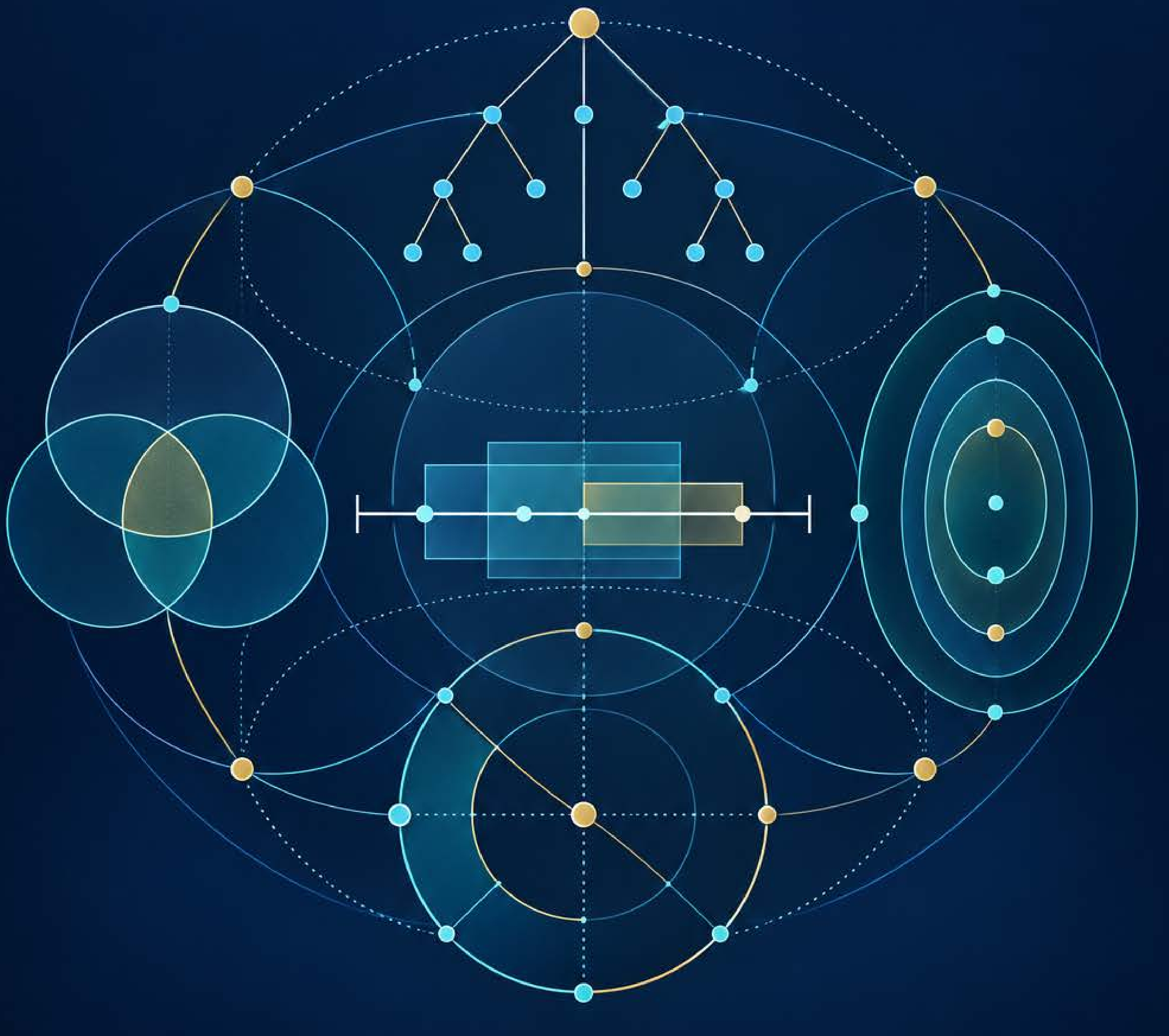
A Comprehensive Survey and Generalization of Linguistic, MOD, Interval, Subset, and Higher-Order Structures



Takaaki Fujita
Florentin Smarandache

A Comprehensive Survey and Generalization of Linguistic, MOD, Interval, Subset, and Higher-Order Structures

Takaaki Fujita **Florentin Smarandache**

Neutrosophic Science International Association (NSIA)
Publishing House
2026

Editor:

Neutrosophic Science International Association (NSIA)
Publishing House
https://fs.unm.edu/NSIA/

Division of Mathematics and Sciences
University of New Mexico
705 Gurley Ave., Gallup Campus
NM 87301, United States of America

University of Guayaquil
Av. Kennedy and Av. Delta
"Dr. Salvador Allende" University Campus
Guayaquil 090514, Ecuador

Peer-Reviewers:

Mukhtar Ahmad
Algebra Research Group, Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung (ITB), Bandung 40132, Indonesia
Email: *30125701@mahasiswa.itb.ac.id*

Mohamed Elhoseny
American University in the Emirates, Dubai, UAE
Email: *mohamed.elhoseny@aue.ae*

Tarek Zayed
Department of Building and Real Estate, The Hong Kong Polytechnic University, Hung Hom, 8 Kowloon, Hong Kong, China
Email: *tarek.zayed@polyu.edu.hk*

ISBN 978-197250253-2

9 781972 502532

Contents

The remainder of this book is organized as follows.

Chapter 1

Introduction

1.1 Linguistic Structure

Natural language is a fundamental framework for communication and knowledge representation. Its mathematical and computational study plays an important role in artificial intelligence, natural language processing, and large language models [1–10].

Because linguistic information often involves vagueness, ambiguity, and incomplete information, several uncertainty-based frameworks have been developed. Representative examples include Fuzzy Natural Language [11–14], Neutrosophic Natural Language [15], Hesitant Fuzzy Linguistic Sets [16–18], Neutrosophic Linguistic Sets [19–21], Linguistic Graphs [22–24], and Linguistic Matrices [25, 26].

This work reviews representative linguistic structures and their mathematical generalizations within a common framework.

1.2 MOD Structure and Interval Structure

MOD structures are based on modular arithmetic and naturally describe finite cyclic systems [27–29]. They include MOD groupoids, vector spaces, functions, graphs, HyperGraphs, SuperHyperGraphs, Cayley graphs [30], circulant graphs, cyclic codes, and related structures over

$$\mathbb{Z}_n.$$

Interval structures represent quantities by ranges rather than exact values and provide useful models for bounded variability and uncertainty [31, 32]. Representative examples include interval groupoids, semigroups, semirings, matrices, fuzzy and neutrosophic interval structures, and interval-valued uncertainty models.

Higher-order extensions, including HyperInterval-Valued Uncertain Sets, SuperHyperInterval-Valued Uncertain Sets, and n-interval structures, are also considered.

1.3 Subset Structure and Higher-Order Generalizations

Subset Structures replace individual elements by nonempty subsets and lift ordinary operations to the subset level.

Let $X \neq \varnothing$ and let
$$\omega : X^k \longrightarrow X$$
be a k-ary operation. Define
$$\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\varnothing\}.$$
The induced subset operation is
$$\widehat{\omega} : \big(\mathcal{P}^*(X)\big)^k \longrightarrow \mathcal{P}^*(X),$$
where
$$\widehat{\omega}(A_1, \dots, A_k) = \{\omega(a_1, \dots, a_k) : a_i \in A_i\}.$$

This differs from a HyperStructure, whose hyperoperation typically has the form
$$h : X^k \longrightarrow \mathcal{P}^*(X).$$
Thus, Subset Structures use subsets as inputs, whereas HyperStructures introduce set-valued outputs.

Higher levels are obtained recursively by
$$\mathcal{P}^{*(0)}(X) = X, \qquad \mathcal{P}^{*(n+1)}(X) = \mathcal{P}^*\big(\mathcal{P}^{*(n)}(X)\big).$$
Hence,
$$X \longrightarrow \mathcal{P}^*(X) \longrightarrow \mathcal{P}^{*(2)}(X) \longrightarrow \cdots$$
provides a natural hierarchy of nested subset structures related to higher-order and SuperHyperStructures.

1.4 MultiStructure and Iterated MultiStructure

MultiStructures combine several distinguishable mathematical structures within a common multi-component framework while preserving the identity of each component.

Definition 1.4.1 (MultiStructure)**.** Let
$$\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m, \qquad m \geq 2,$$
be mathematical structures of the same or mutually compatible type. A *MultiStructure* is the indexed m-component structure
$$\mathcal{M} = (\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_m).$$
The index of each component is regarded as part of the structure, so that the constituent structures remain distinguishable even when some of them have overlapping or identical underlying sets.

When a set-theoretic realization is required, the underlying components may equivalently be represented by the tagged disjoint union

$$\bigsqcup_{i=1}^{m} \{i\} \times S_i,$$

where S_i denotes the underlying set of $\mathcal{S}_i$.

The cases $m = 2$ and $m = 3$ are referred to as a *BiStructure* and a *TriStructure*, respectively. More generally, the MultiStructure viewpoint provides a common framework for organizing mathematical objects composed of several distinguishable structural components. Related examples include Multimatrices, MultiVector Spaces, MultiIntervals, multilayer systems, and multipolar uncertainty models.

The construction can be iterated to introduce hierarchical depth.

Definition 1.4.2 (Iterated MultiStructure)**.** Let

$$\mathfrak{S}^{(0)}$$

be a class of basic mathematical structures. For $n \geq 0$, define recursively

$$\mathfrak{S}^{(n+1)} = \bigcup_{m \geq 2} \left(\mathfrak{S}^{(n)} \right)^m .$$

Thus, an element of $\mathfrak{S}^{(n+1)}$ has the form

$$\mathcal{M}^{(n+1)} = \left(\mathcal{S}_1^{(n)}, \mathcal{S}_2^{(n)}, \ldots, \mathcal{S}_m^{(n)} \right),$$

where

$$m \geq 2, \qquad \mathcal{S}_i^{(n)} \in \mathfrak{S}^{(n)}.$$

For $n \geq 1$, an element of $\mathfrak{S}^{(n)}$ is called an *Iterated MultiStructure of level n*.

Hence, an ordinary MultiStructure introduces several distinguishable components at one level, whereas an Iterated MultiStructure recursively combines such components and thereby introduces hierarchical organization. Schematically,

$$\mathfrak{S}^{(0)} \longrightarrow \mathfrak{S}^{(1)} \longrightarrow \mathfrak{S}^{(2)} \longrightarrow \cdots .$$

1.5 Meta, Complex, and Real Structures

The survey also considers MetaStructures, in which entire mathematical structures are treated as objects, together with their iterated forms. In addition, representative Complex and Real Structures are examined to illustrate algebraic, geometric, and scalar-valued realizations of the general framework.

1.6 Our Contributions

This work provides a unified survey of linguistic, MOD, interval, subset, multi-component, meta, complex, and real structures, together with several higher-order generalizations.

The principal contributions are summarized as follows.

1. We review representative linguistic structures, including fuzzy, neutrosophic, graph-based, and matrix-based models.
2. We organize representative MOD and interval structures and their algebraic, graph-theoretic, and uncertainty-based extensions.
3. We formulate a unified power-set-based framework for Subset Structures and clarify their distinction from HyperStructures.
4. We investigate iterated subset constructions and their connections with higher-order and SuperHyperStructures.
5. We present BiStructures, TriStructures, MultiStructures, Iterated MultiStructures, and MetaStructures within a common hierarchical viewpoint.
6. We discuss representative complex and real structures, together with structural relationships, reductions, closure properties, and well-definedness.

Overall, this organization provides a common mathematical perspective for linguistic information, modular systems, interval mathematics, subset constructions, MultiStructures, MetaStructures, HyperStructures, and higher-order mathematical structures.

Chapter 2

Preliminaries

In this chapter, we briefly review the basic definitions, terminology, and notation used throughout this book.

2.1 Fuzzy Set

A Fuzzy Set assigns each element a single membership degree in $[0,1]$ [33–35]. Related concepts such as Fuzzy Soft Sets [36,37] , Fuzzy HyperSoft Sets [38–40], HyperFuzzy Sets [41,42], and Fuzzy Rough Sets [43–45] are also known. The definitions and concrete examples are presented below.

Definition 2.1.1 (Fuzzy Set)**.** [33] Let X be a nonempty set. A *fuzzy set* A on X is characterized by its membership function

$$\mu_A : X \to [0,1].$$

That is, the fuzzy set A is defined as

$$A = \{(x, \mu_A(x)) \mid x \in X\},$$

where $\mu_A(x)$ represents the degree to which the element $x \in X$ belongs to the set A.

2.2 Intuitionistic fuzzy set

An intuitionistic fuzzy set assigns each element membership and nonmembership degrees [46–49]. Related concepts such as Intuitionistic Fuzzy Soft Sets [50, 51] and Intuitionistic Fuzzy Rough Sets [52–54] are also known. The definitions and concrete examples are presented below.

Definition 2.2.1 (Intuitionistic fuzzy set)**.** [46] Let E be a nonempty set. An *intuitionistic fuzzy set* (IFS) A on E is given by

$$A = \big\{ \langle x, \mu_A(x), \nu_A(x) \rangle : x \in E \big\},$$

where

$$\mu_A, \nu_A : E \longrightarrow [0,1]$$

are, respectively, the membership and non–membership functions, and for every $x \in E$ one has

$$0 \le \mu_A(x) + \nu_A(x) \le 1.$$

The quantity

$$\pi_A(x) := 1 - \mu_A(x) - \nu_A(x)$$

represents the hesitation degree at x.

The usual fuzzy set notion is recovered in the special case $\nu_A(x) = 1 - \mu_A(x)$ for all $x \in E$, that is, when $\pi_A(x) = 0$ for every x.

2.3 Neutrosophic Set

A Neutrosophic Set assigns to each element three independent membership degrees— truth (T), indeterminacy (I), and falsity (F)—each taking values in $[0, 1]$, thereby enabling flexible modeling of uncertainty [55–59]. Neutrosophic Sets are widely recognized as generalizations of Fuzzy Sets and Intuitionistic Fuzzy Sets, and they offer a highly adaptable framework by explicitly accommodating the component I. Related concepts such as Neutrosophic Soft Sets [60–62], Neutrosophic Rough Sets [63], and Neutrosophic HyperSoft Sets [64–66] are also known. The definitions and concrete examples are presented below.

Definition 2.3.1 (Neutrosophic Set)**.** [67] Let X be a non-empty set. A *Neutrosophic Set (NS)* A on X is characterized by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Example 2.3.2 (Real-Life Example of a Neutrosophic Set)**.** Consider the evaluation of whether a restaurant is suitable for recommendation. Let

$$X = \{R_1, R_2, R_3\}$$

be a set of restaurants. Define a neutrosophic set A representing "restaurants recommended by customers" by

$$A = \left\{ \begin{array}{c} (R_1,\ 0.80,\ 0.10,\ 0.20), \\ (R_2,\ 0.60,\ 0.30,\ 0.40), \\ (R_3,\ 0.30,\ 0.20,\ 0.70) \end{array} \right\}.$$

For example, for restaurant R_2,

$$T_A(R_2) = 0.60, \qquad I_A(R_2) = 0.30, \qquad F_A(R_2) = 0.40.$$

Here, 0.60 represents the degree of evidence supporting the recommendation, 0.30 represents indeterminate or conflicting information, and 0.40 represents the degree of evidence against the recommendation.

Thus, the neutrosophic representation can simultaneously express positive, indeterminate, and negative evaluations arising from real-world customer opinions.

2.4 Uncertain Set

An Uncertain Set assigns to each element a degree taken from a prescribed uncertainty model, thereby providing a common framework that includes fuzzy, intuitionistic fuzzy, neutrosophic, and plithogenic settings [68–73].

Definition 2.4.1 (Uncertain Model)**.** [73] Let U denote the class of all *uncertain models.* Each $M \in U$ is determined by the following data:

- a nonempty set $\mathrm{Dom}(M) \subseteq [0, 1]^k$ of *admissible degree tuples*, for some fixed integer $k \ge 1$;
- a collection of model-dependent algebraic or geometric constraints imposed on elements of $\mathrm{Dom}(M)$ (for example, $\mu + \nu \le 1$ in the intuitionistic fuzzy setting, or $0 \le T + I + F \le 3$ in the neutrosophic setting).

Typical examples are as follows:

- Fuzzy model: $\mathrm{Dom}(M) = [0, 1]$;
- Intuitionistic fuzzy model:
$$\mathrm{Dom}(M) = \{(\mu, \nu) \in [0, 1]^2 : \mu + \nu \leq 1\};$$
- Neutrosophic model:
$$\mathrm{Dom}(M) = \{(T, I, F) \in [0, 1]^3 : 0 \leq T + I + F \leq 3\};$$
- Plithogenic model, together with many further extensions.

Definition 2.4.2 (Uncertain Set (U-Set))**.** [73] Let X be a nonempty universe, and let M be a fixed uncertain model with degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$. An *Uncertain Set of type* M (briefly, a *U-Set*) on X is a pair
$$\mathcal{U} = (X, \mu_M),$$
where
$$\mu_M : X \longrightarrow \mathrm{Dom}(M)$$
is called the *uncertainty-degree function* (or membership map) of $\mathcal{U}$.

For each $x \in X$, the value $\mu_M(x) \in \mathrm{Dom}(M)$ records the degree, or collection of degrees, with which x belongs to the uncertain set under the model M.

Remark 2.4.3. Several familiar structures appear as special cases:

- If M is the fuzzy model with $\mathrm{Dom}(M) = [0, 1]$, then $\mu_M : X \to [0, 1]$ is an ordinary fuzzy membership function, and $\mathcal{U}$ is a fuzzy set.
- If M is a neutrosophic model, then $\mu_M(x) = (T(x), I(x), F(x))$, and $\mathcal{U}$ becomes a neutrosophic set.
- Other choices of M recover intuitionistic fuzzy sets, picture fuzzy sets, plithogenic sets, and many related models.

Example 2.4.4 (Real-Life Example of an Uncertain Set)**.** Consider a university selecting students for a scholarship. Let
$$X = \{s_1, s_2, s_3\}$$
be the set of applicants.

Assume that the uncertainty model M evaluates each applicant by two components:
$$\mathrm{Dom}(M) = [0, 1]^2,$$
where the first component represents academic suitability and the second represents financial need.

Define
$$\mu_M : X \longrightarrow [0, 1]^2$$
by
$$\mu_M(s_1) = (0.90, 0.70),$$
$$\mu_M(s_2) = (0.75, 0.95),$$

and

$$\mu_M(s_3) = (0.60, 0.50).$$

Then

$$\mathcal{U} = (X, \mu_M)$$

is an Uncertain Set of type M.

For example,

$$\mu_M(s_2) = (0.75, 0.95)$$

indicates that applicant s_2 has an academic suitability degree of 0.75 and a financial-need degree of 0.95.

Thus, the U-Set framework can represent several uncertainty-related evaluation components simultaneously within a single mathematical model.

As indicated above, this framework admits many different specializations. For reference, Table 2.1 lists representative families of uncertainty sets (U-Sets), organized according to the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0, 1]^k$ (cf. [37, 74]).

Table 2.1: A catalogue of uncertainty-set families (U-Sets) organized by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ [74].

k	note	Representative U-Set model(s) whose degree-domain is a subset of $[0,1]^k$
1		Fuzzy Set [33, 75–77]; N-Fuzzy Set [78–80]; Shadowed Set [81–83]
2		Intuitionistic Fuzzy Set [46,49]; Intuitionistic Evidence Set; Interval-Valued Fuzzy Set [84,85]; Pythagorean Fuzzy Set [86,87]; Fermatean Fuzzy Set [88,89]; q-Rung Orthopair Fuzzy Set [90,91], Vague Set [92,93]; Bipolar Fuzzy Set (two-component description) [94,95]; Variable Fuzzy Set [96–98]; Paraconsistent Fuzzy Set [99,100]; Bifuzzy Set [101,102]
3		Single-Valued Neutrosophic Set [57, 67, 103–106]; Picture Fuzzy Set [107, 108] (Ternary Fuzzy Set [109]); Three-way Fuzzy Set [110,111]; Inconsistent Intuitionistic Fuzzy Set [112]; Cubic Set [113,114]; Spherical Fuzzy Set [115,116]; T-Spherical Fuzzy Set [117,118]; Tripolar Fuzzy Set (three-component formalisms) [119–122]; Kleene Three-Valued Set; Neutrosophic Vague Set [123,124]
4		Quadripartitioned Neutrosophic Set [125,126]; Interval-Valued Intuitionistic Fuzzy Set [127, 128]; Interval-Valued Pythagorean Fuzzy Set [129, 130]; Support Neutrosophic Set [131–134]; Local-Neutrosophic Set [135]; Linear Diophantine Fuzzy Set [136,137]; Double-Valued Neutrosophic Set [138–140]; Dual Hesitant Fuzzy Set [141,142]; Bipolar Intuitionistic Fuzzy Set [143–145]; Bipolar Vague Set [146,147]; Quadripolar Fuzzy Set [148]; Ambiguous Set [149–151]; Belnap four-valued Set; Turiyam Neutrosophic Set [152–155]
5		Pentapartitioned Neutrosophic Set [156–158]; Triple-Valued Neutrosophic Set [159–161]
6		Hexapartitioned Neutrosophic Set; Tripolar Intuitionistic Fuzzy Set [120]; Bipolar Neutrosophic Set [162–164]; Quadruple-Valued Neutrosophic Set [160, 165]; Intuitionistic fuzzy Cubic Set [166–169]; Interval-Valued Neutrosophic Set [170, 171];
7		Heptapartitioned Neutrosophic Set [172,173]; Quintuple-Valued Neutrosophic Set [160, 174, 175]
8		Bipolar Quadripartitioned Neutrosophic Set; Octapartitioned Neutrosophic Set [176, 177]
9		Nonapartitioned Neutrosophic Set [176, 177] ; Neutrosophic Cubic Set [178–182]; Tripolar Neutrosophic Set [183]
n	$(n \geq 1)$	Multi-valued (Fuzzy) Sets [184]; Plithogenic Sets [185, 186], MultiFuzzy Set [187]; n-valued (Łukasiewicz) Set; Multipolar Fuzzy Sets [188–190]; n-Refined Fuzzy Set [191, 192]
$2n$	$(n \geq 1)$	n-Refined Intuitionistic Fuzzy Set [192]; Multipolar Intuitionistic Fuzzy Sets [193–195], Multi-Intuitionistic Fuzzy Set [187]
$3n$	$(n \geq 1)$	n-Refined Neutrosophic Set [192, 196]; Multipolar Neutrosophic Sets [197, 198]; Multi-Neutrosophic Set [187, 199]

Reading guide. Within the U-Set framework [73], each model M is determined by a degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ together with a membership map $\mu_M : X \to \mathrm{Dom}(M)$. Accordingly, the table groups representative families by the ambient dimension k, that is, by the number of numerical components assigned to each element.

(a) A common viewpoint in the literature is that neutrosophic sets form a broad umbrella encompassing several earlier multi-component fuzzy models and their generalizations; see [200].

(b) Ambiguous sets are often described as subclasses of certain four-component neutrosophic families; see [125,126,151].

(c) Turiyam neutrosophic sets are reported as subclasses of quadripartitioned neutrosophic sets; see [201].

Chapter 3

Linguistic Structures

This chapter introduces and reviews several representative linguistic structures.

3.1 Natural Language

Natural language is a human communication system using vocabulary, grammar, and semantics to form context-dependent expressions for communication and interpretation [202–205]. We briefly recall the notions of formal language, word, and natural language used throughout this work.

Definition 3.1.1 (Formal Language)**.** [206] Let Σ be a finite alphabet and let Σ^* denote the set of all finite strings over Σ. A *formal language* over Σ is any set

$$\mathcal{L} \subseteq \Sigma^*.$$

Additional syntactic or grammatical rules may be used to determine which strings belong to $\mathcal{L}$.

Definition 3.1.2 (Word)**.** (cf. [206, 207]) Let Σ be a finite alphabet. A *word* over Σ is an element

$$w \in \Sigma^*, \qquad \Sigma^* = \bigcup_{n=0}^{\infty} \Sigma^n.$$

The length of w is denoted by $|w|$. The empty word $\varepsilon \in \Sigma^*$ satisfies $|\varepsilon| = 0$.

Definition 3.1.3 (Natural Language)**.** (cf. [3, 4, 208]) For mathematical purposes, a *natural language* is modeled as a structure

$$\mathcal{N} = (\Sigma, W, G, \mathrm{Sem}),$$

where Σ is an alphabet, $W \subseteq \Sigma^*$ is a lexicon, G is a grammatical system determining a set of well-formed expressions $\mathrm{Sent}(G) \subseteq \Sigma^*$, and

$$\mathrm{Sem} : \mathrm{Sent}(G) \longrightarrow M$$

is a semantic mapping into a domain of meanings M. The grammar and semantics may be ambiguous or context-dependent.

Example 3.1.4 (A Simple Natural Language)**.** Consider a small natural language for describing animal actions. Let

$$\Sigma = \{\texttt{cats}, \texttt{dogs}, \texttt{chase}, \texttt{mice}\},$$

and let the lexicon be

$$W = \{\texttt{cats}, \texttt{dogs}, \texttt{chase}, \texttt{mice}\}.$$

Let the grammatical system G be determined by the rule

$$\text{Sentence} \longrightarrow \text{Noun Verb Noun},$$

where

$$\text{Noun} \in \{\texttt{cats}, \texttt{dogs}, \texttt{mice}\}, \qquad \text{Verb} = \texttt{chase}.$$

For example,

$$\text{Sent}(G) = \{\texttt{cats chase mice}, \texttt{dogs chase cats}, \ldots\}.$$

Let the semantic domain be

$$M = \{\text{Chase}(a, b) : a, b \in \{\text{cats}, \text{dogs}, \text{mice}\}\}.$$

The semantic mapping is defined, for instance, by

$$\text{Sem}(\texttt{cats chase mice}) = \text{Chase}(\text{cats}, \text{mice}),$$

and

$$\text{Sem}(\texttt{dogs chase cats}) = \text{Chase}(\text{dogs}, \text{cats}).$$

Hence,

$$\mathcal{N} = (\Sigma, W, G, \text{Sem})$$

provides a simple mathematical model of a natural language in which grammatical expressions are mapped to their corresponding meanings.

3.2 Hyperword and Hyperlanguage

A hyperword is a nonempty set of words, while a hyperlanguage is a collection of hyperwords [209].

Definition 3.2.1 (Hyperword and Hyperlanguage)**.** (cf. [209]) Let Σ be a finite alphabet.

A *hyperword* over Σ is a nonempty subset

$$H \in \mathcal{P}(\Sigma^*) \setminus \{\emptyset\}.$$

A *hyperlanguage* over Σ is a collection of hyperwords; hence

$$\mathcal{H} \subseteq \mathcal{P}(\Sigma^*) \setminus \{\emptyset\}.$$

Definition 3.2.2 (Natural Hyperword and Natural Hyperlanguage)**.** (cf. [3, 210]) Let $\mathcal{W}$ be the set of well-formed expressions of a natural language.

A *natural hyperword* is a nonempty subset

$$H \in \mathcal{P}(\mathcal{W}) \setminus \{\emptyset\}.$$

A *natural hyperlanguage* is a collection of natural hyperwords:

$$\mathcal{HL}_{\text{nat}} \subseteq \mathcal{P}(\mathcal{W}) \setminus \{\emptyset\}.$$

Example 3.2.3 (Natural Hyperword and Natural Hyperlanguage)**.** Let $\mathcal{W}$ be a set of well-formed English expressions containing

$$\mathcal{W} = \{\texttt{It is raining}, \texttt{Rain is falling}, \texttt{The weather is rainy},$$

$$\texttt{It is sunny}, \texttt{The sun is shining}, \texttt{It is cloudy}\}.$$

Consider the following subsets of $\mathcal{W}$:

$$H_1 = \{\texttt{It is raining}, \texttt{Rain is falling}, \texttt{The weather is rainy}\},$$

$$H_2 = \{\texttt{It is sunny}, \texttt{The sun is shining}\},$$

and

$$H_3 = \{\texttt{It is cloudy}\}.$$

Since

$$H_1, H_2, H_3 \in \mathcal{P}(\mathcal{W}) \setminus \{\emptyset\},$$

each H_i is a natural hyperword. In particular, H_1 groups several well-formed expressions conveying closely related meanings about rainy weather.

The collection

$$\mathcal{HL}_{\text{nat}} = \{H_1, H_2, H_3\}$$

satisfies

$$\mathcal{HL}_{\text{nat}} \subseteq \mathcal{P}(\mathcal{W}) \setminus \{\emptyset\}.$$

Therefore, $\mathcal{HL}_{\text{nat}}$ is a natural hyperlanguage.

3.3 n-Superhyperword and n-Superhyperlanguage

The notions of hyperword and hyperlanguage can be extended through iterated power-set constructions [211–213].

Definition 3.3.1 (n-Superhyperword and n-Superhyperlanguage)**.** [214] Let Σ be a finite alphabet and define

$$\mathcal{P}^0(\Sigma^*) := \Sigma^*, \qquad \mathcal{P}^{k+1}(\Sigma^*) := \mathcal{P}\big(\mathcal{P}^k(\Sigma^*)\big), \qquad k \geq 0.$$

For $n \geq 1$, an *n-superhyperword* over Σ is a nonempty element

$$H \in \mathcal{P}^n(\Sigma^*) \setminus \{\emptyset\}.$$

An *n-superhyperlanguage* is a collection of n-superhyperwords:

$$\mathcal{L}_n \subseteq \mathcal{P}^n(\Sigma^*) \setminus \{\emptyset\}.$$

For $n = 1$, these notions reduce to hyperwords and hyperlanguages.

Definition 3.3.2 (Natural n-Superhyperword and Natural n-Superhyperlanguage)**.** Let $\mathcal{W}$ be the set of well-formed expressions of a natural language, and define

$$\mathcal{P}^0(\mathcal{W}) := \mathcal{W}, \qquad \mathcal{P}^{k+1}(\mathcal{W}) := \mathcal{P}\big(\mathcal{P}^k(\mathcal{W})\big), \qquad k \geq 0.$$

For $n \geq 1$, a *natural n-superhyperword* is a nonempty element

$$H \in \mathcal{P}^n(\mathcal{W}) \setminus \{\emptyset\}.$$

A *Natural n-Superhyperlanguage* is a collection of such objects:

$$\mathcal{L}_n^{\mathrm{nat}} \subseteq \mathcal{P}^n(\mathcal{W}) \setminus \{\emptyset\}.$$

For $n = 1$, the construction reduces to a natural hyperword and a natural hyperlanguage.

Example 3.3.3 (Natural 2-Superhyperword and Natural 2-Superhyperlanguage)**.** Let

$$\mathcal{W} = \{\texttt{It is raining}, \texttt{Rain is falling}, \texttt{It is sunny}, \texttt{The sun is shining}, \texttt{It is cloudy}\}$$

be a set of well-formed English expressions.

Define the natural hyperwords

$$H_{\mathrm{rain}} = \{\texttt{It is raining}, \texttt{Rain is falling}\},$$

$$H_{\mathrm{sun}} = \{\texttt{It is sunny}, \texttt{The sun is shining}\},$$

and

$$H_{\mathrm{cloud}} = \{\texttt{It is cloudy}\}.$$

Since

$$H_{\mathrm{rain}}, H_{\mathrm{sun}}, H_{\mathrm{cloud}} \in \mathcal{P}(\mathcal{W}),$$

consider

$$S_1 = \{H_{\mathrm{rain}}, H_{\mathrm{cloud}}\}$$

and
$$S_2 = \{H_{\mathrm{sun}}, H_{\mathrm{cloud}}\}.$$

Then
$$S_1, S_2 \in \mathcal{P}^2(\mathcal{W}) \setminus \{\emptyset\}.$$
Hence, S_1 and S_2 are natural 2-superhyperwords.

For example,
$$S_1 = \{\{\texttt{It is raining}, \texttt{Rain is falling}\}, \{\texttt{It is cloudy}\}\}.$$

The collection
$$\mathcal{L}_2^{\mathrm{nat}} = \{S_1, S_2\}$$
satisfies
$$\mathcal{L}_2^{\mathrm{nat}} \subseteq \mathcal{P}^2(\mathcal{W}) \setminus \{\emptyset\}.$$
Therefore, $\mathcal{L}_2^{\mathrm{nat}}$ is a Natural 2-Superhyperlanguage.

3.4 Fuzzy Language

Fuzzy sets generalize classical sets by assigning each element a membership degree between zero and one, thereby representing partial membership and uncertainty [33,35,77]. Related concepts, including HyperFuzzy Sets [41, 215], SuperHyperFuzzy Sets [42, 216, 217], and Fuzzy Soft Sets [218, 219], are also known. Fuzzy Natural Language models linguistic expressions through graded memberships, representing vagueness and partial truth while enabling nuanced interpretation and context-sensitive acceptability [11, 13, 220].

Definition 3.4.1 (Fuzzy Language)**.** (cf. [221–223]) Let Σ be a finite alphabet. Denote by Σ^* (resp. Σ^ω) the set of all finite (resp. right-infinite) words over Σ.

(i) Fuzzy set. A fuzzy set A on a universe X is given by its membership function
$$\mu_A : X \longrightarrow [0,1], \qquad \mu_A(x) \text{ is the degree of } x \text{ belonging to } A.$$

(ii) Fuzzy language on finite words. A (finite-word) fuzzy language is a map
$$r : \Sigma^* \longrightarrow [0,1], \qquad r(w) \text{ is the degree that } w \text{ is accepted.}$$

(iii) Fuzzy language on infinite words. A (infinite-word) fuzzy language is a map
$$r_\omega : \Sigma^\omega \longrightarrow [0,1], \qquad r_\omega(w) \text{ is the degree that } w \text{ is accepted.}$$

(iv) Support. The (finite-word) support and the ω-support are
$$\mathrm{supp}(r) := \{\, w \in \Sigma^* \mid r(w) > 0 \,\}, \qquad \mathrm{supp}(r_\omega) := \{\, w \in \Sigma^\omega \mid r_\omega(w) > 0 \,\}.$$
These notions model uncertainty by assigning to each word a graded acceptance in $[0,1]$.

Definition 3.4.2 (Fuzzy Natural Language)**.** (cf. [12, 224]) A *Fuzzy Natural Language* formalizes linguistic uncertainty with fuzzy sets. Fix a finite vocabulary Σ. Define

$$\mathcal{L}_F = (\Sigma,\ \mathcal{M}_F,\ \mathcal{P}_F,\ \mathcal{S}_F),$$

where

- $\mathcal{M}_F : \Sigma^* \to [0,1]$ assigns to each string w a degree $\mathcal{M}_F(w)$ measuring how strongly w belongs to the language/domain (e.g., in a temperature domain, $\mathcal{M}_F(\text{“warm”}) = 0.8$).
- $\mathcal{P}_F : \Sigma^* \times \Sigma^* \to [0,1]$ is a fuzzy similarity/proximity (e.g., $\mathcal{P}_F(\text{“warm”}, \text{“cozy”}) = 0.7$).
- $\mathcal{S}_F$ is a set of fuzzy constraints (syntactic/semantic rules); each rule ρ is given a degree $\mathcal{S}_F(\rho) \in [0,1]$ indicating admissibility of a construction.

When context variables (e.g., a measured temperature T, a wait time t) are present, $\mathcal{M}_F$ can be instantiated from fuzzy predicates (membership profiles) over those variables and then mapped to strings in Σ^*.

Example 3.4.3 (A Simple Fuzzy Natural Language for Temperature Description)**.** Consider a fuzzy natural language for describing temperature conditions. Let

$$\Sigma = \{\texttt{cold}, \texttt{cool}, \texttt{warm}, \texttt{hot}, \texttt{very}\}.$$

Define

$$\mathcal{L}_F = (\Sigma, \mathcal{M}_F, \mathcal{P}_F, \mathcal{S}_F).$$

Suppose the measured temperature is

$$T = 24^\circ\text{C}.$$

The fuzzy membership component may assign

$$\mathcal{M}_F(\texttt{cold}) = 0.1, \qquad \mathcal{M}_F(\texttt{cool}) = 0.4,$$

$$\mathcal{M}_F(\texttt{warm}) = 0.8, \qquad \mathcal{M}_F(\texttt{hot}) = 0.3.$$

Thus, the expression `warm` is the most strongly supported linguistic description of the observed temperature.

A fuzzy similarity function may be specified, for example, by

$$\mathcal{P}_F(\texttt{warm}, \texttt{hot}) = 0.6,$$

$$\mathcal{P}_F(\texttt{warm}, \texttt{cool}) = 0.5, \qquad \mathcal{P}_F(\texttt{cold}, \texttt{hot}) = 0.1.$$

Consider also the fuzzy rule

$$\rho : \quad \texttt{very} + \texttt{warm} \longrightarrow \texttt{very warm},$$

with admissibility degree

$$\mathcal{S}_F(\rho) = 0.9.$$

Hence, at

$$T = 24°\text{C},$$

the system can represent the statement

`The temperature is warm`

with degree

$$0.8,$$

while simultaneously preserving graded similarity between related linguistic expressions and fuzzy admissibility of linguistic constructions. Therefore,

$$\mathcal{L}_F = (\Sigma, \mathcal{M}_F, \mathcal{P}_F, \mathcal{S}_F)$$

provides a concrete example of a Fuzzy Natural Language.

3.5 Neutrosophic Natural Language

Neutrosophic sets extend the fuzzy-set viewpoint by separating uncertainty into three components: truth, indeterminacy, and falsity [57, 59, 74, 225, 226]. A Neutrosophic Natural Language represents linguistic expressions through degrees of truth, indeterminacy, and falsity, enabling explicit modeling of ambiguity, uncertainty, and contradictory meanings [15].

Definition 3.5.1 (Neutrosophic set)**.** [67] Let X be a nonempty set. A *neutrosophic set* A on X is specified by three functions

$$T_A : X \to [0,1], \qquad I_A : X \to [0,1], \qquad F_A : X \to [0,1],$$

where, for each $x \in X$, the numbers $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent, respectively, the degrees of truth-membership, indeterminacy, and falsity-membership of x in A. They satisfy the admissibility constraint

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3 \qquad (x \in X).$$

Definition 3.5.2 (Neutrosophic Natural Language)**.** [15] A *Neutrosophic Natural Language* is a natural language

$$\mathcal{N}_{\text{Neu}} = (\Sigma, N, P, S, M_{\text{Neu}})$$

equipped with a neutrosophic semantic mapping

$$M_{\text{Neu}} : L(\mathcal{N}_{\text{Neu}}) \longrightarrow [0,1]^3,$$

where, for each linguistic expression w,

$$M_{\text{Neu}}(w) = \big(T(w), I(w), F(w)\big).$$

Here $T(w)$, $I(w)$, and $F(w)$ denote the degrees of truth, indeterminacy, and falsity associated with the interpretation of w, respectively, with

$$0 \le T(w) + I(w) + F(w) \le 3.$$

Example 3.5.3 (Neutrosophic Interpretation of a Weather Statement)**.** Consider the linguistic expression

$$w_1 = \texttt{``It will rain tomorrow.''}$$

in a weather-related natural language.

Suppose that the available meteorological information moderately supports rain, while substantial uncertainty remains because the forecast is not fully reliable. Define

$$M_{\text{Neu}}(w_1) = (0.65, 0.25, 0.20).$$

Thus,

$$T(w_1) = 0.65, \qquad I(w_1) = 0.25, \qquad F(w_1) = 0.20.$$

Here, the truth degree 0.65 indicates relatively strong support for the statement, the indeterminacy degree 0.25 represents uncertainty in the forecast, and the falsity degree 0.20 represents evidence against rain.

Moreover,

$$0 \le 0.65 + 0.25 + 0.20 = 1.10 \le 3.$$

Hence, the neutrosophic semantic mapping provides a simultaneous representation of support, uncertainty, and opposition concerning the linguistic expression.

3.6 Plithogenic Natural Language

A Plithogenic Set [227–229] provides a mathematical framework for representing elements through attribute-dependent appurtenance degrees together with contradiction degrees among attribute values. Depending on the chosen appurtenance domain, the framework can incorporate fuzzy [33, 230], intuitionistic fuzzy [49, 231], neutrosophic [57, 59], and related uncertainty representations.

Definition 3.6.1 (Plithogenic Set)**.** [185, 227] Let S be a universe and let

$$\varnothing \neq P \subseteq S.$$

Let v be an attribute and let

$$V_v \neq \varnothing$$

be the set of possible values of v.

Let

$$D \subseteq [0,1]^s, \qquad s \ge 1,$$

be an admissible appurtenance-degree domain.

A *Plithogenic Set* is a structure

$$PS = (P, v, V_v, d_v, c_v),$$

where

$$d_v : P \times V_v \longrightarrow D$$

is the *Degree of Appurtenance Function (DAF)*, and

$$c_v : V_v \times V_v \longrightarrow [0,1]$$

is the *Degree of Contradiction Function (DCF)*.

For $x \in P$ and $a \in V_v$,

$$d_v(x, a)$$

represents the appurtenance information of x with respect to the attribute value a.

The contradiction function is assumed to satisfy

$$c_v(a, a) = 0$$

for every $a \in V_v$. When a symmetric contradiction model is used, one additionally assumes

$$c_v(a, b) = c_v(b, a) \qquad (a, b \in V_v).$$

Remark 3.6.2. Different versions of Plithogenic Sets may employ different admissible domains for the appurtenance information. In the finite-dimensional numerical setting considered here,

$$D \subseteq [0,1]^s.$$

In particular, fuzzy, intuitionistic fuzzy, and neutrosophic appurtenance information may be represented by suitable choices of D.

Definition 3.6.3 (Plithogenic Fuzzy Set)**.** [227, 232, 233] A *Plithogenic Fuzzy Set* is a Plithogenic Set

$$PS = (P, v, V_v, d_v, c_v)$$

whose appurtenance domain is

$$D = [0,1].$$

Thus,

$$d_v : P \times V_v \longrightarrow [0,1].$$

For $x \in P$ and $a \in V_v$, write

$$\mu_P(x \mid a) := d_v(x,a).$$

Then

$$\mu_P(x \mid a) \in [0,1]$$

is the fuzzy appurtenance degree of x with respect to the attribute value a, whereas

$$c_v(a,b) \in [0,1]$$

measures the contradiction between the attribute values $a, b \in V_v$.

A Plithogenic Language applies the same principle to linguistic expressions by associating them with attribute-dependent appurtenance information and explicit contradiction relations among linguistic attribute values.

Definition 3.6.4 (Multi-Attribute Plithogenic Structure)**.** Let

$$\mathcal{A} = \{v_1, \dots, v_m\}, \qquad m \geq 1,$$

be a finite set of attributes.

For every $i \in \{1, \dots, m\}$, let

$$V_i$$

be the nonempty value set of the attribute v_i, let

$$D_i \subseteq [0,1]^{s_i}, \qquad s_i \geq 1,$$

be its appurtenance-degree domain, and let

$$d_i : X \times V_i \longrightarrow D_i$$

be its appurtenance function.

Furthermore, let

$$c_i : V_i \times V_i \longrightarrow [0,1]$$

be a contradiction function satisfying

$$c_i(a,a) = 0$$

for every $a \in V_i$, and, when symmetry is assumed,

$$c_i(a,b) = c_i(b,a).$$

The structure

$$\mathfrak{P} = (X, \mathcal{A}, \{V_i\}_{i=1}^m, \{D_i\}_{i=1}^m, \{d_i\}_{i=1}^m, \{c_i\}_{i=1}^m)$$

is called a *multi-attribute plithogenic structure* on X.

Definition 3.6.5 (Plithogenic Language)**.** [209, 234] Let Σ be a nonempty vocabulary and let

$$\Omega \subseteq \Sigma^*$$

be a nonempty set of admissible linguistic expressions.

Let

$$\mathcal{A} = \{v_1, \ldots, v_m\}$$

be a finite set of linguistic attributes. For each v_i, let V_i be its value set and let

$$d_i : \Omega \times V_i \longrightarrow D_i$$

and

$$c_i : V_i \times V_i \longrightarrow [0, 1]$$

be the corresponding appurtenance and contradiction functions.

A *Plithogenic Language* is the structure

$$\mathcal{L}_P = \left(\Sigma, \Omega, \mathcal{A}, \{V_i\}_{i=1}^m, \{D_i\}_{i=1}^m, \{d_i\}_{i=1}^m, \{c_i\}_{i=1}^m\right).$$

Thus, for every linguistic expression

$$w \in \Omega$$

and every admissible attribute value

$$a \in V_i,$$

the value

$$d_i(w, a) \in D_i$$

describes the appurtenance information of w with respect to a, while

$$c_i(a, b)$$

represents the contradiction between two values $a, b \in V_i$ of the same attribute.

Definition 3.6.6 (Plithogenic Natural Language)**.** [209, 234] Let

$$\mathcal{L}_P = (\Sigma, \Omega, \mathcal{A}, \{V_i\}_{i=1}^m, \{D_i\}_{i=1}^m, \{d_i\}_{i=1}^m, \{c_i\}_{i=1}^m)$$

be a Plithogenic Language.

A *Plithogenic Natural Language* is an enriched linguistic structure

$$\mathcal{L}_{PL} = \left(\mathcal{L}_P, \mathcal{M}_{PL}, \mathcal{R}_{PL}, \mathcal{C}_{PL}\right),$$

where:

- $\mathcal{M}_{PL}$ is a plithogenic semantic representation derived from the attribute-dependent appurtenance functions d_i. In particular, after choosing reference values

  $$r_i \in V_i, \qquad i = 1, \ldots, m,$$

  one may define

  $$\mathcal{M}_{PL}(w) = \big(d_1(w, r_1), \ldots, d_m(w, r_m)\big).$$

- $$\mathcal{R}_{PL} : \Omega \times \Omega \longrightarrow D_R$$
  is a prescribed plithogenic relation, similarity, or compatibility mapping, where D_R is an appropriate comparison domain. Its construction may use both the appurtenance functions d_i and the contradiction functions c_i.
- $\mathcal{C}_{PL}$ is a family of linguistic rules or constraints, together with an acceptability mapping
  $$\alpha_{PL} : \mathcal{C}_{PL} \longrightarrow [0, 1].$$
  For each rule
  $$\rho \in \mathcal{C}_{PL},$$
  the quantity
  $$\alpha_{PL}(\rho)$$
  represents its graded linguistic acceptability or validity.

Thus, a Plithogenic Natural Language combines linguistic expressions, attribute-dependent uncertainty information, contradiction among attribute values, linguistic relations, and graded linguistic rules within a common mathematical structure.

Remark 3.6.7. Suitable specializations of the appurtenance domains and contradiction functions allow fuzzy, intuitionistic fuzzy, neutrosophic, and related language models to be represented or embedded within the plithogenic framework. The precise reduction depends on the selected degree domains, attributes, and aggregation rules.

Example 3.6.8 (A Plithogenic Natural Language for Weather Description)**.** Consider a Plithogenic Natural Language for describing weather conditions. Let

$$\Sigma = \{\texttt{comfortable}, \texttt{pleasant}, \texttt{hot}, \texttt{cold}, \texttt{humid}, \texttt{dry}\}$$

and let

$$\Omega \subseteq \Sigma^*$$

contain the linguistic expressions considered below.

Let the set of attributes be

$$\mathcal{A} = \{v_T, v_H, v_W\},$$

where

$$v_T = \text{temperature}, \qquad v_H = \text{humidity}, \qquad v_W = \text{wind}.$$

Take the corresponding attribute-value sets to be

$$V_T = \{\texttt{cold}, \texttt{moderate}, \texttt{hot}\},$$

$$V_H = \{\texttt{dry}, \texttt{moderate}, \texttt{humid}\},$$

and

$$V_W = \{\texttt{calm}, \texttt{light}, \texttt{strong}\}.$$

For simplicity, let the three appurtenance domains be

$$D_T = D_H = D_W = [0, 1].$$

Thus,

$$d_T : \Omega \times V_T \to [0, 1],$$

$$d_H : \Omega \times V_H \to [0,1],$$

and

$$d_W : \Omega \times V_W \to [0,1].$$

Consider the linguistic expression

$$w = \texttt{comfortable}.$$

Choose the reference attribute values

$$r_T = \texttt{moderate}, \qquad r_H = \texttt{moderate}, \qquad r_W = \texttt{light}.$$

Suppose that

$$d_T(w, \texttt{moderate}) = 0.80,$$
$$d_H(w, \texttt{moderate}) = 0.70,$$

and

$$d_W(w, \texttt{light}) = 0.90.$$

Then the corresponding plithogenic semantic profile is

$$\mathcal{M}_{PL}(w) = (0.80, 0.70, 0.90).$$

The three components describe the compatibility of `comfortable` with moderate temperature, moderate humidity, and light wind, respectively.

Now define contradiction functions on the individual attribute-value domains. For example, suppose that

$$c_T(\texttt{hot}, \texttt{cold}) = 1,$$

and

$$c_H(\texttt{humid}, \texttt{dry}) = 0.8.$$

Assuming symmetry,

$$c_T(\texttt{cold}, \texttt{hot}) = 1$$

and

$$c_H(\texttt{dry}, \texttt{humid}) = 0.8.$$

Thus, strongly conflicting linguistic values can be represented explicitly within their respective attribute domains.

Consider another linguistic expression

$$w' = \texttt{pleasant}.$$

Suppose that a prescribed plithogenic similarity mapping satisfies

$$\mathcal{R}_{PL}(\texttt{comfortable}, \texttt{pleasant}) = (0.90, 0.85, 0.80).$$

This vector may represent similarity with respect to temperature, humidity, and wind characteristics.

Finally, consider the linguistic rule

$$\rho : \quad \texttt{moderate temperature} + \texttt{moderate humidity} + \texttt{light wind} \longrightarrow \texttt{comfortable weather}.$$

Let

$$\rho \in \mathcal{C}_{PL}$$

and assign the graded acceptability

$$\alpha_{PL}(\rho) = 0.88.$$

Therefore,

$$\mathcal{L}_{PL} = (\mathcal{L}_P, \mathcal{M}_{PL}, \mathcal{R}_{PL}, \mathcal{C}_{PL})$$

provides a Plithogenic Natural Language for weather description, combining attribute-dependent appurtenance information, contradiction degrees among attribute values, linguistic similarity, and graded linguistic rules.

3.7 Meta-Natural Language

A Meta-Natural Language is a higher-level framework that organizes multiple natural languages as objects and defines operations among them systematically [235].

Definition 3.7.1 (Meta-Natural Language). [235] Let NL be the class of natural-language structures

$$L = (\Sigma, W, G, \mathrm{Sem}).$$

A *Meta-Natural Language* is a pair

$$\mathfrak{L} = \big(\mathcal{U}, \{\Phi_\lambda\}_{\lambda\in\Lambda}\big),$$

where $\varnothing \neq \mathcal{U} \subseteq \mathsf{NL}$ is a collection of natural languages and, for each $\lambda \in \Lambda$,

$$\Phi_\lambda : \mathcal{U}^{k_\lambda} \longrightarrow \mathcal{U}$$

is a meta-operation combining or transforming natural languages.

Example 3.7.2 (A Translation-Based Meta-Natural Language). Consider two natural-language structures

$$L_{\mathrm{EN}} = (\Sigma_{\mathrm{EN}}, W_{\mathrm{EN}}, G_{\mathrm{EN}}, \mathrm{Sem}_{\mathrm{EN}})$$

and

$$L_{\mathrm{JA}} = (\Sigma_{\mathrm{JA}}, W_{\mathrm{JA}}, G_{\mathrm{JA}}, \mathrm{Sem}_{\mathrm{JA}}),$$

representing simplified English and Japanese, respectively.

Let

$$\mathcal{U} = \{L_{\mathrm{EN}}, L_{\mathrm{JA}}\}.$$

Define a meta-operation

$$\Phi_{\mathrm{tr}} : \mathcal{U} \longrightarrow \mathcal{U}$$

by

$$\Phi_{\mathrm{tr}}(L_{\mathrm{EN}}) = L_{\mathrm{JA}}, \qquad \Phi_{\mathrm{tr}}(L_{\mathrm{JA}}) = L_{\mathrm{EN}}.$$

For example, the English expression

$$w_{\mathrm{EN}} = \texttt{``It is raining.''}$$

may correspond under the translation process to the Japanese expression

$$w_{\mathrm{JA}} = \texttt{``Ame ga futte iru.''}$$

with approximately preserved meaning,

$$\mathrm{Sem}_{\mathrm{EN}}(w_{\mathrm{EN}}) = \mathrm{Sem}_{\mathrm{JA}}(w_{\mathrm{JA}}).$$

Hence,

$$\mathfrak{L} = \big(\mathcal{U}, \{\Phi_{\mathrm{tr}}\}\big)$$

is a Meta-Natural Language in which entire natural-language structures, rather than individual words or sentences, are treated as objects of a meta-level transformation.

3.8 Linguistic Graph

A *Linguistic Graph* is a graph equipped with linguistic information on its vertices and/or edges, enabling qualitative descriptions of entities and their relationships [26, 236].

Let

$$\mathcal{T}_V \quad \text{and} \quad \mathcal{T}_E$$

be nonempty sets of linguistic terms for vertices and edges, respectively. Let

$$\mathcal{X}_V \subseteq \mathcal{T}_V^{n_V}, \qquad \mathcal{X}_E \subseteq \mathcal{T}_E^{n_E},$$

where

$$n_V, n_E \geq 1,$$

be the corresponding linguistic state spaces.

Definition 3.8.1 (Linguistic Graph)**.** Let

$$G = (V, E)$$

be a finite simple undirected graph, where

$$\varnothing \neq V$$

and

$$E \subseteq \big\{\{u, v\} \mid u, v \in V,\ u \neq v\big\}.$$

A *Linguistic Graph* is a structure

$$G_{\mathrm{L}} = (V, E, \lambda_V, \lambda_E),$$

where

$$\lambda_V : V \longrightarrow \mathcal{X}_V$$

is a vertex-labeling function and

$$\lambda_E : E \longrightarrow \mathcal{X}_E$$

is an edge-labeling function.

For every vertex $v \in V$,

$$\lambda_V(v) = \big(\ell_{V,1}(v), \ldots, \ell_{V,n_V}(v)\big), \qquad \ell_{V,i}(v) \in \mathcal{T}_V,$$

and for every edge $e \in E$,

$$\lambda_E(e) = \big(\ell_{E,1}(e), \ldots, \ell_{E,n_E}(e)\big), \qquad \ell_{E,j}(e) \in \mathcal{T}_E.$$

Thus, the vertices and edges of the underlying graph may carry qualitative linguistic information. When

$$n_V = n_E = 1,$$

each vertex and edge is assigned a single linguistic term.

Remark 3.8.2. If linguistic information is assigned only to vertices, the structure may be written as

$$G_{\mathrm{L}} = (V, E, \lambda_V), \qquad \lambda_V : V \to \mathcal{X}_V,$$

and is called a *vertex-labeled Linguistic Graph.*

Similarly, if linguistic information is assigned only to edges, one may use

$$G_{\mathrm{L}} = (V, E, \lambda_E), \qquad \lambda_E : E \to \mathcal{X}_E,$$

yielding an *edge-labeled Linguistic Graph.*

Example 3.8.3 (A Linguistic Graph for Social Activity)**.** Consider a small social network with

$$V = \{v_1, v_2, v_3, v_4\},$$

where each vertex represents an individual.

Let

$$E = \big\{e_{12}, e_{23}, e_{34}, e_{14}\big\},$$

where

$$e_{12} = \{v_1, v_2\}, \qquad e_{23} = \{v_2, v_3\},$$
$$e_{34} = \{v_3, v_4\}, \qquad e_{14} = \{v_1, v_4\}.$$

Thus, two vertices are adjacent whenever the corresponding individuals are directly connected in the social network.

For the vertex information, let

$$\mathcal{T}_V = \{\texttt{low}, \texttt{medium}, \texttt{high}\},$$

and take

$$\mathcal{X}_V = \mathcal{T}_V.$$

The linguistic terms in $\mathcal{T}_V$ represent the activity levels of the individuals.

Define

$$\lambda_V : V \longrightarrow \mathcal{X}_V$$

by

$$\lambda_V(v_1) = \texttt{high}, \qquad \lambda_V(v_2) = \texttt{medium},$$
$$\lambda_V(v_3) = \texttt{low}, \qquad \lambda_V(v_4) = \texttt{medium}.$$

For the edge information, let

$$\mathcal{T}_E = \{\texttt{weak}, \texttt{moderate}, \texttt{strong}\},$$

and take

$$\mathcal{X}_E = \mathcal{T}_E.$$

These terms represent the qualitative strength of the social relationships.

Define

$$\lambda_E : E \longrightarrow \mathcal{X}_E$$

by

$$\lambda_E(e_{12}) = \texttt{strong}, \qquad \lambda_E(e_{23}) = \texttt{moderate},$$
$$\lambda_E(e_{34}) = \texttt{weak}, \qquad \lambda_E(e_{14}) = \texttt{strong}.$$

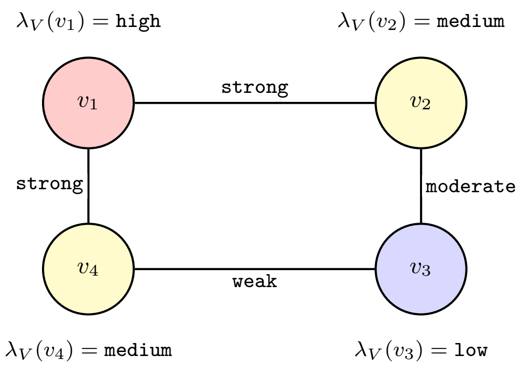


Figure 3.1: A Linguistic Graph with linguistic information assigned to both vertices and edges.

Hence,

$$G_{\mathrm{L}} = (V, E, \lambda_V, \lambda_E)$$

is a Linguistic Graph.

The graph is illustrated in Figure 3.1.

For example,

$$\lambda_V(v_1) = \texttt{high}$$

means that the individual represented by v_1 has a high activity level, whereas

$$\lambda_V(v_3) = \texttt{low}$$

indicates a low activity level.

Similarly,

$$\lambda_E(e_{12}) = \texttt{strong}$$

means that the relationship between v_1 and v_2 is qualitatively strong, while

$$\lambda_E(e_{34}) = \texttt{weak}$$

represents a weak relationship between v_3 and v_4.

Therefore,

$$G_{\mathrm{L}} = (V, E, \lambda_V, \lambda_E)$$

simultaneously represents the underlying graph structure, qualitative information associated with individual vertices, and qualitative information associated with relationships between vertices.

3.9 Linguistic Directed Graph

A Linguistic Directed Graph is a directed graph whose vertices or arcs carry linguistic terms, representing qualitative and directional relationships between entities.

Definition 3.9.1 (Linguistic Directed Graph)**.** Let $\mathcal{X} \subseteq \mathcal{T}^n$ be a linguistic state space. A *Linguistic Directed Graph* is a triple

$$D_{\mathrm{L}} = (V, A, \lambda),$$

where

$$A \subseteq \{(u, v) \in V \times V : u \neq v\}$$

is a set of directed edges and

$$\lambda : V \longrightarrow \mathcal{X}$$

assigns a linguistic value or linguistic vector to each vertex.

An arc $(u, v) \in A$ represents a directed relation from the linguistic state $\lambda(u)$ to the linguistic state $\lambda(v)$.

Example 3.9.2 (A Linguistic Directed Graph for Information Flow)**.** Consider a directed communication network with

$$V = \{v_1, v_2, v_3, v_4\},$$

where each vertex represents a person.

Let the set of directed edges be

$$A = \big\{(v_1, v_2), (v_2, v_3), (v_1, v_4), (v_4, v_3)\big\}.$$

Each arc $(u, v) \in A$ indicates that information is transmitted from person u to person v.

Let the linguistic term set be

$$\mathcal{T} = \{\texttt{low}, \texttt{medium}, \texttt{high}\},$$

and let

$$\mathcal{X} = \mathcal{T}.$$

Define the linguistic labeling function

$$\lambda : V \longrightarrow \mathcal{X}$$

by

$$\lambda(v_1) = \texttt{high}, \qquad \lambda(v_2) = \texttt{medium},$$
$$\lambda(v_3) = \texttt{low}, \qquad \lambda(v_4) = \texttt{medium}.$$

Hence,

$$D_{\mathrm{L}} = (V, A, \lambda)$$

is a Linguistic Directed Graph.

In this example, the linguistic label of each vertex describes the communication activity level of the corresponding person. Thus, v_1 is highly active, v_2 and v_4 are moderately active, and v_3 is less active.

The directed graph can be visualized as shown in Figure 3.2.

For example, the arc

$$(v_1, v_2)$$

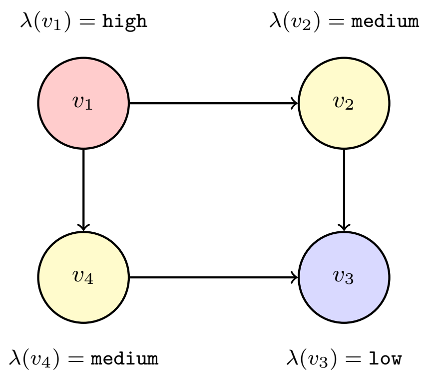


Figure 3.2: A Linguistic Directed Graph representing directed communication relationships together with linguistic vertex labels.

represents a directed relation from the linguistic state

$$\lambda(v_1) = \texttt{high}$$

to

$$\lambda(v_2) = \texttt{medium}.$$

Similarly, the arc

$$(v_4, v_3)$$

represents a directed relation from

$$\lambda(v_4) = \texttt{medium}$$

to

$$\lambda(v_3) = \texttt{low}.$$

Therefore, the structure

$$D_{\mathrm{L}} = (V, A, \lambda)$$

simultaneously represents both the direction of communication and the linguistic information associated with the vertices.

3.10 Linguistic Edge-Weighted Graph

A Linguistic Edge-Weighted Graph assigns linguistic terms or linguistic vectors as edge weights, enabling qualitative representation of relationship strengths or characteristics.

Definition 3.10.1 (Linguistic Edge-Weighted Graph)**.** Let $\mathcal{X} \subseteq \mathcal{T}^n$ be a linguistic state space and let $\mathcal{W} \subseteq \mathcal{T}^m$, $m \geq 1$, be a linguistic weight space. A *Linguistic Edge-Weighted Graph* is a quadruple

$$G_{\mathrm{LW}} = (V, E, \lambda, w),$$

where (V, E, λ) is a Linguistic Graph and

$$w : E \longrightarrow \mathcal{W}$$

assigns a linguistic weight to each edge.

Hence, for every edge $e \in E$,

$$w(e) = \big(\omega_1(e), \ldots, \omega_m(e)\big), \qquad \omega_j(e) \in \mathcal{T},$$

describes the relation between its incident vertices by linguistic terms such as

“near”, “far”, “strong”, “weak”.

For $m = 1$, the edge weight is a single linguistic term.

Example 3.10.2 (A Linguistic Edge-Weighted Graph for a Transportation Network)**.** Consider a transportation network with
$$V = \{v_1, v_2, v_3, v_4\},$$
where the vertices represent four locations, such as stations or terminals.

Let
$$E = \big\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_1, v_4\}\big\}.$$
Thus, two locations are adjacent whenever a direct connection exists between them.

Let the linguistic term set be
$$\mathcal{T} = \{\texttt{low}, \texttt{medium}, \texttt{high}, \texttt{near}, \texttt{moderate}, \texttt{far}\}.$$
Take
$$\mathcal{X} = \{\texttt{low}, \texttt{medium}, \texttt{high}\}, \qquad \mathcal{W} = \{\texttt{near}, \texttt{moderate}, \texttt{far}\}.$$

Define the vertex-labeling function
$$\lambda : V \longrightarrow \mathcal{X}$$
by
$$\lambda(v_1) = \texttt{high}, \qquad \lambda(v_2) = \texttt{medium},$$
$$\lambda(v_3) = \texttt{low}, \qquad \lambda(v_4) = \texttt{medium}.$$

Here, the linguistic value assigned to each vertex may describe, for example, the traffic activity level of the corresponding location.

Next, define the linguistic edge-weight function
$$w : E \longrightarrow \mathcal{W}$$
by
$$w(\{v_1, v_2\}) = \texttt{near}, \qquad w(\{v_2, v_3\}) = \texttt{moderate},$$
$$w(\{v_3, v_4\}) = \texttt{near}, \qquad w(\{v_1, v_4\}) = \texttt{far}.$$

Therefore,
$$G_{\mathrm{LW}} = (V, E, \lambda, w)$$
is a Linguistic Edge-Weighted Graph. A visualization of this graph is shown in Figure 3.3.

For instance,
$$w(\{v_1, v_2\}) = \texttt{near}$$
indicates that the direct connection between v_1 and v_2 is linguistically described as short or close, whereas
$$w(\{v_1, v_4\}) = \texttt{far}$$
indicates a linguistically described long or distant connection.

Similarly, the vertex labels show qualitative activity levels:
$$\lambda(v_1) = \texttt{high}$$
means that location v_1 has a high traffic level, while
$$\lambda(v_3) = \texttt{low}$$
means that location v_3 has a low traffic level.

Hence, the graph
$$G_{\mathrm{LW}} = (V, E, \lambda, w)$$
simultaneously represents linguistic information on vertices and qualitative linguistic weights on edges.

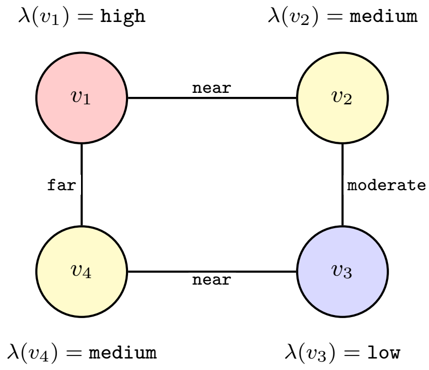


Figure 3.3: A Linguistic Edge-Weighted Graph with linguistic vertex labels and linguistic edge weights.

3.11 Linguistic HyperGraph

A HyperGraph generalizes a graph by allowing each hyperedge to connect any nonempty subset of vertices rather than only vertex pairs [237–240]. A Linguistic HyperGraph is a hypergraph whose vertices or hyperedges are labeled by linguistic terms, enabling qualitative higher-order relational representation.

Definition 3.11.1 (HyperGraph)**.** A *HyperGraph* is a pair

$$H = (V, E),$$

where V is a nonempty set of vertices and

$$E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}$$

is a family of nonempty subsets of V, called *hyperedges*.

Definition 3.11.2 (Linguistic HyperGraph)**.** Let $\mathcal{T}$ be a nonempty set of linguistic terms and let $\mathcal{X} \subseteq \mathcal{T}^m$ be a linguistic state space. A *Linguistic HyperGraph* is a triple

$$H_{\mathrm{L}} = (V, E, \lambda),$$

where (V, E) is a HyperGraph and

$$\lambda : V \longrightarrow \mathcal{X}$$

assigns a linguistic term or linguistic vector to each vertex.

Example 3.11.3 (A Linguistic HyperGraph for Research Collaboration)**.** Consider a research collaboration system with

$$V = \{v_1, v_2, v_3, v_4, v_5\},$$

where each vertex represents a researcher.

Let the hyperedge set be

$$E = \{\{v_1, v_2, v_3\}, \{v_2, v_4, v_5\}, \{v_1, v_4\}\}.$$

Each hyperedge represents a research project involving all researchers contained in that hyperedge. Hence,

$$H = (V, E)$$

is a HyperGraph.

Let the linguistic term set be

$$\mathcal{T} = \{\texttt{junior}, \texttt{intermediate}, \texttt{senior}\},$$

and let

$$\mathcal{X} = \mathcal{T}.$$

Define the linguistic labeling function

$$\lambda : V \longrightarrow \mathcal{X}$$

by

$$\lambda(v_1) = \texttt{senior}, \qquad \lambda(v_2) = \texttt{intermediate},$$

$$\lambda(v_3) = \texttt{junior}, \qquad \lambda(v_4) = \texttt{senior}, \qquad \lambda(v_5) = \texttt{junior}.$$

Therefore,

$$H_{\mathrm{L}} = (V, E, \lambda)$$

is a Linguistic HyperGraph.

In this example, the linguistic label of each vertex describes the academic level of the corresponding researcher. Thus, v_1 and v_4 are senior researchers, v_2 is an intermediate researcher, and v_3 and v_5 are junior researchers. A visualization of this Linguistic HyperGraph is shown in Figure 3.4.

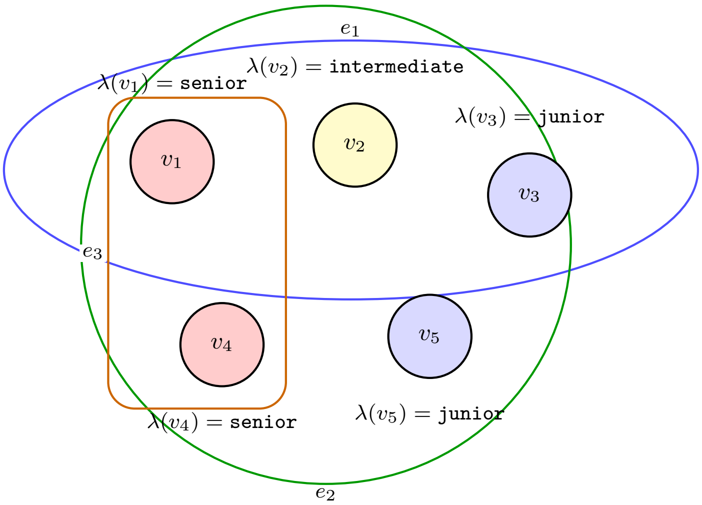


Figure 3.4: A Linguistic HyperGraph with linguistic vertex labels and hyperedges representing higher-order collaborations.

Let

$$e_1 = \{v_1, v_2, v_3\}, \qquad e_2 = \{v_2, v_4, v_5\}, \qquad e_3 = \{v_1, v_4\}.$$

Then e_1 represents a collaborative project involving the three researchers v_1, v_2, and v_3. Their linguistic states are

$$\{\lambda(v_1), \lambda(v_2), \lambda(v_3)\} = \{\texttt{senior}, \texttt{intermediate}, \texttt{junior}\}.$$

Similarly, the hyperedge

$$e_2 = \{v_2, v_4, v_5\}$$

represents another higher-order collaboration involving researchers with linguistic labels

$$\{\lambda(v_2), \lambda(v_4), \lambda(v_5)\} = \{\texttt{intermediate}, \texttt{senior}, \texttt{junior}\}.$$

Finally,

$$e_3 = \{v_1, v_4\}$$

represents a smaller collaboration between two senior researchers.

Hence, the Linguistic HyperGraph

$$H_{\mathrm{L}} = (V, E, \lambda)$$

simultaneously represents higher-order relationships among multiple vertices and qualitative linguistic information associated with those vertices.

3.12 Linguistic SuperHyperGraph

A SuperHyperGraph generalizes a HyperGraph by allowing supervertices and superedges to possess higher-order, recursively structured relationships beyond ordinary vertex–edge incidence [229, 241–244]. A Linguistic SuperHyperGraph assigns linguistic terms or vectors to supervertices or superedges, enabling qualitative representation of complex higher-order relationships.

Definition 3.12.1 (n-SuperHyperGraph)**.** [229] Let V_0 be a nonempty base set and let $n \geq 0$. Define

$$\mathcal{P}^0(V_0) = V_0, \qquad \mathcal{P}^{k+1}(V_0) = \mathcal{P}\big(\mathcal{P}^k(V_0)\big).$$

An *n-SuperHyperGraph* is a pair

$$\mathrm{SHG}^{(n)} = (V, E),$$

where

$$V \subseteq \mathcal{P}^n(V_0), \qquad E \subseteq \mathcal{P}(V) \setminus \{\emptyset\}.$$

The elements of V are called *n-supervertices*, and the elements of E are called *n-superedges*.

Definition 3.12.2 (Linguistic n-SuperHyperGraph)**.** Let $\mathcal{T}$ be a nonempty set of linguistic terms and let $\mathcal{X} \subseteq \mathcal{T}^m$ be a linguistic state space. A *Linguistic n-SuperHyperGraph* is a triple

$$\mathrm{LSHG}^{(n)} = (V, E, \lambda),$$

where (V, E) is an n-SuperHyperGraph and

$$\lambda : V \longrightarrow \mathcal{X}$$

assigns a linguistic term or linguistic vector to each n-supervertex.

Example 3.12.3 (A Linguistic 2-SuperHyperGraph)**.** Let the base vertex set be

$$V_0 = \{a, b, c, d\}.$$

Consider the case

$$n = 2.$$

Define three 2-supervertices by

$$S_1 = \big\{\{a, b\}, \{c\}\big\},$$
$$S_2 = \big\{\{b, c\}, \{d\}\big\},$$

and

$$S_3 = \big\{\{a, d\}\big\}.$$

Since

$$S_1, S_2, S_3 \in \mathcal{P}^2(V_0),$$

let

$$V = \{S_1, S_2, S_3\}.$$

Define the superedge set by

$$E = \{\{S_1, S_2\}, \{S_2, S_3\}, \{S_1, S_2, S_3\}\}.$$

Hence,

$$(V, E)$$

is a 2-SuperHyperGraph.

Let the linguistic term set be

$$\mathcal{T} = \{\texttt{low}, \texttt{medium}, \texttt{high}\},$$

and take

$$\mathcal{X} = \mathcal{T}.$$

Define

$$\lambda : V \longrightarrow \mathcal{X}$$

by

$$\lambda(S_1) = \texttt{high}, \qquad \lambda(S_2) = \texttt{medium}, \qquad \lambda(S_3) = \texttt{low}.$$

Therefore,

$$\mathrm{LSHG}^{(2)} = (V, E, \lambda)$$

is a Linguistic 2-SuperHyperGraph.

For example, the superedge

$$e = \{S_1, S_2, S_3\}$$

connects three 2-supervertices whose linguistic states are

$$\{\texttt{high}, \texttt{medium}, \texttt{low}\}.$$

Thus, this structure simultaneously represents higher-order superhypergraph relationships and linguistic information assigned to individual supervertices.

3.13 Linguistic Maps, Geometry, and Planes

Linguistic Maps associate linguistic terms between linguistic sets, allowing one-to-one, one-to-many, partial, or otherwise nonclassical correspondences between language-based elements systematically [23]. Linguistic Geometry studies geometric relationships among linguistic points, lines, and distances, where comparability and direction may depend on linguistic variables [23]. Linguistic Planes are Cartesian products of linguistic sets or continua, representing points as ordered pairs of linguistic coordinates for analysis [23].

Definition 3.13.1 (Linguistic Map)**.** Let S and T be linguistic sets. A *Linguistic Map* is a set-valued mapping

$$F : S \longrightarrow \mathcal{P}(T),$$

where $F(s)$ is the set of linguistic terms in T associated with $s \in S$. The value $F(s)$ may be empty, a singleton, or contain several linguistic terms. A classical single-valued linguistic map is obtained when

$$|F(s)| = 1 \qquad \text{for every } s \in S.$$

Example 3.13.2 (A Simple Linguistic Map)**.** Let

$$S = \{\texttt{cold}, \texttt{warm}, \texttt{hot}\}$$

and

$$T = \{\texttt{low}, \texttt{medium}, \texttt{high}\}$$

be two linguistic sets.

Define the set-valued linguistic map

$$F : S \longrightarrow \mathcal{P}(T)$$

by

$$F(\texttt{cold}) = \{\texttt{low}\},$$

$$F(\texttt{warm}) = \{\texttt{medium}, \texttt{high}\},$$

and

$$F(\texttt{hot}) = \emptyset.$$

Thus,

$$|F(\texttt{cold})| = 1, \qquad |F(\texttt{warm})| = 2, \qquad |F(\texttt{hot})| = 0.$$

Hence, this example illustrates that a Linguistic Map may assign a singleton, several linguistic terms, or no linguistic term to a given linguistic input.

Definition 3.13.3 (Linguistic Geometry)**.** Let $(L, \preceq)$ be a partially ordered linguistic set and let $\mathcal{D}_L$ be a set of linguistic distance terms. Define

$$C_L = \{(x, y) \in L \times L : x \preceq y \text{ or } y \preceq x\}.$$

A *Linguistic Geometry* is a structure

$$\mathcal{G}_L = (L, \preceq, d_L),$$

where

$$d_L : C_L \longrightarrow \mathcal{D}_L$$

assigns a linguistic distance to each comparable pair of linguistic points. A linguistic line between two points is defined only when the points are comparable.

Example 3.13.4 (A Simple Linguistic Geometry)**.** Let

$$L = \{\texttt{low}, \texttt{medium}, \texttt{high}\}$$

be a totally ordered linguistic set satisfying

$$\texttt{low} \preceq \texttt{medium} \preceq \texttt{high}.$$

Let the set of linguistic distance terms be

$$\mathcal{D}_L = \{\texttt{zero}, \texttt{near}, \texttt{far}\}.$$

Since L is totally ordered, every pair of elements is comparable. Define

$$d_L : C_L \longrightarrow \mathcal{D}_L$$

by

$$d_L(x, x) = \texttt{zero} \qquad \text{for every } x \in L,$$

$$d_L(\texttt{low}, \texttt{medium}) = d_L(\texttt{medium}, \texttt{low}) = \texttt{near},$$
$$d_L(\texttt{medium}, \texttt{high}) = d_L(\texttt{high}, \texttt{medium}) = \texttt{near},$$

and

$$d_L(\texttt{low}, \texttt{high}) = d_L(\texttt{high}, \texttt{low}) = \texttt{far}.$$

Therefore,

$$\mathcal{G}_L = (L, \preceq, d_L)$$

is a Linguistic Geometry.

For example, the linguistic points

$$\texttt{low} \quad \text{and} \quad \texttt{high}$$

are comparable, and their linguistic distance is

$$d_L(\texttt{low}, \texttt{high}) = \texttt{far}.$$

Hence, a linguistic line between these two points is defined.

Definition 3.13.5 (Linguistic Plane)**.** Let L_1 and L_2 be linguistic continua, or totally ordered linguistic sets. A *Linguistic Plane* is the Cartesian product

$$\Pi_L = L_1 \times L_2.$$

Each linguistic point is therefore an ordered pair

$$p = (\ell_1, \ell_2), \qquad \ell_1 \in L_1, \quad \ell_2 \in L_2,$$

whose components are called its linguistic coordinates.

Example 3.13.6 (A Simple Linguistic Plane)**.** Let

$$L_1 = \{\texttt{cold}, \texttt{warm}, \texttt{hot}\}$$

be a totally ordered linguistic set describing temperature, with

$$\texttt{cold} \preceq \texttt{warm} \preceq \texttt{hot}.$$

Let

$$L_2 = \{\texttt{dry}, \texttt{moderate}, \texttt{humid}\}$$

be a totally ordered linguistic set describing humidity, with

$$\texttt{dry} \preceq \texttt{moderate} \preceq \texttt{humid}.$$

The corresponding Linguistic Plane is

$$\Pi_L = L_1 \times L_2.$$

For example,

$$p_1 = (\texttt{cold}, \texttt{dry}),$$
$$p_2 = (\texttt{warm}, \texttt{moderate}),$$

and

$$p_3 = (\texttt{hot}, \texttt{humid})$$

are linguistic points in Π_L.

In particular,

$$p_2 = (\texttt{warm}, \texttt{moderate})$$

describes a state having a linguistically warm temperature and a moderate humidity level.

Thus,

$$\Pi_L = L_1 \times L_2$$

provides a two-dimensional linguistic space in which each point is represented entirely by linguistic coordinates.

3.14 Linguistic Variables and Linguistic Sets

Linguistic variables take qualitative linguistic terms as values, while linguistic sets collect the possible terms associated with each variable.

Definition 3.14.1 (Linguistic Variable)**.** A *linguistic variable* is a variable V whose possible values are expressed by linguistic terms rather than numerical values. Its associated set of linguistic values is denoted by

$$L(V) = \{\ell : \ell \text{ is a linguistic term associated with } V\}.$$

The set $L(V)$ may be finite or infinite and may be ordered, partially ordered, or unordered.

Example 3.14.2 (A Simple Linguistic Variable)**.** Consider the variable

$$V = \text{Temperature}.$$

Instead of assigning numerical values to V, suppose that temperature is described by the linguistic terms

$$L(V) = \{\texttt{very cold}, \texttt{cold}, \texttt{moderate}, \texttt{warm}, \texttt{hot}\}.$$

These terms may be ordered as

$$\texttt{very cold} \preceq \texttt{cold} \preceq \texttt{moderate} \preceq \texttt{warm} \preceq \texttt{hot}.$$

For example, instead of representing the current temperature by a numerical value such as

$$25^\circ\text{C},$$

one may describe it linguistically as

$$V = \texttt{warm}.$$

Hence, V is a Linguistic Variable whose possible values are qualitative linguistic terms rather than numerical quantities.

