vector similarity measures for trapezoidal intuitionistic fuzzy numbers. *Group Decision and Negotiation*, 21(4), 519-530.
- [38] J. Ye (2013). [Interval-valued intuitionistic fuzzy cosine similarity measures for multiple attribute decision-making.](#) *International Journal of General Systems*, 42(8), 883-891.
- [39] J. Ye (2013). [Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment.](#) *International Journal of General Systems*, 42(4), 386-394.
- [40] J. Ye (2013). [Another form of correlation coefficient between single valued neutrosophic sets and its multiple attribute decision-making method.](#) *Neutrosophic Sets and Systems*, 1(1), 8-12.
- [41] J. Ye (2014). [Similarity measures between interval neutrosophic sets and their applications in multicriteria decision-making.](#) *Journal of Intelligent and fuzzy Systems*, 26(1), 165-172.
- [42] J. Ye & Q. S. Zhang, Q. S. (2014). Single valued neutrosophic similarity measures for multiple attribute decision making. *Neutrosophic Sets and Systems*, 2, 48-54.
- [43] J. Ye (2014). [A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets.](#) *Journal of Intelligent & Fuzzy Systems: Applications in Engineering and Technology*, 26(5), 2459-2466.
- [44] J. Ye (2014). [Vector similarity measures of simplified neutrosophic sets and their application in multicriteria decision making.](#) *International Journal of Fuzzy Systems*, 16(2), 204-215.
- [45] J. Ye (2015). [Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses.](#) *Artificial intelligence in medicine*,63(3), 171-179.
- [46] J. Ye. (2014). [Vector Similarity Measures for Hesitant Fuzzy set and their multiple attribute decision making.](#) *Economic Computation & Economic Cybernetics Studies & Research*, 48(4).
- [47] J. Ye (2014) "[Correlation coefficient of dual hesitant fuzzy sets and its application to multiple attribute decision making.](#)" *Applied Mathematical Modelling* 38.2 659-666.
- [48] J. Ye. (2015) "[Multiple-attribute decision-making method under a single-valued neutrosophic hesitant fuzzy environment.](#)" *Journal of Intelligent Systems* 24.1 23-36.
- [49] L. A. Zadeh, (1965) "Fuzzy sets." *Information and control* 8.3 338-353.
- [50] H. Y. Zhang, ,J. Q. Wang & X. H. Chen (2014). Interval neutrosophic sets and their application in multicriteria decision making problems. *The Scientific World Journal*.
- [51] B. Zhu, Z. Xu, and M. Xia *Dual Hesitant Fuzzy Sets* (2012), *Journal of Applied Mathematics* Article ID 879629, 13 pages.
- [52] B. Zhua & Z. Xu (2014). Some results for dual hesitant fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 26, 1657-1668.

TAHIR MAHMOOD

DEPARTMENT OF MATHEMATICS AND STATISTICS, INTERNATIONAL ISLAMIC UNIVERSITY, 44000-ISLAMABAD, PAKISTAN

*E-mail address:* [tahirbakht@yahoo.com](mailto:tahirbakht@yahoo.com)

JUN YE

DEPARTMENT OF ELECTRICAL AND INFORMATION ENGINEERING, SHOAXING UNIVERSITY, CHINA

*E-mail address:* [yehjun@aliyun.com](mailto:yehjun@aliyun.com)

QAISAR KHAN

DEPARTMENT OF MATHEMATICS AND STATISTICS, INTERNATIONAL ISLAMIC UNIVERSITY, 44000-ISLAMABAD, PAKISTAN

*E-mail address:* [qaisarkhan421@gmail.com](mailto:qaisarkhan421@gmail.com)