Definition 3.14.3 (Linguistic Set)**.** Let V be a linguistic variable. A *linguistic set* associated with V is a nonempty set

$$L(V) = \{\ell_i : i \in I\}$$

whose elements ℓ_i are linguistic terms representing possible qualitative values of V.

Example 3.14.4 (A Simple Linguistic Set)**.** Let

$$V = \text{Risk Level}$$

be a Linguistic Variable.

Define its associated Linguistic Set by

$$L(V) = \{\texttt{very low}, \texttt{low}, \texttt{medium}, \texttt{high}, \texttt{very high}\}.$$

Each element of $L(V)$ represents a possible qualitative evaluation of risk. For instance,

$$\texttt{low} \in L(V) \qquad \text{and} \qquad \texttt{high} \in L(V).$$

The set may be ordered as

$$\texttt{very low} \preceq \texttt{low} \preceq \texttt{medium} \preceq \texttt{high} \preceq \texttt{very high}.$$

Therefore,

$$L(V)$$

is a Linguistic Set associated with the Linguistic Variable

$$V = \text{Risk Level}.$$

3.15 Linguistic Multidimensional Space

A Linguistic Multidimensional Space is a Cartesian product of linguistic sets, representing multidimensional objects as vectors of linguistic terms [245].

Definition 3.15.1 (Linguistic Multidimensional Space)**.** [245] Let $V_1, \ldots, V_n$ be linguistic variables with associated linguistic sets

$$L_1, \ldots, L_n.$$

The corresponding *n-dimensional linguistic space* is the Cartesian product

$$\mathcal{L}^{(n)} = L_1 \times L_2 \times \cdots \times L_n.$$

Each element

$$x = (x_1, \ldots, x_n) \in \mathcal{L}^{(n)}$$

is called a *linguistic point* or *linguistic vector*, where

$$x_i \in L_i, \qquad i = 1, \ldots, n.$$

Example 3.15.2 (A Three-Dimensional Linguistic Space)**.** Consider the three linguistic variables

$$V_1 = \text{Temperature}, \qquad V_2 = \text{Humidity}, \qquad V_3 = \text{Wind Speed}.$$

Let their associated linguistic sets be

$$L_1 = \{\texttt{cold}, \texttt{warm}, \texttt{hot}\},$$

$$L_2 = \{\texttt{dry}, \texttt{moderate}, \texttt{humid}\},$$

and

$$L_3 = \{\texttt{low}, \texttt{medium}, \texttt{high}\}.$$

The corresponding three-dimensional linguistic space is

$$\mathcal{L}^{(3)} = L_1 \times L_2 \times L_3.$$

For example,

$$x = (\texttt{warm}, \texttt{humid}, \texttt{low})$$

is an element of

$$\mathcal{L}^{(3)}.$$

Thus, x is a linguistic point, or linguistic vector, representing a weather condition characterized by a warm temperature, high humidity, and low wind speed.

Other linguistic points include

$$y = (\texttt{cold}, \texttt{dry}, \texttt{medium})$$

and

$$z = (\texttt{hot}, \texttt{moderate}, \texttt{high}).$$

Hence,

$$\mathcal{L}^{(3)} = L_1 \times L_2 \times L_3$$

provides a multidimensional qualitative representation of weather conditions using linguistic coordinates.

3.16 Linguistic Matrices and Linguistic Algebraic Structures

Linguistic Matrices are matrices whose entries are linguistic terms, enabling qualitative information to be organized and manipulated in matrix form [25]. Linguistic Algebraic Structures equip linguistic sets with operations such as minimum and maximum, forming semigroups, semivector spaces, and semilinear algebras. Throughout this subsection, let $(L, \leq)$ be a nonempty totally ordered linguistic set or linguistic continuum. Hence, the operations

$$\min, \max : L \times L \longrightarrow L$$

are well defined.

Definition 3.16.1 (Linguistic Matrix)**.** An $m \times n$ *Linguistic Matrix* over L is a matrix

$$A = (a_{ij}) \in L^{m \times n},$$

where

$$a_{ij} \in L, \qquad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

Thus, every entry of A is a linguistic term belonging to L.

Example 3.16.2 (A Simple Linguistic Matrix)**.** Let

$$L = \{\texttt{low}, \texttt{medium}, \texttt{high}\}$$

be a linguistic set.

Consider the matrix

$$A = \begin{pmatrix} \texttt{high} & \texttt{medium} & \texttt{low} \\ \texttt{medium} & \texttt{high} & \texttt{medium} \\ \texttt{low} & \texttt{medium} & \texttt{high} \end{pmatrix}.$$

Since every entry of A belongs to L, we have

$$A \in L^{3\times 3}.$$

Therefore, A is a 3×3 Linguistic Matrix over L.

For example, if the rows represent three alternatives and the columns represent three qualitative criteria, then

$$a_{12} = \texttt{medium}$$

indicates that the first alternative has a medium linguistic evaluation with respect to the second criterion.

Definition 3.16.3 (Linguistic Monoid)**.** Let L be a nonempty set of linguistic terms. A *Linguistic Monoid* is a triple

$$(L, \circ, e_L),$$

where

$$\circ : L \times L \longrightarrow L$$

is a binary operation satisfying the following conditions:

1. **Associativity:**
$$(a \circ b) \circ c = a \circ (b \circ c)$$
for all $a, b, c \in L$;
2. **Identity:** there exists an element $e_L \in L$ such that
$$e_L \circ a = a \circ e_L = a$$
for every $a \in L$.

Thus, a Linguistic Monoid is a monoid whose underlying elements are linguistic terms.

Remark 3.16.4. Let

$$L = \{s_0, s_1, \ldots, s_g\}$$

be a finite totally ordered Linguistic Term Set satisfying

$$s_0 \preceq s_1 \preceq \cdots \preceq s_g.$$

Then

$$(L, \max, s_0)$$

is a commutative Linguistic Monoid, since

$$\max(s_0, s_i) = s_i.$$

Similarly,

$$(L, \min, s_g)$$

is a commutative Linguistic Monoid, since

$$\min(s_g, s_i) = s_i.$$

Definition 3.16.5 (Linguistic Semifield)**.** Let L be a nonempty set of linguistic terms, and let

$$0_L, 1_L \in L, \qquad 0_L \neq 1_L.$$

A *Linguistic Semifield* is a structure

$$\mathbb{L} = (L, \oplus, \otimes, 0_L, 1_L),$$

satisfying the following conditions:

1. $$(L, \oplus, 0_L)$$
   is a commutative monoid;
2. $$(L \setminus \{0_L\}, \otimes, 1_L)$$
   is an abelian group;
3. multiplication distributes over addition:
   $$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c),$$
   $$(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$$
   for all $a, b, c \in L$;
4. 0_L is absorbing with respect to multiplication:
   $$0_L \otimes a = a \otimes 0_L = 0_L$$
   for every $a \in L$.

Thus, a Linguistic Semifield is a semifield whose carrier consists of linguistic terms and whose algebraic operations are defined directly on those terms.

Definition 3.16.6 (Idempotent Linguistic Semifield)**.** A Linguistic Semifield

$$\mathbb{L} = (L, \oplus, \otimes, 0_L, 1_L)$$

is called *idempotent* if

$$a \oplus a = a$$

for every $a \in L$.

Definition 3.16.7 (Linguistic Semigroup)**.** [246–248] A *Linguistic Semigroup* is a pair

$$(L, *),$$

where

$$* : L \times L \longrightarrow L$$

is an associative linguistic binary operation; that is,

$$(x * y) * z = x * (y * z)$$

for all $x, y, z \in L$. In particular,

$$(L, \min) \qquad \text{and} \qquad (L, \max)$$

are Linguistic Semigroups whenever L is totally ordered.

Example 3.16.8 (A Linguistic Semigroup under Maximum)**.** Let

$$L = \{\texttt{low}, \texttt{medium}, \texttt{high}\}$$

be totally ordered by

$$\texttt{low} \preceq \texttt{medium} \preceq \texttt{high}.$$

Define

$$* : L \times L \longrightarrow L$$

by

$$x * y = \max\{x, y\}.$$

For example,

$$\texttt{low} * \texttt{medium} = \texttt{medium},$$

and

$$\texttt{medium} * \texttt{high} = \texttt{high}.$$

Since the maximum operation is associative,

$$\max\{\max\{x, y\}, z\} = \max\{x, \max\{y, z\}\}$$

for all $x, y, z \in L$.

Hence,

$$(L, \max)$$

is a Linguistic Semigroup. In fact, it is also commutative and idempotent because

$$\max\{x, y\} = \max\{y, x\}$$

and

$$\max\{x, x\} = x.$$

Definition 3.16.9 (Linguistic Semivector Space)**.** Let

$$\mathbb{S} = (S, \min, \max)$$

be a Linguistic Semifield and let V be a linguistic set containing S.

A *Linguistic Semivector Space* over $\mathbb{S}$ is defined in either of the following dual forms:

$$(V, \min), \qquad \max\{s, v\} \in V \quad \text{for all } s \in S,\ v \in V,$$

or

$$(V, \max), \qquad \min\{s, v\} \in V \quad \text{for all } s \in S,\ v \in V,$$

where $(V, \min)$ or $(V, \max)$, respectively, is a Linguistic Monoid.

Example 3.16.10 (A Linguistic Semivector Space)**.** Let

$$S = \{\texttt{low}, \texttt{medium}, \texttt{high}\}$$

with

$$\texttt{low} \preceq \texttt{medium} \preceq \texttt{high},$$

and let

$$\mathbb{S} = (S, \min, \max)$$

be a Linguistic Semifield.

Consider the larger linguistic set

$$V = \{\texttt{very low}, \texttt{low}, \texttt{medium}, \texttt{high}, \texttt{very high}\},$$

ordered by

$$\texttt{very low} \preceq \texttt{low} \preceq \texttt{medium} \preceq \texttt{high} \preceq \texttt{very high}.$$

Clearly,

$$S \subseteq V.$$

Consider the structure

$$(V, \min).$$

The operation min is closed and associative on V, and `very high` is its identity element. Thus,

$$(V, \min)$$

is a Linguistic Monoid.

Moreover, for every

$$s \in S \qquad \text{and} \qquad v \in V,$$

we have

$$\max\{s, v\} \in V.$$

For example,

$$\max\{\texttt{medium}, \texttt{very low}\} = \texttt{medium},$$

and

$$\max\{\texttt{high}, \texttt{very high}\} = \texttt{very high}.$$

Therefore,

$$(V, \min)$$

is a Linguistic Semivector Space over

$$\mathbb{S} = (S, \min, \max).$$

Definition 3.16.11 (Linguistic Semilinear Algebra)**.** Let

$$\mathbb{S} = (S, \min, \max)$$

be a Linguistic Semifield. A *Linguistic Semilinear Algebra* over $\mathbb{S}$ is a structure

$$\mathcal{A} = (A, \min, \max)$$

such that both

$$(A, \min) \qquad \text{and} \qquad (A, \max)$$

are Linguistic Semivector Spaces over $\mathbb{S}$ and both operations min and max are closed on A.

Example 3.16.12 (A Linguistic Semilinear Algebra)**.** Let

$$S = \{\texttt{low}, \texttt{medium}, \texttt{high}\}$$

and let

$$\mathbb{S} = (S, \min, \max)$$

be a Linguistic Semifield.

Consider

$$A = \{\texttt{very low}, \texttt{low}, \texttt{medium}, \texttt{high}, \texttt{very high}\},$$

with the total order

$$\texttt{very low} \preceq \texttt{low} \preceq \texttt{medium} \preceq \texttt{high} \preceq \texttt{very high}.$$

Both operations

$$\min : A \times A \longrightarrow A$$

and

$$\max : A \times A \longrightarrow A$$

are closed.

Furthermore,

$$(A, \min)$$

is a Linguistic Semivector Space over $\mathbb{S}$ because

$$\max\{s, a\} \in A$$

for every $s \in S$ and $a \in A$.

Similarly,

$$(A, \max)$$

is a Linguistic Semivector Space over $\mathbb{S}$ because

$$\min\{s, a\} \in A$$

for every $s \in S$ and $a \in A$.

For example,

$$\max\{\texttt{medium}, \texttt{very low}\} = \texttt{medium} \in A,$$

while

$$\min\{\texttt{medium}, \texttt{very high}\} = \texttt{medium} \in A.$$

Consequently,

$$\mathcal{A} = (A, \min, \max)$$

is a Linguistic Semilinear Algebra over

$$\mathbb{S} = (S, \min, \max).$$

3.17 Linguistic Uncertain Set

An Uncertain Set assigns to each element an admissible uncertainty degree determined by a prescribed uncertainty model [249–252]. The framework is designed to provide a common mathematical language for a broad range of uncertainty representations, including fuzzy, intuitionistic fuzzy, neutrosophic, and other related models [68, 73]. The particular semantics of uncertainty are encoded in the model-dependent degree-domain and in the structural conditions imposed on its elements.

Definition 3.17.1 (Linguistic Uncertain Set)**.** Let X be a nonempty set, let M be an uncertain model with

$$\mathrm{Dom}(M) \subseteq [0,1]^k,$$

and let L be a linguistic term set equipped with an order-preserving linguistic scale

$$\iota : L \longrightarrow [0,1].$$

Define

$$\mathrm{Dom}_L(M) = \left\{(a_1,\ldots,a_k) \in L^k \;\middle|\; \big(\iota(a_1),\ldots,\iota(a_k)\big) \in \mathrm{Dom}(M)\right\}.$$

A *Linguistic Uncertain Set* of type (M, L, ι) on X is a pair

$$U_L = (X, \mu_{M,L}),$$

where

$$\mu_{M,L} : X \longrightarrow \mathrm{Dom}_L(M)$$

assigns to each $x \in X$ an admissible tuple of linguistic uncertainty degrees.

Example 3.17.2 (A One-Dimensional Linguistic Uncertain Set)**.** Let

$$X = \{x_1, x_2, x_3\}$$

and consider the uncertain model M with

$$\mathrm{Dom}(M) = [0,1].$$

Let the linguistic term set be

$$L = \{\texttt{very low}, \texttt{low}, \texttt{medium}, \texttt{high}, \texttt{very high}\},$$

ordered by

$$\texttt{very low} \preceq \texttt{low} \preceq \texttt{medium} \preceq \texttt{high} \preceq \texttt{very high}.$$

Define the order-preserving linguistic scale

$$\iota : L \longrightarrow [0,1]$$

by

$$\iota(\texttt{very low}) = 0, \qquad \iota(\texttt{low}) = 0.25,$$
$$\iota(\texttt{medium}) = 0.50, \qquad \iota(\texttt{high}) = 0.75, \qquad \iota(\texttt{very high}) = 1.$$

Since

$$\mathrm{Dom}(M) = [0,1],$$

we have

$$\mathrm{Dom}_L(M) = L.$$

Define

$$\mu_{M,L} : X \longrightarrow L$$

by

$$\mu_{M,L}(x_1) = \texttt{low},$$
$$\mu_{M,L}(x_2) = \texttt{medium},$$

and

$$\mu_{M,L}(x_3) = \texttt{very high}.$$

Thus,

$$U_L = (X, \mu_{M,L})$$

is a Linguistic Uncertain Set.

Its numerical interpretation is

$$\iota(\mu_{M,L}(x_1)) = 0.25, \qquad \iota(\mu_{M,L}(x_2)) = 0.50, \qquad \iota(\mu_{M,L}(x_3)) = 1,$$

all of which belong to $\mathrm{Dom}(M)$.

Example 3.17.3 (A Three-Dimensional Linguistic Uncertain Set)**.** Let

$$X = \{a, b, c\},$$

and consider an uncertain model M whose degree domain is

$$\mathrm{Dom}(M) = \left\{(t, i, f) \in [0,1]^3 : 0 \le t + i + f \le 3\right\}.$$

Let

$$L = \{\texttt{low}, \texttt{medium}, \texttt{high}\}$$

with the order

$$\texttt{low} \preceq \texttt{medium} \preceq \texttt{high}.$$

Define the order-preserving linguistic scale

$$\iota : L \longrightarrow [0,1]$$

by

$$\iota(\texttt{low}) = 0.2, \qquad \iota(\texttt{medium}) = 0.5, \qquad \iota(\texttt{high}) = 0.8.$$

Then

$$\mathrm{Dom}_L(M) = \left\{(\ell_1, \ell_2, \ell_3) \in L^3 : 0 \le \iota(\ell_1) + \iota(\ell_2) + \iota(\ell_3) \le 3\right\}.$$

Define

$$\mu_{M,L} : X \longrightarrow \mathrm{Dom}_L(M)$$

by

$$\mu_{M,L}(a) = (\texttt{high}, \texttt{low}, \texttt{low}),$$
$$\mu_{M,L}(b) = (\texttt{medium}, \texttt{medium}, \texttt{low}),$$

and

$$\mu_{M,L}(c) = (\texttt{low}, \texttt{high}, \texttt{medium}).$$

Their numerical representations are

$$(\iota(\texttt{high}), \iota(\texttt{low}), \iota(\texttt{low})) = (0.8, 0.2, 0.2),$$

$$(\iota(\texttt{medium}), \iota(\texttt{medium}), \iota(\texttt{low})) = (0.5, 0.5, 0.2),$$

and

$$(\iota(\texttt{low}), \iota(\texttt{high}), \iota(\texttt{medium})) = (0.2, 0.8, 0.5).$$

Each of these tuples belongs to

$$\mathrm{Dom}(M).$$

Therefore,

$$U_L = (X, \mu_{M,L})$$

is a Linguistic Uncertain Set with three linguistic uncertainty components for each element.

3.18 HyperLinguistic Fuzzy Set (Proposed in this book)

A Linguistic Fuzzy Set assigns each element a linguistic membership term, providing a qualitative representation of fuzzy membership and uncertainty [253–255]. A HyperLinguistic Fuzzy Set assigns each element a nonempty set of linguistic membership terms, representing hyperlinguistic fuzzy information and uncertainty. Let

$$S = \{s_0, s_1, \ldots, s_\tau\}$$

be a linguistic term set, and let X be a nonempty universe.

Definition 3.18.1 (Linguistic Fuzzy Set)**.** [253,254] A *Linguistic Fuzzy Set* A on X is characterized by a linguistic membership function

$$\mu_A : X \longrightarrow S,$$

where $\mu_A(x) \in S$ represents the linguistic membership degree of x in A. Equivalently,

$$A = \left\{ \big(x, \mu_A(x)\big) : x \in X \right\}.$$

Let S be a nonempty linguistic term set. Define the associated hyperlinguistic domain by

$$\mathcal{H}(S) = \mathcal{P}(S) \setminus \{\emptyset\}.$$

Thus, every

$$H \in \mathcal{H}(S)$$

is a nonempty set of linguistic terms and may be regarded as a *hyperword* over S. More generally, any nonempty family

$$\mathcal{L}_H \subseteq \mathcal{H}(S)$$

is called a *HyperLanguage* over S.

Definition 3.18.2 (HyperLinguistic Fuzzy Set)**.** Let X be a nonempty universe and let

$$\mathcal{L}_H \subseteq \mathcal{P}(S) \setminus \{\emptyset\}$$

be a nonempty HyperLanguage over S. A *HyperLinguistic Fuzzy Set* A_H on X is characterized by a hyperlinguistic membership function

$$\mu_{A_H} : X \longrightarrow \mathcal{L}_H.$$

Hence,

$$A_H = \left\{ \big(x, \mu_{A_H}(x)\big) : x \in X \right\},$$

where

$$\mu_{A_H}(x) \in \mathcal{P}(S) \setminus \{\emptyset\}$$

is a nonempty set of linguistic membership terms associated with x.

Example 3.18.3 (A Simple HyperLinguistic Fuzzy Set)**.** Let

$$X = \{x_1, x_2, x_3\}$$

be a universe, and let

$$S = \{\texttt{low}, \texttt{medium}, \texttt{high}, \texttt{very high}\}$$

be a linguistic term set.

Consider the HyperLanguage

$$\mathcal{L}_H = \{\{\texttt{low}, \texttt{medium}\}, \{\texttt{medium}, \texttt{high}\}, \{\texttt{high}, \texttt{very high}\}\}.$$

Clearly,

$$\mathcal{L}_H \subseteq \mathcal{P}(S) \setminus \{\emptyset\}.$$

Define the hyperlinguistic membership function

$$\mu_{A_H} : X \longrightarrow \mathcal{L}_H$$

by

$$\mu_{A_H}(x_1) = \{\texttt{low}, \texttt{medium}\},$$

$$\mu_{A_H}(x_2) = \{\texttt{medium}, \texttt{high}\},$$

and

$$\mu_{A_H}(x_3) = \{\texttt{high}, \texttt{very high}\}.$$

Therefore,

$$A_H = \{(x_1, \{\texttt{low}, \texttt{medium}\}), (x_2, \{\texttt{medium}, \texttt{high}\}), (x_3, \{\texttt{high}, \texttt{very high}\})\}$$

is a HyperLinguistic Fuzzy Set on X.

For example,

$$\mu_{A_H}(x_2) = \{\texttt{medium}, \texttt{high}\}$$

indicates that the fuzzy membership of x_2 is represented not by a single linguistic term, but by the hyperword consisting of the two linguistic terms `medium` and `high`.

If

$$\iota : S \longrightarrow [0, 1]$$

is a linguistic scale, define its hyperextension by

$$\iota_H(H) = \{\iota(s) : s \in H\}, \qquad H \in \mathcal{H}(S).$$

Thus,

$$\iota_H \circ \mu_{A_H} : X \longrightarrow \mathcal{P}([0, 1]) \setminus \{\emptyset\}$$

provides the corresponding set-valued fuzzy interpretation.

Theorem 3.18.4 (Well-Definedness of a HyperLinguistic Fuzzy Set)**.** *Let X and S be nonempty sets, let*

$$\emptyset \neq \mathcal{L}_H \subseteq \mathcal{P}(S) \setminus \{\emptyset\},$$

and let

$$\mu_{A_H} : X \longrightarrow \mathcal{L}_H.$$

Then A_H is well defined as a HyperLinguistic Fuzzy Set. Moreover, for every linguistic scale

$$\iota : S \longrightarrow [0, 1],$$

the induced mapping

$$\iota_H \circ \mu_{A_H} : X \longrightarrow \mathcal{P}([0, 1]) \setminus \{\emptyset\}$$

is well defined.

Proof. For every $x \in X$,

$$\mu_{A_H}(x) \in \mathcal{L}_H \subseteq \mathcal{P}(S) \setminus \{\emptyset\}.$$

Therefore,

$$\mu_{A_H}(x) \neq \emptyset$$

and every element of $\mu_{A_H}(x)$ belongs to S. Hence the hyperlinguistic membership value of each x is a valid hyperword over S.

Since

$$\iota(s) \in [0, 1] \qquad \text{for every } s \in S,$$

we obtain

$$\iota_H(\mu_{A_H}(x)) = \{\iota(s) : s \in \mu_{A_H}(x)\} \subseteq [0, 1].$$

Because $\mu_{A_H}(x)$ is nonempty, its image under ι is also nonempty. Consequently,

$$\iota_H(\mu_{A_H}(x)) \in \mathcal{P}([0, 1]) \setminus \{\emptyset\}.$$

Thus both the hyperlinguistic membership assignment and its fuzzy interpretation are well defined. □

3.19 SuperHyperLinguistic Fuzzy Set (Proposed in this book)

Let S be a nonempty linguistic term set. Define the iterated power sets recursively by

$$\mathcal{P}^0(S) = S, \qquad \mathcal{P}^{n+1}(S) = \mathcal{P}\big(\mathcal{P}^n(S)\big), \qquad n \geq 0.$$

For $n \geq 1$, define the n-superhyperlinguistic domain by

$$\mathcal{SH}^{(n)}(S) = \mathcal{P}^n(S) \setminus \{\emptyset\}.$$

An element of $\mathcal{SH}^{(n)}(S)$ is called an *n-superhyperword* over S.

A nonempty family

$$\mathcal{L}^{(n)}_{SH} \subseteq \mathcal{SH}^{(n)}(S)$$

is called an *n-SuperHyperLanguage* over S.

Definition 3.19.1 (n-SuperHyperLinguistic Fuzzy Set)**.** Let X be a nonempty universe and let

$$\mathcal{L}_{SH}^{(n)} \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}$$

be a nonempty n-SuperHyperLanguage. An *n-SuperHyperLinguistic Fuzzy Set* $A_{SH}^{(n)}$ on X is characterized by a membership function

$$\mu_{A_{SH}^{(n)}} : X \longrightarrow \mathcal{L}_{SH}^{(n)}.$$

Thus,

$$A_{SH}^{(n)} = \left\{ \left(x, \mu_{A_{SH}^{(n)}}(x) \right) : x \in X \right\},$$

where

$$\mu_{A_{SH}^{(n)}}(x) \in \mathcal{P}^n(S) \setminus \{\emptyset\}$$

is an n-superhyperword representing the higher-order linguistic membership information of x.

Example 3.19.2 (A 2-SuperHyperLinguistic Fuzzy Set)**.** Let

$$X = \{x_1, x_2, x_3\}$$

be a nonempty universe, and let

$$S = \{\texttt{low}, \texttt{medium}, \texttt{high}\}$$

be a linguistic term set.

Consider the case

$$n = 2.$$

Define

$$\mathcal{P}^0(S) = S, \qquad \mathcal{P}^1(S) = \mathcal{P}(S), \qquad \mathcal{P}^2(S) = \mathcal{P}(\mathcal{P}(S)).$$

Let

$$H_1 = \big\{\{\texttt{low}\}, \{\texttt{low}, \texttt{medium}\}\big\},$$

$$H_2 = \big\{\{\texttt{medium}\}, \{\texttt{medium}, \texttt{high}\}\big\},$$

and

$$H_3 = \big\{\{\texttt{high}\}, \{\texttt{low}, \texttt{high}\}\big\}.$$

Since

$$H_1, H_2, H_3 \in \mathcal{P}^2(S) \setminus \{\emptyset\},$$

each H_i is a 2-superhyperword over S.

Define the 2-SuperHyperLanguage

$$\mathcal{L}_{SH}^{(2)} = \{H_1, H_2, H_3\}.$$

Then

$$\mathcal{L}_{SH}^{(2)} \subseteq \mathcal{P}^2(S) \setminus \{\emptyset\}.$$

Now define

$$\mu_{A_{SH}^{(2)}} : X \longrightarrow \mathcal{L}_{SH}^{(2)}$$

by

$$\mu_{A_{SH}^{(2)}}(x_1) = H_1, \qquad \mu_{A_{SH}^{(2)}}(x_2) = H_2, \qquad \mu_{A_{SH}^{(2)}}(x_3) = H_3.$$

Hence,

$$A_{SH}^{(2)} = \{(x_1, H_1), (x_2, H_2), (x_3, H_3)\}$$

is a 2-SuperHyperLinguistic Fuzzy Set.

For example,

$$\mu_{A_{SH}^{(2)}}(x_2) = \left\{\{\texttt{medium}\}, \{\texttt{medium}, \texttt{high}\}\right\}$$

represents the membership information of x_2 by a higher-order linguistic object containing two linguistic hyperwords.

Thus, the membership structure follows the hierarchy

$$\text{linguistic term} \longrightarrow \text{hyperword} \longrightarrow \text{2-superhyperword}.$$

Let

$$\iota : S \longrightarrow [0,1]$$

be a linguistic scale. Define recursively

$$\iota^{(0)} = \iota$$

and

$$\iota^{(r+1)}(A) = \left\{\iota^{(r)}(a) : a \in A\right\}, \qquad A \in \mathcal{P}^{r+1}(S).$$

Then

$$\iota^{(n)} : \mathcal{P}^n(S) \longrightarrow \mathcal{P}^n([0,1])$$

gives the numerical fuzzy interpretation of the n-superhyperlinguistic structure.

Theorem 3.19.3 (Well-Definedness of an n-SuperHyperLinguistic Fuzzy Set)**.** *Let X and S be nonempty, let $n \geq 1$, and suppose that*

$$\emptyset \neq \mathcal{L}_{SH}^{(n)} \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}.$$

If

$$\mu_{A_{SH}^{(n)}} : X \longrightarrow \mathcal{L}_{SH}^{(n)},$$

then $A_{SH}^{(n)}$ is a well-defined n-SuperHyperLinguistic Fuzzy Set.

Moreover, if

$$\iota : S \longrightarrow [0,1],$$

then

$$\iota^{(n)} \circ \mu_{A_{SH}^{(n)}} : X \longrightarrow \mathcal{P}^n([0,1]) \setminus \{\emptyset\}$$

is well defined.

Proof. For every $x \in X$,

$$\mu_{A_{SH}^{(n)}}(x) \in \mathcal{L}_{SH}^{(n)} \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}.$$

Hence each membership value is a valid nonempty n-superhyperword over S.

We prove recursively that

$$\iota^{(r)} : \mathcal{P}^r(S) \longrightarrow \mathcal{P}^r([0,1])$$

is well defined for every $r \geq 0$.

For $r = 0$, this follows immediately from

$$\iota : S \longrightarrow [0, 1].$$

Assume that $\iota^{(r)}$ is well defined. For

$$A \in \mathcal{P}^{r+1}(S),$$

every $a \in A$ belongs to $\mathcal{P}^r(S)$, and therefore

$$\iota^{(r)}(a) \in \mathcal{P}^r([0, 1]).$$

Consequently,

$$\iota^{(r+1)}(A) = \{\iota^{(r)}(a) : a \in A\} \in \mathcal{P}^{r+1}([0, 1]).$$

Thus the claim follows by induction.

Finally,

$$\mu_{A^{(n)}_{SH}}(x) \neq \emptyset,$$

so its image under $\iota^{(n)}$ is also nonempty. Therefore,

$$\iota^{(n)}\left(\mu_{A^{(n)}_{SH}}(x)\right) \in \mathcal{P}^n([0, 1]) \setminus \{\emptyset\}.$$

Hence the n-SuperHyperLinguistic Fuzzy Set and its induced higher-order fuzzy interpretation are both well defined. □

3.20 Linguistic Term Set

A Linguistic Term Set is a collection of qualitative linguistic labels representing the possible values of a linguistic variable [256–260]. Several related concepts have also been studied, including Hesitant Fuzzy Linguistic Term Sets [261–264] and Probabilistic Linguistic Term Sets [265–268].

Definition 3.20.1 (Linguistic Term Set)**.** (cf. [256, 257]) A *Linguistic Term Set* is a nonempty set

$$S = \{s_i : i \in I\},$$

whose elements s_i are linguistic terms representing qualitative values of a linguistic variable.

If S is ordered, we write

$$s_i \preceq s_j$$

whenever s_i represents a linguistic value not greater than s_j.

Example 3.20.2 (A Linguistic Term Set)**.** Let

$$V = \text{Temperature}$$

be a linguistic variable.

Consider the linguistic term set

$$S = \{\texttt{very cold}, \texttt{cold}, \texttt{moderate}, \texttt{warm}, \texttt{hot}\}.$$

The elements of S describe qualitative temperature levels. They may be ordered as

$$\texttt{very cold} \preceq \texttt{cold} \preceq \texttt{moderate} \preceq \texttt{warm} \preceq \texttt{hot}.$$

Hence,

$$S$$

is a Linguistic Term Set associated with the linguistic variable $V = \text{Temperature}$.

For reference, a comparison between a Linguistic Set and a Linguistic Term Set is presented in Table 3.1.

Table 3.1: Comparison between a Linguistic Set and a Linguistic Term Set.

Aspect	**Linguistic Set**	**Linguistic Term Set**
Basic form	$L(V) = \{\ell_i : i \in I\}$	$S = \{s_i : i \in I\}$
Main interpretation	A set of linguistic values associated with a linguistic variable V.	A collection of linguistic terms representing qualitative values.
Emphasis	Association with a particular linguistic variable or domain.	Organization and representation of linguistic labels or terms.
Example	$L(\text{Risk}) = \{\texttt{low}, \texttt{medium}, \texttt{high}\}$	$S = \{\texttt{low}, \texttt{medium}, \texttt{high}\}$
Relation	May be regarded as the value set of a linguistic variable.	May serve as the linguistic term domain defining that value set.

3.21 HyperLinguistic Term Set (Proposed in this book)

A HyperLinguistic Term Set is a collection of nonempty subsets of linguistic terms, enabling grouped or multiple qualitative linguistic descriptions.

Definition 3.21.1 (HyperLinguistic Term Set)**.** Let S be a Linguistic Term Set. A *hyperlinguistic term* over S is a nonempty subset

$$H \in \mathcal{P}(S) \setminus \{\emptyset\}.$$

A *HyperLinguistic Term Set* over S is a nonempty family

$$\mathcal{H}_L \subseteq \mathcal{P}(S) \setminus \{\emptyset\}.$$

Thus, each element of $\mathcal{H}_L$ consists of one or more linguistic terms from S.

Example 3.21.2 (A HyperLinguistic Term Set)**.** Let

$$S = \{\texttt{low}, \texttt{medium}, \texttt{high}\}$$

be a Linguistic Term Set.

Define

$$H_1 = \{\texttt{low}\},$$

$$H_2 = \{\texttt{low}, \texttt{medium}\},$$

and

$$H_3 = \{\texttt{medium}, \texttt{high}\}.$$

Since

$$H_1, H_2, H_3 \in \mathcal{P}(S) \setminus \{\emptyset\},$$

each H_i is a hyperlinguistic term over S.

Therefore,

$$\mathcal{H}_L = \{H_1, H_2, H_3\}$$

satisfies

$$\mathcal{H}_L \subseteq \mathcal{P}(S) \setminus \{\emptyset\}.$$

Hence,

$$\mathcal{H}_L = \{\{\texttt{low}\}, \{\texttt{low}, \texttt{medium}\}, \{\texttt{medium}, \texttt{high}\}\}$$

is a HyperLinguistic Term Set.

3.22 SuperHyperLinguistic Term Set (Proposed in this book)

A SuperHyperLinguistic Term Set consists of higher-order collections of linguistic terms formed through iterated power-set constructions for hierarchical representation.

Definition 3.22.1 (n-SuperHyperLinguistic Term Set)**.** Let S be a Linguistic Term Set and define the iterated power sets by

$$\mathcal{P}^0(S) = S, \qquad \mathcal{P}^{n+1}(S) = \mathcal{P}\big(\mathcal{P}^n(S)\big), \qquad n \geq 0.$$

For $n \geq 1$, an *n-superhyperlinguistic term* over S is a nonempty element

$$H^{(n)} \in \mathcal{P}^n(S) \setminus \{\emptyset\}.$$

An *n-SuperHyperLinguistic Term Set* over S is a nonempty collection

$$\mathcal{SH}_L^{(n)} \subseteq \mathcal{P}^n(S) \setminus \{\emptyset\}.$$

For $n = 1$, this construction reduces to a HyperLinguistic Term Set.

Example 3.22.2 (A 2-SuperHyperLinguistic Term Set)**.** Let

$$S = \{\texttt{low}, \texttt{medium}, \texttt{high}\}$$

be a Linguistic Term Set.

Consider

$$n = 2.$$

Define the following 2-superhyperlinguistic terms:

$$K_1 = \{\{\texttt{low}\}, \{\texttt{low}, \texttt{medium}\}\},$$

$$K_2 = \{\{\texttt{medium}\}, \{\texttt{medium}, \texttt{high}\}\},$$

and

$$K_3 = \{\{\texttt{high}\}, \{\texttt{low}, \texttt{high}\}\}.$$

Since

$$K_1, K_2, K_3 \in \mathcal{P}^2(S) \setminus \{\emptyset\},$$

each K_i is a 2-superhyperlinguistic term over S.

Define
$$\mathcal{SH}_L^{(2)} = \{K_1, K_2, K_3\}.$$
Then
$$\mathcal{SH}_L^{(2)} \subseteq \mathcal{P}^2(S) \setminus \{\emptyset\}.$$

Therefore,
$$\mathcal{SH}_L^{(2)}$$
is a 2-SuperHyperLinguistic Term Set.

For example,
$$K_2 = \{\{\texttt{medium}\}, \{\texttt{medium}, \texttt{high}\}\}$$
is a higher-order linguistic object consisting of two hyperlinguistic terms.

3.23 Uncertain Linguistic Term Set (Proposed in this book)

An Uncertain Linguistic Term Set represents imprecise linguistic assessments using intervals of ordered linguistic terms bounded by lower and upper linguistic values.

Definition 3.23.1 (Uncertain Linguistic Term Set)**.** Let
$$S = \{s_0, s_1, \dots, s_g\}$$
be a finite totally ordered Linguistic Term Set satisfying
$$s_0 \preceq s_1 \preceq \cdots \preceq s_g.$$

An *uncertain linguistic term* over S is an interval
$$[s_i, s_j], \qquad 0 \le i \le j \le g,$$
where s_i and s_j are respectively the lower and upper linguistic bounds.

The *Uncertain Linguistic Term Set* generated by S is
$$\mathcal{U}(S) = \{[s_i, s_j] : 0 \le i \le j \le g\}.$$

The degenerate case
$$[s_i, s_i]$$
is identified with the ordinary linguistic term s_i.

Example 3.23.2 (An Uncertain Linguistic Term Set for Risk Assessment)**.** Let
$$S = \{s_0, s_1, s_2, s_3, s_4\} = \{\texttt{very low}, \texttt{low}, \texttt{medium}, \texttt{high}, \texttt{very high}\}$$
be a totally ordered Linguistic Term Set satisfying
$$s_0 \preceq s_1 \preceq s_2 \preceq s_3 \preceq s_4.$$

The corresponding Uncertain Linguistic Term Set is

$$\mathcal{U}(S) = \{[s_i, s_j] : 0 \le i \le j \le 4\}.$$

For example,

$$u_1 = [s_1, s_2] = [\texttt{low}, \texttt{medium}],$$

$$u_2 = [s_2, s_4] = [\texttt{medium}, \texttt{very high}],$$

and

$$u_3 = [s_3, s_3] = [\texttt{high}, \texttt{high}]$$

are uncertain linguistic terms in $\mathcal{U}(S)$.

The term

$$u_1 = [\texttt{low}, \texttt{medium}]$$

represents an uncertain risk assessment lying linguistically between `low` and `medium`, while

$$u_2 = [\texttt{medium}, \texttt{very high}]$$

represents a wider range of uncertainty.

Since

$$u_3 = [s_3, s_3],$$

it is a degenerate uncertain linguistic term and is identified with the ordinary linguistic term

$$s_3 = \texttt{high}.$$

Thus, $\mathcal{U}(S)$ provides interval-valued linguistic descriptions for representing imprecise qualitative assessments.

3.24 Linguistic Automaton

A Linguistic Automaton incorporates linguistic information into states, transitions, or processing rules to model qualitative computational behavior and decision processes [269]. Let L be a nonempty Linguistic Term Set.

Definition 3.24.1 (Linguistic Automaton)**.** A *Linguistic Automaton* is a structure

$$\mathcal{A}_L = (Q, \Sigma, \delta, q_0, F, \lambda),$$

where Q is a set of states, Σ is an input alphabet,

$$\delta : Q \times \Sigma \longrightarrow \mathcal{P}(Q)$$

is a transition mapping, $q_0 \in Q$ is an initial state, $F \subseteq Q$ is a set of final states, and

$$\lambda : Q \cup (Q \times \Sigma \times Q) \longrightarrow L$$

assigns linguistic information to states or transitions.

Thus, the behavior of the automaton may be described using linguistic terms such as

$$\texttt{low}, \quad \texttt{medium}, \quad \texttt{high}.$$

Example 3.24.2 (A Simple Linguistic Automaton)**.** Let

$$Q = \{q_0, q_1, q_2\}, \qquad \Sigma = \{a, b\},$$

and let the Linguistic Term Set be

$$L = \{\texttt{low}, \texttt{medium}, \texttt{high}\}.$$

Consider the transition

$$q_0 \xrightarrow{a} q_1$$

with linguistic value

$$\lambda(q_0, a, q_1) = \texttt{high},$$

and the transition

$$q_1 \xrightarrow{b} q_2$$

with

$$\lambda(q_1, b, q_2) = \texttt{medium}.$$

Thus, the automaton describes its transition behavior using linguistic terms such as `high` and `medium`.

3.25 Linguistic Finite Automaton

A Finite Automaton is a computational model with finitely many states and transitions, processing input symbols according to predefined transition rules [270–272]. A Linguistic Finite Automaton uses finitely many states and linguistic transition values to represent qualitative state changes and symbolic processing [273]. Let L be a nonempty Linguistic Term Set.

Definition 3.25.1 (Linguistic Finite Automaton)**.** [273] A *Linguistic Finite Automaton* is a tuple

$$\mathcal{A}_{LF} = (Q, \Sigma, \delta_L, q_0, F),$$

where Q and Σ are finite nonempty sets, $q_0 \in Q$, $F \subseteq Q$, and

$$\delta_L : Q \times \Sigma \times Q \longrightarrow L$$

assigns a linguistic transition value to each possible transition.

If

$$\iota : L \longrightarrow [0, 1]$$

is a linguistic scale, then

$$\iota \circ \delta_L : Q \times \Sigma \times Q \longrightarrow [0, 1]$$

gives the corresponding numerical transition degrees.

Example 3.25.2 (A Simple Linguistic Finite Automaton)**.** Let

$$Q = \{q_0, q_1\}, \qquad \Sigma = \{0, 1\},$$

and

$$L = \{\texttt{weak}, \texttt{moderate}, \texttt{strong}\}.$$

Define the linguistic transition function by

$$\delta_L(q_0, 1, q_1) = \texttt{strong},$$

$$\delta_L(q_0, 0, q_0) = \texttt{moderate},$$

and

$$\delta_L(q_1, 1, q_1) = \texttt{weak}.$$

Hence,

$$\mathcal{A}_{LF} = (Q, \Sigma, \delta_L, q_0, \{q_1\})$$

is a Linguistic Finite Automaton with linguistically evaluated transitions.

3.26 Linguistic Petri Net

A Petri net is a mathematical model of concurrent systems using places, transitions, and tokens to represent states and events [274, 275]. Related concepts such as Fuzzy Petri Nets [276–278] and Neutrosophic Petri Nets [279–281] are also known. A Linguistic Petri Net integrates linguistic terms into places, transitions, or rules for qualitative knowledge representation, reasoning, and dynamic modeling [282–287]. Let L be a nonempty Linguistic Term Set.

Definition 3.26.1 (Linguistic Petri Net)**.** A *Linguistic Petri Net* is a structure

$$\mathcal{N}_L = (P, T, I, O, M_0, \lambda_P, \lambda_T),$$

where P is a finite set of places, T is a finite set of transitions,

$$I : P \times T \longrightarrow \mathbb{N}_0, \qquad O : T \times P \longrightarrow \mathbb{N}_0$$

are the input and output incidence functions, and M_0 is the initial marking.

The mappings

$$\lambda_P : P \longrightarrow L$$

and

$$\lambda_T : T \longrightarrow L$$

assign linguistic information to places and transitions, respectively.

Hence, propositions, truth assessments, transition conditions, or rule strengths can be represented by linguistic terms.

Example 3.26.2 (A Simple Linguistic Petri Net)**.** Let

$$P = \{p_1, p_2\}, \qquad T = \{t_1\},$$

where p_1 represents an input condition and p_2 an output condition.

Let

$$I(p_1, t_1) = 1, \qquad O(t_1, p_2) = 1.$$

Take the Linguistic Term Set

$$L = \{\texttt{low}, \texttt{medium}, \texttt{high}\}.$$

Define

$$\lambda_P(p_1) = \texttt{high}, \qquad \lambda_P(p_2) = \texttt{medium},$$

and

$$\lambda_T(t_1) = \texttt{high}.$$

Thus, the transition t_1 represents a linguistically strong rule connecting a highly supported input condition to a medium output condition.

3.27 Linguistic Dynamic System

A Linguistic Dynamic System represents states, inputs, and transitions through linguistic values, enabling qualitative modeling of evolving processes over time [288–292]. Let L_X and L_U be linguistic state and input spaces, respectively.

Definition 3.27.1 (Linguistic Dynamic System)**.** A *Linguistic Dynamic System* is a discrete-time system

$$\mathcal{D}_L = (L_X, L_U, \Phi),$$

where

$$\Phi : L_X \times L_U \longrightarrow L_X$$

is a linguistic state-transition mapping.

The evolution of the system is given by

$$x_{t+1} = \Phi(x_t, u_t),$$

where

$$x_t \in L_X, \qquad u_t \in L_U.$$

Thus, the states, inputs, and evolution laws of the system may be expressed using linguistic or fuzzy linguistic information instead of purely numerical quantities.

Example 3.27.2 (A Simple Linguistic Dynamic System)**.** Let the linguistic state space be

$$L_X = \{\texttt{low}, \texttt{medium}, \texttt{high}\},$$

and the linguistic input space be

$$L_U = \{\texttt{decrease}, \texttt{stable}, \texttt{increase}\}.$$

Define the transition mapping

$$\Phi : L_X \times L_U \longrightarrow L_X$$

by, for example,

$$\Phi(\texttt{medium}, \texttt{increase}) = \texttt{high},$$

$$\Phi(\texttt{high}, \texttt{decrease}) = \texttt{medium},$$

and

$$\Phi(\texttt{low}, \texttt{stable}) = \texttt{low}.$$

Hence, if

$$x_t = \texttt{medium} \quad \text{and} \quad u_t = \texttt{increase},$$

then

$$x_{t+1} = \Phi(x_t, u_t) = \texttt{high}.$$

This gives a simple Linguistic Dynamic System.

3.28 Linguistic Preference Relation

A Linguistic Preference Relation assigns linguistic terms to pairs of alternatives, representing qualitative preference strengths and comparative judgments between alternatives [293–296]. Related concepts, such as Fuzzy Linguistic Preference Relations [297–299] and Neutrosophic Linguistic Preference Relations [300, 301], are also known.

Let

$$X = \{x_1, \dots, x_n\}$$

be a finite set of alternatives, and let

$$L = \{\ell_0, \dots, \ell_g\}$$

be an ordered Linguistic Term Set.

Definition 3.28.1 (Linguistic Preference Relation)**.** [293, 294] A *Linguistic Preference Relation* on X is a mapping

$$R_L : X \times X \longrightarrow L,$$

where

$$R_L(x_i, x_j) = \ell_{ij}$$

represents the linguistic preference of x_i over x_j.

Equivalently, it may be represented by the linguistic matrix

$$R_L = (\ell_{ij})_{n\times n}.$$

If $N : L \to L$ is a linguistic negation operator, reciprocity may additionally be imposed by

$$R_L(x_j, x_i) = N\big(R_L(x_i, x_j)\big).$$

Example 3.28.2 (A Simple Linguistic Preference Relation)**.** Let

$$X = \{x_1, x_2, x_3\}$$

be a set of alternatives, and let

$$L = \{\texttt{very low}, \texttt{low}, \texttt{equal}, \texttt{high}, \texttt{very high}\}$$

be an ordered Linguistic Term Set.

Define a Linguistic Preference Relation

$$R_L : X \times X \longrightarrow L$$

by

$$R_L(x_1, x_2) = \texttt{high},$$
$$R_L(x_1, x_3) = \texttt{very high},$$

and

$$R_L(x_2, x_3) = \texttt{low}.$$

For example,

$$R_L(x_1, x_2) = \texttt{high}$$

means that x_1 is linguistically preferred to x_2 with a high preference level.

The relation may be represented by a linguistic preference matrix

$$R_L = \begin{pmatrix} \texttt{equal} & \texttt{high} & \texttt{very high} \\ \texttt{low} & \texttt{equal} & \texttt{low} \\ \texttt{very low} & \texttt{high} & \texttt{equal} \end{pmatrix}.$$

3.29 Linguistic Distribution Assessment

A Linguistic Distribution Assessment represents an evaluation by assigning symbolic proportions to linguistic terms, capturing distributed qualitative judgments across an ordered linguistic term set effectively (cf. [302–304]).

Definition 3.29.1 (Linguistic Distribution Assessment)**.** Let

$$S = \{s_0, s_1, \dots, s_g\}$$

be a Linguistic Term Set. A *Linguistic Distribution Assessment (LDA)* on S is a set

$$D = \{(s_i, \alpha_i) : i = 0, 1, \dots, g\},$$

where

$$\alpha_i \geq 0, \qquad \sum_{i=0}^{g} \alpha_i = 1.$$

Here, α_i denotes the symbolic proportion associated with the linguistic term s_i.

Example 3.29.2 (A Linguistic Distribution Assessment)**.** Let

$$S = \{s_0, s_1, s_2, s_3, s_4\} = \{\text{very low}, \text{low}, \text{medium}, \text{high}, \text{very high}\}$$

be an ordered Linguistic Term Set.

Suppose that the evaluation of an alternative is represented by

$$D = \{(s_0, 0.05), (s_1, 0.10), (s_2, 0.25), (s_3, 0.40), (s_4, 0.20)\}.$$

Since

$$0.05 + 0.10 + 0.25 + 0.40 + 0.20 = 1,$$

the set D is a Linguistic Distribution Assessment on S.

For example, the pair

$$(s_3, 0.40) = (\text{high}, 0.40)$$

indicates that the linguistic term high has symbolic proportion 0.40 in the assessment.

3.30 2-Tuple Linguistic Representation Model

The 2-Tuple Linguistic Representation Model expresses linguistic information as a linguistic term paired with a symbolic translation, preserving precision during aggregation and computation accurately reliably [305–311].

Definition 3.30.1 (2-Tuple Linguistic Representation Model)**.** Let

$$S = \{s_0, s_1, \dots, s_g\}$$

be an ordered Linguistic Term Set. A *2-tuple linguistic representation* is an ordered pair

$$(s_i, \alpha),$$

where

$$s_i \in S, \qquad \alpha \in [-0.5, 0.5)$$

is the symbolic translation associated with s_i.

For any

$$\beta \in [0, g],$$

the corresponding linguistic 2-tuple is defined by

$$\Delta(\beta) = (s_i, \alpha), \qquad i = \text{round}(\beta), \qquad \alpha = \beta - i.$$

Example 3.30.2 (A 2-Tuple Linguistic Representation)**.** Let

$$S = \{s_0, s_1, s_2, s_3, s_4\} = \{\text{very low}, \text{low}, \text{medium}, \text{high}, \text{very high}\}$$

be an ordered Linguistic Term Set.

Suppose that an aggregated linguistic evaluation is represented numerically by

$$\beta = 2.30.$$

Then

$$i = \text{round}(2.30) = 2$$

and

$$\alpha = \beta - i = 2.30 - 2 = 0.30.$$

Therefore,

$$\Delta(2.30) = (s_2, 0.30) = (\text{medium}, 0.30).$$

Thus, the linguistic assessment is represented by the basic linguistic term medium together with the symbolic translation 0.30.

3.31 Linguistic Z-Number

A Z-number represents uncertain information using two components: a fuzzy restriction and a measure of reliability associated with that restriction [312–315]. A Linguistic Z-Number represents an assessment together with its linguistic reliability, allowing qualitative information and confidence in that information to be modeled simultaneously effectively clearly [316–319].

Definition 3.31.1 (Linguistic Z-Number)**.** Let

$$S_A = \{s_0, s_1, \ldots, s_m\}$$

and

$$S_B = \{r_0, r_1, \ldots, r_n\}$$

be Linguistic Term Sets. A *Linguistic Z-Number* is an ordered pair

$$Z = (A, B), \qquad A \in S_A, \qquad B \in S_B,$$

where A represents a linguistic assessment or restriction and B represents the linguistic reliability, certainty, or credibility associated with A.

Example 3.31.2 (A Linguistic Z-Number)**.** Let

$$S_A = \{\text{very low}, \text{low}, \text{medium}, \text{high}, \text{very high}\}$$

be a Linguistic Term Set for assessments, and let

$$S_B = \{\text{very unreliable}, \text{unreliable}, \text{moderately reliable}, \text{reliable}, \text{very reliable}\}$$

be a Linguistic Term Set for reliability.

Consider

$$A = \text{high} \qquad \text{and} \qquad B = \text{reliable}.$$

Then

$$Z = (A, B) = (\text{high}, \text{reliable})$$

is a Linguistic Z-Number.

Here, A = high represents the linguistic assessment, while B = reliable expresses the linguistic reliability associated with that assessment.

3.32 Probabilistic Linguistic Term Set

A Probabilistic Linguistic Term Set assigns probabilities to selected linguistic terms, representing uncertain qualitative assessments with explicitly weighted linguistic information [265–267, 320].

Definition 3.32.1 (Probabilistic Linguistic Term Set)**.** Let

$$S = \{s_0, s_1, \dots, s_g\}$$

be a linguistic term set. A *Probabilistic Linguistic Term Set* (PLTS) is a finite collection

$$L(p) = \{s_{\alpha_k}(p_k) \mid k = 1, \dots, m\},$$

where

$$s_{\alpha_k} \in S, \qquad p_k \geq 0, \qquad \sum_{k=1}^{m} p_k \leq 1.$$

Here, p_k represents the probability associated with the linguistic term s_{α_k}.

Example 3.32.2 (Probabilistic Linguistic Term Set)**.** Let

$$S = \{s_0, s_1, s_2, s_3, s_4\} = \{\text{very low}, \text{low}, \text{medium}, \text{high}, \text{very high}\}.$$

Consider

$$L(p) = \{s_2(0.3), s_3(0.5), s_4(0.2)\}.$$

Since

$$0.3 + 0.5 + 0.2 = 1,$$

$L(p)$ is a valid Probabilistic Linguistic Term Set. It represents an assessment that is medium with probability 0.3, high with probability 0.5, and very high with probability 0.2.

3.33 Unbalanced Linguistic Term Set

An Unbalanced Linguistic Term Set contains ordered linguistic labels whose semantic meanings are unevenly distributed across the underlying evaluation scale [321–323].

Definition 3.33.1 (Unbalanced Linguistic Term Set)**.** [321] Let

$$S = \{s_0, s_1, \dots, s_g\}$$

be an ordered linguistic term set. It is called an *Unbalanced Linguistic Term Set* if the linguistic terms are not uniformly and symmetrically distributed over their underlying semantic domain.

Example 3.33.2 (Unbalanced Linguistic Term Set)**.** Consider the ordered linguistic term set

$$S = \{s_0, s_1, s_2, s_3, s_4\} = \{\text{very low}, \text{low}, \text{medium}, \text{high}, \text{very high}\}.$$

Suppose its semantic positions on $[0, 1]$ are

$$0, \qquad 0.15, \qquad 0.50, \qquad 0.80, \qquad 1.$$

The successive distances are

$$0.15, \qquad 0.35, \qquad 0.30, \qquad 0.20,$$

which are not equal. Therefore, the linguistic terms are not uniformly distributed, and S is an Unbalanced Linguistic Term Set.

3.34 Linguistic Hierarchy

A Linguistic Hierarchy organizes linguistic term sets into multiple granularity levels, enabling consistent transformation between coarse and refined linguistic representations (cf. [257, 324]).

Definition 3.34.1 (Linguistic Hierarchy)**.** A *Linguistic Hierarchy* is a family

$$LH = \bigcup_{t \in T} \ell(t, n(t)),$$

where

$$\ell(t, n(t)) = \left\{ s_0^{(t)}, s_1^{(t)}, \ldots, s_{n(t)-1}^{(t)} \right\}$$

is a linguistic term set of granularity $n(t)$ at level t, with

$$n(t+1) > n(t).$$

Thus, successive levels provide increasingly refined linguistic representations.

Example 3.34.2 (Linguistic Hierarchy)**.** Consider the two linguistic levels

$$\ell(1,3) = \{s_0^{(1)}, s_1^{(1)}, s_2^{(1)}\} = \{\text{low}, \text{medium}, \text{high}\},$$

and

$$\ell(2,5) = \{s_0^{(2)}, s_1^{(2)}, s_2^{(2)}, s_3^{(2)}, s_4^{(2)}\}$$

with

$$\ell(2,5) = \{\text{very low}, \text{low}, \text{medium}, \text{high}, \text{very high}\}.$$

Since

$$3 < 5,$$

the second level provides a finer linguistic granularity than the first one. Hence

$$LH = \ell(1,3) \cup \ell(2,5)$$

forms a simple Linguistic Hierarchy.

3.35 Linguistic Scale Function

A Linguistic Scale Function maps ordered linguistic terms to numerical semantic values, preserving their ordering and representing linguistic meaning quantitatively (cf. [325–327]).

Definition 3.35.1 (Linguistic Scale Function)**.** Let

$$S = \{s_0, s_1, \ldots, s_{2g}\}$$

be an ordered linguistic term set. A *Linguistic Scale Function* is a mapping

$$f : S \longrightarrow [0,1]$$

defined by

$$f(s_i) = \theta_i, \qquad 0 \le \theta_0 < \theta_1 < \cdots < \theta_{2g} \le 1.$$

Thus, f assigns an ordered numerical semantic value to each linguistic term.

Example 3.35.2 (Linguistic Scale Function)**.** Let

$$S = \{s_0, s_1, s_2, s_3, s_4\} = \{\text{very low}, \text{low}, \text{medium}, \text{high}, \text{very high}\}.$$

Define

$$f : S \longrightarrow [0,1]$$

by

$$f(s_0) = 0, \qquad f(s_1) = 0.25, \qquad f(s_2) = 0.50,$$
$$f(s_3) = 0.75, \qquad f(s_4) = 1.$$

Since

$$0 < 0.25 < 0.50 < 0.75 < 1,$$

f preserves the ordering of the linguistic terms and is therefore a Linguistic Scale Function.

3.36 HyperLinguistic Graph (Proposed in this book)

A HyperLinguistic Graph assigns nonempty sets of linguistic terms to vertices and edges of a graph, allowing multiple linguistic descriptions for each graph element.

Definition 3.36.1 (HyperLinguistic Graph)**.** Let

$$G = (V, E)$$

be a graph and let S be a nonempty Linguistic Term Set. Define

$$\mathcal{H}(S) = \mathcal{P}(S) \setminus \{\varnothing\}.$$

A *HyperLinguistic Graph* is a structure

$$G_{\mathrm{HL}} = (V, E, \lambda_V, \lambda_E),$$

where

$$\lambda_V : V \longrightarrow \mathcal{H}(S), \qquad \lambda_E : E \longrightarrow \mathcal{H}(S).$$

Thus, each vertex and edge is assigned a nonempty set of linguistic terms.

Example 3.36.2 (HyperLinguistic Graph)**.** Let

$$V = \{u, v, w\}, \qquad E = \{\{u, v\}, \{v, w\}\},$$

and let

$$S = \{\text{low}, \text{medium}, \text{high}\}.$$

Define the vertex assignments by

$$\lambda_V(u) = \{\text{low}, \text{medium}\},$$

$$\lambda_V(v) = \{\text{medium}, \text{high}\}, \qquad \lambda_V(w) = \{\text{high}\},$$

and the edge assignments by

$$\lambda_E(\{u, v\}) = \{\text{medium}\},$$

$$\lambda_E(\{v, w\}) = \{\text{medium}, \text{high}\}.$$

Since every assigned value is a nonempty subset of S,

$$\lambda_V : V \longrightarrow \mathcal{H}(S), \qquad \lambda_E : E \longrightarrow \mathcal{H}(S)$$

are well defined. Therefore,

$$G_{\mathrm{HL}} = (V, E, \lambda_V, \lambda_E)$$

is a HyperLinguistic Graph.

Remark 3.36.3. A HyperLinguistic Graph should be distinguished from a Linguistic HyperGraph. In a HyperLinguistic Graph, the underlying combinatorial structure remains an ordinary graph, while the linguistic labels are set-valued. In a Linguistic HyperGraph, the underlying incidence structure itself is a hypergraph.

3.37 SuperHyperLinguistic Graph (Proposed in this book)

A SuperHyperLinguistic Graph extends a HyperLinguistic Graph by assigning higher-order, recursively nested linguistic information to its vertices and edges.

Definition 3.37.1 (n-SuperHyperLinguistic Graph)**.** Let

$$G = (V, E)$$

be a graph and let S be a nonempty Linguistic Term Set. Define recursively

$$\mathcal{P}^0(S) = S, \qquad \mathcal{P}^{n+1}(S) = \mathcal{P}\big(\mathcal{P}^n(S)\big).$$

For $n \geq 1$, an *n-SuperHyperLinguistic Graph* is a structure

$$G^{(n)}_{\mathrm{SHL}} = (V, E, \lambda^{(n)}_V, \lambda^{(n)}_E),$$

where

$$\lambda^{(n)}_V : V \longrightarrow \mathcal{P}^n(S) \setminus \{\varnothing\},$$

and

$$\lambda^{(n)}_E : E \longrightarrow \mathcal{P}^n(S) \setminus \{\varnothing\}.$$

Thus, each vertex and edge carries an n-level nested linguistic description. For $n = 1$, this construction reduces to a HyperLinguistic Graph.

Example 3.37.2 (2-SuperHyperLinguistic Graph)**.** Let

$$V = \{u, v\}, \qquad E = \{\{u, v\}\},$$

and let

$$S = \{\text{low}, \text{medium}, \text{high}\}.$$

Consider $n = 2$, so that

$$\mathcal{P}^2(S) = \mathcal{P}\big(\mathcal{P}(S)\big).$$

Define the vertex assignments by

$$\lambda^{(2)}_V(u) = \big\{\{\text{low}\}, \{\text{low}, \text{medium}\}\big\},$$

and

$$\lambda^{(2)}_V(v) = \big\{\{\text{medium}\}, \{\text{medium}, \text{high}\}\big\}.$$

Define the edge assignment by

$$\lambda^{(2)}_E(\{u, v\}) = \big\{\{\text{medium}\}, \{\text{low}, \text{high}\}\big\}.$$

Each assigned value is a nonempty element of

$$\mathcal{P}^2(S).$$

Hence,

$$\lambda^{(2)}_V : V \longrightarrow \mathcal{P}^2(S) \setminus \{\varnothing\},$$

and

$$\lambda^{(2)}_E : E \longrightarrow \mathcal{P}^2(S) \setminus \{\varnothing\}$$

are well defined. Therefore,

$$G^{(2)}_{\mathrm{SHL}} = \Big(V, E, \lambda^{(2)}_V, \lambda^{(2)}_E\Big)$$

is a 2-SuperHyperLinguistic Graph.

Chapter 4

Some MOD Structures

In this chapter, we present several representative MOD structures.

4.1 Modulo Operation

A modulo operation returns the remainder obtained when one integer is divided by a positive integer according to Euclidean division.

Definition 4.1.1 (Modulo Operation)**.** Let $a \in \mathbb{Z}$ and $n \in \mathbb{Z}_{>0}$. The *modulo operation* assigns to a the unique remainder

$$r \in \{0, 1, \ldots, n-1\}$$

such that

$$a = qn + r$$

for some $q \in \mathbb{Z}$. We write

$$a \bmod n = r.$$

Example 4.1.2 (Example of a Modulo Operation)**.** Let

$$a = 17 \qquad \text{and} \qquad n = 5.$$

By Euclidean division,

$$17 = 3 \cdot 5 + 2.$$

Therefore,

$$17 \bmod 5 = 2.$$

Hence, the modulo operation returns 2, which is the remainder obtained when 17 is divided by 5.

4.2 MOD Groupoid

A groupoid is a nonempty set equipped with a binary operation, without requiring associativity, identity elements, or inverses in general [328]. Related concepts, such as Matroids [329], Antimatroids [330], Fuzzy Groupoids [328], HyperGroupoids [331–333], and SuperHyperGroupoids [334], are also known. A MOD groupoid is a set of residues modulo n equipped with a closed binary operation that is not necessarily associative.

Definition 4.2.1 (MOD Groupoid)**.** Let

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}, \qquad n \geq 2,$$

and let

$$t, u \in \mathbb{Z}_n.$$

Define a binary operation

$$* : \mathbb{Z}_n \times \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$$

by

$$a * b = ta + ub \pmod{n}.$$

Then

$$(\mathbb{Z}_n, *)$$

is called a *MOD groupoid*. The operation is closed on $\mathbb{Z}_n$, while associativity is not required.

Example 4.2.2 (Example of a MOD Groupoid)**.** Let

$$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\},$$

and choose

$$t = 2, \qquad u = 3.$$

Define

$$a * b = 2a + 3b \pmod{5}.$$

For example,

$$1 * 4 = 2(1) + 3(4) = 14 \equiv 4 \pmod{5}.$$

Similarly,

$$3 * 2 = 2(3) + 3(2) = 12 \equiv 2 \pmod{5}.$$

Since

$$a * b \in \mathbb{Z}_5$$

for every

$$a, b \in \mathbb{Z}_5,$$

the structure

$$(\mathbb{Z}_5, *)$$

is a MOD groupoid.

4.3 MOD Vector Space

A MOD vector space is a vector space over a finite field, with vector operations performed componentwise modulo a prime.

Definition 4.3.1 (MOD Vector Space)**.** Let p be a prime and let

$$\mathbb{F}_p = \mathbb{Z}_p = \{0, 1, \ldots, p-1\}$$

be the finite field of integers modulo p. For $d \geq 1$, define

$$V = \mathbb{F}_p^d.$$

Vector addition and scalar multiplication are defined componentwise modulo p:

$$(x_1, \ldots, x_d) + (y_1, \ldots, y_d) = (x_1 + y_1, \ldots, x_d + y_d) \pmod{p},$$

and

$$\lambda(x_1, \ldots, x_d) = (\lambda x_1, \ldots, \lambda x_d) \pmod{p}, \qquad \lambda \in \mathbb{F}_p.$$

Then V is called a *MOD vector space* over $\mathbb{F}_p$.

Example 4.3.2 (Example of a MOD Vector Space)**.** Let

$$p = 5$$

and consider the vector space

$$V = \mathbb{F}_5^2.$$

Take

$$x = (1, 4), \qquad y = (3, 2), \qquad \lambda = 3.$$

Their vector sum is

$$x + y = (1 + 3,\ 4 + 2) = (4, 6) \equiv (4, 1) \pmod 5.$$

Also,

$$\lambda x = 3(1, 4) = (3, 12) \equiv (3, 2) \pmod 5.$$

Since both vector addition and scalar multiplication are performed componentwise modulo 5, and their results remain in

$$\mathbb{F}_5^2,$$

the set

$$V = \mathbb{F}_5^2$$

is a MOD vector space over $\mathbb{F}_5$.

4.4 MOD Complex Functions

A MOD Complex Function maps modular complex numbers to modular complex numbers using arithmetic interpreted modulo a fixed integer m [335].

Definition 4.4.1 (MOD Complex Function)**.** Let $m \geq 2$, and let

$$C_n(m) = \{a + bi_F \mid a, b \in [0, m)\}, \qquad i_F^2 = m - 1,$$

be the MOD complex plane. A *MOD complex function* is a mapping

$$f : C_n(m) \longrightarrow C_n(m)$$

of the form

$$f(z) = u(x, y) + i_F v(x, y), \qquad z = x + i_F y,$$

where the real-valued components are interpreted in the MOD interval $[0, m)$, with the corresponding arithmetic taken modulo m.

Example 4.4.2 (Example of a MOD Complex Function)**.** Let

$$m = 5$$

and consider the MOD complex plane

$$C_n(5) = \{a + bi_F \mid a, b \in [0, 5)\}, \qquad i_F^2 = 4.$$

Define

$$f : C_n(5) \longrightarrow C_n(5)$$

by

$$f(z) = 2z + 1,$$

where all component arithmetic is performed modulo 5.

For
$$z = 3 + 4i_F,$$
we obtain
$$f(z) = 2(3 + 4i_F) + 1 = 7 + 8i_F.$$
Reducing both components modulo 5,
$$7 \equiv 2 \pmod 5, \qquad 8 \equiv 3 \pmod 5.$$
Hence,
$$f(3 + 4i_F) = 2 + 3i_F \in C_n(5).$$
Thus, f is a MOD complex function.

4.5 Fuzzy MOD Functions

A *Fuzzy MOD Function* maps values in the fuzzy MOD interval $[0, 1)$ to values in $[0, 1)$, with arithmetic interpreted modulo 1.

Definition 4.5.1 (Fuzzy MOD Function)**.** Let
$$R_n(1) = [0, 1)$$
be the fuzzy MOD interval. A *fuzzy MOD function* is a mapping
$$f : R_n(1) \longrightarrow R_n(1),$$
where the arithmetic involved in f is performed modulo 1.

In particular, a polynomial
$$f(x) = \sum_{k=0}^{r} a_k x^k, \qquad a_k \in [0, 1),$$
is called a *fuzzy MOD polynomial function*, with all arithmetic interpreted modulo 1.

Example 4.5.2 (Example of a Fuzzy MOD Function)**.** Consider the fuzzy MOD interval
$$R_n(1) = [0, 1).$$
Define
$$f : R_n(1) \longrightarrow R_n(1)$$
by
$$f(x) = (2x + 0.3) \bmod 1.$$
For example, if
$$x = 0.6,$$
then
$$f(0.6) = (2(0.6) + 0.3) \bmod 1 = 1.5 \bmod 1 = 0.5.$$
Since
$$0.5 \in [0, 1),$$
the value remains in the fuzzy MOD interval. Thus, f is a fuzzy MOD function.

4.6 Neutrosophic MOD Functions

A *Neutrosophic MOD Function* maps modular neutrosophic numbers involving indeterminacy I, with coefficients and arithmetic interpreted modulo a fixed integer [335].

Definition 4.6.1 (Neutrosophic MOD Function)**.** Let $m \geq 2$, and define the MOD neutrosophic plane by

$$R_n^I(m) = \left\{a + bI \mid a, b \in [0, m),\ I^2 = I\right\}.$$

A *neutrosophic MOD function* is a function defined over $R_n^I(m)$. In particular, a polynomial

$$f(x) = \sum_{k=0}^{r} a_k x^k, \qquad a_k \in R_n^I(m),$$

is called a *neutrosophic MOD polynomial function*, where all coefficient arithmetic is performed modulo m.

Example 4.6.2 (Example of a Neutrosophic MOD Function)**.** Let

$$m = 5$$

and consider

$$R_n^I(5) = \{a + bI \mid a, b \in [0, 5),\ I^2 = I\}.$$

Define

$$f : R_n^I(5) \longrightarrow R_n^I(5)$$

by

$$f(x) = 2x + (1 + I),$$

where all coefficient arithmetic is performed modulo 5.

For

$$x = 3 + 4I,$$

we obtain

$$f(3 + 4I) = 2(3 + 4I) + (1 + I) = 7 + 9I.$$

Reducing the coefficients modulo 5,

$$7 \equiv 2 \pmod 5, \qquad 9 \equiv 4 \pmod 5.$$

Therefore,

$$f(3 + 4I) = 2 + 4I \in R_n^I(5).$$

Hence, f is a neutrosophic MOD function.

4.7 MOD Real Plane

A MOD real plane represents ordered pairs from a bounded semi-open interval, providing a modulo-based analogue of the real plane [336–339].

Definition 4.7.1 (MOD Real Plane)**.** Let $m \geq 1$. The *MOD real plane* associated with the MOD interval $[0, m)$ is defined by

$$R_n(m) = [0, m) \times [0, m) = \{(a, b) \mid a, b \in [0, m)\}.$$

4.8 MOD Fuzzy Real Plane

A MOD fuzzy real plane is the unit interval case of a MOD real plane, with coordinates restricted to $[0, 1)$.

Definition 4.8.1 (MOD Fuzzy Real Plane)**.** The *MOD fuzzy real plane* is the MOD real plane corresponding to $m = 1$, namely

$$R_n(1) = [0, 1) \times [0, 1) = \{(a, b) \mid a, b \in [0, 1)\}.$$

Example 4.8.2 (Example of a MOD Fuzzy Real Plane)**.** Consider the MOD fuzzy real plane

$$R_n(1) = [0, 1) \times [0, 1).$$

For example, the point

$$P = (0.3, 0.8)$$

belongs to $R_n(1)$, since

$$0 \leq 0.3 < 1 \qquad \text{and} \qquad 0 \leq 0.8 < 1.$$

Hence,

$$(0.3, 0.8) \in R_n(1).$$

Thus, P is an element of the MOD fuzzy real plane.

4.9 MOD Neutrosophic Plane

A MOD neutrosophic plane represents numbers combining real and indeterminate components within a bounded MOD interval, satisfying the idempotent relation [336, 338, 339].

Definition 4.9.1 (MOD Neutrosophic Plane)**.** Let $m \geq 1$. The *MOD neutrosophic plane* is defined by

$${}^{I}R_n(m) = \{a + bI \mid a, b \in [0, m),\ I^2 = I\},$$

where I denotes the neutrosophic indeterminacy.

Example 4.9.2 (Example of a MOD Neutrosophic Plane)**.** Let

$$m = 5.$$

Then the MOD neutrosophic plane is

$${}^{I}R_n(5) = \{a + bI \mid a, b \in [0, 5),\ I^2 = I\}.$$

Consider the neutrosophic number

$$x = 2 + 3I.$$

Since

$$2, 3 \in [0, 5),$$

we have

$$2 + 3I \in {}^{I}R_n(5).$$

Moreover, using

$$I^2 = I,$$

we obtain

$$(2 + 3I)^2 = 4 + 12I + 9I^2 = 4 + 21I.$$

Interpreting the coefficients modulo 5,

$$4 + 21I \equiv 4 + I \pmod 5.$$

Hence, the MOD neutrosophic structure preserves the real and indeterminate components under modular arithmetic.

4.10 MOD Complex Plane

A MOD complex plane consists of bounded complex numbers whose real and imaginary components lie within a common MOD interval [338].

Definition 4.10.1 (MOD Complex Plane)**.** [338] Let $m \geq 2$. The *MOD complex plane* associated with the MOD interval $[0, m)$ is defined by

$$C_n(m) = \{a + bi_F \mid a, b \in [0, m),\ i_F^2 = m - 1\},$$

where i_F denotes the MOD imaginary unit.

Example 4.10.2 (Example of a MOD Complex Plane)**.** Let

$$m = 5.$$

Then the corresponding MOD complex plane is

$$C_n(5) = \{a + bi_F \mid a, b \in [0, 5),\ i_F^2 = 4\}.$$

Consider the MOD complex number

$$z = 2 + 3i_F.$$

Since

$$2, 3 \in [0, 5),$$

we have

$$z = 2 + 3i_F \in C_n(5).$$

Moreover,

$$z^2 = (2 + 3i_F)^2 = 4 + 12i_F + 9i_F^2.$$

Using

$$i_F^2 = 4,$$

we obtain

$$z^2 = 40 + 12i_F.$$

Reducing the coefficients modulo 5,

$$40 \equiv 0 \pmod 5, \qquad 12 \equiv 2 \pmod 5.$$

Hence,

$$z^2 \equiv 2i_F \pmod 5,$$

which again belongs to $C_n(5)$.

4.11 MOD Graph

A MOD graph is a graph whose vertices are selected from the integers modulo n, forming a modular vertex structure [340].

Definition 4.11.1 (MOD Graph)**.** [340] Let

$$\mathbb{Z}_n = \{0, 1, \ldots, n - 1\}, \qquad n \geq 2.$$

A *MOD graph* is a graph

$$G = (V, E)$$

whose vertex set satisfies

$$\varnothing \neq V \subseteq \mathbb{Z}_n.$$

In particular, V may be the whole set $\mathbb{Z}_n$. Thus, a MOD graph is a graph whose vertices are selected from the integers modulo n.

Example 4.11.2 (Example of a MOD Graph)**.** Let

$$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}.$$

Consider the graph

$$G = (V, E),$$

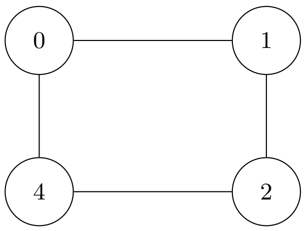
where

$$V = \{0, 1, 2, 4\} \subseteq \mathbb{Z}_5$$

and

$$E = \big\{\{0, 1\}, \{1, 2\}, \{2, 4\}, \{0, 4\}\big\}.$$

Thus, G consists of four vertices selected from the residue class $\mathbb{Z}_5$. The corresponding graph is illustrated below.



In this graph,

$$0 \sim 1, \qquad 1 \sim 2, \qquad 2 \sim 4, \qquad 4 \sim 0,$$

so G forms the cycle

$$0 - 1 - 2 - 4 - 0.$$

Notice that $3 \in \mathbb{Z}_5$ is not required to be a vertex, since the definition only requires

$$V \subseteq \mathbb{Z}_5.$$

Because every vertex of G belongs to $\mathbb{Z}_5$, the graph G is a MOD graph.

4.12 MOD HyperGraph (Proposed in this book)

A HyperGraph generalizes a graph by allowing each edge to connect any number of vertices, representing higher-order relationships among entities [237, 239, 240, 341, 342]. A MOD HyperGraph is a hypergraph whose vertices are selected from the integers modulo n, while each hyperedge is a nonempty subset of the modular vertex set.

Definition 4.12.1 (MOD HyperGraph)**.** Let

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}, \qquad n \geq 2.$$

A *MOD HyperGraph* is a hypergraph

$$H = (V, E)$$

such that

$$\varnothing \neq V \subseteq \mathbb{Z}_n$$

and

$$E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}.$$

Thus, every hyperedge

$$e \in E$$

is a nonempty subset of V, whose elements are integers modulo n.

Example 4.12.2 (Example of a MOD HyperGraph)**.** Let

$$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}.$$

Consider the hypergraph

$$H = (V, E),$$

where

$$V = \{0, 1, 2, 3, 5\} \subseteq \mathbb{Z}_6$$

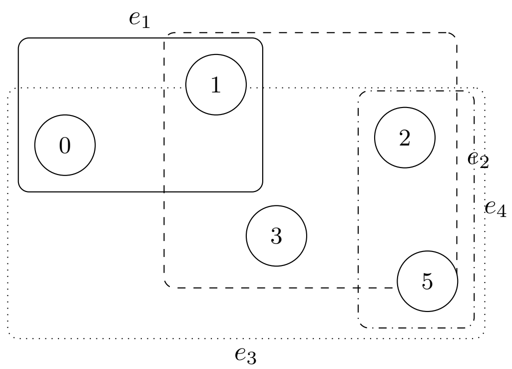
and

$$E = \{e_1, e_2, e_3, e_4\},$$

with

$$e_1 = \{0, 1\}, \qquad e_2 = \{1, 2, 3\}, \qquad e_3 = \{0, 3, 5\}, \qquad e_4 = \{2, 5\}.$$

Thus, H has five vertices selected from $\mathbb{Z}_6$ and four hyperedges. The corresponding hypergraph is illustrated below.



In this figure, the hyperedge e_1 joins the vertices 0 and 1, the hyperedge e_2 contains the three vertices $1, 2, 3$, the hyperedge e_3 contains $0, 3, 5$, and the hyperedge e_4 contains $2, 5$.

Since each hyperedge is a nonempty subset of V and

$$V \subseteq \mathbb{Z}_6,$$

the hypergraph H is a MOD HyperGraph.

Note that the element $4 \in \mathbb{Z}_6$ does not need to belong to V; it is sufficient that the vertex set be a subset of $\mathbb{Z}_6$.

Theorem 4.12.3 (Well-Definedness of a MOD HyperGraph)**.** *Let*

$$\varnothing \neq V \subseteq \mathbb{Z}_n$$

and

$$E \subseteq \mathcal{P}(V) \setminus \{\varnothing\}.$$

Then

$$H = (V, E)$$

is a well-defined hypergraph.

Proof. Since

$$V \subseteq \mathbb{Z}_n,$$

the set V is finite and well-defined. Moreover, for every

$$e \in E,$$

we have

$$\varnothing \neq e \subseteq V.$$

Hence every hyperedge consists only of vertices belonging to V. Therefore the incidence structure

$$H = (V, E)$$

satisfies the defining conditions of a hypergraph and is well-defined. □

4.13 MOD SuperHyperGraph (Proposed in this book)

A SuperHyperGraph generalizes hypergraphs by allowing vertices and hyperedges to be nested collections, representing higher-order and hierarchical relationships among entities [229, 243, 244, 343–345]. A MOD SuperHyperGraph is a higher-order hypergraph whose supervertices are nonempty subsets of $\mathbb{Z}_n$ and whose superhyperedges are nonempty collections of such supervertices.

Definition 4.13.1 (MOD SuperHyperGraph)**.** Let

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}, \qquad n \geq 2.$$

Let

$$\mathcal{V} \subseteq \mathcal{P}(\mathbb{Z}_n) \setminus \{\varnothing\}$$

be a nonempty collection of *supervertices*. A *MOD SuperHyperGraph* is a pair

$$SH = (\mathcal{V}, \mathcal{E}),$$

where

$$\mathcal{E} \subseteq \mathcal{P}(\mathcal{V}) \setminus \{\varnothing\}.$$

Thus, every supervertex

$$A \in \mathcal{V}$$

is a nonempty subset of $\mathbb{Z}_n$, and every superhyperedge

$$E \in \mathcal{E}$$

is a nonempty collection of supervertices.

Example 4.13.2 (Example 1 of a MOD SuperHyperGraph)**.** Let

$$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}.$$

Consider the collection of supervertices

$$\mathcal{V} = \{A_1, A_2, A_3\},$$

where

$$A_1 = \{0, 1\}, \qquad A_2 = \{2, 3\}, \qquad A_3 = \{4, 5\}.$$

Clearly,

$$\mathcal{V} \subseteq \mathcal{P}(\mathbb{Z}_6) \setminus \{\varnothing\}.$$

Define the collection of superhyperedges by

$$\mathcal{E} = \{\{A_1, A_2\}, \{A_2, A_3\}, \{A_1, A_2, A_3\}\}.$$

Since every element of $\mathcal{E}$ is a nonempty subset of $\mathcal{V}$, we have

$$\mathcal{E} \subseteq \mathcal{P}(\mathcal{V}) \setminus \{\varnothing\}.$$

Therefore,

$$SH = (\mathcal{V}, \mathcal{E})$$

is a MOD SuperHyperGraph.

Example 4.13.3 (Example 2 of a MOD SuperHyperGraph)**.** Let

$$\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}.$$

Define

$$\mathcal{V} = \big\{B_1, B_2, B_3, B_4\big\},$$

where

$$B_1 = \{0, 2\}, \qquad B_2 = \{1, 3, 5\}, \qquad B_3 = \{2, 4, 6\}, \qquad B_4 = \{0, 6\}.$$

Then

$$\mathcal{V} \subseteq \mathcal{P}(\mathbb{Z}_7) \setminus \{\varnothing\}.$$

Let

$$\mathcal{E} = \big\{\{B_1, B_4\}, \{B_1, B_2, B_3\}, \{B_2, B_3, B_4\}\big\}.$$

Each superhyperedge is a nonempty collection of supervertices, so

$$\mathcal{E} \subseteq \mathcal{P}(\mathcal{V}) \setminus \{\varnothing\}.$$

Hence,

$$SH = (\mathcal{V}, \mathcal{E})$$

is a MOD SuperHyperGraph over $\mathbb{Z}_7$.

Theorem 4.13.4 (Well-Definedness of a MOD SuperHyperGraph)**.** *Let*

$$\varnothing \neq \mathcal{V} \subseteq \mathcal{P}(\mathbb{Z}_n) \setminus \{\varnothing\}$$

and

$$\mathcal{E} \subseteq \mathcal{P}(\mathcal{V}) \setminus \{\varnothing\}.$$

Then

$$SH = (\mathcal{V}, \mathcal{E})$$

is a well-defined SuperHyperGraph.

Proof. For every

$$A \in \mathcal{V},$$

we have

$$\varnothing \neq A \subseteq \mathbb{Z}_n,$$

so each supervertex is a well-defined collection of modular vertices. Furthermore, for every

$$E \in \mathcal{E},$$

we have

$$\varnothing \neq E \subseteq \mathcal{V}.$$

Hence every superhyperedge consists entirely of well-defined supervertices. Therefore

$$SH = (\mathcal{V}, \mathcal{E})$$

is a well-defined SuperHyperGraph. □

Remark 4.13.5. If every supervertex is a singleton,

$$\mathcal{V} = \big\{\{v\} \mid v \in V\big\}$$

for some

$$V \subseteq \mathbb{Z}_n,$$

then the MOD SuperHyperGraph reduces naturally to a MOD HyperGraph.

4.14 MOD Natural Neutrosophic Graph

A Neutrosophic Graph assigns truth, indeterminacy, and falsity degrees to vertices and edges, enabling flexible representation of uncertain, incomplete, inconsistent, and complex relational information systems [59,346–348]. A MOD natural neutrosophic graph has vertices drawn from modular neutrosophic elements combining ordinary residues with indeterminacy represented by I [340].

Definition 4.14.1 (MOD Natural Neutrosophic Graph)**.** [340] Let

$$^{I}\mathbb{Z}_n = \{a + bI \mid a, b \in \mathbb{Z}_n,\ I^2 = I\}$$

be a MOD natural neutrosophic set, where I represents indeterminacy. A *MOD natural neutrosophic graph* is a graph

$$G = (V, E)$$

whose vertex set satisfies

$$\varnothing \neq V \subseteq {}^{I}\mathbb{Z}_n.$$

Hence, its vertices may contain both ordinary modulo integers and neutrosophic elements involving I.

Example 4.14.2 (Example of a MOD Natural Neutrosophic Graph)**.** Let

$$n = 5$$

and consider the MOD natural neutrosophic set

$$^{I}\mathbb{Z}_5 = \{a + bI \mid a, b \in \mathbb{Z}_5,\ I^2 = I\}.$$

Define the vertex set

$$V = \{0,\ 1 + I,\ 2 + 3I,\ 4I\}.$$

Here,

$$0 = 0 + 0I, \qquad 1 + I = 1 + 1I, \qquad 2 + 3I = 2 + 3I, \qquad 4I = 0 + 4I,$$

so each vertex belongs to

$$^{I}\mathbb{Z}_5.$$

Hence,

$$V \subseteq {}^{I}\mathbb{Z}_5.$$

Let the edge set be

$$E = \big\{\{0, 1 + I\}, \{1 + I, 2 + 3I\}, \{2 + 3I, 4I\}, \{0, 4I\}\big\}.$$

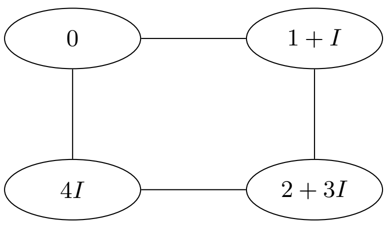
Thus,

$$G = (V, E)$$

is a graph whose vertices are taken from

$$^{I}\mathbb{Z}_5.$$

The graph is illustrated below.



In this example, the adjacencies are

$$0 \sim 1+I, \qquad 1+I \sim 2+3I, \qquad 2+3I \sim 4I, \qquad 4I \sim 0.$$

Hence, G forms a cycle on four vertices:

$$0 - (1+I) - (2+3I) - 4I - 0.$$

Since every vertex of G belongs to

$${}^{I}\mathbb{Z}_5,$$

the graph

$$G = (V, E)$$

is a MOD natural neutrosophic graph over

$${}^{I}\mathbb{Z}_5.$$

4.15 MOD Bipartite Graph

A bipartite graph is a graph whose vertex set is partitioned into two disjoint subsets, with edges joining only vertices from different subsets [349–351]. Related concepts such as fuzzy bipartite graphs [352,353], bipartite hypergraphs [354,355], and neutrosophic bipartite graphs [356,357] are also known. A MOD bipartite graph is a bipartite graph in which the edges connect two disjoint vertex classes and are considered within a modular framework [340].

Definition 4.15.1 (MOD Bipartite Graph)**.** [340] Let

$$G = (U \cup W, E)$$

be a bipartite graph, where

$$U \cap W = \varnothing$$

and every edge joins a vertex of U to a vertex of W. Let

$$\omega : E \longrightarrow \mathbb{Z}_n$$

be an edge-weight function. Then

$$G = (U \cup W, E, \omega)$$

is called a *MOD bipartite graph* if every edge weight is taken from $\mathbb{Z}_n$.

Example 4.15.2 (Example of a MOD Bipartite Graph)**.** Let

$$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}.$$

Consider the two disjoint vertex sets

$$U = \{u_1, u_2\}, \qquad W = \{w_1, w_2, w_3\},$$

with

$$U \cap W = \varnothing.$$

Let

$$E = \big\{\{u_1, w_1\}, \{u_1, w_3\}, \{u_2, w_2\}, \{u_2, w_3\}\big\}.$$

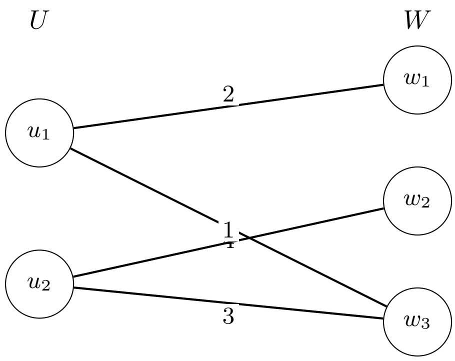
Every edge has one endpoint in U and the other in W. Therefore,

$$G = (U \cup W, E)$$

is a bipartite graph.

Define the MOD edge-weight function
$$\omega : E \longrightarrow \mathbb{Z}_5$$
by
$$\omega(\{u_1, w_1\}) = 2, \qquad \omega(\{u_1, w_3\}) = 4,$$
and
$$\omega(\{u_2, w_2\}) = 1, \qquad \omega(\{u_2, w_3\}) = 3.$$

The resulting MOD bipartite graph is illustrated below.



In the figure, the left vertex class is
$$U = \{u_1, u_2\},$$
while the right vertex class is
$$W = \{w_1, w_2, w_3\}.$$
There are no edges between two vertices of U or between two vertices of W. Hence, the bipartite structure is preserved.

Moreover, the edge weights are
$$\omega(u_1w_1) = 2, \qquad \omega(u_1w_3) = 4, \qquad \omega(u_2w_2) = 1, \qquad \omega(u_2w_3) = 3,$$
and therefore
$$\omega(E) = \{1, 2, 3, 4\} \subseteq \mathbb{Z}_5.$$

Since every edge weight belongs to
$$\mathbb{Z}_5,$$
the weighted bipartite graph
$$G = (U \cup W, E, \omega)$$
is a MOD bipartite graph over $\mathbb{Z}_5$.

4.16 MOD Cognitive Map

A cognitive map is a directed graph representing causal relationships among concepts, with weighted edges indicating the strength and direction [358–361]. Related concepts, such as Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps, are also known [362–366]. A MOD Cognitive Map represents causal relationships among concepts using directed edges weighted by elements of $\mathbb{Z}_n$ in modular arithmetic [367].

Definition 4.16.1 (MOD Cognitive Map)**.** Let

$$C_1, C_2, \ldots, C_m$$

be a finite set of concepts, and let

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}, \qquad n \geq 2.$$

A *MOD Cognitive Map* (MOD-CM) is a weighted directed graph

$$G = (V, E, \omega),$$

where

$$V = \{C_1, C_2, \ldots, C_m\}$$

and

$$\omega : E \longrightarrow \mathbb{Z}_n.$$

Each directed edge $C_i \to C_j$ represents a causal relationship between concepts, and its weight

$$e_{ij} = \omega(C_i, C_j) \in \mathbb{Z}_n$$

indicates the corresponding MOD causal influence.

The associated adjacency MOD matrix is

$$M = (e_{ij})_{m \times m},$$

where e_{ij} is the weight of the directed edge from C_i to C_j.

Example 4.16.2 (Example of a MOD Cognitive Map)**.** Let

$$V = \{C_1, C_2, C_3\},$$

where

$$C_1 = \text{Advertising}, \qquad C_2 = \text{Customer Interest}, \qquad C_3 = \text{Sales}.$$

Let

$$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}.$$

Consider the directed edge set

$$E = \{(C_1, C_2), (C_2, C_3), (C_1, C_3)\}.$$

Define the MOD edge-weight function

$$\omega : E \longrightarrow \mathbb{Z}_5$$

by

$$\omega(C_1, C_2) = 2, \qquad \omega(C_2, C_3) = 3, \qquad \omega(C_1, C_3) = 1.$$

Thus, the corresponding MOD cognitive map is

$$G = (V, E, \omega).$$

Its adjacency MOD matrix is

$$M = \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here, for example,

$$\omega(C_1, C_2) = 2$$

represents a MOD causal influence from advertising to customer interest, while

$$\omega(C_2, C_3) = 3$$

represents a MOD causal influence from customer interest to sales.

Since every edge weight belongs to

$$\mathbb{Z}_5,$$

the weighted directed graph

$$G = (V, E, \omega)$$

is a MOD Cognitive Map over $\mathbb{Z}_5$.

4.17 Cayley Graph over $\mathbb{Z}_n$

A *Cayley graph over* $\mathbb{Z}_n$ is a graph whose vertices are residues, with adjacency determined by specified modular group differences [368–371]. Related concepts, such as Fuzzy Cayley Graphs [372, 373] and Cayley HyperGraphs [374–376], are also known.

Definition 4.17.1 (Cayley Graph over $\mathbb{Z}_n$)**.** [368, 369] Let

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$$

be the cyclic group under addition modulo n, and let

$$S \subseteq \mathbb{Z}_n \setminus \{0\}.$$

The *Cayley graph over* $\mathbb{Z}_n$ is the graph

$$\mathrm{Cay}(\mathbb{Z}_n, S)$$

with vertex set $\mathbb{Z}_n$, where x is adjacent to y whenever

$$y - x \in S \pmod{n}.$$

When $S = -S$, the graph is undirected and is a circulant graph.

Example 4.17.2 (Example of a Cayley Graph over $\mathbb{Z}_n$)**.** Let

$$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$$

and choose

$$S = \{1, 5\}.$$

Since

$$5 \equiv -1 \pmod{6},$$

we have

$$S = -S.$$

Consider the Cayley graph

$$G = \mathrm{Cay}(\mathbb{Z}_6, S).$$

Two vertices $x, y \in \mathbb{Z}_6$ are adjacent whenever

$$y - x \equiv 1 \quad \text{or} \quad y - x \equiv 5 \pmod 6.$$

Thus, the edge set is

$$E = \{\{0,1\},\{1,2\},\{2,3\},\{3,4\},\{4,5\},\{5,0\}\}.$$

Therefore,

$$G = \mathrm{Cay}(\mathbb{Z}_6, \{1,5\})$$

is a Cayley graph over $\mathbb{Z}_6$. Since

$$S = -S,$$

the graph is undirected and forms the cycle graph

$$C_6.$$

4.18 Circulant Graph

A *circulant graph* is a Cayley graph of a cyclic group, where adjacency depends only on modular differences between vertices [377–380].

Definition 4.18.1 (Circulant Graph)**.** Let

$$\mathbb{Z}_n = \{0, 1, \dots, n-1\}, \qquad n \geq 2,$$

and let

$$S \subseteq \mathbb{Z}_n \setminus \{0\}.$$

The *circulant graph*

$$\mathrm{Cay}(\mathbb{Z}_n, S)$$

has vertex set $\mathbb{Z}_n$, with an edge from i to j whenever

$$j - i \pmod n \in S.$$

If

$$S = -S,$$

the graph is undirected. Thus, a circulant graph is a Cayley graph of the cyclic group $\mathbb{Z}_n$.

Example 4.18.2 (Example of a Circulant Graph)**.** Let

$$\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}$$

and choose

$$S = \{1, 3, 5, 7\}.$$

Since

$$-1 \equiv 7, \qquad -3 \equiv 5 \pmod 8,$$

we have

$$S = -S.$$

Consider the circulant graph

$$G = \mathrm{Cay}(\mathbb{Z}_8, S).$$

Two vertices $i, j \in \mathbb{Z}_8$ are adjacent whenever

$$j - i \pmod 8 \in \{1, 3, 5, 7\}.$$

For example, vertex 0 is adjacent to

$$1,\ 3,\ 5,\ 7,$$

because

$$1 - 0 \equiv 1, \quad 3 - 0 \equiv 3, \quad 5 - 0 \equiv 5, \quad 7 - 0 \equiv 7 \pmod 8.$$

Similarly, vertex 2 is adjacent to

$$1,\ 3,\ 5,\ 7.$$

Thus the graph is undirected and each vertex has degree 4.

Therefore,

$$G = \mathrm{Cay}\big(\mathbb{Z}_8, \{1, 3, 5, 7\}\big)$$

is a circulant graph.

4.19 Cyclic Code over $\mathbb{Z}_n$

A *cyclic code over* $\mathbb{Z}_n$ is a linear code closed under cyclic coordinate shifts, represented algebraically by quotient-ring ideals [381–383].

Definition 4.19.1 (Cyclic Code over $\mathbb{Z}_n$)**.** Let

$$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}, \qquad n \geq 2,$$

and let $m \geq 1$. A *cyclic code of length* m *over* $\mathbb{Z}_n$ is a $\mathbb{Z}_n$-submodule

$$C \subseteq \mathbb{Z}_n^m$$

such that

$$(c_0, c_1, \ldots, c_{m-1}) \in C$$

implies

$$(c_{m-1}, c_0, \ldots, c_{m-2}) \in C.$$

Equivalently, identifying

$$(c_0, \ldots, c_{m-1}) \longleftrightarrow c_0 + c_1 x + \cdots + c_{m-1} x^{m-1},$$

a cyclic code is an ideal of

$$\mathbb{Z}_n[x]/(x^m - 1).$$

Example 4.19.2 (Example of a Cyclic Code over $\mathbb{Z}_2$)**.** Let

$$\mathbb{Z}_2 = \{0, 1\}$$

and consider codes of length

$$m = 3.$$

Define

$$C = \{(0,0,0), (1,1,1)\} \subseteq \mathbb{Z}_2^3.$$

The set C is a $\mathbb{Z}_2$-submodule of $\mathbb{Z}_2^3$. Moreover, the cyclic shift of

$$(1,1,1)$$

is

$$(1,1,1),$$

and the cyclic shift of

$$(0,0,0)$$

is

$$(0,0,0).$$

Hence C is closed under cyclic coordinate shifts.

Under the correspondence

$$(c_0, c_1, c_2) \longleftrightarrow c_0 + c_1x + c_2x^2,$$

the nonzero codeword corresponds to

$$g(x) = 1 + x + x^2.$$

Thus,

$$C = \langle 1 + x + x^2 \rangle$$

is an ideal of

$$\mathbb{Z}_2[x]/(x^3 - 1).$$

Therefore,

$$C = \{(0,0,0), (1,1,1)\}$$

is a cyclic code of length 3 over $\mathbb{Z}_2$.

4.20 $\mathbb{Z}_n$-Module

A $\mathbb{Z}_n$-module is an algebraic structure whose elements admit addition and scalar multiplication by residues modulo n.

Definition 4.20.1 ($\mathbb{Z}_n$-Module)**.** Let

$$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}, \qquad n \geq 2.$$

A $\mathbb{Z}_n$*-module* is an abelian group

$$(M, +)$$

equipped with a scalar multiplication

$$\mathbb{Z}_n \times M \longrightarrow M, \qquad (a, x) \longmapsto ax,$$

satisfying the usual module axioms for all $a, b \in \mathbb{Z}_n$ and $x, y \in M$.

Example 4.20.2 (Example of a $\mathbb{Z}_n$-Module)**.** Let

$$n = 6$$

and consider

$$M = \mathbb{Z}_6^2.$$

The addition in M is defined componentwise modulo 6, and scalar multiplication by elements of $\mathbb{Z}_6$ is also performed componentwise.

For example, let

$$x = (2, 5), \qquad y = (4, 3), \qquad a = 3 \in \mathbb{Z}_6.$$

Then

$$x + y = (2 + 4, 5 + 3) = (6, 8) \equiv (0, 2) \pmod 6,$$

and

$$ax = 3(2, 5) = (6, 15) \equiv (0, 3) \pmod 6.$$

Thus,

$$M = \mathbb{Z}_6^2$$

is a $\mathbb{Z}_6$-module.

4.21 Matrix Ring $M_k(\mathbb{Z}_n)$

The matrix ring $M_k(\mathbb{Z}_n)$ consists of square matrices whose entries are residues modulo n [384–386].

Definition 4.21.1 (Matrix Ring $M_k(\mathbb{Z}_n)$)**.** Let $n \geq 2$ and $k \geq 1$. Define

$$M_k(\mathbb{Z}_n) = \{A = (a_{ij})_{k \times k} \mid a_{ij} \in \mathbb{Z}_n\}.$$

With entrywise matrix addition and ordinary matrix multiplication, where all entries are computed modulo n,

$$M_k(\mathbb{Z}_n)$$

forms a ring called the *matrix ring over* $\mathbb{Z}_n$.

Example 4.21.2 (Example of the Matrix Ring $M_2(\mathbb{Z}_5)$)**.** Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix} \in M_2(\mathbb{Z}_5).$$

Their sum is

$$A + B = \begin{pmatrix} 5 & 3 \\ 5 & 7 \end{pmatrix} \equiv \begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix} \pmod 5.$$

Their product is

$$AB = \begin{pmatrix} 1 \cdot 4 + 2 \cdot 2 & 1 \cdot 1 + 2 \cdot 3 \\ 3 \cdot 4 + 4 \cdot 2 & 3 \cdot 1 + 4 \cdot 3 \end{pmatrix} = \begin{pmatrix} 8 & 7 \\ 20 & 15 \end{pmatrix}.$$

Reducing modulo 5,

$$AB \equiv \begin{pmatrix} 3 & 2 \\ 0 & 0 \end{pmatrix} \pmod 5.$$

Hence both addition and multiplication remain in

$$M_2(\mathbb{Z}_5).$$

4.22 Polynomial Ring $\mathbb{Z}_n[x]$

The polynomial ring $\mathbb{Z}_n[x]$ consists of polynomials whose coefficients are residues modulo n [387,388].

Definition 4.22.1 (Polynomial Ring)**.** Let

$$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}.$$

The *polynomial ring over* $\mathbb{Z}_n$ is

$$\mathbb{Z}_n[x] = \{a_0 + a_1x + \cdots + a_mx^m \mid m \geq 0,\ a_i \in \mathbb{Z}_n\},$$

with the usual polynomial addition and multiplication, where coefficient arithmetic is performed modulo n.

Example 4.22.2 (Example of a Polynomial Ring)**.** Consider the polynomials

$$f(x) = 2 + 3x + x^2$$

and

$$g(x) = 4 + 2x$$

in

$$\mathbb{Z}_5[x].$$

Their sum is

$$f(x) + g(x) = 6 + 5x + x^2 \equiv 1 + x^2 \pmod 5.$$

Their product is

$$\begin{aligned} f(x)g(x) &= (2 + 3x + x^2)(4 + 2x) \\ &= 8 + 16x + 10x^2 + 2x^3. \end{aligned}$$

Reducing the coefficients modulo 5,

$$f(x)g(x) \equiv 3 + x + 2x^3 \pmod 5.$$

Therefore,

$$f(x) + g(x), \quad f(x)g(x) \in \mathbb{Z}_5[x].$$

4.23 Unitary Cayley Graph

A unitary Cayley graph connects two residues whenever their difference is invertible modulo n [389–391].

Definition 4.23.1 (Unitary Cayley Graph)**.** Let

$$U_n = \{a \in \mathbb{Z}_n \mid \gcd(a, n) = 1\}$$

be the group of units of $\mathbb{Z}_n$. The *unitary Cayley graph* is

$$X_n = \mathrm{Cay}(\mathbb{Z}_n, U_n),$$

with vertex set $\mathbb{Z}_n$, where distinct vertices x and y are adjacent if and only if

$$x - y \in U_n.$$

Equivalently,

$$\gcd(x - y, n) = 1.$$

Example 4.23.2 (Example of a Unitary Cayley Graph)**.** Let

$$n = 8.$$

The units of $\mathbb{Z}_8$ are

$$U_8 = \{1, 3, 5, 7\}.$$

Hence the unitary Cayley graph is

$$X_8 = \mathrm{Cay}\big(\mathbb{Z}_8, \{1, 3, 5, 7\}\big).$$

Two distinct vertices $x, y \in \mathbb{Z}_8$ are adjacent whenever

$$\gcd(x - y, 8) = 1.$$

For example,

$$\gcd(0 - 1, 8) = 1,$$

so 0 is adjacent to 1. Similarly,

$$0$$

is adjacent to

$$1, 3, 5, 7.$$

However,

$$\gcd(0 - 2, 8) = 2,$$

so 0 is not adjacent to 2.

Thus,

$$X_8$$

is a unitary Cayley graph over $\mathbb{Z}_8$.

4.24 Zero-Divisor Graph

A zero-divisor graph represents the multiplicative relationships among the nonzero zero divisors of a commutative ring [392–395]. Related concepts, including Zero-Divisor HyperGraphs [396, 397], Fuzzy Zero-Divisor Graphs [398, 399], and Neutrosophic Zero-Divisor Graphs [400–402], are also known.

Definition 4.24.1 (Zero-Divisor Graph of $\mathbb{Z}_n$)**.** Let

$$Z^*(\mathbb{Z}_n) = \{a \in \mathbb{Z}_n \setminus \{0\} \mid ab = 0 \text{ for some } b \neq 0\}.$$

The *zero-divisor graph* of $\mathbb{Z}_n$, denoted by

$$\Gamma(\mathbb{Z}_n),$$

is the simple graph with vertex set

$$Z^*(\mathbb{Z}_n),$$

where two distinct vertices a and b are adjacent if and only if

$$ab = 0 \quad \text{in } \mathbb{Z}_n.$$

Example 4.24.2 (Example of a Zero-Divisor Graph of $\mathbb{Z}_8$)**.** Consider the commutative ring

$$\mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\},$$

with addition and multiplication taken modulo 8.

The nonzero zero divisors of $\mathbb{Z}_8$ are

$$Z^*(\mathbb{Z}_8) = \{2, 4, 6\}.$$

Indeed, each of these elements annihilates a nonzero element modulo 8. For example,

$$2 \cdot 4 = 8 \equiv 0 \pmod 8,$$

$$4 \cdot 6 = 24 \equiv 0 \pmod 8.$$

The zero-divisor graph

$$\Gamma(\mathbb{Z}_8)$$

therefore has vertex set

$$V\big(\Gamma(\mathbb{Z}_8)\big) = \{2, 4, 6\}.$$

Two distinct vertices $x, y \in Z^*(\mathbb{Z}_8)$ are adjacent whenever

$$xy \equiv 0 \pmod 8.$$

Now,

$$2 \cdot 4 = 8 \equiv 0 \pmod 8,$$

so 2 and 4 are adjacent. Similarly,

$$4 \cdot 6 = 24 \equiv 0 \pmod 8,$$

so 4 and 6 are adjacent.

On the other hand,

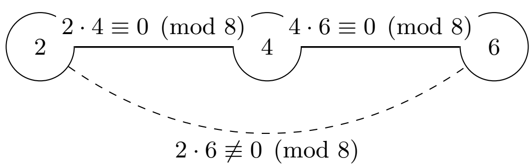
$$2 \cdot 6 = 12 \equiv 4 \not\equiv 0 \pmod 8,$$

and hence 2 and 6 are not adjacent.

Therefore,

$$E\big(\Gamma(\mathbb{Z}_8)\big) = \big\{\{2, 4\}, \{4, 6\}\big\}.$$

The corresponding zero-divisor graph is illustrated below.



The solid edges represent pairs whose product is zero modulo 8, whereas the dashed curve indicates that 2 and 6 are not adjacent. Consequently,

$$\Gamma(\mathbb{Z}_8) \cong P_3,$$

and the zero-divisor graph is the path

$$2 - 4 - 6.$$

4.25 c-MOD Zero-Divisor Graph (Proposed in this book)

A c-MOD Zero-Divisor Graph connects distinct nonzero zero divisors when c times their product is congruent to zero modulo n.

Definition 4.25.1 (c-MOD Zero-Divisor Graph)**.** Let

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}, \qquad n \geq 2,$$

and let

$$c \in \mathbb{Z}_n \setminus \{0\}.$$

Denote the set of nonzero zero divisors of $\mathbb{Z}_n$ by

$$Z^*(\mathbb{Z}_n) = Z(\mathbb{Z}_n) \setminus \{0\}.$$

The *c-MOD Zero-Divisor Graph* of $\mathbb{Z}_n$, denoted by

$$\Gamma_c(\mathbb{Z}_n),$$

is the simple graph with vertex set

$$V\big(\Gamma_c(\mathbb{Z}_n)\big) = Z^*(\mathbb{Z}_n),$$

where two distinct vertices $x, y \in Z^*(\mathbb{Z}_n)$ are adjacent if and only if

$$cxy \equiv 0 \pmod{n}.$$

Example 4.25.2 (A c-MOD Zero-Divisor Graph for Cyclic Maintenance Scheduling)**.** Consider a manufacturing plant whose daily operation is divided into 12 cyclic time slots. Suppose that periodic maintenance patterns of machines are encoded by elements of

$$\mathbb{Z}_{12} = \{0, 1, \ldots, 11\}.$$

The numerical values are regarded as modular cycle codes rather than literal numbers of hours.

The nonzero zero divisors of $\mathbb{Z}_{12}$ are

$$Z^*(\mathbb{Z}_{12}) = \{2, 3, 4, 6, 8, 9, 10\}.$$

Associate these codes with seven machines

$$M_2, M_3, M_4, M_6, M_8, M_9, M_{10},$$

where the subscript denotes the modular maintenance-cycle code assigned to each machine.

Let

$$c = 2.$$

The parameter c may be interpreted as a synchronization factor accounting for two operating shifts. Two machines M_x and M_y are regarded as *synchronization-compatible* whenever their adjusted combined cycle satisfies

$$2xy \equiv 0 \pmod{12}.$$

Thus, the corresponding 2-MOD Zero-Divisor Graph is

$$\Gamma_2(\mathbb{Z}_{12}),$$

with vertex set

$$V\big(\Gamma_2(\mathbb{Z}_{12})\big) = \{2, 3, 4, 6, 8, 9, 10\}.$$

For example, consider the machines M_2 and M_3. Since

$$2(2)(3) = 12 \equiv 0 \pmod{12},$$

we have

$$\{2, 3\} \in E\big(\Gamma_2(\mathbb{Z}_{12})\big).$$

Hence, the maintenance patterns represented by M_2 and M_3 are synchronization-compatible under the factor $c = 2$.

Similarly,

$$2(3)(4) = 24 \equiv 0 \pmod{12},$$

so

$$\{3, 4\} \in E\big(\Gamma_2(\mathbb{Z}_{12})\big).$$

On the other hand,

$$2(2)(4) = 16 \not\equiv 0 \pmod{12},$$

and therefore

$$\{2, 4\} \notin E\big(\Gamma_2(\mathbb{Z}_{12})\big).$$

Thus, the maintenance patterns represented by M_2 and M_4 are not synchronization-compatible according to this modular criterion.

For this example, the complete edge set is

$$\begin{aligned} E\big(\Gamma_2(\mathbb{Z}_{12})\big) = \big\{&\{2,3\}, \{2,6\}, \{2,9\}, \{3,4\}, \{3,6\}, \{3,8\}, \{3,10\}, \\ &\{4,6\}, \{4,9\}, \{6,8\}, \{6,9\}, \{6,10\}, \{8,9\}, \{9,10\}\big\}. \end{aligned}$$

Consequently, the graph provides a modular network representation of pairwise synchronization relationships among cyclic maintenance patterns. An edge indicates that the two corresponding maintenance-cycle codes, after adjustment by the synchronization factor $c = 2$, return to the zero residue modulo 12.

This example illustrates how a c-MOD Zero-Divisor Graph can be used as an abstract mathematical model for cyclic scheduling, synchronization, and compatibility analysis in periodically operating systems.

4.26 Integral Circulant Graph

An integral circulant graph is a circulant graph whose adjacency eigenvalues are integers [403–406].

Definition 4.26.1 (Integral Circulant Graph)**.** Let $n \geq 2$, and let

$$D$$

be a set of proper positive divisors of n. The *integral circulant graph*

$$\mathrm{ICG}_n(D)$$

has vertex set

$$\mathbb{Z}_n,$$

where two distinct vertices $x, y \in \mathbb{Z}_n$ are adjacent if and only if

$$\gcd(x - y, n) \in D.$$

Such a graph is circulant and has only integer adjacency eigenvalues.

Example 4.26.2 (Example of an Integral Circulant Graph)**.** Let

$$n = 8$$

and choose

$$D = \{1, 2\}.$$

Consider the integral circulant graph

$$G = \mathrm{ICG}_8(\{1, 2\}).$$

Its vertex set is

$$V = \mathbb{Z}_8 = \{0, 1, 2, 3, 4, 5, 6, 7\}.$$

Two distinct vertices x, y are adjacent whenever

$$\gcd(x - y, 8) \in \{1, 2\}.$$

For example,

$$\gcd(1 - 0, 8) = 1,$$

so

$$\{0, 1\} \in E.$$

Also,

$$\gcd(2 - 0, 8) = 2,$$

so

$$\{0, 2\} \in E.$$

In contrast,

$$\gcd(4 - 0, 8) = 4 \notin D,$$

so

$$\{0, 4\} \notin E.$$

Thus,

$$\mathrm{ICG}_8(\{1, 2\})$$

is an integral circulant graph determined by the divisor set

$$D = \{1, 2\}.$$

4.27 HyperMOD Structure (Proposed in this book)

A *hyperstructure* replaces an ordinary single-valued binary operation by a set-valued hyperoperation [212, 334]. A HyperMOD Structure combines this principle with arithmetic modulo a fixed positive integer.

Definition 4.27.1 (Hyperstructure)**.** [212, 334] Let H be a nonempty set, and define

$$\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\varnothing\}.$$

A *hyperoperation* on H is a mapping

$$\circ : H \times H \longrightarrow \mathcal{P}^*(H).$$

The pair

$$(H, \circ)$$

is called a *hyperstructure*. Thus, for every $a, b \in H$, the product

$$a \circ b$$

is a nonempty subset of H.

Definition 4.27.2 (HyperMOD Structure)**.** Let

$$\mathbb{Z}_m = \{0, 1, \ldots, m-1\}, \qquad m \geq 2,$$

and let

$$\varnothing \neq H \subseteq \mathbb{Z}_m.$$

Let

$$\varnothing \neq \Lambda \subseteq \mathbb{Z}_m \times \mathbb{Z}_m.$$

For $a, b \in H$, define

$$a \circ_\Lambda b = \{(ta + ub) \bmod m \mid (t, u) \in \Lambda\}.$$

Assume that H is closed under these modular hyperproducts; that is,

$$a \circ_\Lambda b \subseteq H \qquad \text{for all } a, b \in H.$$

Since $\Lambda \neq \varnothing$, every set $a \circ_\Lambda b$ is nonempty. Hence,

$$\circ_\Lambda : H \times H \longrightarrow \mathcal{P}^*(H)$$

is a well-defined hyperoperation.

The hyperstructure

$$(H, \circ_\Lambda)$$

is called a *HyperMOD Structure* over $\mathbb{Z}_m$ associated with Λ.

Remark 4.27.3. If

$$H = \mathbb{Z}_m,$$

then the closure condition is automatic, because

$$(ta + ub) \bmod m \in \mathbb{Z}_m$$

for all

$$a, b, t, u \in \mathbb{Z}_m.$$

Example 4.27.4 (Example of a HyperMOD Structure)**.** Let

$$H = \mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$$

and choose

$$\Lambda = \{(1, 1), (1, 2)\}.$$

Define

$$a \circ_\Lambda b = \{(a + b) \bmod 5,\ (a + 2b) \bmod 5\}\,.$$

Because

$$H = \mathbb{Z}_5,$$

both

$$(a + b) \bmod 5 \quad \text{and} \quad (a + 2b) \bmod 5$$

belong to H for every $a, b \in H$. Moreover, since $\Lambda \neq \varnothing$,

$$a \circ_\Lambda b \neq \varnothing.$$

Thus,

$$\circ_\Lambda : H \times H \longrightarrow \mathcal{P}^*(H)$$

is a well-defined hyperoperation.

For example,

$$1 \circ_\Lambda 2 = \{(1 + 2) \bmod 5,\ (1 + 2 \cdot 2) \bmod 5\}\,.$$

Hence,

$$1 \circ_\Lambda 2 = \{3, 0\}.$$

Similarly,

$$3 \circ_\Lambda 4 = \{(3 + 4) \bmod 5,\ (3 + 2 \cdot 4) \bmod 5\} = \{2, 1\}.$$

Therefore,

$$(\mathbb{Z}_5, \circ_\Lambda)$$

is a HyperMOD Structure.

4.28 SuperHyperMOD Structure (Proposed in this book)

A *SuperHyperstructure* extends a hyperstructure to higher-order objects obtained through iterated power-set constructions [334, 407, 408]. A SuperHyperMOD Structure combines this higher-order organization with modular arithmetic.

Definition 4.28.1 (Iterated Nonempty Powersets)**.** Let S be a nonempty set. Define recursively

$$\mathcal{P}_*^0(S) = S$$

and

$$\mathcal{P}_*^{r+1}(S) = \mathcal{P}^*\big(\mathcal{P}_*^r(S)\big), \qquad r \geq 0,$$

where

$$\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\varnothing\}.$$

Thus,

$$\mathcal{P}_*^1(S) = \mathcal{P}(S) \setminus \{\varnothing\},$$

and

$$\mathcal{P}_*^2(S) = \mathcal{P}^*\big(\mathcal{P}^*(S)\big).$$

Definition 4.28.2 (r-SuperHyperstructure)**.** Let S be a nonempty set and let $r \geq 1$. Suppose that

$$\varnothing \neq \mathcal{H} \subseteq \mathcal{P}_*^r(S).$$

An *r-SuperHyperstructure* on S is a pair

$$(\mathcal{H}, \star),$$

where

$$\star : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{P}^*(\mathcal{H})$$

is a hyperoperation.

The elements of $\mathcal{H}$ are called *r-super-elements*.

To define modular operations consistently at every level, we first introduce their recursive lifts.

Definition 4.28.3 (Lifted Modular Combination)**.** Let

$$m \geq 2, \qquad (t, u) \in \mathbb{Z}_m \times \mathbb{Z}_m.$$

Define mappings

$$C_{t,u}^{(r)} : \mathcal{P}_*^r(\mathbb{Z}_m) \times \mathcal{P}_*^r(\mathbb{Z}_m) \longrightarrow \mathcal{P}_*^r(\mathbb{Z}_m)$$

recursively as follows.

At level $r = 0$, define

$$C_{t,u}^{(0)}(a, b) = (ta + ub) \bmod m, \qquad a, b \in \mathbb{Z}_m.$$

For $r \geq 0$ and

$$A, B \in \mathcal{P}_*^{r+1}(\mathbb{Z}_m),$$

define

$$C_{t,u}^{(r+1)}(A, B) = \left\{ C_{t,u}^{(r)}(a, b) \;\middle|\; a \in A,\ b \in B \right\}.$$

Remark 4.28.4. The lifted modular combination is well defined at every level. Indeed, if

$$A, B \in \mathcal{P}_*^{r+1}(\mathbb{Z}_m),$$

then $A \neq \varnothing$ and $B \neq \varnothing$. Hence

$$C_{t,u}^{(r+1)}(A, B) \neq \varnothing.$$

Moreover, every element of this set belongs to

$$\mathcal{P}_*^r(\mathbb{Z}_m),$$

so

$$C_{t,u}^{(r+1)}(A, B) \in \mathcal{P}_*^{r+1}(\mathbb{Z}_m).$$

Definition 4.28.5 (r-SuperHyperMOD Structure)**.** Let

$$\mathbb{Z}_m = \{0, 1, \ldots, m-1\}, \qquad m \geq 2,$$

and let

$$r \geq 1.$$

Suppose that

$$\varnothing \neq \mathcal{H} \subseteq \mathcal{P}_*^r(\mathbb{Z}_m).$$

Let

$$\varnothing \neq \Lambda \subseteq \mathbb{Z}_m \times \mathbb{Z}_m.$$

For

$$A, B \in \mathcal{H},$$

define

$$A \star_\Lambda B = \left\{ C^{(r)}_{t,u}(A, B) \;\middle|\; (t, u) \in \Lambda \right\}.$$

Assume that $\mathcal{H}$ is closed under every lifted modular combination; that is,

$$C^{(r)}_{t,u}(A, B) \in \mathcal{H}$$

for all

$$A, B \in \mathcal{H}$$

and all

$$(t, u) \in \Lambda.$$

Then

$$A \star_\Lambda B \in \mathcal{P}^*(\mathcal{H})$$

for all $A, B \in \mathcal{H}$, and therefore

$$\star_\Lambda : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{P}^*(\mathcal{H})$$

is a well-defined hyperoperation.

The structure

$$(\mathcal{H}, \star_\Lambda)$$

is called an r-*SuperHyperMOD Structure* over $\mathbb{Z}_m$.

Remark 4.28.6 (Reduction to the HyperMOD Case)**.** If the preceding construction is extended to $r = 0$, then

$$\mathcal{P}^0_*(\mathbb{Z}_m) = \mathbb{Z}_m$$

and

$$C^{(0)}_{t,u}(a, b) = (ta + ub) \bmod m.$$

Consequently,

$$a \star_\Lambda b = \{(ta + ub) \bmod m \mid (t, u) \in \Lambda\},$$

which is precisely the HyperMOD construction. Thus, HyperMOD Structures constitute the level-0 case of the SuperHyperMOD hierarchy.

Example 4.28.7 (Example of a 1-SuperHyperMOD Structure)**.** Let

$$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$$

and take

$$\mathcal{H} = \mathcal{P}^1_*(\mathbb{Z}_5) = \mathcal{P}(\mathbb{Z}_5) \setminus \{\varnothing\}.$$

Thus, the members of $\mathcal{H}$ are all nonempty subsets of $\mathbb{Z}_5$.

Choose

$$\Lambda = \{(1, 1), (1, 2)\}.$$

For arbitrary

$$A, B \in \mathcal{H},$$

we have

$$C^{(1)}_{1,1}(A, B) = \{(a + b) \bmod 5 \mid a \in A,\ b \in B\},$$

and

$$C_{1,2}^{(1)}(A,B) = \{(a+2b) \bmod 5 \mid a \in A,\ b \in B\}.$$

Since A and B are nonempty, both sets are nonempty. Furthermore, every element of either set belongs to $\mathbb{Z}_5$. Hence,

$$C_{1,1}^{(1)}(A,B),\ C_{1,2}^{(1)}(A,B) \in \mathcal{H}.$$

Therefore, $\mathcal{H}$ is closed under the required lifted modular combinations.

Now let

$$A = \{0,1\}, \qquad B = \{2\}.$$

Then

$$C_{1,1}^{(1)}(A,B) = \{(a+b) \bmod 5 \mid a \in A,\ b \in B\} = \{2,3\}.$$

Similarly,

$$C_{1,2}^{(1)}(A,B) = \{(a+2b) \bmod 5 \mid a \in A,\ b \in B\} = \{4,0\}.$$

Hence,

$$A \star_\Lambda B = \big\{\{2,3\},\{0,4\}\big\}.$$

Since

$$\{2,3\},\{0,4\} \in \mathcal{H},$$

we obtain

$$A \star_\Lambda B \in \mathcal{P}^*(\mathcal{H}).$$

Therefore,

$$(\mathcal{H}, \star_\Lambda)$$

is a 1-SuperHyperMOD Structure over $\mathbb{Z}_5$.

4.29 Zero-HyperDivisor Graph (Proposed in this book)

A Zero-HyperDivisor Graph extends the ordinary zero-divisor graph by replacing individual divisors with nonempty subsets under a powerset-based hyperstructure.

Definition 4.29.1 (Zero-HyperDivisor Graph)**.** Let R be a commutative ring and let

$$\mathcal{P}_*(R) = \mathcal{P}(R) \setminus \{\varnothing\}.$$

Define the hyperproduct

$$\circ : \mathcal{P}_*(R) \times \mathcal{P}_*(R) \longrightarrow \mathcal{P}_*(R)$$

by

$$A \circ B = \{ab \mid a \in A,\ b \in B\}.$$

Thus,

$$(\mathcal{P}_*(R), \circ)$$

is a powerset-based multiplicative hyperstructure.

A nonzero set

$$A \in \mathcal{P}_*(R) \setminus \{\{0\}\}$$

is called a *zero-hyperdivisor* if there exists

$$B \in \mathcal{P}_*(R) \setminus \{\{0\}\}$$

such that

$$A \circ B = \{0\}.$$

Let

$$Z_H^*(R) = \{A \in \mathcal{P}_*(R) \setminus \{\{0\}\} \mid A \circ B = \{0\} \text{ for some } B \neq \{0\}\}.$$

The *Zero-HyperDivisor Graph* of R, denoted by

$$\Gamma_H(R),$$

is the simple graph with vertex set

$$V(\Gamma_H(R)) = Z_H^*(R),$$

where two distinct vertices A and B are adjacent if and only if

$$A \circ B = \{0\}.$$

Example 4.29.2 (Example of a Zero-HyperDivisor Graph)**.** Let

$$R = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}.$$

Consider the nonempty subsets

$$A = \{0, 2\}, \qquad B = \{0, 3\}.$$

Using the hyperproduct

$$A \circ B = \{ab \pmod 6 \mid a \in A,\ b \in B\},$$

we obtain

$$A \circ B = \{0 \cdot 0,\ 0 \cdot 3,\ 2 \cdot 0,\ 2 \cdot 3\} \pmod 6.$$

Since

$$2 \cdot 3 = 6 \equiv 0 \pmod 6,$$

it follows that

$$A \circ B = \{0\}.$$

Moreover,

$$A \neq \{0\}, \qquad B \neq \{0\}.$$

Hence A and B are zero-hyperdivisors and are adjacent in the Zero-HyperDivisor Graph

$$\Gamma_H(\mathbb{Z}_6).$$

Thus,

$$\{A, B\} \in E\big(\Gamma_H(\mathbb{Z}_6)\big).$$

4.30 Zero-SuperHyperDivisor Graph (Proposed in this book)

A Zero-SuperHyperDivisor Graph extends the zero-hyperdivisor construction to iterated powerset levels using a SuperHyperstructure.

Definition 4.30.1 (Zero-SuperHyperDivisor Graph)**.** Let R be a commutative ring. Define the nonempty iterated powersets by

$$\mathcal{P}_*^0(R) = R,$$

and

$$\mathcal{P}_*^{k+1}(R) = \mathcal{P}_*\big(\mathcal{P}_*^k(R)\big), \qquad k \geq 0.$$

Define recursively the lifted products

$$\odot_k : \mathcal{P}_*^k(R) \times \mathcal{P}_*^k(R) \longrightarrow \mathcal{P}_*^k(R)$$

by

$$a \odot_0 b = ab$$

and, for $k \geq 0$,

$$\mathcal{A} \odot_{k+1} \mathcal{B} = \{A \odot_k B \mid A \in \mathcal{A},\ B \in \mathcal{B}\}.$$

Define the zero super-element recursively by

$$\mathbf{0}_0 = 0, \qquad \mathbf{0}_{k+1} = \{\mathbf{0}_k\}.$$

For $n \geq 2$, a nonzero super-element

$$\mathcal{A} \in \mathcal{P}_*^n(R) \setminus \{\mathbf{0}_n\}$$

is called a *zero-superhyperdivisor of order n* if there exists

$$\mathcal{B} \in \mathcal{P}_*^n(R) \setminus \{\mathbf{0}_n\}$$

such that

$$\mathcal{A} \odot_n \mathcal{B} = \mathbf{0}_n.$$

Let

$$Z^*_{SH,n}(R) = \{\mathcal{A} \in \mathcal{P}_*^n(R) \setminus \{\mathbf{0}_n\} \mid \mathcal{A} \odot_n \mathcal{B} = \mathbf{0}_n \text{ for some } \mathcal{B} \neq \mathbf{0}_n\}.$$

The *Zero-SuperHyperDivisor Graph of order n* is the graph

$$\Gamma^{(n)}_{SH}(R)$$

with

$$V\big(\Gamma^{(n)}_{SH}(R)\big) = Z^*_{SH,n}(R),$$

where two distinct vertices $\mathcal{A}$ and $\mathcal{B}$ are adjacent if and only if

$$\mathcal{A} \odot_n \mathcal{B} = \mathbf{0}_n.$$

Example 4.30.2 (Example of a Zero-SuperHyperDivisor Graph)**.** Let

$$R = \mathbb{Z}_6$$

and consider the order-2 super-elements

$$\mathcal{A} = \big\{\{0,2\}, \{2\}\big\}$$

and

$$\mathcal{B} = \big\{\{0,3\}, \{3\}\big\}.$$

Clearly,

$$\mathcal{A}, \mathcal{B} \in \mathcal{P}_*^2(\mathbb{Z}_6).$$

At the first powerset level, define

$$A \odot_1 B = \{ab \pmod 6 \mid a \in A,\ b \in B\}.$$

We have

$$\{0,2\} \odot_1 \{0,3\} = \{0\},$$
$$\{0,2\} \odot_1 \{3\} = \{0\},$$
$$\{2\} \odot_1 \{0,3\} = \{0\},$$

and

$$\{2\} \odot_1 \{3\} = \{0\}.$$

Therefore,

$$\begin{aligned}\mathcal{A} \odot_2 \mathcal{B} &= \{A \odot_1 B \mid A \in \mathcal{A},\ B \in \mathcal{B}\} \\ &= \{\{0\}\}.\end{aligned}$$

Since the zero super-element of order 2 is

$$\mathbf{0}_2 = \{\{0\}\},$$

we obtain

$$\mathcal{A} \odot_2 \mathcal{B} = \mathbf{0}_2.$$

Moreover,

$$\mathcal{A} \neq \mathbf{0}_2, \qquad \mathcal{B} \neq \mathbf{0}_2.$$

Hence $\mathcal{A}$ and $\mathcal{B}$ are zero-superhyperdivisors of order 2, and they are adjacent in

$$\Gamma_{SH}^{(2)}(\mathbb{Z}_6).$$

Remark 4.30.3. Under these definitions, the Zero-HyperDivisor Graph is the first powerset-level case of the Zero-SuperHyperDivisor construction:

$$\Gamma_H(R) = \Gamma_{SH}^{(1)}(R).$$

Thus, the hierarchy may be viewed as

$$\text{Zero-Divisor Graph} \longrightarrow \text{Zero-HyperDivisor Graph} \longrightarrow \text{Zero-SuperHyperDivisor Graph}.$$

4.31 MOD Rectangular Natural Fuzzy Numbers

MOD Rectangular Natural Fuzzy Numbers represent ordered pairs of fuzzy modular values with distinct moduli, providing rectangular finite structures for fuzzy algebraic computations and modeling.

Definition 4.31.1 (MOD Rectangular Natural Fuzzy Number)**.** Let

$${}^F\mathbb{Z}_n$$

denote a MOD natural fuzzy number system modulo n. For integers

$$m, n \geq 2, \qquad m \neq n,$$

define

$${}^F\mathbb{Z}_m \times {}^F\mathbb{Z}_n = \left\{(a,b) : a \in {}^F\mathbb{Z}_m,\ b \in {}^F\mathbb{Z}_n\right\}.$$

An element

$$(a,b) \in {}^F\mathbb{Z}_m \times {}^F\mathbb{Z}_n$$

is called a *MOD Rectangular Natural Fuzzy Number*.

Example 4.31.2 (MOD Rectangular Natural Fuzzy Number)**.** Let

$$m = 5, \qquad n = 3.$$

Consider

$${}^{F}\mathbb{Z}_5 \times {}^{F}\mathbb{Z}_3.$$

For illustration, write a fuzzy modular value as

$$(r, \mu),$$

where r is a residue and $\mu \in [0, 1]$ is its membership degree. Take

$$a = (2, 0.8) \in {}^{F}\mathbb{Z}_5$$

and

$$b = (1, 0.6) \in {}^{F}\mathbb{Z}_3.$$

Then

$$x = (a, b) = \big((2, 0.8), (1, 0.6)\big) \in {}^{F}\mathbb{Z}_5 \times {}^{F}\mathbb{Z}_3$$

is a MOD Rectangular Natural Fuzzy Number.

4.32 Mod Rectangular Natural Neutrosophic Numbers

MOD Rectangular Natural Neutrosophic Numbers represent ordered pairs of neutrosophic modular values with distinct moduli, forming rectangular structures for algebraic operations and uncertainty representation purposes [409].

Definition 4.32.1 (MOD Rectangular Natural Neutrosophic Number)**.** [409] Let

$${}^{I}\mathbb{Z}_n$$

denote the MOD natural neutrosophic number system modulo n. For integers

$$m, n \geq 2, \qquad m \neq n,$$

define

$${}^{I}\mathbb{Z}_m \times {}^{I}\mathbb{Z}_n = \left\{(a, b) : a \in {}^{I}\mathbb{Z}_m,\ b \in {}^{I}\mathbb{Z}_n\right\}.$$

An element

$$(a, b) \in {}^{I}\mathbb{Z}_m \times {}^{I}\mathbb{Z}_n$$

is called a *MOD Rectangular Natural Neutrosophic Number*.

Example 4.32.2 (MOD Rectangular Natural Neutrosophic Number)**.** Let

$$m = 6, \qquad n = 2.$$

Consider the MOD rectangular natural neutrosophic number system

$$M = {}^{I}\mathbb{Z}_6 \times {}^{I}\mathbb{Z}_2.$$

Choose

$$a \in {}^{I}\mathbb{Z}_6, \qquad b \in {}^{I}\mathbb{Z}_2.$$

For example, using I to denote indeterminacy, take

$$a = 3 + 2I, \qquad b = 1 + I,$$

with the coefficients interpreted modulo 6 and modulo 2, respectively. Then

$$x = (3 + 2I,\ 1 + I) \in {}^{I}\mathbb{Z}_6 \times {}^{I}\mathbb{Z}_2$$

is a MOD Rectangular Natural Neutrosophic Number.

4.33 Cyclic Group

A Cyclic Group is a group generated by a single element, so every group element is a power of that generator.

Definition 4.33.1 (Cyclic Group)**.** A group G is called a *Cyclic Group* if there exists an element $g \in G$ such that
$$G = \langle g \rangle = \{g^n \mid n \in \mathbb{Z}\}.$$
The element g is called a generator of G.

Example 4.33.2 (Cyclic Group)**.** Consider the group

$$G = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$$

under addition modulo 6. The element 1 generates the whole group, since

$$\langle 1 \rangle = \{0, 1, 2, 3, 4, 5\}.$$

Indeed,
$$0 = 0 \cdot 1, \quad 1 = 1 \cdot 1, \quad 2 = 2 \cdot 1, \quad \ldots, \quad 5 = 5 \cdot 1 \pmod 6.$$
Hence $\mathbb{Z}_6$ is a Cyclic Group generated by 1.

4.34 Residue Class Ring

A Residue Class Ring consists of congruence classes modulo an integer, equipped with addition and multiplication performed modulo that integer.

Definition 4.34.1 (Residue Class Ring)**.** Let $n \geq 2$. The *Residue Class Ring modulo* n is

$$\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{[0], [1], \ldots, [n-1]\},$$

equipped with addition and multiplication modulo n.

Example 4.34.2 (Residue Class Ring)**.** Consider the residue class ring modulo 5:

$$\mathbb{Z}_5 = \{[0], [1], [2], [3], [4]\}.$$

For example,
$$[3] + [4] = [7] = [2]$$
and
$$[3] \cdot [4] = [12] = [2]$$
in $\mathbb{Z}_5$. Thus, addition and multiplication are performed modulo 5.

4.35 Unit Group

A Unit Group consists of all invertible elements of a ring, forming a group under the ring's multiplication operation naturally.

Definition 4.35.1 (Unit Group)**.** Let R be a ring with identity. The *Unit Group* of R is

$$R^\times = \{u \in R \mid \exists\, v \in R,\ uv = vu = 1\},$$

equipped with ring multiplication.

Example 4.35.2 (Unit Group)**.** Consider the ring

$$R = \mathbb{Z}_8.$$

Its invertible elements are

$$R^\times = \{[1], [3], [5], [7]\}.$$

Indeed,

$$[3][3] = [9] = [1], \qquad [5][5] = [25] = [1],$$

and

$$[7][7] = [49] = [1]$$

in $\mathbb{Z}_8$. Therefore,

$$\mathbb{Z}_8^\times = \{[1], [3], [5], [7]\}$$

forms a Unit Group under multiplication modulo 8.

4.36 Finite Field

A Finite Field is a field containing finitely many elements, necessarily having prime-power cardinality and supporting ordinary field operations.

Definition 4.36.1 (Finite Field)**.** A *Finite Field* is a field

$$\mathbb{F}_q$$

having finitely many elements, where

$$q = p^m$$

for some prime p and integer $m \geq 1$.

Example 4.36.2 (Finite Field)**.** Consider

$$\mathbb{F}_5 = \{0, 1, 2, 3, 4\},$$

with addition and multiplication modulo 5. Since 5 is prime, every nonzero element has a multiplicative inverse:

$$1^{-1} = 1, \qquad 2^{-1} = 3, \qquad 3^{-1} = 2, \qquad 4^{-1} = 4.$$

For example,

$$2 \cdot 3 = 6 \equiv 1 \pmod{5}.$$

Hence $\mathbb{F}_5$ is a Finite Field with five elements.

Chapter 5

Interval Structures

In this chapter, we survey and investigate various interval structures.

5.1 Polynomial Interval Groupoid

A polynomial interval groupoid consists of interval-valued polynomials equipped with a closed coefficientwise binary operation, not necessarily associative in general [410, 411].

Definition 5.1.1 (Polynomial Interval Groupoid)**.** Let R be a ring or a modular ring such as $\mathbb{Z}_n$, and let

$$\mathcal{I}(R) = \{[0,a] \mid a \in R\}.$$

Consider the set of interval-valued polynomials

$$\mathcal{P}_{\mathcal{I}}(R) = \left\{ \sum_{k=0}^{m} [0,a_k]x^k \;\middle|\; a_k \in R,\ m \geq 0 \right\}.$$

For fixed $t, u \in R$, define

$$\left(\sum_{k=0}^{m} [0,a_k]x^k\right) * \left(\sum_{k=0}^{m} [0,b_k]x^k\right) = \sum_{k=0}^{m} [0, ta_k + ub_k]x^k,$$

where missing coefficients are taken to be zero and, for $R = \mathbb{Z}_n$, the endpoint arithmetic is performed modulo n. Then

$$\left(\mathcal{P}_{\mathcal{I}}(R), *\right)$$

is called a *polynomial interval groupoid*, provided that $*$ is closed on $\mathcal{P}_{\mathcal{I}}(R)$.

Example 5.1.2 (Example of a Polynomial Interval Groupoid)**.** Let

$$R = \mathbb{Z}_5 = \{0, 1, 2, 3, 4\},$$

and choose

$$t = 2, \qquad u = 1.$$

Consider

$$p(x) = [0,1] + [0,3]x + [0,2]x^2$$

and

$$q(x) = [0,4] + [0,1]x + [0,3]x^2.$$

Using

$$p(x) * q(x) = \sum_{k=0}^{2} [0, 2a_k + b_k] x^k$$

with endpoint arithmetic modulo 5, we obtain

$$p(x) * q(x) = [0, 2(1) + 4] + [0, 2(3) + 1]x + [0, 2(2) + 3]x^2.$$

Hence,

$$p(x) * q(x) = [0, 1] + [0, 2]x + [0, 2]x^2,$$

because

$$6 \equiv 1, \qquad 7 \equiv 2 \pmod 5.$$

Therefore,

$$p(x) * q(x) \in \mathcal{P}_{\mathcal{I}}(\mathbb{Z}_5),$$

illustrating closure of the polynomial interval groupoid.

5.2 Interval Matrix Groupoid

An interval matrix groupoid consists of interval-valued matrices equipped with a closed entrywise binary operation, without requiring associativity in general.

Definition 5.2.1 (Interval Matrix Groupoid)**.** [410] Let R be a ring or a modular ring, and let

$$\mathcal{I}(R) = \{[0, a] \mid a \in R\}.$$

For positive integers m, n, define

$$\mathcal{M}_{m,n}(\mathcal{I}(R)) = \{([0, a_{ij}])_{m \times n} \mid a_{ij} \in R\}.$$

For fixed $t, u \in R$, define

$$A * B = tA + uB,$$

that is,

$$([0, a_{ij}]) * ([0, b_{ij}]) = ([0, ta_{ij} + ub_{ij}]).$$

If $R = \mathbb{Z}_q$, all endpoint arithmetic is taken modulo q. Then

$$\big(\mathcal{M}_{m,n}(\mathcal{I}(R)), *\big)$$

is called an *interval matrix groupoid*.

Example 5.2.2 (Example of an Interval Matrix Groupoid)**.** Let

$$R = \mathbb{Z}_5$$

and consider

$$A = \begin{pmatrix} [0,1] & [0,2] \\ [0,3] & [0,4] \end{pmatrix}, \qquad B = \begin{pmatrix} [0,4] & [0,1] \\ [0,2] & [0,3] \end{pmatrix}.$$

Choose

$$t = 2, \qquad u = 1,$$

and define

$$A * B = 2A + B.$$

Then

$$A * B = \begin{pmatrix} [0, 2(1) + 4] & [0, 2(2) + 1] \\ [0, 2(3) + 2] & [0, 2(4) + 3] \end{pmatrix}.$$

Reducing the endpoints modulo 5, we obtain

$$A * B = \begin{pmatrix} [0,1] & [0,0] \\ [0,3] & [0,1] \end{pmatrix}.$$

Thus,

$$A * B \in \mathcal{M}_{2,2}\big(\mathcal{I}(\mathbb{Z}_5)\big),$$

so the operation remains closed in the interval matrix groupoid.

5.3 Fuzzy Interval Groupoid

A fuzzy interval groupoid consists of intervals with endpoints in the unit interval, equipped with a closed nonassociative binary operation.

Definition 5.3.1 (Fuzzy Interval Groupoid)**.** Let

$$\mathcal{I}_F = \{[0, a] \mid a \in [0, 1]\}$$

be the set of fuzzy intervals. Let

$$\lambda, \mu \in [0, 1], \qquad \lambda + \mu \leq 1.$$

Define a binary operation

$$* : \mathcal{I}_F \times \mathcal{I}_F \longrightarrow \mathcal{I}_F$$

by

$$[0, a] * [0, b] = [0, \lambda a + \mu b].$$

Then

$$(\mathcal{I}_F, *)$$

is called a *fuzzy interval groupoid*. Associativity of $*$ is not required.

Example 5.3.2 (Example of a Fuzzy Interval Groupoid)**.** Let

$$\mathcal{I}_F = \{[0, a] \mid a \in [0, 1]\},$$

and choose

$$\lambda = 0.4, \qquad \mu = 0.5.$$

Since

$$\lambda + \mu = 0.9 \leq 1,$$

the operation

$$[0, a] * [0, b] = [0, 0.4a + 0.5b]$$

is well-defined.

Consider

$$[0, 0.6], \qquad [0, 0.8] \in \mathcal{I}_F.$$

Then

$$[0, 0.6] * [0, 0.8] = [0, 0.4(0.6) + 0.5(0.8)].$$

Therefore,

$$[0, 0.6] * [0, 0.8] = [0, 0.24 + 0.40] = [0, 0.64].$$

Since

$$0.64 \in [0, 1],$$

we have

$$[0, 0.64] \in \mathcal{I}_F.$$

Hence, this gives a concrete example of the fuzzy interval groupoid operation.

Theorem 5.3.3 (Well-Definedness of the Fuzzy Interval Groupoid)**.** *The binary operation $*$ in the above definition is well-defined on $\mathcal{I}_F$. Hence,*

$$(\mathcal{I}_F, *)$$

is a well-defined groupoid.

Proof. Let

$$[0, a], [0, b] \in \mathcal{I}_F.$$

Then

$$0 \le a, b \le 1.$$

Since

$$\lambda, \mu \ge 0, \qquad \lambda + \mu \le 1,$$

we have

$$0 \le \lambda a + \mu b \le \lambda + \mu \le 1.$$

Therefore,

$$[0, \lambda a + \mu b] \in \mathcal{I}_F.$$

Thus $*$ is closed on $\mathcal{I}_F$. Moreover, each pair $([0, a], [0, b])$ determines a unique interval

$$[0, \lambda a + \mu b].$$

Hence $*$ is a well-defined binary operation, and consequently

$$(\mathcal{I}_F, *)$$

is a well-defined fuzzy interval groupoid. □

5.4 Neutrosophic Interval Groupoid

A neutrosophic interval groupoid consists of intervals with neutrosophic endpoints, equipped with a closed binary operation involving indeterminacy explicitly algebraically [412].

Definition 5.4.1 (Neutrosophic Interval Groupoid)**.** Let R be a ring and let

$$R(I) = \{a + bI \mid a, b \in R,\ I^2 = I\}$$

be its neutrosophic extension. Define the neutrosophic interval set

$$\mathcal{I}_N(R) = \{[0, \alpha] \mid \alpha \in R(I)\}.$$

For fixed $t, u \in R$, define

$$[0, \alpha] * [0, \beta] = [0, t\alpha + u\beta], \qquad \alpha, \beta \in R(I).$$

If $R = \mathbb{Z}_n$, the coefficients of the neutrosophic endpoints are computed modulo n. Then

$$\big(\mathcal{I}_N(R), *\big)$$

is called a *neutrosophic interval groupoid*, provided that the operation $*$ is closed on $\mathcal{I}_N(R)$.

Example 5.4.2 (Example of a Neutrosophic Interval Groupoid)**.** Let

$$R = \mathbb{Z}_5$$

and consider

$$\mathcal{I}_N(\mathbb{Z}_5) = \{[0, a + bI] \mid a, b \in \mathbb{Z}_5,\ I^2 = I\}.$$

Choose

$$t = 2, \qquad u = 3,$$

and let

$$\alpha = 1 + 2I, \qquad \beta = 2 + I.$$

Then

$$[0, \alpha] * [0, \beta] = [0, 2\alpha + 3\beta].$$

Thus,

$$2\alpha + 3\beta = 2(1 + 2I) + 3(2 + I) = 8 + 7I.$$

Reducing the coefficients modulo 5,

$$8 + 7I \equiv 3 + 2I \pmod 5.$$

Therefore,

$$[0, 1 + 2I] * [0, 2 + I] = [0, 3 + 2I].$$

Since

$$3 + 2I \in \mathbb{Z}_5(I),$$

the resulting interval belongs to $\mathcal{I}_N(\mathbb{Z}_5)$.

5.5 Interval Semigroup

An interval semigroup is a set of intervals equipped with a closed associative binary operation inherited from a semigroup structure [413, 414].

Definition 5.5.1 (Interval Semigroup)**.** [414] Let $S \subseteq [0, \infty)$ be a nonempty set equipped with a closed associative binary operation

$$\circ : S \times S \longrightarrow S.$$

Define

$$\mathcal{I}(S) = \{[0, a] \mid a \in S\}.$$

For $[0, a], [0, b] \in \mathcal{I}(S)$, define

$$[0, a] \odot [0, b] = [0, a \circ b].$$

Then

$$\big(\mathcal{I}(S), \odot\big)$$

is called an *interval semigroup*.

Example 5.5.2 (Example of an Interval Semigroup)**.** Let

$$S = \mathbb{R}_{\geq 0}$$

with ordinary multiplication. Since multiplication is closed and associative on S, define

$$\mathcal{I}(S) = \{[0, a] \mid a \in \mathbb{R}_{\geq 0}\}$$

with

$$[0, a] \odot [0, b] = [0, ab].$$

For example,

$$[0, 2] \odot [0, 3] = [0, 2 \cdot 3] = [0, 6].$$

Since

$$6 \in \mathbb{R}_{\geq 0},$$

we have

$$[0, 6] \in \mathcal{I}(S).$$

Hence,

$$\big(\mathcal{I}(S), \odot\big)$$

is an interval semigroup.

5.6 Interval Polynomial Semigroup

An interval polynomial semigroup consists of interval-valued polynomials closed under an associative polynomial operation, typically induced by interval coefficient multiplication [413, 414].

Definition 5.6.1 (Interval Polynomial Semigroup)**.** Let $R \subseteq [0, \infty)$ be a commutative semiring, and let

$$\mathcal{I}(R) = \{[0, a] \mid a \in R\}.$$

Define the set of interval polynomials by

$$\mathcal{I}(R)[x] = \left\{ \sum_{i=0}^{m} [0, a_i] x^i \;\middle|\; a_i \in R,\; m \geq 0 \right\}.$$

For

$$p(x) = \sum_{i=0}^{m} [0, a_i] x^i, \qquad q(x) = \sum_{j=0}^{n} [0, b_j] x^j,$$

define

$$p(x) \odot q(x) = \sum_{k=0}^{m+n} \left[0, \sum_{i+j=k} a_i b_j \right] x^k.$$

Then

$$\big(\mathcal{I}(R)[x], \odot\big)$$

is called an *interval polynomial semigroup.*

Example 5.6.2 (Example of an Interval Polynomial Semigroup)**.** Let

$$R = \mathbb{R}_{\geq 0}$$

and consider the interval polynomials

$$p(x) = [0, 1] + [0, 2]x$$

and

$$q(x) = [0, 3] + [0, 1]x.$$

Their product is

$$p(x) \odot q(x) = [0, 1 \cdot 3] + [0, 1 \cdot 1 + 2 \cdot 3]x + [0, 2 \cdot 1]x^2.$$

Therefore,

$$p(x) \odot q(x) = [0, 3] + [0, 7]x + [0, 2]x^2.$$

Since all endpoints belong to

$$\mathbb{R}_{\geq 0},$$

we have

$$p(x) \odot q(x) \in \mathcal{I}(R)[x].$$

Thus, this gives a concrete multiplication in an interval polynomial semigroup.

5.7 Fuzzy Interval Semigroup

A fuzzy interval semigroup consists of unit-interval values combined through an associative fuzzy operation, such as a t-norm, ensuring closure.

Definition 5.7.1 (Fuzzy Interval Semigroup)**.** Let

$$\mathcal{I}_F = \{[0,a] \mid a \in [0,1]\}$$

be the set of fuzzy intervals, and let

$$T : [0,1] \times [0,1] \longrightarrow [0,1]$$

be a t-norm. Define

$$* : \mathcal{I}_F \times \mathcal{I}_F \longrightarrow \mathcal{I}_F$$

by

$$[0,a] * [0,b] = [0, T(a,b)].$$

Then

$$(\mathcal{I}_F, *)$$

is called a *fuzzy interval semigroup.*

Example 5.7.2 (Example of a Fuzzy Interval Semigroup)**.** Let

$$\mathcal{I}_F = \{[0,a] \mid a \in [0,1]\},$$

and choose the product t-norm

$$T(a,b) = ab.$$

The corresponding operation is

$$[0,a] * [0,b] = [0, ab].$$

For example,

$$[0,0.6] * [0,0.8] = [0, 0.6 \cdot 0.8] = [0, 0.48].$$

Since

$$0.48 \in [0,1],$$

we have

$$[0, 0.48] \in \mathcal{I}_F.$$

Hence, the product t-norm provides a concrete example of a fuzzy interval semigroup operation.

Theorem 5.7.3 (Well-Definedness of the Fuzzy Interval Semigroup)**.** *The operation $*$ defined above is a well-defined associative binary operation on $\mathcal{I}_F$. Hence,*

$$(\mathcal{I}_F, *)$$

is a well-defined semigroup.

Proof. Let

$$[0,a], [0,b] \in \mathcal{I}_F.$$

Since T is a t-norm,

$$T(a,b) \in [0,1].$$

Therefore,

$$[0, T(a,b)] \in \mathcal{I}_F,$$

so $*$ is closed and well-defined.

Moreover, for any

$$[0,a], [0,b], [0,c] \in \mathcal{I}_F,$$

the associativity of T gives

$$T(T(a,b),c) = T(a, T(b,c)).$$

Hence,

$$([0,a] * [0,b]) * [0,c] = [0,a] * ([0,b] * [0,c]).$$

Thus $*$ is associative, and consequently

$$(\mathcal{I}_F, *)$$

is a well-defined fuzzy interval semigroup. □

5.8 Neutrosophic Interval Semigroup

A neutrosophic interval semigroup consists of neutrosophic intervals closed under an associative operation respecting the indeterminacy relation algebraically throughout [412].

Definition 5.8.1 (Neutrosophic Interval Semigroup)**.** Let R be a commutative semiring, and define its neutrosophic extension by

$$R(I) = \{a + bI \mid a, b \in R,\ I^2 = I\}.$$

Let

$$\mathcal{I}_N(R) = \{[0, \alpha] \mid \alpha \in R(I)\}$$

be the set of formal neutrosophic intervals. For

$$[0, \alpha], [0, \beta] \in \mathcal{I}_N(R),$$

define

$$[0, \alpha] \odot [0, \beta] = [0, \alpha\beta],$$

where multiplication in $R(I)$ is determined by

$$I^2 = I.$$

Then

$$\big(\mathcal{I}_N(R), \odot\big)$$

is called a *neutrosophic interval semigroup.*

Example 5.8.2 (Example of a Neutrosophic Interval Semigroup)**.** Let

$$R = \mathbb{R}_{\geq 0}$$

and consider its neutrosophic extension

$$R(I) = \{a + bI \mid a, b \in \mathbb{R}_{\geq 0},\ I^2 = I\}.$$

Take

$$\alpha = 1 + 2I, \qquad \beta = 3 + I.$$

Then

$$[0, \alpha] \odot [0, \beta] = [0, \alpha\beta].$$

Using $I^2 = I$, we obtain

$$\begin{aligned} \alpha\beta &= (1 + 2I)(3 + I) \\ &= 3 + I + 6I + 2I^2 \\ &= 3 + 9I. \end{aligned}$$

Therefore,

$$[0, 1 + 2I] \odot [0, 3 + I] = [0, 3 + 9I].$$

Since

$$3 + 9I \in R(I),$$

the resulting interval belongs to

$$\mathcal{I}_N(R).$$

Thus, this is a concrete operation in a neutrosophic interval semigroup.

5.9 Interval Semiring

An interval semiring is an interval-valued algebraic structure with associative addition and multiplication, additive identity, and distributivity between both operations.

Definition 5.9.1 (Interval Semiring)**.** Let

$$(S, +, \cdot, 0, 1)$$

be a commutative semiring whose elements are nonnegative in a compatible order. Define

$$\mathcal{I}(S) = \{[0, a] \mid a \in S\}.$$

For

$$[0, a], [0, b] \in \mathcal{I}(S),$$

define

$$[0, a] \oplus [0, b] = [0, a + b]$$

and

$$[0, a] \otimes [0, b] = [0, ab].$$

Then

$$\big(\mathcal{I}(S), \oplus, \otimes, [0, 0], [0, 1]\big)$$

is called an *interval semiring.*

Example 5.9.2 (Example of an Interval Semiring)**.** Let

$$S = \mathbb{N}_0 = \{0, 1, 2, \dots\}$$

with the usual addition and multiplication. Then

$$\mathcal{I}(S) = \{[0, a] \mid a \in \mathbb{N}_0\}.$$

Consider

$$[0, 2], [0, 3] \in \mathcal{I}(S).$$

Their interval sum is

$$[0, 2] \oplus [0, 3] = [0, 2 + 3] = [0, 5],$$

and their interval product is

$$[0, 2] \otimes [0, 3] = [0, 2 \cdot 3] = [0, 6].$$

Since

$$5, 6 \in \mathbb{N}_0,$$

both

$$[0, 5], [0, 6] \in \mathcal{I}(S).$$

Hence,

$$\big(\mathcal{I}(\mathbb{N}_0), \oplus, \otimes, [0, 0], [0, 1]\big)$$

is an interval semiring.

5.10 Interval Semifield

An interval semifield is an interval semiring whose nonzero elements possess multiplicative inverses, while additive inverses are generally not required.

Definition 5.10.1 (Interval Semifield)**.** [414] Let

$$(K, +, \cdot, 0, 1)$$

be a commutative semifield, that is, a commutative semiring in which every nonzero element has a multiplicative inverse. Define

$$\mathcal{I}(K) = \{[0, a] \mid a \in K\}.$$

For

$$[0, a], [0, b] \in \mathcal{I}(K),$$

define

$$[0, a] \oplus [0, b] = [0, a + b]$$

and

$$[0, a] \otimes [0, b] = [0, ab].$$

Then

$$\big(\mathcal{I}(K), \oplus, \otimes, [0, 0], [0, 1]\big)$$

is called an *interval semifield*. For every

$$[0, a] \neq [0, 0],$$

its multiplicative inverse is

$$[0, a]^{-1} = [0, a^{-1}],$$

since

$$[0, a] \otimes [0, a^{-1}] = [0, 1].$$

Example 5.10.2 (Example of an Interval Semifield)**.** Let

$$K = \mathbb{R}_{\geq 0}$$

with the usual addition and multiplication. Every nonzero element $a \in K$ has the multiplicative inverse

$$a^{-1} = \frac{1}{a} \in K.$$

Thus,

$$\mathcal{I}(K) = \{[0, a] \mid a \in \mathbb{R}_{\geq 0}\}.$$

For example,

$$[0, 2] \oplus [0, 3] = [0, 5]$$

and

$$[0, 2] \otimes [0, 3] = [0, 6].$$

Moreover, the multiplicative inverse of

$$[0, 2]$$

is

$$[0, 2]^{-1} = \left[0, \frac{1}{2}\right],$$

because

$$[0, 2] \otimes \left[0, \frac{1}{2}\right] = [0, 1].$$

Hence,

$$\big(\mathcal{I}(\mathbb{R}_{\geq 0}), \oplus, \otimes, [0, 0], [0, 1]\big)$$

provides a concrete example of an interval semifield.

5.11 Interval-Valued Fuzzy Set

An interval-valued fuzzy set assigns each element a closed interval in $[0, 1]$, representing lower and upper possible membership degrees explicitly [415–419].

Definition 5.11.1 (Interval-Valued Fuzzy Set)**.** Let X be a nonempty set, and let

$$\mathcal{I}([0,1]) = \{[a,b] \mid 0 \leq a \leq b \leq 1\}.$$

An *Interval-Valued Fuzzy Set* A on X is characterized by a mapping

$$\mu_A : X \longrightarrow \mathcal{I}([0,1]).$$

Thus, for each $x \in X$,

$$\mu_A(x) = [\mu_A^L(x), \mu_A^U(x)], \qquad 0 \leq \mu_A^L(x) \leq \mu_A^U(x) \leq 1,$$

where the interval represents the possible membership degrees of x.

Example 5.11.2 (Example of an Interval-Valued Fuzzy Set)**.** Let

$$X = \{x_1, x_2, x_3\}$$

be a set of three products, and suppose that the interval-valued membership degrees describe their possible degrees of customer satisfaction.

Define an interval-valued fuzzy set A by

$$\mu_A(x_1) = [0.70, 0.90],$$

$$\mu_A(x_2) = [0.40, 0.60],$$

and

$$\mu_A(x_3) = [0.80, 0.95].$$

Thus,

$$A = \{(x_1, [0.70, 0.90]), (x_2, [0.40, 0.60]), (x_3, [0.80, 0.95])\}.$$

For example,

$$\mu_A(x_1) = [0.70, 0.90]$$

means that the possible membership degree of x_1 in the set of highly satisfactory products lies between 0.70 and 0.90. Hence, A is an interval-valued fuzzy set on X.

5.12 Interval-Valued Soft Set

An Interval-Valued Soft Set assigns each parameter a lower and upper subset of the universe, representing bounded parametric uncertainty explicitly [420, 421].

Definition 5.12.1 (Interval-Valued Soft Set)**.** Let X be a nonempty universe, let E be a set of parameters, and let

$$A \subseteq E.$$

Denote by $\mathrm{IVS}(X)$ the family of all interval-valued sets

$$[L, U], \qquad L \subseteq U \subseteq X.$$

An *Interval-Valued Soft Set* over X is a mapping

$$F_A : A \longrightarrow \mathrm{IVS}(X),$$

such that, for each $e \in A$,

$$F_A(e) = \left[F_A^-(e), F_A^+(e)\right], \qquad F_A^-(e) \subseteq F_A^+(e) \subseteq X.$$

Thus, each parameter is associated with a lower and an upper approximation of a subset of X.

Example 5.12.2 (Interval-Valued Soft Set)**.** Let

$$X = \{x_1, x_2, x_3\}, \qquad A = \{e_1, e_2\}.$$

Define

$$F_A(e_1) = \big[\{x_1\}, \{x_1, x_2\}\big],$$

and

$$F_A(e_2) = \big[\{x_2\}, \{x_2, x_3\}\big].$$

Since

$$\{x_1\} \subseteq \{x_1, x_2\}, \qquad \{x_2\} \subseteq \{x_2, x_3\},$$

the mapping

$$F_A : A \longrightarrow \mathrm{IVS}(X)$$

defines an Interval-Valued Soft Set over X.

5.13 Interval-Valued Neutrosophic Set

An interval-valued neutrosophic set assigns each element truth, indeterminacy, and falsity intervals, allowing uncertainty within all three membership components simultaneously [422–426].

Definition 5.13.1 (Interval-Valued Neutrosophic Set)**.** [427–429] Let X be a nonempty set. An *Interval-Valued Neutrosophic Set* A on X is characterized by three interval-valued mappings

$$T_A, I_A, F_A : X \longrightarrow \mathcal{I}([0,1]),$$

where, for each $x \in X$,

$$T_A(x) = [T_A^L(x), T_A^U(x)],$$
$$I_A(x) = [I_A^L(x), I_A^U(x)],$$

and

$$F_A(x) = [F_A^L(x), F_A^U(x)].$$

These intervals represent the truth, indeterminacy, and falsity degrees of x, respectively.

Example 5.13.2 (Example of an Interval-Valued Neutrosophic Set)**.** Let

$$X = \{x_1, x_2\}$$

be a set of two alternatives. Define an interval-valued neutrosophic set A on X by

$$T_A(x_1) = [0.70, 0.85], \qquad I_A(x_1) = [0.10, 0.20], \qquad F_A(x_1) = [0.15, 0.25],$$

and

$$T_A(x_2) = [0.40, 0.60], \qquad I_A(x_2) = [0.20, 0.35], \qquad F_A(x_2) = [0.30, 0.45].$$

Equivalently,

$$A = \{(x_1, [0.70, 0.85], [0.10, 0.20], [0.15, 0.25]),$$
$$(x_2, [0.40, 0.60], [0.20, 0.35], [0.30, 0.45])\}.$$

For instance,

$$T_A(x_1) = [0.70, 0.85]$$

represents the possible truth degree of x_1,

$$I_A(x_1) = [0.10, 0.20]$$

represents its possible indeterminacy degree, and

$$F_A(x_1) = [0.15, 0.25]$$

represents its possible falsity degree.

Therefore, A is an interval-valued neutrosophic set on X.

5.14 Interval-Valued Uncertain Set

An interval-valued uncertain set assigns each element admissible intervals within a model-dependent degree-domain, representing bounded uncertainty in component values explicitly [74].

Definition 5.14.1 (Interval-Valued Uncertain Set)**.** [74] Let X be a nonempty set, and let M be an uncertain model with degree-domain

$$D_M \subseteq [0,1]^s, \qquad s \geq 1.$$

Define

$$\mathcal{I}(D_M) = \left\{ \big([a_1, b_1], \ldots, [a_s, b_s]\big) \;\middle|\; \prod_{i=1}^{s} [a_i, b_i] \subseteq D_M \right\}.$$

An *Interval-Valued Uncertain Set* of type M on X is a pair

$$U_M^{\mathrm{IV}} = (X, \mu_M^{\mathrm{IV}}),$$

where

$$\mu_M^{\mathrm{IV}} : X \longrightarrow \mathcal{I}(D_M).$$

Thus, each $x \in X$ is assigned an admissible interval-valued uncertainty degree determined by the underlying uncertain model M.

5.15 HyperInterval-Valued Uncertain Set

A HyperInterval-Valued Uncertain Set assigns each element a nonempty family of admissible uncertainty-degree intervals determined by the underlying uncertain model [430].

Let M be an uncertain model with degree-domain

$$D_M \subseteq [0,1]^k, \qquad k \geq 1,$$

equipped with the componentwise order. Define

$$\mathcal{I}(D_M) = \{[\mathbf{a}, \mathbf{b}]_{D_M} \mid \mathbf{a}, \mathbf{b} \in D_M,\ \mathbf{a} \leq \mathbf{b}\},$$

where

$$[\mathbf{a}, \mathbf{b}]_{D_M} = \{\mathbf{d} \in D_M \mid \mathbf{a} \leq \mathbf{d} \leq \mathbf{b}\}.$$

Definition 5.15.1 (HyperInterval-Valued Uncertain Set)**.** Let $X \neq \varnothing$. A *HyperInterval-Valued Uncertain Set* of type M on X is a mapping

$$H_M : X \longrightarrow \mathcal{P}^*\big(\mathcal{I}(D_M)\big),$$

where

$$\mathcal{P}^*(S) = \mathcal{P}(S) \setminus \{\varnothing\}.$$

Thus, each $x \in X$ is assigned a nonempty family of admissible uncertainty-degree intervals.

5.16 SuperHyperInterval-Valued Uncertain Set

A SuperHyperInterval-Valued Uncertain Set assigns each element hierarchically iterated nonempty nested families of admissible uncertainty-degree intervals across multiple structural levels [430]. Define the iterated nonempty powersets recursively by

$$\mathcal{P}^{*0}(S) = S$$

and

$$\mathcal{P}^{*(r+1)}(S) = \mathcal{P}^{*}\left(\mathcal{P}^{*r}(S)\right), \qquad r \geq 0.$$

Definition 5.16.1 (SuperHyperInterval-Valued Uncertain Set)**.** Let $X \neq \varnothing$ and $n \geq 1$. A *SuperHyperInterval-Valued Uncertain Set* of type M and order n on X is a mapping

$$SH_M^{(n)} : X \longrightarrow \mathcal{P}^{*(n-1)}\big(\mathcal{I}(D_M)\big).$$

Hence, each $x \in X$ is assigned an $(n-1)$-level hierarchical family of admissible uncertainty-degree intervals.

5.17 n-Interval Semigroup

An n-interval semigroup combines n interval semigroups into a componentwise product structure whose operation remains closed and associative throughout components [431].

Definition 5.17.1 (n-Interval Semigroup)**.** Let

$$(S_i, *_i), \qquad i = 1, \ldots, n,$$

be interval semigroups, where $n \geq 2$. Define

$$S = S_1 \times \cdots \times S_n.$$

For

$$x = (x_1, \ldots, x_n), \qquad y = (y_1, \ldots, y_n) \in S,$$

define the componentwise operation

$$x * y = (x_1 *_1 y_1, \ldots, x_n *_n y_n).$$

Then

$$(S, *)$$

is called an *n-interval semigroup*.

Example 5.17.2 (Example of a 2-Interval Semigroup)**.** Let

$$S_1 = \{[0, a] \mid a \in \mathbb{N}_0\}$$

with the operation

$$[0, a] *_1 [0, b] = [0, a + b],$$

and let

$$S_2 = \{[0, a] \mid a \in \mathbb{N}_0\}$$

with the operation

$$[0, a] *_2 [0, b] = [0, ab].$$

Both $(S_1, *_1)$ and $(S_2, *_2)$ are interval semigroups, since ordinary addition and multiplication on $\mathbb{N}_0$ are closed and associative.

Define

$$S = S_1 \times S_2.$$

Consider

$$x = ([0,2],[0,3]), \qquad y = ([0,4],[0,5]).$$

Then

$$\begin{aligned} x * y &= \big([0,2] *_1 [0,4], [0,3] *_2 [0,5]\big) \\ &= \big([0, 2+4], [0, 3 \cdot 5]\big) \\ &= ([0,6],[0,15]). \end{aligned}$$

Since

$$([0,6],[0,15]) \in S_1 \times S_2,$$

the structure

$$(S, *)$$

is a 2-interval semigroup.

5.18 n-Interval Groupoid

An n-interval groupoid combines n interval groupoids through componentwise operations, requiring closure in each component but does not require associativity in general [431].

Definition 5.18.1 (n-Interval Groupoid)**.** Let

$$(G_i, *_i), \qquad i = 1, \ldots, n,$$

be interval groupoids, where $n \geq 2$. Define

$$G = G_1 \times \cdots \times G_n.$$

For

$$x = (x_1, \ldots, x_n), \qquad y = (y_1, \ldots, y_n) \in G,$$

define

$$x * y = (x_1 *_1 y_1, \ldots, x_n *_n y_n).$$

Then

$$(G, *)$$

is called an *n-interval groupoid*. Associativity is not required.

Example 5.18.2 (Example of a 2-Interval Groupoid)**.** Let

$$G_1 = \{[0,a] \mid a \in \mathbb{Z}_5\}$$

with

$$[0,a] *_1 [0,b] = [0, a+b \pmod 5],$$

and let

$$G_2 = \{[0,a] \mid a \in \mathbb{Z}_5\}$$

with

$$[0,a] *_2 [0,b] = [0, 2a+b \pmod 5].$$

Both operations are closed on their respective interval sets.

Define

$$G = G_1 \times G_2.$$

Take

$$x = ([0,1],[0,3]), \qquad y = ([0,4],[0,2]).$$

Then
$$\begin{aligned} x * y &= \big([0,1] *_1 [0,4], [0,3] *_2 [0,2]\big) \\ &= \big([0, 1+4 \pmod 5], [0, 2(3)+2 \pmod 5]\big) \\ &= ([0,0],[0,3]). \end{aligned}$$
Hence,
$$x * y \in G.$$

Moreover, the second component need not be associative. For example,
$$(1 *_2 2) *_2 3 = 4 *_2 3 \equiv 1 \pmod 5,$$
whereas
$$1 *_2 (2 *_2 3) = 1 *_2 2 \equiv 4 \pmod 5.$$
Thus associativity is not required, and
$$(G, *)$$
is a 2-interval groupoid.

5.19 n-Interval Group

An n-interval group combines n interval groups componentwise, inheriting closure, associativity, identities, and inverses from every constituent group structure directly [431].

Definition 5.19.1 (n-Interval Group)**.** Let
$$(G_i, *_i), \qquad i = 1, \ldots, n,$$
be interval groups, where $n \geq 2$. Define
$$G = G_1 \times \cdots \times G_n$$
with the componentwise operation
$$(x_1, \ldots, x_n) * (y_1, \ldots, y_n) = (x_1 *_1 y_1, \ldots, x_n *_n y_n).$$
Then
$$(G, *)$$
is called an *n-interval group.* Its identity is
$$e = (e_1, \ldots, e_n),$$
and the inverse of
$$x = (x_1, \ldots, x_n)$$
is
$$x^{-1} = (x_1^{-1}, \ldots, x_n^{-1}).$$

Example 5.19.2 (Example of a 2-Interval Group)**.** Let
$$G_1 = \{[0,a] \mid a \in \mathbb{Z}_3\}$$
with
$$[0,a] *_1 [0,b] = [0, a+b \pmod 3],$$
and let
$$G_2 = \{[0,a] \mid a \in \mathbb{Z}_5\}$$

with

$$[0,a] *_2 [0,b] = [0, a+b \pmod 5].$$

Then $(G_1, *_1)$ and $(G_2, *_2)$ are interval groups isomorphic to the additive groups

$$\mathbb{Z}_3 \qquad \text{and} \qquad \mathbb{Z}_5,$$

respectively.

Define

$$G = G_1 \times G_2.$$

For

$$x = ([0,1],[0,4]), \qquad y = ([0,2],[0,3]),$$

we obtain

$$\begin{aligned} x * y &= \big([0, 1+2 \pmod 3], [0, 4+3 \pmod 5]\big) \\ &= ([0,0],[0,2]). \end{aligned}$$

The identity element is

$$e = ([0,0],[0,0]).$$

For example, the inverse of

$$x = ([0,1],[0,4])$$

is

$$x^{-1} = ([0,2],[0,1]),$$

because

$$1 + 2 \equiv 0 \pmod 3$$

and

$$4 + 1 \equiv 0 \pmod 5.$$

Therefore,

$$x * x^{-1} = ([0,0],[0,0]) = e.$$

Hence,

$$(G, *)$$

is a 2-interval group.

5.20 Interval Arithmetic

Interval arithmetic performs numerical operations on intervals so that the resulting interval contains all possible values of the corresponding operation.

Definition 5.20.1 (Interval Arithmetic)**.** Let

$$X = [a,b], \qquad Y = [c,d]$$

be closed real intervals. For a binary operation $\circ$, define

$$X \circ Y = \{x \circ y \mid x \in X,\ y \in Y\}.$$

In particular,

$$\begin{aligned} X + Y &= [a+c, b+d], \\ X - Y &= [a-d, b-c], \end{aligned}$$

and

$$XY = \big[\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}\big].$$

These operations constitute *interval arithmetic*.

Example 5.20.2 (Example of Interval Arithmetic)**.** Let

$$X = [1, 3] \qquad \text{and} \qquad Y = [2, 4].$$

Using interval arithmetic, their sum is

$$X + Y = [1 + 2, 3 + 4] = [3, 7].$$

Their difference is

$$X - Y = [1 - 4, 3 - 2] = [-3, 1].$$

For multiplication,

$$XY = \big[\min\{1 \cdot 2, 1 \cdot 4, 3 \cdot 2, 3 \cdot 4\}, \max\{1 \cdot 2, 1 \cdot 4, 3 \cdot 2, 3 \cdot 4\}\big].$$

Hence,

$$XY = [2, 12].$$

Thus, interval arithmetic provides intervals containing all possible results of the corresponding operations.

5.21 Interval Matrix

An interval matrix is a matrix whose entries are intervals rather than single numerical values (cf. [432–434]).

Definition 5.21.1 (Interval Matrix)**.** Let

$$\mathcal{I}(\mathbb{R}) = \{[a, b] \mid a, b \in \mathbb{R},\ a \leq b\}.$$

An *interval matrix* is a matrix

$$A = (A_{ij})_{m \times n}$$

such that

$$A_{ij} \in \mathcal{I}(\mathbb{R})$$

for every i and j. Equivalently,

$$A_{ij} = [\underline{a}_{ij}, \overline{a}_{ij}], \qquad \underline{a}_{ij} \leq \overline{a}_{ij}.$$

Example 5.21.2 (Example of an Interval Matrix)**.** Consider the matrix

$$A = \begin{pmatrix} [1, 2] & [0, 3] \\ [-1, 1] & [2, 4] \end{pmatrix}.$$

Each entry of A is a closed real interval. For example,

$$A_{11} = [1, 2], \qquad A_{12} = [0, 3], \qquad A_{21} = [-1, 1], \qquad A_{22} = [2, 4].$$

Therefore,

$$A \in \mathcal{I}(\mathbb{R})^{2 \times 2},$$

and A is an interval matrix.

5.22 Interval Graph

An interval graph represents intersections among intervals on the real line [435–437].

Definition 5.22.1 (Interval Graph)**.** Let

$$\mathcal{F} = \{I_v \mid v \in V\}$$

be a finite family of closed intervals on the real line. The *interval graph* associated with $\mathcal{F}$ is the graph

$$G = (V, E),$$

where two distinct vertices $u, v \in V$ are adjacent if and only if

$$I_u \cap I_v \neq \varnothing.$$

Thus,

$$\{u, v\} \in E \iff I_u \cap I_v \neq \varnothing.$$

Example 5.22.2 (Example of an Interval Graph)**.** Let

$$V = \{v_1, v_2, v_3, v_4\}$$

and assign to the vertices the following closed intervals:

$$I_{v_1} = [0, 2], \qquad I_{v_2} = [1, 3], \qquad I_{v_3} = [2.5, 4], \qquad I_{v_4} = [5, 6].$$

Recall that two distinct vertices v_i and v_j are adjacent precisely when their corresponding intervals intersect:

$$\{v_i, v_j\} \in E \iff I_{v_i} \cap I_{v_j} \neq \varnothing.$$

In the present example,

$$I_{v_1} \cap I_{v_2} = [1, 2] \neq \varnothing,$$

and

$$I_{v_2} \cap I_{v_3} = [2.5, 3] \neq \varnothing.$$

Hence,

$$\{v_1, v_2\}, \{v_2, v_3\} \in E.$$

On the other hand,

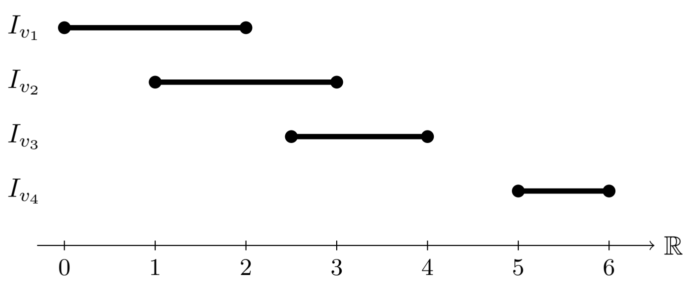
$$I_{v_1} \cap I_{v_3} = \varnothing,$$

and

$$I_{v_4} \cap I_{v_i} = \varnothing \qquad (i = 1, 2, 3).$$

Therefore, v_4 is an isolated vertex.

The interval representation is illustrated below.



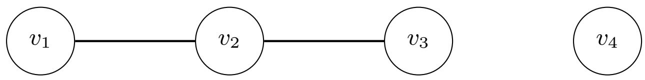
Thus, the corresponding interval graph is

$$G = (V, E),$$

where

$$E = \big\{\{v_1, v_2\}, \{v_2, v_3\}\big\}.$$

Its graph representation is shown below.



Consequently, the vertices v_1, v_2, v_3 induce the path

$$v_1 - v_2 - v_3,$$

while v_4 is isolated. Hence,

$$G \cong P_3 \cup K_1.$$

This example demonstrates that adjacency in an interval graph is determined exactly by the nonempty intersection of the intervals assigned to its vertices.

5.23 Interval Hypergraph

An interval hypergraph is a hypergraph whose hyperedges form consecutive blocks under a suitable linear ordering of the vertices (cf. [438–440]).

Definition 5.23.1 (Interval Hypergraph)**.** Let

$$H = (V, E)$$

be a finite hypergraph. The hypergraph H is called an *interval hypergraph* if there exists a linear ordering

$$v_1 < v_2 < \cdots < v_n$$

of V such that every hyperedge $e \in E$ has the form

$$e = \{v_i, v_{i+1}, \ldots, v_j\}$$

for some

$$1 \le i \le j \le n.$$

Thus, every hyperedge consists of consecutive vertices in the ordering.

Example 5.23.2 (Example of an Interval Hypergraph)**.** Let

$$V = \{v_1, v_2, v_3, v_4, v_5\}$$

be a vertex set equipped with the linear ordering

$$v_1 < v_2 < v_3 < v_4 < v_5.$$

Define the hyperedge set

$$E = \big\{e_1, e_2, e_3\big\},$$

where

$$e_1 = \{v_1, v_2\}, \qquad e_2 = \{v_2, v_3, v_4\}, \qquad e_3 = \{v_4, v_5\}.$$

Each hyperedge consists of consecutive vertices with respect to the given linear ordering. More precisely,

$$e_1 = \{v_i \mid 1 \le i \le 2\},$$

$$e_2 = \{v_i \mid 2 \le i \le 4\},$$

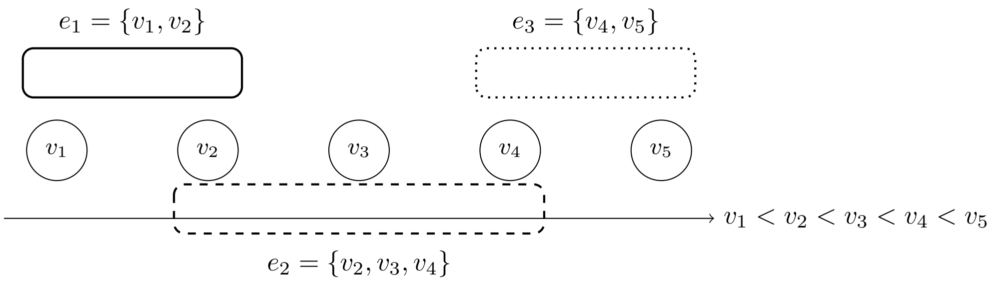
and

$$e_3 = \{v_i \mid 4 \le i \le 5\}.$$

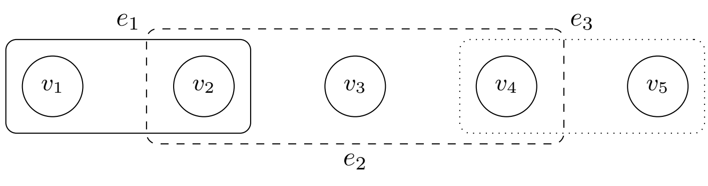
The interval structure of the hyperedges can be represented by the index intervals

$$e_1 \leftrightarrow [1, 2], \qquad e_2 \leftrightarrow [2, 4], \qquad e_3 \leftrightarrow [4, 5].$$

The corresponding interval representation is illustrated below.



Alternatively, the same hypergraph can be visualized directly as a hypergraph whose hyperedges surround consecutive blocks of vertices.



Thus, every hyperedge forms an interval of consecutive vertices in the ordering

$$v_1 < v_2 < v_3 < v_4 < v_5.$$

Therefore,

$$H = (V, E)$$

is an interval hypergraph.

5.24 Interval Order

An interval order is a partially ordered set representable by precedence relations between intervals on the real line.

Definition 5.24.1 (Interval Order)**.** Let

$$(P, \prec)$$

be a partially ordered set. It is called an *interval order* if each element $x \in P$ can be assigned a real interval

$$I_x = [a_x, b_x]$$

such that, for all distinct $x, y \in P$,

$$x \prec y \quad \Longleftrightarrow \quad b_x < a_y.$$

Hence, x precedes y precisely when the interval representing x lies entirely to the left of the interval representing y.

Example 5.24.2 (Example of an Interval Order)**.** Let

$$P = \{x_1, x_2, x_3\}$$

and assign the intervals

$$I_{x_1} = [0, 2], \qquad I_{x_2} = [1, 3], \qquad I_{x_3} = [4, 5].$$

Since

$$2 < 4,$$

we have

$$x_1 \prec x_3.$$

Similarly, since

$$3 < 4,$$

we have

$$x_2 \prec x_3.$$

However,

$$I_{x_1} \cap I_{x_2} \neq \varnothing,$$

so neither

$$x_1 \prec x_2$$

nor

$$x_2 \prec x_1$$

holds.

Hence, the partial order

$$x_1 \prec x_3, \qquad x_2 \prec x_3,$$

with x_1 and x_2 incomparable, is an interval order.

5.25 HyperInterval Graph (Proposed in this book)

A *HyperInterval Graph* extends an ordinary graph by assigning a nonempty family of admissible intervals to each vertex and edge. Thus, interval-valued information may be represented in a set-valued rather than a single-interval form.

Let $\mathcal{I}$ be a nonempty universe of admissible intervals, and define

$$\mathcal{P}^*(\mathcal{I}) = \mathcal{P}(\mathcal{I}) \setminus \{\varnothing\}.$$

Definition 5.25.1 (HyperInterval Graph)**.** Let

$$G = (V, E)$$

be a graph and let $\mathcal{I} \neq \varnothing$ be a prescribed universe of admissible intervals.

A *HyperInterval Graph* is a structure

$$G_{\mathrm{HI}} = (V, E, \mathrm{HI}_V, \mathrm{HI}_E),$$

where

$$\mathrm{HI}_V : V \longrightarrow \mathcal{P}^*(\mathcal{I})$$

and

$$\mathrm{HI}_E : E \longrightarrow \mathcal{P}^*(\mathcal{I}).$$

Thus, for every vertex $v \in V$ and every edge $e \in E$,

$$\varnothing \neq \mathrm{HI}_V(v) \subseteq \mathcal{I}, \qquad \varnothing \neq \mathrm{HI}_E(e) \subseteq \mathcal{I}.$$

Hence, each vertex and edge is assigned a nonempty family of admissible intervals.

Example 5.25.2 (Example of a HyperInterval Graph)**.** Let

$$G = (V, E),$$

where

$$V = \{v_1, v_2, v_3\}$$

and

$$E = \{e_{12}, e_{23}\}, \qquad e_{12} = \{v_1, v_2\}, \qquad e_{23} = \{v_2, v_3\}.$$

Let the universe of admissible intervals be

$$\mathcal{I} = \{[0, 1], [1, 2], [2, 3], [3, 4]\}.$$

Define

$$\mathrm{HI}_V : V \longrightarrow \mathcal{P}^*(\mathcal{I})$$

by

$$\mathrm{HI}_V(v_1) = \{[0, 1], [1, 2]\},$$

$$\mathrm{HI}_V(v_2) = \{[1, 2], [2, 3], [3, 4]\},$$

and

$$\mathrm{HI}_V(v_3) = \{[2, 3]\}.$$

Similarly, define

$$\mathrm{HI}_E : E \longrightarrow \mathcal{P}^*(\mathcal{I})$$

by

$$\mathrm{HI}_E(e_{12}) = \{[0,1],[1,2]\},$$

and

$$\mathrm{HI}_E(e_{23}) = \{[2,3],[3,4]\}.$$

Each assigned value is a nonempty subset of $\mathcal{I}$. For instance,

$$\mathrm{HI}_V(v_2) = \{[1,2],[2,3],[3,4]\} \in \mathcal{P}^*(\mathcal{I}),$$

and

$$\mathrm{HI}_E(e_{23}) = \{[2,3],[3,4]\} \in \mathcal{P}^*(\mathcal{I}).$$

Therefore,

$$G_{\mathrm{HI}} = (V, E, \mathrm{HI}_V, \mathrm{HI}_E)$$

is a HyperInterval Graph.

Unlike a graph in which each vertex or edge is assigned only one interval, the present structure assigns a nonempty family of intervals to every vertex and edge.

5.26 SuperHyperInterval Graph (Proposed in this book)

A *SuperHyperInterval Graph* extends a HyperInterval Graph by allowing interval information to be organized into recursively nested nonempty families. This construction provides a natural hierarchy of higher-order interval-valued information.

Definition 5.26.1 (Iterated Nonempty Powersets of Intervals)**.** Let $\mathcal{I} \neq \varnothing$ be a universe of admissible intervals. Define recursively

$$\mathcal{P}_*^0(\mathcal{I}) = \mathcal{I},$$

and, for every $r \geq 0$,

$$\mathcal{P}_*^{r+1}(\mathcal{I}) = \mathcal{P}\left(\mathcal{P}_*^r(\mathcal{I})\right) \setminus \{\varnothing\}.$$

In particular,

$$\mathcal{P}_*^1(\mathcal{I}) = \mathcal{P}^*(\mathcal{I}),$$

and

$$\mathcal{P}_*^2(\mathcal{I}) = \mathcal{P}^*\left(\mathcal{P}^*(\mathcal{I})\right).$$

Definition 5.26.2 (n-SuperHyperInterval Graph)**.** Let

$$G = (V, E)$$

be a graph, let $\mathcal{I} \neq \varnothing$ be a universe of admissible intervals, and let

$$n \geq 1.$$

An *n-SuperHyperInterval Graph* is a structure

$$G_{\mathrm{SHI}}^{(n)} = \left(V, E, \mathrm{SHI}_V^{(n)}, \mathrm{SHI}_E^{(n)}\right),$$

where

$$\mathrm{SHI}_V^{(n)} : V \longrightarrow \mathcal{P}_*^n(\mathcal{I})$$

and

$$\mathrm{SHI}_E^{(n)} : E \longrightarrow \mathcal{P}_*^n(\mathcal{I}).$$

Thus, every vertex and edge is assigned an n-level nonempty nested family of admissible intervals.

Remark 5.26.3 (Reduction to the HyperInterval Graph)**.** For

$$n = 1,$$

we have

$$\mathcal{P}_*^1(\mathcal{I}) = \mathcal{P}^*(\mathcal{I}).$$

Consequently,

$$\mathrm{SHI}_V^{(1)} : V \longrightarrow \mathcal{P}^*(\mathcal{I}), \qquad \mathrm{SHI}_E^{(1)} : E \longrightarrow \mathcal{P}^*(\mathcal{I}),$$

which is precisely the structure of a HyperInterval Graph.

Hence, a HyperInterval Graph is the order-1 case of the SuperHyperInterval Graph hierarchy.

Example 5.26.4 (Example of a 2-SuperHyperInterval Graph)**.** Let

$$G = (V, E),$$

where

$$V = \{v_1, v_2, v_3\}$$

and

$$E = \{e_{12}, e_{23}\}, \qquad e_{12} = \{v_1, v_2\}, \qquad e_{23} = \{v_2, v_3\}.$$

Let

$$\mathcal{I} = \{[0,1], [1,2], [2,3], [3,4]\}$$

be the universe of admissible intervals.

For $n = 2$,

$$\mathcal{P}_*^2(\mathcal{I}) = \mathcal{P}^*\left(\mathcal{P}^*(\mathcal{I})\right).$$

Thus, an element of

$$\mathcal{P}_*^2(\mathcal{I})$$

is a nonempty family whose members are themselves nonempty families of intervals from $\mathcal{I}$.

Define

$$\mathrm{SHI}_V^{(2)} : V \longrightarrow \mathcal{P}_*^2(\mathcal{I})$$

by

$$\mathrm{SHI}_V^{(2)}(v_1) = \{\{[0,1], [1,2]\}, \{[2,3]\}\},$$

$$\mathrm{SHI}_V^{(2)}(v_2) = \{\{[1,2], [2,3]\}, \{[2,3], [3,4]\}\},$$

and

$$\mathrm{SHI}_V^{(2)}(v_3) = \{\{[0,1]\}, \{[3,4]\}\}.$$

Similarly, define

$$\mathrm{SHI}_E^{(2)} : E \longrightarrow \mathcal{P}_*^2(\mathcal{I})$$

by

$$\mathrm{SHI}_E^{(2)}(e_{12}) = \{\{[0,1],[1,2]\},\{[1,2],[2,3]\}\},$$

and

$$\mathrm{SHI}_E^{(2)}(e_{23}) = \{\{[2,3]\},\{[2,3],[3,4]\}\}.$$

For example,

$$\{[0,1],[1,2]\} \in \mathcal{P}_*^1(\mathcal{I})$$

and

$$\{[2,3]\} \in \mathcal{P}_*^1(\mathcal{I}).$$

Therefore,

$$\mathrm{SHI}_V^{(2)}(v_1) = \{\{[0,1],[1,2]\},\{[2,3]\}\} \in \mathcal{P}_*^2(\mathcal{I}).$$

The same argument applies to every value of

$$\mathrm{SHI}_V^{(2)} \quad \text{and} \quad \mathrm{SHI}_E^{(2)}.$$

Hence,

$$G_{\mathrm{SHI}}^{(2)} = \left(V, E, \mathrm{SHI}_V^{(2)}, \mathrm{SHI}_E^{(2)}\right)$$

is a 2-SuperHyperInterval Graph.

The hierarchy represented in this example is

$$\text{interval} \longrightarrow \text{nonempty family of intervals} \longrightarrow \text{nonempty family of nonempty families of intervals.}$$

Thus, the graph carries second-order hierarchical interval information on both its vertices and edges.

5.27 Interval Vector

An Interval Vector is a vector whose components are intervals, representing bounded uncertainty or variability within each coordinate of space.

Definition 5.27.1 (Interval Vector)**.** Let

$$\mathbb{IR} = \{[\underline{x},\overline{x}] \mid \underline{x},\overline{x} \in \mathbb{R},\ \underline{x} \le \overline{x}\}.$$

An *Interval Vector* is a vector

$$\mathbf{x} = ([\,\underline{x}_1,\overline{x}_1\,],\ldots,[\,\underline{x}_n,\overline{x}_n\,]) \in \mathbb{IR}^n.$$

Example 5.27.2 (Interval Vector)**.** Consider

$$\mathbf{x} = ([1,2],[-1,1],[3,5]).$$

Since each component is a closed real interval, we have

$$\mathbf{x} \in \mathbb{IR}^3.$$

Hence $\mathbf{x}$ is an Interval Vector.

5.28 Interval Linear System

An Interval Linear System is a linear equation system whose coefficients or right-hand sides are intervals rather than exact values.

Definition 5.28.1 (Interval Linear System)**.** An *Interval Linear System* is a system

$$\mathbf{A}x = \mathbf{b},$$

where

$$\mathbf{A} \in \mathbb{IR}^{m\times n}, \qquad \mathbf{b} \in \mathbb{IR}^{m},$$

are an interval matrix and an interval vector, respectively.

Example 5.28.2 (Interval Linear System)**.** Consider

$$\mathbf{A}x = \mathbf{b},$$

where

$$\mathbf{A} = \begin{pmatrix} [1,2] & [0,1] \\ [2,3] & [1,2] \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} [3,4] \\ [5,7] \end{pmatrix}.$$

Thus,

$$\begin{cases} [1,2]x_1 + [0,1]x_2 = [3,4], \\ [2,3]x_1 + [1,2]x_2 = [5,7]. \end{cases}$$

This is an Interval Linear System.

5.29 Interval Polynomial

An Interval Polynomial is a family of polynomials whose coefficients vary independently within prescribed intervals, representing uncertainty in polynomial parameters.

Definition 5.29.1 (Interval Polynomial)**.** An *Interval Polynomial* of degree at most n is specified by interval coefficients

$$\mathbf{a}_0, \ldots, \mathbf{a}_n \in \mathbb{IR}$$

and represents the family

$$\mathbf{P} = \left\{ \sum_{k=0}^{n} a_k x^k \;\middle|\; a_k \in \mathbf{a}_k \right\}.$$

Example 5.29.2 (Interval Polynomial)**.** Let

$$\mathbf{a}_0 = [1,2], \qquad \mathbf{a}_1 = [-1,1], \qquad \mathbf{a}_2 = [2,3].$$

Then the Interval Polynomial

$$\mathbf{P} = \left\{ a_0 + a_1 x + a_2 x^2 \;\middle|\; a_0 \in [1,2],\ a_1 \in [-1,1],\ a_2 \in [2,3] \right\}$$

represents a family of quadratic polynomials. For example,

$$p(x) = 1.5 + 0.5x + 2.5x^2$$

belongs to $\mathbf{P}$.

5.30 Interval-Valued Function

An Interval-Valued Function maps each input element to an interval, allowing outputs to represent bounded ranges rather than single values.

Definition 5.30.1 (Interval-Valued Function)**.** Let X be a nonempty set. An *Interval-Valued Function* is a mapping

$$F : X \longrightarrow \mathbb{IR},$$

where

$$F(x) = [\underline{F}(x), \overline{F}(x)]$$

for every $x \in X$.

Example 5.30.2 (Interval-Valued Function)**.** Let

$$X = \mathbb{R}$$

and define

$$F : X \longrightarrow \mathbb{IR}$$

by

$$F(x) = [x - 1, x + 1].$$

For example,

$$F(2) = [1, 3].$$

Hence F is an Interval-Valued Function.

5.31 Interval-Valued Relation

An Interval-Valued Relation assigns an interval-valued degree to each ordered pair, representing uncertain strength of relationships between elements in sets.

Definition 5.31.1 (Interval-Valued Relation)**.** Let X and Y be nonempty sets. An *Interval-Valued Relation* from X to Y is a mapping

$$R : X \times Y \longrightarrow \mathcal{I}([0, 1]),$$

where $\mathcal{I}([0, 1])$ denotes the set of all closed subintervals of $[0, 1]$. Thus,

$$R(x, y) = [\underline{R}(x, y), \overline{R}(x, y)].$$

Example 5.31.2 (Interval-Valued Relation)**.** Let

$$X = \{x_1, x_2\}, \qquad Y = \{y_1, y_2\}.$$

Define

$$R : X \times Y \longrightarrow \mathcal{I}([0, 1])$$

by

$$R(x_1, y_1) = [0.7, 0.9], \qquad R(x_1, y_2) = [0.3, 0.5],$$

$$R(x_2, y_1) = [0.4, 0.6], \qquad R(x_2, y_2) = [0.8, 1].$$

Thus, R is an Interval-Valued Relation from X to Y.

5.32 Interval-Valued Intuitionistic Fuzzy Set

An Interval-Valued Intuitionistic Fuzzy Set assigns interval membership and nonmembership degrees to each element while satisfying intuitionistic consistency constraints simultaneously [441–445].

Definition 5.32.1 (Interval-Valued Intuitionistic Fuzzy Set)**.** Let X be a nonempty set. An *Interval-Valued Intuitionistic Fuzzy Set* A on X is characterized by

$$M_A, N_A : X \longrightarrow \mathcal{I}([0,1]),$$

where

$$M_A(x) = [\mu_A^-(x), \mu_A^+(x)], \qquad N_A(x) = [\nu_A^-(x), \nu_A^+(x)],$$

such that

$$\mu_A^+(x) + \nu_A^+(x) \leq 1$$

for every $x \in X$.

Example 5.32.2 (Interval-Valued Intuitionistic Fuzzy Set)**.** Let

$$X = \{x_1, x_2\}.$$

Define

$$M_A(x_1) = [0.5, 0.6], \qquad N_A(x_1) = [0.2, 0.3],$$

and

$$M_A(x_2) = [0.3, 0.5], \qquad N_A(x_2) = [0.2, 0.4].$$

Then

$$0.6 + 0.3 = 0.9 \leq 1$$

and

$$0.5 + 0.4 = 0.9 \leq 1.$$

Therefore, A is an Interval-Valued Intuitionistic Fuzzy Set on X.

5.33 Interval Type-2 Fuzzy Set

A Type-2 Fuzzy Set assigns each element a membership function, allowing uncertainty in the membership degrees themselves to be represented [446–448]. Related concepts, including Type-3 Fuzzy Sets [449–451], Type-n Fuzzy Sets, and Type-2 Neutrosophic Sets [452–454], have also been studied. An Interval Type-2 Fuzzy Set represents each primary membership degree by an interval, with uniform secondary membership over the corresponding interval domain [455, 456].

Definition 5.33.1 (Interval Type-2 Fuzzy Set)**.** [456, 457] Let X be a nonempty universe. An *Interval Type-2 Fuzzy Set* $\widetilde{A}$ is characterized by interval-valued primary memberships

$$J_x \subseteq [0,1], \qquad x \in X,$$

with secondary membership function

$$\mu_{\widetilde{A}}(x, u) = 1$$

for every

$$u \in J_x.$$

Equivalently,

$$\widetilde{A} = \{((x,u), 1) \mid x \in X,\ u \in J_x\}.$$

Example 5.33.2 (Interval Type-2 Fuzzy Set)**.** Let

$$X = \{x_1, x_2\}$$

and define the primary membership intervals

$$J_{x_1} = [0.4, 0.7], \qquad J_{x_2} = [0.6, 0.9].$$

Let

$$\mu_{\widetilde{A}}(x, u) = 1$$

for every $u \in J_x$. Then

$$\widetilde{A} = \{((x_1, u), 1) \mid u \in [0.4, 0.7]\} \cup \{((x_2, u), 1) \mid u \in [0.6, 0.9]\}.$$

Hence $\widetilde{A}$ is an Interval Type-2 Fuzzy Set.

5.34 Proper Interval Graph

A Proper Interval Graph is an interval graph admitting an interval representation where no representing interval properly contains another interval [458–461].

Definition 5.34.1 (Proper Interval Graph)**.** A graph

$$G = (V, E)$$

is a *Proper Interval Graph* if there exists a family of intervals

$$\{I_v\}_{v \in V}$$

such that

$$\{u, v\} \in E \iff I_u \cap I_v \neq \varnothing,$$

and no interval properly contains another interval.

Example 5.34.2 (Proper Interval Graph)**.** Let

$$V = \{v_1, v_2, v_3\}$$

and assign the intervals

$$I_{v_1} = [0, 2], \qquad I_{v_2} = [1, 3], \qquad I_{v_3} = [2.5, 4.5].$$

Then

$$I_{v_1} \cap I_{v_2} \neq \varnothing, \qquad I_{v_2} \cap I_{v_3} \neq \varnothing,$$

but

$$I_{v_1} \cap I_{v_3} = \varnothing.$$

Thus,

$$E = \{\{v_1, v_2\}, \{v_2, v_3\}\}.$$

Since no interval properly contains another, $G = (V, E)$ is a Proper Interval Graph.

5.35 Unit Interval Graph

A Unit Interval Graph is an interval graph representable by equal-length intervals, with adjacency determined exactly by nonempty interval intersections [462–465].

Definition 5.35.1 (Unit Interval Graph)**.** A graph

$$G = (V, E)$$

is a *Unit Interval Graph* if there exists a family of unit-length intervals

$$\{I_v\}_{v \in V}$$

such that

$$\{u, v\} \in E \iff I_u \cap I_v \neq \varnothing.$$

Example 5.35.2 (Unit Interval Graph)**.** Let

$$V = \{v_1, v_2, v_3\}$$

and assign the unit intervals

$$I_{v_1} = [0, 1], \qquad I_{v_2} = [0.5, 1.5], \qquad I_{v_3} = [1.2, 2.2].$$

Then

$$I_{v_1} \cap I_{v_2} \neq \varnothing, \qquad I_{v_2} \cap I_{v_3} \neq \varnothing,$$

while

$$I_{v_1} \cap I_{v_3} = \varnothing.$$

Therefore,

$$E = \{\{v_1, v_2\}, \{v_2, v_3\}\},$$

and $G = (V, E)$ is a Unit Interval Graph.

Chapter 6

Subset Structures (Power Structures)

This chapter discusses Subset Structures.

6.1 Subset Structure (Power Structure)

This section introduces several types of Subset Structures. Subset Structure is viewed here as a common umbrella terminology encompassing classical power/complex constructions and related subset-based structures.

6.1.1 Subset Topological Spaces

Subset Topological Spaces organize families of subsets using union, intersection, and inherited semigroup operations to create ordinary and special structures [466–468].

Definition 6.1.1 (Subset Topological Spaces)**.** Let $(P, *)$ be a semigroup, and let

$$\mathcal{S} = \mathcal{P}(P) \setminus \{\varnothing\}, \qquad \mathcal{S}' = \mathcal{S} \cup \{\varnothing\} = \mathcal{P}(P).$$

For $A, B \subseteq P$, define the induced set operation

$$A * B = \{a * b : a \in A,\ b \in B\}.$$

The following structures are called the *subset topological spaces* associated with $(P, *)$:

$$T_o = (\mathcal{S}', \cup, \cap),$$

$$T_\cup = (\mathcal{S}, \cup, *), \qquad T_\cap = (\mathcal{S}', \cap, *).$$

Here, T_o is called the ordinary subset topological space, while $T_\cup$ and $T_\cap$ are called special subset topological spaces.

Example 6.1.2 (Subset Topological Space)**.** Let

$$P = \mathbb{Z}_2 = \{0, 1\}$$

with addition modulo 2, denoted by $*$. Then

$$\mathcal{S} = \big\{\{0\}, \{1\}, \{0, 1\}\big\},$$

and

$$\mathcal{S}' = \{\varnothing, \{0\}, \{1\}, \{0,1\}\}.$$

For example, if

$$A = \{0,1\}, \qquad B = \{1\},$$

then

$$A * B = \{0+1, 1+1\} = \{0,1\},$$

where addition is taken modulo 2. Moreover,

$$A \cup B = \{0,1\}, \qquad A \cap B = \{1\}.$$

Thus the associated subset topological structures are

$$T_o = (\mathcal{S}', \cup, \cap), \qquad T_\cup = (\mathcal{S}, \cup, *), \qquad T_\cap = (\mathcal{S}', \cap, *).$$

6.1.2 Subset Groupoids

Subset Groupoids extend a groupoid operation to nonempty subsets by combining every pair of elements and collecting all resulting products [469].

Definition 6.1.3 (Subset Groupoid)**.** Let $(G, *)$ be a groupoid, and let

$$\mathcal{S} = \mathcal{P}(G) \setminus \{\varnothing\}$$

be the family of all nonempty subsets of G. For $A, B \in \mathcal{S}$, define

$$A * B = \{a * b : a \in A,\ b \in B\}.$$

Then

$$(\mathcal{S}, *)$$

is called the *subset groupoid* induced by $(G, *)$.

Example 6.1.4 (Subset Groupoid)**.** Let

$$G = \mathbb{Z}_3 = \{0,1,2\}$$

and define

$$a * b = a - b \pmod 3.$$

Then $(G, *)$ is a groupoid. Consider

$$A = \{0,1\}, \qquad B = \{1,2\}.$$

The induced subset product is

$$A * B = \{0*1, 0*2, 1*1, 1*2\}.$$

Since

$$0*1 = 2, \qquad 0*2 = 1, \qquad 1*1 = 0, \qquad 1*2 = 2,$$

we obtain

$$A * B = \{0,1,2\} = G.$$

Hence this gives a concrete operation in the induced Subset Groupoid.

6.1.3 Subset Semirings

Subset Semirings extend semiring addition and multiplication to subsets, producing set-valued operations and yielding Type I or Type II structures [470].

Definition 6.1.5 (Subset Semiring)**.** Let

$$(R, +, \cdot)$$

be a semiring, and let

$$\mathcal{S} = \mathcal{P}(R)$$

be the collection of all subsets of R. For $A, B \in \mathcal{S}$, define

$$A \oplus B = \{a + b : a \in A,\ b \in B\},$$

and

$$A \otimes B = \{a \cdot b : a \in A,\ b \in B\}.$$

Then

$$(\mathcal{S}, \oplus, \otimes)$$

is called the *Subset Semiring* induced by R. If R is a ring, it is called a *Subset Semiring of Type I*; if R is a semiring, it is called a *Subset Semiring of Type II.*

Example 6.1.6 (Subset Semiring)**.** Let

$$R = \{0, 1\}$$

be the Boolean semiring, where

$$a + b = \max\{a, b\}, \qquad a \cdot b = \min\{a, b\}.$$

Consider

$$\mathcal{S} = \mathcal{P}(R) \setminus \{\varnothing\} = \big\{\{0\}, \{1\}, \{0, 1\}\big\}.$$

Let

$$A = \{0, 1\}, \qquad B = \{1\}.$$

Then

$$A \oplus B = \{0 + 1, 1 + 1\} = \{1\},$$

and

$$A \otimes B = \{0 \cdot 1, 1 \cdot 1\} = \{0, 1\}.$$

Thus the Boolean semiring operations induce corresponding operations on the nonempty subsets of R.

6.1.4 Subset Semilinear Algebras

Subset Semilinear Algebras are subset semivector spaces over semifields equipped with an associative induced product closed within the subset family [471].

Definition 6.1.7 (Subset Semilinear Algebra)**.** Let F be a semifield and let P be a set equipped with addition and an associative product $\cdot$. Let

$$\mathcal{S} = \mathcal{P}(P)$$

be a subset semivector space over F.

For $A, B \in \mathcal{S}$, define the induced product

$$A \odot B = \{a \cdot b : a \in A,\ b \in B\}.$$

If

$$A \odot B \in \mathcal{S} \qquad \text{for all } A, B \in \mathcal{S},$$

and $(\mathcal{S}, \odot)$ is a semigroup, then

$$(\mathcal{S}, +, \odot)$$

is called a *Subset Semilinear Algebra* over the semifield F.

Example 6.1.8 (Subset Semilinear Algebra)**.** Let

$$F = \mathbb{R}_{\geq 0}$$

and consider the family

$$\mathcal{S} = \{[0, a] : a \in \mathbb{R}_{\geq 0}\}.$$

For

$$A = [0, a], \qquad B = [0, b],$$

define

$$A + B = [0, a + b],$$

and, for $\lambda \in F$,

$$\lambda A = [0, \lambda a].$$

The induced product is

$$A \odot B = \{xy : x \in A,\ y \in B\} = [0, ab].$$

For example, if

$$A = [0, 2], \qquad B = [0, 3],$$

then

$$A + B = [0, 5], \qquad A \odot B = [0, 6].$$

Hence $\mathcal{S}$ is closed under the induced addition, scalar multiplication, and associative product.

6.1.5 Subset Interval Groupoids

Subset Interval Groupoids extend groupoid operations coordinatewise to interval elements and then to nonempty families of intervals under induced multiplication [472].

Definition 6.1.9 (Subset Interval Groupoid)**.** Let $(G, *)$ be a groupoid, and define the associated interval groupoid

$$\mathcal{I}(G) = \{[a, b] : a, b \in G\},$$

with the coordinatewise operation

$$[a, b] * [c, d] = [a * c, b * d].$$

Let

$$\mathcal{S} = \mathcal{P}(\mathcal{I}(G)) \setminus \{\varnothing\}.$$

For $A, B \in \mathcal{S}$, define

$$A * B = \{x * y : x \in A,\ y \in B\}.$$

Then

$$(\mathcal{S}, *)$$

is called the *Subset Interval Groupoid* induced by $(G, *)$.

Example 6.1.10 (Subset Interval Groupoid)**.** Let

$$G = \mathbb{Z}_3 = \{0, 1, 2\}$$

with addition modulo 3. Consider

$$A = \{[0, 1], [1, 2]\}, \qquad B = \{[1, 0]\}.$$

Using the coordinatewise operation

$$[a, b] * [c, d] = [a + c, b + d] \pmod 3,$$

we obtain

$$[0, 1] * [1, 0] = [1, 1]$$

and

$$[1, 2] * [1, 0] = [2, 2].$$

Therefore,

$$A * B = \{[1, 1], [2, 2]\}.$$

Thus $A * B$ is again a nonempty family of interval elements.

6.1.6 Subset Vertex Graphs for Social Networks

Subset Vertex Graphs represent subsets as vertices, connecting through shared elements in Type I or strict inclusion in Type II [473].

Definition 6.1.11 (Subset Vertex Graph)**.** Let S be a nonempty set and let

$$V \subseteq \mathcal{P}(S) \setminus \{\varnothing\}.$$

A *Subset Vertex Graph* is a graph

$$G = (V, E)$$

whose vertices are subsets of S, with adjacency determined by a set-theoretic relation.

For a *Type-I Subset Vertex Graph*,

$$\{A, B\} \in E \quad \Longleftrightarrow \quad A \neq B \text{ and } A \cap B \neq \varnothing.$$

For a *Type-II Subset Vertex Graph*, G is directed and

$$(A, B) \in E \quad \Longleftrightarrow \quad A \subsetneq B.$$

In both cases, loops are excluded.

Example 6.1.12 (Subset Vertex Graph)**.** Let

$$S = \{1, 2, 3\}$$

and consider

$$V = \big\{\{1\}, \{1, 2\}, \{2, 3\}\big\}.$$

For the Type-I Subset Vertex Graph, two distinct vertices are adjacent whenever they have a nonempty intersection. Since

$$\{1\} \cap \{1, 2\} = \{1\},$$

and

$$\{1, 2\} \cap \{2, 3\} = \{2\},$$

while

$$\{1\} \cap \{2, 3\} = \varnothing,$$

the edge set is

$$E = \big\{\{\{1\}, \{1, 2\}\}, \{\{1, 2\}, \{2, 3\}\}\big\}.$$

For the Type-II Subset Vertex Graph,

$$\{1\} \subsetneq \{1, 2\},$$

so there is a directed edge

$$\{1\} \longrightarrow \{1, 2\}.$$

No other strict inclusion relation occurs among these vertices.

6.1.7 Subset Polynomial Semirings and Subset Matrix Semirings

Subset Polynomial Semirings consist of polynomials whose coefficients are nonempty subsets, with addition and multiplication induced from the semiring operations [474].

Definition 6.1.13 (Subset Polynomial Semiring)**.** Let

$$(R, +, \cdot)$$

be a semiring, and let

$$\mathcal{S} = \mathcal{P}(R) \setminus \{\varnothing\}.$$

For $A, B \in \mathcal{S}$, define

$$A \oplus B = \{a + b : a \in A,\ b \in B\}, \qquad A \otimes B = \{ab : a \in A,\ b \in B\}.$$

The *Subset Polynomial Semiring* over R is

$$\mathcal{S}[x] = \left\{ \sum_{i=0}^{n} A_i x^i : n \geq 0,\ A_i \in \mathcal{S} \right\},$$

where polynomial addition and multiplication are induced by $\oplus$ and $\otimes$.

Example 6.1.14 (Subset Polynomial Semiring)**.** Let

$$R = \mathbb{N}_0$$

with the usual addition and multiplication. Consider

$$p(x) = \{1, 2\} + \{0, 1\}x$$

and

$$q(x) = \{2\} + \{1\}x.$$

For the constant coefficients,

$$\{1, 2\} \oplus \{2\} = \{3, 4\},$$

while

$$\{0, 1\} \oplus \{1\} = \{1, 2\}.$$

Hence

$$p(x) \oplus q(x) = \{3, 4\} + \{1, 2\}x.$$

For multiplication,

$$\{1, 2\} \otimes \{2\} = \{2, 4\},$$

$$(\{1, 2\} \otimes \{1\}) \oplus (\{0, 1\} \otimes \{2\}) = \{1, 2, 3, 4\},$$

and

$$\{0, 1\} \otimes \{1\} = \{0, 1\}.$$

Therefore,

$$p(x) \otimes q(x) = \{2, 4\} + \{1, 2, 3, 4\}x + \{0, 1\}x^2.$$

Subset Matrix Semirings consist of matrices with subset entries, where addition and multiplication are performed entrywise using induced semiring operations [474].

Definition 6.1.15 (Subset Matrix Semiring)**.** Let

$$(R, +, \cdot)$$

be a semiring, and let

$$\mathcal{S} = \mathcal{P}(R) \setminus \{\varnothing\},$$

where, for $A, B \in \mathcal{S}$,

$$A \oplus B = \{a + b : a \in A,\ b \in B\}, \qquad A \otimes B = \{ab : a \in A,\ b \in B\}.$$

For positive integers m, n, define

$$\mathcal{M}_{m,n}(\mathcal{S}) = \{[A_{ij}] : A_{ij} \in \mathcal{S}\}.$$

For

$$A = [A_{ij}], \qquad B = [B_{ij}] \in \mathcal{M}_{m,n}(\mathcal{S}),$$

define the entrywise operations

$$A \oplus B = [A_{ij} \oplus B_{ij}], \qquad A \otimes B = [A_{ij} \otimes B_{ij}].$$

Then

$$\big(\mathcal{M}_{m,n}(\mathcal{S}), \oplus, \otimes\big)$$

is called the *Subset Matrix Semiring* induced by R.

Example 6.1.16 (Subset Matrix Semiring)**.** Let

$$R = \mathbb{N}_0$$

with the usual addition and multiplication, and consider

$$A = \begin{bmatrix} \{0,1\} & \{1\} \\ \{2\} & \{0,2\} \end{bmatrix}, \qquad B = \begin{bmatrix} \{1\} & \{0,1\} \\ \{1,2\} & \{2\} \end{bmatrix}.$$

Using entrywise induced addition, we obtain

$$A \oplus B = \begin{bmatrix} \{1,2\} & \{1,2\} \\ \{3,4\} & \{2,4\} \end{bmatrix}.$$

Similarly, using entrywise induced multiplication,

$$A \otimes B = \begin{bmatrix} \{0,1\} & \{0,1\} \\ \{2,4\} & \{0,4\} \end{bmatrix}.$$

Thus both operations produce matrices whose entries are again nonempty subsets of R.

6.1.8 Subset Non Associative Semirings

Subset Non-Associative Semirings extend semiring operations to subset families while preserving distributivity but allowing multiplication to fail the associative law [475].

Definition 6.1.17 (Subset Non-Associative Semiring)**.** Let

$$(R, +, \cdot)$$

be a non-associative semiring, and let

$$\mathcal{S} \subseteq \mathcal{P}(R)$$

be a nonempty family of subsets of R. For $A, B \in \mathcal{S}$, define the induced operations

$$A \oplus B = \{a + b : a \in A,\ b \in B\},$$

and

$$A \otimes B = \{a \cdot b : a \in A,\ b \in B\}.$$

If $\mathcal{S}$ is closed under $\oplus$ and $\otimes$, $(\mathcal{S}, \oplus)$ is a commutative monoid, and $\otimes$ distributes over $\oplus$, while $\otimes$ is not required to be associative, then

$$(\mathcal{S}, \oplus, \otimes)$$

is called a *Subset Non-Associative Semiring*.

Example 6.1.18 (Subset Non-Associative Semiring)**.** Let

$$R = \mathbb{F}_2 e_1 \oplus \mathbb{F}_2 e_2 = \{0, e_1, e_2, e_1 + e_2\},$$

with the usual vector addition over $\mathbb{F}_2$. Define multiplication bilinearly by

$$e_1 \cdot e_1 = e_2, \qquad e_2 \cdot e_1 = e_1, \qquad e_1 \cdot e_2 = e_2 \cdot e_2 = 0.$$

Then multiplication is distributive over addition but is not associative, since

$$(e_1 \cdot e_1) \cdot e_1 = e_2 \cdot e_1 = e_1,$$

whereas

$$e_1 \cdot (e_1 \cdot e_1) = e_1 \cdot e_2 = 0.$$

Let

$$\mathcal{S} = \big\{\{x\} : x \in R\big\}.$$

For singleton subsets, define

$$\{x\} \oplus \{y\} = \{x + y\}, \qquad \{x\} \otimes \{y\} = \{x \cdot y\}.$$

Then $\mathcal{S}$ is closed under both operations. Moreover,

$$(\{e_1\} \otimes \{e_1\}) \otimes \{e_1\} = \{e_1\},$$

while

$$\{e_1\} \otimes (\{e_1\} \otimes \{e_1\}) = \{0\}.$$

Hence the induced subset multiplication is non-associative.

6.1.9 Power Semigroup

Power Semigroups extend a semigroup operation to nonempty subsets by collecting all pairwise products, preserving associativity under the induced operation [476–478].

Definition 6.1.19 (Power Semigroup)**.** Let

$$(S, \cdot)$$

be a semigroup, and let

$$\mathcal{P}^*(S) = \mathcal{P}(S) \setminus \{\varnothing\}$$

be the family of all nonempty subsets of S.

For $A, B \in \mathcal{P}^*(S)$, define

$$A \odot B = \{a \cdot b : a \in A,\ b \in B\}.$$

Then

$$\big(\mathcal{P}^*(S), \odot\big)$$

is called the *Power Semigroup* of S.

Example 6.1.20 (Power Semigroup)**.** Let

$$S = \mathbb{Z}_3 = \{0, 1, 2\}$$

with addition modulo 3. Then $(S, +)$ is a semigroup and

$$\mathcal{P}^*(S) = \big\{\{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, S\big\}.$$

Consider

$$A = \{0, 1\}, \qquad B = \{1, 2\}.$$

Their power-semigroup product is

$$A \odot B = \{a + b \pmod 3 : a \in A,\ b \in B\}.$$

Hence

$$A \odot B = \{1, 2, 2, 0\} = \{0, 1, 2\} = S.$$

Thus $\mathcal{P}^*(S)$, equipped with the induced operation, forms the Power Semigroup of $\mathbb{Z}_3$.

6.1.10 Power Algebra

Power Algebra extends every operation of an algebra to nonempty subsets by applying the operation elementwise to all possible tuples [479, 480].

Definition 6.1.21 (Power Algebra)**.** Let

$$\mathbf{A} = (A, \Omega)$$

be an algebra, and let

$$\mathcal{P}^*(A) = \mathcal{P}(A) \setminus \{\varnothing\}.$$

For each n-ary operation

$$\omega : A^n \to A$$

in Ω, define

$$\widehat{\omega}(A_1, \ldots, A_n) = \{\omega(a_1, \ldots, a_n) : a_i \in A_i\}.$$

Then

$$\mathcal{P}^*(\mathbf{A}) = \big(\mathcal{P}^*(A), \widehat{\Omega}\big)$$

is called the *Power Algebra* of $\mathbf{A}$.

Example 6.1.22 (Power Algebra)**.** Consider the Boolean algebra

$$\mathbf{A} = (\{0, 1\}, \vee, \wedge).$$

Its nonempty power set is

$$\mathcal{P}^*(A) = \big\{\{0\}, \{1\}, \{0, 1\}\big\}.$$

Let

$$X = \{0, 1\}, \qquad Y = \{1\}.$$

The induced join operation gives

$$\widehat{\vee}(X, Y) = \{x \vee y : x \in X,\ y \in Y\} = \{1\},$$

while the induced meet operation gives

$$\widehat{\wedge}(X, Y) = \{x \wedge y : x \in X,\ y \in Y\} = \{0, 1\}.$$

Therefore,

$$\big(\mathcal{P}^*(A), \widehat{\vee}, \widehat{\wedge}\big)$$

is the Power Algebra induced by $\mathbf{A}$.

6.1.11 Complex Algebra of Subalgebras

Complex Algebra of Subalgebras consists of nonempty subalgebras closed under complex operations inherited from power algebra and forms a subalgebra [481, 482].

Definition 6.1.23 (Complex Algebra of Subalgebras)**.** Let

$$\mathbf{A} = (A, \Omega)$$

be an algebra, and let

$$\mathrm{Sub}^*(\mathbf{A})$$

denote the family of all nonempty subalgebras of $\mathbf{A}$.

If $\mathrm{Sub}^*(\mathbf{A})$ is closed under all complex operations induced from Ω, then

$$\big(\mathrm{Sub}^*(\mathbf{A}), \widehat{\Omega}\big)$$

is called the *Complex Algebra of Subalgebras* of $\mathbf{A}$.

Example 6.1.24 (Complex Algebra of Subalgebras)**.** Consider the algebra

$$\mathbf{A} = (\mathbb{Z}_4, +, -, 0),$$

where addition is taken modulo 4. Its subgroups, and hence its nonempty subalgebras, are

$$H_0 = \{0\}, \qquad H_1 = \{0, 2\}, \qquad H_2 = \mathbb{Z}_4.$$

Thus

$$\mathrm{Sub}^*(\mathbf{A}) = \{H_0, H_1, H_2\}.$$

For example, the complex sum of H_0 and H_1 is

$$H_0 \widehat{+} H_1 = \{a + b \pmod 4 : a \in H_0,\ b \in H_1\} = H_1,$$

and

$$H_1 \widehat{+} H_2 = H_2.$$

Moreover,

$$-H_i = H_i \qquad (i = 0, 1, 2).$$

Hence the family of subalgebras is closed under the induced complex operations and forms a Complex Algebra of Subalgebras.

6.1.12 Closure System

A Closure System is a family of subsets containing the universe and remaining closed under arbitrary intersections of its members (cf. [483]).

Definition 6.1.25 (Closure System)**.** Let X be a set. A family

$$\mathcal{C} \subseteq \mathcal{P}(X)$$

is called a *Closure System* on X if

$$X \in \mathcal{C}$$

and, for every family

$$\{C_i\}_{i \in I} \subseteq \mathcal{C},$$

one has

$$\bigcap_{i \in I} C_i \in \mathcal{C}.$$

Thus, a Closure System is a family of subsets closed under arbitrary intersections.

Example 6.1.26 (Closure System)**.** Let

$$X = \{1, 2, 3\}$$

and define

$$\mathcal{C} = \big\{X, \{1, 2\}, \{2, 3\}, \{2\}\big\}.$$

Clearly,

$$X \in \mathcal{C}.$$

Moreover,

$$\{1, 2\} \cap \{2, 3\} = \{2\} \in \mathcal{C},$$

and intersections involving $\{2\}$ remain equal to $\{2\}$. The intersection of the empty family is X, which also belongs to $\mathcal{C}$.

Therefore, $\mathcal{C}$ is closed under arbitrary intersections and is a Closure System on X.

6.1.13 Subgroup Lattice

A Subgroup Lattice organizes all subgroups of a group under inclusion, with intersection as meet and generated subgroup as join [484–487].

Definition 6.1.27 (Subgroup Lattice)**.** Let G be a group, and let

$$\mathrm{Sub}(G) = \{H \subseteq G : H \leq G\}$$

be the set of all subgroups of G.

For $H, K \in \mathrm{Sub}(G)$, define

$$H \wedge K = H \cap K$$

and

$$H \vee K = \langle H \cup K \rangle,$$

where $\langle H \cup K \rangle$ denotes the subgroup generated by $H \cup K$. Then

$$\big(\mathrm{Sub}(G), \wedge, \vee\big)$$

is called the *Subgroup Lattice* of G.

Example 6.1.28 (Subgroup Lattice)**.** Let

$$G = \mathbb{Z}_6$$

under addition modulo 6. Consider the subgroups

$$H = \{0, 3\}, \qquad K = \{0, 2, 4\}.$$

Their meet is

$$H \wedge K = H \cap K = \{0\}.$$

Their join is

$$H \vee K = \langle H \cup K \rangle.$$

Since 2 and 3 together generate $\mathbb{Z}_6$,

$$H \vee K = \mathbb{Z}_6.$$

Thus, in the subgroup lattice of $\mathbb{Z}_6$,

$$\{0, 3\} \wedge \{0, 2, 4\} = \{0\}, \qquad \{0, 3\} \vee \{0, 2, 4\} = \mathbb{Z}_6.$$

6.1.14 Set-Inclusion Graph

Set-Inclusion Graphs connect subsets of two prescribed sizes whenever one subset is contained within another, forming a bipartite graph structure [488, 489].

Definition 6.1.29 (Set-Inclusion Graph)**.** Let

$$[n] = \{1, 2, \ldots, n\}, \qquad 1 \leq k < \ell \leq n - 1.$$

The *Set-Inclusion Graph* $G(n, k, \ell)$ is the graph with vertex set

$$V = \binom{[n]}{k} \cup \binom{[n]}{\ell},$$

where two distinct vertices $A, B \in V$ are adjacent if and only if

$$A \subset B \qquad \text{or} \qquad B \subset A.$$

Example 6.1.30 (Set-Inclusion Graph)**.** Consider

$$G(3,1,2).$$

Since

$$[3] = \{1,2,3\},$$

the vertex set is

$$V = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}.$$

Two vertices are adjacent whenever one is contained in the other. Therefore,

$$E = \left\{ \begin{array}{c} \{\{1\},\{1,2\}\},\{\{1\},\{1,3\}\}, \\ \{\{2\},\{1,2\}\},\{\{2\},\{2,3\}\}, \\ \{\{3\},\{1,3\}\},\{\{3\},\{2,3\}\} \end{array} \right\}.$$

Thus $G(3,1,2)$ is a bipartite graph whose two parts consist of the 1-subsets and 2-subsets of [3], respectively.

6.1.15 Kneser Graph

Kneser Graphs represent fixed-size subsets as vertices, joining two vertices exactly when the corresponding subsets are mutually disjoint within the universe [407, 490–493].

Definition 6.1.31 (Kneser Graph)**.** Let

$$[n] = \{1,2,\ldots,n\}, \qquad n \geq 2k.$$

The *Kneser Graph* $KG(n,k)$ is the graph with vertex set

$$V = \binom{[n]}{k},$$

where two distinct vertices $A, B \in V$ are adjacent if and only if

$$A \cap B = \varnothing.$$

Example 6.1.32 (Kneser Graph)**.** Consider the Kneser graph

$$KG(5,2).$$

Its vertex set consists of all 2-element subsets of

$$[5] = \{1,2,3,4,5\},$$

so

$$V = \binom{[5]}{2}.$$

For example, the vertex

$$A = \{1,2\}$$

is adjacent precisely to the 2-subsets disjoint from A, namely

$$\{3,4\}, \qquad \{3,5\}, \qquad \{4,5\}.$$

Hence,

$$\{1,2\} \sim \{3,4\}, \qquad \{1,2\} \sim \{3,5\}, \qquad \{1,2\} \sim \{4,5\}.$$

On the other hand,

$$\{1,2\} \not\sim \{1,3\}$$

because

$$\{1,2\} \cap \{1,3\} = \{1\} \neq \varnothing.$$

The graph $KG(5,2)$ is the well-known Petersen graph.

6.1.16 Power-Set Lattice

A Power-Set Lattice is the lattice of all subsets of a set, ordered by inclusion, with union and intersection operations.

Definition 6.1.33 (Power-Set Lattice)**.** Let X be a set. The *Power-Set Lattice* of X is

$$\big(\mathcal{P}(X), \subseteq\big),$$

where the meet and join are given by

$$A \wedge B = A \cap B, \qquad A \vee B = A \cup B.$$

Example 6.1.34 (Power-Set Lattice)**.** Let

$$X = \{a, b\}.$$

Then

$$\mathcal{P}(X) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}.$$

For example, let

$$A = \{a\}, \qquad B = \{b\}.$$

Their meet and join are

$$A \wedge B = A \cap B = \varnothing,$$

and

$$A \vee B = A \cup B = \{a, b\}.$$

Therefore,

$$\big(\mathcal{P}(X), \subseteq\big)$$

is a Power-Set Lattice.

6.1.17 Intersection Graph of Sets

An Intersection Graph of Sets represents sets as vertices, connecting two distinct vertices exactly when their corresponding sets intersect nontrivially.

Definition 6.1.35 (Intersection Graph of Sets)**.** Let

$$\mathcal{F} = \{A_i \mid i \in I\}$$

be a family of nonempty sets. The *Intersection Graph* of $\mathcal{F}$ is the graph

$$G = (V, E),$$

where

$$V = \mathcal{F},$$

and, for distinct $A, B \in \mathcal{F}$,

$$\{A, B\} \in E \iff A \cap B \neq \varnothing.$$

Example 6.1.36 (Intersection Graph of Sets)**.** Let

$$\mathcal{F} = \{A_1, A_2, A_3\},$$

where

$$A_1 = \{1, 2\}, \qquad A_2 = \{2, 3\}, \qquad A_3 = \{4, 5\}.$$

We have

$$A_1 \cap A_2 = \{2\} \neq \varnothing,$$

whereas
$$A_1 \cap A_3 = \varnothing, \qquad A_2 \cap A_3 = \varnothing.$$
Hence, the corresponding Intersection Graph is
$$G = (V, E),$$
with
$$V = \{A_1, A_2, A_3\}$$
and
$$E = \{\{A_1, A_2\}\}.$$
Thus, only A_1 and A_2 are adjacent.

6.2 Subset Structure versus HyperStructure and More

This section discusses the mathematical relationship between subset structures, hyperstructures, iterated subset structures, and SuperHyperStructures. Although these concepts are closely related through power-set constructions, they differ essentially in their carriers, operations, and levels of set-valuedness.

6.2.1 Subset Structure

Let X be a nonempty set and define
$$\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\varnothing\}.$$
Suppose that
$$\omega : X^k \longrightarrow X$$
is a k-ary operation on X. The operation ω can be naturally extended to nonempty subsets of X by
$$\widehat{\omega} : \big(\mathcal{P}^*(X)\big)^k \longrightarrow \mathcal{P}^*(X),$$
where
$$\widehat{\omega}(A_1, \ldots, A_k) = \{\omega(a_1, \ldots, a_k) : a_i \in A_i,\ 1 \leq i \leq k\}.$$
Definition 6.2.1 (Subset Structure)**.** Let
$$\mathcal{A} = \big(X, \{\omega_\lambda\}_{\lambda \in \Lambda}\big)$$
be an algebraic structure. The corresponding *Subset Structure* is the structure
$$\mathcal{P}^*(\mathcal{A}) = (\mathcal{P}^*(X), \{\widehat{\omega}_\lambda\}_{\lambda \in \Lambda}),$$
where every operation ω_λ is extended to subsets pointwise.

Thus, a Subset Structure replaces individual elements of X by nonempty subsets of X, while the original operations are inherited through elementwise evaluation.

For example, if
$$(X, *)$$
is a groupoid, then
$$A \widehat{*} B = \{a * b : a \in A,\ b \in B\}$$
gives the corresponding Subset Groupoid.

Similarly, subset semigroups, subset semirings, subset matrix semirings, and subset polynomial semirings can be viewed as particular instances of this general construction.

6.2.2 Subset Structure V.S. HyperStructure

A Subset Structure should be distinguished from a HyperStructure. The main difference is the location at which the set-valuedness occurs.

Definition 6.2.2 (HyperStructure)**.** Let X be a nonempty set. A k-ary *hyperoperation* on X is a mapping

$$h : X^k \longrightarrow \mathcal{P}^*(X).$$

A structure equipped with one or more hyperoperations is called a *HyperStructure*.

For an ordinary operation,

$$\omega : X^k \longrightarrow X,$$

each tuple of elements produces exactly one element. In contrast, a hyperoperation

$$h : X^k \longrightarrow \mathcal{P}^*(X)$$

may produce several elements simultaneously.

A Subset Structure instead has

$$\widehat{\omega} : \big(\mathcal{P}^*(X)\big)^k \longrightarrow \mathcal{P}^*(X).$$

Hence, its inputs are themselves subsets of X.

The essential distinction can therefore be written as

$$\boxed{\text{HyperStructure: } X^k \longrightarrow \mathcal{P}^*(X)}$$

whereas

$$\boxed{\text{Subset Structure: } \big(\mathcal{P}^*(X)\big)^k \longrightarrow \mathcal{P}^*(X).}$$

Moreover, for the singleton embedding

$$\iota : X \longrightarrow \mathcal{P}^*(X), \qquad \iota(x) = \{x\},$$

the subset extension of an ordinary operation satisfies

$$\widehat{\omega}\big(\{x_1\}, \ldots, \{x_k\}\big) = \{\omega(x_1, \ldots, x_k)\}.$$

Thus, the operation remains effectively single-valued on singleton subsets.

This property distinguishes an induced Subset Structure from a genuine HyperStructure, since a general hyperoperation may satisfy

$$|h(x_1, \ldots, x_k)| > 1.$$

Nevertheless, every hyperoperation can itself be extended to subsets. If

$$h : X^k \longrightarrow \mathcal{P}^*(X),$$

define

$$\widehat{h} : \big(\mathcal{P}^*(X)\big)^k \longrightarrow \mathcal{P}^*(X)$$

by

$$\widehat{h}(A_1, \ldots, A_k) = \bigcup_{a_1 \in A_1, \ldots, a_k \in A_k} h(a_1, \ldots, a_k).$$

Therefore, subset constructions and hyperoperations can be combined naturally.

Consequently, a Subset Structure and a HyperStructure are not identical concepts, although both are naturally connected through the power-set operator.

6.2.3 Iterated Subset Structure

The subset construction can be repeated recursively.

Define
$$\mathcal{P}^{*(0)}(X) = X$$
and, for every $n \geq 0$,
$$\mathcal{P}^{*(n+1)}(X) = \mathcal{P}^{*}\left(\mathcal{P}^{*(n)}(X)\right).$$

Thus,
$$\mathcal{P}^{*(1)}(X) = \mathcal{P}^{*}(X),$$
$$\mathcal{P}^{*(2)}(X) = \mathcal{P}^{*}\left(\mathcal{P}^{*}(X)\right),$$
and more generally
$$\mathcal{P}^{*(n)}(X)$$
consists of n-level nested nonempty subsets.

For example,
$$x \in X,$$
$$\{x_1, x_2\} \in \mathcal{P}^{*(1)}(X),$$
and
$$\big\{\{x_1, x_2\}, \{x_3\}\big\} \in \mathcal{P}^{*(2)}(X).$$

Definition 6.2.3 (Iterated Subset Structure)**.** Let
$$\mathcal{A} = \big(X, \{\omega_\lambda\}_{\lambda\in\Lambda}\big)$$
be an algebraic structure. For $n \geq 1$, the *n-th Iterated Subset Structure* is
$$\mathcal{P}^{*(n)}(\mathcal{A}) = \left(\mathcal{P}^{*(n)}(X), \{\omega_\lambda^{[n]}\}_{\lambda\in\Lambda}\right),$$
where the operations are defined recursively by
$$\omega_\lambda^{[0]} = \omega_\lambda,$$
and
$$\omega_\lambda^{[n+1]}(A_1, \ldots, A_k) = \left\{\omega_\lambda^{[n]}(a_1, \ldots, a_k) : a_i \in A_i\right\}.$$

Hence there is a hierarchy
$$X \longrightarrow \mathcal{P}^{*}(X) \longrightarrow \mathcal{P}^{*(2)}(X) \longrightarrow \cdots \longrightarrow \mathcal{P}^{*(n)}(X).$$

The canonical singleton embeddings
$$\sigma_n : \mathcal{P}^{*(n)}(X) \longrightarrow \mathcal{P}^{*(n+1)}(X)$$
are given by
$$\sigma_n(A) = \{A\}.$$

There is also a natural flattening operation. For $n \geq 1$, define recursively

$$\mu_1 : \mathcal{P}^{*(1)}(X) \longrightarrow \mathcal{P}^*(X)$$

by

$$\mu_1(A) = A,$$

and

$$\mu_{n+1}(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} \mu_n(A).$$

Thus,

$$\mu_n : \mathcal{P}^{*(n)}(X) \longrightarrow \mathcal{P}^*(X)$$

removes the intermediate nesting levels.

For example,

$$\mu_2\left(\{\{a,b\},\{b,c\}\}\right) = \{a,b,c\}.$$

Therefore, flattening preserves the underlying elements but generally loses hierarchical information.

6.2.4 Iterated Subset Structure V.S. SuperHyperStructure

Iterated Subset Structures provide a natural mathematical framework for describing objects containing nested collections of objects. This idea is closely related to SuperHyperStructures, where super-elements may themselves possess internal set-valued structure.

To make the distinction precise, let

$$V^{(n)} \subseteq \mathcal{P}^{*(n)}(X).$$

Definition 6.2.4 (n-Level SuperHyperStructure)**.** An *n-Level SuperHyperStructure* over X is a structure

$$\mathcal{H}^{(n)} = \left(V^{(n)}, \{h_\lambda^{(n)}\}_{\lambda \in \Lambda}\right),$$

where

$$V^{(n)} \subseteq \mathcal{P}^{*(n)}(X)$$

and each k_λ-ary superhyperoperation is a mapping

$$h_\lambda^{(n)} : \left(V^{(n)}\right)^{k_\lambda} \longrightarrow \mathcal{P}^*\left(V^{(n)}\right).$$

Thus, two different mechanisms are involved.

An Iterated Subset Structure introduces hierarchical depth:

$$X \to \mathcal{P}^*(X) \to \mathcal{P}^{*(2)}(X) \to \cdots.$$

A HyperStructure introduces multivaluedness:

$$X^k \to \mathcal{P}^*(X).$$

A SuperHyperStructure may combine both:

$$\left(\mathcal{P}^{*(n)}(X)\right)^k \longrightarrow \mathcal{P}^*\left(\mathcal{P}^{*(n)}(X)\right) = \mathcal{P}^{*(n+1)}(X).$$

Therefore, schematically,

$$\boxed{\text{Iterated Subset Structure} = \text{nested set-valued carrier}}$$

whereas

$$\boxed{\text{SuperHyperStructure} = \text{nested carrier} + \text{hyper-valued interaction.}}$$

In particular, an n-th Iterated Subset Structure has operations of the form

$$\omega^{[n]} : \left(\mathcal{P}^{*(n)}(X)\right)^k \longrightarrow \mathcal{P}^{*(n)}(X),$$

whereas an n-level SuperHyperStructure may have

$$h^{(n)} : \left(\mathcal{P}^{*(n)}(X)\right)^k \longrightarrow \mathcal{P}^{*(n+1)}(X).$$

Accordingly, the power-set iteration

$$X \longrightarrow \mathcal{P}^*(X) \longrightarrow \mathcal{P}^{*(2)}(X) \longrightarrow \cdots$$

provides a natural bridge from ordinary structures to increasingly hierarchical structures. Hyperoperations provide an independent generalization by replacing single-valued outputs with set-valued outputs. Their combination yields a natural abstract framework for SuperHyperStructures.

Chapter 7

BiStructure, TriStructure, and MultiStructure

This chapter examines BiStructures, TriStructures, and MultiStructures.

7.1 Introduction to BiStructure and MultiStructure

This section first provides an overview of BiStructure and MultiStructure. The concepts collected in this chapter are not necessarily instances of the generic MultiStructure definition; some are included because they exhibit intrinsic binary, ternary, or multiple structural organization.

7.1.1 BiStructure

BiStructures combine two distinguishable mathematical structures into a two-component framework while preserving the identity of each component.

Definition 7.1.1 (BiStructure)**.** Let

$$\mathcal{S}_1 \qquad \text{and} \qquad \mathcal{S}_2$$

be two mathematical structures of the same or compatible type. A *BiStructure* is the ordered two-component structure

$$\mathcal{B} = (\mathcal{S}_1, \mathcal{S}_2).$$

The component positions are regarded as part of the structure, so that $\mathcal{S}_1$ and $\mathcal{S}_2$ remain distinguishable even when their underlying sets overlap or coincide.

If a set-theoretic realization is required, the two components may be represented by the tagged disjoint union

$$\{1\} \times S_1 \ \sqcup \ \{2\} \times S_2,$$

where S_i denotes the underlying set of $\mathcal{S}_i$.

7.1.2 TriStructure

TriStructures combine three distinguishable mathematical structures into a three-component framework while preserving the identity of each component.

Definition 7.1.2 (TriStructure)**.** Let

$$\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$$

be three mathematical structures of the same or compatible type. A *TriStructure* is the ordered three-component structure

$$\mathcal{T} = (\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3).$$

The component positions are regarded as part of the structure, so that the three constituent structures remain distinguishable even when some of their underlying sets overlap or coincide.

If a set-theoretic realization is required, the components may be represented by the tagged disjoint union

$$\bigsqcup_{i=1}^{3} \{i\} \times S_i,$$

where S_i denotes the underlying set of $\mathcal{S}_i$.

7.1.3 MultiStructure

MultiStructures combine several compatible mathematical structures into a unified multi-component framework while preserving the identity of each individual component.

Definition 7.1.3 (MultiStructure)**.** Let

$$\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n, \qquad n \geq 2,$$

be mathematical structures of the same or mutually compatible type. A *MultiStructure* is the indexed n-component structure

$$\mathcal{M} = (\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_n).$$

The index of each component is regarded as part of the structure, so that the constituent structures remain distinguishable even when some of them have identical or overlapping underlying sets.

If a set-theoretic realization is required, the components may be represented by the tagged disjoint union

$$\bigsqcup_{i=1}^{n} \{i\} \times S_i,$$

where S_i denotes the underlying set of $\mathcal{S}_i$.

7.1.4 Iterated MultiStructure

Iterated MultiStructures recursively combine MultiStructures from one level to form higher-level structures, thereby introducing hierarchical organization while preserving the constituent components.

Definition 7.1.4 (Iterated MultiStructure)**.** Let

$$\mathfrak{S}^{(0)}$$

be a class of basic mathematical structures. For $k \geq 0$, define recursively

$$\mathfrak{S}^{(k+1)} = \bigcup_{n \geq 2} \left(\mathfrak{S}^{(k)}\right)^n.$$

Thus, every element of $\mathfrak{S}^{(k+1)}$ has the form

$$\mathcal{M}^{(k+1)} = \left(\mathcal{S}_1^{(k)}, \mathcal{S}_2^{(k)}, \ldots, \mathcal{S}_n^{(k)}\right),$$

where

$$n \geq 2, \qquad \mathcal{S}_i^{(k)} \in \mathfrak{S}^{(k)}.$$

For $k \geq 1$, an element of $\mathfrak{S}^{(k)}$ is called an *Iterated MultiStructure of level k*.

Thus, a MultiStructure combines several distinguishable structures at a single level, whereas an Iterated MultiStructure recursively combines structures from preceding levels and therefore introduces hierarchical depth.

7.2 Concepts of BiStructure

This section discusses the concepts of BiStructure.

7.2.1 Bimatrices

Bimatrices combine two distinct matrices into a single formal structure, enabling paired representation and comparison of matrix-based information or systems [494, 495].

Definition 7.2.1 (Bimatrix)**.** Let

$$A_1 = [a_{ij}] \qquad \text{and} \qquad A_2 = [b_{ij}]$$

be two distinct matrices. A *bimatrix* is the formal combination

$$A_B = A_1 \cup A_2, \qquad A_1 \neq A_2,$$

where $\cup$ is used only as a notational symbol joining the two component matrices.

If both A_1 and A_2 are $m \times n$ matrices, then

$$A_B = A_1 \cup A_2$$

is called an *$m \times n$ rectangular bimatrix*.

7.2.2 Bipartite Graph

Bipartite Graphs partition vertices into two disjoint classes, with every edge connecting vertices belonging to different classes and never within one [496–499].

Definition 7.2.2 (Bipartite Graph)**.** A graph

$$G = (V, E)$$

is called a *bipartite graph* if its vertex set can be partitioned into two disjoint sets

$$V = V_1 \cup V_2, \qquad V_1 \cap V_2 = \varnothing,$$

such that every edge joins a vertex in V_1 to a vertex in V_2.

7.2.3 Bilinear Algebra

Bilinear Algebras combine two distinct linear algebras defined over the same field into a unified two-component algebraic structure [500].

Definition 7.2.3 (Bilinear Algebra)**.** Let

$$V = V_1 \cup V_2,$$

where V_1 and V_2 are linear algebras over the same field F. Then V is called a *Bilinear Algebra* (or *Linear Bialgebra*) over F.

7.2.4 Bivector Space

Bivector Spaces combine two vector spaces over the same field into a unified structure preserving their individual vector-space components [500, 501].

Definition 7.2.4 (Bivector Space)**.** Let

$$V = V_1 \cup V_2,$$

where V_1 and V_2 are vector spaces over the same field F. Then V is called a *Bivector Space* over F.

7.2.5 Bipolar Fuzzy Set

Bipolar Fuzzy Sets represent positive and negative membership information simultaneously using degrees from the intervals $[0, 1]$ and $[-1, 0]$ [95].

Definition 7.2.5 (Bipolar Fuzzy Set)**.** Let X be a nonempty set. A *Bipolar Fuzzy Set* A on X is defined by

$$A = \left\{ \left(x, \mu_A^+(x), \mu_A^-(x)\right) : x \in X \right\},$$

where

$$\mu_A^+ : X \to [0, 1], \qquad \mu_A^- : X \to [-1, 0].$$

Here, $\mu_A^+(x)$ represents the positive membership degree of x, whereas $\mu_A^-(x)$ represents its negative membership degree.

7.2.6 Bipolar Neutrosophic Set

Bipolar Neutrosophic Sets represent positive and negative truth, indeterminacy, and falsity degrees simultaneously for properties and their corresponding counter-properties [60, 502].

Definition 7.2.6 (Bipolar Neutrosophic Set)**.** Let X be a nonempty set. A *Bipolar Neutrosophic Set* A on X is represented by

$$A = \left\{ \left(x, T_A^+(x), I_A^+(x), F_A^+(x), T_A^-(x), I_A^-(x), F_A^-(x)\right) : x \in X \right\},$$

where

$$T_A^+, I_A^+, F_A^+ : X \to [0, 1]$$

and

$$T_A^-, I_A^-, F_A^- : X \to [-1, 0].$$

The positive components describe truth, indeterminacy, and falsity with respect to a property, while the negative components describe the corresponding degrees with respect to its counter-property.

7.2.7 Bipolar Uncertain Set

Bipolar Uncertain Sets represent uncertain information from positive and negative perspectives, allowing paired evaluations associated with properties and corresponding counter-properties.

Definition 7.2.7 (Bipolar Uncertain Set)**.** Let X be a nonempty universe, and let

$$\Pi = \{\pi_+, \pi_-\}$$

be two distinct poles. Let M be an uncertain model with a nonempty degree-domain

$$D_M.$$

A *Bipolar Uncertain Set* of type M on X is a tuple

$$U_M^\Pi = \left(X, M, \Pi, \mu_M^\Pi\right),$$

where

$$\mu_M^\Pi : X \longrightarrow D_M^2$$

is defined by

$$\mu_M^\Pi(x) = \left(\mu_M^+(x), \mu_M^-(x)\right), \qquad x \in X,$$

with

$$\mu_M^+(x), \mu_M^-(x) \in D_M.$$

Here, $\mu_M^+(x)$ and $\mu_M^-(x)$ represent the uncertainty degrees of x with respect to the positive and negative poles, respectively.

7.2.8 Bigroup

Bigroups combine two distinct group structures into a single two-component algebraic structure, while retaining the group properties of each component [503–507].

Definition 7.2.8 (Bigroup)**.** [507] Let

$$G = G_1 \cup G_2,$$

where

$$(G_1, *_1) \qquad \text{and} \qquad (G_2, *_2)$$

are two distinct groups. Then

$$G = (G_1, *_1) \cup (G_2, *_2)$$

is called a *Bigroup.*

Example 7.2.9 (Bigroup)**.** Let

$$G_1 = (\mathbb{Z}_4, +_4), \qquad G_2 = (\mathbb{Z}_3, +_3),$$

where $+_n$ denotes addition modulo n. Since both G_1 and G_2 are groups, the two-component structure

$$G = (\mathbb{Z}_4, +_4) \cup (\mathbb{Z}_3, +_3)$$

is a Bigroup.

7.2.9 Bisemigroup

Bisemigroups combine two distinct semigroups into a two-component algebraic structure, with each component independently satisfying associativity under its corresponding operation [507].

Definition 7.2.10 (Bisemigroup)**.** Let

$$S = S_1 \cup S_2,$$

where

$$(S_1, *_1) \qquad \text{and} \qquad (S_2, *_2)$$

are two distinct semigroups. Then

$$S = (S_1, *_1) \cup (S_2, *_2)$$

is called a *Bisemigroup*.

Example 7.2.11 (Bisemigroup)**.** Let

$$S_1 = (\mathbb{N}_0, +)$$

and

$$S_2 = (\{0, 1\}, \max).$$

Since addition on $\mathbb{N}_0$ and the maximum operation on $\{0, 1\}$ are associative,

$$S = (\mathbb{N}_0, +) \cup (\{0, 1\}, \max)$$

is a Bisemigroup.

7.2.10 Biinterval

Biintervals combine two distinct intervals into a two-component interval structure, allowing two ranges of values to be represented simultaneously [508].

Definition 7.2.12 (Biinterval)**.** Let

$$I_1 = [a_1, b_1] \qquad \text{and} \qquad I_2 = [a_2, b_2]$$

be two distinct intervals. The two-component interval structure

$$I_B = I_1 \cup I_2$$

is called a *Biinterval*.

Example 7.2.13 (Biinterval)**.** Consider the two distinct intervals

$$I_1 = [0, 1], \qquad I_2 = [2, 4].$$

Then

$$I_B = [0, 1] \cup [2, 4]$$

is a Biinterval consisting of the two interval components I_1 and I_2.

7.2.11 Biring

Birings combine two distinct rings into a two-component algebraic structure, with each component independently satisfying the corresponding ring axioms [507].

Definition 7.2.14 (Biring)**.** Let
$$R = R_1 \cup R_2,$$
where
$$(R_1, +_1, \cdot_1) \qquad \text{and} \qquad (R_2, +_2, \cdot_2)$$
are two distinct rings. Then
$$R = (R_1, +_1, \cdot_1) \cup (R_2, +_2, \cdot_2)$$
is called a *Biring*.

Example 7.2.15 (Biring)**.** Let
$$R_1 = (\mathbb{Z}_4, +_4, \cdot_4), \qquad R_2 = (\mathbb{Z}_6, +_6, \cdot_6),$$
where the operations are taken modulo 4 and 6, respectively. Since R_1 and R_2 are rings,
$$R = (\mathbb{Z}_4, +_4, \cdot_4) \cup (\mathbb{Z}_6, +_6, \cdot_6)$$
is a Biring.

7.2.12 Bimodule

Bimodules are abelian groups carrying compatible left and right module actions, allowing two rings to act simultaneously from opposite sides on the same structure consistently.

Definition 7.2.16 (Bimodule)**.** Let R and S be rings. An (R, S)-*bimodule* is an abelian group M that is simultaneously a left R-module and a right S-module such that
$$(rm)s = r(ms)$$
for all $r \in R$, $m \in M$, and $s \in S$.

Example 7.2.17 (Bimodule)**.** Let
$$R = S = \mathbb{Z}, \qquad M = \mathbb{Z}.$$
Define the left and right actions by ordinary multiplication:
$$r \cdot m = rm, \qquad m \cdot s = ms.$$
Then, for all $r, m, s \in \mathbb{Z}$,
$$(r \cdot m) \cdot s = (rm)s = r(ms) = r \cdot (m \cdot s).$$
Hence $\mathbb{Z}$ is a
$$(\mathbb{Z}, \mathbb{Z})\text{-bimodule.}$$

7.2.13 Bitopological Space

Bitopological Spaces equip one underlying set with two topologies, enabling simultaneous study of two notions of openness, continuity, separation, and convergence on it at once [509–511].

Definition 7.2.18 (Bitopological Space)**.** Let X be a nonempty set, and let τ_1 and τ_2 be topologies on X. Then

$$(X, \tau_1, \tau_2)$$

is called a *Bitopological Space*.

Example 7.2.19 (Bitopological Space)**.** Let

$$X = \{a, b, c\},$$

and define

$$\tau_1 = \{\varnothing, X, \{a\}\},$$

and

$$\tau_2 = \{\varnothing, X, \{b, c\}\}.$$

Both τ_1 and τ_2 are topologies on X. Therefore,

$$(X, \tau_1, \tau_2)$$

is a Bitopological Space.

7.2.14 Bilattice

Bilattices equip one set with two lattice orders, providing two independent ordering perspectives that can represent distinct notions such as truth and information simultaneously together (cf. [512, 513]).

Definition 7.2.20 (Bilattice)**.** A *Bilattice* is a structure

$$B = (B, \leq_1, \leq_2)$$

such that

$$(B, \leq_1) \qquad \text{and} \qquad (B, \leq_2)$$

are lattices.

Example 7.2.21 (Bilattice)**.** Let

$$B = \{0, 1\}^2.$$

Define two partial orders by

$$(a, b) \leq_1 (c, d) \quad \Longleftrightarrow \quad a \leq c \text{ and } b \leq d,$$

and

$$(a, b) \leq_2 (c, d) \quad \Longleftrightarrow \quad a \leq c \text{ and } b \geq d.$$

Both

$$(B, \leq_1) \qquad \text{and} \qquad (B, \leq_2)$$

are lattices. Hence

$$(B, \leq_1, \leq_2)$$

is a Bilattice.

7.2.15 Biset

Bisets are sets equipped with compatible left and right group actions, allowing two groups to act simultaneously while preserving the interaction between both actions consistently.

Definition 7.2.22 (Biset)**.** Let G and H be groups. A (G, H)-*biset* is a set X equipped with a left action of G and a right action of H such that

$$(gx)h = g(xh)$$

for all $g \in G$, $x \in X$, and $h \in H$.

Example 7.2.23 (Biset)**.** Let

$$G = H = X = \mathbb{Z}_2.$$

Define the left and right actions by

$$g \cdot x = g + x \pmod 2,$$

and

$$x \cdot h = x + h \pmod 2.$$

Then

$$(g \cdot x) \cdot h = g + x + h = g \cdot (x \cdot h).$$

Therefore X is a

$$(\mathbb{Z}_2, \mathbb{Z}_2)\text{-biset}.$$

7.2.16 Bicategory

Bicategories generalize categories by including objects, morphisms, and morphisms between morphisms, with associativity and identity laws holding coherently up to natural isomorphisms rather than strictly (cf. [514–516]).

Definition 7.2.24 (Bicategory)**.** A *Bicategory* $\mathcal{B}$ consists of objects $A, B, \ldots$, hom-categories

$$\mathcal{B}(A, B),$$

composition functors

$$\circ : \mathcal{B}(B, C) \times \mathcal{B}(A, B) \longrightarrow \mathcal{B}(A, C),$$

and identity 1-morphisms 1_A, where associativity and identity laws hold up to coherent natural isomorphisms satisfying the pentagon and triangle axioms.

Example 7.2.25 (Bicategory)**.** The category

$$\mathbf{Cat}$$

provides a standard example of a Bicategory. Its

- objects are small categories,
- 1-morphisms are functors,
- 2-morphisms are natural transformations.

Composition of functors and natural transformations gives the required bicategorical composition. In fact, **Cat** is a strict 2-category and hence, in particular, a Bicategory.

7.2.17 Bialgebra

Bialgebras combine algebra and coalgebra structures on one vector space, requiring multiplication and comultiplication to satisfy compatibility conditions linking both structures in a coherent way (cf. [517, 518]).

Definition 7.2.26 (Bialgebra)**.** Let K be a field. A *Bialgebra* over K is a vector space B equipped with an algebra structure

$$(B, m, \eta)$$

and a coalgebra structure

$$(B, \Delta, \varepsilon)$$

such that

$$\Delta : B \to B \otimes B \qquad \text{and} \qquad \varepsilon : B \to K$$

are algebra homomorphisms.

Example 7.2.27 (Bialgebra)**.** Let K be a field and consider the polynomial algebra

$$B = K[x].$$

Define the comultiplication and counit on the generator x by

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \qquad \varepsilon(x) = 0,$$

with

$$\Delta(1) = 1 \otimes 1, \qquad \varepsilon(1) = 1,$$

and extend Δ and ε as algebra homomorphisms. Then

$$(K[x], m, \eta, \Delta, \varepsilon)$$

is a Bialgebra over K.

7.2.18 Bimodule Category

Bimodule Categories are categories equipped with compatible left and right actions of two monoidal categories, extending the bimodule concept from algebraic objects to categorical settings (cf. [519, 520]).

Definition 7.2.28 (Bimodule Category)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories. A $(\mathcal{C}, \mathcal{D})$-*bimodule category* is a category $\mathcal{M}$ equipped with actions

$$\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}, \qquad \triangleleft : \mathcal{M} \times \mathcal{D} \to \mathcal{M},$$

together with coherent natural isomorphisms

$$(c \triangleright m) \triangleleft d \cong c \triangleright (m \triangleleft d).$$

Example 7.2.29 (Bimodule Category)**.** Let

$$\mathcal{C} = \mathcal{D} = \mathcal{M} = \mathbf{Vect}_K,$$

where $\mathbf{Vect}_K$ is the monoidal category of vector spaces over a field K. Define

$$V \triangleright M = V \otimes_K M,$$

and

$$M \triangleleft W = M \otimes_K W.$$

The canonical associativity isomorphism

$$(V \otimes_K M) \otimes_K W \cong V \otimes_K (M \otimes_K W)$$

gives the required compatibility. Hence

$$\mathbf{Vect}_K$$

is a

$$(\mathbf{Vect}_K, \mathbf{Vect}_K)\text{-bimodule category.}$$

7.2.19 Bilinear Map

A Bilinear Map is a function linear in each of two arguments separately, while mapping pairs into a vector space.

Definition 7.2.30 (Bilinear Map)**.** Let V, W, and U be vector spaces over a field $\mathbb{K}$. A map

$$B : V \times W \longrightarrow U$$

is called *bilinear* if it is linear in each argument separately.

Example 7.2.31 (Bilinear Map)**.** Let

$$V = W = U = \mathbb{R}$$

and define

$$B : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

by

$$B(x, y) = xy.$$

For any $x_1, x_2, y \in \mathbb{R}$ and $\lambda \in \mathbb{R}$,

$$B(x_1 + x_2, y) = (x_1 + x_2)y = B(x_1, y) + B(x_2, y),$$

and

$$B(\lambda x_1, y) = \lambda B(x_1, y).$$

Similarly, B is linear in the second argument. Hence $B(x, y) = xy$ is a Bilinear Map.

7.2.20 Bifunctor

A Bifunctor is a functor defined on a product of two categories, acting functorially and consistently in both variables separately.

Definition 7.2.32 (Bifunctor)**.** Let $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{E}$ be categories. A *Bifunctor* is a functor

$$F : \mathcal{C} \times \mathcal{D} \longrightarrow \mathcal{E}.$$

Thus, F is functorial in both variables.

Example 7.2.33 (Bifunctor)**.** Let

$$\mathbf{Vect}_{\mathbb{K}}$$

be the category of vector spaces over a field $\mathbb{K}$. Define

$$F : \mathbf{Vect}_{\mathbb{K}} \times \mathbf{Vect}_{\mathbb{K}} \longrightarrow \mathbf{Vect}_{\mathbb{K}}$$

by

$$F(V, W) = V \otimes_{\mathbb{K}} W.$$

For linear maps

$$f : V \to V', \qquad g : W \to W',$$

define

$$F(f, g) = f \otimes g : V \otimes W \longrightarrow V' \otimes W'$$

by

$$(f \otimes g)(v \otimes w) = f(v) \otimes g(w).$$

This construction preserves identities and compositions. Therefore, the tensor product defines a Bifunctor.

7.2.21 Bigraded Algebra

A Bigraded Algebra decomposes into components indexed by two degrees, with multiplication sending component pairs into their summed-degree component naturally.

Definition 7.2.34 (Bigraded Algebra)**.** Let $\mathbb{K}$ be a field. A *Bigraded Algebra* is a $\mathbb{K}$-algebra A with a decomposition

$$A = \bigoplus_{(p,q)\in\mathbb{N}^2} A_{p,q}$$

such that

$$A_{p,q}A_{r,s} \subseteq A_{p+r,q+s}$$

for all $p, q, r, s \in \mathbb{N}$.

Example 7.2.35 (Bigraded Algebra)**.** Let

$$A = \mathbb{K}[x, y]$$

be the polynomial algebra in two variables. Define

$$A_{p,q} = \mathbb{K}x^p y^q, \qquad p, q \in \mathbb{N}.$$

Then

$$A = \bigoplus_{(p,q)\in\mathbb{N}^2} A_{p,q}.$$

Moreover,

$$(x^p y^q)(x^r y^s) = x^{p+r} y^{q+s} \in A_{p+r,q+s}.$$

Hence

$$A_{p,q}A_{r,s} \subseteq A_{p+r,q+s},$$

so $\mathbb{K}[x, y]$ is a Bigraded Algebra.

7.2.22 Bicomplex Numbers

Bicomplex Numbers extend complex numbers using two commuting imaginary units, producing four-dimensional hypercomplex numbers with algebraic properties and zero divisors.

Definition 7.2.36 (Bicomplex Numbers)**.** Let i and j be commuting imaginary units satisfying

$$i^2 = j^2 = -1, \qquad ij = ji.$$

A *Bicomplex Number* is an element

$$z = z_1 + jz_2, \qquad z_1, z_2 \in \mathbb{C}(i).$$

Equivalently,

$$z = a + bi + cj + dij, \qquad a, b, c, d \in \mathbb{R}.$$

Example 7.2.37 (Bicomplex Number)**.** Let

$$i^2 = j^2 = -1, \qquad ij = ji.$$

Consider

$$z = 1 + 2i + 3j + 4ij.$$

Equivalently,

$$z = (1 + 2i) + j(3 + 4i).$$

Since

$$1 + 2i,\ 3 + 4i \in \mathbb{C}(i),$$

we have

$$z \in \mathbb{C}(i) + j\mathbb{C}(i).$$

Therefore,

$$z = 1 + 2i + 3j + 4ij$$

is a Bicomplex Number.

7.2.23 Bipartite Molecular Graph

A Molecular Graph represents a molecule as a graph whose vertices denote atoms and edges denote chemical bonds between them [521, 522]. Related concepts, including Molecular HyperGraphs [523–526], Molecular Fuzzy Graphs [527], Molecular SuperHyperGraphs [528, 529], and Chemical SuperHyperGraphs, have also been studied. A Bipartite Molecular Graph is a molecular graph whose atoms can be partitioned into two disjoint classes such that every bond joins atoms from different classes [530, 531].

Definition 7.2.38 (Bipartite Molecular Graph)**.** Let

$$G = (V, E)$$

be a molecular graph, where V represents the atoms and E represents the bonds of a molecule. The graph G is called a *Bipartite Molecular Graph* if there exist disjoint subsets $V_1, V_2 \subseteq V$ such that

$$V = V_1 \sqcup V_2$$

and

$$E \subseteq \big\{\{u, v\} \mid u \in V_1,\ v \in V_2\big\}.$$

Thus, no edge joins two vertices belonging to the same part. Equivalently, the molecular graph contains no odd cycle.

Example 7.2.39 (Bipartite Molecular Graph)**.** Consider the molecular graph of water,

$$\mathrm{H}_1 - \mathrm{O} - \mathrm{H}_2.$$

Let

$$V = \{\mathrm{O}, \mathrm{H}_1, \mathrm{H}_2\}$$

and

$$E = \big\{\{\mathrm{O}, \mathrm{H}_1\}, \{\mathrm{O}, \mathrm{H}_2\}\big\}.$$

Define

$$V_1 = \{\mathrm{O}\}, \qquad V_2 = \{\mathrm{H}_1, \mathrm{H}_2\}.$$

Then

$$V = V_1 \sqcup V_2,$$

and every edge joins a vertex in V_1 to a vertex in V_2. Therefore,

$$G = (V, E)$$

is a Bipartite Molecular Graph.

7.3 Concepts of TriStructure

This section examines the concepts of TriStructure.

7.3.1 Tripolar Fuzzy Set

Tripolar Fuzzy Sets assign positive, neutral, and negative membership degrees to each element, representing information from three distinct perspectives simultaneously [119–122, 532].

Definition 7.3.1 (Tripolar Fuzzy Set)**.** Let X be a nonempty set. A *Tripolar Fuzzy Set* A on X is defined by

$$A = \left\{ \big(x, \rho_A(x), \nu_A(x), \delta_A(x)\big) : x \in X \right\},$$

where

$$\rho_A, \nu_A : X \to [0,1], \qquad \delta_A : X \to [-1,0],$$

and

$$0 \le \rho_A(x) + \nu_A(x) \le 1 \qquad (x \in X).$$

Here, $\rho_A(x)$, $\nu_A(x)$, and $\delta_A(x)$ represent the positive, neutral, and counter-property membership degrees of x, respectively.

Example 7.3.2 (Tripolar Fuzzy Set)**.** Let

$$X = \{x_1, x_2\}.$$

Define a Tripolar Fuzzy Set A on X by

$$A = \{(x_1, 0.6, 0.2, -0.3), (x_2, 0.4, 0.5, -0.7)\}.$$

Thus,

$$\rho_A(x_1) = 0.6, \qquad \nu_A(x_1) = 0.2, \qquad \delta_A(x_1) = -0.3,$$

and

$$\rho_A(x_2) = 0.4, \qquad \nu_A(x_2) = 0.5, \qquad \delta_A(x_2) = -0.7.$$

Since

$$0.6 + 0.2 = 0.8 \le 1 \qquad \text{and} \qquad 0.4 + 0.5 = 0.9 \le 1,$$

all required conditions are satisfied. Hence A is a Tripolar Fuzzy Set on X.

7.3.2 Tripolar Neutrosophic Set

Tripolar Neutrosophic Sets assign positive, neutral, and negative truth, indeterminacy, and falsity degrees, representing uncertainty from three distinct perspectives simultaneously [183].

Definition 7.3.3 (Tripolar Neutrosophic Set)**.** Let X be a nonempty set and let

$$P = \{+, 0, -\}.$$

A *Tripolar Neutrosophic Set* A on X is a mapping

$$\tau_A : X \to [0,1]^9$$

such that, for every $x \in X$,

$$\tau_A(x) = \Big((T_A^+(x), I_A^+(x), F_A^+(x)), (T_A^0(x), I_A^0(x), F_A^0(x)), (T_A^-(x), I_A^-(x), F_A^-(x)) \Big),$$

where

$$T_A^p, I_A^p, F_A^p : X \to [0,1], \qquad p \in P.$$

Thus, each pole $p \in \{+, 0, -\}$ assigns a neutrosophic triple consisting of truth, indeterminacy, and falsity degrees.

Example 7.3.4 (Tripolar Neutrosophic Set)**.** Let

$$X = \{x_1, x_2\}.$$

Define a Tripolar Neutrosophic Set A on X by

$$\tau_A(x_1) = \Big((0.8, 0.1, 0.2), (0.4, 0.5, 0.3), (0.2, 0.6, 0.7) \Big),$$

and

$$\tau_A(x_2) = \Big((0.6, 0.2, 0.3), (0.5, 0.4, 0.2), (0.3, 0.5, 0.8)\Big).$$

For example, for x_1,

$$(T_A^+, I_A^+, F_A^+) = (0.8, 0.1, 0.2),$$
$$(T_A^0, I_A^0, F_A^0) = (0.4, 0.5, 0.3),$$

and

$$(T_A^-, I_A^-, F_A^-) = (0.2, 0.6, 0.7).$$

Since every component belongs to $[0, 1]$, A is a Tripolar Neutrosophic Set on X.

7.3.3 Tripolar Uncertain Set

Tripolar Uncertain Sets assign uncertainty degrees to positive, neutral, and negative poles, representing each element from three distinct perspectives simultaneously.

Definition 7.3.5 (Tripolar Uncertain Set)**.** Let X be a nonempty universe, and let

$$\Pi = \{\pi_+, \pi_0, \pi_-\}$$

be three distinct poles. Let M be an uncertain model with a nonempty degree-domain

$$D_M.$$

A *Tripolar Uncertain Set* of type M on X is a tuple

$$U_M^\Pi = \left(X, M, \Pi, \mu_M^\Pi\right),$$

where

$$\mu_M^\Pi : X \longrightarrow D_M^3$$

is defined by

$$\mu_M^\Pi(x) = \left(\mu_M^+(x), \mu_M^0(x), \mu_M^-(x)\right), \qquad x \in X,$$

with

$$\mu_M^+(x), \mu_M^0(x), \mu_M^-(x) \in D_M.$$

Here, the three components represent the uncertainty degrees of x with respect to the positive, neutral, and negative poles, respectively.

7.3.4 Tripartite Graph

Tripartite Graphs partition vertices into three independent sets, with every edge connecting vertices that belong to two distinct vertex parts [533–535].

Definition 7.3.6 (Tripartite Graph)**.** A graph

$$G = (V, E)$$

is called a *Tripartite Graph* if its vertex set can be partitioned as

$$V = V_1 \sqcup V_2 \sqcup V_3$$

such that every edge has endpoints in two distinct parts. Equivalently, each V_i is an independent set.

Example 7.3.7 (Tripartite Graph)**.** Let

$$V_1 = \{a, b\}, \qquad V_2 = \{c, d\}, \qquad V_3 = \{e, f\},$$

and let

$$V = V_1 \sqcup V_2 \sqcup V_3.$$

Define

$$E = \big\{\{a, c\}, \{a, e\}, \{b, d\}, \{b, f\}, \{c, e\}, \{d, f\}\big\}.$$

Since every edge joins vertices belonging to two distinct parts and no edge connects vertices within the same V_i, the graph

$$G = (V, E)$$

is a Tripartite Graph. A graphical representation of G is shown in Figure 7.1.

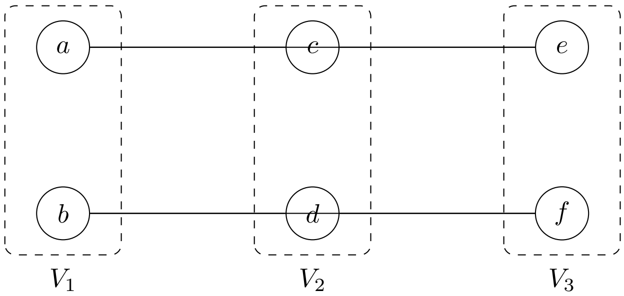


Figure 7.1: A Tripartite Graph with vertex partition $V = V_1 \sqcup V_2 \sqcup V_3$.

7.3.5 Trilinear Map

Trilinear Maps are mappings of three vector variables that remain linear with respect to each argument when the others are fixed [536, 537].

Definition 7.3.8 (Trilinear Map)**.** Let V_1, V_2, V_3 and W be vector spaces over a field F. A map

$$T : V_1 \times V_2 \times V_3 \longrightarrow W$$

is called *trilinear* if it is linear in each argument separately.

Example 7.3.9 (Trilinear Map)**.** Let

$$V_1 = V_2 = V_3 = W = \mathbb{R}.$$

Define

$$T : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$$

by

$$T(x, y, z) = xyz.$$

For fixed values of any two arguments, T is linear in the remaining argument. Hence T is a Trilinear Map.

7.3.6 Trilinear Form

Trilinear Forms are scalar-valued mappings of three vector arguments that are linear in each argument separately over a common field.

Definition 7.3.10 (Trilinear Form)**.** Let V be a vector space over a field F. A *Trilinear Form* on V is a map

$$T : V \times V \times V \longrightarrow F$$

that is linear in each of its three arguments separately.

Example 7.3.11 (Trilinear Form)**.** Let

$$V = \mathbb{R}^2.$$

For

$$u = (u_1, u_2), \qquad v = (v_1, v_2), \qquad w = (w_1, w_2),$$

define

$$T : V \times V \times V \longrightarrow \mathbb{R}$$

by

$$T(u, v, w) = u_1 v_1 w_1 + u_2 v_2 w_2.$$

The map T is linear in each of its three arguments separately. Therefore, T is a Trilinear Form on $\mathbb{R}^2$.

7.3.7 Tricategory

Tricategories generalize bicategories by incorporating higher morphisms, with composition, associativity, and identity laws satisfied through coherent higher-dimensional equivalences and isomorphisms (cf. [538–543]).

Definition 7.3.12 (Tricategory)**.** A *Tricategory* $\mathcal{T}$ consists of objects and, for every pair of objects A, B, a hom-bicategory

$$\mathcal{T}(A, B),$$

together with composition and identity structures whose associativity and unit laws hold up to coherent equivalences and higher isomorphisms.

Example 7.3.13 (Tricategory)**.** A standard example is the strict 3-category

$$\mathbf{2Cat},$$

whose

- objects are strict 2-categories,
- 1-morphisms are strict 2-functors,
- 2-morphisms are 2-natural transformations,
- 3-morphisms are modifications.

Since every strict 3-category is, in particular, a Tricategory,

$$\mathbf{2Cat}$$

provides an example of a Tricategory.

7.3.8 Trigroup

A Trigroup combines three distinct group structures into a single three-component algebraic framework while preserving each individual group component (cf. [544]).

Definition 7.3.14 (Trigroup)**.** Let
$$G = G_1 \cup G_2 \cup G_3,$$
where
$$(G_1, *_1), \qquad (G_2, *_2), \qquad (G_3, *_3)$$
are three distinct groups. Then
$$G = (G_1, *_1) \cup (G_2, *_2) \cup (G_3, *_3)$$
is called a *Trigroup.*

Example 7.3.15 (Trigroup)**.** Consider the three cyclic groups
$$G_1 = (\mathbb{Z}_2, +_2), \qquad G_2 = (\mathbb{Z}_3, +_3), \qquad G_3 = (\mathbb{Z}_5, +_5),$$
where $+_n$ denotes addition modulo n. Then the three-component structure
$$G = (\mathbb{Z}_2, +_2) \cup (\mathbb{Z}_3, +_3) \cup (\mathbb{Z}_5, +_5)$$
is a Trigroup.

7.3.9 Trisemigroup

Trisemigroups are sets equipped with three associative binary operations satisfying specified compatibility identities that govern interactions among the three operations (cf. [544]).

Definition 7.3.16 (Trisemigroup)**.** A *Trisemigroup* is a set S equipped with three associative binary operations
$$\dashv, \qquad \vdash, \qquad \perp$$
such that, for all $x, y, z \in S$,
$$\begin{aligned}
x \dashv (y \vdash z) &= x \dashv (y \dashv z),\\
(x \dashv y) \vdash z &= (x \vdash y) \vdash z,\\
x \vdash (y \dashv z) &= (x \vdash y) \dashv z,\\
x \dashv (y \perp z) &= x \dashv (y \dashv z),\\
(x \perp y) \vdash z &= (x \vdash y) \vdash z,\\
x \vdash (y \perp z) &= (x \vdash y) \perp z,\\
x \perp (y \dashv z) &= (x \perp y) \dashv z,\\
x \perp (y \vdash z) &= (x \dashv y) \perp z.
\end{aligned}$$

Example 7.3.17 (Trisemigroup)**.** Let
$$S = \mathbb{N}_0$$
and define the three binary operations by
$$x \dashv y = x + y, \qquad x \vdash y = x + y, \qquad x \perp y = x + y.$$
Since ordinary addition is associative and all three operations coincide, every required mixed compatibility identity reduces to
$$x + (y + z) = (x + y) + z.$$
Therefore,
$$(\mathbb{N}_0, \dashv, \vdash, \perp)$$
is a Trisemigroup.

7.3.10 Trialgebra

Trialgebras are vector spaces equipped with three bilinear associative operations satisfying compatibility identities that collectively define a trisemigroup-like algebraic structure (cf. [545, 546]).

Definition 7.3.18 (Associative Trialgebra)**.** Let F be a field. An *Associative Trialgebra* is an F-vector space A equipped with three bilinear operations

$$\dashv, \qquad \vdash, \qquad \perp : A \times A \longrightarrow A,$$

such that each operation is associative and the underlying set

$$(A, \dashv, \vdash, \perp)$$

is a Trisemigroup.

Example 7.3.19 (Associative Trialgebra)**.** Let

$$A = \mathbb{R}$$

be regarded as a vector space over $\mathbb{R}$. Define three bilinear operations by

$$x \dashv y = xy, \qquad x \vdash y = xy, \qquad x \perp y = xy.$$

Each operation is bilinear and associative. Moreover, because the three operations coincide, all Trisemigroup compatibility identities follow from

$$x(yz) = (xy)z.$$

Hence

$$(\mathbb{R}, \dashv, \vdash, \perp)$$

is an Associative Trialgebra over $\mathbb{R}$.

7.3.11 Filippov Algebra

A Filippov Algebra is a vector space with an alternating multilinear bracket satisfying the generalized Jacobi identity known as Filippov identity [547–550].

Definition 7.3.20 (Filippov Algebra)**.** Let $\mathbb{K}$ be a field and let $n \geq 2$. A *Filippov Algebra*, or *n-Lie Algebra*, is a vector space L equipped with an alternating n-linear bracket

$$[\cdot, \dots, \cdot] : L^n \longrightarrow L$$

satisfying the Filippov identity

$$[x_1, \dots, x_{n-1}, [y_1, \dots, y_n]] = \sum_{i=1}^{n} [y_1, \dots, [x_1, \dots, x_{n-1}, y_i], \dots, y_n].$$

Example 7.3.21 (Filippov Algebra)**.** Let

$$L = \mathbb{R}^4$$

with basis

$$\{e_1, e_2, e_3, e_4\}.$$

Define an alternating trilinear bracket by

$$[e_i, e_j, e_k] = \sum_{\ell=1}^{4} \varepsilon_{ijk\ell} e_\ell,$$

where $\varepsilon_{ijk\ell}$ is the Levi–Civita symbol. For example,

$$[e_1, e_2, e_3] = e_4, \qquad [e_1, e_2, e_4] = -e_3.$$

This bracket satisfies the Filippov identity, so L is a 3-Lie Algebra, and hence a Filippov Algebra.

7.3.12 Ternary Semigroup

A Ternary Semigroup is a nonempty set with an associative ternary operation combining three elements into another element of the set [551–553].

Definition 7.3.22 (Ternary Semigroup)**.** A *Ternary Semigroup* is a nonempty set S equipped with a ternary operation

$$f : S^3 \longrightarrow S$$

satisfying

$$f(f(a,b,c),d,e) = f(a,f(b,c,d),e) = f(a,b,f(c,d,e))$$

for all $a,b,c,d,e \in S$.

Example 7.3.23 (Ternary Semigroup)**.** Let

$$S = \mathbb{N}$$

and define

$$f : S^3 \longrightarrow S$$

by

$$f(a,b,c) = a + b + c.$$

Then, for all $a,b,c,d,e \in S$,

$$f(f(a,b,c),d,e) = a + b + c + d + e,$$

$$f(a,f(b,c,d),e) = a + b + c + d + e,$$

and

$$f(a,b,f(c,d,e)) = a + b + c + d + e.$$

Hence $(\mathbb{N}, f)$ is a Ternary Semigroup.

7.3.13 Ternary Group

A Ternary Group is an associative ternary system where each one-variable equation has a unique solution when other elements are fixed [554–556].

Definition 7.3.24 (Ternary Group)**.** A *Ternary Group* is a ternary semigroup (G, f) such that each equation

$$f(x,a,b) = c, \qquad f(a,x,b) = c, \qquad f(a,b,x) = c$$

has a unique solution $x \in G$ for all $a,b,c \in G$.

Example 7.3.25 (Ternary Group)**.** Let

$$G = \mathbb{Z}$$

and define

$$f : G^3 \longrightarrow G$$

by

$$f(a,b,c) = a - b + c.$$

Then

$$f(f(a,b,c),d,e) = f(a,f(b,c,d),e) = f(a,b,f(c,d,e)) = a - b + c - d + e.$$

Moreover, each equation

$$f(x,a,b) = c, \qquad f(a,x,b) = c, \qquad f(a,b,x) = c$$

has a unique solution in $\mathbb{Z}$. For example,

$$f(a,x,b) = c$$

gives

$$x = a + b - c.$$

Therefore, $(\mathbb{Z}, f)$ is a Ternary Group.

7.3.14 Ternary Algebra

A Ternary Algebra is a vector space equipped with a trilinear operation combining three elements into another element within it [557–559].

Definition 7.3.26 (Ternary Algebra)**.** Let $\mathbb{K}$ be a field. A *Ternary Algebra* is a vector space A over $\mathbb{K}$ equipped with a trilinear operation

$$[\cdot,\cdot,\cdot] : A \times A \times A \longrightarrow A.$$

Thus, for all $x, y, z \in A$, the product

$$[x, y, z] \in A$$

is linear in each argument separately.

Example 7.3.27 (Ternary Algebra)**.** Let

$$A = \mathbb{R}$$

and define

$$[\cdot,\cdot,\cdot] : A^3 \longrightarrow A$$

by

$$[x, y, z] = xyz.$$

For example,

$$[2, 3, 4] = 24.$$

Since ordinary multiplication in $\mathbb{R}$ is linear in each variable when the other variables are fixed, the operation

$$[x, y, z] = xyz$$

is trilinear. Hence $\mathbb{R}$ equipped with this operation is a Ternary Algebra.

7.3.15 Ternary Ring

A Ternary Ring is an additive abelian group equipped with distributive, associative ternary multiplication combining three elements into one element [560–562].

Definition 7.3.28 (Ternary Ring)**.** A *Ternary Ring* is an abelian group $(R, +)$ equipped with a ternary multiplication

$$[\cdot,\cdot,\cdot] : R^3 \longrightarrow R$$

that is distributive over addition in each argument and satisfies ternary associativity:

$$[[a, b, c], d, e] = [a, [b, c, d], e] = [a, b, [c, d, e]]$$

for all $a, b, c, d, e \in R$.

Example 7.3.29 (Ternary Ring)**.** Let

$$R = \mathbb{Z}$$

with the usual addition, and define the ternary multiplication by

$$[a, b, c] = abc.$$

It is distributive in each argument. Moreover,

$$[[a, b, c], d, e] = (abc)de = abcde,$$

$$[a, [b, c, d], e] = a(bcd)e = abcde,$$

and

$$[a, b, [c, d, e]] = ab(cde) = abcde.$$

Thus,

$$[[a, b, c], d, e] = [a, [b, c, d], e] = [a, b, [c, d, e]].$$

Therefore, $(\mathbb{Z}, +, [\cdot,\cdot,\cdot])$ is a Ternary Ring.

7.4 Concepts of MultiStructure

This section examines the concepts of MultiStructure.

7.4.1 Multimatrix

Multimatrices combine multiple matrices into one formal structure, preserving each component while allowing unified representation and analysis of matrix systems.

Definition 7.4.1 (Multimatrix)**.** Let

$$A_1, A_2, \ldots, A_n, \qquad n \geq 2,$$

be matrices. A *Multimatrix* is the formal n-component matrix structure

$$A_M = A_1 \cup A_2 \cup \cdots \cup A_n,$$

where $\cup$ denotes the formal combination of the component matrices. If all A_i are $m \times p$ matrices, then A_M is called an $m \times p$ *rectangular multimatrix*.

Example 7.4.2 (Multimatrix)**.** Consider the three matrices

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then

$$A_M = A_1 \cup A_2 \cup A_3$$

is a Multimatrix. Since all three component matrices are 2×2, A_M is a 2×2 rectangular Multimatrix.

7.4.2 MultiVector Space

MultiVector Spaces combine multiple vector spaces over the same field into a unified structure consisting of several distinct vector-space components.

Definition 7.4.3 (MultiVector Space)**.** Let

$$V = V_1 \cup V_2 \cup \cdots \cup V_n, \qquad n \geq 3,$$

where each V_i is a vector space over the same field F. Then V is called a *MultiVector Space*, or an *n-Vector Space*, over F.

Example 7.4.4 (MultiVector Space)**.** Let

$$V_1 = \mathbb{R}, \qquad V_2 = \mathbb{R}^2, \qquad V_3 = \mathbb{R}^3,$$

each regarded as a vector space over the field $\mathbb{R}$. Then the formal three-component structure

$$V = V_1 \cup V_2 \cup V_3 = \mathbb{R} \cup \mathbb{R}^2 \cup \mathbb{R}^3$$

is a MultiVector Space over $\mathbb{R}$.

7.4.3 MultiInterval

MultiIntervals combine multiple interval components into a single finite structure while preserving each interval as a distinguishable constituent of the representation.

Definition 7.4.5 (MultiInterval)**.** Let

$$I_1, I_2, \ldots, I_n, \qquad n \geq 2,$$

be intervals. A *MultiInterval* is a finite collection of these interval components, represented by

$$\mathcal{I} = I_1 \cup I_2 \cup \cdots \cup I_n.$$

When the interval components are required to remain distinguishable, $\mathcal{I}$ is regarded as an n-component interval structure.

Example 7.4.6 (MultiInterval)**.** Consider the four intervals

$$I_1 = [0, 1], \qquad I_2 = [2, 4], \qquad I_3 = [5, 7], \qquad I_4 = [8, 10].$$

Then

$$\mathcal{I} = I_1 \cup I_2 \cup I_3 \cup I_4$$

is a four-component MultiInterval, where the individual interval components are retained as distinguishable parts of the structure.

7.4.4 Multipartite Graph

A Multipartite Graph partitions its vertices into multiple disjoint sets, with edges allowed only between vertices belonging to different parts [563–566].

Definition 7.4.7 (Multipartite Graph)**.** Let

$$G = (V, E)$$

be a graph. Then G is called a *k-partite graph* if its vertex set can be partitioned into k pairwise disjoint sets

$$V = V_1 \cup V_2 \cup \cdots \cup V_k, \qquad V_i \cap V_j = \varnothing \quad (i \neq j),$$

such that no two vertices belonging to the same part V_i are adjacent. Equivalently, every edge of G has its endpoints in two distinct parts.

Example 7.4.8 (Multipartite Graph)**.** Let

$$V_1 = \{a, b\}, \qquad V_2 = \{c, d\}, \qquad V_3 = \{e\}, \qquad V_4 = \{f, g\},$$

and set

$$V = V_1 \sqcup V_2 \sqcup V_3 \sqcup V_4.$$

Define

$$E = \big\{\{a, c\}, \{a, e\}, \{b, f\}, \{c, e\}, \{d, g\}, \{e, f\}\big\}.$$

Every edge has endpoints belonging to distinct parts, and no two vertices within the same V_i are adjacent. Therefore,

$$G = (V, E)$$

is a 4-partite graph. A graphical representation of G is shown in Figure 7.2.

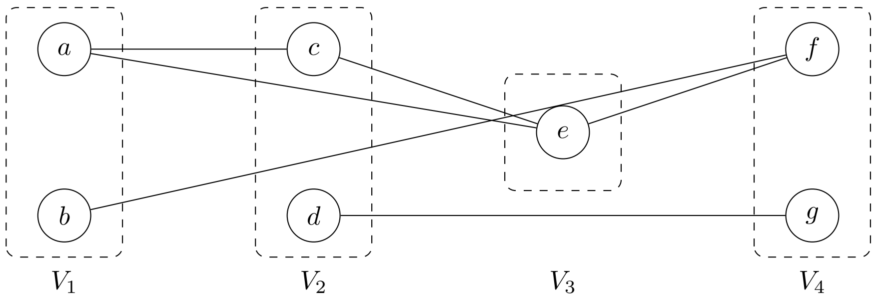


Figure 7.2: A 4-partite graph with vertex partition $V = V_1 \sqcup V_2 \sqcup V_3 \sqcup V_4$.

7.4.5 MultiGraph

MultiGraphs allow multiple edges, and possibly loops, between vertices by assigning a nonnegative integer multiplicity to each unordered vertex pair [407].

Definition 7.4.9 (MultiGraph)**.** Let V be a finite set, and let

$$\binom{V}{2}_m$$

denote the set of unordered pairs of vertices with repetition allowed. A *MultiGraph* is a pair

$$G = (V, \mu),$$

where

$$\mu : \binom{V}{2}_m \longrightarrow \mathbb{N}_0$$

assigns to each pair $\{u, v\}$ its edge multiplicity. Thus, multiple edges and loops are allowed.

Example 7.4.10 (MultiGraph)**.** Let

$$V = \{u, v, w\}.$$

Define the edge multiplicity function by

$$\mu(\{u, v\}) = 3, \qquad \mu(\{v, w\}) = 2, \qquad \mu(\{u, u\}) = 1,$$

and

$$\mu(\{u, w\}) = \mu(\{v, v\}) = \mu(\{w, w\}) = 0.$$

Thus, there are three parallel edges between u and v, two parallel edges between v and w, and one loop at u. Hence

$$G = (V, \mu)$$

is a MultiGraph. A graphical representation of G is shown in Figure 7.3.

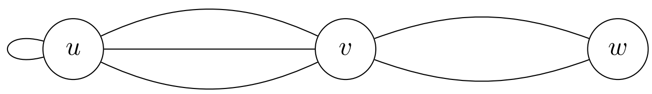


Figure 7.3: A MultiGraph with three parallel edges between u and v, two parallel edges between v and w, and one loop at u.

7.4.6 Iterated MultiGraph

Iterated MultiGraphs generalize multigraphs by using iterated multisets as vertices while retaining edge multiplicities between pairs of such vertices [407].

Definition 7.4.11 (Iterated MultiGraph)**.** Let X be a nonempty set, and let $\mathcal{M}(X)$ denote the set of all finite multisets over X. Define recursively

$$\mathcal{M}^0(X) = X, \qquad \mathcal{M}^{n+1}(X) = \mathcal{M}\big(\mathcal{M}^n(X)\big).$$

An *Iterated MultiGraph of order n* is a MultiGraph

$$G^{(n)} = (V^{(n)}, \mu^{(n)}),$$

where

$$V^{(n)} \subseteq \mathcal{M}^n(X)$$

is a finite vertex set and

$$\mu^{(n)} : \binom{V^{(n)}}{2}_m \longrightarrow \mathbb{N}_0$$

assigns an edge multiplicity to each unordered pair of vertices. Thus, the vertices are n-fold iterated multisets over X.

Example 7.4.12 (Iterated MultiGraph)**.** Let

$$X = \{a, b\}$$

and consider the multisets

$$M_1 = \{\{a, a\}\}, \qquad M_2 = \{\{a, b\}\}$$

in $\mathcal{M}(X)$. Define

$$U = \{\{M_1, M_2\}\}, \qquad W = \{\{M_2, M_2\}\},$$

so that

$$U, W \in \mathcal{M}^2(X).$$

Let

$$V^{(2)} = \{U, W\},$$

and define

$$\mu^{(2)}(\{U, W\}) = 2, \qquad \mu^{(2)}(\{U, U\}) = 1, \qquad \mu^{(2)}(\{W, W\}) = 0.$$

Thus, there are two parallel edges between U and W, one loop at U, and no loop at W. Hence

$$G^{(2)} = \left(V^{(2)}, \mu^{(2)}\right)$$

is an Iterated MultiGraph of order 2. A graphical representation is shown in Figure 7.4.

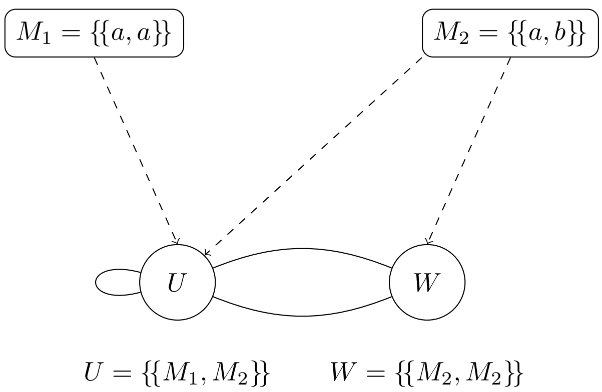


Figure 7.4: An Iterated MultiGraph of order 2, where the vertices $U, W \in \mathcal{M}^2(X)$ are second-order multisets, with two parallel edges between U and W and one loop at U.

7.4.7 Multipolar Fuzzy Set

Multipolar Fuzzy Sets represent each element through multiple membership degrees, allowing several distinct poles or perspectives to be modeled simultaneously [567–569].

Definition 7.4.13 (Multipolar Fuzzy Set)**.** Let X be a nonempty set and let $m \geq 2$. A *Multipolar Fuzzy Set A* on X is defined by a mapping

$$\mu_A : X \longrightarrow [0,1]^m,$$

where

$$\mu_A(x) = \big(\mu_A^{(1)}(x), \mu_A^{(2)}(x), \ldots, \mu_A^{(m)}(x)\big).$$

Each component

$$\mu_A^{(j)} : X \longrightarrow [0,1], \qquad 1 \leq j \leq m,$$

represents the membership degree of x with respect to the j-th pole.

Example 7.4.14 (Multipolar Fuzzy Set)**.** Let

$$X = \{x_1, x_2\}$$

and let $m = 3$. Define

$$\mu_A(x_1) = (0.8, 0.4, 0.2), \qquad \mu_A(x_2) = (0.3, 0.7, 0.6).$$

Equivalently,

$$\mu_A : X \longrightarrow [0,1]^3.$$

Thus, x_1 has membership degrees 0.8, 0.4, and 0.2 with respect to the three poles, while x_2 has membership degrees 0.3, 0.7, and 0.6. Hence A is a 3-polar Fuzzy Set on X.

7.4.8 Multipolar Neutrosophic Set

Multipolar Neutrosophic Sets assign multiple truth, indeterminacy, and falsity triples to each element, representing uncertainty from several distinct poles simultaneously [570–573].

Definition 7.4.15 (Multipolar Neutrosophic Set)**.** Let X be a nonempty set and let $m \geq 2$. A *Multipolar Neutrosophic Set A* on X assigns to each $x \in X$ an m-tuple of neutrosophic values

$$A(x) = \big((T_A^{(1)}(x), I_A^{(1)}(x), F_A^{(1)}(x)), \ldots, (T_A^{(m)}(x), I_A^{(m)}(x), F_A^{(m)}(x))\big),$$

where

$$T_A^{(j)}, I_A^{(j)}, F_A^{(j)} : X \longrightarrow [0,1], \qquad 1 \leq j \leq m.$$

For each pole j, the values $T_A^{(j)}(x)$, $I_A^{(j)}(x)$, and $F_A^{(j)}(x)$ represent the truth, indeterminacy, and falsity degrees of x, respectively.

Example 7.4.16 (Multipolar Neutrosophic Set)**.** Let

$$X = \{x_1, x_2\}$$

and let $m = 3$. Define

$$A(x_1) = \Big((0.8, 0.1, 0.2), (0.5, 0.3, 0.4), (0.2, 0.6, 0.7)\Big),$$

and

$$A(x_2) = \Big((0.6, 0.2, 0.3), (0.4, 0.5, 0.2), (0.7, 0.1, 0.4)\Big).$$

For example, the three neutrosophic values associated with x_1 are

$$(T_A^{(1)}, I_A^{(1)}, F_A^{(1)}) = (0.8, 0.1, 0.2),$$

$$(T_A^{(2)}, I_A^{(2)}, F_A^{(2)}) = (0.5, 0.3, 0.4),$$

and

$$(T_A^{(3)}, I_A^{(3)}, F_A^{(3)}) = (0.2, 0.6, 0.7).$$

Since every truth, indeterminacy, and falsity component belongs to $[0,1]$, A is a 3-polar Neutrosophic Set on X.

7.4.9 Multipolar Uncertain Set

An m-Polar Uncertain Set assigns to each element one admissible uncertainty degree for each of m distinct poles.

Definition 7.4.17 (m-Polar Uncertain Set)**.** Let X be a nonempty universe, let

$$m \geq 1,$$

and let

$$\Pi = \{\pi_1, \pi_2, \dots, \pi_m\}$$

be a set of m distinct poles. Let M be an uncertain model with a nonempty degree-domain

$$D_M \subseteq [0,1]^s, \qquad s \geq 1.$$

An *m-Polar Uncertain Set* of type M on X is a tuple

$$U_M^{\Pi} = \left(X, M, \Pi, \mu_M^{\Pi}\right),$$

where

$$\mu_M^{\Pi} : X \longrightarrow D_M^m$$

is the m-polar uncertainty-degree function.

For every $x \in X$,

$$\mu_M^{\Pi}(x) = \big(\mu_{M,1}(x), \mu_{M,2}(x), \dots, \mu_{M,m}(x)\big),$$

where

$$\mu_{M,r}(x) \in D_M, \qquad r = 1, \dots, m.$$

Here, $\mu_{M,r}(x)$ represents the uncertainty degree of x with respect to the r-th pole π_r.

7.4.10 Iterated Multipolar Fuzzy Set (Proposed in this book)

Iterated Multipolar Fuzzy Sets recursively assign multipolar fuzzy values at successive levels, creating hierarchical membership structures with multiple poles at each depth.

Definition 7.4.18 (Iterated Multipolar Fuzzy Set)**.** Let X be a nonempty set and let $m \geq 2$. Define recursively

$$D_m^{(0)} = [0,1], \qquad D_m^{(n+1)} = \left(D_m^{(n)}\right)^m, \qquad n \geq 0.$$

An *Iterated m-Polar Fuzzy Set of depth n* on X is a mapping

$$\mu_A^{(m,n)} : X \longrightarrow D_m^{(n)}.$$

Equivalently, for $n \geq 1$,

$$\mu_A^{(m,n)}(x) = \big(d_1(x), d_2(x), \dots, d_m(x)\big),$$

where

$$d_j(x) \in D_m^{(n-1)}, \qquad j = 1, \dots, m.$$

Thus, each pole at depth n contains an m-polar fuzzy value of depth $n-1$.

Remark 7.4.19. For $n = 1$,

$$D_m^{(1)} = [0,1]^m,$$

so an Iterated m-Polar Fuzzy Set reduces to an ordinary m-Polar Fuzzy Set.

7.4.11 Multiset

Multisets generalize ordinary sets by assigning each element a nonnegative integer multiplicity, thereby allowing repeated occurrences while retaining a precise mathematical representation of general collections.

Definition 7.4.20 (Multiset)**.** Let X be a set. A *Multiset* M over X is specified by a multiplicity function

$$m_M : X \longrightarrow \mathbb{N}_0,$$

where $m_M(x)$ denotes the number of occurrences of x in M.

Example 7.4.21 (Multiset)**.** Let

$$X = \{a, b, c\}.$$

Define the multiplicity function

$$m_M : X \longrightarrow \mathbb{N}_0$$

by

$$m_M(a) = 3, \qquad m_M(b) = 2, \qquad m_M(c) = 1.$$

Then the corresponding Multiset can be written as

$$M = \{\{a, a, a, b, b, c\}\}.$$

Thus, a, b, and c occur with multiplicities 3, 2, and 1, respectively.

7.4.12 Multihypergraph

Multihypergraphs extend hypergraphs by allowing repeated hyperedges and, in some formulations, repeated vertices within hyperedges, thereby representing higher order relations with multiplicity information explicitly included [574, 575].

Definition 7.4.22 (Multihypergraph)**.** [575] Let V be a finite set and let $\mathcal{M}(V)$ denote the family of finite multisets over V. A *Multihypergraph* is a pair

$$H = (V, \mathcal{E}),$$

where $\mathcal{E}$ is a multiset of elements of $\mathcal{M}(V)$. Thus, vertices may occur repeatedly within a hyperedge, and hyperedges may also have multiplicities.

Example 7.4.23 (Multihypergraph)**.** Let

$$V = \{a, b, c\}.$$

Consider the multihyperedges

$$e_1 = \{\{a, a, b\}\}, \qquad e_2 = \{\{b, c\}\}.$$

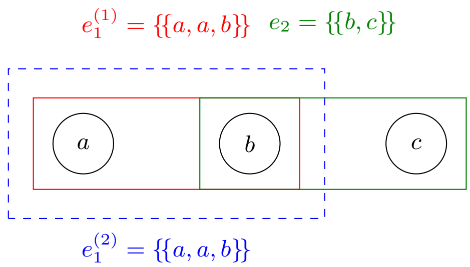
Define the multiset of hyperedges by

$$\mathcal{E} = \{\{e_1, e_1, e_2\}\}.$$

Thus, the hyperedge e_1 appears twice, while e_2 appears once. Hence

$$H = (V, \mathcal{E})$$

is a Multihypergraph. In particular, the vertex a occurs twice in e_1, and the hyperedge e_1 itself has multiplicity 2. A graphical representation is shown in Figure 7.5.



In each copy of e_1, vertex a has multiplicity 2.

Figure 7.5: A Multihypergraph with multihyperedges $\mathcal{E} = \{\{e_1, e_1, e_2\}\}$, where $e_1 = \{\{a, a, b\}\}$ appears twice and $e_2 = \{\{b, c\}\}$ appears once.

7.4.13 Multilayer Network

Multilayer Networks represent entities across multiple layers, allowing connections within and between layers and supporting analysis of systems with several interacting relational dimensions simultaneously together [576–578].

Definition 7.4.24 (Multilayer Network)**.** Let V be a set of entities and let

$$L_1, \ldots, L_d$$

be sets of elementary layers. A *Multilayer Network* is a structure

$$M = (V_M, E_M, V, \{L_1, \ldots, L_d\}),$$

where

$$V_M \subseteq V \times L_1 \times \cdots \times L_d$$

is the set of node-layer tuples and

$$E_M \subseteq V_M \times V_M$$

is the set of edges between them.

Example 7.4.25 (Multilayer Network)**.** Let

$$V = \{a, b\}, \qquad L_1 = \{\text{online}, \text{offline}\}.$$

Define the node-layer set

$$V_M = \{(a, \text{online}), (b, \text{online}), (a, \text{offline}), (b, \text{offline})\}.$$

Let

$$E_M = \big\{ \big((a, \text{online}), (b, \text{online})\big), \big((a, \text{offline}), (b, \text{offline})\big), \big((a, \text{online}), (a, \text{offline})\big) \big\}.$$

Then

$$M = (V_M, E_M, V, \{L_1\})$$

is a Multilayer Network containing both intralayer and interlayer connections.

7.4.14 Multiplex Network

Multiplex Networks represent the same set of nodes across multiple layers, where each layer captures a distinct type of relation among those nodes simultaneously separately [579–582].

Definition 7.4.26 (Multiplex Network)**.** Let V be a common node set and let L be a set of layers. A *Multiplex Network* is a structure

$$M = (V, \{E_\ell\}_{\ell \in L}),$$

where

$$E_\ell \subseteq \binom{V}{2}$$

is the edge set associated with layer ℓ. Thus, the same nodes may have different relations in different layers.

Example 7.4.27 (Multiplex Network)**.** Let

$$V = \{a, b, c\}$$

and let

$$L = \{\ell_1, \ell_2\},$$

where ℓ_1 represents friendship and ℓ_2 represents professional collaboration. Define

$$E_{\ell_1} = \{\{a, b\}, \{b, c\}\},$$

and

$$E_{\ell_2} = \{\{a, c\}, \{b, c\}\}.$$

Then

$$M = (V, \{E_{\ell_1}, E_{\ell_2}\})$$

is a Multiplex Network in which the same node set participates in two different relational layers.

7.4.15 Multisorted Algebra

Multisorted Algebras use multiple carrier sets, called sorts, with operations whose input and output types are explicitly specified across different sorts within one algebraic system [583, 584].

Definition 7.4.28 (Multisorted Algebra)**.** Let S be a set of sorts. A *Multisorted Algebra* consists of a family of carrier sets

$$\{A_s\}_{s \in S}$$

together with operations of the form

$$f : A_{s_1} \times \cdots \times A_{s_n} \longrightarrow A_t,$$

where

$$s_1, \ldots, s_n, t \in S.$$

Thus, each operation has specified input and output sorts.

Example 7.4.29 (Multisorted Algebra)**.** Let the set of sorts be

$$S = \{\mathsf{Nat}, \mathsf{Bool}\},$$

with carrier sets

$$A_{\mathsf{Nat}} = \mathbb{N}_0, \qquad A_{\mathsf{Bool}} = \{0, 1\}.$$

Consider the operations

$$+ : A_{\mathsf{Nat}} \times A_{\mathsf{Nat}} \longrightarrow A_{\mathsf{Nat}},$$

and

$$\text{isZero} : A_{\mathsf{Nat}} \longrightarrow A_{\mathsf{Bool}},$$

where

$$\text{isZero}(n) = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Then

$$\mathcal{A} = (\{A_{\mathsf{Nat}}, A_{\mathsf{Bool}}\}, +, \text{isZero})$$

is a Multisorted Algebra with two distinct sorts.

7.4.16 Multilinear Map

A Multilinear Map is a function linear in each argument separately while allowing multiple vector-space inputs and one output simultaneously (cf. [585–587]).

Definition 7.4.30 (Multilinear Map)**.** Let $V_1, \dots, V_n$ and W be vector spaces over a field $\mathbb{K}$. A map

$$f : V_1 \times \cdots \times V_n \longrightarrow W$$

is called *multilinear* if it is linear in each argument separately.

Example 7.4.31 (Multilinear Map)**.** Let

$$V_1 = V_2 = \mathbb{R}^2, \qquad W = \mathbb{R}.$$

Define

$$f : V_1 \times V_2 \longrightarrow \mathbb{R}$$

by

$$f((x_1, x_2), (y_1, y_2)) = x_1 y_1 + x_2 y_2.$$

For example,

$$f((1, 2), (3, 4)) = 1 \cdot 3 + 2 \cdot 4 = 11.$$

Since f is linear in each argument separately, it is a Multilinear Map.

7.4.17 Multilinear Form

A Multilinear Form is a multilinear map from several vector arguments into the underlying scalar field of the vector space.

Definition 7.4.32 (Multilinear Form)**.** Let V be a vector space over a field $\mathbb{K}$. An *n-linear form* is a multilinear map

$$\varphi : V^n \longrightarrow \mathbb{K}.$$

Example 7.4.33 (Multilinear Form)**.** Let

$$V = \mathbb{R}^3$$

and define

$$\varphi : V^3 \longrightarrow \mathbb{R}$$

by

$$\varphi(x, y, z) = x_1 y_2 z_3.$$

For

$$x = (1, 2, 3), \quad y = (2, 4, 1), \quad z = (3, 1, 5),$$

we obtain

$$\varphi(x, y, z) = 1 \cdot 4 \cdot 5 = 20.$$

Since φ is linear in each argument, it is a Multilinear Form.

7.4.18 Tensor

A Tensor is an element of a tensor product space representing multilinear relationships among vectors, covectors, or other tensorial components.

Definition 7.4.34 (Tensor)**.** Let $V_1, \ldots, V_n$ be vector spaces over a field $\mathbb{K}$. A *tensor* is an element

$$T \in V_1 \otimes_{\mathbb{K}} \cdots \otimes_{\mathbb{K}} V_n.$$

A tensor of the form

$$v_1 \otimes \cdots \otimes v_n$$

is called a simple or decomposable tensor.

Example 7.4.35 (Tensor)**.** Let

$$V = W = \mathbb{R}^2$$

with standard basis

$$e_1 = (1, 0), \qquad e_2 = (0, 1).$$

Then

$$T = e_1 \otimes e_2 \in V \otimes_{\mathbb{R}} W$$

is a simple tensor. Thus, T is a Tensor in

$$\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{R}^2.$$

7.4.19 Tensor Algebra

A Tensor Algebra combines tensor powers of a vector space into a graded algebra under multiplication induced by tensor products (cf. [588]).

Definition 7.4.36 (Tensor Algebra)**.** Let V be a vector space over a field $\mathbb{K}$. The *Tensor Algebra* of V is

$$T(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n},$$

where

$$V^{\otimes 0} = \mathbb{K}.$$

Multiplication is induced by the tensor product

$$V^{\otimes p} \times V^{\otimes q} \longrightarrow V^{\otimes (p+q)}.$$

Example 7.4.37 (Tensor Algebra)**.** Let

$$V = \mathbb{R}e$$

be a one-dimensional real vector space. Its tensor algebra is

$$T(V) = \mathbb{R} \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \cdots .$$

For example,

$$e \cdot (e \otimes e) = e \otimes e \otimes e \in V^{\otimes 3}.$$

Hence $T(V)$ is a Tensor Algebra.

7.4.20 Multigraded Algebra

A Multigraded Algebra decomposes into components indexed by multiple degrees, with multiplication sending homogeneous components into components of summed degree (cf. [589, 590]).

Definition 7.4.38 (Multigraded Algebra)**.** Let $\mathbb{K}$ be a field and let $r \geq 1$. A *Multigraded Algebra* is a $\mathbb{K}$-algebra A admitting a decomposition

$$A = \bigoplus_{\alpha \in \mathbb{N}^r} A_\alpha$$

such that

$$A_\alpha A_\beta \subseteq A_{\alpha+\beta}$$

for all $\alpha, \beta \in \mathbb{N}^r$.

Example 7.4.39 (Multigraded Algebra)**.** Let

$$A = \mathbb{K}[x, y].$$

Define

$$A_{p,q} = \mathbb{K}x^p y^q, \qquad (p, q) \in \mathbb{N}^2.$$

Then

$$A = \bigoplus_{(p,q) \in \mathbb{N}^2} A_{p,q}.$$

Moreover,

$$(x^p y^q)(x^r y^s) = x^{p+r} y^{q+s} \in A_{p+r,q+s}.$$

Therefore, $\mathbb{K}[x, y]$ is a Multigraded Algebra.

7.4.21 Multicategory

A Multicategory generalizes a category by allowing morphisms with multiple input objects and one output object, together with associative composition (cf. [591–593]).

Definition 7.4.40 (Multicategory)**.** A *Multicategory* $\mathcal{C}$ consists of a class of objects and, for objects $X_1, \ldots, X_n, Y$, sets of multimorphisms

$$\mathcal{C}(X_1, \ldots, X_n; Y),$$

together with identity multimorphisms and associative, unital composition of multimorphisms.

Example 7.4.41 (Multicategory)**.** Let the objects be vector spaces over a field $\mathbb{K}$, and define

$$\mathcal{C}(V_1, \ldots, V_n; W)$$

to be the set of all multilinear maps

$$V_1 \times \cdots \times V_n \longrightarrow W.$$

For example, the map

$$f : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}, \qquad f(x, y) = xy,$$

is a multimorphism

$$f \in \mathcal{C}(\mathbb{R}, \mathbb{R}; \mathbb{R}).$$

Composition of multilinear maps gives the required multicategory composition. Hence this forms a Multicategory.

7.4.22 n-ary Semigroup

An n-ary Semigroup is a nonempty set with an associative operation combining n elements into one element of the set [594–597].

Definition 7.4.42 (n-ary Semigroup)**.** Let $n \geq 2$. An *n-ary Semigroup* is a nonempty set S equipped with an n-ary operation

$$f : S^n \longrightarrow S$$

satisfying generalized associativity; namely, for all $x_1, \dots, x_{2n-1} \in S$ and $1 \leq i, j \leq n$,

$$\begin{aligned} &f(x_1, \dots, x_{i-1}, f(x_i, \dots, x_{i+n-1}), x_{i+n}, \dots, x_{2n-1}) \\ &\quad = f(x_1, \dots, x_{j-1}, f(x_j, \dots, x_{j+n-1}), x_{j+n}, \dots, x_{2n-1}). \end{aligned}$$

Example 7.4.43 (n-ary Semigroup)**.** Let

$$S = \mathbb{N}$$

and define

$$f : S^n \longrightarrow S$$

by

$$f(x_1, \dots, x_n) = x_1 + \cdots + x_n.$$

Any nesting of f on

$$x_1, \dots, x_{2n-1}$$

gives

$$x_1 + x_2 + \cdots + x_{2n-1}.$$

Therefore, f satisfies generalized associativity, and

$$(\mathbb{N}, f)$$

is an n-ary Semigroup.

7.4.23 n-ary Group

An n-ary Group is an associative n-ary system where equations are uniquely solvable in any argument when others are fixed [598–600].

Definition 7.4.44 (n-ary Group)**.** An *n-ary Group* is an n-ary semigroup (G, f) such that, for every $i \in \{1, \dots, n\}$, the equation

$$f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n) = b$$

has a unique solution $x \in G$ for arbitrary

$$a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n, b \in G.$$

Example 7.4.45 (n-ary Group)**.** Let

$$G = \mathbb{Z}$$

and define

$$f(x_1, \dots, x_n) = x_1 + \cdots + x_n.$$

This operation is n-ary associative. Moreover, the equation

$$f(a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n) = b$$

has the unique solution

$$x = b - \sum_{j \neq i} a_j.$$

Hence

$$(\mathbb{Z}, f)$$

is an n-ary Group.

7.4.24 Iterated Multiary Semigroup (Proposed in this book)

An Iterated Multiary Semigroup recursively composes an n-ary associative operation, producing higher-arity associative operations on the same underlying set systematically.

Definition 7.4.46 (Iterated Multiary Semigroup)**.** Let (S, f) be an n-ary semigroup, where

$$f : S^n \longrightarrow S, \qquad n \geq 2.$$

Define recursively

$$f^{[1]} = f,$$

and, for $r \geq 1$,

$$f^{[r+1]}(x_1, \ldots, x_{(r+1)(n-1)+1}) = f\left(f^{[r]}(x_1, \ldots, x_{r(n-1)+1}), x_{r(n-1)+2}, \ldots, x_{(r+1)(n-1)+1}\right).$$

Then

$$f^{[r]} : S^{\,r(n-1)+1} \longrightarrow S$$

is called the r-th iterated multiary operation. The structure

$$\mathcal{S}^{(\infty)} = \left(S, \{f^{[r]}\}_{r\geq 1}\right)$$

is called an *Iterated Multiary Semigroup.*

Example 7.4.47 (Iterated Multiary Semigroup)**.** Let

$$S = \mathbb{N}$$

and let

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i.$$

Then its r-th iterated operation is

$$f^{[r]}(x_1, \ldots, x_{r(n-1)+1}) = \sum_{i=1}^{r(n-1)+1} x_i.$$

For example,

$$f^{[2]}(x_1, \ldots, x_{2n-1}) = x_1 + \cdots + x_{2n-1}.$$

Thus,

$$\left(\mathbb{N}, \{f^{[r]}\}_{r\geq 1}\right)$$

is an Iterated Multiary Semigroup.

7.4.25 Iterated Multiary Group (Proposed in this book)

An Iterated Multiary Group recursively composes an n-ary group operation, preserving unique solvability while generating higher-arity group operations systematically consistently.

Definition 7.4.48 (Iterated Multiary Group)**.** Let (G, f) be an n-ary group. For each $r \geq 1$, define the iterated operation

$$f^{[r]} : G^{\,r(n-1)+1} \longrightarrow G$$

recursively by

$$f^{[1]} = f$$

and

$$f^{[r+1]}(x_1, \ldots, x_{(r+1)(n-1)+1}) = f\left(f^{[r]}(x_1, \ldots, x_{r(n-1)+1}), x_{r(n-1)+2}, \ldots, x_{(r+1)(n-1)+1}\right).$$

If every equation

$$f^{[r]}(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_m) = b, \qquad m = r(n-1)+1,$$

has a unique solution $x \in G$ in each argument position, then

$$\mathcal{G}^{(\infty)} = \Big(G, \{f^{[r]}\}_{r\geq 1}\Big)$$

is called an *Iterated Multiary Group*.

Example 7.4.49 (Iterated Multiary Group)**.** Let

$$G = \mathbb{Z}$$

with the n-ary operation

$$f(x_1, \ldots, x_n) = x_1 + \cdots + x_n.$$

For every $r \geq 1$,

$$f^{[r]}(x_1, \ldots, x_m) = x_1 + \cdots + x_m, \qquad m = r(n-1)+1.$$

The equation

$$f^{[r]}(a_1, \ldots, a_{i-1}, x, a_{i+1}, \ldots, a_m) = b$$

has the unique solution

$$x = b - \sum_{j\neq i} a_j.$$

Hence

$$\Big(\mathbb{Z}, \{f^{[r]}\}_{r\geq 1}\Big)$$

is an Iterated Multiary Group.

7.4.26 S-MultiSpace

An S-Multispace combines multiple mathematical spaces, possibly carrying different structures, into one framework while preserving the individual nature of its component spaces and associated operations [601–604, 604–606].

Definition 7.4.50 (S-Multispace)**.** Let

$$(X_i, \mathcal{S}_i), \qquad i = 1, \ldots, n, \quad n \geq 2,$$

be mathematical spaces, where $\mathcal{S}_i$ denotes the structure defined on X_i. An *S-Multispace* is the structure

$$\mathcal{M} = \bigcup_{i=1}^{n} (X_i, \mathcal{S}_i),$$

whose underlying space is

$$X = \bigcup_{i=1}^{n} X_i.$$

Thus, an S-Multispace combines several spaces, possibly equipped with different mathematical structures, within a single framework.

Example 7.4.51 (S-Multispace)**.** Let

$$(X_1, \mathcal{S}_1) = (\mathbb{R}, \tau_{\mathrm{std}})$$

be the real line with its usual topology, and let

$$(X_2, \mathcal{S}_2) = (\mathbb{Z}, +)$$

be the additive group of integers. Then

$$\mathcal{M} = (\mathbb{R}, \tau_{\mathrm{std}}) \cup (\mathbb{Z}, +)$$

is an S-Multispace combining a topological space and a group structure.

7.4.27 S-MultiHyperSpace (Proposed in this book)

An S-MultiHyperSpace combines multiple hyper-structured spaces whose elements are nonempty subsets of underlying spaces, preserving distinct component structures within one unified multi-space framework and interaction.

Definition 7.4.52 (S-MultiHyperSpace)**.** Let

$$(X_i, \mathcal{S}_i), \qquad i = 1, \ldots, n, \quad n \geq 2,$$

be mathematical spaces. For each i, let

$$\mathcal{H}_i \subseteq \mathcal{P}(X_i) \setminus \{\varnothing\}$$

be a nonempty family of subsets of X_i, equipped with a structure $\mathcal{S}_i^H$ induced or defined on $\mathcal{H}_i$. An *S-MultiHyperSpace* is the structure

$$\mathcal{M}_H = \bigcup_{i=1}^{n} (\mathcal{H}_i, \mathcal{S}_i^H).$$

Thus, each component of an S-MultiHyperSpace consists of hyper-elements represented by nonempty subsets of an underlying space.

Example 7.4.53 (S-MultiHyperSpace)**.** Let

$$X_1 = \{a, b\}, \qquad X_2 = \{1, 2, 3\},$$

and define

$$\mathcal{H}_1 = \big\{\{a\}, \{a, b\}\big\},$$
$$\mathcal{H}_2 = \big\{\{1, 2\}, \{2, 3\}\big\}.$$

Equip both families with the inclusion relation $\subseteq$. Then

$$\mathcal{M}_H = (\mathcal{H}_1, \subseteq) \cup (\mathcal{H}_2, \subseteq)$$

is an S-MultiHyperSpace whose elements are nonempty subsets of the underlying spaces.

7.4.28 S-MultiSuperHyperSpace (Proposed in this book)

An S-MultiSuperHyperSpace combines multiple recursively nested superhyper spaces, allowing higher-order collections of underlying elements while preserving distinct component structures within a unified multi-space framework hierarchically.

Definition 7.4.54 (S-MultiSuperHyperSpace)**.** Let

$$(X_i, \mathcal{S}_i), \qquad i = 1, \ldots, n, \quad n \geq 2,$$

be mathematical spaces, and define recursively

$$\mathcal{P}^1(X_i) = \mathcal{P}(X_i), \qquad \mathcal{P}^{r+1}(X_i) = \mathcal{P}\big(\mathcal{P}^r(X_i)\big).$$

For an integer $r \geq 2$, let

$$\mathcal{SH}_i^{(r)} \subseteq \mathcal{P}^r(X_i)$$

be a nonempty family equipped with a corresponding structure $\mathcal{S}_i^{SH}$. An *S-MultiSuperHyperSpace of level* r is

$$\mathcal{M}_{SH}^{(r)} = \bigcup_{i=1}^{n} \left(\mathcal{SH}_i^{(r)}, \mathcal{S}_i^{SH}\right).$$

Thus, its components may consist of recursively nested collections of elements of the underlying spaces.

Example 7.4.55 (S-MultiSuperHyperSpace). Let

$$X_1 = \{a, b\}, \qquad X_2 = \{1, 2\}.$$

For level $r = 2$, define

$$\mathcal{SH}_1^{(2)} = \{\{\{a\}, \{a, b\}\}\} \subseteq \mathcal{P}^2(X_1),$$

and

$$\mathcal{SH}_2^{(2)} = \{\{\{1\}, \{1, 2\}\}\} \subseteq \mathcal{P}^2(X_2).$$

Equip the components with the inclusion structure. Then

$$\mathcal{M}_{SH}^{(2)} = (\mathcal{SH}_1^{(2)}, \subseteq) \cup (\mathcal{SH}_2^{(2)}, \subseteq)$$

is an S-MultiSuperHyperSpace of level 2.

7.4.29 S-Iterated Multispace (Proposed in this book)

An S-Iterated Multispace recursively combines S-Multispaces from preceding levels, producing hierarchical spaces whose components may themselves consist of multiple structured mathematical spaces at deeper levels.

Definition 7.4.56 (S-Iterated Multispace). Let $\mathfrak{S}^{(0)}$ be a nonempty class of mathematical spaces. Define recursively, for $r \geq 0$,

$$\mathfrak{S}^{(r+1)} = \left\{ \bigcup_{i=1}^{n} \mathcal{X}_i \;\middle|\; n \geq 2,\ \mathcal{X}_i \in \mathfrak{S}^{(r)} \right\}.$$

An element

$$\mathcal{M}^{(r)} \in \mathfrak{S}^{(r)}, \qquad r \geq 1,$$

is called an *S-Iterated Multispace of level r*.

Thus, each level recursively combines several multispaces constructed at the preceding level.

Example 7.4.57 (S-Iterated Multispace). Let the level-0 spaces be

$$\mathcal{X}_1 = (\mathbb{R}, +), \qquad \mathcal{X}_2 = (\mathbb{R}, \tau_{\mathrm{std}}),$$

$$\mathcal{X}_3 = (\mathbb{Z}, +), \qquad \mathcal{X}_4 = (\mathbb{Z}, \leq).$$

Define two level-1 S-Multispaces by

$$\mathcal{M}_1 = \mathcal{X}_1 \cup \mathcal{X}_2, \qquad \mathcal{M}_2 = \mathcal{X}_3 \cup \mathcal{X}_4.$$

Then

$$\mathcal{M}^{(2)} = \mathcal{M}_1 \cup \mathcal{M}_2$$

is an S-Iterated Multispace of level 2.

7.4.30 Euclidean Multispace (Proposed in this book)

A Euclidean Multispace combines several Euclidean spaces, possibly of different dimensions, within one framework while preserving their individual Euclidean structures.

Definition 7.4.58 (Euclidean Multispace)**.** Let

$$\mathbb{R}^{n_i}, \qquad i = 1, \ldots, m, \quad m \geq 2,$$

be Euclidean spaces, possibly of different dimensions. A *Euclidean Multispace* is the union

$$\mathcal{E} = \bigcup_{i=1}^{m} \mathbb{R}^{n_i},$$

where each component $\mathbb{R}^{n_i}$ is equipped with its standard Euclidean metric

$$d_i(x, y) = \left(\sum_{k=1}^{n_i} (x_k - y_k)^2 \right)^{1/2}.$$

Thus, a Euclidean Multispace combines several Euclidean spaces within a single multi-space framework.

Example 7.4.59 (Euclidean Multispace)**.** Consider

$$\mathcal{E} = \mathbb{R}^2 \cup \mathbb{R}^3.$$

On $\mathbb{R}^2$, use

$$d_1(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

and on $\mathbb{R}^3$, use

$$d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

Thus,

$$\mathcal{E} = (\mathbb{R}^2, d_1) \cup (\mathbb{R}^3, d_2)$$

is a Euclidean Multispace.

7.4.31 Baire Multispace (Proposed in this book)

A Baire Multispace combines multiple Baire spaces within one framework, while each component preserves the Baire property under its topology.

Definition 7.4.60 (Baire Multispace)**.** Let

$$(X_i, \tau_i), \qquad i = 1, \ldots, m, \quad m \geq 2,$$

be Baire spaces. A *Baire Multispace* is the multi-space

$$\mathcal{B} = \bigcup_{i=1}^{m} (X_i, \tau_i),$$

where each component (X_i, τ_i) is a Baire space; that is, every countable intersection of open dense subsets of X_i is dense in X_i.

Example 7.4.61 (Baire Multispace)**.** Consider

$$\mathcal{B} = (\mathbb{R}, \tau_{\text{std}}) \cup (\mathbb{R}^2, \tau_{\text{std}}).$$

Both

$$\mathbb{R} \qquad \text{and} \qquad \mathbb{R}^2$$

with their usual topologies are complete metrizable spaces and hence Baire spaces. Therefore, $\mathcal{B}$ is a Baire Multispace.

7.4.32 Multi-Metric Space

A Multi-Metric Space combines several metric spaces, each carrying its own metric, within one unified framework for representing distance structures.

Definition 7.4.62 (Multi-Metric Space)**.** Let

$$\widetilde{M} = \bigcup_{i=1}^{m} M_i, \qquad m \geq 2,$$

where each

$$(M_i, \rho_i)$$

is a metric space. Then

$$\widetilde{M} = \bigcup_{i=1}^{m} (M_i, \rho_i)$$

is called a *Multi-Metric Space*.

Example 7.4.63 (Multi-Metric Space)**.** Let

$$M_1 = \mathbb{R}, \qquad M_2 = \mathbb{R}^2,$$

with metrics

$$\rho_1(x, y) = |x - y|$$

and

$$\rho_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

respectively. Then

$$\widetilde{M} = (M_1, \rho_1) \cup (M_2, \rho_2)$$

is a Multi-Metric Space.

7.4.33 MultiGroup

A MultiGroup combines several group structures together with specified compatibility relations between their operations (cf. [607]).

Definition 7.4.64 (MultiGroup)**.** Let

$$\mathcal{G} = \{(G_i, \circ_i) \mid 1 \leq i \leq n\}, \qquad n \geq 2,$$

be a family of groups, and let

$$\mathcal{D} \subseteq \{(i, j) \mid 1 \leq i, j \leq n,\ i \neq j\}$$

be a set of ordered pairs specifying which operations are required to satisfy cross-distributive laws.

The family $\mathcal{G}$ is called a *MultiGroup* if, for every $(i, j) \in \mathcal{D}$,

$$x \circ_i (y \circ_j z) = (x \circ_i y) \circ_j (x \circ_i z)$$

and

$$(y \circ_j z) \circ_i x = (y \circ_i x) \circ_j (z \circ_i x)$$

whenever all the expressions involved are defined.

Thus, the ordered pair $(i, j) \in \mathcal{D}$ means that the operation $\circ_i$ distributes over the operation $\circ_j$.

Example 7.4.65 (MultiGroup)**.** Consider the field

$$\mathbb{F}_3 = \{0, 1, 2\}.$$

Let

$$G_1 = (\mathbb{F}_3, +)$$

be its additive group and

$$G_2 = (\mathbb{F}_3^\times, \cdot), \qquad \mathbb{F}_3^\times = \{1, 2\},$$

be its multiplicative group.

Take

$$\mathcal{G} = \{G_1, G_2\}$$

and specify the compatibility direction

$$\mathcal{D} = \{(2, 1)\},$$

meaning that multiplication is required to distribute over addition whenever all the expressions involved are defined.

In particular, for

$$x, y, z \in \mathbb{F}_3^\times$$

such that

$$y + z \in \mathbb{F}_3^\times,$$

the distributive laws inherited from the field $\mathbb{F}_3$ give

$$x \cdot (y + z) = x \cdot y + x \cdot z$$

and

$$(y + z) \cdot x = y \cdot x + z \cdot x.$$

For example, take

$$x = 2, \qquad y = z = 1.$$

Since

$$1 + 1 = 2 \in \mathbb{F}_3^\times,$$

all the relevant expressions are defined, and

$$2 \cdot (1 + 1) = 2 \cdot 2 = 4 \equiv 1 \pmod 3,$$

while

$$2 \cdot 1 + 2 \cdot 1 = 2 + 2 = 4 \equiv 1 \pmod 3.$$

Similarly,

$$(1 + 1) \cdot 2 = 2 \cdot 2 \equiv 1 \pmod 3,$$

and

$$1 \cdot 2 + 1 \cdot 2 = 2 + 2 \equiv 1 \pmod 3.$$

Hence

$$\mathcal{G} = \{(\mathbb{F}_3, +), (\mathbb{F}_3^\times, \cdot)\}$$

provides a simple example of a MultiGroup with multiplication distributing over addition whenever the cross-operations are defined.

7.4.34 Multi-Ring

A Multi-Ring combines several ring structures together with compatible associativity and distributivity conditions between their labeled operations (cf. [608]).

Definition 7.4.66 (Multi-Ring)**.** Let

$$\mathcal{R} = \{(R_i, +_i, \times_i) \mid 1 \leq i \leq m\}, \qquad m \geq 2,$$

be a family of rings.

The family $\mathcal{R}$ is called a *Multi-Ring* if, for distinct indices i and j, the cross-operations satisfy

$$(x +_i y) +_j z = x +_i (y +_j z),$$

$$(x \times_i y) \times_j z = x \times_i (y \times_j z),$$

together with the left and right distributive laws

$$x \times_i (y +_j z) = (x \times_i y) +_j (x \times_i z),$$

and

$$(y +_j z) \times_i x = (y \times_i x) +_j (z \times_i x),$$

whenever all the expressions involved are defined.

Example 7.4.67 (Multi-Ring)**.** Let

$$R_1 = R_2 = \mathbb{Z},$$

and consider two labeled copies of the ordinary ring structure:

$$+_1 = +_2 = +, \qquad \times_1 = \times_2 = \cdot.$$

Then

$$(R_1, +_1, \times_1) \qquad \text{and} \qquad (R_2, +_2, \times_2)$$

are rings.

Since the labeled operations coincide with ordinary addition and multiplication on $\mathbb{Z}$, the cross-associativity identities become

$$(x + y) + z = x + (y + z),$$

and

$$(xy)z = x(yz).$$

Likewise, the cross-distributive identities become

$$x(y + z) = xy + xz$$

and

$$(y + z)x = yx + zx.$$

All these identities hold for every

$$x, y, z \in \mathbb{Z}.$$

Therefore,

$$\mathcal{R} = \{(\mathbb{Z}, +_1, \times_1), (\mathbb{Z}, +_2, \times_2)\}$$

is a simple labeled example of a Multi-Ring.

7.4.35 n-Ary Algebra

An n-ary algebra generalizes an ordinary binary algebra by replacing the binary product with an n-linear operation satisfying generalized associativity [609–611].

Definition 7.4.68 (n-Ary Algebra)**.** Let k be a commutative ring and let A be a k-module. For an integer $n \geq 2$, an *n-ary algebra* is a pair

$$(A, \mu),$$

where

$$\mu : A^n \longrightarrow A$$

is an n-linear operation satisfying

$$\begin{aligned}
&\mu\big(\mu(x_1, \dots, x_n), x_{n+1}, \dots, x_{2n-1}\big) \\
&= \mu\big(x_1, \mu(x_2, \dots, x_{n+1}), x_{n+2}, \dots, x_{2n-1}\big) \\
&= \cdots = \mu\big(x_1, \dots, x_{n-1}, \mu(x_n, \dots, x_{2n-1})\big)
\end{aligned}$$

for all

$$x_1, \dots, x_{2n-1} \in A.$$

This condition is called *n-ary associativity.* For $n = 2$, the definition reduces to that of an ordinary associative algebra.

Example 7.4.69 (3-Ary Algebra)**.** Let

$$A = \mathbb{R}$$

and define

$$\mu : A^3 \longrightarrow A$$

by

$$\mu(x, y, z) = xyz.$$

The map μ is trilinear. Moreover, for all $x_1, \dots, x_5 \in \mathbb{R}$,

$$\mu(\mu(x_1, x_2, x_3), x_4, x_5) = x_1x_2x_3x_4x_5,$$

$$\mu(x_1, \mu(x_2, x_3, x_4), x_5) = x_1x_2x_3x_4x_5,$$

and

$$\mu(x_1, x_2, \mu(x_3, x_4, x_5)) = x_1x_2x_3x_4x_5.$$

Hence

$$(\mathbb{R}, \mu)$$

is a 3-ary algebra.

7.4.36 n-Ary Hypergroup

An n-ary hypergroup generalizes an ordinary hypergroup by replacing the binary hyperoperation with an n-ary set-valued operation satisfying generalized associativity and solvability in every argument [612–615].

Definition 7.4.70 (n-Ary Hypergroup)**.** [612, 613] Let $H \neq \varnothing$ and let

$$f : H^n \longrightarrow \mathcal{P}^*(H)$$

be an n-ary hyperoperation, where

$$\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\varnothing\}.$$

The pair (H, f) is called an *n-ary hypergroup* if the following conditions hold.

First, f is associative; that is, for every

$$x_1, \dots, x_{2n-1} \in H$$

and every $i, j \in \{1, \dots, n\}$,

$$\begin{aligned} &f\big(x_1, \dots, x_{i-1}, f(x_i, \dots, x_{i+n-1}), x_{i+n}, \dots, x_{2n-1}\big) \\ &= f\big(x_1, \dots, x_{j-1}, f(x_j, \dots, x_{j+n-1}), x_{j+n}, \dots, x_{2n-1}\big). \end{aligned}$$

Second, for every $i \in \{1, \dots, n\}$ and every

$$a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n, b \in H,$$

there exists $x_i \in H$ such that

$$b \in f(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_n).$$

Thus, every position of the n-ary hyperoperation is solvable with respect to any prescribed output element.

For nonempty subsets $A_1, \dots, A_n \subseteq H$, extend f by

$$f(A_1, \dots, A_n) = \bigcup_{a_i \in A_i} f(a_1, \dots, a_n).$$

Example 7.4.71 (3-Ary Hypergroup)**.** Let

$$H = \mathbb{Z}_3 = \{0, 1, 2\}$$

and define

$$f : H^3 \longrightarrow \mathcal{P}^*(H)$$

by

$$f(x, y, z) = \{\, (x + y + z) \bmod 3 \,\}.$$

For example,

$$f(1, 2, 2) = \{2\}.$$

Since addition modulo 3 is associative,

$$f(f(x_1, x_2, x_3), x_4, x_5) = f(x_1, f(x_2, x_3, x_4), x_5) = f(x_1, x_2, f(x_3, x_4, x_5)).$$

Furthermore, each argument is solvable. For instance, given $a, b, c \in \mathbb{Z}_3$, the equation

$$c \in f(a, x, b)$$

has the solution

$$x \equiv c - a - b \pmod 3.$$

Therefore,

$$(H, f)$$

is a 3-ary hypergroup.

7.4.37 n-ary Fuzzy Relation

An n-ary fuzzy relation assigns each tuple from a Cartesian product a membership degree in $[0,1]$, representing graded relational strength among n components simultaneously [616, 617].

Definition 7.4.72 (n-ary Fuzzy Relation)**.** Let $X_1,\dots,X_n$ be nonempty sets. An *n-ary Fuzzy Relation* is a fuzzy subset of

$$X_1 \times \cdots \times X_n,$$

represented by a membership function

$$R : X_1 \times \cdots \times X_n \longrightarrow [0,1].$$

For each $(x_1,\dots,x_n)$, the value

$$R(x_1,\dots,x_n)$$

denotes the degree to which the tuple satisfies the relation.

Example 7.4.73 (3-ary Fuzzy Relation)**.** Let

$$X_1 = X_2 = X_3 = [0,1].$$

Define

$$R : X_1 \times X_2 \times X_3 \longrightarrow [0,1]$$

by

$$R(x,y,z) = \min\{x,y,z\}.$$

For example,

$$R(0.8, 0.6, 0.9) = 0.6.$$

Thus, the tuple

$$(0.8, 0.6, 0.9)$$

belongs to the fuzzy relation with membership degree 0.6. Hence R is a 3-ary Fuzzy Relation.

7.4.38 n-ary Function

An n-ary function maps each ordered n-tuple of input values to exactly one output value, generalizing unary and binary functions to multiple arguments within a specified codomain (cf. [408, 618, 619]).

Definition 7.4.74 (n-ary Function)**.** Let $X_1,\dots,X_n$ and Y be nonempty sets. An *n-ary Function* is a mapping

$$f : X_1 \times \cdots \times X_n \longrightarrow Y.$$

Thus, f assigns a unique output

$$f(x_1,\dots,x_n) \in Y$$

to each n-tuple $(x_1,\dots,x_n)$.

Example 7.4.75 (3-ary Function)**.** Let

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R}$$

be defined by

$$f(x,y,z) = x + y + z.$$

For example,

$$f(1,2,3) = 6.$$

Thus, each ordered triple

$$(x,y,z) \in \mathbb{R}^3$$

is assigned exactly one real number. Therefore, f is a 3-ary Function.

7.4.39 MultiInterval-valued Fuzzy Set (Proposed in this book)

A MultiInterval-valued Fuzzy Set assigns each element a finite nonempty family of membership intervals, representing several admissible fuzzy degrees or evidence ranges simultaneously within $[0,1]$.

Let
$$\mathcal{L} = \{[\alpha, \beta] \subseteq [0,1] \mid 0 \leq \alpha \leq \beta \leq 1\}.$$

Definition 7.4.76 (MultiInterval-valued Fuzzy Set)**.** Let $X \neq \varnothing$. A *MultiInterval-valued Fuzzy Set* (MIVFS) on X is a mapping
$$A : X \longrightarrow \mathcal{P}^*_{\mathrm{fin}}(\mathcal{L}),$$
where
$$\mathcal{P}^*_{\mathrm{fin}}(\mathcal{L}) = \{S \subseteq \mathcal{L} \mid S \neq \varnothing,\ |S| < \infty\}.$$
Thus, each $x \in X$ is assigned a finite nonempty family
$$A(x) = \{[\alpha_j(x), \beta_j(x)] \mid j \in J_x\}$$
of admissible fuzzy membership intervals.

Example 7.4.77 (MultiInterval-valued Fuzzy Set)**.** Let
$$X = \{x_1, x_2\}.$$
Define
$$A : X \longrightarrow \mathcal{P}^*_{\mathrm{fin}}(\mathcal{L})$$
by
$$A(x_1) = \{[0.2, 0.4], [0.6, 0.8]\},$$
and
$$A(x_2) = \{[0.3, 0.5], [0.7, 0.9]\}.$$
Each $A(x_i)$ is a finite nonempty family of intervals contained in $[0,1]$. Therefore,
$$A$$
is a MultiInterval-valued Fuzzy Set on X.

7.4.40 MultiInterval-valued Neutrosophic Set (Proposed in this book)

A MultiInterval-valued Neutrosophic Set assigns each element finite nonempty families of intervals for truth, indeterminacy, and falsity, representing multiple admissible neutrosophic evaluations simultaneously within $[0,1]$.

Definition 7.4.78 (MultiInterval-valued Neutrosophic Set)**.** Let $X \neq \varnothing$. A *MultiInterval-valued Neutrosophic Set* (MIVNS) on X is a triple
$$A = (T_A, I_A, F_A),$$
where
$$T_A, I_A, F_A : X \longrightarrow \mathcal{P}^*_{\mathrm{fin}}(\mathcal{L}).$$
Hence, each $x \in X$ is assigned finite nonempty families of intervals for its truth, indeterminacy, and falsity degrees, respectively.

Example 7.4.79 (MultiInterval-valued Neutrosophic Set)**.** Let

$$X = \{x\}.$$

Define

$$T_A(x) = \{[0.6, 0.8], [0.8, 0.9]\},$$
$$I_A(x) = \{[0.1, 0.2], [0.2, 0.3]\},$$

and

$$F_A(x) = \{[0.1, 0.3], [0.3, 0.4]\}.$$

Each component is a finite nonempty family of intervals in $[0, 1]$. Hence

$$A = (T_A, I_A, F_A)$$

is a MultiInterval-valued Neutrosophic Set on X.

7.4.41 Iterated MultiInterval-valued Fuzzy Set (Proposed in this book)

An Iterated MultiInterval-valued Fuzzy Set recursively combines MultiInterval-valued fuzzy membership structures, allowing each element to possess hierarchical collections of interval-based membership information across levels successively.

For $n \geq 1$, define recursively

$$\mathcal{M}_1 = \mathcal{P}^*_{\text{fin}}(\mathcal{L}),$$

and

$$\mathcal{M}_{n+1} = \bigcup_{m \geq 2} (\mathcal{M}_n)^m.$$

Definition 7.4.80 (Iterated MultiInterval-valued Fuzzy Set)**.** Let $X \neq \varnothing$ and $n \geq 1$. An *Iterated MultiInterval-valued Fuzzy Set of level* n is a mapping

$$A^{(n)} : X \longrightarrow \mathcal{M}_n.$$

Thus, $A^{(1)}$ is an ordinary MultiInterval-valued Fuzzy Set, whereas higher levels recursively combine several lower-level MultiInterval membership structures.

Example 7.4.81 (Iterated MultiInterval-valued Fuzzy Set)**.** Consider level 2. Let

$$X = \{x_1, x_2\}$$

and define the following elements of $\mathcal{M}_1$:

$$H_1 = \{[0.2, 0.4], [0.5, 0.7]\},$$
$$H_2 = \{[0.3, 0.6]\},$$

and

$$H_3 = \{[0.6, 0.8], [0.8, 0.9]\}.$$

Define

$$A^{(2)} : X \longrightarrow \mathcal{M}_2$$

by

$$A^{(2)}(x_1) = (H_1, H_2),$$

and

$$A^{(2)}(x_2) = (H_2, H_3).$$

Since

$$H_1, H_2, H_3 \in \mathcal{M}_1,$$

we have

$$(H_1, H_2), (H_2, H_3) \in \mathcal{M}_2.$$

Therefore,

$$A^{(2)}$$

is an Iterated MultiInterval-valued Fuzzy Set of level 2.

7.4.42 Iterated MultiInterval-valued Neutrosophic Set (Proposed in this book)

An Iterated MultiInterval-valued Neutrosophic Set recursively organizes truth, indeterminacy, and falsity MultiInterval structures into hierarchical levels, enabling nested neutrosophic uncertainty representation for each element systematically.

Definition 7.4.82 (Iterated MultiInterval-valued Neutrosophic Set)**.** Let $X \neq \varnothing$ and $n \geq 1$. An *Iterated MultiInterval-valued Neutrosophic Set of level* n is a triple

$$A^{(n)} = \left(T_A^{(n)}, I_A^{(n)}, F_A^{(n)}\right),$$

where

$$T_A^{(n)}, I_A^{(n)}, F_A^{(n)} : X \longrightarrow \mathcal{M}_n.$$

Thus, the truth, indeterminacy, and falsity components are represented by recursively nested Multi-Interval structures of level n.

Example 7.4.83 (Iterated MultiInterval-valued Neutrosophic Set)**.** Consider level 2 and let

$$X = \{x\}.$$

Define

$$H_1 = \{[0.6, 0.8], [0.8, 0.9]\}, \qquad H_2 = \{[0.4, 0.6]\},$$
$$H_3 = \{[0.1, 0.3], [0.3, 0.4]\}, \qquad H_4 = \{[0.2, 0.5]\},$$

where

$$H_1, H_2, H_3, H_4 \in \mathcal{M}_1.$$

Define

$$T_A^{(2)}(x) = (H_1, H_2),$$
$$I_A^{(2)}(x) = (H_2, H_4),$$

and

$$F_A^{(2)}(x) = (H_3, H_4).$$

Since each ordered pair belongs to

$$\mathcal{M}_2,$$

the triple

$$A^{(2)} = \left(T_A^{(2)}, I_A^{(2)}, F_A^{(2)}\right)$$

is an Iterated MultiInterval-valued Neutrosophic Set of level 2.

7.4.43 Multipartite Molecular Graph (Proposed in this book)

A Multipartite Molecular Graph is a molecular graph whose atoms are partitioned into several disjoint classes, with bonds occurring only between atoms from different classes.

Definition 7.4.84 (Multipartite Molecular Graph)**.** Let

$$G = (V, E)$$

be a molecular graph. The graph G is called a *k-partite Molecular Graph*, where $k \geq 2$, if its vertex set can be partitioned as

$$V = V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k,$$

where each $V_i \neq \varnothing$, and

$$\{u, v\} \in E \quad \Longrightarrow \quad u \in V_i,\ v \in V_j \quad \text{for some } i \neq j.$$

Thus, no chemical bond joins two atoms belonging to the same part. A molecular graph admitting such a partition is called a *Multipartite Molecular Graph*.

Example 7.4.85 (Multipartite Molecular Graph)**.** Consider the molecular graph of carbon dioxide,

$$\mathrm{O}_1 = \mathrm{C} = \mathrm{O}_2.$$

Let

$$V = \{\mathrm{C}, \mathrm{O}_1, \mathrm{O}_2\}$$

and

$$E = \big\{\{\mathrm{C}, \mathrm{O}_1\}, \{\mathrm{C}, \mathrm{O}_2\}\big\}.$$

Define the partition

$$V_1 = \{\mathrm{C}\}, \qquad V_2 = \{\mathrm{O}_1\}, \qquad V_3 = \{\mathrm{O}_2\}.$$

Then

$$V = V_1 \sqcup V_2 \sqcup V_3,$$

and every edge joins vertices belonging to different parts. Therefore, the molecular graph

$$G = (V, E)$$

is a 3-partite Molecular Graph.

7.4.44 Molecular MultiGraph

A Molecular MultiGraph represents a molecule by allowing multiple edges between the same pair of atoms, thereby modeling multiple bonds explicitly.

Definition 7.4.86 (Molecular MultiGraph)**.** A *Molecular MultiGraph* is a multigraph

$$G = (V, E, \mu),$$

where V is the set of atoms, E is the set of unordered pairs of atoms connected by chemical bonds, and

$$\mu : E \longrightarrow \mathbb{N}$$

assigns to each adjacent pair its bond multiplicity.

Thus,

$$\mu(\{u, v\}) = 1, 2, 3, \ldots$$

may represent a single, double, triple, or higher-order bond between the atoms u and v.

Example 7.4.87 (Molecular MultiGraph)**.** Consider the carbon monoxide molecule

$$\mathrm{C} \equiv \mathrm{O},$$

whose carbon and oxygen atoms are connected by a triple bond.

Let

$$V = \{\mathrm{C}, \mathrm{O}\}, \qquad E = \big\{\{\mathrm{C}, \mathrm{O}\}\big\},$$

and define

$$\mu : E \longrightarrow \mathbb{N}$$

by

$$\mu(\{\mathrm{C}, \mathrm{O}\}) = 3.$$

The value 3 represents the triple bond between the carbon and oxygen atoms. Hence,

$$G = (V, E, \mu)$$

is a Molecular MultiGraph.

Chapter 8

MetaStructures

This chapter discusses MetaStructures.

8.1 MetaStructures and Iterated MetaStructures

MetaStructures treat entire mathematical structures as objects and use meta-operations to combine them, producing new structures within a unified framework [620]. Iterated MetaStructures repeatedly apply the MetaStructure construction, creating hierarchical levels where structures themselves become objects of progressively higher recursive meta-structures.

Definition 8.1.1 (MetaStructure)**.** [620] Let Σ be a fixed mathematical signature, and let Str_Σ denote the class of all Σ-structures. A *MetaStructure* is a structure

$$\mathcal{M} = (\mathcal{U}, \{\Phi_\lambda\}_{\lambda\in\Lambda}),$$

where

$$\varnothing \neq \mathcal{U} \subseteq \mathrm{Str}_\Sigma$$

is a collection of mathematical structures, and each meta-operation

$$\Phi_\lambda : \mathcal{U}^{k_\lambda} \longrightarrow \mathcal{U}$$

maps a finite tuple of structures to another structure in $\mathcal{U}$. Thus, a MetaStructure treats entire mathematical structures as objects and combines them through meta-level operations.

Definition 8.1.2 (Iterated MetaStructure)**.** Let Σ be a fixed mathematical signature. Define recursively

$$\mathcal{U}^{(0)} = \mathrm{Str}_\Sigma,$$

and, for $n \geq 0$, let

$$\mathcal{U}^{(n+1)}$$

be a nonempty collection of MetaStructures whose objects belong to $\mathcal{U}^{(n)}$.

An *n-th Iterated MetaStructure* is a MetaStructure

$$\mathcal{M}^{(n)} = \left(\mathcal{U}^{(n)}, \{\Phi_\lambda^{(n)}\}_{\lambda\in\Lambda}\right),$$

where each

$$\Phi_\lambda^{(n)} : \left(\mathcal{U}^{(n)}\right)^{k_\lambda} \longrightarrow \mathcal{U}^{(n)}$$

is a meta-operation at level n.

Thus, Iterated MetaStructures recursively form higher-order structures whose objects are themselves structures from the preceding meta-level.

8.2 Concepts for MetaStructures

This section presents several concepts related to MetaStructures.

8.2.1 Meta-Graph

A Meta-Graph is a graph whose vertices are themselves graphs, with edges representing prescribed relations between those constituent graphs directly [621–625].

Definition 8.2.1 (Meta-Graph)**.** Let $\mathfrak{G}$ be a nonempty collection of graphs and let $\mathcal{R}$ be a family of binary relations on $\mathfrak{G}$. A *Meta-Graph* is a graph

$$M = (V_M, E_M),$$

where

$$V_M \subseteq \mathfrak{G}.$$

Thus, each meta-vertex is itself a graph, and a meta-edge connects two graphs whenever they satisfy a prescribed relation in $\mathcal{R}$.

Example 8.2.2 (Meta-Graph)**.** Let

$$G_1 = (\{a, b\}, \{\{a, b\}\}), \qquad G_2 = (\{x, y, z\}, \{\{x, y\}, \{y, z\}, \{z, x\}\}).$$

Thus, G_1 is a graph consisting of a single edge, whereas G_2 is a triangle. Define

$$V_M = \{G_1, G_2\}, \qquad E_M = \big\{\{G_1, G_2\}\big\}.$$

Then

$$M = (V_M, E_M)$$

is a Meta-Graph whose meta-vertices are the graphs G_1 and G_2. A graphical representation is shown in Figure 8.1.

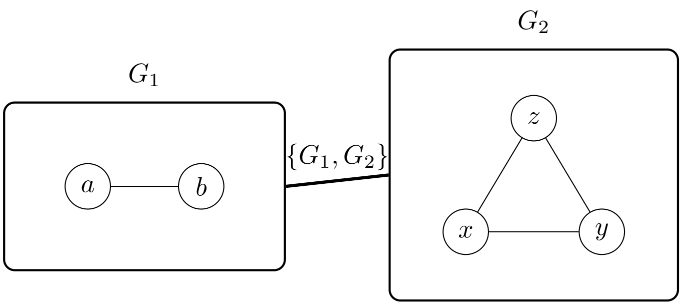


Figure 8.1: A Meta-Graph with two meta-vertices G_1 and G_2, where each meta-vertex is itself a graph.

8.2.2 Meta-HyperGraph

A Meta-HyperGraph is a hypergraph whose vertices are hypergraphs and whose meta-hyperedges can simultaneously connect multiple constituent hypergraphs within it [624].

Definition 8.2.3 (Meta-HyperGraph)**.** [624] Let $\mathfrak{H}$ be a nonempty collection of hypergraphs. A *Meta-HyperGraph* is a hypergraph

$$M_H = (V_M, E_M),$$

where

$$\varnothing \neq V_M \subseteq \mathfrak{H}, \qquad E_M \subseteq \mathcal{P}(V_M) \setminus \{\varnothing\}.$$

Thus, each meta-vertex is itself a hypergraph, and each meta-hyperedge may connect several hypergraphs simultaneously.

Example 8.2.4 (Meta-HyperGraph)**.** Let H_1, H_2, H_3 be three hypergraphs and set

$$V_M = \{H_1, H_2, H_3\}, \qquad E_M = \big\{\{H_1, H_2, H_3\}\big\}.$$

Then

$$M_H = (V_M, E_M)$$

is a Meta-HyperGraph having one meta-hyperedge that simultaneously connects the three hypergraphs.

8.2.3 Meta-SuperHyperGraph

A Meta-SuperHyperGraph is a SuperHyperGraph whose supervertices are nested collections of SuperHyperGraphs, enabling hierarchical relations among higher-order graph structures themselves [624].

Definition 8.2.5 (Meta-SuperHyperGraph)**.** [624] Let $\mathfrak{S}$ be a nonempty collection of SuperHyperGraphs and let $n \geq 1$. An *n-Meta-SuperHyperGraph* is a SuperHyperGraph

$$M_{SH}^{(n)} = (V_M, E_M),$$

where

$$V_M \subseteq \mathcal{P}^n(\mathfrak{S}), \qquad E_M \subseteq \mathcal{P}(V_M) \setminus \{\varnothing\}.$$

Hence, its supervertices are nested collections of SuperHyperGraphs, allowing hierarchical relations among higher-order graph structures.

Example 8.2.6 (Meta-SuperHyperGraph)**.** Let

$$\mathfrak{S} = \{S_1, S_2\},$$

where S_1 and S_2 are SuperHyperGraphs. For $n = 1$, define

$$V_M = \big\{\{S_1\}, \{S_2\}, \{S_1, S_2\}\big\} \subseteq \mathcal{P}(\mathfrak{S}),$$

and

$$E_M = \Big\{\big\{\{S_1\}, \{S_1, S_2\}\big\}\Big\}.$$

Then

$$M_{SH}^{(1)} = (V_M, E_M)$$

is a 1-Meta-SuperHyperGraph.

8.2.4 Iterated Meta-Graph

An Iterated Meta-Graph recursively forms graph hierarchies whose vertices are Meta-Graphs from preceding levels, creating progressively deeper graph-of-graphs structures recursively [624].

Definition 8.2.7 (Iterated Meta-Graph)**.** [624] Let $\mathfrak{G}^{(0)}$ be a nonempty collection of graphs and define recursively

$$\mathfrak{G}^{(t+1)} = \Big\{M \mid M \text{ is a Meta-Graph whose vertices belong to } \mathfrak{G}^{(t)}\Big\}.$$

For $t \geq 1$, an *Iterated Meta-Graph of depth t* is a graph

$$M^{(t)} = \Big(V^{(t)}, E^{(t)}\Big),$$

where

$$V^{(t)} \subseteq \mathfrak{G}^{(t)}.$$

Thus, its vertices are recursively constructed meta-graphs from the preceding hierarchical level.

Example 8.2.8 (Iterated Meta-Graph)**.** Let the ordinary graphs be

$$G_1 = \big(\{a, b\}, \{\{a, b\}\}\big),$$

$$G_2 = \big(\{c, d, e\}, \{\{c, d\}, \{d, e\}, \{e, c\}\}\big),$$

and

$$G_3 = \big(\{f, g, h\}, \{\{f, g\}, \{g, h\}\}\big).$$

Define two Meta-Graphs by

$$M_1 = (\{G_1, G_2\}, \{\{G_1, G_2\}\})$$

and

$$M_2 = (\{G_2, G_3\}, \{\{G_2, G_3\}\}) .$$

Now define

$$M^{(2)} = (\{M_1, M_2\}, \{\{M_1, M_2\}\}) .$$

Then $M^{(2)}$ is an Iterated Meta-Graph of depth 2, since its vertices M_1 and M_2 are themselves Meta-Graphs, whose vertices are ordinary graphs. A graphical representation is shown in Figure 8.2.

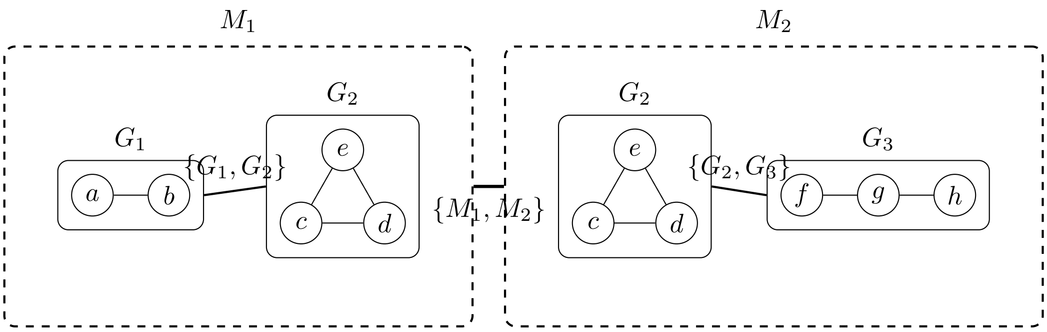


Figure 8.2: An Iterated Meta-Graph $M^{(2)}$ of depth 2. The second-level vertices M_1 and M_2 are Meta-Graphs, while their meta-vertices G_1, G_2, G_3 are ordinary graphs.

8.2.5 Category of Structures

A category of structures consists of mathematical structures as objects and structure-preserving maps as morphisms, equipped with associative composition and identity morphisms for every object [626].

Definition 8.2.9 (Category of Structures)**.** Let Σ be a fixed mathematical signature. A *Category of Structures*, denoted by

$$\mathbf{Str}_\Sigma,$$

is a category whose objects are Σ-structures and whose morphisms are structure-preserving maps between them. Thus,

$$\mathrm{Ob}(\mathbf{Str}_\Sigma) = \{\Sigma\text{-structures}\},$$

with associative composition of morphisms and an identity morphism for each object.

Example 8.2.10 (Category of Structures)**.** Consider the category

$$\mathbf{Grp}$$

of groups. Its objects include

$$\mathbb{Z}_2 \qquad \text{and} \qquad \mathbb{Z}_4,$$

and the map

$$f : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4, \qquad f([x]_2) = [2x]_4,$$

is a group homomorphism and hence a morphism in **Grp**. Thus, **Grp** is a Category of Structures.

8.2.6 2-Category

A 2-category consists of objects, morphisms between objects, and 2-morphisms between morphisms, equipped with associative vertical and horizontal compositions satisfying identity and interchange laws coherently [627–631].

Definition 8.2.11 (2-Category). [627–629] A *2-category* $\mathcal{C}$ consists of objects

$$A, B, C, \ldots,$$

1-morphisms

$$f : A \longrightarrow B,$$

and 2-morphisms

$$\alpha : f \Longrightarrow g$$

between parallel 1-morphisms $f, g : A \to B$. The 2-morphisms admit vertical and horizontal compositions satisfying associativity, identity, and interchange laws. Equivalently, for every pair of objects A, B, the collection

$$\mathcal{C}(A, B)$$

forms a category.

Example 8.2.12 (2-Category). The category

$$\mathbf{Cat}$$

forms a 2-category in which objects are categories, 1-morphisms are functors

$$F : \mathcal{C} \longrightarrow \mathcal{D},$$

and 2-morphisms are natural transformations

$$\alpha : F \Longrightarrow G$$

between parallel functors $F, G : \mathcal{C} \to \mathcal{D}$.

8.2.7 Internal Category

An internal category is a category defined inside a category, using objects of objects and arrows, together with source, target, identity, and composition morphisms satisfying axioms [632–634].

Definition 8.2.13 (Internal Category). Let $\mathcal{C}$ be a category with the required pullbacks. An *Internal Category* in $\mathcal{C}$ consists of two objects

$$C_0, C_1 \in \mathrm{Ob}(\mathcal{C}),$$

together with source and target morphisms

$$s, t : C_1 \longrightarrow C_0,$$

an identity morphism

$$i : C_0 \longrightarrow C_1,$$

and a composition morphism

$$m : C_1 \times_{C_0} C_1 \longrightarrow C_1,$$

satisfying the usual associativity and identity axioms internally in $\mathcal{C}$.

Example 8.2.14 (Internal Category). Consider an internal category in **Set** with

$$C_0 = \{0, 1\}$$

and

$$C_1 = \{\mathrm{id}_0, \mathrm{id}_1, a\},$$

where

$$s(a) = 0, \qquad t(a) = 1.$$

The maps

$$s, t : C_1 \to C_0$$

give the source and target, while the identity and composition maps are defined by the usual categorical rules. Hence this is an Internal Category in **Set**.

8.2.8 System of Systems

A System of Systems is an integrated arrangement of independently functioning systems whose interactions, coordination, or dependencies produce capabilities beyond those of individual constituent systems [635–640].

Definition 8.2.15 (System of Systems)**.** [635] Let $\{S_i\}_{i\in I}$ be a family of systems. A *System of Systems* is a structure

$$\mathcal{S} = (\{S_i\}_{i\in I}, \mathcal{R}),$$

where each S_i is itself a system and

$$\mathcal{R} \subseteq \bigcup_{i,j\in I} S_i \times S_j$$

represents interactions, dependencies, or coordination relations among the constituent systems. Thus, the basic components of $\mathcal{S}$ are themselves systems.

Example 8.2.16 (System of Systems)**.** Let

$$S_1 = (X_1, R_1), \qquad X_1 = \{a, b\}, \qquad R_1 = \{(a, b)\},$$

and

$$S_2 = (X_2, R_2), \qquad X_2 = \{c, d\}, \qquad R_2 = \{(c, d)\}.$$

Define an inter-system relation

$$\mathcal{R} = \{(b, c)\}.$$

Then

$$\mathcal{S} = \bigl(\{S_1, S_2\}, \mathcal{R}\bigr)$$

is a simple System of Systems consisting of two interacting constituent systems.

8.2.9 Network of Networks

A Network of Networks consists of multiple constituent networks connected through inter-network links, allowing interactions among separate networks while preserving their internal vertices and edges [641–646].

Definition 8.2.17 (Network of Networks)**.** [646] Let

$$G_i = (V_i, E_i), \qquad i \in I,$$

be a family of networks. A *Network of Networks* is a structure

$$\mathcal{N} = (\{G_i\}_{i\in I}, E_{\text{inter}}),$$

where

$$E_{\text{inter}} \subseteq \bigcup_{\substack{i,j\in I\\ i\neq j}} (V_i \times V_j)$$

is the set of links connecting vertices belonging to different constituent networks.

The associated total network is

$$G = \left(\bigcup_{i\in I} V_i, \bigcup_{i\in I} E_i \cup E_{\text{inter}}\right).$$

Example 8.2.18 (Network of Networks)**.** Let

$$G_1 = (\{a, b\}, \{\{a, b\}\})$$

and

$$G_2 = (\{c, d\}, \{\{c, d\}\})\,.$$

Define the inter-network edge set by

$$E_{\text{inter}} = \{\{b, c\}\}.$$

Then

$$\mathcal{N} = (\{G_1, G_2\}, E_{\text{inter}})$$

is a Network of Networks. Its associated total network is

$$G = (\{a, b, c, d\}, \{\{a, b\}, \{b, c\}, \{c, d\}\})\,.$$

8.2.10 HyperMeta Graph (Proposed in this book)

A HyperMeta Graph is a graph whose vertices are nonempty collections of graphs, with edges representing relations between these collections.

Definition 8.2.19 (HyperMeta Graph)**.** Let $\mathfrak{G}$ be a nonempty collection of graphs, and define

$$\mathcal{P}^*(\mathfrak{G}) = \mathcal{P}(\mathfrak{G}) \setminus \{\varnothing\}.$$

A *HyperMeta Graph* is a graph

$$M_H = (V_H, E_H),$$

where

$$V_H \subseteq \mathcal{P}^*(\mathfrak{G}),$$

and

$$E_H \subseteq \big\{\{A, B\} \mid A, B \in V_H,\ A \neq B\big\}.$$

Thus, each hyper-meta-vertex is a nonempty collection of graphs, and edges represent prescribed relations between such collections.

Theorem 8.2.20 (Well-Definedness of a HyperMeta Graph)**.** *Let $\mathfrak{G}$ be a nonempty collection of graphs and let*

$$V_H \subseteq \mathcal{P}^*(\mathfrak{G}), \qquad \mathcal{P}^*(\mathfrak{G}) = \mathcal{P}(\mathfrak{G}) \setminus \{\varnothing\}.$$

Suppose

$$E_H \subseteq \big\{\{A, B\} \mid A, B \in V_H,\ A \neq B\big\}.$$

Then

$$M_H = (V_H, E_H)$$

is a well-defined simple graph whose vertices are nonempty collections of graphs.

Proof. Since

$$V_H \subseteq \mathcal{P}^*(\mathfrak{G}),$$

every element of V_H is a nonempty collection of graphs from $\mathfrak{G}$.

Moreover, every element of E_H has the form

$$\{A, B\}, \qquad A, B \in V_H, \quad A \neq B.$$

Hence each edge joins two distinct vertices of V_H, so no loops occur and all edges are valid two-element subsets of V_H.

Therefore,

$$M_H = (V_H, E_H)$$

is a well-defined simple graph, and hence a well-defined HyperMeta Graph. □

8.2.11 Meta-Fuzzy Set

A Meta-Fuzzy Set assigns membership degrees to entire fuzzy sets, enabling higher-level representation and analysis of collections of fuzzy information [647].

Definition 8.2.21 (Meta-Fuzzy Set)**.** [647] Let X be a nonempty set, and let

$$\mathrm{Fuz}(X) = \{\mu \mid \mu : X \to [0,1]\}$$

denote the collection of all fuzzy sets on X. A *Meta-Fuzzy Set* on X is a mapping

$$\mu^\sharp : \mathrm{Fuz}(X) \longrightarrow [0,1].$$

Thus, $\mu^\sharp$ assigns a membership degree to each fuzzy set rather than to each individual element of X.

Example 8.2.22 (Meta-Fuzzy Set)**.** Let

$$X = \{x_1, x_2\}.$$

Consider the fuzzy set $\mu \in \mathrm{Fuz}(X)$ given by

$$\mu(x_1) = 0.8, \qquad \mu(x_2) = 0.4.$$

Define the Meta-Fuzzy Set

$$\mu^\sharp : \mathrm{Fuz}(X) \longrightarrow [0,1]$$

by

$$\mu^\sharp(\nu) = \frac{\nu(x_1) + \nu(x_2)}{2}$$

for every $\nu \in \mathrm{Fuz}(X)$. Then

$$\mu^\sharp(\mu) = \frac{0.8 + 0.4}{2} = 0.6.$$

Thus, the fuzzy set μ itself has the meta-membership degree 0.6.

8.2.12 Meta-Neutrosophic Set

A Meta-Neutrosophic Set assigns truth, indeterminacy, and falsity degrees to entire neutrosophic sets, enabling higher-level evaluation of neutrosophic information structures [647].

Definition 8.2.23 (Meta-Neutrosophic Set)**.** [647] Let X be a nonempty set, and let

$$\mathrm{Neu}(X) = \{A = (T_A, I_A, F_A) \mid T_A, I_A, F_A : X \to [0,1]\}$$

be the collection of neutrosophic sets on X. A *Meta-Neutrosophic Set* on X is a mapping

$$N^\sharp : \mathrm{Neu}(X) \longrightarrow [0,1]^3,$$

defined by

$$N^\sharp(A) = \big(T^\sharp(A), I^\sharp(A), F^\sharp(A)\big),$$

where

$$T^\sharp, I^\sharp, F^\sharp : \mathrm{Neu}(X) \longrightarrow [0,1].$$

Hence,

$$0 \le T^\sharp(A) + I^\sharp(A) + F^\sharp(A) \le 3$$

for every $A \in \mathrm{Neu}(X)$.

Example 8.2.24 (Meta-Neutrosophic Set)**.** Let

$$X = \{x_1, x_2\},$$

and consider the neutrosophic set

$$A = (T_A, I_A, F_A)$$

defined by

$$(T_A(x_1), I_A(x_1), F_A(x_1)) = (0.8, 0.2, 0.1),$$

and

$$(T_A(x_2), I_A(x_2), F_A(x_2)) = (0.6, 0.4, 0.3).$$

Define

$$N^\sharp(A) = \big(T^\sharp(A), I^\sharp(A), F^\sharp(A)\big)$$

by

$$T^\sharp(A) = \frac{T_A(x_1) + T_A(x_2)}{2},$$

$$I^\sharp(A) = \frac{I_A(x_1) + I_A(x_2)}{2},$$

and

$$F^\sharp(A) = \frac{F_A(x_1) + F_A(x_2)}{2}.$$

Hence,

$$N^\sharp(A) = (0.7, 0.3, 0.2)\,.$$

Since

$$0.7 + 0.3 + 0.2 = 1.2 \le 3,$$

this is a valid Meta-Neutrosophic evaluation of A.

Chapter 9

Concepts for Complex Structures

This chapter explains several concepts related to Complex Structures. The term Complex Structure is used in a broad book-specific sense in this chapter.

9.1 About Complex Structures

First, we provide an overview of Complex Structures.

9.1.1 Complex Structures

A Complex Structure is a mathematical structure whose elements lie in the complex numbers, equipped with specified operations and relations.

Definition 9.1.1 (Complex Structure)**.** Let $\mathbb{C}$ denote the field of complex numbers. A *Complex Structure* is a mathematical structure

$$\mathcal{C} = (X, \Omega, \mathcal{R}),$$

where

$$\varnothing \neq X \subseteq \mathbb{C},$$

Ω is a family of operations on X, and $\mathcal{R}$ is a family of relations on X. Thus, the underlying elements of the structure are complex numbers.

9.1.2 Quaternion Structures

A quaternion number is a four-dimensional hypercomplex number comprising one real component and three imaginary components with generally noncommutative multiplication [648–650]. Quaternion Structures are mathematical structures whose elements are quaternions, equipped with specified operations and relations that may reflect quaternionic noncommutative multiplication behavior.

Definition 9.1.2 (Quaternion Structure)**.** Let

$$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\}$$

be the quaternion algebra, where

$$i^2 = j^2 = k^2 = ijk = -1.$$

A *Quaternion Structure* is a mathematical structure

$$\mathcal{H} = (X, \Omega, \mathcal{R}),$$

where

$$\varnothing \neq X \subseteq \mathbb{H},$$

and Ω and $\mathcal{R}$ are respectively families of operations and relations defined on X.

9.1.3 Octonion Structures

An octonion number is an eight-dimensional hypercomplex number comprising one real component and seven imaginary components with noncommutative, nonassociative multiplication [651,652]. Octonion Structures are mathematical structures whose elements are octonions, equipped with specified operations and relations that may reflect noncommutative and nonassociative multiplication behavior.

Definition 9.1.3 (Octonion Structure)**.** Let $\mathbb{O}$ denote the algebra of octonions,

$$\mathbb{O} = \left\{ \sum_{r=0}^{7} a_r e_r \;\middle|\; a_r \in \mathbb{R} \right\}, \qquad e_0 = 1.$$

An *Octonion Structure* is a mathematical structure

$$\mathcal{O} = (X, \Omega, \mathcal{R}),$$

where

$$\varnothing \neq X \subseteq \mathbb{O},$$

and Ω and $\mathcal{R}$ are respectively families of operations and relations on X. The operations may inherit the noncommutative and nonassociative properties of octonion multiplication.

9.2 Concepts for Complex Structures

This section presents several concepts related to Complex Structures.

9.2.1 Finite Complex Numbers

Finite Complex Numbers have the form

$$a + bi_F$$

over $\mathbb{Z}_n$, where

$$i_F^2 \equiv -1 \pmod{n},$$

and all coefficient arithmetic is performed modulo n [653–655].

Definition 9.2.1 (Finite Complex Numbers)**.** Let

$$\mathbb{Z}_n = \{0, 1, \ldots, n-1\}, \qquad n \geq 2,$$

and let i_F be a finite imaginary unit satisfying

$$i_F^2 = n - 1 \equiv -1 \pmod{n}.$$

The set of *Finite Complex Numbers modulo n* is defined by

$$\mathbb{C}(\mathbb{Z}_n) = \{a + bi_F \mid a, b \in \mathbb{Z}_n\}.$$

All arithmetic on the coefficients is performed modulo n.

Example 9.2.2 (Finite Complex Numbers)**.** Let $n = 5$. Then

$$\mathbb{C}(\mathbb{Z}_5) = \{a + bi_F \mid a, b \in \mathbb{Z}_5,\ i_F^2 = 4\}.$$

For example,

$$z = 2 + 3i_F$$

is a finite complex number modulo 5.

9.2.2 Complex-Valued Graphs

Complex-Valued Graphs assign complex numbers to vertices and edges as weights or attributes, thereby extending graphs with magnitude and phase information [656].

Definition 9.2.3 (Complex-Valued Graph)**.** Let

$$G = (V, E)$$

be a graph. A *Complex-Valued Graph* is a structure

$$G_{\mathbb{C}} = (V, E, \omega_V, \omega_E),$$

where

$$\omega_V : V \longrightarrow \mathbb{C}, \qquad \omega_E : E \longrightarrow \mathbb{C}.$$

Thus, each vertex and edge is assigned a complex-valued weight or attribute.

Example 9.2.4 (Complex-Valued Graph)**.** Let

$$V = \{u, v\}, \qquad E = \{\{u, v\}\}.$$

Define the complex-valued vertex weights by

$$\omega_V(u) = 1 + i, \qquad \omega_V(v) = 2 - i,$$

and the edge weight by

$$\omega_E(\{u, v\}) = 3 + 2i.$$

Then

$$G_{\mathbb{C}} = (V, E, \omega_V, \omega_E)$$

is a Complex-Valued Graph. A graphical representation is shown in Figure 9.1.

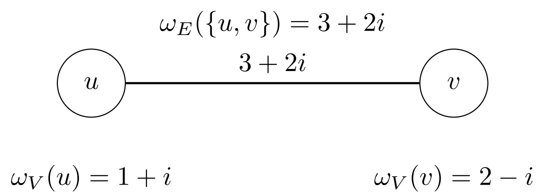


Figure 9.1: A Complex-Valued Graph with complex values assigned to both vertices and the edge.

9.2.3 Quaternion-Valued Graphs (Proposed in this book)

Quaternion-Valued Graphs assign quaternion numbers to vertices and edges, enabling multidimensional noncommutative weights, attributes, and interactions within graph-based models.

Definition 9.2.5 (Quaternion-Valued Graph)**.** Let

$$G = (V, E)$$

be a graph and let $\mathbb{H}$ denote the quaternion algebra. A *Quaternion-Valued Graph* is a structure

$$G_{\mathbb{H}} = (V, E, \omega_V, \omega_E),$$

where

$$\omega_V : V \longrightarrow \mathbb{H}, \qquad \omega_E : E \longrightarrow \mathbb{H}.$$

Thus, vertices and edges carry quaternion-valued weights or attributes.

Example 9.2.6 (Quaternion-Valued Graph)**.** Let

$$V = \{u, v\}, \qquad E = \{\{u, v\}\}.$$

Define

$$\omega_V(u) = 1 + i, \qquad \omega_V(v) = 2 + j,$$

and

$$\omega_E(\{u, v\}) = 1 + i + j + k.$$

Then

$$G_{\mathbb{H}} = (V, E, \omega_V, \omega_E)$$

is a Quaternion-Valued Graph.

9.2.4 Octonion-Valued Graphs (Proposed in this book)

Octonion-Valued Graphs assign octonions to vertices and edges, representing higher-dimensional weights and interactions with potentially noncommutative and nonassociative algebraic behavior.

Definition 9.2.7 (Octonion-Valued Graph)**.** Let

$$G = (V, E)$$

be a graph and let $\mathbb{O}$ denote the octonion algebra. An *Octonion-Valued Graph* is a structure

$$G_{\mathbb{O}} = (V, E, \omega_V, \omega_E),$$

where

$$\omega_V : V \longrightarrow \mathbb{O}, \qquad \omega_E : E \longrightarrow \mathbb{O}.$$

Thus, vertices and edges carry octonion-valued weights or attributes, possibly reflecting the noncommutative and nonassociative properties of octonion multiplication.

Example 9.2.8 (Octonion-Valued Graph)**.** Let

$$V = \{u, v\}, \qquad E = \{\{u, v\}\}.$$

Define

$$\omega_V(u) = 1 + e_1, \qquad \omega_V(v) = 2 + e_2,$$

and

$$\omega_E(\{u, v\}) = 1 + e_1 + e_3.$$

Then

$$G_{\mathbb{O}} = (V, E, \omega_V, \omega_E)$$

is an Octonion-Valued Graph.

9.2.5 MOD Complex graph

A MOD Complex Graph uses finite complex numbers modulo n as vertex values, edge weights, or both within a graph [340].

Definition 9.2.9 (MOD Complex Graph)**.** Let

$$\mathbb{C}(\mathbb{Z}_n) = \{a + bi_F \mid a, b \in \mathbb{Z}_n,\ i_F^2 \equiv -1 \pmod n\}, \qquad n \geq 2.$$

A *MOD Complex Graph* is a graph

$$G = (V, E)$$

in which the vertices are selected from

$$V \subseteq \mathbb{C}(\mathbb{Z}_n),$$

and/or the edges are assigned MOD complex weights through a map

$$\omega : E \longrightarrow \mathbb{C}(\mathbb{Z}_n).$$

Thus, vertex values, edge weights, or both are finite complex numbers defined modulo n.

Example 9.2.10 (MOD Complex Graph)**.** Let

$$\mathbb{C}(\mathbb{Z}_5) = \{a + bi_F \mid a, b \in \mathbb{Z}_5,\ i_F^2 = 4\}.$$

Take

$$V = \{1 + i_F,\ 2 + 3i_F\}, \qquad E = \big\{\{1 + i_F, 2 + 3i_F\}\big\},$$

and define the edge weight by

$$\omega\big(\{1 + i_F, 2 + 3i_F\}\big) = 4 + 2i_F.$$

Since both vertices and the edge weight belong to $\mathbb{C}(\mathbb{Z}_5)$, the structure

$$G_{\mathrm{MC}} = (V, E, \omega)$$

is a MOD Complex Graph. A graphical representation is shown in Figure 9.2.

9.2.6 Complex Fuzzy Set

A Complex Fuzzy Set assigns each element a complex-valued membership grade combining an amplitude in

$$[0, 1]$$

with a real-valued phase component [657–659].

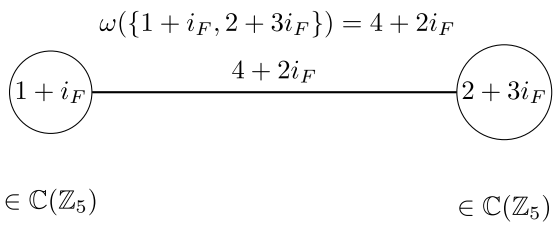


Figure 9.2: A MOD Complex Graph over $\mathbb{C}(\mathbb{Z}_5)$, with edge weight $4 + 2i_F$.

Definition 9.2.11 (Complex Fuzzy Set)**.** Let X be a nonempty universe. A *Complex Fuzzy Set* A on X is characterized by a complex-valued membership function

$$\mu_A : X \longrightarrow \mathbb{D},$$

where

$$\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}.$$

For each $x \in X$,

$$\mu_A(x) = r_A(x)e^{i\theta_A(x)},$$

where

$$r_A(x) \in [0, 1], \qquad \theta_A(x) \in \mathbb{R}.$$

Here, $r_A(x)$ represents the membership amplitude and $\theta_A(x)$ represents the phase.

Example 9.2.12 (Complex Fuzzy Set)**.** Let

$$X = \{x_1, x_2\}.$$

Define

$$\mu_A(x_1) = 0.7e^{i\pi/3}, \qquad \mu_A(x_2) = 0.4e^{i\pi}.$$

Then

$$A = \Big\{(x_1, 0.7e^{i\pi/3}), (x_2, 0.4e^{i\pi})\Big\}$$

is a Complex Fuzzy Set on X.

9.2.7 Complex Neutrosophic Set

A Complex Neutrosophic Set assigns complex-valued truth, indeterminacy, and falsity grades, each represented through independent amplitude and phase components [170, 660–663].

Definition 9.2.13 (Complex Neutrosophic Set)**.** Let X be a nonempty universe. A *Complex Neutrosophic Set* A on X is characterized by three complex-valued functions

$$T_A, I_A, F_A : X \longrightarrow \mathbb{D},$$

where

$$\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}.$$

For each $x \in X$,

$$T_A(x) = t_A(x)e^{i\alpha_A(x)},$$
$$I_A(x) = u_A(x)e^{i\beta_A(x)},$$

and

$$F_A(x) = f_A(x)e^{i\gamma_A(x)},$$

where

$$t_A(x), u_A(x), f_A(x) \in [0, 1], \qquad \alpha_A(x), \beta_A(x), \gamma_A(x) \in \mathbb{R},$$

with

$$0 \leq t_A(x) + u_A(x) + f_A(x) \leq 3.$$

Example 9.2.14 (Complex Neutrosophic Set)**.** Let

$$X = \{x\}.$$

Define

$$T_A(x) = 0.7e^{i\pi/3}, \qquad I_A(x) = 0.2e^{i\pi/2}, \qquad F_A(x) = 0.1e^{i\pi}.$$

Then

$$A = \left\{ \left(x, 0.7e^{i\pi/3}, 0.2e^{i\pi/2}, 0.1e^{i\pi} \right) \right\}$$

is a Complex Neutrosophic Set.

9.2.8 Complex Uncertain Set

A Complex Uncertain Set assigns each element a complex-valued uncertainty tuple whose modulus satisfies the constraints of the underlying uncertain model.

Definition 9.2.15 (Complex Uncertain Set)**.** Let X be a nonempty universe, and let M be an uncertain model with

$$\mathrm{Dom}(M) \subseteq [0,1]^k, \qquad k \geq 1.$$

Let

$$\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\},$$

and define the induced complex degree-domain by

$$\mathrm{Dom}_{\mathbb{C}}(M) = \left\{ (z_1, \dots, z_k) \in \mathbb{D}^k \;\middle|\; (|z_1|, \dots, |z_k|) \in \mathrm{Dom}(M) \right\}.$$

A *Complex Uncertain Set* of type M on X is a pair

$$U^{\mathbb{C}} = (X, \mu_M^{\mathbb{C}}),$$

where

$$\mu_M^{\mathbb{C}} : X \longrightarrow \mathrm{Dom}_{\mathbb{C}}(M).$$

Thus, each element is assigned a complex-valued uncertainty tuple whose modulus tuple satisfies the underlying uncertain model M.

Example 9.2.16 (Complex Uncertain Set)**.** Let

$$X = \{x\}$$

and consider the uncertain model

$$\mathrm{Dom}(M) = \{(a,b) \in [0,1]^2 \mid a + b \leq 1\}.$$

Define

$$\mu_M^{\mathbb{C}}(x) = \left(0.6e^{i\pi/3}, 0.3e^{i\pi/2} \right).$$

Since

$$0.6 + 0.3 \leq 1,$$

this defines a Complex Uncertain Set of type M.

9.2.9 Complex Vector Space

A Complex Vector Space is a vector space over the complex numbers, supporting vector addition and complex scalar multiplication satisfying standard axioms [664, 665].

Definition 9.2.17 (Complex Vector Space)**.** A *Complex Vector Space* is a vector space

$$V$$

over the field of complex numbers $\mathbb{C}$. Thus, scalar multiplication is a map

$$\mathbb{C} \times V \longrightarrow V$$

satisfying the usual vector-space axioms.

Example 9.2.18 (Complex Vector Space)**.** Consider

$$V = \mathbb{C}^2.$$

For

$$u = (1, i), \qquad v = (2, -i), \qquad \lambda = 1 + i,$$

both

$$u + v \in \mathbb{C}^2$$

and

$$\lambda u \in \mathbb{C}^2.$$

Hence $\mathbb{C}^2$ is a Complex Vector Space.

9.2.10 Complex Algebra

A Complex Algebra is a complex vector space equipped with a bilinear multiplication that combines elements while respecting complex scalar multiplication [666–668].

Definition 9.2.19 (Complex Algebra)**.** A *Complex Algebra* is a complex vector space A equipped with a $\mathbb{C}$-bilinear multiplication

$$m : A \times A \longrightarrow A, \qquad (a, b) \longmapsto ab.$$

If

$$(ab)c = a(bc)$$

for all $a, b, c \in A$, then A is an associative complex algebra.

Example 9.2.20 (Complex Algebra)**.** Consider

$$A = M_2(\mathbb{C}),$$

the set of all 2×2 complex matrices. With ordinary matrix addition, scalar multiplication, and matrix multiplication,

$$M_2(\mathbb{C})$$

is an associative Complex Algebra.

9.2.11 Complex Matrix

A Complex Matrix is a rectangular array whose entries are complex numbers, supporting standard matrix operations such as addition and multiplication (cf. [669]).

Definition 9.2.21 (Complex Matrix)**.** A *Complex Matrix* is a matrix

$$A = (a_{ij}) \in M_{m,n}(\mathbb{C}),$$

whose entries satisfy

$$a_{ij} \in \mathbb{C}, \qquad 1 \le i \le m, \quad 1 \le j \le n.$$

Example 9.2.22 (Complex Matrix)**.** The matrix

$$A = \begin{pmatrix} 1+i & 2 \\ -i & 3+2i \end{pmatrix} \in M_2(\mathbb{C})$$

is a Complex Matrix.

9.2.12 Complex Hilbert Space

A Complex Hilbert Space is a complete complex inner-product space whose induced norm provides geometric and analytic structure for vectors (cf. [670–673]).

Definition 9.2.23 (Complex Hilbert Space)**.** [670,671] A *Complex Hilbert Space* is a complex vector space H equipped with an inner product

$$\langle \cdot, \cdot \rangle : H \times H \longrightarrow \mathbb{C}$$

such that H is complete with respect to the norm

$$\|x\| = \sqrt{\langle x, x \rangle}.$$

Example 9.2.24 (Complex Hilbert Space)**.** Consider

$$H = \mathbb{C}^2$$

with inner product

$$\langle x, y \rangle = x_1 \overline{y_1} + x_2 \overline{y_2}.$$

Then $\mathbb{C}^2$ is a finite-dimensional Complex Hilbert Space.

9.2.13 Complex Manifold

A Complex Manifold is a manifold locally modeled on complex Euclidean space, with holomorphic transition maps between overlapping coordinate charts [674–678].

Definition 9.2.25 (Complex Manifold)**.** A *Complex Manifold* of complex dimension n is a manifold M locally modeled on $\mathbb{C}^n$ such that all transition maps

$$\varphi_\beta \circ \varphi_\alpha^{-1}$$

between overlapping coordinate charts are holomorphic.

Example 9.2.26 (Complex Manifold)**.** The complex plane

$$M = \mathbb{C}$$

is a Complex Manifold of complex dimension 1, with the global coordinate chart

$$\varphi : \mathbb{C} \longrightarrow \mathbb{C}, \qquad \varphi(z) = z.$$

9.2.14 Almost Complex Structure

An Almost Complex Structure is a smooth bundle endomorphism of the tangent bundle whose square equals the negative identity map everywhere (cf. [679]).

Definition 9.2.27 (Almost Complex Structure)**.** Let M be a smooth manifold. An *Almost Complex Structure* on M is a smooth bundle endomorphism

$$J : TM \longrightarrow TM$$

satisfying

$$J^2 = -\operatorname{Id}_{TM}.$$

Example 9.2.28 (Almost Complex Structure)**.** Let

$$M = \mathbb{R}^2.$$

Define

$$J : T\mathbb{R}^2 \longrightarrow T\mathbb{R}^2$$

by

$$J(x, y) = (-y, x).$$

Then

$$J^2(x, y) = (-x, -y),$$

so

$$J^2 = -\operatorname{Id}.$$

Hence J defines an Almost Complex Structure on $\mathbb{R}^2$.

9.2.15 Complex Lie Group

A Complex Lie Group is simultaneously a group and complex manifold whose multiplication and inversion maps are holomorphic functions globally [680–682].

Definition 9.2.29 (Complex Lie Group)**.** A *Complex Lie Group* is a group G that is also a complex manifold such that the multiplication

$$G \times G \longrightarrow G, \qquad (g, h) \longmapsto gh,$$

and inversion

$$G \longrightarrow G, \qquad g \longmapsto g^{-1},$$

are holomorphic maps.

Example 9.2.30 (Complex Lie Group)**.** Consider

$$\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$$

under multiplication. The maps

$$m(z, w) = zw$$

and

$$\iota(z) = z^{-1}$$

are holomorphic. Hence

$$(\mathbb{C}^\times, \cdot)$$

is a Complex Lie Group.

9.2.16 Complex Dynamical System

A Complex Dynamical System studies repeated iteration of holomorphic or meromorphic maps on complex spaces and their resulting long-term behavior [683–685].

Definition 9.2.31 (Complex Dynamical System)**.** Let X be a complex manifold and let

$$f : X \longrightarrow X$$

be a holomorphic map. A *Complex Dynamical System* is the pair

$$(X, f),$$

whose dynamics are determined by the iterates

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}, \qquad n \geq 0.$$

Example 9.2.32 (Complex Dynamical System)**.** Let

$$X = \mathbb{C}$$

and define

$$f : \mathbb{C} \longrightarrow \mathbb{C}, \qquad f(z) = z^2.$$

Then

$$(\mathbb{C}, f)$$

is a Complex Dynamical System, with iterates

$$f^n(z) = z^{2^n}.$$

9.2.17 Complex Normed Space

A Complex Normed Space is a complex vector space equipped with a norm measuring vector magnitude and satisfying norm axioms.

Definition 9.2.33 (Complex Normed Space)**.** A *Complex Normed Space* is a complex vector space V equipped with a norm

$$\| \cdot \| : V \longrightarrow [0, \infty)$$

satisfying

$$\|\lambda x\| = |\lambda| \, \|x\|$$

for all $\lambda \in \mathbb{C}$ and $x \in V$, together with the usual norm axioms.

Example 9.2.34 (Complex Normed Space)**.** Let

$$V = \mathbb{C}^2$$

and define

$$\|(z_1, z_2)\| = \sqrt{|z_1|^2 + |z_2|^2}.$$

For example, if

$$z = (1 + i, 2),$$

then

$$\|z\| = \sqrt{|1 + i|^2 + |2|^2} = \sqrt{2 + 4} = \sqrt{6}.$$

Thus,

$$(\mathbb{C}^2, \| \cdot \|)$$

is a Complex Normed Space.

9.2.18 Complex Banach Space

A Complex Banach Space is a complete complex normed vector space in which every Cauchy sequence converges to an element (cf. [686–688]).

Definition 9.2.35 (Complex Banach Space)**.** A *Complex Banach Space* is a complex normed space

$$(V, \|\cdot\|)$$

that is complete with respect to the metric

$$d(x, y) = \|x - y\|.$$

Example 9.2.36 (Complex Banach Space)**.** Consider

$$V = \mathbb{C}$$

with the norm

$$\|z\| = |z|.$$

Every Cauchy sequence in $\mathbb{C}$ converges to an element of $\mathbb{C}$. Therefore,

$$(\mathbb{C}, |\cdot|)$$

is a Complex Banach Space.

9.2.19 Hermitian Vector Space

A Hermitian Vector Space is a complex vector space equipped with a Hermitian inner product satisfying conjugate symmetry and positivity [689–691].

Definition 9.2.37 (Hermitian Vector Space)**.** A *Hermitian Vector Space* is a complex vector space V equipped with a Hermitian inner product

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{C}$$

satisfying

$$\langle x, y \rangle = \overline{\langle y, x \rangle}, \qquad \langle x, x \rangle > 0$$

for every nonzero $x \in V$.

Example 9.2.38 (Hermitian Vector Space)**.** Let

$$V = \mathbb{C}^2$$

and define

$$\langle x, y \rangle = x_1 \overline{y_1} + x_2 \overline{y_2}.$$

For

$$x = (1, i), \qquad y = (2, 1),$$

we obtain

$$\langle x, y \rangle = 1 \cdot 2 + i \cdot 1 = 2 + i.$$

Moreover,

$$\langle x, x \rangle = |1|^2 + |i|^2 = 2 > 0.$$

Hence $\mathbb{C}^2$ with this inner product is a Hermitian Vector Space.

9.2.20 Hermitian Matrix

A Hermitian Matrix is a complex square matrix equal to its conjugate transpose and admits unitary diagonalization with real eigenvalues [692–694].

Definition 9.2.39 (Hermitian Matrix)**.** A complex square matrix

$$A \in M_n(\mathbb{C})$$

is called a *Hermitian Matrix* if

$$A^* = A,$$

where

$$A^* = \overline{A}^T$$

is the conjugate transpose of A.

Example 9.2.40 (Hermitian Matrix)**.** Consider

$$A = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}.$$

Its conjugate transpose is

$$A^* = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix} = A.$$

Therefore, A is a Hermitian Matrix.

9.2.21 Unitary Matrix

A Unitary Matrix is a complex square matrix whose inverse equals its conjugate transpose, preserving inner products, lengths, and orthogonality [695–698].

Definition 9.2.41 (Unitary Matrix)**.** A complex square matrix

$$U \in M_n(\mathbb{C})$$

is called a *Unitary Matrix* if

$$U^*U = UU^* = I_n.$$

Equivalently,

$$U^{-1} = U^*.$$

Example 9.2.42 (Unitary Matrix)**.** Consider

$$U = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

Then

$$U^* = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix},$$

and

$$U^*U = UU^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

Hence U is a Unitary Matrix.

9.2.22 Unitary Group

The Unitary Group consists of all unitary matrices of a fixed degree under matrix multiplication. It forms a compact real Lie group [699–701].

Definition 9.2.43 (Unitary Group)**.** The *Unitary Group* of degree n is defined by

$$U(n) = \{U \in M_n(\mathbb{C}) \mid U^*U = UU^* = I_n\},$$

where U^* denotes the conjugate transpose of U. Equivalently,

$$U^{-1} = U^*.$$

The group operation is ordinary matrix multiplication.

Example 9.2.44 (Unitary Group)**.** Consider

$$U(2) = \{U \in M_2(\mathbb{C}) \mid U^*U = I_2\}.$$

For example, let

$$U = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}.$$

Then

$$U^* = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix},$$

and hence

$$U^*U = UU^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

Therefore,

$$U \in U(2).$$

Moreover, if $U, V \in U(2)$, then

$$(UV)^*(UV) = V^*U^*UV = V^*V = I_2,$$

so

$$UV \in U(2).$$

Thus, $U(2)$ is a Unitary Group under matrix multiplication.

9.2.23 Complex Lie Algebra

A Complex Lie Algebra is a complex vector space with a bilinear bracket satisfying antisymmetry and Jacobi identity for elements [702–704].

Definition 9.2.45 (Complex Lie Algebra)**.** A *Complex Lie Algebra* is a complex vector space $\mathfrak{g}$ equipped with a bilinear bracket

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

satisfying

$$[x, x] = 0$$

and the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Example 9.2.46 (Complex Lie Algebra)**.** Let

$$\mathfrak{g} = \mathfrak{gl}_2(\mathbb{C}) = M_2(\mathbb{C}),$$

and define the bracket

$$[A, B] = AB - BA.$$

For example, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then

$$[A, B] = AB - BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

The commutator bracket is complex bilinear, antisymmetric, and satisfies the Jacobi identity. Therefore, $\mathfrak{gl}_2(\mathbb{C})$ is a Complex Lie Algebra.

9.2.24 Complex Projective Space

Complex Projective Space is the space of complex lines through the origin in a complex vector space identified under scaling [705–707].

Definition 9.2.47 (Complex Projective Space)**.** The *Complex Projective Space* of dimension n is

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} \setminus \{0\}\right) / \sim,$$

where

$$z \sim w \iff w = \lambda z$$

for some

$$\lambda \in \mathbb{C}^\times.$$

Example 9.2.48 (Complex Projective Space)**.** Consider

$$\mathbb{CP}^1 = \left(\mathbb{C}^2 \setminus \{0\}\right) / \sim .$$

The vectors

$$(1, i) \qquad \text{and} \qquad (2, 2i)$$

represent the same projective point because

$$(2, 2i) = 2(1, i), \qquad 2 \in \mathbb{C}^\times.$$

Thus,

$$[1 : i] = [2 : 2i]$$

is a point of $\mathbb{CP}^1$.

9.2.25 Complex Torus

A Complex Torus is a compact complex manifold obtained as a quotient of complex Euclidean space by a full lattice [708–710].

Definition 9.2.49 (Complex Torus)**.** Let

$$\Lambda \subset \mathbb{C}^n$$

be a lattice of rank $2n$. A *Complex Torus* of complex dimension n is the quotient

$$T = \mathbb{C}^n / \Lambda.$$

Example 9.2.50 (Complex Torus)**.** Let

$$\Lambda = \mathbb{Z} + i\mathbb{Z} = \{m + ni \mid m, n \in \mathbb{Z}\} \subset \mathbb{C}.$$

Since Λ is a lattice of rank 2, the quotient

$$T = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$$

is a one-dimensional Complex Torus. For example,

$$z \sim z + 1 \sim z + i$$

in T for every $z \in \mathbb{C}$.

9.2.26 Complex Soft Set

A Complex Soft Set assigns to each parameter a complex fuzzy set, thereby representing both membership magnitude and phase information [711, 712].

Definition 9.2.51 (Complex Soft Set)**.** Let U be a nonempty universe, E a set of parameters, and

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| \le 1\}.$$

Let

$$\mathrm{CFS}(U) = \{\mu \mid \mu : U \to \mathbb{D}\}$$

denote the family of all complex fuzzy sets on U. For a nonempty set $A \subseteq E$, a *Complex Soft Set* over U is a pair

$$(F, A),$$

where

$$F : A \longrightarrow \mathrm{CFS}(U).$$

Equivalently, for each $e \in A$ and $x \in U$,

$$F(e)(x) = r_e(x)e^{i\theta_e(x)},$$

where

$$r_e(x) \in [0, 1], \qquad \theta_e(x) \in [0, 2\pi].$$

Example 9.2.52 (Complex Soft Set)**.** Let

$$U = \{x_1, x_2, x_3\}, \qquad E = \{e_1, e_2\},$$

and take

$$A = \{e_1, e_2\} \subseteq E.$$

Define

$$F : A \longrightarrow \mathrm{CFS}(U)$$

by

$$F(e_1)(x_1) = 0.8e^{i\pi/6}, \qquad F(e_1)(x_2) = 0.5e^{i\pi/3}, \qquad F(e_1)(x_3) = 0.3e^{i\pi/2},$$

and

$$F(e_2)(x_1) = 0.6e^{i\pi/4}, \qquad F(e_2)(x_2) = 0.9e^{i\pi/8}, \qquad F(e_2)(x_3) = 0.4e^{i\pi}.$$

Since all amplitudes belong to $[0, 1]$, every value lies in

$$\mathbb{D} = \{z \in \mathbb{C} \mid |z| \le 1\}.$$

Hence, for each $e \in A$,

$$F(e) : U \longrightarrow \mathbb{D}$$

is a complex fuzzy set. Therefore,

$$(F, A)$$

is a Complex Soft Set over U.

Chapter 10

Real Structure

In this chapter, we investigate Real Structures.

10.1 Real Structure

In this section, we examine the fundamental definition of a Real Structure.

Definition 10.1.1 (Real Structure)**.** Let $\mathbb{R}$ denote the set of real numbers. A *Real Structure* is a mathematical structure

$$\mathcal{S} = (X, \mathcal{F}),$$

where X is a nonempty set and

$$\mathcal{F} = \{f_i\}_{i \in I}$$

is a family of real-valued functions

$$f_i : X^{n_i} \longrightarrow \mathbb{R}, \qquad n_i \geq 1.$$

Thus, the elements or relations of $\mathcal{S}$ are characterized by quantities taking values in $\mathbb{R}$.

10.2 Some Concepts for Real Structure

In this section, we explain some concepts related to Real Structures.

10.2.1 Fuzzy Real Number

A fuzzy real number represents an imprecise real quantity through a normalized fuzzy set with bounded support and interval alpha-cuts (cf. [713–715]).

Definition 10.2.1 (Fuzzy Real Number)**.** A *fuzzy real number* is a fuzzy set

$$A : \mathbb{R} \to [0, 1]$$

such that:

1. A is normal;
2. for every $\alpha \in (0,1]$, the α-cut
$$A_\alpha = \{x \in \mathbb{R} \mid A(x) \geq \alpha\}$$
is a closed interval;
3. the support
$$\operatorname{supp}(A) = \{x \in \mathbb{R} \mid A(x) > 0\}$$
is bounded.

Example 10.2.2 (Fuzzy Real Number)**.** Define $A : \mathbb{R} \to [0,1]$ by

$$A(x) = \begin{cases} \dfrac{x}{2}, & 0 \leq x \leq 2, \\ \dfrac{4-x}{2}, & 2 < x \leq 4, \\ 0, & \text{otherwise.} \end{cases}$$

Then $A(2) = 1$, so A is normal. Moreover, for every $\alpha \in (0,1]$,

$$A_\alpha = [2\alpha,\, 4 - 2\alpha],$$

which is a closed interval, and
$$\operatorname{supp}(A) = (0,4)$$
is bounded. Hence A is a fuzzy real number.

10.2.2 Neutrosophic Real Number

A neutrosophic real number combines real coefficients with an indeterminacy element I, satisfying $I^2 = I$, to represent values (cf. [716–719]).

Definition 10.2.3 (Neutrosophic Real Number)**.** A *neutrosophic real number* is a number of the form
$$x = a + bI, \qquad a, b \in \mathbb{R},$$
where I represents indeterminacy and satisfies

$$I^2 = I.$$

The set of all neutrosophic real numbers is denoted by

$$\mathbb{R}(I) = \{a + bI \mid a, b \in \mathbb{R},\ I^2 = I\}.$$

Example 10.2.4 (Neutrosophic Real Number)**.** Consider

$$x = 3 + 2I, \qquad I^2 = I.$$

Since $3, 2 \in \mathbb{R}$, we have
$$x \in \mathbb{R}(I).$$
For example,
$$x^2 = (3 + 2I)^2 = 9 + 12I + 4I^2 = 9 + 16I.$$
Thus, $3 + 2I$ is a neutrosophic real number.

10.2.3 Real Vector Space

A Real Vector Space is a vector space over the real numbers, supporting vector addition and real scalar multiplication operations (cf. [720, 721]).

Definition 10.2.5 (Real Vector Space)**.** A *Real Vector Space* is a vector space

$$V$$

over the field of real numbers $\mathbb{R}$.

Example 10.2.6 (Real Vector Space)**.** Consider

$$V = \mathbb{R}^2.$$

For

$$u = (1, 2), \qquad v = (3, -1), \qquad \lambda = 2,$$

we have

$$u + v = (4, 1) \in V$$

and

$$\lambda u = (2, 4) \in V.$$

Hence $\mathbb{R}^2$ is a real vector space under the usual operations.

10.2.4 Real Algebra

A Real Algebra is a real vector space equipped with a bilinear multiplication that combines elements while respecting scalar multiplication (cf. [722–724]).

Definition 10.2.7 (Real Algebra)**.** A *Real Algebra* is a real vector space A equipped with an $\mathbb{R}$-bilinear multiplication

$$A \times A \longrightarrow A, \qquad (a, b) \longmapsto ab.$$

Example 10.2.8 (Real Algebra)**.** Let

$$A = M_2(\mathbb{R})$$

with the usual matrix addition, scalar multiplication, and matrix multiplication. For example,

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

satisfy

$$A_1 A_2 = A_2 \in M_2(\mathbb{R}).$$

Thus, $M_2(\mathbb{R})$ is a real algebra.

10.2.5 Real Matrix

A Real Matrix is a rectangular array whose entries are real numbers, supporting standard matrix addition, multiplication, and transpose operations (cf. [725]).

Definition 10.2.9 (Real Matrix)**.** A *Real Matrix* is a matrix

$$A = (a_{ij}) \in M_{m,n}(\mathbb{R}),$$

whose entries satisfy

$$a_{ij} \in \mathbb{R}.$$

Example 10.2.10 (Real Matrix)**.** The matrix

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \end{pmatrix}$$

has only real entries. Therefore,

$$A \in M_{2,3}(\mathbb{R}),$$

and A is a real matrix.

10.2.6 Real-Valued Function

A Real-Valued Function maps each element of a domain to a real number, providing numerical values for quantities or relationships (cf. [726–728]).

Definition 10.2.11 (Real-Valued Function)**.** Let X be a nonempty set. A *Real-Valued Function* is a mapping

$$f : X \longrightarrow \mathbb{R}.$$

Example 10.2.12 (Real-Valued Function)**.** Let

$$X = \mathbb{R}$$

and define

$$f : X \to \mathbb{R}, \qquad f(x) = x^2 - 2x + 1.$$

For example,

$$f(3) = 4.$$

Hence f is a real-valued function.

10.2.7 Real Inner Product Space

A Real Inner Product Space is a real vector space equipped with a symmetric positive-definite bilinear inner product on vectors (cf. [729–731]).

Definition 10.2.13 (Real Inner Product Space)**.** A *Real Inner Product Space* is a real vector space V equipped with an inner product

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R}$$

that is bilinear, symmetric, and positive definite.

Example 10.2.14 (Real Inner Product Space)**.** Let

$$V = \mathbb{R}^2$$

and define

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2.$$

For

$$x = (1, 2), \qquad y = (3, 4),$$

we obtain

$$\langle x, y \rangle = 1 \cdot 3 + 2 \cdot 4 = 11.$$

Thus, $\mathbb{R}^2$ with the standard inner product is a real inner product space.

10.2.8 Real Normed Space

A Real Normed Space is a real vector space equipped with a norm measuring vector magnitude and satisfying norm axioms (cf. [732]).

Definition 10.2.15 (Real Normed Space)**.** A *Real Normed Space* is a real vector space V equipped with a norm

$$\| \cdot \| : V \longrightarrow [0, \infty)$$

satisfying the usual norm axioms.

Example 10.2.16 (Real Normed Space)**.** Let

$$V = \mathbb{R}^2$$

and define

$$\|(x_1, x_2)\|_1 = |x_1| + |x_2|.$$

For example,

$$\|(2, -3)\|_1 = 5.$$

Therefore,

$$(\mathbb{R}^2, \|\cdot\|_1)$$

is a real normed space.

10.2.9 Real Banach Space

A Real Banach Space is a complete real normed vector space in which every Cauchy sequence converges to an element (cf. [733–735]).

Definition 10.2.17 (Real Banach Space)**.** A *Real Banach Space* is a real normed space

$$(V, \|\cdot\|)$$

that is complete with respect to the metric

$$d(x, y) = \|x - y\|.$$

Example 10.2.18 (Real Banach Space)**.** Consider

$$(\mathbb{R}, |\cdot|).$$

Since every Cauchy sequence of real numbers converges to a real number, the normed space

$$(\mathbb{R}, |\cdot|)$$

is complete. Hence it is a real Banach space.

10.2.10 Real Hilbert Space

A Real Hilbert Space is a complete real inner-product space whose induced norm provides geometric and analytic structure for vectors [736].

Definition 10.2.19 (Real Hilbert Space)**.** A *Real Hilbert Space* is a real inner product space H that is complete with respect to the norm

$$\|x\| = \sqrt{\langle x, x\rangle}.$$

Example 10.2.20 (Real Hilbert Space)**.** Let

$$\ell^2(\mathbb{R}) = \left\{ x = (x_n)_{n\geq 1} \;\middle|\; \sum_{n=1}^{\infty} x_n^2 < \infty \right\}.$$

Define

$$\langle x, y\rangle = \sum_{n=1}^{\infty} x_n y_n.$$

Since $\ell^2(\mathbb{R})$ is complete under the induced norm, it is a real Hilbert space.

10.2.11 Euclidean Space

Euclidean Space is a real coordinate space equipped with the standard inner product, providing notions of distance, angle, and geometry [737–740].

Definition 10.2.21 (Euclidean Space)**.** The *n-dimensional Euclidean Space* is

$$\mathbb{R}^n$$

equipped with the standard inner product

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i.$$

Example 10.2.22 (Euclidean Space)**.** Consider the Euclidean space

$$\mathbb{R}^3.$$

For

$$x = (1, 2, 3), \qquad y = (2, 0, 1),$$

the standard inner product is

$$\langle x, y\rangle = 1 \cdot 2 + 2 \cdot 0 + 3 \cdot 1 = 5.$$

Thus, $\mathbb{R}^3$ with its standard inner product is a Euclidean space.

10.2.12 Real Manifold

A Real Manifold is a topological space locally modeled on Euclidean space, with compatible coordinate charts describing its smooth structure.

Definition 10.2.23 (Real Manifold)**.** A *Real Manifold* of dimension n is a topological manifold M locally modeled on

$$\mathbb{R}^n,$$

with coordinate transition maps having the required smoothness.

Example 10.2.24 (Real Manifold)**.** The unit circle

$$S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

is locally homeomorphic to open subsets of

$$\mathbb{R}.$$

Therefore, S^1 is a one-dimensional real manifold.

10.2.13 Real Lie Group

A Real Lie Group is simultaneously a group and smooth real manifold whose multiplication and inversion operations are smooth maps [741–744].

Definition 10.2.25 (Real Lie Group)**.** A *Real Lie Group* is a group G that is also a smooth real manifold such that

$$G \times G \longrightarrow G, \qquad (g, h) \longmapsto gh,$$

and

$$G \longrightarrow G, \qquad g \longmapsto g^{-1},$$

are smooth maps.

Example 10.2.26 (Real Lie Group)**.** Consider

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \,\middle|\, \theta \in \mathbb{R} \right\}.$$

Under matrix multiplication, $SO(2)$ is a group and a smooth real manifold, and its multiplication and inversion are smooth. Hence $SO(2)$ is a real Lie group.

10.2.14 Real Lie Algebra

A Real Lie Algebra is a real vector space with a bilinear antisymmetric bracket satisfying the Jacobi identity for elements [745, 746].

Definition 10.2.27 (Real Lie Algebra)**.** A *Real Lie Algebra* is a real vector space $\mathfrak{g}$ equipped with a bilinear bracket

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$$

that is antisymmetric and satisfies the Jacobi identity

$$[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0.$$

Example 10.2.28 (Real Lie Algebra)**.** Consider

$$\mathfrak{so}(2) = \left\{ \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix} \,\middle|\, a \in \mathbb{R} \right\}.$$

With the Lie bracket

$$[A,B] = AB - BA,$$

$\mathfrak{so}(2)$ is a real Lie algebra. In this case,

$$[A,B] = 0$$

for all $A, B \in \mathfrak{so}(2)$.

10.2.15 Real Projective Space

Real Projective Space consists of real lines through the origin in Euclidean space, identifying nonzero vectors differing by scalar multiplication [747–749].

Definition 10.2.29 (Real Projective Space)**.** The *Real Projective Space* of dimension n is

$$\mathbb{RP}^n = \left(\mathbb{R}^{n+1} \setminus \{0\}\right) / \sim,$$

where

$$x \sim y \iff y = \lambda x$$

for some

$$\lambda \in \mathbb{R} \setminus \{0\}.$$

Example 10.2.30 (Real Projective Space)**.** Consider

$$\mathbb{RP}^1 = (\mathbb{R}^2 \setminus \{0\}) / \sim .$$

The vectors

$$(1,2) \qquad \text{and} \qquad (-1,-2)$$

represent the same projective point because

$$(-1,-2) = -1(1,2).$$

Thus,

$$[1:2] = [-1:-2] \in \mathbb{RP}^1.$$

10.2.16 Symmetric Matrix

A Symmetric Matrix is a real square matrix equal to its transpose, possessing real eigenvalues and always an orthonormal eigenbasis [750–752].

Definition 10.2.31 (Symmetric Matrix)**.** A real square matrix

$$A \in M_n(\mathbb{R})$$

is called a *Symmetric Matrix* if

$$A^T = A.$$

Example 10.2.32 (Symmetric Matrix)**.** Consider

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

Since

$$A^T = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = A,$$

the matrix A is symmetric.

10.2.17 Orthogonal Matrix

An Orthogonal Matrix is a real square matrix whose transpose equals its inverse, thereby preserving lengths, angles, and inner products [753–755].

Definition 10.2.33 (Orthogonal Matrix)**.** A real square matrix

$$Q \in M_n(\mathbb{R})$$

is called an *Orthogonal Matrix* if

$$Q^T Q = QQ^T = I_n.$$

Equivalently,

$$Q^{-1} = Q^T.$$

Example 10.2.34 (Orthogonal Matrix)**.** Consider

$$Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$Q^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$Q^T Q = QQ^T = I_2.$$

Hence Q is an orthogonal matrix.

10.2.18 Orthogonal Group

The Orthogonal Group consists of orthogonal matrices of fixed size under multiplication, forming a Lie group preserving Euclidean inner products [756–758].

Definition 10.2.35 (Orthogonal Group)**.** The *Orthogonal Group* of degree n is

$$O(n) = \left\{Q \in M_n(\mathbb{R}) \mid Q^T Q = I_n\right\},$$

equipped with matrix multiplication.

Example 10.2.36 (Orthogonal Group)**.** Consider

$$O(2) = \left\{Q \in M_2(\mathbb{R}) \mid Q^T Q = I_2\right\}.$$

For example,

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in O(2),$$

since

$$Q^T Q = I_2.$$

The set $O(2)$ is closed under matrix multiplication and inversion, and therefore forms the orthogonal group of degree 2.

Chapter 11

Conclusion

This book has presented a systematic survey of linguistic, MOD, interval, subset, multi-component, meta, complex, and real structures, together with several uncertainty-based and higher-order generalizations. Linguistic structures provide mathematical frameworks for representing natural language, qualitative information, ambiguity, and uncertainty through fuzzy, neutrosophic, graph-theoretic, and related approaches. MOD structures describe finite cyclic systems based on modular arithmetic, whereas interval structures represent bounded, imprecise, and uncertain information through ranges of admissible values.

Subset Structures extend ordinary mathematical structures by replacing elements with subsets and lifting operations to the subset level, thereby establishing natural relationships with HyperStructures and higher-order constructions. MultiStructures generalize classical structures by combining multiple distinguishable components, while Iterated MultiStructures introduce an additional hierarchical organization. MetaStructures provide a further level of abstraction by treating entire mathematical structures as objects. Representative Complex and Real Structures also illustrate important algebraic, geometric, and scalar-based realizations within the broader framework.

By organizing these diverse concepts within a common mathematical perspective, this book clarifies their definitions, structural relationships, special cases, reductions, and generalizations. Future research may further investigate interactions among linguistic, modular, interval-valued, subset-based, multi-component, meta, uncertainty-oriented, Hyper-, and SuperHyperStructures. Further developments may also explore applications in discrete mathematics, algebra, artificial intelligence, computational modeling, network science, and complex information systems.

Disclaimer

Funding

This study was conducted without any financial support from external organizations or grants.

Acknowledgments

We would like to express our sincere gratitude to everyone who provided valuable insights, support, and encouragement throughout this research. We also extend our thanks to the readers for their interest and to the authors of the referenced works, whose scholarly contributions have greatly influenced this study. Lastly, we are deeply grateful to the publishers and reviewers who facilitated the dissemination of this work.

Data Availability

Since this research is purely theoretical and mathematical, no empirical data or computational analysis was utilized. Researchers are encouraged to expand upon these findings with data-oriented or experimental approaches in future studies.

Ethical Statement

As this study does not involve experiments with human participants or animals, no ethical approval was required.

Conflicts of Interest

The authors declare that they have no conflicts of interest related to the content or publication of this book.

Code Availability

No code or software was developed for this study.

Use of Generative AI and AI-Assisted Tools

The authors used generative AI and AI-assisted tools only for limited language-editing tasks, such as checking English grammar and improving readability. The mathematical definitions, results, interpretations, and final verification remain the responsibility of the authors.

Disclaimer (Others)

This work presents theoretical ideas and frameworks that have not yet been empirically validated. Readers are encouraged to explore practical applications and further refine these concepts. Although care has been taken to ensure accuracy and appropriate citations, any errors or oversights are unintentional. The perspectives and interpretations expressed herein are solely those of the authors and do not necessarily reflect the viewpoints of their affiliated institutions.

Appendix (List of Tables)

Appendix (List of Figures)

Bibliography

[1] Brojo Kishore Mishra and Raghvendra Kumar. *Natural language processing in artificial intelligence*. CRC Press, 2020.

[2] Dioneia Motta Monte-Serrat and Carlo Cattani. *The natural language for artificial intelligence*. Elsevier, 2021.

[3] KR1442 Chowdhary. Natural language processing. *Fundamentals of artificial intelligence*, pages 603–649, 2020.

[4] Steven Bird, Ewan Klein, and Edward Loper. *Natural language processing with Python: analyzing text with the natural language toolkit*. " O'Reilly Media, Inc.", 2009.

[5] Diksha Khurana, Aditya Koli, Kiran Khatter, and Sukhdev Singh. Natural language processing: state of the art, current trends and challenges. *Multimedia tools and applications*, 82(3):3713–3744, 2023.

[6] Aravind K Joshi. Natural language processing. *Science*, 253(5025):1242–1249, 1991.

[7] Xinyi Li, Sai Wang, Siqi Zeng, Yu Wu, and Yi Yang. A survey on llm-based multi-agent systems: workflow, infrastructure, and challenges. *Vicinagearth*, 1(1):9, 2024.

[8] Yingxuan Yang, Qiuying Peng, Jun Wang, Ying Wen, and Weinan Zhang. Llm-based multi-agent systems: Techniques and business perspectives. *arXiv preprint arXiv:2411.14033*, 2024.

[9] Junda He, Christoph Treude, and David Lo. Llm-based multi-agent systems for software engineering: Literature review, vision, and the road ahead. *ACM Transactions on Software Engineering and Methodology*, 34(5):1–30, 2025.

[10] Dino Nejašmić, Saša Mladenović, and Andrina Granić. Interaction with llm-based systems: A structured review and taxonomy of mechanisms and autonomy. *Applied Sciences*, 16(10):5001, 2026.

[11] Jing Xie and Liang Zhao. Research on resume screening and career matching model based on fuzzy natural language processing. *Journal of Computational Methods in Sciences and Engineering*, page 14727978251366558, 2025.

[12] Ingo Glöckner and Alois Knoll. A formal theory of fuzzy natural language quantification and its role in granular computing. In *Granular computing: An emerging paradigm*, pages 215–256. Springer, 2001.

[13] Noor Hanoon Haroon, Zainab R Abdulsada, Haider Sharif, Wasan Saad Ahmed, Munqith Saleem, and Israa Abed Jawad. Social media analysis using fuzzy natural language processing with an extension of semantic queries. In *2023 Annual International Conference on Emerging Research Areas: International Conference on Intelligent Systems (AICERA/ICIS)*, pages 1–6. IEEE, 2023.

[14] Ryosuke Fujioka, Shun'ichi Tano, Wataru Okamoto, Atsushi Inoue, and T Iwatani. Treatment of fuzziness in natural language by fuzzy lingual system-flins. In *Proceedings of 1995 IEEE International Conference on Fuzzy Systems.*, volume 2, pages 1045–1050. IEEE, 1995.

[15] Diego Fernando Coka Flores, Ignacio Fernando Barcos Arias, María Elena Infante Miranda, and Omar Mar Cornelio. Applying neutrosophic natural language processing to analyze complex phenomena in interdisciplinary contexts. *Neutrosophic Sets and Systems*, 74(1):26, 2024.

[16] Rosa M Rodriguez, Luis Martinez, and Francisco Herrera. Hesitant fuzzy linguistic term sets for decision making. *IEEE Transactions on fuzzy systems*, 20(1):109–119, 2011.

[17] Huchang Liao, Zeshui Xu, Enrique Herrera-Viedma, and Francisco Herrera. Hesitant fuzzy linguistic term set and its application in decision making: a state-of-the-art survey. *International Journal of Fuzzy Systems*, 20(7):2084–2110, 2018.

[18] Cuiping Wei, Na Zhao, and Xijin Tang. Operators and comparisons of hesitant fuzzy linguistic term sets. *IEEE Transactions on Fuzzy Systems*, 22(3):575–585, 2013.

[19] Ying-Ying Li, Hongyu Zhang, and Jian-Qiang Wang. Linguistic neutrosophic sets and their application in multicriteria decision-making problems. *International Journal for Uncertainty Quantification*, 7(2), 2017.

[20] Muhammad Saqlain, Poom Kumam, and Wiyada Kumam. Neutrosophic linguistic valued hypersoft set with application: Medical diagnosis and treatment. *Neutrosophic sets and systems*, 63:130–152, 2024.

[21] Hüseyin Kamacı. Linguistic single-valued neutrosophic soft sets with applications in game theory. *International Journal of Intelligent Systems*, 36(8):3917–3960, 2021.

[22] Hongming Zhang, Daniel Khashabi, Yangqiu Song, and Dan Roth. Transomcs: From linguistic graphs to commonsense knowledge. *arXiv preprint arXiv:2005.00206*, 2020.

[23] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Linguistic Geometry and its Applications*. Infinite Study, 2022.

[24] Vincent Archer. Mulling: multilevel linguistic graphs for knowledge extraction. In *Proceedings of TextGraphs-5-2010 Workshop on Graph-based Methods for Natural Language Processing*, pages 69–73, 2010.

[25] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Linguistic matrices*. Infinite Study, 2022.

[26] WB Vasantha Kandasamy, Florentin Smarandache, and Indra Venkatbabu. *Innovative uses of matrices*. Infinite Study, 2012.

[27] Markus Müller-Olm and Helmut Seidl. Analysis of modular arithmetic. In *European Symposium on Programming*, pages 46–60. Springer, 2005.

[28] Fangan Yssouf Dosso, Nadia El Mrabet, Nicolas Méloni, François Palma, and Pascal Véron. Friendly primes for efficient modular arithmetic using the polynomial modular number system. *Journal of cryptographic engineering*, 15(3):18, 2025.

[29] Markus Müller-Olm and Helmut Seidl. Analysis of modular arithmetic. *ACM Transactions on Programming Languages and Systems (TOPLAS)*, 29(5):29–es, 2007.

[30] Cai Heng Li. On isomorphisms of finite cayley graphs—a survey. *Discrete mathematics*, 256(1-2):301–334, 2002.

[31] Peter C Fishburn. Interval representations for interval orders and semiorders. *Journal of Mathematical Psychology*, 10(1):91–105, 1973.

[32] Yutaka Nakamura. Real interval representations. *Journal of Mathematical Psychology*, 46(2):140–177, 2002.

[33] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[34] Hans-Jürgen Zimmermann. *Fuzzy set theory—and its applications*. Springer Science & Business Media, 2011.

[35] Azriel Rosenfeld. Fuzzy graphs. In *Fuzzy sets and their applications to cognitive and decision processes*, pages 77–95. Elsevier, 1975.

[36] Takaaki Fujita and Florentin Smarandache. *A Comprehensive Survey of Soft Graphs and Rough Graphs*. Neutrosophic Science International Association (NSIA) Publishing House, 2026.

[37] Takaaki Fujita and Florentin Smarandache. *A Comprehensive and Updated Survey of Fuzzy, Neutrosophic, and Uncertain Sets*. Neutrosophic Science International Association (NSIA) Publishing House, 2026.

[38] Aram Noori Qadir, Atiqe Ur Rahman, and Pishtiwan Othman Sabir. An integrated approach of complex fuzzy hypersoft set and possibility degree setting to manage uncertainties associated with multi-attribute decision-making. *Journal of Uncertain Systems*, 2026.

[39] Zaffar Ahmed Shaikh, Muhammad Naveed Jafar, Ali Elrashidi, and Hamiden Abd El-Wahed Khalifa. Smart city traffic management using dynamic fuzzy hypersoft set algorithms. *Engineering, Technology & Applied Science Research*, 16(4):37935–37941, 2026.

[40] Muhammad Naveed Jafar, Muhammad Ahmad, and Kainat Muniba. Evaluation of advanced drinking water treatment technologies using pythagorean fuzzy hypersoft set-based multi-criteria. *IETI Transactions on Data Analysis and Forecasting*, 4(1):56–65, 2026.

[41] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[42] Takaaki Fujita. Hyperfuzzy and superhyperfuzzy group decision-making. *Spectrum of Decision Making and Applications*, pages 1–18, 2027.

[43] Changzhong Wang, Changyue Wang, Yuhua Qian, and Qiangkui Leng. Feature selection based on weighted fuzzy rough sets. *IEEE Transactions on Fuzzy Systems*, 32(7):4027–4037, 2024.

[44] Xinyu Su, Zhong Yuan, Baiyang Chen, Dezhong Peng, Hongmei Chen, and Yingke Chen. Detecting anomalies with granular-ball fuzzy rough sets. *Information Sciences*, 678:121016, 2024.

[45] Saba Ayub, Muhammad Shabir, Muhammad Riaz, Waqas Mahmood, Darko Bozanic, and Dragan Marinkovic. Linear diophantine fuzzy rough sets: A new rough set approach with decision making. *Symmetry*, 14(3):525, 2022.

[46] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[47] Krassimir Atanassov. Intuitionistic fuzzy sets. *International journal bioautomation*, 20:1, 2016.

[48] Krassimir T Atanassov and G Gargov. *Intuitionistic fuzzy logics*. Springer, 2017.

[49] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[50] Thanh Nha Nguyen and Quang-Thinh Bui. A framework of near-intuitionistic fuzzy soft sets with applications to uncertainty-based diagnosis. *Journal of Intelligent & Fuzzy Systems*, page 18758967261435701, 2026.

[51] Ema Carnia, Sukono, Moch Panji Agung Saputra, Mugi Lestari, Audrey Ariij Sya'imaa HS, Astrid Sulistya Azahra, and Mohd Zaki Awang Chek. Integrated framework of generalized interval-valued hesitant intuitionistic fuzzy soft sets with the ahp for investment decision-making under uncertainty. *Mathematics*, 13(19):3188, 2025.

[52] Neelam Kumari, Varsha Gautam, and Shivam Shreevastava. A review on rough, fuzzy rough and intuitionistic fuzzy rough sets based techniques for attribute reduction. *International Journal of System Assurance Engineering and Management*, pages 1–15, 2026.

[53] Shivani Singh and Tanmoy Som. Intuitionistic fuzzy rough sets: Theory to practice. *Mathematics in Computational Science and Engineering*, pages 91–133, 2022.

[54] Zhengliang Chen. A novel intuitionistic fuzzy rough sets-based clustering model based on aczel–alsina aggregation operators. *Symmetry*, 16(10):1292, 2024.

[55] Florentin Smarandache and Maissam Jdid. *An Overview of Neutrosophic and Plithogenic Theories and Applications.* Infinite Study, 2023.

[56] Florentin Smarandache and NM Gallup. Generalization of the intuitionistic fuzzy set to the neutrosophic set. In *International Conference on Granular Computing*, pages 8–42. Citeseer, 2006.

[57] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets.* Infinite study, 2010.

[58] Takaaki Fujita and Smarandache Florentin. *A Review and Introduction to Neutrosophic Applications across Various Scientific Fields.* Neutrosophic Science International Association (NSIA) Publishing House, 2025.

[59] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, 10:86–101, 2016.

[60] Nenad Stojanović, Marina Svičević, Tatjana Stojanović, Nemanja Vučićević, and Filip Andrić. A novel decision-making algorithm based on interval-valued bipolar neutrosophic soft sets. *Information*, 2026.

[61] MN Bharathi and G Jayalalitha. Neutrosophic soft sets and topsis mcdm using iteration method in one and two dimensions. In *AIP Conference Proceedings*, page 030022. AIP Publishing LLC, 2026.

[62] Irina Gladysheva, Raya Karlibaeva, Bobur Mirzayev, Tukhtabek Rakhimov, and Rustem Shichiyakh. An efficient approach for customer attrition prediction using regression-based feature reduction and neutrosophic soft set in big data ecosystems. *Engineering, Technology & Applied Science Research*, 16(3):36427–36431, 2026.

[63] Takaaki Fujita and Florentin Smarandache. *Handbook of Rough Set Extensions and Uncertainty Models.* Infinite Study, 2026.

[64] Muhammad Tahir and Muhammad Imran Shahid. An intelligent industry 5.0 logistics decision system: circular supply chain management with fermatean neutrosophic hypersoft sets and machine learning. *Journal of Contemporary Decision Science*, 2(1):260–286, 2026.

[65] Zaffar Ahmed Shaikh, Muhammad Naveed Jafar, Ali Elrashidi, Kainat Muniba, and Muhammad Saeed. Ranking cities for air quality and livability using dynamic single-valued neutrosophic hypersoft set similarity measures. *Engineering, Technology & Applied Science Research*, 16(5):39099–39108, 2026.

[66] Eman Hussein, Yousef Al-Qudah, HM Jaradat, Faisal Al-Sharqi, Saja Elnajar, and Ali Jaradat. Similarity measure and sine exponential measure of possibility interval-valued neutrosophic hypersoft sets and their. *Neutrosophic Sets and Systems*, page 41, 2025.

[67] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[68] Takaaki Fujita and Florentin Smarandache. *A comprehensive survey of set-theoretic concepts related to fuzzy, neutrosophic, and uncertain sets.* Infinite Study, 2026.

[69] Takaaki Fujita and Florentin Smarandache. *AI, Hyperstructural, SuperHyperstructural, Fuzzy, and Uncertain Analogues of the P versus NP Problem.* Neutrosophic Science International Association (NSIA) Publishing House, 2026.

[70] Takaaki Fujita and Florentin Smarandache. *Fuzzy, Neutrosophic, and Uncertain Graph Theory: Properties and Applications (II): Uncertain Planar Graph Theory.* Infinite Study, 2026.

[71] Takaaki Fujita and Florentin Smarandache. *Survey of Fuzzy and Uncertain Logical Operators.* Infinite Study, 2026.

[72] Takaaki Fujita and Florentin Smarandache. *Survey of Various Fuzzy and Uncertain Decision-Making Methods.* Infinite Study, 2026.

[73] Takaaki Fujita and Florentin Smarandache. A unified framework for u-structures and functorial structure: Managing super, hyper, superhyper, tree, and forest uncertain over/under/off models. *Neutrosophic Sets and Systems*, 91:337–380, 2025.

[74] Takaaki Fujita and Florentin Smarandache. *A Dynamic Survey of Fuzzy, Intuitionistic Fuzzy, Neutrosophic, Plithogenic, and Extensional Sets.* Neutrosophic Science International Association (NSIA), 2025.

[75] Mohammed A. Al Shumrani, Muhammad Gulistan, and Tahir Abbas. Multi-criteria decision making approach for solar energy implementation using n-cubic fuzzy interaction aggregation operators. *Scientific Reports*, 16, 2025.

[76] Sheikh Rashid, Muhammad Gulistan, Tahir Abbas, Kholood Mohammad Alsager, and Muhammad Rahim. Multi-attribute decision-making methods to low-carbon green supply chain management in n-cubic fuzzy aggregation operators. *International Journal of Fuzzy Systems*, 28:1629 – 1641, 2025.

[77] John N Mordeson and Premchand S Nair. *Fuzzy graphs and fuzzy hypergraphs*, volume 46. Physica, 2012.

[78] P. R. Kavyasree and B. Surender Reddy. N-cubic sets applied to linear spaces. *Kragujevac Journal of Mathematics*, 2022.

[79] Gundeti Soujanya and B. Surender Reddy. N-cubic picture fuzzy linear spaces. *Indian Journal Of Science And Technology*, 2024.

[80] Young Bae Jun, Seok Zun Song, and Seon Jeong Kim. N-hyper sets. *Mathematics*, 2018.

[81] Witold Pedrycz. Shadowed sets: representing and processing fuzzy sets. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 28(1):103–109, 1998.

[82] Mingjie Cai, Qingguo Li, and Guangming Lang. Shadowed sets of dynamic fuzzy sets. *Granular Computing*, 2:85 – 94, 2016.

[83] Gianpiero Cattaneo and Davide Ciucci. Shadowed sets and related algebraic structures. *Fundamenta Informaticae*, 55(3-4):255–284, 2003.

[84] RA Borzooei and Hossein Rashmanlou. Cayley interval-valued fuzzy graphs. *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, 78(3):83–94, 2016.

[85] Madhumangal Pal, Sovan Samanta, and Hossein Rashmanlou. Some results on interval-valued fuzzy graphs. *International Journal of Computer Science and Electronics Engineering*, 3(3):205–211, 2015.

[86] Xindong Peng. New operations for interval-valued pythagorean fuzzy set. *Scientia Iranica*, 26:1049–1076, 2018.

[87] Weijian Ren and Yuhong Du. Development and application of an interval-valued pythagorean fuzzy set-based grade evaluation method to foreign fiber content. *Journal of Natural Fibers*, 20, 2023.

[88] Manish Kumar Gunjan, Amal Kumar Adak, and Davood Darvishi Salookolaei. Fermatean fuzzy artinian and noetherian rings. *Asian Research Journal of Mathematics*, 21(5):179–190, 2025.

[89] Fei Gao, Meihong Han, Siyang Wang, and Jie Gao. A novel fermatean fuzzy bwm-vikor based multi-criteria decision-making approach for selecting health care waste treatment technology. *Eng. Appl. Artif. Intell.*, 127:107451, 2024.

[90] Ping Wang, Jie Wang, Guiwu Wei, and Cun Wei. Similarity measures of q-rung orthopair fuzzy sets based on cosine function and their applications. *Mathematics*, 2019.

[91] Muhmamet Deveci, Dragan Pamucar, Ilgin Gokasar, Mario Köppen, and Brij Bhooshan Gupta. Personal mobility in metaverse with autonomous vehicles using q-rung orthopair fuzzy sets based opa-rafsi model. *IEEE Transactions on Intelligent Transportation Systems*, 24:15642–15651, 2023.

[92] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[93] Humberto Bustince and P Burillo. Vague sets are intuitionistic fuzzy sets. *Fuzzy sets and systems*, 79(3):403–405, 1996.

[94] Juanjuan Lu, Linli Zhu, and Wei Gao. Structured representation of fuzzy data by bipolar fuzzy hypergraphs. In *International Conference on Machine Learning for Cyber Security*, pages 663–676. Springer, 2022.

[95] Wen-Ran Zhang. Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. *NAFIPS/IFIS/NASA '94. Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference. The Industrial Fuzzy Control and Intellige*, pages 305–309, 1994.

[96] Chen Shouyu and Guo Yu. Variable fuzzy sets and its application in comprehensive risk evaluation for flood-control engineering system. *Fuzzy Optimization and Decision Making*, 5(2):153–162, 2006.

[97] Wen-chuan Wang, Dong-mei Xu, Kwok-wing Chau, and Guan-jun Lei. Assessment of river water quality based on theory of variable fuzzy sets and fuzzy binary comparison method. *Water resources management*, 28(12):4183–4200, 2014.

[98] Yunhai Fang, Xilai Zheng, Hui Peng, Huan Wang, and Jia Xin. A new method of the relative membership degree calculation in variable fuzzy sets for water quality assessment. *Ecological indicators*, 98:515–522, 2019.

[99] Vasile Patrascu. A new penta-valued logic based knowledge representation. *arXiv preprint arXiv:1502.05562*, 2015.

[100] Vasile Patrascu. Similarity, cardinality and entropy for bipolar fuzzy set in the framework of penta-valued representation. *arXiv preprint arXiv:1506.02060*, 2015.

[101] Yong-Uk Cho, Young-Bae Jun, and Seok-Zun Song. Bifuzzy ideals of pseudo mv-algebras. *JOURNAL OF APPLIED MATHEMATICS AND COMPUTING*, 22(1/2):475, 2006.

[102] Tadeusz Gerstenkorn and Jacek Mańko. Bifuzzy probabilistic sets. *Fuzzy sets and systems*, 71(2):207–214, 1995.

[103] Muhammad Akram. Single-valued neutrosophic planar graphs. *International Journal of Algebra*, 5:157, 2016.

[104] Raja Muhammad Hashim, Muhammad Gulistan, Musaed A. Alhussein, Khursheed Aurangzeb, and Adnan Khurshid. A decision-making system for managing renewable energy alternatives and strategy using triangular neutrosophic bipolar fuzzy topsis. *International Journal of Information Technology & Decision Making*, 2025.

[105] Pu Ji, Jian qiang Wang, and Hong yu Zhang. Frank prioritized bonferroni mean operator with single-valued neutrosophic sets and its application in selecting third-party logistics providers. *Neural Computing and Applications*, 30:799–823, 2018.

[106] Mohammad Hamidi and Arsham Borumand Saeid. Achievable single-valued neutrosophic graphs in wireless sensor networks. *New Mathematics and Natural Computation*, 14(02):157–185, 2018.

[107] Bui Cong Cuong. Picture fuzzy sets. *Journal of Computer Science and Cybernetics*, 30:409, 2015.

[108] Sankar Das, Ganesh Ghorai, and Madhumangal Pal. Picture fuzzy tolerance graphs with application. *Complex & Intelligent Systems*, 8(1):541–554, 2022.

[109] Chao Wang, Minghu Ha, and Xiaowei Liu. A mathematical model of ternary fuzzy set for voting. *Journal of Intelligent & Fuzzy Systems*, 29:2381 – 2386, 2015.

[110] Jingqian Wang, Xiaohong Zhang, and Qingqing Hu. Three-way fuzzy sets and their applications (ii). *Axioms*, 11(10):532, 2022.

[111] Qingqing Hu and Xiaohong Zhang. Three-way fuzzy sets and their applications (iii). *Axioms*, 12:57, 2023.

[112] CJ Hinde, RS Patching, and SA McCoy. Inconsistent intuitionistic fuzzy sets and mass assignment. *Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets, Generalized Nets and Related Topics. Foundations*, pages 133–153, 2007.

[113] G Muhiuddin, Sun Shin Ahn, Chang Su Kim, and Young Bae Jun. Stable cubic sets. *J. Comput. Anal. Appl*, 23(5):802–819, 2017.

[114] Madad Khan, Young Bae Jun, Muhammad Gulistan, and Naveed Yaqoob. The generalized version of jun's cubic sets in semigroups. *Journal of Intelligent & Fuzzy Systems*, 28(2):947–960, 2015.

[115] Muhammad Akram, Danish Saleem, and Talal Al-Hawary. Spherical fuzzy graphs with application to decision-making. *Mathematical and Computational Applications*, 25(1):8, 2020.

[116] Fatma Kutlu Gündoğdu and Cengiz Kahraman. Spherical fuzzy sets and spherical fuzzy topsis method. *Journal of intelligent & fuzzy systems*, 36(1):337–352, 2019.

[117] Khushbakhat Asif, Muhammad Kamran Jamil, Hanen Karamti, Muhammad Azeem, and Kifayat Ullah. Randić energies for t-spherical fuzzy hamacher graphs and their applications in decision making for business plans. *Computational and Applied Mathematics*, 42:1–32, 2023.

[118] Abhishek Guleria and Rakesh Kumar Bajaj. T-spherical fuzzy graphs: operations and applications in various selection processes. *Arabian Journal for Science and Engineering*, 45(3):2177–2193, 2020.

[119] M Murali Krishna Rao and B Venkateswarlu. Tripolar fuzzy interior ideals of a γ-semiring. *Asia Pacific Journal of Management*, 5(2):192–207, 2018.

[120] Takaaki Fujita. A mathematical study of tripolar intuitionistic fuzzy graphs. *International Journal of Computer Application (IJCA)*, 16(4):16/1–16/11, 2026.

[121] M Murali Krishna Rao, B Venkateswarlu, and Y Adi Narayana. Tripolar fuzzy soft ideals and tripolar fuzzy soft interior ideals over semiring. *Italian journal of pure and applied Mathematics*, page 731743, 2019.

[122] M Murali Krishna Rao. Tripolar fuzzy interior ideals of γ-semigroup. *Ann. Fuzzy Math. Inform*, 15(2):199–206, 2018.

[123] Shawkat Alkhazaleh. Neutrosophic vague set theory. *Critical Review*, 10:29–39, 2015.

[124] Ashraf Al-Quran, Hazwani Hashim, and Lazim Abdullah. A hybrid approach of interval neutrosophic vague sets and dematel with new linguistic variable. *Symmetry*, 12(2):275, 2020.

[125] R Radha, A Stanis Arul Mary, and Florentin Smarandache. Quadripartitioned neutrosophic pythagorean soft set. *International Journal of Neutrosophic Science (IJNS) Volume 14, 2021*, page 11, 2021.

[126] Satham Hussain, Jahir Hussain, Isnaini Rosyida, and Said Broumi. Quadripartitioned neutrosophic soft graphs. In *Handbook of Research on Advances and Applications of Fuzzy Sets and Logic*, pages 771–795. IGI Global, 2022.

[127] Mouhamed Bayane Bouraima, Ertugrul Ayyıldız, Melike Erdogan, and Dragan Pamucar. An interval-valued intuitionistic fuzzy group decision model for evaluation of cross-border railway development. *Cognitive Computation*, 18(1):22, 2026.

[128] Ravindar Raj Chinnappan and Dhanasekar Sundaram. Image enhancement framework combining interval-valued intuitionistic fuzzy sets and fractional sobel operator. *Artificial Intelligence Review*, 58(9):291, 2025.

[129] Yukun Hu, Hong Wang, Chunhui Li, Ang Xu, Li Yi, Chao Liang, Bingyang Xi, and Fang Dai. An integrated approach for design concept evaluation based on interval-valued pythagorean fuzzy set and consensus model. *Kybernetes*, 55(8):3611–3636, 2026.

[130] Yoojeong Oh, Hans Pasman, Safyan Akram Khan, and Sunhwa Park. Hydrogen storage selection for saudi arabia: A multi-criteria decision making under interval-valued pythagorean fuzzy environment. *International Journal of Hydrogen Energy*, 105:1281–1293, 2025.

[131] Sudan Jha, Raghvendra Kumar, Ishaani Priyadarshini, Florentin Smarandache, Hoang Viet Long, et al. Neutrosophic image segmentation with dice coefficients. *Measurement*, 134:762–772, 2019.

[132] Takaaki Fujita and Florentin Smarandache. *Some types of hyperneutrosophic set (5): Support, paraconsistent, faillibilist, and others.* Infinite Study, 2025.

[133] Takaaki Fujita and Florentin Smarandache. *Neutrosophy Revisited: Formalizing Core Concepts from Nidus idearum (Book Series) and Related Research.* Infinite Study, 2025.

[134] Nguyen Xuan Thao, Florentin Smarandache, and Nguyen Van Dinh. *Support-neutrosophic set: a new concept in soft computing.* Infinite Study, 2017.

[135] Takaaki Fujita and Florentin Smarandache. Local-neutrosophic logic and local-neutrosophic sets: Incorporating locality with applications. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 51, 2025.

[136] Abdul Wahab Mustafa, Zia Bashir, Jawad Ali, and Muhammed I Syam. A unified framework of information measures for complex linear diophantine fuzzy sets with application to chronic kidney disease. *Scientific Reports*, 2026.

[137] Masooma Raza Hashmi, Muhammad Riaz, Arshid Mahmood, Muhammad Ajmaeen, and Muhammad Aslam. Optimizing performance metrics with new distance and similarity measures using control parameters of linear diophantine fuzzy sets. *Engineering Applications of Artificial Intelligence*, 162:112557, 2025.

[138] Ilanthenral Kandasamy. Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm. *Journal of Intelligent systems*, 27(2):163–182, 2018.

[139] Qiuyan Zhao and Wentao Li. Incorporating intelligence in multiple-attribute decision-making using algorithmic framework and double-valued neutrosophic sets: Varied applications to employment quality evaluation for university graduates. *Neutrosophic Sets and Systems*, 76:59–78, 2025.

[140] Qaisar Khan, Peide Liu, and Tahir Mahmood. Some generalized dice measures for double-valued neutrosophic sets and their applications. *Mathematics*, 6(7):121, 2018.

[141] Bin Zhu and Zeshui Xu. Some results for dual hesitant fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 26(4):1657–1668, 2014.

[142] José Carlos R Alcantud, Gustavo Santos-García, Xindong Peng, and Jianming Zhan. Dual extended hesitant fuzzy sets. *Symmetry*, 11(5):714, 2019.

[143] D. Amsaveni M.Divya. Bipolar intuitionistic fuzzy hyper soft set. *International Journal of Applied Mathematics*, 2025.

[144] Bavanari Satyanarayana, Shake Baji, D. Ramesh, Aiyared Iampan, and Gadde Sambasiva Rao. Bipolar intuitionistic fuzzy ideals in bck-algebras. *European Journal of Pure and Applied Mathematics*, 2026.

[145] Abd Ulazeez M. Alkouri and Zeyad Alshboul. Environmental impact assessment using bipolar complex intuitionistic fuzzy sets. *Soft Computing*, 29:3795 – 3809, 2025.

[146] A Iampan, T Eswarlal, et al. Bipolar vague relations and their application to real-world problems. *IAENG International Journal of Computer Science*, 51(9):1316, 2024.

[147] U Venkata Kalyani and T Eswarlal. Bipolar vague cosets. *Adv. Math., Sci. J*, 9:6777–6787, 2020.

[148] M Balamurugan, Khalil H Hakami, Moin A Ansari, Anas Al-Masarwah, and K Loganathan. Quadri-polar fuzzy fantastic ideals in bci-algebras: A topsis framework and application. *European Journal of Pure and Applied Mathematics*, 17(4):3129–3155, 2024.

[149] Florentin Smarandache. Ambiguous set is a subclass of the double refined indeterminacy neutrosophic set, and of the refined neutrosophic set in general. *Neutrosophic Sets & Systems*, 58, 2023.

[150] Pritpal Singh et al. Ambiguous set theory: A new approach to deal with unconsciousness and ambiguousness of human perception. *Full Length Article*, 5(1):52–2, 2023.

[151] Florentin Smarandache and F Smarandache. Ambiguous set brings nothing new. *nidus idearum*, page 24, 2024.

[152] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[153] Takaaki Fujita. Review of rough turiyam neutrosophic directed graphs and rough pentapartitioned neutrosophic directed graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:48–79, 2025.

[154] Takaaki Fujita and Florentin Smarandache. Pythagorean, fermatean, and complex turiyam neutrosophic graphs. *SciNexuses*, 2:39–63, 2025.

[155] Prem Kumar Singh, Naveen Surathu, Ghattamaneni Surya Prakash, et al. Turiyam based four way unknown profile characterization on social networks. *Full Length Article*, 10(2):27–7, 2024.

[156] Rama Mallick and Surapati Pramanik. Pentapartitioned neutrosophic set and its properties. *Neutrosophic Sets and Systems*, 35:49, 2020.

[157] Radha R. and Stanis Arul Mary. Pentapartitioned neutrosophic pythagorean set. *International Research Journal on Advanced Science Hub*, 2021.

[158] Suman Das, Rakhal Das, and Binod Chandra Tripathy. Topology on rough pentapartitioned neutrosophic set. *Iraqi Journal of Science*, 2022.

[159] Jiarong Jia. Triple-valued neutrosophic off for public digital cultural service level evaluation in craft and art museums. *Neutrosophic Sets and Systems*, 88(1):44, 2025.

[160] Takaaki Fujita. Triple-valued neutrosophic set, quadruple-valued neutrosophic set, quintuple-valued neutrosophic set, and double-valued indetermsoft set. *Neutrosophic Systems with Applications*, 25(5):3, 2025.

[161] Hongxin Wang. Professional identity formation in traditional chinese medicine students: An educational perspective using triple-valued neutrosophic set. *Neutrosophic Sets and Systems*, 88:83–92, 2025.

[162] Aliya Fahmi, Aziz Khan, and Thabet Abdeljawad. A novel approach to group decision-making using generalized bipolar neutrosophic sets. *PLOS One*, 20, 2025.

[163] Chunfang Liu. A novel distance measure of bipolar neutrosophic sets with an application in pattern classification. *Symmetry*, 2025.

[164] Lazim Abdullah and Nor Liyana Amalini Mohd Kamal. Application of dematel based on bipolar neutrosophic sets for sustainable agriculture practices. *Neutrosophic Systems with Applications*, 2025.

[165] Fang Sun, Mifeng Ren, Yujing Shi, and Xuanbai Feng. Quadruple-valued neutrosophic offset for maintenance platforms design quality evaluation of digital monitoring and remote operation in new energy power generation systems. *Neutrosophic Sets and Systems*, 88:940–953, 2025.

[166] M. Priyadharshini and D. Jayanthi. Similarity measures on cubic intuitionistic fuzzy sets. *Notes on Intuitionistic Fuzzy Sets*, 2025.

[167] Priyadharshini Manoharan, Jayanthi Duraisamy, and Saranya Manoharan. Cardinality and relative cardinality on cubic intuitionistic fuzzy sets. *International Journal of Information Technology*, 16:4059 – 4068, 2024.

[168] Harish Garg and Gagandeep Kaur. Novel distance measures for cubic intuitionistic fuzzy sets and their applications to pattern recognitions and medical diagnosis. *Granular Computing*, 5:169 – 184, 2018.

[169] Muhammad Sajid, Khuram Ali Khan, Atiqe Ur Rahman, Sanaa Ahmed Bajri, Alhanouf Alburaikan, and Hamiden Abd El-Wahed Khalifa. A novel algorithmic multi-attribute decision-making framework for solar panel selection using modified aggregations of cubic intuitionistic fuzzy hypersoft set. *Heliyon*, 10, 2024.

[170] Maha Mohammed Saeed, Sami Ullah Khan, Fatima Suriyya, Arif Mehmood, and Jamil J Hamja. Interval-valued complex neutrosophic sets and complex neutrosophic soft topological spaces. *International Journal of Analysis and Applications*, 23:132–132, 2025.

[171] Samaneh Salari and Ali Karimi. Introducing an integrated approach for fire safety assessment in healthcare facilities by interval valued neutrosophic-ahp and fuzzy inference system. *Heliyon*, 11(2), 2025.

[172] Raed Hatamleh, Nasir Odat, Abdallah Al-Husban, Arif Mehmood Khattak, Alaa M Abd El-latif, Husham M Attaalfadeel, Walid Abdelfattah, and Ahmad M Abdel-Mageed. Mapping love: A heptapartitioned neutrosophic machine learning study of university students' romantic sensations. *European Journal of Pure and Applied Mathematics*, 18(3):6484–6484, 2025.

[173] Maha Mohammed Saeed, Raed Hatamleh, Hamza Ali Abujabal, Yahya Khan, Abdallah Al-Husban, Amy A Laja, Sulfaisa MM Pangilan, Cris L Armada, Jamil J Hamja, and Arif Mehmood Khattak. Heptapartitioned neutrosophic soft topologies and machine learning techniques for exploring romantic feelings. *European Journal of Pure and Applied Mathematics*, 18(3):6221–6221, 2025.

[174] Wenwen Meng. Quintuple-valued neutrosophic offset for quality evaluation of cross-border e-commerce talent training based on artificial intelligence. *Neutrosophic Sets and Systems*, 88(1):54, 2025.

[175] Yuan Sun. Assessing the performance of basic education informatization in the digital age using quintuple-valued neutrosophic set. *Neutrosophic Sets and Systems*, 88(1):18, 2025.

[176] Takaaki Fujita, Raed Hatamleh, and Ahmed Heilat. Advanced partitioned neutrosophic offsets, oversets, and undersets: Modeling "neither agree nor disagree" and beyond. *Statistics, Optimization & Information Computing*, 2026.

[177] Takaaki Fujita. Advanced partitioned neutrosophic sets: Formalization of hexa-, hepta-, octa-, nona-, and deca-partitioned structures. *Abhath Journal of Basic and Applied Sciences*, 4(2):40–60, 2025.

[178] CV Malavika, Kalaivani Chandran, Sofia Jennifer John, and Florentin Smarandache. Python based tool for neutrosophic cubic sets and their operations. *Journal of Fuzzy Extension and Applications*, 7(2):592–614, 2026.

[179] Majid Khan, Muhammad Gulistan, and Mohammed M. Ali Al-Shamiri. The approach of induced generalized neutrosophic cubic shapley choquet integral aggregation operators via the codas method to solve distance-based multicriteria decision-making problems. *Journal of Mathematics*, 2022, 2022.

[180] Majid Khan, Muhammad Gulistan, Musaed A. Alhussein, Khursheed Aurangzeb, and Adnan Khurshid. Navigating ambiguity: A novel neutrosophic cubic shapley normalized weighted bonferroni mean aggregation operator with application in the investment environment. *Heliyon*, 10, 2024.

[181] Yi-Ming Li, Majid Khan, Adnan Khurshid, Muhammad Gulistan, Ateeq ur Rehman, Mumtaz Ali, Shahab Abdulla, and Aitazaz Ahsan Farooque. Designing pentapartitioned neutrosophic cubic set aggregation operator-based air pollution decision-making model. *Complex & Intelligent Systems*, pages 1–18, 2023.

[182] Adil Darvesh, Khalid Naeem, and Syed Bilal Hussain Shah. Scrutiny of neutrosophic cubic fuzzy data with different parametric values using bonferroni weighted mean: A robust mcdm approach. *Decision making advances*, 3(1):245–265, 2025.

[183] Takaaki Fujita. A mathematical study of tripolar neutrosophic graphs. *International Journal of Research in Engineering & Science (IJRES)*, 16(4):13, 2026.

[184] Giovanni Panti. Multi-valued logics. In *Quantified representation of uncertainty and imprecision*, pages 25–74. Springer, 1998.

[185] Florentin Smarandache. Plithogeny, plithogenic set, logic, probability, and statistics. *arXiv preprint arXiv:1808.03948*, 2018.

[186] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Plithogenic Graphs*. Infinite Study, 2020.

[187] Fujita Takaaki and Arif Mehmood. Iterative multifuzzy set, iterative multineutrosophic set, iterative multisoft set, and multiplithogenic sets. *Neutrosophic Computing and Machine Learning*, 41:1–30, 2025.

[188] Hilah Awad Alharbi and Kholood Mohammad Alsager. Fermatean m-polar fuzzy soft rough sets with application to medical diagnosis. *AIMS Mathematics*, 10(6):14314–14346, 2025.

[189] Harish Garg, Muhammad Riaz, Muhammad Abdullah Khokhar, and Maryam Saba. Correlation measures for cubic m-polar fuzzy sets with applications. *Mathematical Problems in Engineering*, 2021.

[190] Mohammad Mohseni Takallo, Rajab Ali Borzooei, Young Bae Jun, and Sun Shin Ahn. An a-ideal of bci-algebras in connection with multipolar fuzzy sets. *New Math. Nat. Comput.*, 17:553–570, 2021.

[191] Muhammad Arshad, Muhammad Saeed, and Atiqe Ur Rahman. Convexity cum concavity on refined fuzzy set with some properties. *Uncertainty Discourse and Applications*, 1(1):140–150, 2024.

[192] Florentin Smarandache. n-valued refined neutrosophic logic and its applications to physics. *Infinite study*, 4:143–146, 2013.

[193] Rajab Ali Borzooei, Hee Sik Kim, Young Bae Jun, and Sun Shin Ahn. On multipolar intuitionistic fuzzy b-algebras. *Mathematics*, 2020.

[194] Lian Chen, Aysha Khan, Maryam Akhoundi, AliAsghar Talebi, G. Muhiuddin, and Syed Bahman Hosseinzadeh Sadati. A study on m-polar interval-valued intuitionistic fuzzy graphs with application in management. *Journal of Mathematics*, 2022.

[195] K Sankar and D Ezhilmaran. Morphism of m-polar intuitionistic fuzzy graphs. *Journal of Computational and Theoretical Nanoscience*, 15(6):2277–2282, 2018.

[196] Vakkas Uluçay. Some concepts on interval-valued refined neutrosophic sets and their applications. *Journal of Ambient Intelligence and Humanized Computing*, 12(7):7857–7872, 2021.

[197] Khalid Naeem and Bijan Divvaz. Information measures for madm under m-polar neutrosophic environment. *Granular Computing*, 8(3):597–616, 2023.

[198] Basavaraj V Hiremath, Durga Nagarajan, Hossein Rashmanlou, Farshid Mofidnakhaei, et al. m-polar quadripartitioned neutrosophic graphs with applications in decision-making for mobile network selection. *Neutrosophic Sets & Systems*, 82:458, 2025.

[199] Florentin Smarandache. *Introduction to the multineutrosophic set.* Infinite Study, 2023.

[200] Florentin Smarandache. Neutrosophic set is a generalization of intuitionistic fuzzy set, inconsistent intuitionistic fuzzy set (picture fuzzy set, ternary fuzzy set), pythagorean fuzzy set, spherical fuzzy set, and q-rung orthopair fuzzy set, while neutrosophication is a generalization of regret theory, grey system theory, and three-ways decision (revisited). *Journal of New Theory*, 29:1–31, 2019.

[201] Florentin Smarandache. *Nidus Idearum. Scilogs, XIII: Structure/NeutroStructure/AntiStructure.* Infinite Study, 2024.

[202] Nadia Mushtaq Gardazi, Ali Daud, Muhammad Kamran Malik, Amal Bukhari, Tariq Alsahfi, and Bader Alshemaimri. Bert applications in natural language processing: a review. *Artificial Intelligence Review*, 58(6):166, 2025.

[203] Xue Jiang, Weiren Wang, Shaohan Tian, Hao Wang, Turab Lookman, and Yanjing Su. Applications of natural language processing and large language models in materials discovery. *npj Computational Materials*, 11(1):79, 2025.

[204] Partha Pakray, Alexander Gelbukh, and Sivaji Bandyopadhyay. Natural language processing applications for low-resource languages. *Natural Language Processing*, 31(2):183–197, 2025.

[205] Simona Frenda, Gavin Abercrombie, Valerio Basile, Alessandro Pedrani, Raffaella Panizzon, Alessandra Teresa Cignarella, Cristina Marco, and Davide Bernardi. Perspectivist approaches to natural language processing: a survey: S. frenda et al. *Language Resources and Evaluation*, 59(2):1719–1746, 2025.

[206] Stefano Crespi Reghizzi, Luca Breveglieri, and Angelo Morzenti. *Formal languages and compilation.* Springer, 2013.

[207] Michael A Harrison. *Introduction to formal language theory.* Addison-Wesley Longman Publishing Co., Inc., 1978.

[208] Christopher D Manning. *Foundations of statistical natural language processing.* The MIT Press, 1999.

[209] Takaaki Fujita. Theoretical interpretations of large uncertain and hyper language models: Advancing natural uncertain and hyper language processing. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 245, 2025.

[210] Borzoo Bonakdarpour and Sarai Sheinvald. Finite-word hyperlanguages. *Information and Computation*, 295:104944, 2023.

[211] Takaaki Fujita. A concise review on various concepts of superhyperstructures. *SuperHyperTopologies and SuperHyper-Structures with their Applications*, page 97, 2025.

[212] Takaaki Fujita and Florentin Smarandache. *Hyperstructures and Superhyperstructures: Theory and Selected Applications.* Infinite Study, 2026.

[213] Takaaki Fujita. Natural n-superhyper plithogenic language. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, page 294, 2025.

[214] Takaaki Fujita. Exploration of graph classes and concepts for superhypergraphs and n-th power mathematical structures. *Advancing Uncertain Combinatorics through Graphization, Hyperization, and Uncertainization: Fuzzy, Neutrosophic, Soft, Rough, and Beyond*, 3(4):512, 2025.

[215] Takaaki Fujita. A short note on hyperfuzzy directed graphs. *International Journal of Emerging Trends in Engineering and Development (IJETED)*, 16(4):18/1–18/7, 2026.

[216] Takaaki Fujita, Iqbal M Batiha, Nidal Anakira, Mohammad S Hijazi, Areen Al-Khateeb, and Tala Sasa. Hyperfuzzy and superhyperfuzzy weighted averages. *Statistics, Optimization & Information Computing*, 15(5):4021–4033, 2026.

[217] Takaaki Fujita. The hyperfuzzy vikor and hyperfuzzy dematel methods for multi-criteria decision-making. *Spectrum of Decision Making and Applications*, 3(1):292–315, 2026.

[218] Bekir Tanay and M Burc Kandemir. Topological structure of fuzzy soft sets. *Computers & Mathematics with Applications*, 61(10):2952–2957, 2011.

[219] Jiří Močkoř and Petr Hurtík. Approximations of fuzzy soft sets by fuzzy soft relations with image processing application. *Soft Computing*, 25(10):6915–6925, 2021.

[220] Abdulaziz S Alkabaa, Osman Taylan, Hanan S Alqabbaa, and Bulent Guloglu. Assessing occupational work-related stress and anxiety of healthcare staff during covid-19 using fuzzy natural language-based association rule mining. In *Healthcare*, page 1745, 2025.

[221] José R González de Mendívil and Jose R Garitagoitia. Fuzzy languages with infinite range accepted by fuzzy automata: Pumping lemma and determinization procedure. *Fuzzy Sets and Systems*, 249:1–26, 2014.

[222] José R González de Mendívil. A generalization of myhill–nerode theorem for fuzzy languages. *Fuzzy Sets and Systems*, 301:103–115, 2016.

[223] George Rahonis. Fuzzy languages. In *Handbook of Weighted Automata*, pages 481–517. Springer, 2009.

[224] Naeemeh Adel. *Fuzzy natural language similarity measures through computing with words.* PhD thesis, Manchester Metropolitan University, 2022.

[225] Said Broumi, Mohamed Talea, Assia Bakali, Florentin Smarandache, and PK Kishore Kumar. Shortest path problem on single valued neutrosophic graphs. In *2017 international symposium on networks, computers and communications (ISNCC)*, pages 1–6. IEEE, 2017.

[226] Muhammad Saqlain, José M Merigó, Vladimir Simic, and Broumi Said. Twenty-five years of neutrosophic sets: A bibliometric analysis. *Applied Research Advances*, pages 1–22, 2026.

[227] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[228] Fazeelat Sultana, Muhammad Gulistan, Mumtaz Ali, Naveed Yaqoob, Muhammad Khan, Tabasam Rashid, and Tauseef Ahmed. A study of plithogenic graphs: applications in spreading coronavirus disease (covid-19) globally. *Journal of ambient intelligence and humanized computing*, 14(10):13139–13159, 2023.

[229] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[230] Talal Al-Hawary. Complete fuzzy graphs. *International Journal of Mathematical Combinatorics*, 4:26, 2011.

[231] Muhammad Akram, Bijan Davvaz, and Feng Feng. Intuitionistic fuzzy soft k-algebras. *Mathematics in Computer Science*, 7:353–365, 2013.

[232] Raúl González Salas, Jhael Mikaela Gómez Ashqui, and Marco Paúl Medina Valencia. Assessing viral susceptibility in tilapia species using plithogenic fuzzy soft set: a multi-expert approach for tilv. *Neutrosophic Sets and Systems*, 89:185–196, 2025.

[233] Nivetha Martin, Seyyed Ahmad Edalpathnath, Florentin Smarandache, et al. Extended plithogenic hypersoft sets in decision making on electric vehicle charging infrastructure. *Journal of Fuzzy Extension and Applications*, page e226517, 2025.

[234] Takaaki Fujita. Plithogenic fuzzy natural language and its upside-down logic: A framework for contradiction-aware reasoning, 2025.

[235] Takaaki Fujita. Meta-natural language (natural language of natural languages), 2025.

[236] WB Vasantha Kandasam, K Ilanthenral, and Florentin Smarandache. *Linguistic Functions*. Infinite Study, 2022.

[237] Yifan Feng, Haoxuan You, Zizhao Zhang, Rongrong Ji, and Yue Gao. Hypergraph neural networks. In *Proceedings of the AAAI conference on artificial intelligence*, pages 3558–3565, 2019.

[238] Derun Cai, Moxian Song, Chenxi Sun, Baofeng Zhang, Shenda Hong, and Hongyan Li. Hypergraph structure learning for hypergraph neural networks. In *IJCAI*, pages 1923–1929, 2022.

[239] George Karypis, Rajat Aggarwal, Vipin Kumar, and Shashi Shekhar. Multilevel hypergraph partitioning: Application in vlsi domain. In *Proceedings of the 34th annual Design Automation Conference*, pages 526–529, 1997.

[240] George Karypis. Multilevel hypergraph partitioning. In *Multilevel Optimization in VLSICAD*, pages 125–154. Springer, 2003.

[241] Ehab Roshdy, Marwa Khashaba, and Mariam Emad Ahmed Ali. Neutrosophic super-hypergraph fusion for proactive cyberattack countermeasures: A soft computing framework. *Neutrosophic Sets and Systems*, 94(1):15, 2025.

[242] Takaaki Fujita and Florentin Smarandache. *Representing higher-order networks: A survey of graph-based frameworks*. Infinite Study, 2026.

[243] Masoud Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50:480–487, 2022.

[244] Florentin Smarandache. *Introduction to the n-SuperHyperGraph-the most general form of graph today*. Infinite Study, 2022.

[245] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Linguistic Multidimensional Spaces*. Infinite Study, 2023.

[246] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Linguistic Semilinear Algebras and Linguistic Semivector Spaces*. Infinite Study, 2022.

[247] WB Vasantha Kandasamy, Florentin Smarandache, and K Amal. *Fuzzy linguistic topological spaces*. Infinite Study, 2012.

[248] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Special Subset Linguistic Topological Spaces*. Infinite Study, 2023.

[249] Takaaki Fujita and Florentin Smarandache. *Survey of Fuzzy and Uncertain Logical Operators*. Infinite Study, 2026.

[250] Takaaki Fujita. A proposed mathematical framework for uncertain it service management and agile it service management. *Neutrosophic Knowledge*, 11:45–59, 2026.

[251] Takaaki Fujita. Operations, similarity measures, and rough-set approximations for uncertain sets. *Vietnam Journal of Science, Technology and Engineering*, 2026.

[252] Takaaki Fujita and Florentin Smarandache. *Survey of Various Fuzzy and Uncertain Decision-Making Methods*. Infinite Study, 2026.

[253] Badredine Arfi. *Linguistic fuzzy logic methods in social sciences*, volume 253. Springer, 2010.

[254] Badredine Arfi. Fuzzy decision making in politics: A linguistic fuzzy-set approach (lfsa). *Political Analysis*, 13(1):23–56, 2005.

[255] Shahzaib Ashraf. Picture fuzzy linguistic sets and their applications for multi-attribute group. *Nucleus*, 2018.

[256] Chengzu Bai, Ren Zhang, Longxia Qian, and Yaning Wu. Comparisons of probabilistic linguistic term sets for multi-criteria decision making. *Knowledge-Based Systems*, 119:284–291, 2017.

[257] Francisco Herrera, Enrique Herrera-Viedma, and Luis Martínez. A fuzzy linguistic methodology to deal with unbalanced linguistic term sets. *IEEE Transactions on fuzzy Systems*, 16(2):354–370, 2008.

[258] Huchang Liao, Zeshui Xu, Xiao-Jun Zeng, and José M Merigó. Qualitative decision making with correlation coefficients of hesitant fuzzy linguistic term sets. *Knowledge-Based Systems*, 76:127–138, 2015.

[259] Francisco Herrera, Enrique Herrera-Viedma, and Luis Martı?nez. A fusion approach for managing multi-granularity linguistic term sets in decision making. *Fuzzy sets and systems*, 114(1):43–58, 2000.

[260] Francisco Rodrigues Lima-Junior, Mery Ellen Brandt de Oliveira, and Carlos Henrique Lopes Resende. An overview of applications of hesitant fuzzy linguistic term sets in supply chain management: the state of the art and future directions. *Mathematics*, 11(13):2814, 2023.

[261] Walaa Abuasaker, Mónica Sánchez, Jennifer Nguyen, Nil Agell, Núria Agell, and Francisco J Ruiz. A comparative analysis of european media coverage of the israel–gaza war using hesitant fuzzy linguistic term sets. *Machine Learning and Knowledge Extraction*, 7(1):8, 2025.

[262] Yuxin Zhao, Nan Jiang, Yongxu He, and Xiong Deng. Entropy measures of multigranular unbalanced hesitant fuzzy linguistic term sets for multiple criteria decision making. *Information Sciences*, 686:121346, 2025.

[263] Junji Hao, Bingfeng Ge, Yuming Huang, Zeqiang Hou, Tianjiao Yang, and Wanying Wei. Hierarchical modeling and analysis of an international conflict based on hesitant fuzzy linguistic term sets. *Systems*, 13(7):557, 2025.

[264] Yuting Chen and Liuxin Chen. Research on non-cooperative behavior management in group consensus under double hierarchy hesitant fuzzy linguistic term sets. In *Proceedings of the 2025 5th International Conference on Applied Mathematics, Modelling and Intelligent Computing*, pages 535–540, 2025.

[265] Han Yang and Gaili Xu. Tourism attraction selection driven by online tourist reviews: A novel multi-attribute decision making method based on the evidence theory and probabilistic linguistic term sets. *Applied Soft Computing*, 177:113243, 2025.

[266] Ren Wang and Dan Peng. A system-based evaluation and feedback mechanism for green quality of metropolis energy big data using probabilistic linguistic term sets. *Scientific Reports*, 2026.

[267] Bodian Li, Tong Zhou, Qian Xiao, Kunzhi Zhong, and Xunqian Xu. Risk assessment indicator weighting for deep foundation pit construction using dual probabilistic linguistic term sets. *Buildings*, 16(13):2568, 2026.

[268] Tianqing Shen and Long Sun. Esg-oriented supplier selection under uncertainty: A todim decision framework integrating probabilistic linguistic term sets, prospect theory and variable-weight consensus. *Process Integration and Optimization for Sustainability*, pages 1–24, 2026.

[269] RG Piotrowski and YURI TAMBOVTSEV. Development of a linguistic automaton on the basis of statistics of speech. *Literary and linguistic computing*, 9(4):291–294, 1994.

[270] Michael O Rabin and Dana Scott. Finite automata and their decision problems. *IBM journal of research and development*, 3(2):114–125, 1959.

[271] Mark V Lawson. Finite automata. In *Handbook of networked and embedded control systems*, pages 117–143. Springer, 2005.

[272] Pierre Fraigniaud, David Ilcinkas, Guy Peer, Andrzej Pelc, and David Peleg. Graph exploration by a finite automaton. *Theoretical Computer Science*, 345(2-3):331–344, 2005.

[273] Supriya Raheja, Reena Dhadich, and Smita Rajpal. An alternative interpretation of linguistic variables as linguistic finite automata. *arXiv preprint arXiv:1112.3435*, 2011.

[274] James L Peterson. Petri nets. *ACM Computing Surveys (CSUR)*, 9(3):223–252, 1977.

[275] Wolfgang Reisig. Petri nets. In *Modeling in Systems Biology: The Petri Net Approach*, pages 37–56. Springer, 2016.

[276] Hu-Chen Liu, Jian-Xin You, ZhiWu Li, and Guangdong Tian. Fuzzy petri nets for knowledge representation and reasoning: A literature review. *Engineering Applications of Artificial Intelligence*, 60:45–56, 2017.

[277] Carl G Looney. Fuzzy petri nets for rule-based decisionmaking. *IEEE Transactions on Systems, Man, and Cybernetics*, 18(1):178–183, 1988.

[278] Witold Pedrycz and Fernando Gomide. A generalized fuzzy petri net model. *IEEE transactions on fuzzy systems*, 2(4):295–301, 1994.

[279] Weichao Yue, Xiaoxue Wan, Sanyi Li, Hangli Ren, and Hui He. Simplified neutrosophic petri nets used for identification of superheat degree: W. yue et al. *International Journal of Fuzzy Systems*, 24(8):3431–3455, 2022.

[280] Takaaki Fujita and Florentin Smarandache. *A Survey of Fuzzy and Uncertain Concepts in Applied Mathematics*. Infinite Study, 2025.

[281] Jamal Khudair Madhloom, Zainab Hammoodi Noori, Sif K Ebis, Oday A Hassen, and Saad M Darwish. An information security engineering framework for modeling packet filtering firewall using neutrosophic petri nets. *Computers*, 12(10):202, 2023.

[282] Weichao Yue, Weihua Gui, and Yongfang Xie. Experiential knowledge representation and reasoning based on linguistic petri nets with application to aluminum electrolysis cell condition identification. *Information Sciences*, 529:141–165, 2020.

[283] Xun Mou, Ling-Xiang Mao, Hu-Chen Liu, and MengChu Zhou. Spherical linguistic petri nets for knowledge representation and reasoning under large group environment. *IEEE Transactions on Artificial Intelligence*, 3(3):402–413, 2022.

[284] Hua Shi, Hu-Chen Liu, Jing-Hui Wang, and Xun Mou. New linguistic z-number petri nets for knowledge acquisition and representation under large group environment: H. shi et al. *International Journal of Fuzzy Systems*, 24(8):3483–3500, 2022.

[285] Weichao Yue, Mengqi Yu, Fang Wang, Sanyi Li, Xiaoxue Wan, and Xiaofang Chen. Dynamic adaptive minimum adjustment consensus-based probabilistic double hierarchy linguistic petri nets for superheat degree recognition. *Expert Systems with Applications*, page 132184, 2026.

[286] Weichao Yue, Lingfeng Hou, Xiaoxue Wan, Xiaofang Chen, and Weihua Gui. Superheat degree recognition of aluminum electrolysis cell using unbalance double hierarchy hesitant linguistic petri nets. *IEEE Transactions on Instrumentation and Measurement*, 72:1–15, 2023.

[287] Weichao Yue, Lingfeng Hou, Xiaoxue Wan, Yongfang Xie, Xiaofang Chen, and Weihua Gui. Consensus-based probabilistic hesitant intuitionistic linguistic petri nets for knowledge-intensive work of superheat degree identification. *Advanced Engineering Informatics*, 59:102261, 2024.

[288] Fei-Yue Wang. On the abstraction of conventional dynamic systems: from numerical analysis to linguistic analysis. *Information Sciences*, 171(1-3):233–259, 2005.

[289] Chengdong Li, Jianqiang Yi, Yisheng Lv, and Peiyong Duan. A hybrid learning method for the data-driven design of linguistic dynamic systems. *IEEE/CAA Journal of Automatica Sinica*, 6(6):1487–1498, 2019.

[290] LI Cheng-Dong, Gui-Qing ZHANG, WANG Hui-Dong, and REN Wei-Na. Properties and data-driven design of perceptual reasoning method based linguistic dynamic systems. *Acta Automatica Sinica*, 40(10):2221–2232, 2014.

[291] Guo-Rong Cai, Shui-Li Chen, and Wen-Zhong Guo. Fuzzy neural network structure of linguistic dynamic systems based on nonlinear particle swarm optimization. In *2008 3rd international conference on intelligent system and knowledge engineering*, volume 1, pages 886–891. IEEE, 2008.

[292] Zhao Liang. Overview of linguistic dynamic systems based on perception information. In *Proceedings of the 31st Chinese Control Conference*, pages 7519–7524. IEEE, 2012.

[293] Francisco Herrera and Enrique Herrera-Viedma. Choice functions and mechanisms for linguistic preference relations. *European Journal of Operational Research*, 120(1):144–161, 2000.

[294] Zeshui Xu. A method based on linguistic aggregation operators for group decision making with linguistic preference relations. *Information sciences*, 166(1-4):19–30, 2004.

[295] Zeshui Xu. Incomplete linguistic preference relations and their fusion. *Information Fusion*, 7(3):331–337, 2006.

[296] Yucheng Dong, Yinfeng Xu, and Hongyi Li. On consistency measures of linguistic preference relations. *European Journal of Operational Research*, 189(2):430–444, 2008.

[297] Hai Wang and Zeshui Xu. Some consistency measures of extended hesitant fuzzy linguistic preference relations. *Information Sciences*, 297:316–331, 2015.

[298] Ying-Hsiu Chen, Tien-Chin Wang, and Chao-Yen Wu. Multi-criteria decision making with fuzzy linguistic preference relations. *Applied Mathematical Modelling*, 35(3):1322–1330, 2011.

[299] Zhibin Wu and Jiuping Xu. Managing consistency and consensus in group decision making with hesitant fuzzy linguistic preference relations. *Omega*, 65:28–40, 2016.

[300] Sui-Zhi Luo, Peng-Fei Cheng, Jian-Qiang Wang, and Yuan-Ji Huang. Selecting project delivery systems based on simplified neutrosophic linguistic preference relations. *Symmetry*, 9(8):151, 2017.

[301] Bui Cong Cuong. Pareto solution in neutrosophic set setting for multiple criteria decision making problems. In *Fuzzy Multi-criteria Decision-Making Using Neutrosophic Sets*, pages 371–415. Springer, 2018.

[302] Shengbao Yao and Miao Gu. An influence network-based consensus model for large-scale group decision making with linguistic information. *International Journal of Computational Intelligence Systems*, 15(1):3, 2022.

[303] Yucheng Dong, Yuzhu Wu, Hengjie Zhang, and Guiqing Zhang. Multi-granular unbalanced linguistic distribution assessments with interval symbolic proportions. *Knowledge-Based Systems*, 82:139–151, 2015.

[304] Zhen Zhang, Chonghui Guo, and Luis Martínez. Managing multigranular linguistic distribution assessments in large-scale multiattribute group decision making. *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 47(11):3063–3076, 2016.

[305] Francisco Herrera and Luis Martínez. A 2-tuple fuzzy linguistic representation model for computing with words. *IEEE Transactions on fuzzy systems*, 8(6):746–752, 2000.

[306] Jin-Hsien Wang and Jongyun Hao. A new version of 2-tuple fuzzy linguistic representation model for computing with words. *IEEE transactions on fuzzy systems*, 14(3):435–445, 2006.

[307] Ta-Chun Wen, Kuei-Hu Chang, and Hsin-Hung Lai. Integrating the 2-tuple linguistic representation and soft set to solve supplier selection problems with incomplete information. *Engineering Applications of Artificial Intelligence*, 87:103248, 2020.

[308] Chenliang Li and Xiaobing Yu. Consensus reaching model for counter-intuitive in d–s evidence theory and application under 2-tuple linguistic representation. *Engineering Applications of Artificial Intelligence*, 112:104832, 2022.

[309] Julio Alfredo Telot González, Aylín Estrada Velazco, Maylí Estopiñán Lantigua, and Camilo Armando Suárez Álvarez. Evaluation of an accreditation variable for university institutions using 2 tuple linguistic representation model. In *Conferencia Cientifica Internacional Uciencia*, pages 277–289. Springer, 2021.

[310] Juan Carlos Leyva-López, Jesús Jaime Solano-Noriega, Jorge Anselmo Rodríguez-Castro, and Pedro J Sánchez. An extension of the electre iii method based on the 2-tuple linguistic representation model for dealing with heterogeneous information. *Computación y Sistemas*, 28(3):1655–1677, 2024.

[311] Stephen G Matthews, Mario A Gongora, Adrian A Hopgood, and Samad Ahmadi. Temporal fuzzy association rule mining with 2-tuple linguistic representation. In *2012 IEEE International Conference on Fuzzy Systems*, pages 1–8. IEEE, 2012.

[312] Takaaki Fujita and Florentin Smarandache. *Survey of Z-Number, D-Number, and ZD-Number.* Infinite Study, 2026.

[313] Mujahid Abdullahi, Tahir Ahmad, and Vinod Ramachandran. A review on some arithmetic concepts of z-number and its application to real-world problems. *International Journal of Information Technology & Decision Making*, 19(04):1091–1122, 2020.

[314] Kai-wen Shen and Jian-qiang Wang. Z-vikor method based on a new comprehensive weighted distance measure of z-number and its application. *IEEE Transactions on Fuzzy Systems*, 26(6):3232–3245, 2018.

[315] Lotfi A Zadeh. The concept of a z-number-a new direction in uncertain computation. In *2011 IEEE International Conference on Information Reuse & Integration*, pages xxii–xxiii. IEEE, 2011.

[316] Jian-qiang Wang, Yong-xi Cao, and Hong-yu Zhang. Multi-criteria decision-making method based on distance measure and choquet integral for linguistic z-numbers. *Cognitive computation*, 9(6):827–842, 2017.

[317] Shaikh Ibrahim Abdullah, Kajal De, Antonios Kalampakas, Sovan Samanta, and Tofigh Allahviranloo. Social networks based on linguistic z numbers and comparisons of centrality measures. In *Management of Uncertainty Using Linguistic Z-Numbers: Applications for Decision-Making, Granular Computing and Social Networks*, pages 241–264. Springer, 2024.

[318] Qinghua Liu, Jiadui Chen, Yongming Wu, and Kai Yang. Linguistic z-numbers and cloud model weighted ranking technology and its application in concept evaluation of information axiom. *The Journal of Supercomputing*, 78(5):6061–6089, 2022.

[319] Yi Mao, Yaning Xu, and Yuezhong Fan. A sequential three-way decision model based on linguistic z-numbers. *Plos one*, 20(5):e0320350, 2025.

[320] Zhaxi Pahua, Haidong Zhang, Yizhu Cairang, and Yanping He. Three-way decision method integrating probabilistic linguistic term sets and prospect theory in large-scale scenarios. *Engineering Applications of Artificial Intelligence*, 161:112053, 2025.

[321] Tanya Malhotra and Anjana Gupta. A new 2-tuple linguistic approach for unbalanced linguistic term sets. *IEEE Transactions on Fuzzy Systems*, 29(8):2158–2168, 2020.

[322] Tanya Malhotra and Anjana Gupta. Probabilistic multiplicative unbalanced linguistic term set and its application in matrix games. *International Journal of Machine Learning and Cybernetics*, 14(4):1253–1283, 2023.

[323] Parul Chauhan, Anjana Gupta, and Tanya Malhotra. A novel cloud model based on multiplicative unbalanced linguistic term set: P. chauhan et al. *The Journal of Supercomputing*, 79(14):16378–16408, 2023.

[324] Yucheng Dong, Cong-Cong Li, and Francisco Herrera. Connecting the linguistic hierarchy and the numerical scale for the 2-tuple linguistic model and its use to deal with hesitant unbalanced linguistic information. *Information Sciences*, 367:259–278, 2016.

[325] Zia Bashir, Jawad Ali, and Tabasam Rashid. Consensus-based robust decision making methods under a novel study of probabilistic uncertain linguistic information and their application in forex investment: Z. bashir et al. *Artificial Intelligence Review*, 54(3):2091–2132, 2021.

[326] Omar Barukab, Asghar Khan, and Sher Afzal Khan. Fermatean fuzzy linguistic term set based on linguistic scale function with dombi aggregation operator and their application to multi criteria group decision–making problem. *Heliyon*, 10(17), 2024.

[327] Soumi Manna, Tanushree Mitra Basu, and Shyamal Kumar Mondal. A new algorithmic approach to linguistic valued soft multi-criteria group decision-making problems using linguistic scale function. *Applied Soft Computing*, 83:105651, 2019.

[328] Nabil L Youssef and KA Dib. A new approach to fuzzy groupoids. *Fuzzy sets and systems*, 49(3):381–392, 1992.

[329] James G Oxley. *Matroid theory*, volume 3. Oxford University Press, USA, 2006.

[330] Hua Mao and Sanyang Liu. On antimatroids of infinite character. *Mathematica Pannonica*, 23(2):257–266, 2012.

[331] Murat Alp and Mohammad Ali Dehghanizadeh. Some results on the soft hg-hypergroupoids. *Contemporary Mathematics*, pages 238–250, 2026.

[332] Kazem Hamidizadeh, Saeed Mirvakili, Rauofeh Manaviyat, and Hossein Naraghi. A comparative analysis of hypergroupoids derived from metric spaces. *International Journal of Nonlinear Analysis and Applications*, 2025.

[333] Mohammad Munir, Nasreen Kausar, Rukhshanda Anjum, Qingbing Xu, Waqas Ahmad, et al. Hypergroupoids as tools for studying blood group genetics. *International Journal of Fuzzy Logic and Intelligent Systems*, 21(2):135–144, 2021.

[334] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[335] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *MOD Functions: A New Approach to Function Theory.* Infinite Study, 2015.

[336] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Problems on MOD Structures.* Infinite Study, 2016.

[337] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *MOD Pseudo Linear Algebras*. Infinite Study, 2015.

[338] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Algebraic Structures on MOD Planes*. Infinite Study, 2015.

[339] Florentin Smarandache, WB Kandasamy, and K Ilanthenral. *MOD Planes: A New Dimension to Modulo Theory*. EuropaNova, Brussels, 2015.

[340] Florentin Smarandache, WB Kandasamy, and K Ilanthenral. *MOD Graphs*. EuropaNova, Brussels, 2016.

[341] Alain Bretto. Hypergraph theory. *An introduction. Mathematical Engineering. Cham: Springer*, 1, 2013.

[342] Claude Berge. *Hypergraphs: combinatorics of finite sets*, volume 45. Elsevier, 1984.

[343] Takaaki Fujita and Florentin Smarandache. *Theoretical Foundations of Superhypergraph and Plithogenic Graph Neural Networks*. Neutrosophic Science International Association (NSIA) Publishing House, 2026.

[344] Takaaki Fujita. Modeling hierarchical and higher-order uncertainty in it systems management using strong intuitionistic fuzzy hypergraphs and superhypergraphs. *Journal of Engineering Management and Systems Science*, 1(1):1–25, 2026.

[345] Takaaki Fujita. Recursive intuitionistic fuzzy superhypergraphs. *Journal of Analytical Uncertainty*, 1(2):52–64, 2026.

[346] M Myvizhi, Ahmed Abdelhafeez, Mohamed Refaat Abdellah, Mohamed Eassa, Mohamed S Sawah, Hussam Elbehiery, et al. Single-valued neutrosophic graph with heptapartitioend structure. *Neutrosophic Sets and Systems*, 80:728–748, 2025.

[347] Liangsong Huang, Yu Hu, Yuxia Li, PK Kishore Kumar, Dipak Koley, and Arindam Dey. A study of regular and irregular neutrosophic graphs with real life applications. *Mathematics*, 7(6):551, 2019.

[348] Mohammed Alqahtani, M Kaviyarasu, Anas Al-Masarwah, and M Rajeshwari. Application of complex neutrosophic graphs in hospital infrastructure design. *Mathematics*, 12(5):719, 2024.

[349] Armen S Asratian, Tristan MJ Denley, and Roland Häggkvist. *Bipartite graphs and their applications*, volume 131. Cambridge university press Cambridge, 1998.

[350] Georgios A Pavlopoulos, Panagiota I Kontou, Athanasia Pavlopoulou, Costas Bouyioukos, Evripides Markou, and Pantelis G Bagos. Bipartite graphs in systems biology and medicine: a survey of methods and applications. *GigaScience*, 7(4):giy014, 2018.

[351] Jean-Loup Guillaume and Matthieu Latapy. Bipartite graphs as models of complex networks. *Physica A: Statistical Mechanics and its Applications*, 371(2):795–813, 2006.

[352] Md Enamul Haque and Mehmet Engin Tozal. Identification of fraudulent healthcare claims using fuzzy bipartite knowledge graphs. *IEEE Transactions on Services Computing*, 16(6):3931–3945, 2023.

[353] S Mathew and MS Sunitha. Some remarks on fuzzy bipartite graphs. *Advances in Fuzzy Sets and Systems*, 10(1):45–56, 2011.

[354] Govind Sharma, Swyam Prakash Singh, V Susheela Devi, and M Narasimha Murty. The cat set on the mat: Cross attention for set matching in bipartite hypergraphs. *arXiv preprint arXiv:2111.00243*, 2021.

[355] Takaaki Fujita and Florentin Smarandache. *HyperGraph and SuperHyperGraph Theory with Applications (VI): Graph Structure (Path, Tree, Cycle, Planarity, Bipartite, and More)*. Infinite Study, 2026.

[356] A Balakrishnan and K Thiruganasambandam. Neutrosophic topologized bipartite graph. *Reliability: Theory & Applications*, 20(4 (89)):105–113, 2025.

[357] Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Neutrosophic graphs: a new dimension to graph theory*. Infinite Study, 2015.

[358] Michael Peer, Iva K Brunec, Nora S Newcombe, and Russell A Epstein. Structuring knowledge with cognitive maps and cognitive graphs. *Trends in cognitive sciences*, 25(1):37–54, 2021.

[359] Roger M Downs and David Stea. Cognitive maps and spatial behaviour: Process and products. *The map reader: Theories of mapping practice and cartographic representation*, pages 312–317, 2011.

[360] Bart Kosko. Fuzzy cognitive maps. *International journal of man-machine studies*, 24(1):65–75, 1986.

[361] Stephen Kaplan. Cognitive maps in perception and thought. *Image and environment: Cognitive mapping and spatial behavior*, pages 63–78, 1973.

[362] Antonios Paraskevas and Michael Madas. Neutrosophic cognitive maps. *Soft Computing and Machine Learning: A Fuzzy and Neutrosophic View of Reality*, page 27, 2025.

[363] Atharva Rajesh Shekatkar, Aditya Deodeshmukh, and Ilanthenral Kandasamy. Refined neutrosophic cognitive maps: A novel framework for modeling indeterminate causal relationships. *International Journal of Fuzzy Systems*, pages 1–24, 2025.

[364] Salah Hasan Al-subhi, Elpiniki I Papageorgiou, Pedro Piñero Pérez, Gaafar Sadeq S Mahdi, and Luis Alvarado Acuna. Triangular neutrosophic cognitive map for multistage sequential decision-making problems: Sh al-subhi et al. *International Journal of Fuzzy Systems*, 23(3):657–679, 2021.

[365] Rui Wang. A neutrosophic cognitive map-based modeling framework for analyzing the indeterminate impact of exercise on university students' mental health. *Neutrosophic Sets and Systems*, 86(1):50, 2025.

[366] WB Vasantha Kandasamy and Florentin Smarandache. Fuzzy cognitive maps and neutrosophic cognitive maps. *Book Review by Milan Mares: Kybernetika*, 40(1):151, 2004.

[367] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *MOD Cognitive Maps models and MOD natural neutrosophic Cognitive Maps models*. Infinite Study, 2015.

[368] Khodakhast Bibak, Bruce M Kapron, and Venkatesh Srinivasan. The cayley graphs associated with some quasi-perfect lee codes are ramanujan graphs. *IEEE Transactions on Information Theory*, 62(11):6355–6358, 2016.

[369] Wei Li and Kai Yang. Odd cycles in symmetric cayley graphs on prime cyclic groups. *arXiv preprint arXiv:2606.24426*, 2026.

[370] K Wendy Tang and Bruce W Arden. Representations of borel cayley graphs. *SIAM Journal on Discrete Mathematics*, 6(4):655–676, 1993.

[371] Paula M Chiapparoli and Ricardo A Podestá. Isospectral cayley graphs with even and odd spectrum. *arXiv preprint arXiv:2601.05510*, 2026.

[372] Alexander M Banaru, Daria A Banaru, and Sergey M Aksenov. Groupoid of intermolecular contacts and its fuzzy cayley graph. *Moscow University Chemistry Bulletin*, 78(3):103–113, 2023.

[373] MH Shahzamanian and Bijan Davvaz. Roughness in fuzzy cayley graphs. *Bulletin of the Karaganda University. Mathematics Series*, 112(4):105–118, 2023.

[374] Naparat Sripaisan and Yotsanan Meemark. Algebraic degree of spectra of cayley hypergraphs. *Discrete Applied Mathematics*, 316:87–94, 2022.

[375] Jaeun Lee and Young Soo Kwon. Cayley hypergraphs and cayley hypermaps. *Discrete Mathematics*, 313(4):540–549, 2013.

[376] Tatiana B Jajcayová and Robert Jajcay. Generalizations of cayley graphs to uniform hypergraphs. *The Art of Discrete and Applied Mathematics*, 2024.

[377] Brian Alspach and Torrence D Parsons. Isomorphism of circulant graphs and digraphs. *Discrete Mathematics*, 25(2):97–108, 1979.

[378] Clemens Heuberger. On planarity and colorability of circulant graphs. *Discrete Mathematics*, 268(1-3):153–169, 2003.

[379] Brian Alspach, Joy Morris, and V Vilfred. Self-complementary circulant graphs. *Ars Combinatoria*, 53:187–192, 1999.

[380] Muhammad Salman, Imran Javaid, and Muhammad Anwar Chaudhry. Resolvability in circulant graphs. *Acta Mathematica Sinica, English Series*, 28(9):1851–1864, 2012.

[381] A Robert Calderbank and Neil JA Sloane. Modular and p-adic cyclic codes. *Designs, codes and Cryptography*, 6(1):21–35, 1995.

[382] Steven T Dougherty and Young Ho Park. On modular cyclic codes. *Finite Fields and Their Applications*, 13(1):31–57, 2007.

[383] Pramod Kanwar and Sergio R Lopez-Permouth. Cyclic codes over the integers modulopm. *Finite fields and their applications*, 3(4):334–352, 1997.

[384] SK Rososhek. Modified matrix modular cryptosystems. *British Journal of Mathematics & Computer Science*, 5(5):613, 2015.

[385] Dinesh Khurana and TY Lam. Generalized commutators in matrix rings. *Linear and Multilinear Algebra*, 60(7):797–827, 2012.

[386] Jacob TB Beard Jr and Robert M McConnel. Matrix fields over the integers modulo m. *Linear Algebra and Its Applications*, 14(2):95–105, 1976.

[387] Mi-Ying Miryam Huang, Xinyu Mao, and Jiapeng Zhang. Sublinear proofs over polynomial rings. *Cryptology ePrint Archive*, 2025.

[388] Dikran Karagueuzian and Peter Symonds. The module structure of a group action on a polynomial ring: a finiteness theorem. *Journal of the American Mathematical Society*, 20(4):931–967, 2007.

[389] Reza Akhtar, Megan Boggess, Tiffany Jackson-Henderson, Isidora Jiménez, Rachel Karpman, Amanda Kinzel, and Dan Pritikin. On the unitary cayley graph of a finite ring. *the electronic journal of combinatorics*, pages R117–R117, 2009.

[390] Dariush Kiani and Mohsen Molla Haji Aghaei. On the unitary cayley graph of a ring. *the electronic journal of combinatorics*, pages P10–P10, 2012.

[391] Dariush Kiani, Mohsen Molla Haji Aghaei, Yotsanan Meemark, and Borworn Suntornpoch. Energy of unitary cayley graphs and gcd-graphs. *Linear algebra and its applications*, 435(6):1336–1343, 2011.

[392] David F Anderson, Michael C Axtell, and Joe A Stickles Jr. Zero-divisor graphs in commutative rings. *Commutative algebra: Noetherian and non-Noetherian perspectives*, pages 23–45, 2010.

[393] Shane P Redmond. An ideal-based zero-divisor graph of a commutative ring. *Communications in Algebra*, 31(9):4425–4443, 2003.

[394] S Akbari and A Mohammadian. On the zero-divisor graph of a commutative ring. *Journal of algebra*, 274(2):847–855, 2004.

[395] Frank DeMeyer and Lisa DeMeyer. Zero divisor graphs of semigroups. *Journal of Algebra*, 283(1):190–198, 2005.

[396] T Tamizh Chelvam, Krishnan Selvakumar, and V Ramanathan. On the planarity of the k-zero-divisor hypergraphs. *AKCE International Journal of Graphs and Combinatorics*, 12(2-3):169–176, 2015.

[397] Krishnan Selvakumar and V Ramanathan. Classification of nonlocal rings with genus one 3-zero-divisor hypergraphs. *Communications in Algebra*, 45(1):275–284, 2017.

[398] Nasir Ali, Hafiz Muhammad Afzal Siddiqui, Muhammad Imran Qureshi, Manal Elzain Mohamed Abdalla, NS Abd EL-Gawaad, and Fikadu Tesgera Tolasa. On study of multiset dimension in fuzzy zero divisor graphs associated with commutative rings. *International Journal of Computational Intelligence Systems*, 17(1):298, 2024.

[399] Christy P Antony and Shiny Jose. Domination parameters in intuitionistic fuzzy zero-divisor graphs. *New Mathematics and Natural Computation*, 2026.

[400] Said Broumi et al. Decomposition of neutrosophic zero-divisor graph. *Neutrosophic Sets and Systems*, 83(1):9, 2025.

[401] K Gunasekar and P Muralikrishna. Graph decomposition techniques in neutrosophic zero divisor models of commutative ring. *Neutrosophic Sets and Systems*, 97(1):5, 2026.

[402] Takaaki Fujita and Florentin Smarandache. *A reconsideration of advanced concepts in neutrosophic graphs: Smart, zero divisor, layered, weak, semi, and chemical graphs*. Infinite Study, 2024.

[403] Wasin So. Integral circulant graphs. *Discrete mathematics*, 306(1):153–158, 2006.

[404] Nitin Saxena, Simone Severini, and Igor E Shparlinski. Parameters of integral circulant graphs and periodic quantum dynamics. *International Journal of Quantum Information*, 5(03):417–430, 2007.

[405] Aleksandar Ilić. Distance spectra and distance energy of integral circulant graphs. *Linear algebra and its applications*, 433(5):1005–1014, 2010.

[406] Aleksandar Ilić and Milan Bašić. On the chromatic number of integral circulant graphs. *Computers & Mathematics with Applications*, 60(1):144–150, 2010.

[407] Takaaki Fujita and Florentin Smarandache. *Representing Higher-Order Networks: A Survey of Graph-Based Frameworks*. Neutrosophic Science International Association (NSIA) Publishing House, 2026.

[408] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions*. Infinite Study, 2023.

[409] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Mod Rectangular Natural Neutrosophic Numbers*. Infinite Study, 2018.

[410] Florentin Smarandache and WB Kandasamy. *Subset Interval Groupoids*. Educational Publisher Inc., Ohio, 2013.

[411] WB Vasantha Kandasamy, Florentin Smarandache, and Moon Kumar Chetry. *Interval Groupoids*. Infinite Study, 2010.

[412] WB Vasantha Kandasamy and Florentin Smarandache. *Neutrosophic Interval Bialgebraic Structures*. Infinite Study, 2011.

[413] Florentin Smarandache and WB Kandasamy. *INTERVAL LINEAR ALGEBRA*. Kappa & Omega, Glenndale, 2010.

[414] Florentin Smarandache and WB Kandasamy. *INTERVAL SEMIRINGS*. Kappa & Omega, Glenndale, 2011.

[415] Barbara Pękala. Uncertainty data in interval-valued fuzzy set theory. *Properties, Algorithms and Applications, Studies in Fuzziness and Soft Computing*, 367, 2019.

[416] Chris Cornelis, Glad Deschrijver, and Etienne E Kerre. Advances and challenges in interval-valued fuzzy logic. *Fuzzy sets and systems*, 157(5):622–627, 2006.

[417] Didier Dubois and Henri Prade. Interval-valued fuzzy sets, possibility theory and imprecise probability. In *EUSFLAT Conf.*, pages 314–319, 2005.

[418] Glad Deschrijver and Chris Cornelis. Representability in interval-valued fuzzy set theory. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 15(03):345–361, 2007.

[419] Wenyi Zeng and Yu Shi. Note on interval-valued fuzzy set. In *International Conference on Fuzzy Systems and Knowledge Discovery*, pages 20–25. Springer, 2005.

[420] JG Lee, G Senel, YB Jun, Fadhil Abbas, and K Hur. Topological structures via interval-valued soft sets. *Ann. Fuzzy Math. Inform*, 22(2):133–169, 2021.

[421] Jong Il Baek, Tareq M Al-shami, Saeid Jafari, Minseok Cheong, and Kul Hur. New interval-valued soft separation axioms. *Axioms*, 13(7):493, 2024.

[422] ShuoYan Chou, Truong Thi Thuy Duong, and Nguyen Xuan Thao. Renewable energy selection based on a new entropy and dissimilarity measure on an interval-valued neutrosophic set. *Journal of Intelligent & Fuzzy Systems*, 40(6):11375–11392, 2021.

[423] Eda Bolturk and Cengiz Kahraman. Interval-valued neutrosophic ahp with possibility degree method. *International Journal of the Analytic Hierarchy Process*, 10(3), 2018.

[424] Muhammad Ihsan, Muhammad Saeed, Atiqe Ur Rahman, Mazin Abed Mohammed, Karrar Hameed Abdulkaree, Abed Saif Alghawli, and MA Al-Qaness. An innovative decision-making framework for supplier selection based on a hybrid interval-valued neutrosophic soft expert set. *AIMS Mathematics*, 8(9):22127–22161, 2023.

[425] Mengran Sun, Xin Ding, Yushui Geng, and Jing Zhao. Topsis method based on entropy weight for interval-valued neutrosophic numbers and its application in multi-attribute group decision-making. In *2024 IEEE International Conference on Systems, Man, and Cybernetics (SMC)*, pages 149–154. IEEE, 2024.

[426] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Interval valued neutrosophic graphs. *Critical Review, XII*, 2016:5–33, 2016.

[427] Lei Yang, Junyan Niu, and Jiangfeng Li. Interval-valued neutrosophic sets-based ivns-ram method for enhanced security evaluation in computer networks. *Neutrosophic Sets and Systems*, page 479, 2025.

[428] Liguo Yang, Min Zhao, and Jiarui Shao. A multi-criteria framework for supplier assessment based on interval-valued neutrosophic sets to improve supply chain transparency. *International Journal of Fuzzy Systems*, pages 1–18, 2025.

[429] J Jayasudha and C Kowsalyaharishanthi. An application of interval valued neutrosophic soft multisets in mcdm. *Sigma: Journal of Engineering & Natural Sciences/Mühendislik ve Fen Bilimleri Dergisi*, 43(3):922, 2025.

[430] Takaaki Fujita, Ahmed Heilat, and Ajoy kanti Das. Hyperinterval-valued and superhyperinterval-valued fuzzy/neutrosophic set. *Statistics, Optimization & Information Computing*, 16(2):1003–1032, 2026.

[431] Florentin Smarandache and WB Kandasamy. *INTERVAL ALGEBRAIC BISTRUCTURES*. THE EDUCATIONAL PUBLISHER INC, Ohio, 2011.

[432] T Nirmala, D Datta, HS Kushwaha, and K Ganesan. Inverse interval matrix: A new approach. *Applied Mathematical Sciences*, 5(13):607–624, 2011.

[433] David Hartman, Milan Hladík, and David Říha. Computing the spectral decomposition of interval matrices and a study on interval matrix powers. *Applied Mathematics and Computation*, 403:126174, 2021.

[434] Behnam Hashemi and Mehdi Dehghan. Results concerning interval linear systems with multiple right-hand sides and the interval matrix equation ax= b. *Journal of computational and applied mathematics*, 235(9):2969–2978, 2011.

[435] Martin Charles Golumbic. Interval graphs and related topics. *Discrete Mathematics*, 55(2):113–121, 1985.

[436] Delbert Fulkerson and Oliver Gross. Incidence matrices and interval graphs. *Pacific journal of mathematics*, 15(3):835–855, 1965.

[437] William T Trotter Jr and Frank Harary. On double and multiple interval graphs. *Journal of Graph Theory*, 3(3):205–211, 1979.

[438] John I Moore Jr. Interval hypergraphs and d-interval hypergraphs. *Discrete Mathematics*, 17(2):173–179, 1977.

[439] Ron Aharoni, Ron Holzman, and Shira Zerbib. Edge-covers in d-interval hypergraphs. *Discrete & Computational Geometry*, 58(3):650–662, 2017.

[440] Elias Dahlhaus. Improved efficient parallel algorithms to recognize interval graphs and interval hypergraphs. In *Proceedings of the Thirtieth Hawaii International Conference on System Sciences*, volume 1, pages 172–181. IEEE, 1997.

[441] Krassimir T. Atanassov. On interval valued intuitionistic fuzzy sets. *Interval-Valued Intuitionistic Fuzzy Sets*, 2019.

[442] Ravindar Raj Chinnappan and Dhanasekar Sundaram. Image enhancement framework combining interval-valued intuitionistic fuzzy sets and fractional sobel operator. *Artificial Intelligence Review*, 58, 2025.

[443] Gajalaxmi Pandaram, Jayanthi Duraisamy, and Priyadharshini Manoharan. Distance measures on circular interval-valued intuitionistic fuzzy sets. *Intelligent Decision Technologies*, 20:230 – 245, 2025.

[444] Krassimir T. Atanassov. Interval-valued intuitionistic fuzzy sets. In *Studies in Fuzziness and Soft Computing*, 2019.

[445] Jinxia Huo, Weidong Zhang, and Zhenmin Chen. Enhanced decision-making through an intelligent algorithmic approach for multiple-attribute college english teaching quality evaluation with interval-valued intuitionistic fuzzy sets. *International Journal of Knowledge-Based and Intelligent Engineering Systems*, 28:279 – 294, 2024.

[446] Nilesh N. Karnik and Jerry M. Mendel. Centroid of a type-2 fuzzy set. *Inf. Sci.*, 132:195–220, 2001.

[447] Surajit Dan, Mohuya B. Kar, Saibal Majumder, Bikashkoli Roy, Samarjit Kar, and Dragan Stevan Pamucar. Intuitionistic type-2 fuzzy set and its properties. *Symmetry*, 11:808, 2019.

[448] Sarah Greenfield and Francisco Chiclana. Type-reduced set structure and the truncated type-2 fuzzy set. *Fuzzy Sets Syst.*, 352:119–141, 2018.

[449] Rafik A. Aliev, Rahib Hidayat Abiyev, Rafig R. Aliyev, and Sanan Abizada. Similarity measures for interval type-3 fuzzy sets. *Inf. Sci.*, 709:121991, 2025.

[450] Turki Younis Abdalla, Abdulkareem Y. Abdalla, and Adala M. Chyaid. Type-3 fuzzy system-based intelligent control approaches and applications. *Advances in Fuzzy Systems*, 2025, 2025.

[451] Anup Kumar, Ritu Raj, and Ardashir Mohammadzadeh. Recent advancements in type-3 fuzzy logic systems: A comprehensive review. *IEEE Transactions on Emerging Topics in Computational Intelligence*, 9:2676–2689, 2025.

[452] Mohamed Abdel-Basset, Nihal N. Mostafa, Karam M. Sallam, Ibrahim Elgendi, and Kumudumalee Munasinghe. Enhanced covid-19 x-ray image preprocessing schema using type-2 neutrosophic set. *Appl. Soft Comput.*, 123:108948, 2022.

[453] Siti Nur Idara Rosli, Mohammad Izat Emir Zulkifly, and Broumi Said. Implementing 3d modeling: Interpolated b-spline surfaces enhanced with interval type-2 neutrosophic set theory. *Warisan Journal of Mathematical Sciences and Engineering*, 2025.

[454] Marwan H. Hassan, Saad M. Darwish, and Saleh M. Elkaffas. Type-2 neutrosophic set and their applications in medical databases deadlock resolution. *Computers, Materials & Continua*, 2023.

[455] Xianliang Liu and Shu ping Wan. Combinatorial iterative algorithms for computing the centroid of an interval type-2 fuzzy set. *IEEE Transactions on Fuzzy Systems*, 28:607–617, 2020.

[456] Yifan Liu, Hui Yin, Baizhan Xia, Dejie Yu, and Ye-Hwa Chen. Interval type-2 fuzzy set-theoretic control design for uncertain dynamical systems. *International Journal of Fuzzy Systems*, 26:1069 – 1087, 2024.

[457] Xianliang Liu and Yicheng Lin. New efficient algorithms for the centroid of an interval type-2 fuzzy set. *Inf. Sci.*, 570:468–486, 2021.

[458] Joel E. Cohen. The asymptotic probability that a random graph is a unit interval graph, indifference graph, or proper interval graph. *Discret. Math.*, 40:21–24, 1982.

[459] Celina M. Herrera de Figueiredo, João Meidanis, and Célia Picinin de Mello. A linear-time algorithm for proper interval graph recognition. *Inf. Process. Lett.*, 56:179–184, 1995.

[460] Haim Kaplan, Ron Shamir, and Robert Endre Tarjan. Tractability of parameterized completion problems on chordal, strongly chordal, and proper interval graphs. *SIAM J. Comput.*, 28:1906–1922, 1999.

[461] Ashwin Jacob, Arpit Kumar, and Diptapriyo Majumdar. A polynomial kernel for vertex deletion to the scattered class of proper interval graph and trees. *ArXiv*, abs/2605.02399, 2026.

[462] Israel R. Curbelo and Hannah R. Malko. On the on-line coloring of unit interval graphs with proper interval representation. *Discret. Math. Theor. Comput. Sci.*, 27, 2024.

[463] Peng Li, Tong Liu, and Yangjing Long. On the unit interval graphs which are paired k-disjoint path coverable. *Discret. Math. Algorithms Appl.*, 16:2350105:1–2350105:12, 2023.

[464] Bartlomiej Bosek and Anna Zych-Pawlewicz. Recoloring unit interval graphs with logarithmic recourse budget. *ArXiv*, abs/2202.08006, 2022.

[465] Ignaz Rutter, Darren Strash, Peter Stumpf, and Michael Vollmer. Simultaneous representation of proper and unit interval graphs. *Algorithmica*, 87:783 – 811, 2019.

[466] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *MOD Natural Neutrosophic Subset Topological Spaces and Kakutani's Theorem.* Infinite Study, 2013.

[467] Florentin Smarandache. *Special Subset Linguistic Topological Spaces.* Infinite Study, 2023.

[468] Florentin Smarandache and WB Kandasamy. *Special type of subset topological spaces.* Educational Publisher Inc., Ohio, 2013.

[469] WB Vasantha Kandasamy and Florentin Smarandache. *Subset Groupoids.* Infinite Study, 2013.

[470] Florentin Smarandache and WB Kandasamy. *Subset Semirings.* Educational Publisher Inc., Ohio, 2013.

[471] WB Vasantha Kandasamy and Florentin Smarandache. *Subset Semilinear Algebras.* Educational Publisher Incorporated, 2013.

[472] Florentin Smarandache and WB Kandasamy. *Subset Interval Groupoids.* Educational Publisher Inc., Ohio, 2013.

[473] Florentin Smarandache, WB Kandasamy, and K Ilanthenral. *Subset Vertex Graphs for Social Networks.* EuropaNova, Brussels, 2018.

[474] Florentin Smarandache and WB Kandasamy. *Subset Polynomial Semirings and Subset Matrix Semirings.* Educational Publisher Inc., Ohio, 2013.

[475] WB Vasantha Kandasamy and Florentin Smarandache. *Subset Non Associative Semirings.* Infinite Study, 2013.

[476] Donald J McCarthy and David L Hayes. Subgroups of the power semigroup of a group. *Journal of Combinatorial Theory, Series A*, 14(2):173–186, 1973.

[477] Shuolin Liu and Salvatore Tringali. Power semigroups and two rigidity theorems for groups. *arXiv preprint arXiv:2606.01917*, 2026.

[478] Jean-Eric Pin. Power semigroups and related varieties of finite semigroups. In *Semigroups and Their Applications: Proceedings of the International Conference "Algebraic Theory of Semigroups and Its Applications" held at the California State University, Chico, April 10–12, 1986*, pages 139–152. Springer, 1987.

[479] Agata Pilitowska and Anna Zamojska-Dzienio. On some congruences of power algebras. *Central European Journal of Mathematics*, 10(3):987–1003, 2012.

[480] Chris Brink. Power structures. *Algebra Universalis*, 30(2):177–216, 1993.

[481] Kira Adaricheva, Agata Pilitowska, and David Stanovskỳ. Complex algebras of subalgebras. *Algebra and logic*, 47(6):367–383, 2008.

[482] Ivica Bošnjak and Rozálija Madarász. Some results on complex algebras of subalgebras. *Novi Sad Journal of Mathematics*, 237(2):231–240, 2007.

[483] Luigi Santocanale. The equational theory of the natural join and inner union is decidable. In *International Conference on Foundations of Software Science and Computation Structures*, pages 494–510. Springer, 2018.

[484] Roland Schmidt. *Subgroup lattices of groups.* Walter de Gruyter, 1994.

[485] Muzhi Wang, A-Ming Liu, and Alexander N Skiba. On two classes of sublattices of the subgroup lattice of a finite group. *Mathematics*, 14(2):328, 2026.

[486] Zhang Chi and Alexander N Skiba. On two sublattices of the subgroup lattice of a finite group. *Journal of Group Theory*, 22(6):1035–1047, 2019.

[487] John Shareshian and Russ Woodroofe. A new subgroup lattice characterization of finite solvable groups. *Journal of Algebra*, 351(1):448–458, 2012.

[488] Xueyi Huang, Qiongxiang Huang, and Jianfeng Wang. The spectrum and automorphism group of the set-inclusion graph. In *Algebra Colloquium*, pages 497–506. World Scientific, 2021.

[489] KG Sreekumar, V Sindhu, Suresh G Singh, and Ismail Naci Cangul. Spectrum of bipartite kneser type-k graphs. *Palestine Journal of Mathematics*, 14:51, 2025.

[490] Hou Tin Chau, David Ellis, Ehud Friedgut, and Noam Lifshitz. On the maximum degree of induced subgraphs of the kneser graph. *arXiv preprint arXiv:2312.06370*, 2023.

[491] Ya-Chen Chen. Triangle-free hamiltonian kneser graphs. *Journal of Combinatorial Theory, Series B*, 89(1):1–16, 2003.

[492] Tristan Denley. The odd girth of the generalised kneser graph. *European Journal of Combinatorics*, 18(6):607–611, 1997.

[493] Torsten Mütze, Jerri Nummenpalo, and Bartosz Walczak. Sparse kneser graphs are hamiltonian. In *Proceedings of the 50th Annual ACM SIGACT Symposium on Theory of Computing*, pages 912–919, 2018.

[494] WB Vasantha Kandasamy, Florentin Smarandache, and K Ilanthenral. *Introduction to linear Bialgebra*, volume 8. Infinite Study, 2005.

[495] Florentin Smarandache, WB Kandasamy, and K Ilanthenral. *Applications of Bimatrices to some Fuzzy and Neutrosophic Models*. Hexis, Phoenix, 2005.

[496] Jiangxia Cao, Xixun Lin, Shu Guo, Luchen Liu, Tingwen Liu, and Bin Wang. Bipartite graph embedding via mutual information maximization. In *Proceedings of the 14th ACM international conference on web search and data mining*, pages 635–643, 2021.

[497] Hongyuan Zha, Xiaofeng He, Chris Ding, Horst Simon, and Ming Gu. Bipartite graph partitioning and data clustering. In *Proceedings of the tenth international conference on Information and knowledge management*, pages 25–32, 2001.

[498] Lowell W Beineke, Frank Harary, and John W Moon. On the thickness of the complete bipartite graph. In *Mathematical Proceedings of the Cambridge Philosophical Society*, pages 01–05. Cambridge University Press, 1964.

[499] Wei Xia, Quanxue Gao, Qianqian Wang, Xinbo Gao, Chris Ding, and Dacheng Tao. Tensorized bipartite graph learning for multi-view clustering. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 45(4):5187–5202, 2022.

[500] WB Vasantha Kandasamy and Florentin Smarandache. *Neutrosophic Bilinear Algebras and their Generalizations*. Infinite Study, 2010.

[501] WB Vasantha Kandasamy, Florentin Smarandache, and K Ilanthenral. *Introduction to Bimatrices*, volume 6. Infinite Study, 2005.

[502] Jianming Zhan, Muhammad Akram, and Muzzamal Sitara. Novel decision-making method based on bipolar neutrosophic information. *Soft Computing*, 23:9955 – 9977, 2018.

[503] WB Vasantha Kandasamy, M Khoshnevisan, and K Ilanthenral. Smarandache representation and its applications. *Scientia Magna*, 2(4):59, 2006.

[504] Florentin Smarandache and WB Kandasamy. *Neutrosophic Interval Bialgebraic Structures*. ZIP PUBLISHING, OHIO, 2011.

[505] Florentin Smarandache, WB Kandasamy, and K Ilanthenral. *SET LINEAR ALGEBRA AND SET FUZZY LINEAR ALGEBRA*. InfoLearnQuest, Ann Arbor, 2008.

[506] WB Vasantha Kandasamy and Florentin Smarandache. *Some neutrosophic algebraic structures and neutrosophic N-algebraic structures*. Infinite Study, 2006.

[507] WB Vasantha Kandasamy. *Bilagebraic Structures and Smarandache Bialgebraic Structures*. Infinite Study, 2003.

[508] Florentin Smarandache and WB Kandasamy. *INTERVAL ALGEBRAIC BISTRUCTURES*. THE EDUCATIONAL PUBLISHER INC, Ohio, 2011.

[509] Badri Dvalishvili. *Bitopological spaces: theory, relations with generalized algebraic structures and applications*, volume 199. Elsevier, 2005.

[510] JC143169 Kelly. Bitopological spaces. *Proceedings of the London Mathematical Society*, 3(1):71–89, 1963.

[511] Nongluk Viriyapong, Supannee Sompong, and Chawalit Boonpok. bitopological spaces. *Computer Science*, 19(3):855–860, 2024.

[512] Melvin Fitting. Bilattices and the theory of truth. *Journal of Philosophical Logic*, 18(3):225–256, 1989.

[513] Matthew L Ginsberg. Bilattices and modal operators. *Journal of Logic and Computation*, 1(1):41–69, 1990.

[514] Michael Stay. Compact closed bicategories. *arXiv preprint arXiv:1301.1053*, 2013.

[515] Michael A Shulman. Framed bicategories and monoidal fibrations. *arXiv preprint arXiv:0706.1286*, 2007.

[516] Aurelio Carboni, Scott Johnson, Ross Street, and Dominic Verity. Modulated bicategories. *Journal of Pure and Applied Algebra*, 94(3):229–282, 1994.

[517] S-n A Joni and G-C Rota. Coalgebras and bialgebras in combinatorics. *Studies in Applied Mathematics*, 61(2):93–139, 1979.

[518] Warren D Nichols. Bialgebras of type one. *Communications in Algebra*, 6(15):1521–1552, 1978.

[519] Justin Greenough. Monoidal 2-structure of bimodule categories. *Journal of Algebra*, 324(8):1818–1859, 2010.

[520] Jürgen Fuchs, Gregor Schaumann, and Christoph Schweigert. A trace for bimodule categories. *Applied Categorical Structures*, 25(2):227–268, 2017.

[521] Jiaxuan You, Bowen Liu, Rex Ying, Vijay S. Pande, and Jure Leskovec. Graph convolutional policy network for goal-directed molecular graph generation. In *Neural Information Processing Systems*, 2018.

[522] Nenad Trinajstic. *Chemical graph theory*. CRC press, 2018.

[523] Guillermo Restrepo. Higher order structures in chemistry: hypergraphs reshape the molecule and the reaction. *Digital Discovery*, 5(3):973–982, 2026.

[524] Elena V Konstantinova and Vladimir A Skorobogatov. Molecular hypergraphs: The new representation of nonclassical molecular structures with polycentric delocalized bonds. *Journal of chemical information and computer sciences*, 35(3):472–478, 1995.

[525] Ahsanur Rahman, Christopher L Poirel, David J Badger, and TM Murali. Reverse engineering molecular hypergraphs. In *Proceedings of the ACM Conference on Bioinformatics, Computational Biology and Biomedicine*, pages 68–75, 2012.

[526] Junwu Chen and Philippe Schwaller. Molecular hypergraph neural networks. *The Journal of Chemical Physics*, 160(14), 2024.

[527] Takaaki Fujita. Molecular fuzzy graphs, hypergraphs, and superhypergraphs. *Journal of Intelligent Decision and Computational Modelling*, 1(3):158–171, 2025.

[528] Takaaki Fujita, Ajoy Kanti Das, and Suman Das. Molecular line hypergraphs and superhypergraphs. *Neutrosophic Knowledge*, 9:31–65, 2026.

[529] Takaaki Fujita. An introduction and reexamination of molecular hypergraph and molecular n-superhypergraph. *Asian Journal of Physical and Chemical Sciences*, 13(3):1–38, 2025.

[530] Wei Gao, Juan Luis García Guirao, and Hualong Wu. Nordhaus–gaddum type inequalities for some distance-based indices of bipartite molecular graphs. *Journal of Mathematical Chemistry*, 58(7):1345–1352, 2020.

[531] Ivan Gutman and S Radenkovic. A lower bound for the estrada index of bipartite molecular graphs. *Kragujevac J. Sci*, 29:67–72, 2007.

[532] Takaaki Fujita. On tripolar fuzzy offgraph. *INTERNATIONAL JOURNAL OF COMPUTER APPLICATION (IJCA)*, 16(4):13, 2026.

[533] Xueting Liao, Danyang Zheng, and Xiaojun Cao. Coronavirus pandemic analysis through tripartite graph clustering in online social networks. *Big Data Mining and Analytics*, 4(4):242–251, 2021.

[534] Zhenhua Meng and Fanshen Meng. From criteria to ranking: Targeting-aware tripartite graph learning for multi-criteria recommendation. In *Proceedings of the ACM Web Conference 2026*, pages 6421–6432, 2026.

[535] Kelong Mao, Xi Xiao, Jieming Zhu, Biao Lu, Ruiming Tang, and Xiuqiang He. Item tagging for information retrieval: A tripartite graph neural network based approach. In *Proceedings of the 43rd international ACM SIGIR conference on research and development in information retrieval*, pages 2327–2336, 2020.

[536] Ming-Deh A Huang. Trilinear maps for cryptography ii. *arXiv preprint arXiv:1810.03646*, 2018.

[537] Ming-Deh A Huang. Weil descent and cryptographic trilinear maps. *arXiv preprint arXiv:1908.06891*, 2019.

[538] Robert Gordon, Anthony John Power, and Ross Street. *Coherence for tricategories*, volume 558. American Mathematical Soc., 1995.

[539] Nick Gurski. Biequivalences in tricategories. *arXiv preprint arXiv:1102.0979*, 2011.

[540] Nils Carqueville, Catherine Meusburger, and Gregor Schaumann. 3-dimensional defect tqfts and their tricategories. *Advances in Mathematics*, 364:107024, 2020.

[541] Richard Garner and Nick Gurski. The low-dimensional structures formed by tricategories. In *Mathematical Proceedings of the Cambridge Philosophical Society*, pages 551–589. Cambridge University Press, 2009.

[542] Eugenia Cheng and Nick Gurski. The periodic table of n-categories ii: degenerate tricategories. *Cahiers de topologie et géométrie différentielle catégoriques*, 52(2):82–125, 2011.

[543] Alexander E Hoffnung. Spans in 2-categories: A monoidal tricategory. *arXiv preprint arXiv:1112.0560*, 2011.

[544] Guy R Biyogmam and Calvin Tcheka. From trigroups to leibniz 3-algebras. *arXiv preprint arXiv:1904.12030*, 2019.

[545] Jean-Louis Loday and María O Ronco. Trialgebras and families of polytopes. *arXiv preprint math/0205043*, 2002.

[546] Bouzid Mosbahi, Imed Basdouri, and Ahmed Zahari. Classification, derivations and centroids of low-dimensional associative trialgebras. *Galois Journal of Algebra*, 1(1):118–136, 2025.

[547] Ivan Kaygorodov and Yury Volkov. Degenerations of filippov algebras. *Journal of Mathematical Physics*, 61(2), 2020.

[548] Alexandre P Pojidaev. Enveloping algebras of filippov algebras. *Communications in Algebra*, 31(2):883–900, 2003.

[549] Hamid Darabi, Farshid Saeedi, and Mehdi Eshrati. A characterization of finite dimensional nilpotent filippov algebras. *Journal of Geometry and Physics*, 101:100–107, 2016.

[550] Ivan Borisovich Kaygorodov. (n+ 1)-ary derivations of semisimple filippov algebras. *Mathematical Notes*, 96(1):208–216, 2014.

[551] Antoni Chronowski. Ternary semigroups and ternary algebras. In *Functional equations in mathematical analysis*, pages 371–416. Springer, 2011.

[552] Anak Nongmanee, Kanruethai Jeenkaew, and Sorasak Leeratanavalee. Power ternary semigroups on ternary semihypergroups. *Thai Journal of Mathematics*, 24(2):141–157, 2026.

[553] Anak Nongmanee and Sorasak Leeratanavalee. Ternary menger algebras: A generalization of ternary semigroups. *Mathematics*, 9(5):553, 2021.

[554] Anak Nongmanee and Sorasak Leeratanavalee. Quaternary rectangular bands and representations of ternary semigroups. *Thai Journal of Mathematics*, 20(2):729–745, 2022.

[555] TK Dutta, S Kar, and K Das. Power ternary semirings. *Afrika Matematika*, 26(7):1483–1494, 2015.

[556] H Maschke. On ternary substitution-groups of finite order which leave a triangle unchanged. *American Journal of Mathematics*, 17(2):168–184, 1895.

[557] Nadezda Bazunova, Andrzej Borowiec, and Richard Kerner. Universal differential calculus on ternary algebras. *Letters in Mathematical Physics*, 67(3):195–206, 2004.

[558] John R Faulkner. A construction of lie algebras from a class of ternary algebras. *AMERICAN MATHEMATICAL SOCIETY*, 155(2), 1971.

[559] Hammimi Ataguema and Abdenacer Makhlouf. Deformations of ternary algebras. *J. Gen. Lie Theory Appl*, 1(1):41–55, 2007.

[560] William G Lister. Ternary rings. *Transactions of the American Mathematical Society*, 154:37–55, 1971.

[561] Heinrich Zettl. A characterization of ternary rings of operators. *Advances in Mathematics*, 48(2):117–143, 1983.

[562] Dalibor Kluckỳ and Libuše Marková. Ternary rings with zero associated to translation planes. *Czechoslovak Mathematical Journal*, 23(4):617–628, 1973.

[563] Irène Charon and Olivier Hudry. Optimal clustering of multipartite graphs. *Discrete applied mathematics*, 156(8):1330–1341, 2008.

[564] Michael Ferrara, Michael S Jacobson, Florian Pfender, and Paul S Wenger. Graph saturation in multipartite graphs. *arXiv preprint arXiv:1408.3137*, 2014.

[565] Allan Lo, Andrew Treglown, and Yi Zhao. Complete subgraphs in a multipartite graph. *Combinatorics, Probability and Computing*, 31(6):1092–1101, 2022.

[566] Ya-Lei Jin and Xiao-Dong Zhang. Complete multipartite graphs are determined by their distance spectra. *Linear Algebra and its Applications*, 448:285–291, 2014.

[567] Krishna Rani Maity and Madhumangal Pal. A novel approach on m-polar fuzzy set with dombi power aggregation operators to solve multi-attribute decision-making problems. *TWMS Journal of Applied and Engineering Mathematics*, 15(9):2313–2330, 2025.

[568] Damla Yılmaz, Hasret Yazarlı, and Young Bae Jun. m-polar fuzzy d-ideals on d-algebras. *Journal of Mathematical Extension*, 19, 2025.

[569] Madan Jagtap and Prasad Karande. The m-polar fuzzy topsis hybrid method for manufacturing decisions. *Scientific Reports*, 16(1):4479, 2026.

[570] Limin Wu. Multipolar interval-valued neutrosophic soft set for integrating sustainability into logistics: A performance-based evaluation of green supply chains. *Neutrosophic Sets and Systems*, 88(1):65, 2025.

[571] Asmaa Ghasoob Raoof, Qays Hatem Imran, S Khalil, Ali Younis Shakir, et al. On new classes of supra soft connectedness via m-polar neutrosophic topological spaces. *Neutrosophic Sets and Systems*, 94(1):12, 2025.

[572] Basavaraj V Hiremath, Durga Nagarajan, Satham Hussain S, Hossein Rashmanlou, and Farshid Mofidnakhaei. m-polar quadripartitioned neutrosophic graphs with applications in decision-making for mobile network selection. *Neutrosophic Sets and Systems*, 82(1):29, 2025.

[573] Atiqa Siraj, Khalid Naeem, and Broumi Said. Pythagorean m-polar fuzzy neutrosophic metric spaces. *Neutrosophic Sets and Systems*, 53(1):33, 2023.

[574] Takaaki Fujita and Florentin Smarandache. *A concise study of some superhypergraph classes*. Infinite Study, 2025.

[575] Jordan Barrett, Paweł Prałat, Aaron Smith, and François Théberge. Counting simplicial pairs in hypergraphs. *Journal of Complex Networks*, 13(4):cnaf021, 2025.

[576] Ginestra Bianconi. *Multilayer networks: structure and function*. Oxford university press, 2018.

[577] Mikko Kivelä, Alex Arenas, Marc Barthelemy, James P Gleeson, Yamir Moreno, and Mason A Porter. Multilayer networks. *Journal of complex networks*, 2(3):203–271, 2014.

[578] Manlio De Domenico. More is different in real-world multilayer networks. *Nature Physics*, 19(9):1247–1262, 2023.

[579] Emanuele Cozzo, Guilherme Ferraz De Arruda, Francisco Aparecido Rodrigues, and Yamir Moreno. *Multiplex networks: basic formalism and structural properties*, volume 2. Springer, 2018.

[580] Federico Battiston, Vincenzo Nicosia, and Vito Latora. The new challenges of multiplex networks: Measures and models. *The European Physical Journal Special Topics*, 226(3):401–416, 2017.

[581] Federico Battiston, Vincenzo Nicosia, and Vito Latora. Structural measures for multiplex networks. *Physical Review E*, 89(3):032804, 2014.

[582] Emanuele Cozzo, Mikko Kivelä, Manlio De Domenico, Albert Solé-Ribalta, Alex Arenas, Sergio Gómez, Mason A Porter, and Yamir Moreno. Structure of triadic relations in multiplex networks. *New Journal of Physics*, 17(7):073029, 2015.

[583] Thodsaporn Kumduang and Khwancheewa Wattanatripop. Multisorted algebras of trees of a weakly fixed variable. *European Journal of Pure and Applied Mathematics*, 18(2):5835–5835, 2025.

[584] Erkko Lehtonen, Reinhard Pöschel, and Tamás Waldhauser. Reflection-closed varieties of multisorted algebras and minor identities. *Algebra universalis*, 79(3):70, 2018.

[585] Jean-Sébastien Coron, Tancrede Lepoint, and Mehdi Tibouchi. Practical multilinear maps over the integers. In *Annual Cryptology Conference*, pages 476–493. Springer, 2013.

[586] Craig Gentry, Sergey Gorbunov, and Shai Halevi. Graph-induced multilinear maps from lattices. In *Theory of Cryptography Conference*, pages 498–527. Springer, 2015.

[587] Sanjam Garg, Craig Gentry, and Shai Halevi. Candidate multilinear maps from ideal lattices. In *Annual international conference on the theory and applications of cryptographic techniques*, pages 1–17. Springer, 2013.

[588] Fredrik Kjolstad, Shoaib Kamil, Stephen Chou, David Lugato, and Saman Amarasinghe. The tensor algebra compiler. *Proceedings of the ACM on Programming Languages*, 1(OOPSLA):1–29, 2017.

[589] Ivan V Arzhantsev and Jürgen Hausen. On the multiplication map of a multigraded algebra. *Mathematical Research Letters*, 14(1):129–136, 2007.

[590] Hailong Dao, Ezra Miller, Jonathan Montaño, Christopher O'Neill, and Kevin Woods. Quasipolynomial behavior via constructibility in multigraded algebra. *arXiv preprint arXiv:2512.18536*, 2025.

[591] Maria Manuel Clementino and Walter Tholen. Metric, topology and multicategory—a common approach. *Journal of Pure and Applied Algebra*, 179(1-2):13–47, 2003.

[592] Claudio Hermida. Representable multicategories. *Advances in Mathematics*, 151(2):164–225, 2000.

[593] Claudio Pisani. Unbiased multicategory theory. *arXiv preprint arXiv:2409.10150*, 2024.

[594] Miguel Couceiro, Jimmy Devillet, Jean-Luc Marichal, and Pierre Mathonet. Reducibility of n-ary semigroups: from quasitriviality towards idempotency. *Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry*, 63(1):149–166, 2022.

[595] Wiesław Dudek. Idempotents in n-ary semigroups. *Southeast Asian Bulletin of Mathematics*, 2001.

[596] Jukkrit Daengsaen and Sorasak Leeratanavalee. Semilattices of simple and regular n-ary semigroups. In *Semigroup Forum*, pages 294–314. Springer, 2023.

[597] Jukkrit Daengsaen and Sorasak Leeratanavalee. Regularities in ordered n-ary semihypergroups. *Mathematics*, 9(16):1857, 2021.

[598] Alexander M Gal'mak. N-ary groups. *arXiv preprint arXiv:1111.7191*, 2011.

[599] Wieslaw A Dudek. Remarks to glazek's results on n-ary groups. *arXiv preprint arXiv:0704.2749*, 2007.

[600] Wieslaw Dudek. On some old and new problems in n-ary groups. *Quasigroups and Related Systems*, 8(1):15–36, 2001.

[601] Florentin Smarandache. *Infinitesimal Punctures, 3: Infinitesimally Punctured Structures. S-MultiSpace and S-MultiStructure, Hybrid Variational Principles, and Topological Matter.* Infinite Study, 2026.

[602] MAO Linfan. Disentangling smarandache multispace and multisystem with information decoding. *International Journal of Mathematical Combinatorics*, 1:1, 2022.

[603] MAO Linfan. Smarandache geometry. *Smarandache Multispace & Multistructure*, 28:15, 2013.

[604] Florentin Smarandache. Neutrosophic transdisciplinarity—multi-space & multi-structure. *Smarandache Multispace & Multistructure*, 28:22, 2013.

[605] Florentin Smarandache. *S-denying a theory.* Infinite Study, 2013.

[606] MAO Linfan. Reality with smarandachely denied axiom. *Mathematical Combinatorics*, 3:1–19, 2021.

[607] Linfan Mao. On algebraic multi-group spaces. *arXiv preprint math/0510427*, 2005.

[608] Linfan Mao. On algebraic multi-ring spaces. *arXiv preprint math/0510478*, 2005.

[609] Josi A de Azcárraga and Josi M Izquierdo. n-ary algebras: A review with applications. *Journal of Physics A: Mathematical and Theoretical*, 43(29):293001, 2010.

[610] Hammimi Ataguema, Abdenacer Makhlouf, and Sergei Silvestrov. Generalization of n-ary nambu algebras and beyond. *Journal of Mathematical Physics*, 50(8), 2009.

[611] Renate Carlsson. N-ary algebras. *Nagoya Mathematical Journal*, 78:45–56, 1980.

[612] B Davvaz and T Vougiouklis. n-ary hypergroups. *Iranian Journal of Science*, 30(2):165–174, 2006.

[613] Bijan Davvaz, Wieslaw A Dudek, and S Mirvakili. Neutral elements, fundamental relations and n-ary hypersemigroups. *International Journal of Algebra and Computation*, 19(04):567–583, 2009.

[614] Yunqiang Yin, Jianming Zhan, and Piergiulio Corsini. Fuzzy roughness of n-ary hypergroups based on a complete residuated lattice. *Neural Computing and Applications*, 20(1):41–57, 2011.

[615] Violeta Leoreanu Fotea. Fuzzy rough n-ary subhypergroups. *Iranian Journal of Fuzzy Systems*, 5(3):45–56, 2008.

[616] Abbas Amini, Narjes Firouzkouhi, Marziyeh Nazari, Nader Ghareeb, Chun Cheng, and Bijan Davvaz. Artificial intelligent global online learning (gol) theory by generalized n-ary fuzzy relation. *Artificial Intelligence Review*, 57(3):68, 2024.

[617] Deepak Kartikey. *Fuzzy Sets and Their Applications.* BR Publications, 2026.

[618] Guillaume Allais. Generic level polymorphic n-ary functions. In *Proceedings of the 4th ACM SIGPLAN International Workshop on Type-Driven Development*, pages 14–26, 2019.

[619] Ivica Bošnjak and Rozália Madarász. On power structures. *Algebra and Discrete Mathematics*, (2):14–35, 2003.

[620] Takaaki Fujita. Metastructure, meta-hyperstructure, and meta-superhyper structure. *Journal of Computers and Applications*, 1(1):1–22, 2025.

[621] Takaaki Fujita, Ajoy Kanti Das, Suman Das, Sankar Prasad Mondal, and Volkan Duran. On edge-metagraphs. *Galoitica: Journal of Mathematical Structures and Applications*, 13(2):10–16, 2026.

[622] Takaaki Fujita, Ajoy Kanti Das, Suman Das, and Sankar Prasad Mondal. Meta-fuzzy graph, meta-neutrosophic graph, meta-digraph, and meta multigraph with some applications. *Neutrosophic Knowledge*, 9:93–101, 2026.

[623] Takaaki Fujita, Ajoy Kanti Das, and Suman Das. Meta-graphs and iterated meta-graphs: A novel framework for hierarchical network analysis and complex system modeling. *Applied Research Advances*, pages 1–12, 2026.

[624] Takaaki Fujita. Metahypergraphs, metasuperhypergraphs, and iterated metagraphs: Modeling graphs of graphs, hypergraphs of hypergraphs, superhypergraphs of superhypergraphs, and beyond. *Current Research in Interdisciplinary Studies*, 4(5):1–23, 2025.

[625] Takaaki Fujita. Molecular metagraph and molecular iterated metagraph in chemistry and biochemistry. *Journal of Applied Mathematics and Symbolic Science*, 2(1):01–15, 2026.

[626] Jiří Adámek, Horst Herrlich, and George E Strecker. Abstract and concrete categories: the joy of cats. *Repr. Theory Appl. Categ.*, page 1, 2006.

[627] G Max Kelly and Ross Street. Review of the elements of 2-categories. In *Category Seminar: Proceedings Sydney Category Theory Seminar 1972/1973*, pages 75–103. Springer, 2006.

[628] Stephen Lack. A 2-categories companion. In *Towards higher categories*, pages 105–191. Springer, 2009.

[629] John C Baez and Martin Neuchl. Higher dimensional algebra: I. braided monoidal 2-categories. *Advances in Mathematics*, 121(2):196–244, 1996.

[630] Mitchell Buckley. Fibred 2-categories and bicategories. *Journal of pure and applied algebra*, 218(6):1034–1074, 2014.

[631] María Emilia Descotte, Eduardo Julio Dubuc, and Martin Szyld. Sigma limits in 2-categories and flat pseudofunctors. *Advances in Mathematics*, 333:266–313, 2018.

[632] Francis Borceux. *Handbook of categorical algebra: Categories and structures*, volume 2. Cambridge University Press, 1994.

[633] Enrico Ghiorzi. Internal enriched categories. *Applied Categorical Structures*, 30(5):947–968, 2022.

[634] Claudio Hermida and Paulo Mateus. Paracategories i: internal paracategories and saturated partial algebras. *Theoretical Computer Science*, 309(1-3):125–156, 2003.

[635] Mark W Maier. Architecting principles for systems-of-systems. *Systems Engineering: The Journal of the International Council on Systems Engineering*, 1(4):267–284, 1998.

[636] Claus Ballegaard Nielsen, Peter Gorm Larsen, John Fitzgerald, Jim Woodcock, and Jan Peleska. Systems of systems engineering: Basic concepts, model-based techniques, and research directions. *ACM Computing Surveys (CSUR)*, 48(2):1–41, 2015.

[637] Charles B Keating and Polinpapilinho F Katina. Systems of systems engineering: prospects and challenges for the emerging field. *International Journal of System of Systems Engineering*, 2(2-3):234–256, 2011.

[638] Judith S Dahmann and Kristen J Baldwin. Understanding the current state of us defense systems of systems and the implications for systems engineering. In *2008 2nd Annual IEEE Systems Conference*, pages 1–7. IEEE, 2008.

[639] Judith S Dahmann. Systems of systems characterization and types. *Systems of Systems Engineering for NATO Defence Applications (STO-EN-SCI-276)*, pages 1–14, 2015.

[640] Daniel A DeLaurentis. A taxonomy-based perspective for systems of systems design methods. In *2005 IEEE international conference on systems, man and cybernetics*, volume 1, pages 86–91. IEEE, 2005.

[641] Jianxi Gao, Daqing Li, and Shlomo Havlin. From a single network to a network of networks. *National Science Review*, 1(3):346–356, 2014.

[642] S Havlin, DY Kenett, A Bashan, J Gao, and HE Stanley. Vulnerability of network of networks. *The European Physical Journal Special Topics*, 223(11):2087–2106, 2014.

[643] Gaogao Dong, Yanting Luo, Yangyang Liu, Fan Wang, Huanmei Qin, and André LM Vilela. Percolation behaviors of a network of networks under intentional attack with limited information. *Chaos, Solitons & Fractals*, 159:112147, 2022.

[644] Jianxi Gao, Sergey V Buldyrev, H Eugene Stanley, Xiaoming Xu, and Shlomo Havlin. Percolation of a general network of networks. *Physical Review E—Statistical, Nonlinear, and Soft Matter Physics*, 88(6):062816, 2013.

[645] Shangjia Dong, Haizhong Wang, Alireza Mostafizi, and Xuan Song. A network-of-networks percolation analysis of cascading failures in spatially co-located road-sewer infrastructure networks. *Physica A: Statistical Mechanics and its Applications*, 538:122971, 2020.

[646] Jianxi Gao, Sergey V Buldyrev, H Eugene Stanley, and Shlomo Havlin. Networks formed from interdependent networks. *Nature physics*, 8(1):40–48, 2012.

[647] Takaaki Fujita. Metafuzzy, metaneutrosophic, metasoft, and metarough set. *Journal of Bangladesh Academy of Sciences*, 50(1):107–145, 2026.

[648] Ramakrishnan Mukundan. Quaternions. In *Advanced methods in computer graphics: with examples in OpenGL*, pages 77–112. Springer, 2012.

[649] Titouan Parcollet, Mohamed Morchid, and Georges Linares. A survey of quaternion neural networks: T. parcollet et al. *Artificial Intelligence Review*, 53(4):2957–2982, 2020.

[650] JP Ward. From complex numbers to quaternions. *Teaching Mathematics and its Applications: An International Journal of the IMA*, 16(2):79–84, 1997.

[651] Ricardo Augusto Watanabe, Cibele Cristina Trinca Watanabe, and Estevão Esmi. Fuzzy octonion numbers: Some analytical properties. In *International Fuzzy Systems Association World Congress*, pages 716–726. Springer, 2019.

[652] Roman Jacome, Kumar Vijay Mishra, Brian M Sadler, and Henry Arguello. Octonion phase retrieval. *IEEE Signal Processing Letters*, 31:1615–1619, 2024.

[653] Florentin Smarandache and WB Kandasamy. *Finite Neutrosophic Complex Numbers.* Zip Publishing, Ohio, 2011.

[654] Florentin Smarandache and WB Kandasamy. *Exploring the Extension of Natural Operations on Intervals, Matrices and Complex Numbers.* Zip Publishing, Ohio, 2012.

[655] Florentin Smarandache and WB Kandasamy. *Non Associative Algebraic Structures Using Finite Complex Numbers.* Zip Publishing, Ohio, 2012.

[656] WB Vasantha Kandasamy, K Ilanthenral, and Florentin Smarandache. *Complex valued graphs for soft computing.* Infinite Study, 2017.

[657] Dan E Tamir, Naphtali D Rishe, and Abraham Kandel. Complex fuzzy sets and complex fuzzy logic an overview of theory and applications. *Fifty years of fuzzy logic and its applications*, pages 661–681, 2015.

[658] Omolbanin Yazdanbakhsh and Scott Dick. A systematic review of complex fuzzy sets and logic. *Fuzzy sets and systems*, 338:1–22, 2018.

[659] Xueling Ma, Jianming Zhan, Madad Khan, Muhammad Zeeshan, Saima Anis, and Abdul Sami Awan. Complex fuzzy sets with applications in signals. *Computational and Applied Mathematics*, 38(4):150, 2019.

[660] R Rajalakshmi, M Gayathri Lakshmi, Mana Donganont, Prasanta Raut, and Bhabani S Mohanty. Application of complex neutrosophic sets to real-world decision making problems. *Neutrosophic Sets and Systems*, 91(1):15, 2025.

[661] RK Saini, F Smarandache, Moh Kasim, and Ashik Ahirwar. Decision-making modeling in agro-food systems using type-2 interval-valued linguistic complex neutrosophic sets. *Neutrosophic Sets & Systems*, 87:948, 2025.

[662] Luu Quoc Dat, Nguyen Tho Thong, Mumtaz Ali, Florentin Smarandache, Mohamed Abdel-Basset, Hoang Viet Long, et al. Linguistic approaches to interval complex neutrosophic sets in decision making. *IEEE access*, 7:38902–38917, 2019.

[663] Said Broumi, Assia Bakali, Mohamed Talea, Florentin Smarandache, Prem Kumar Singh, Vakkas Uluçay, and Mohsin Khan. Bipolar complex neutrosophic sets and its application in decision making problem. In *Fuzzy multi-criteria decision-making using neutrosophic sets*, pages 677–710. Springer, 2018.

[664] Volker Scheidemann. *Introduction to complex analysis in several variables.* Springer, 2005.

[665] Meighan I Dillon. *Linear algebra: vector spaces and linear transformations*, volume 57. American Mathematical Society, 2022.

[666] Robert M Yamaleev. Complex algebras on n-order polynomials and generalizations of trigonometry, oscillator model and hamilton dynamics. *Advances in Applied Clifford Algebras*, 15(1):123–150, 2005.

[667] Leonard Eugene Dickson. *Algebras and their arithmetics*, volume 15. University of Chicago Press, 1923.

[668] Daniel Alfsmann, Heinz G Göckler, Stephen J Sangwine, and Todd A Ell. Hypercomplex algebras in digital signal processing: Benefits and drawbacks. In *2007 15th European Signal Processing Conference*, pages 1322–1326. IEEE, 2007.

[669] Aiguo Wu. On con-numbers and con-matrices with applications to control systems. *Journal of Systems Science and Complexity*, 35(1):32–57, 2022.

[670] Thomas Vetterlein. Orthogonality spaces of finite rank and the complex hilbert spaces. *International Journal of Geometric Methods in Modern Physics*, 16(05):1950080, 2019.

[671] Raphael Gervais Lavoie, Louis Marchildon, and Dominic Rochon. Infinite-dimensional bicomplex hilbert spaces. *Annals of Functional Analysis*, 1(2):75–91, 2010.

[672] MP Vaughan. Time evolution of quantum gates and the necessity of complex numbers. *arXiv preprint arXiv:2604.17086*, 2026.

[673] Earl Berkson. A characterization of complex hilbert spaces. *Bulletin of the London Mathematical Society*, 2(3):313–315, 1970.

[674] Eugenio Calabi. Construction and properties of some 6-dimensional almost complex manifolds. *Transactions of the American Mathematical Society*, 87(2):407–438, 1958.

[675] SS Chern and JK Moser. Real hypersurfaces in complex manifolds. *A MATHEMATICIAN ANDUIS MATHEMATICAL WORK*, page 385, 1974.

[676] Friedrich Hirzebruch. Some problems on differentiable and complex manifolds. *Annals of Mathematics*, 60(2):213–236, 1954.

[677] László Lempert. Loop spaces as complex manifolds. *Journal of Differential Geometry*, 38(3):519–543, 1993.

[678] Newton S Hawley. A theorem on compact complex manifolds. *Annals of Mathematics*, 52(3):637–641, 1950.

[679] LN APOSTOLOVA, MS MARINOV, and KP PETROV. Theorems for extension on manifolds with almost complex structures. In *Perspectives Of Complex Analysis, Differential Geometry And Mathematical Physics*, pages 36–44. World Scientific, 2001.

[680] Boris Rosenfeld and Bill Wiebe. *Geometry of Lie groups.* Springer Science & Business Media, 2013.

[681] Alistair Savage. Introduction to lie groups. *Course notes of MAT*, 1411, 2015.

[682] Pavel Etingof. Lie groups and lie algebras. *arXiv preprint arXiv:2201.09397*, pages 18–34, 2022.

[683] Heinz-Otto Peitgen and Peter H Richter. *The beauty of fractals: images of complex dynamical systems*. Springer Science & Business Media, 1986.

[684] John Erik Fornæss and Nessim Sibony. Complex dynamics in higher dimension. *Several complex variables (Berkeley, CA, 1995–1996)*, 37:273–296, 1999.

[685] Gaven J Martin. Complex dynamical systems. *International Journal of Mathematical Education in Science and Technology*, 25(6):879–897, 1994.

[686] Seán Dineen. Extreme integral polynomials on a complex banach space. *Mathematica Scandinavica*, 92(1):129–140, 2003.

[687] Angus E Taylor. Analysis in complex banach spaces. *Bull. Amer. Math. Soc.*, 49(12):652–669, 1943.

[688] Mark Elin, Simeon Reich, and David Shoikhet. Complex dynamical systems and the geometry of domains in banach spaces. *Dissertationes Mathematicae*, 427:1–62, 2004.

[689] Andrew J Nicas. Classifying pairs of lagrangians in a hermitian vector space. *Topology and its Applications*, 42(1):71–81, 1991.

[690] Elisha Falbel, Jean-Pierre Marco, and Florent Schaffhauser. Classifying triples of lagrangians in a hermitian vector space. *Topology and its Applications*, 144(1-3):1–27, 2004.

[691] Massimo Vaccaro. Subspaces of a para-quaternionic hermitian vector space. *International Journal of Geometric Methods in Modern Physics*, 8(07):1487–1506, 2011.

[692] Peter Lancaster and Leiba Rodman. Canonical forms for hermitian matrix pairs under strict equivalence and congruence. *SIAM Review*, 47(3):407–443, 2005.

[693] Richard Bellman and Ky Fan. On systems of linear inequalities in hermitian matrix variables. *Convexity*, 7:1–11, 1963.

[694] Wenbao Ai, Yongwei Huang, and Shuzhong Zhang. New results on hermitian matrix rank-one decomposition. *Mathematical programming*, 128(1):253–283, 2011.

[695] Paolo Rossi, Massimo Campostrini, and Ettore Vicari. The large-n expansion of unitary-matrix models. *Physics reports*, 302(4):143–209, 1998.

[696] Sameer Murthy. Unitary matrix models, free fermion ensembles, and the giant graviton expansion. *arXiv preprint arXiv:2202.06897*, 2022.

[697] Vipul Periwal and Danny Shevitz. Exactly solvable unitary matrix models: multicritical potentials and correlations. *Nuclear Physics B*, 344(3):731–746, 1990.

[698] S Kharchev and A Mironov. Integrable structures of unitary matrix models. *International Journal of Modern Physics A*, 7(20):4803–4824, 1992.

[699] Jean Dieudonné. On the structure of unitary groups. *Transactions of the American Mathematical Society*, 72(3):367–385, 1952.

[700] John F Adams. On chern characters and the structure of the unitary group. In *Mathematical Proceedings of the Cambridge Philosophical Society*, pages 189–199. Cambridge University Press, 1961.

[701] Josef Paldus. Unitary group approach to many-electron correlation problem. In *The Unitary Group for the evaluation of electronic energy matrix elements*, pages 1–50. Springer, 1981.

[702] Libor Šnobl and Pavel Winternitz. *Classification and identification of Lie algebras*, volume 33. American Mathematical Society Providence, RI, 2014.

[703] Dimitrij Akhiezer. *Lie group actions in complex analysis*. Springer Science & Business Media, 2012.

[704] V Jurdjevic. Kowalewski top and complex lie algebras. *Analysis and Mathematical Physics*, 11(4):173, 2021.

[705] MICHAEL EASTWOOD and Justin Sawon. complex projective space. *Invitations to geometry and topology*, page 126, 2002.

[706] Shoshichi Kobayashi and Takushiro Ochiai. Characterizations of complex projective spaces and hyperquadrics. *Journal of Mathematics of Kyoto University*, 13(1):31–47, 1973.

[707] Makoto Kimura. Real hypersurfaces and complex submanifolds in complex projective space. *Transactions of the American Mathematical Society*, 296(1):137–149, 1986.

[708] Atsushi Shimizu. On complex tori with many endomorphisms. *Tsukuba Journal of Mathematics*, 8(2):297–318, 1984.

[709] Frans Oort and Yuri G Zarhin. Complex tori. *Indagationes Mathematicae*, 7(4):473–487, 1996.

[710] Donghoon Jang. Four dimensional almost complex torus manifolds. *arXiv preprint arXiv:2310.11024*, 2023.

[711] Takaaki Fujita and Florentin Smarandache. A dynamic survey of soft set theory and its extensions. *arXiv preprint arXiv:2602.21268*, 2026.

[712] Hamed Ahmadzade, Rong Gao, and Habib Naderi. On the convergence of complex uncertain random sequences. *Journal of Intelligent & Fuzzy Systems*, 35(2):2459–2467, 2018.

[713] Siegfried Gähler and Werner Gähler. Fuzzy real numbers. *Fuzzy Sets and Systems*, 66(2):137–158, 1994.

[714] Binod Chandra Tripathy and Amar Jyoti Dutta. Bounded variation double sequence space of fuzzy real numbers. *Computers & Mathematics with Applications*, 59(2):1031–1037, 2010.

[715] Omar A AbuAarqob, Nabil T Shawagfeh, and Omar A AbuGhneim. Functions defined on fuzzy real numbers according to zadehs extension. In *International Mathematical Forum*, pages 763–776, 2008.

[716] Mohammad Abobala. Foundations of neutrosophic number theory. *Neutrosophic Sets and Systems*, 39(1):10, 2021.

[717] Mohammad Abobala and Ahmad Hatip. An algebraic approach to neutrosophic euclidean geometry. *Neutrosophic Sets and Systems*, 43(1):10, 2021.

[718] Mohammad Abobala. Neutrosophic real inner product spaces. *Neutrosophic Sets and Systems*, 43:225–246, 2021.

[719] Abdallah Shihadeh, Khaled Ahmad Mohammad Matarneh, Raed Hatamleh, Randa Bashir Yousef Hijazeen, Mowafaq Omar Al-Qadri, and Abdallah Al-Husban. An example of two-fold fuzzy algebras based on neutrosophic real numbers. *Neutrosophic Sets and Systems*, 67:169–178, 2024.

[720] John Stillwell. Vectors and euclidean spaces. In *The Four Pillars of Geometry*, pages 65–87. Springer, 2005.

[721] Christopher Terence John Dodson and Timothy Poston. Real vector spaces. In *Tensor Geometry: The Geometric Viewpoint and its Uses*, pages 18–42. Springer, 1991.

[722] Abraham Adrian Albert. Absolute valued real algebras. *Annals of Mathematics*, 48(2):495–501, 1947.

[723] Jie Li. The derived-discrete algebras over the real numbers. *Algebras and Representation Theory*, 26(4):1141–1162, 2023.

[724] Tomer Moran, Shay Moran, and Shlomo Moran. Elementary derivations of the real composition algebras. *arXiv preprint arXiv:2001.07521*, 2020.

[725] Yu Chen. Matrix representations of the real numbers. *Linear Algebra and Its Applications*, 536:174–185, 2018.

[726] Guillaume Melquiond. Proving bounds on real-valued functions with computations. In *International Joint Conference on Automated Reasoning*, pages 2–17. Springer, 2008.

[727] Kalvis Apsītis, Rūsiņš Freivalds, and Carl H Smith. On the inductive inference of real valued functions. In *Proceedings of the eighth annual conference on Computational learning theory*, pages 170–177, 1995.

[728] Eiju Hirowatari and Setsuo Arikawa. Inferability of recursive real-valued functions. In *International Workshop on Algorithmic Learning Theory*, pages 18–31. Springer, 1997.

[729] Walter Benz. *Classical geometries in modern contexts: geometry of real inner product spaces*. Springer, 2007.

[730] Nikolaos A Papadopoulos and Florian Scheck. Positive operators–isometries–real inner product spaces. In *Linear Algebra for Physics*, pages 323–342. Springer, 2024.

[731] Abraham A Ungar. Extension of the unit disk gyrogroup into the unit ball of any real inner product space. *Journal of mathematical analysis and applications*, 202(3):1040–1057, 1996.

[732] Keiko Narita, Noboru Endou, and Yasunari Shidama. Bidual spaces and reflexivity of real normed spaces. *Formalized Mathematics*, 22(4):303–311, 2014.

[733] Pascal Bégout. The dual space of a complex banach space restricted to the field of real numbers. *arXiv preprint arXiv:2206.09630*, 2022.

[734] Reinhard Wolf. Averaging distances in real quasihypermetric banach spaces of finite dimension. *Israel Journal of Mathematics*, 110(1):125–151, 1999.

[735] Richard M Aron and Petr Hájek. Odd degree polynomials on real banach spaces. *Positivity*, 11(1):143–153, 2007.

[736] Bancha Panyanak, Chainarong Khunpanuk, Nattawut Pholasa, and Nuttapol Pakkaranang. Dynamical inertial extragradient techniques for solving equilibrium and fixed-point problems in real hilbert spaces. *Journal of Inequalities and Applications*, 2023:1–36, 2023.

[737] Yanhong Fei, Yingjie Liu, Chentao Jia, Zhengyu Li, Xian Wei, and Mingsong Chen. A survey of geometric optimization for deep learning: From euclidean space to riemannian manifold. *ACM Computing Surveys*, 57:1 – 37, 2023.

[738] Yurun Tian, Bin Fan, and Fuchao Wu. L2-net: Deep learning of discriminative patch descriptor in euclidean space. *2017 IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, pages 6128–6136, 2017.

[739] Guangyi Zhang and Ali Etemad. Spatio-temporal eeg representation learning on riemannian manifold and euclidean space. *IEEE Transactions on Emerging Topics in Computational Intelligence*, 8:1469–1483, 2020.

[740] Agata Darmochwał. The euclidean space. *Formalized Mathematics*, 2(4):599–603, 1991.

[741] Arkadij L Onishchik and Ernest B Vinberg. *Lie groups and algebraic groups*. Springer Science & Business Media, 2012.

[742] Farzad Shahi. Real lie groups of finite dimension. *arXiv preprint arXiv:2108.13793*, 2021.

[743] Grzegorz Jagiella. Definable topological dynamics and real lie groups. *Mathematical Logic Quarterly*, 61(1-2):45–55, 2015.

[744] Hirotaka Fujimoto. On holomorphic maps into a real lie group of holomorphic transformations. *Nagoya Mathematical Journal*, 40:139–146, 1970.

[745] Evelyn Weimar-Woods. The three-dimensional real lie algebras and their contractions. *Journal of mathematical physics*, 32(8):2028–2033, 1991.

[746] Kenji Atsuyama. Another construction of real simple lie algebras. *Kodai Mathematical Journal*, 6(1):122–133, 1983.

[747] Ulrik Buchholtz and Egbert Rijke. The real projective spaces in homotopy type theory. In *2017 32nd Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–8. IEEE, 2017.

[748] PRAVANJAN KUMAR RANA, TUSHAR KANTI GUHA, and BHASKAR MANDAL. Study of real projective space and its applications. *BULLETIN OF THE CALCUTTA MATHEMATICAL SOCIETY*, 117(3):334–339, 2025.

[749] Simone Giombi, Himanshu Khanchandani, and Xinan Zhou. Aspects of cfts on real projective space. *Journal of Physics A: Mathematical and Theoretical*, 54(2):024003, 2021.

[750] Beresford N Parlett. Symmetric matrix pencils. *Journal of Computational and Applied Mathematics*, 38(1-3):373–385, 1991.

[751] Rosemary Anne Bailey and JC Gower. Approximating a symmetric matrix. *Psychometrika*, 55(4):665–675, 1990.

[752] Zhang Yi, Yan Fu, and Hua Jin Tang. Neural networks based approach for computing eigenvectors and eigenvalues of symmetric matrix. *Computers & Mathematics with Applications*, 47(8-9):1155–1164, 2004.

[753] Ann Sinap and Walter Van Assche. Orthogonal matrix polynomials and applications. *Journal of Computational and Applied Mathematics*, 66(1-2):27–52, 1996.

[754] Andre I. Khuri and IJ Good. The parameterization of orthogonal matrices: A review mainly for statisticians. *South African Statistical Journal*, 23(2):231–250, 1989.

[755] Raymond EAC Paley. On orthogonal matrices. *Journal of mathematics and Physics*, 12(1-4):311–320, 1933.

[756] Alan T James. Normal multivariate analysis and the orthogonal group. *The Annals of Mathematical Statistics*, 25(1):40–75, 1954.

[757] Tsuneo Tamagawa. On the structure of orthogonal groups. *American Journal of Mathematics*, 80(1):191–197, 1958.

[758] Gustav Lehrer and Ruibin Zhang. The second fundamental theorem of invariant theory for the orthogonal group. *Annals of Mathematics*, pages 2031–2054, 2012.

A Comprehensive Survey and Generalization of Linguistic, MOD, Interval, Subset, and Higher-Order Structures

Takaaki Fujita [1] * and Florentin Smarandache[2]
[1] Independent Researcher, Tokyo, Japan.
Email: Takaaki.fujita060@gmail.com
[2] University of New Mexico, Gallup Campus, NM 87301, USA.
Email: fsmarandache@gmail.com

Abstract

Mathematical structures provide systematic frameworks for representing linguistic information, modular systems, interval-valued data, subset-based constructions, and higher-order relationships. In this work, we present a unified survey and systematic generalization of linguistic, MOD, interval, subset, multi-component, meta, complex, and real structures, together with several uncertainty-oriented and higher-order extensions.

Linguistic structures are examined from natural and formal language models to fuzzy, neutrosophic, graph-based, and related mathematical representations. MOD structures are organized through algebraic and combinatorial constructions based on modular arithmetic, including graphs, HyperGraphs, and SuperHyperGraphs. Interval structures are investigated as models of bounded variability, imprecision, and uncertainty through algebraic, graph-theoretic, and interval-valued extensions.

We further study Subset Structures obtained by lifting ordinary operations to families of nonempty subsets, and clarify their distinction from and relationship with HyperStructures. Iterated power-set constructions are used to generate hierarchical subset structures and to explain their connections with higher-order and SuperHyperStructures. BiStructures, TriStructures, MultiStructures, and Iterated MultiStructures are considered as systematic multi-component and hierarchical extensions of ordinary mathematical structures. MetaStructures and Iterated MetaStructures provide an additional level of abstraction by treating entire mathematical structures as objects, while complex and real structures illustrate scalar-based and algebraic realizations of the general framework.

Particular attention is given to definitions, structural relationships, special cases, reductions, closure properties, and well-definedness. Overall, the proposed organization provides a common mathematical perspective for comparing diverse constructions and developing further models in uncertainty theory, discrete mathematics, algebra, network science, and higher-order mathematical structures.

Keywords: Linguistic Structures, MOD Structures, Interval Structures, Subset Structures, MultiStructures, MetaStructures, HyperStructures, SuperHyperStructures, Complex Structures, Real Structures, Uncertainty Modeling, Higher-Order Structures

ABSTRACT

This book presents a unified survey and systematic generalization of mathematical structures used to represent linguistic information, modular systems, interval-valued data, uncertainty, and higher-order relationships. It examines linguistic, MOD, interval, subset, multi-component, meta, complex, and real structures, together with fuzzy, neutrosophic, graph-based, HyperStructure, and SuperHyperStructure extensions.

By clarifying definitions, structural relationships, reductions, closure properties, and well-definedness, the volume establishes a common framework for comparing diverse constructions and developing new models in uncertainty theory, discrete mathematics, algebra, network science, and higher-order mathematical systems.

ISBN 978-197250253-2

9 781972 502532